TUT4

STA380 Review for Term Test 1.

(1) $f_Y(y) = \begin{cases} \frac{1}{8}, & 0 < y \le 2 \\ \frac{y}{8}, & 2 < y \le 4 \\ 0, & \text{ow.} \end{cases}$

$E[Y] = \int y \cdot f(y)\, dy$

$E[Y] = \int_0^2 y \cdot \frac{1}{8}\, dy + \int_2^4 y \cdot \frac{y}{8}\, dy$

Review Uniform pdf : $f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{ow} \end{cases}$

$\int_a^b \frac{1}{b-a} \cdot x \cdot dx$

$E[Y] = 2 \int_0^2 \frac{1}{2-0} \cdot \frac{y}{8}\, dy + 2 \int_2^4 \frac{1}{4-2} \cdot \frac{y^2}{8}\, dy$

$= 2 \cdot E_{U(0,2)}\left[\frac{Y}{8}\right] + 2 \cdot E_{U(2,4)}\left[\frac{Y^2}{8}\right]$

Algorithm:

$U_1 \sim \text{Uniform}(0,2) \Rightarrow$ runif(n, 0, 2)

$U_2 \sim \text{Uniform}(2,4) \Rightarrow$ runif(n, 2, 4)

② Inverse - transformation Method.

Ex. $X \sim$ Exponential (λ)

pdf. $f_X(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) Find $F_X(x)$. CDF of X.

$$F_X(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda t} dt$$

$$= -e^{-\lambda t} \Big|_0^x$$

$$= -(e^{-\lambda x} - e^{-\lambda \cdot 0})$$

$$= -(e^{-\lambda x} - 1) = 1 - e^{-\lambda x}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

(b) derive $F_X^{-1}(x)$

For $x \geq 0$, set $U = 1 - e^{-\lambda x} \Rightarrow e^{-\lambda x} = 1 - U$

$$-\lambda x = \ln(1-U)$$

$$x = \frac{\ln(1-U)}{-\lambda}$$

$1 - x > 0$

$-x > -1 \Rightarrow x < 1$

$$F_X^{-1}(x) = \frac{\ln(1-x)}{-\lambda}, \quad 0 \leq x < 1$$

(c) Algorithm to generate a sample of size $n = 10^4$

For each random variate required,

- Generate a random variable $u \sim \text{Uniform}(0,1)$
- Deliver $x = \dfrac{\ln(1-u)}{-\lambda}$

(d) R codes

```
n <- 10^4
u <- runif(n)
x <- ln(1-u) / (-λ)
```

runif(n, 0, 1).

Unifor (4,7)

runif(n, 4, 7)

③ Acceptance - Rejection Method.

Practice A, Q2.

Review:

$0 < \frac{f(x)}{cg(x)} \le 1$ for all x.

$\downarrow$ upper bound.

$c \Rightarrow$ max

accept rate $\Rightarrow \mathbb{P}(\text{accept}) = \frac{1}{c}$

$$f(x) = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} e^{-x}, \quad x>0$$

$$g(x) = \frac{2}{3} e^{-\frac{2}{3}x}, \quad x>0$$

(a) $c = \max\left\{ \frac{f(x)}{g(x)} \right\}$

$$\frac{f(x)}{g(x)} = \frac{\frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} e^{-x}}{\frac{2}{3} e^{-\frac{2}{3}x}}$$

$$= \frac{3}{\sqrt{\pi}} x^{\frac{1}{2}} e^{-x+\frac{2}{3}x} = \frac{3}{\sqrt{\pi}} x^{\frac{1}{2}} \cdot e^{-\frac{x}{3}}$$

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{3}{\sqrt{\pi}} \cdot \frac{1}{2} x^{-\frac{1}{2}} \cdot e^{-\frac{x}{3}} + \frac{3}{\sqrt{\pi}} x^{\frac{1}{2}} e^{-\frac{x}{3}} \cdot \left(-\frac{1}{3}\right)$$

$$= \frac{3}{\sqrt{\pi}} e^{-\frac{x}{3}} \left(\frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{3} x^{\frac{1}{2}} \right) \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{3} = 0 \Rightarrow \frac{1}{2\sqrt{x}} \cancel{\lessgtr} \frac{\sqrt{x}}{3}$$

$$\Rightarrow 2 \cdot x = 3 \Rightarrow x = \frac{3}{2}$$

$$c = \frac{f\left(\frac{3}{2}\right)}{g\left(\frac{3}{2}\right)} = \frac{3}{\sqrt{\pi}} \cdot \left(\frac{3}{2}\right)^{\frac{1}{2}} \cdot e^{-\left(\frac{3}{2 \cdot 3}\right)} \approx 1.2573$$

