STA380 Term Test 1 Practice Version A

Solutions to the practice tests will not be posted. Students are encouraged to ask questions on Piazza or during office hours; however, they should include evidence of their own attempts when doing so. The primary purpose of providing the practice test is to help students become familiar with the format, length, and style of the questions, as well as to offer an opportunity for additional practice. Several questions are directly from the lecture slides, practice problems, and quizzes. The difficulty of the practice test and the actual test may differ.

Question 1

Let X be a continuous random variable with the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{8} & 0 < x \leq 2, \\ \frac{x}{8} & 2 < x \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) **[2 points]** Find $F_X(x)$, the cumulative distribution function of X. **Show all steps. Circle the final answer.**

(b) **[2 points]** Derive $F_X^{-1}(x)$. **Show all steps. Circle the final answer.**

(c) **[2 points]** Write an algorithm to generate a sample of size $n = 10^4$ from the distribution of X using the *inverse transform method*. **Be precise.**

(d) **[2 points]** Write out the `R` code for the algorithm.

Question 2

In this question, we want to use the *acceptance-rejection* method to generate a sample from a random variable X with the following target probability density function:

$$f(x) = \frac{2}{\sqrt{\pi}} x^{1/2} e^{-x}, \quad x > 0.$$

Let the trial/candidate probability density function be:

$$g(x) = \frac{2}{3} e^{-2x/3}, \quad x > 0.$$

(a) **[3 points]** What is $c = \max\left\{\frac{f(x)}{g(x)}\right\}$, where $f(x)$ is the target distribution and $g(x)$ is the trial distribution? **You do <u>not</u> need to show that the value you found is a maximum.**

(b) **[2 points]** Write out the *acceptance-rejection* algorithm to generate a sample from X. **Be precise.**

(c) **[3 points]** Write out the `R` code for the algorithm. Assume $n = 10^4$.

(d) **[2 points]** If the first uniform in the acceptance-rejection algorithm was $U = 0.62$ and the first candidate draw was $Y = 0.02$. Would you accept or reject this candidate? **You must specify *accept* or *reject*. Show all steps.**

Question 3

[2 points] If $Z \sim Normal(0, 1)$ and $W \sim \chi^2_{(v)}$ where v represents the degrees of freedom, and Z, W are independent then

$$\frac{Z}{\sqrt{W/v}} \sim t_{(v)}$$

Where $t_{(v)}$ represents the Student t distribution with v degrees of freedom. Use `rnorm()` and `rchisq(n, df)` to generate $n = 10^4$ samples from the $t_{(15)}$ distribution.

In `rchisq(n, df)`, `n` represents the number of observations and `df` represents the degrees of freedom.

Question 4

Prove the following:

(a) **[3 points]** If X is a continuous random variable with cdf $F_X(x)$ then $U := F_X(X) \sim Uniform(0, 1)$.

(b) **[3 points]** Let $U \sim Uniform(0, 1)$. Define $X = F^{-1}(U)$, where F is a cdf. Then, F is the cdf of X.

Question 5

Suppose $X_1 \sim Normal(0, 1)$, $X_2 \sim Normal(3, 1)$, and X_1, X_2 are independent. Consider the following questions:

(a) **[2 points]** Is the following mixture a valid cumulative distribution function? Why or why not? **Justify your answer.**

$$F_X(x) := 0.3F_{X_1}(x) + 0.9F_{X_2}(x)$$

(b) **[2 points]** Write `R` code to generate $n = 10^4$ samples of the previous mixture "cumulative" distribution function (regardless if it's a valid cumulative distribution function).

(c) **[2 points]** Is the following convolution a valid random variable? Why or why not? **Justify your answer.**

$$S := 0.3X_1 + 0.9X_2$$

(d) **[2 points]** Write `R` code to generate $n = 10^4$ samples of the previous convolution (regardless if it's a valid random variable or not).

Question 6

The probability density function (pdf) of the inverse-gamma distribution is given by

$$g_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}(1/x)^{\alpha+1}\exp(-\beta/x).$$

Suppose $\alpha = 2, \beta = 3$.

(a) **[3 points]** Provide a detailed algorithm to obtain a Monte Carlo estimate of $\mathbb{P}(2 \leq X \leq 5)$ using a Monte Carlo sample size of $m = 10^4$. **Show all steps. Be precise.**

(b) **[2 points]** Write out the `R` code for the algorithm.