STA380 Term Test 1 Practice Version B

Solutions to the practice tests will not be posted. Students are encouraged to ask questions on Piazza or during office hours; however, they should include evidence of their own attempts when doing so. The primary purpose of providing the practice test is to help students become familiar with the format, length, and style of the questions, as well as to offer an opportunity for additional practice. Several questions are directly from the lecture slides, practice problems, and quizzes. The difficulty of the practice test and the actual test may differ.

Question 1

Let X be a continuous random variable with the following probability mass function:

x	1	2	3	4
$p_X(x)$	0.2	0.5	0.2	0.1

(a) **[2 points]** Write an algorithm to generate a sample of size $n = 10^4$ from the distribution of X using the *inverse transform method*. **Be precise.**

(b) **[2 points]** Write out the `R` code for the algorithm.

(c) Find the following:

(i) **[1 point]** $F_X^{-1}(0.1)$.

(ii) **[1 point]** $F_X^{-1}(0.4)$.

(iii) **[1 point]** $F_X^{-1}(0.7)$.

(iv) **[1 point]** $F_X^{-1}(1)$.

Question 2

In this question, we want to use the *acceptance-rejection* method to generate a sample from a random variable X with the following target probability density function:

$$f_X(x) = \begin{cases} 5x^4 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let the trial/candidate distribution be $Uniform(0, 1)$.

(a) **[3 points]** What is $c = \max\left\{\frac{f(x)}{g(x)}\right\}$, where $f(x)$ is the target distribution and $g(x)$ is the trial distribution? **You <u>should</u> show that the value you found is a maximum.**

(b) **[2 points]** Write out the *acceptance-rejection* algorithm to generate a sample from X. **Be precise.**

(c) **[3 points]** Write out the `R` code for the algorithm. Assume $n = 10^4$.

(d) **[2 points]** If the first uniform in the acceptance-rejection algorithm was $U = 0.23$ and the first candidate draw was $Y = 0.74$. Would you accept or reject this candidate? **You must specify *accept* or *reject*. Show all steps.**

Question 3

[2 points] Suppose $W_1 \sim \chi^2_{(v_1)}$, $W_2 \sim \chi^2_{(v_2)}$, and W_1, W_2 are independent. Then,

$$\frac{W_1/v_1}{W_2/v_2} \sim F_{v_1,v_2}$$

Where F_{v_1,v_2} is the F distribution with degrees of freedom v_1, v_2. Use `rchisq(n, df)` to generate $n = 10^4$ samples from the $F_{3,7}$ distribution.

In `rchisq(n, df)`, `n` represents the number of observations and `df` represents the degrees of freedom.

Question 4

Prove the following:

(a) **[2 points]** Given the acceptance-rejection algorithm, prove that the accepted sample has the same distribution as X, assuming that X is *discrete.*

(b) **[3 points]** Given the acceptance-rejection algorithm, prove that the accepted sample has the same distribution as X, assuming that X is *continuous.*

(c) **[1 points]** Given the acceptance-rejection algorithm, show that that $c = \max\left\{\frac{f(x)}{g(x)}\right\} \geq 1$.

Question 5

Suppose $X_1 \sim Exponential(\beta = 3)$, $X_2 \sim Exponential(\beta = 3)$, and X_1, X_2 are independent and β represents the *scale* parameter. Consider the following questions:

(a) **[2 points]** Explicitly write out the following mixture cumulative distribution function:

$$F_X(x) := 0.3F_{X_1}(x_1) + 0.7F_{X_2}(x_2)$$

(b) **[2 points]** Write `R` code to generate $n = 10^4$ samples of the previous cumulative distribution function. You may use `rexp(n, rate)` where `n` represents the number of observations and `rate` represents the rate parameter. Note that the scale parameter is equal to $\frac{1}{\text{rate}}$.

(c) **[2 points]** What is the distribution of the following convolution function? **Justify your answer.**

$$S := X_1 + X_2$$

(d) **[2 points]** Write `R` code to generate $n = 10^4$ samples of the previous convolution using `rexp(n, rate)` as defined earlier.

Question 6

Consider:

$$\theta = \int_{0.3}^{0.7} \frac{\Gamma(5)}{\Gamma(2)\Gamma(3)} x(1-x)^2 dx$$

(a) **[3 points]** Provide a detailed algorithm to obtain a Monte Carlo estimate of θ using a Monte Carlo sample size of $m = 10^4$. **Show all steps. Be precise.**

(b) **[2 points]** Write out the `R` code for the algorithm.