1. Consider the following probability density function:

$$f_Y(y) = \begin{cases} \frac{1}{8} & 0 < y \le 2, \\ \frac{y}{8} & 2 < y \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Use the simple Monte Carlo estimator to **estimate** the expected value. Write the algorithm and the corresponding R codes. Use $n = 10^4$. Compare it to the true value.

key idea: Monte Carlo estimator is a estimator that use random sampling to approximate a quantity that is difficult or impossible to compute.

$$E[Y] = \int y \cdot f(y) \, dy$$

$$E[Y] = \int_0^2 y \cdot \frac{1}{8} \, dy + \int_2^4 y \cdot \frac{y}{8} \, dy$$

$$= 2 \int_0^2 \frac{1}{2-0} \cdot \frac{y}{8} \, dy + 2 \int_2^4 \frac{1}{4-2} \cdot \frac{y^2}{8} \, dy$$

$$= 2 \, E_{U_{(0,2)}}\left[\frac{Y}{8}\right] + 2 \, E_{U_{(2,4)}}\left[\frac{Y^2}{8}\right]$$

Review: Uniform pdf :
$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Algorithm: ① Generate $u_{11} \dots u_{1m} \sim$ Uniform $(0,2)$

② Generate $u_{21} \dots u_{2m} \sim$ Uniform $(2,4)$

③ Deliver $\widehat{E[g(Y)]} = \frac{2}{m} \sum_{i=1}^{m} \frac{u_{1i}}{8} + \frac{2}{m} \sum_{i=1}^{m} \frac{u_{2i}^2}{8}$

2. Consider the Weibull distribution:

$$f_X(x) = \begin{cases} \frac{\alpha}{\lambda} \left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-(x/\lambda)^{\alpha}} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Where $\alpha$ represents the shape parameter and $\lambda$ represents the scale parameter. Let $X \sim Weibull(\alpha = 2, \lambda = 3)$. Use the **hit or miss** approach to compute $P(0 < X < 3)$.

(a) Write out the algorithm.

(b) Using $n = 10^4$, code the algorithm in R using `rweibull()`. Compare this value with `pweibull()`. *Hint: please use and read* `help(rweibull)` *and* `help(pweibull)` *to understand how this function works.)*

(c) Using $n = 10^4$, code the algorithm in R **without** using `rweibull()`. Compare this value with `pweibull()`. *Hint: in the previous set of practice problems, we generated values from the weibull distrbution using the acceptance-rejection method.*

(a) Algorithm:

① Generate random sample, $X_1, X_2 \cdots\cdots X_n$ from Weibull $(\alpha = 2, \lambda = 3)$.

②. For each $X_i$, we compute.

$$I(0 \leq X_i \leq 3) = \begin{cases} 1 & , 0 \leq X_i \leq 3 \\ 0 & , \text{otherwise} \end{cases}$$

③. $\hat{P}(0 \leq X_i \leq 3) = \frac{1}{n} \sum_{i=1}^{n} I(0 \leq X_i \leq 3)$

These problems are not to be handed in, but they are for extra practice for students to be prepared for term tests.

1. Consider $X \sim Gamma(\alpha = 2, \beta = 2)$ where $\alpha$ is the shape parameter, and $\beta$ is the scale parameter.

    (a) Use the hit or miss method to compute $\mathbb{P}(X < 2)$. You may use `rgamma()` and $n = 10^5$.

    (b) Recall in lecture, we said the Monte Carlo estimate for the variance of an estimator:

    $$\widehat{\mathbb{V}ar[\hat{\theta}]} = \frac{\hat{\sigma}^2}{n} = \frac{\sum_{i=1}^{n}\left[g(x_i) - \overline{g(x)}\right]^2}{n^2}$$

    Use this to construct a 90% confidence interval of this estimate.

$$\hat{\theta} \pm z_{\frac{\alpha}{2}} \cdot sd.$$