# Towards maximal singular curves over finite fields

Annamaria lezzi
(Joint work with Yves Aubry)

Institut des Mathématiques de Marseille, Université d'Aix-Marseille, France

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The quantity  $N_{oldsymbol{q}}(g\,,\,\pi)$ 

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### **Notation**

•  $\mathbb{F}_q$  the finite field with q elements.

• The word "curve" will always stand for an absolutely irreducible projective algebraic curve.

Let X be a curve over  $\mathbb{F}_a$ . We denote by:

- $\mathbb{F}_q(X)$  the function field of X;
- $\sharp X(\mathbb{F}_q)$ , the number of rational points on X over  $\mathbb{F}_q$ ;
- $\tilde{X}$  the normalization of X and  $\nu: \tilde{X} \to X$  the normalization map (regular finite and birational):  $\mathbb{F}_q(X) = \mathbb{F}_q(\tilde{X})$ ;
- g the geometric genus of X, i.e. the genus of  $\tilde{X}$ ;
- $\pi$  the arithmetic genus of X.

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### The arithmetic genus

Let Q be a point on X and let  $\mathcal{O}_Q$  be the local ring in  $\mathbb{F}_q(X)$  associated to Q.

Fact:  $\mathcal{O}_Q$  is integrally closed if and only if Q is a nonsingular point.

Let  $\overline{\mathcal{O}_Q}$  be the integral closure of  $\mathcal{O}_Q$ .  $\overline{\mathcal{O}_Q}/\mathcal{O}_Q$  is a finite dimensional  $\mathbb{F}_q$ -vector space. We define the **degree of singularity of** Q:

$$\delta_{\mathcal{Q}} := \dim_{\mathbb{F}_q} \overline{\mathcal{O}_{\mathcal{Q}}}/\mathcal{O}_{\mathcal{Q}}$$

The **arithmetic genus**  $\pi$  of a curve X is the integer:

$$\pi := g + \sum_{Q \in X} \delta_Q$$
.

- $\pi > g$ ;
- $\pi = g$  if and only if X is a smooth curve;
- If X is a plane curve of degree d,  $\pi = \frac{(d-1)(d-2)}{2}$ .

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### Smooth curves

If X is smooth  $(\pi = g)$ , the integers  $q, \sharp X(\mathbb{F}_q)$  and g satisfy the **Serre-Weil inequality**:

$$|\sharp X(\mathbb{F}_q) - (q+1)| \le g[2\sqrt{q}]$$

Let us denote by

$$N_q(g)$$

the maximal number of rational points over  $\mathbb{F}_q$  that a smooth curve of genus g can have. We have:

$$N_q(g) \leq q + 1 + g[2\sqrt{q}]$$

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### Bounds for singular curves

In 1996, Aubry and Perret find relations between a curve and its normalization:

$$Z_X(T) = Z_{\tilde{X}}(T) \prod_{P \in \operatorname{Sing} X} \left( \frac{\prod_{Q \in \nu^{-1}(P)} (1 - T^{\deg Q})}{1 - T^{\deg P}} \right)$$

$$|\sharp \tilde{X}(\mathbb{F}_q) - \sharp X(\mathbb{F}_q)| \leq \pi - g,$$

As a consequence we get the analogous of Serre-Weil bound for singular curves (**Aubry-Perret bound**):

$$|\sharp X(\mathbb{F}_q)-(q+1)|\leq g[2\sqrt{q}]+\pi-g.$$

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# The quantity $N_q(g,\pi)$

We introduce an analogous quantity of  $N_q(g)$  for singular curves:

#### Definition

For q a power of a prime, g and  $\pi$  non negative integers such that  $\pi \geq g,$  let us define the quantity

$$N_q(g,\pi)$$

as the maximal number of rational points over  $\mathbb{F}_q$  that a curve defined over  $\mathbb{F}_q$  of geometric genus g and arithmetic genus  $\pi$  can have.