(b) Algorithm

- Generate $Y \sim$ Exponential $(\frac{2}{3})$
- Generate $U \sim$ Uniform $(0, 1)$
- if

$$\frac{f(x)}{c\,g(x)}$$

$$u < \frac{\frac{3}{\sqrt{\pi}} x^{\frac{1}{2}} e^{\frac{-x}{3}}}{1.2573}$$

then accept y and set $x = y$.

otherwise, reject y and generate a random variate again.

(c) R check TUT 2. Q3.

(d) if $U = 0.62$ $Y = 0.02$.

would you accept or reject this candidate?

$$0.62 \overset{?}{<} \frac{\frac{3}{\sqrt{\pi}} (0.02)^{\frac{1}{2}} \cdot e^{(-\frac{0.02}{3})}}{1.2573}$$

$$0.62 \not< 0.1891$$ X reject this candidate.

$$0.62 < 0.7$$ accept.

④ Practice A. Q5.

$X_1 \sim$ Normal $(0, 1)$, $X_2 \sim$ Normal $(3, 1)$

$X_1 \perp\!\!\!\perp X_2$.

(a) is the following mixture is valid or not?

$$F_X(x) = 0.3 F_{X_1}(x) + 0.9 F_X(x)$$

No! $0.3 + 0.9 = 1.2 \neq 1$

Therefore this is not valid mixture.

(b) R codes.

0.2 0.2 0.6

```
n <- 10^4
k <- sample (1:2, size = n, replace = TRUE, prob = c(0.3, 0.9))
x <- ifelse (k == 1, x1, x2)
```

★.

$k = 1$ ⓧ

(c) Convolution is valid r.v.?

$$S = 0.3X_1 + 0.9X_2$$

Yes. this is linear combination.

(d) R codes

```
n <- 10^4
x1 <- rnorm(n, mean=0, sd=1)
x2 <- rnorm(n, mean=3, sd=1)

s <- (0.3 * x1) + (0.9 * x2)
```

⑤ Practice A Q6.

$$g_X(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\frac{1}{x}\right)^{\alpha+1} \exp\left(\frac{-\beta}{x}\right)$$

$\alpha = 2, \ \beta = 3$

$$g_X(x) = \frac{3^2}{\Gamma(2)} \left(\frac{1}{x}\right)^{2+1} \cdot e^{\left(\frac{-3}{x}\right)}$$

Review

$\Gamma(n) = (n-1)!$

$\Gamma(2) = (2-1)! = 1$

$\Gamma(1) = 0! = 1$

(a) $\mathbb{P}(2 \leq X \leq 5)$

$$g_X(x) = 9 \left(\frac{1}{x}\right)^3 e^{\left(\frac{-3}{x}\right)}$$

$$\mathbb{P}(2 \leq X \leq 5) = \int_2^5 9 \left(\frac{1}{x}\right)^3 e^{-\left(\frac{3}{x}\right)} dx$$

If $X \sim$ Uniform (a, b) then

$$f_X(x) = \frac{1}{b-a} = \frac{1}{5-2} = \frac{1}{3}, \quad 2 < X < 5$$

$$E[g(X)] = \int_2^5 \frac{1}{3} \left(9 \cdot \left(\frac{1}{x}\right)^3 e^{\left(\frac{-3}{x}\right)}\right) dx$$

$\Rightarrow 3 E_{U(2,5)}[g(X)]$ by SLLN ☆!

Algorithm.

1. generate $u_1 \cdots u_m \sim \text{Uniform}(2,5)$

2. Compute $\widehat{E[g(x)]} = \frac{1}{m} \sum_{i=1}^{m} \left(9 \cdot \left(\frac{1}{u_i}\right)^3 e^{\left(-\frac{3}{u_i}\right)} \right)$

3. deliver $\hat{\theta} = 3\,\widehat{E[g(x)]}$

R codes.

```
n <- 10^4
u <- runif (n, 2, 5)
3* mean( 9* (1/u)^3 * exp(-3/u))
```