STA380 Practice Problems for Preliminary Quiz

These problems are not to be handed in, but they are for extra practice for students to be prepared for the quiz.

1. Let X have the probability density function:

$$f_X(x) = \begin{cases} 3e^{-3x} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Compute $\mathbb{E}[X]$.
   (b) Compute $\mathbb{E}[e^{2x}]$.
   (c) Compute $\mathbb{V}[X]$.

2. Suppose we have a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n$ with the following probability density function:

$$f_X(x) = \frac{2}{\theta} e^{-\frac{2}{\theta}x}, \quad x > 0, \;\; \theta > 0.$$

   Furthermore, suppose my estimator for θ is $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) Compute $\mathbb{E}[\overline{X}]$, the **expected value** of the estimator.
   (b) Compute $\mathbb{V}[\overline{X}]$, the **variance** of the estimator.
   (c) Compute $\mathbb{B}[\overline{X}]$, the **bias** of the estimator. Is this estimator unbiased?
   (d) Compute $MSE[\overline{X}]$, the **mean squared error** of the estimator.

3. Suppose $n = 100$ people attended a screening of the highly anticipated Minecraft movie. For various reasons, some viewers chose to leave before it ended. The average and variance of the time spent watching the movie were 40 minutes and 5 minutes, respectively. Construct a 90% confidence interval to estimate μ, the true mean time at which viewers decide to leave the movie. Use the pivotal quantity:

$$Z = \frac{\overline{x} - \mu}{\sigma_{\overline{x}}}.$$

4. Let $Y_1, Y_2, ..., Y_n$ denote a random sample from a distribution with the following probability density function with parameters $\alpha > 0$ and $\beta > 0$, where β is known.

$$f(y|\alpha, \beta) = \begin{cases} \alpha\beta^{\alpha} y^{-(\alpha+1)} & y \geq \beta \\ 0 & \text{otherwise} \end{cases}$$

   Find the MLE for α.

5. Let $Y_1, ..., Y_n$ be a random sample with the following common probability mass function:

$$f(y) = \begin{cases} \theta(1-\theta)^{y-1} & y = 1, 2, ... \\ 0 & \text{otherwise} \end{cases}$$

   Here, the unknown parameter $0 < \theta < 1$.

   (a) Find the MLE of θ.
   (b) Find the MLE of $\mathbb{E}(Y_1)$.

Below are the solutions.

1. (a) You should identify that this is an $Exponential(\lambda = 3)$ (where λ is the **rate**) and thus the expected value is $\mathbb{E}[X] = \frac{1}{\lambda} = \frac{1}{3}$.

   (b)
$$\begin{aligned}\mathbb{E}[e^{2x}] &= \int_0^\infty e^{2x} 3e^{-3x} dx \\ &= 3\int_0^\infty e^{-x} dx \\ &= -3e^{-x}\Big|_0^\infty \\ &= 3\end{aligned}$$

   (c) Similar to part (a), the variance is $\mathbb{V}[X] = \frac{1}{\lambda^2} = \frac{1}{9}$.

2. (a) Again, one could view this as $Exponential(\lambda = 2/\theta)$ and thus $\mathbb{E}[X] = 1/\lambda = \theta/2$. Furthermore,
$$\begin{aligned}\mathbb{E}[\overline{X}] &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X] \quad \text{by linearity of expectation,} \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\theta}{2}\right) \\ &= \frac{1}{n} n \left(\frac{\theta}{2}\right) \\ &= \left(\frac{\theta}{2}\right)\end{aligned}$$

   (b) Here, $\mathbb{V}[X] = 1/\lambda^2 = \theta^2/4$.
$$\begin{aligned}\mathbb{V}[\overline{X}] &= \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{V}[X] \quad \text{since the samples are i.i.d.,} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\frac{\theta^2}{4} \\ &= \frac{1}{n^2} n \frac{\theta^2}{4} \\ &= \frac{\theta^2}{4n}\end{aligned}$$

   (c)
$$\begin{aligned}\mathbb{B}[\overline{X}] &= \mathbb{E}[\overline{X}] - \theta \\ &= \left(\frac{\theta}{2}\right) - \theta \\ &= \frac{\theta - 2\theta}{2} \\ &= \frac{-\theta}{2}\end{aligned}$$

   This is not unbiased because $\mathbb{B}[\overline{X}] \neq 0$ for $\theta > 0$.

   (d)
$$\begin{aligned}MSE(\overline{X}) &= \mathbb{V}[\overline{X}] + \mathbb{B}[\overline{X}]^2 \\ &= \frac{\theta^2}{4n} + \left(\frac{-\theta}{2}\right)^2 \\ &= \frac{\theta^2}{4n} + \frac{\theta^2}{4}\frac{2n}{2n} \\ &= \frac{\theta^2(1+n)}{4n}\end{aligned}$$

3. The quantity Z has a standard Gaussian distribution where $\mathbb{P}(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha$. Then,

$$\begin{aligned}
1 - \alpha &= \mathbb{P}(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) \\
&= \mathbb{P}\left(-z_{\alpha/2} \leq \frac{\overline{x} - \mu}{\sigma_{\overline{x}}} \leq z_{\alpha/2}\right) \\
&= \mathbb{P}\left(-z_{\alpha/2}\sigma_{\overline{x}} - \overline{x} \leq -\mu \leq z_{\alpha/2}\sigma_{\overline{x}} - \overline{x}\right) \\
&= \mathbb{P}\left(\overline{x} - z_{\alpha/2}\sigma_{\overline{x}} \leq \mu \leq \overline{x} + z_{\alpha/2}\sigma_{\overline{x}}\right)
\end{aligned}$$

Hence use the confidence interval $\overline{x} \pm z_{\alpha/2}\sigma_{\overline{x}}$. Clearly, $n = 100$, $\overline{x} = 40$, $\sigma_{\overline{x}} = \frac{\sqrt{5}}{\sqrt{100}}$, and $\alpha = 0.1$ Therefore,

```
> n <- 100
> xbar <- 40
> sigma <- sqrt(5)/sqrt(n)
> alpha <- 0.1
> z_score <- qnorm(alpha/2, lower.tail = FALSE)
> xbar - z_score*sigma
[1] 39.6322
> xbar + z_score*sigma
[1] 40.3678
```

Hence the 90% confidence interval of μ is (39.6322, 40.3678).

4. Click here.

5. Click here. You do **not** need to worry about finding MOME (the methods of moments estimator).