Question 2

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters μ and σ^2 are unknown. A confidence interval for σ^2 can be found as follows.

We know that $(n-1)S^2/\sigma^2$ is a random variable with a $\chi^2(n-1)$ distribution. Thus we can find constants a and b so that $P((n-1)S^2/\sigma^2 < b) = 0.975$ and $P(a < (n-1)S^2/\sigma^2 < b) = 0.95$.

(a) Show that this second probability statement can be written as:

$$P\left((n-1)S^2/b < \sigma^2 < (n-1)S^2/a\right) = 0.95.$$

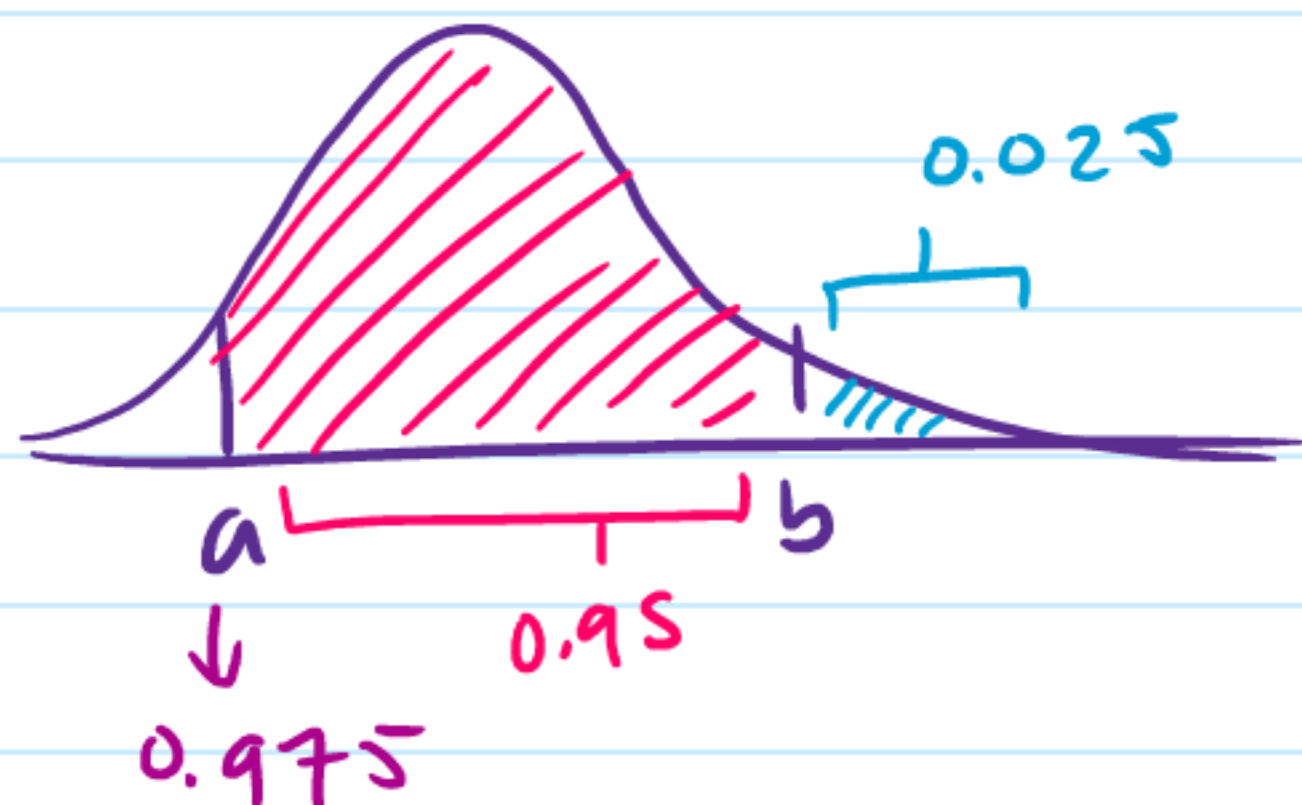(b) If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for σ^2.

a) note that $\quad a < \dfrac{(n-1)S^2}{\sigma^2} < b$

$\Rightarrow \quad \dfrac{1}{a} > \dfrac{\sigma^2}{(n-1)S^2} > \dfrac{1}{b} \quad \Rightarrow \quad \dfrac{(n-1)S^2}{a} > \sigma^2 > \dfrac{(n-1)S^2}{b}$

Hence $\quad 0.95 = P\left(a < \dfrac{(n-1)S^2}{\sigma^2} < b\right) = P\left(\dfrac{(n-1)S^2}{b} < \sigma^2 < \dfrac{(n-1)S^2}{a}\right)$

b) $\quad (n-1)s^2 = (8)(7.93) = 63.44$

note:

$\chi^2_{0.025}$	$\chi^2_{0.010}$	$\chi^2_{0.005}$	df
5.02389	6.63490	7.87944	1
7.37776	9.21034	10.5966	2
9.34840	11.3449	12.8381	3
11.1433	13.2767	14.8602	4
12.8325	15.0863	16.7496	5
14.4494	16.8119	18.5476	6
16.0128	18.4753	20.2777	7
17.5346	20.0902	21.9550	8

for b

df	$\chi^2_{0.995}$	$\chi^2_{0.990}$	$\chi^2_{0.975}$
1	0.0000393	0.0001571	0.0009821
2	0.0100251	0.0201007	0.0506356
3	0.0717212	0.114832	0.215795
4	0.206990	0.297110	0.484419
5	0.411740	0.554300	0.831211
6	0.675727	0.872085	1.237347
7	0.989265	1.239043	1.68987
8	1.344419	1.646482	2.17973

for a

$\dfrac{63.44}{a} = \dfrac{63.44}{2.17973} = 29.1045$

$\dfrac{63.44}{b} = \dfrac{63.44}{17.5346} = 3.618$

Hence we are 95% confident that σ^2 is within $(3.618, 29.1045)$