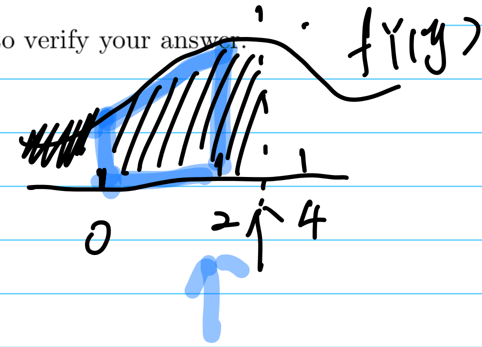
STA380

Week 2.

1. Generate the following probability density function using the inverse-transformation method:

$$f_Y(y) = \begin{cases} \frac{1}{8} & 0 < y \le 2, \\ \frac{y}{8} & 2 < y \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Additionally, graph the histogram against the true density curve to verify your answer.



Sol: ① CDF

for $y \le 0$, $F_Y(y) = 0$

for $x\ y < 2$, $F_Y(y) = P(Y \le y) = \int_0^y \frac{1}{8}\,dt = \frac{1}{8}t\Big|_0^y = \frac{1}{8}y \overset{set}{=} u$

$\Rightarrow y = 8u$ & $0 < 8u \le 2 \Rightarrow 0 < u \le \frac{1}{4}$

for $2 < y \le 4$, $F_Y(y) = F_Y(2) + \int_2^y \frac{t}{8}\,dt =$

$$= \frac{1}{4} + \frac{1}{16}t^2\Big|_2^y = \frac{1}{4} + \frac{1}{16}(y^2 - 4)$$

$$= \frac{y^2}{16} \overset{set}{=} u$$

$y^2 = 16u \Rightarrow y = \pm 4\sqrt{u}$

y can't be $-4\sqrt{u}$? $\quad 2 < y \le 4$.

$\therefore y = 4\sqrt{u}$ only.

$\therefore 2 < 4\sqrt{u} \le 4 \Rightarrow \frac{1}{2} \le \sqrt{u} \le 1$

$\Rightarrow \frac{1}{4} \le u \le 1$

$$x = \begin{cases} 8u, & 0 < u \le \frac{1}{4} \\ 4\sqrt{u}, & \frac{1}{4} < u \le 1 \end{cases}$$

Algorithm:

① Generate a random variable

$u \sim \text{Uniform}(0,1)$

② Delivery $x = \begin{cases} 8u, & 0 < u \le \frac{1}{4} \\ 4\sqrt{u}, & \frac{1}{4} < u \le 1 \end{cases}$

2. Let X be a continuous random variable with the following probability density function:

$$f_X(x) = 5x^4, \quad 0 \le x \le 1$$

Write out the algorithm and code to generate a random variable from the distribution of X using the inverse-transform method.

Sol:

$$F_X(x) = \int_0^x 5t^4\,dt = t^5\Big|_0^x = x^5, \quad 0 \le x \le 1$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^5, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$F_X(x) = x^5 \stackrel{\text{set}}{=} u \quad \Rightarrow \quad x = u^{\frac{1}{5}}$$

$$0 \le u^{\frac{1}{5}} \le 1 \quad \Rightarrow \quad 0 \le u \le 1$$

$$\Rightarrow F_X^{-1}(x) = u^{\frac{1}{5}}, \quad 0 \le u \le 1$$

Algorithm: ① Generate a random variable

$u \sim \text{Uniform}(0, 1)$

② Deliver $x = u^{\frac{1}{5}}$

3. Consider the Weibull distribution:

$$f_X(x) = \begin{cases} \frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-(x/\lambda)^\alpha} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Where α represents the shape parameter and λ represents the scale parameter. Use the acceptance-rejection method to generate a sample from the Weibull distribution where $\alpha = 2$ and $\lambda = 3$. Let the trial distribution be $Exponential(\theta = 3)$ where θ is the scale parameter.

(a) What is $c = \max\left\{\frac{f(x)}{g(x)}\right\}$, where $f(x)$ is the target distribution and $g(x)$ is the trial distribution?

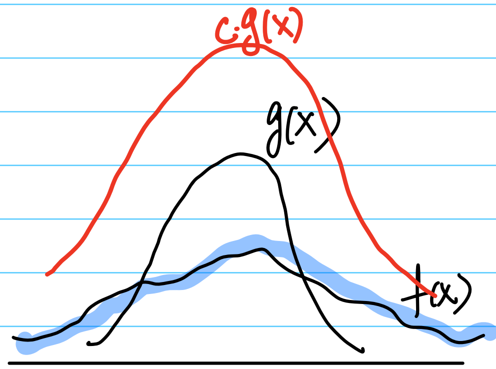
(b) Write out the acceptance-rejection algorithm for this case.

(c) Write out the R code for the acceptance-rejection algorithm.

(d) Assume that you would like to generate 10^4 value from the target distribution. Determine how many draws (iterations), in average, are required from the trial distribution. The answer should be a number.

Sol:

3(a)



$0 < \frac{f(x)}{c\,g(x)} \leq 1$ for all x.

This requires $\boxed{\frac{f(x)}{g(x)}} \leq c$ ← upper bound.

accept rate : $\mathbb{P}(\text{accept}) = \frac{1}{c}$.

(a) $\Rightarrow \frac{f(x)}{g(x)} = \frac{\frac{2}{3}\left(\frac{x}{3}\right)^{2-1} e^{-\left(\frac{x}{3}\right)^2}}{\frac{1}{3} e^{-\frac{x}{3}}} = \frac{2}{3}\, x\, e^{-\left(\frac{x}{3}\right)^2 + \frac{x}{3}}$

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \cdots\cdots = (2x+3)(x-3) = 0$$

$x = -\frac{3}{2}$ & $x = 3$ ✓

$c = \max\left\{\frac{f(x)}{g(x)}\right\} = \frac{2}{3}(3)\exp\left\{-\left(\frac{3}{3}\right)^2 + \frac{3}{3}\right\} = 2$

(b) Algorithm :

① Generate $Y \sim$ Exponential $(\theta = 3)$

② Generate $U \sim$ Uniform $(0, 1)$

③. If

$$u < \frac{1}{3} x \cdot e^{-\left(\frac{x}{3}\right)^2 + \frac{x}{3}} \quad \leftarrow \frac{f(x)}{c \cdot g(x)}$$

accept y, set $x = y$. Otherwise, reject y and generate a random variable again.