STA380 Practice Problems for Quiz 3

These problems are not to be handed in, but they are for extra practice for students to be prepared for the quiz.

1. Consider the following probability density function:

$$f_Y(y) = \begin{cases} \frac{1}{8} & 0 < y \leq 2, \\ \frac{y}{8} & 2 < y \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

   Use the simple Monte Carlo estimator to **estimate** the expected value. Write the algorithm and the corresponding `R` codes. Use $n = 10^4$. Compare it to the true value.

2. Consider the Weibull distribution:

$$f_X(x) = \begin{cases} \frac{\alpha}{\lambda} \left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-(x/\lambda)^{\alpha}} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Where α represents the shape parameter and λ represents the scale parameter. Let $X \sim Weibull(\alpha = 2, \lambda = 3)$. Use the **hit or miss** approach to compute $\mathbb{P}(0 < X < 3)$.

   (a) Write out the algorithm.

   (b) Using $n = 10^4$, code the algorithm in `R` using `rweibull()`. Compare this value with `pweibull()`. *Hint: please use and read* `help(rweibull)` *and* `help(pweibull)` *to understand how this function works.)*

   (c) Using $n = 10^4$, code the algorithm in `R` **without** using `rweibull()`. Compare this value with `pweibull()`. *Hint: in the previous set of practice problems, we generated values from the weibull distrbution using the acceptance-rejection method.*

Below are the solutions.

1. Note that,

$$
\begin{aligned}
\mathbb{E}[Y] &= \int_0^2 y\frac{1}{8}dy + \int_2^4 y\frac{y}{8}dy \\
&= (2-0)\int_0^2 \frac{1}{(2-0)}y\frac{1}{8}dy + (4-2)\int_2^4 \frac{1}{(4-2)}y\frac{y}{8}dy \\
&= 2\mathbb{E}[U_1/8] + 2\mathbb{E}[U_2^2/8]
\end{aligned} \tag{1}
$$

Where $U_1 \sim Uniform(0,2)$ and $U_2 \sim Uniform(2,4)$. Suppose we had a random sample $U_{11}, U_{12}, \ldots, U_{1n} \overset{i.i.d.}{\sim} Uniform(0,2)$ and $U_{21}, U_{22}, \ldots, U_{2n} \overset{i.i.d.}{\sim} Uniform(2,4)$. Using the SLLN, our estimator will be:

$$
\widehat{\mathbb{E}[Y]} = \frac{2}{n}\sum_{i=1}^{n}\frac{u_{1i}}{8} + \frac{2}{n}\sum_{i=1}^{n}\frac{u_{2i}^2}{8}
$$

The algorithm is as follows:

(a) Generate $u_{11}, \ldots, u_{1m} \sim Uniform(0,2)$.

(b) Generate $u_{21}, \ldots, u_{2m} \sim Uniform(2,4)$.

(c) Deliver $\widehat{\mathbb{E}[g(Y)]} = \frac{2}{n}\sum_{i=1}^{n}\frac{u_{1i}}{8} + \frac{2}{n}\sum_{i=1}^{n}\frac{u_{2i}^2}{8}$.

The code for estimation is:

```
n <- 10^4
u1 <- runif(n, 0, 2)
u2 <- runif(n, 2, 4)
est <- 2*mean(u1/8) + 2*mean(u2^2/8)
```

2. (a)
- Generate a random sample $X_1, \ldots, X_n$ from the $Weibull(\alpha = 2, \lambda = 3)$.
- For each observation X_i, compute:

$$
I(0 \le X_i \le 3) = \begin{cases} 1 & 0 < X_i \le 3, \\ 0 & \text{otherwise.} \end{cases}
$$

- Compute $\widehat{\mathbb{P}}(0 \le X_i \le 3) = \frac{1}{n}\sum_{i=1}^{n} I(0 \le X_i \le 3)$.

(b)

```
n <- 10^4
x <- rweibull(n, shape = 2, scale = 3)
prob_est <- sum(x < 3)/n
prob_est

prob_exa <- pweibull(3, shape = 2, scale = 3)

test_that("Ensuring mc estimation of mean is sensible", {
  expect_equal(prob_est, prob_exa, tol = 1e-2)
})
# if you increase n, you can also decrease the above tolerance.
```

(c) In this example, without using `rweibull()` we use the acceptance-rejection method:

```
n <- 10^4
accepted <- numeric(n)
u_accepted <- numeric(n)
i <- 0
iteration <- 0
while(i < n){
  y <- rexp(n = 1, rate = 1/3) # candidate from g
  u <- runif(1) # u ~ uniform(0, 1)
  ftgt <- (1/3) * y * exp(-(y/3)^2 + y/3) # f(x)/cg(x)
```

```
  if(u < ftgt){
    i <- i+1
    accepted[i] <- y
    u_accepted[i] <- u
  }
  iteration <- iteration + 1
}
# below are the samples we are going to use
accepted
prob_est2 <- sum(accepted < 3)/n
prob_est2

test_that("Ensuring mc estimation of mean is sensible", {
  expect_equal(prob_est2, prob_exa, tol = 1e-2)
})
```