Obviously we have

$$N_q(g,g) = N_q(g),$$
  $N_q(g,\pi) \le N_q(g) + \pi - g,$   $N_q(g,\pi) \le q + 1 + g[2\sqrt{q}] + \pi - g.$ 

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# **Terminology**

#### Definition

Let X be a curve over  $\mathbb{F}_q$  of geometric genus g and arithmetic genus  $\pi$ . The curve X is said to be:

(i) an optimal curve if

$$\sharp X(\mathbb{F}_q) = N_q(g,\pi);$$

(ii) a  $\delta$ -optimal curve if

$$\sharp X(\mathbb{F}_q) = N_q(g) + \pi - g = N_q(g) + \delta;$$

(iii) a maximal curve if

$$\sharp X(\mathbb{F}_q) = q + 1 + g[2\sqrt{q}] + \pi - g.$$

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### Fukasawa, Homma and Kim's curve

In 2011, Fukasawa, Homma and Kim consider and study the rational plane curve B over  $\mathbb{F}_q$  defined by the image of

$$\Phi: \quad \mathbb{P}^1 \quad \to \quad \mathbb{P}^2$$

$$(s,t) \quad \mapsto \quad (s^{q+1}, s^q t + s t^q, t^{q+1})$$

B is a maximal singular curve with g=0 and  $\pi=\frac{q^2-q}{2}$ :

$$\sharp B(\mathbb{F}_q) = q+1+rac{{\mathsf q}^2-{\mathsf q}}{2}$$

Remark: For  $P \in \mathbb{P}^1$ ,  $\Phi(P) \in \text{Sing}(B)$  if and only if  $P \in \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q)$ . In this case,  $\Phi^{-1}(\Phi(P)) = \{P, P^q\}$ .

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# $\delta$ -optimal and maximal curves

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#### Proposition

Let X be a curve of geometric genus g and arithmetic genus  $\pi$ . If X is  $\delta$ -optimal (maximal) then

- the normalization  $\tilde{X}$  is an optimal (maximal) curve;
- $\bigcirc$  Sing $(X) \subset X(\mathbb{F}_q)$ ;
- **9** if P is a singular point on X, then  $\nu^{-1}(P) = \{Q\}$ , with Q a point of degree 2 on  $\tilde{X}$ ;
- π − g ≤ B<sub>2</sub>(X̃), where B<sub>2</sub>(X̃) denotes the number of points of degree 2 on X̃;
- **5**  $Z_X(T) = Z_{\tilde{X}}(T)(1+T)^{\pi-g}$ .

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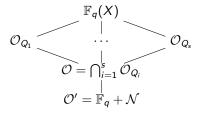
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# Construction of a prescribed singularity

Let start from a smooth curve X over  $\mathbb{F}_q$  and let  $S = \{Q_1, \dots, Q_s\}$  be a non-empty finite set of closed points on X.



 $\mathcal O$  is a semi-local ring with maximal ideals  $\mathcal N_{Q_i}:=\mathcal M_{Q_i}\cap\mathcal O$  for  $i=1,\ldots,s$ .

Let  $n_1,\ldots,n_s$  be s positive integers, let us set  $\mathcal{N}:=\mathcal{N}_{Q_1}^{n_1}\cdots\mathcal{N}_{Q_s}^{n_s}$  and let us consider:

$$\mathcal{O}' := \mathbb{F}_q + \mathcal{N}.$$

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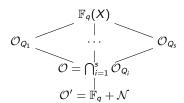
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### Proposition

 $\mathcal{O}' = \mathbb{F}_a + \mathcal{N}$  verifies the following properties:

- Frac( $\mathcal{O}'$ ) =  $\mathbb{F}_q(X)$  and  $\mathcal{O}$  is the integral closure of  $\mathcal{O}'$  in  $\mathbb{F}_q(X)$ .
- ${f 0}$   ${\cal O}'$  is a local ring with maximal ideal  ${\cal N}$  and residual field  ${\cal O}'/{\cal N}\cong {\Bbb F}_q$ .
- **3**  $\mathcal{O}/\mathcal{O}'$  is a  $\mathbb{F}_q$ -vector space such that

$$\mathsf{dim}_{\mathbb{F}_q}(\mathcal{O}/\mathcal{O}') = \sum_{i=1}^s n_i \deg Q_i - 1.$$

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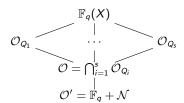
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#### **Theorem**

