Algebraic Curves over Finite Fields

Homework 1

Note: This homework assignment is in constant evolution... Problems will be added as the semester goes on, but once an exercise is posted, it will not change. Discussion of the homework problems with me, or collaboration (in a reasonable degree) with your classmates, is encouraged, but you have to provide a note on which problems you had assistance. The due date of this first homework assignment will be communicated in class.

Ex 1. Let $k = \mathbb{C}$ and let X be the (affine) conic described by the polynomial

$$f(x, y) = x^2 + y^2 - 1 \in k[x, y].$$

(a) Find the rational parametrization

$$\begin{array}{ccc} k & \longrightarrow & X \\ t & \longmapsto & (\varphi(t), \psi(t)) \end{array}$$

obtained via the construction given in class by using the point $(-1, 0) \in X$.

(b) Interpret geometrically the parametrization found in (a) in order to prove the trigonometric identities:

$$\sin(\theta) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}, \quad \cos(\theta) = \frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}, \quad \text{for all } \theta \in (-\pi, \pi) \subset \mathbb{R}.$$

(c) Find the general form for the solution in $\mathbb{Q}^2$ of the equation $x^2 + y^2 - 1 = 0$.

(d) A *Pythagorean triple* is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. Use (c) in order to prove the *Euclid's formula* that generates a Pythagorean triple for each $m, n \in \mathbb{N}$ with $m > n > 0$:

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$