There exists a curve X' defined over  $\mathbb{F}_a$ 

- having X as normalization,
- ② with only one singular point P such that  $\mathcal{O}_P = \mathcal{O}'$  and P is rational.
- **1** P has a degree of singularity equal to  $\sum_{i=1}^{s} n_i \deg Q_i 1$  and

$$\pi(X') = g + \sum_{i=1}^s n_i \deg Q_i - 1.$$

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$$\mathbb{F}_{q}(X)$$

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O = O_{Q}$$

$$O' = \mathbb{F}_{q} + \mathcal{M}_{Q}$$

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- Frac( $\mathcal{O}'$ ) =  $\mathbb{F}_q(X)$  and  $\mathcal{O}$  is the integral closure of  $\mathcal{O}'$  in  $\mathbb{F}_q(X)$ .
- $\textbf{ 0} \quad \textit{o' is a local ring with maximal ideal } \mathcal{N} \text{ and residual field } \mathcal{O'}/\mathcal{N} \cong \mathbb{F}_q.$

$$\mathsf{dim}_{\mathbb{F}_q}(\mathcal{O}/\mathcal{O}') = \mathsf{deg}\; Q - 1.$$

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#### Theorem

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- having X as normalization,
- ② with only one singular point P such that  $\mathcal{O}_P = \mathcal{O}'$  and P is rational.
- $oldsymbol{0}$  P has a degree of singularity equal to  $\deg Q-1$  and

$$\pi(X') = g + \deg Q - 1.$$

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# Singular curves with many points and small $\boldsymbol{\pi}$

#### Theorem

Let X be a smooth curve of genus g defined over  $\mathbb{F}_q$ . Let  $\pi$  be an integer of the form

$$\pi = g + a_2 + 2a_3 + 3a_4 + \cdots + (n-1)a_n$$

with  $0 \le a_i \le B_i(X)$ , where  $B_i(X)$  is the number of closed points of degree i on the curve X. Then there exists a (singular) curve X' over  $\mathbb{F}_q$  of arithmetic genus  $\pi$  such that X is the normalization of X' and

$$\sharp X'(\mathbb{F}_q) = \sharp X(\mathbb{F}_q) + a_2 + a_3 + a_4 + \cdots + a_n.$$

Roughly speaking we can "transform" a point of degree d on a smooth curve in a singular rational one provided that we increase the value of the arithmetic genus of d-1.

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 $\mathcal{X}_q(g)$ : the set of optimal smooth curves X over  $\mathbb{F}_q$  of genus g.

 $B_2(\mathcal{X}_q(g))$ : the maximum number of points of degree 2 on a curve of  $\mathcal{X}_q(g)$ .

#### Theorem

We have:

$$N_q(g,\pi) = N_q(g) + \pi - g \iff g \le \pi \le g + B_2(\mathcal{X}_q(g)).$$

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### The case of rational curves: g=0

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#### Proposition

We have

$$N_q(0,\pi)=q+1+\pi$$

if and only if  $0 \le \pi \le \frac{q^2-q}{2}$ .

Fukasawa, Homma and Kim's curve is an explicit example of this proposition for  $\pi = \frac{q^2 - q}{2}$ .

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### g=1

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#### Proposition

**1** If p does not divide m, or q is a square, or q = p we have:

$$N_q(1,\pi) = q + 1 + [2\sqrt{q}] + \pi - 1$$

if and only if  $1 \leq \pi \leq 1 + \frac{q^2 + q - [2\sqrt{q}]([2\sqrt{q}] + 1)}{2}.$ 

2 In the other cases we have

$$N_q(1,\pi) = q + [2\sqrt{q}] + \pi - 1$$

if and only if  $1 \leq \pi \leq 1 + \frac{q^2 + q + [2\sqrt{q}](1 - [2\sqrt{q}])}{2}$  .

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# Spectrum of maximal curves

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#### Proposition

Assume q is square. If X is a maximal curve defined over  $\mathbb{F}_q$  of geometric genus g and arithmetic genus  $\pi$ , then:

$$\pi \leq g + \frac{q^2 + (2g-1)q - 2g\sqrt{q}(2\sqrt{q}+1)}{2} = (-q - \sqrt{q} + 1)g + \frac{q(q-1)}{2},$$

$$g \leq \frac{\sqrt{q}(\sqrt{q}-1)}{2}$$

and

$$\pi \leq \frac{q(q-1)}{2}$$
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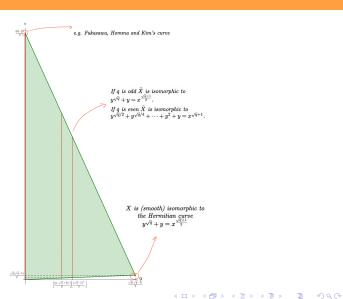
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Thank you very much for the attention!

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