LIMITS INVOLVING INFINITY (Sec. 1.6)

In class 2 we built a table of values for the function $\frac{1}{x^2}$ when x approaches 0:

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000
$\downarrow$ 0	

and we remarked that, while x approaches 0, then $\frac{1}{x^2}$ becomes arbitrarily large.

We denote this situation by:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

$\to$ this means that the values of $\frac{1}{x^2}$ can be made arbitrarily large by taking x sufficiently close to 0 (on either side of 0) but not equal to 0.

Curiosity: The symbol "∞" was introduced by John Wallis in 1655 in his book "De sectionibus conicis".

There are several hypothesis about the origin of this symbol: the most accredited is that ∞ is a variant of a Roman numeral 1,000 (originally CIↃ) which was sometimes used to mean "many".

Analogously the writing:

$$\lim_{x\to a} f(x) = -\infty$$

denotes that the values of $f(x)$ are as large negative as we like for all values of x that are sufficiently close to a, but not equal to a.

Let us consider now the following limit: $\lim_{x\to 0} \frac{1}{x}$.

We note that the output of the function $\frac{1}{x}$ is "very different" when x is close to 0 from the left and from the right

FROM THE LEFT

x	$\frac{1}{x}$
-1	-1
-0.5	-2
-0.2	-5
-0.1	-10
-0.05	-20
-0.01	-100
-0.001	-1000
$\downarrow$	$\downarrow$
0^-	$-\infty$

FROM THE RIGHT

x	$\frac{1}{x}$
1	1
0.5	2
0.2	5
0.1	10
0.05	20
0.01	100
0.001	1000
$\downarrow$	$\downarrow$
0^+	∞

Hence we have:

$$\lim_{x\to 0^-} \frac{1}{x} = \text{"}\frac{1}{0^-}\text{"} = -\infty$$

a positive quantity divided by a very small negative quantity gives a very large negative number

$$\lim_{x\to 0^+} \frac{1}{x} = \text{"}\frac{1}{0^+}\text{"} = \infty$$

a positive quantity divided by a very small positive quantity gives a very large positive number

$$\left.\begin{array}{l}\lim_{x\to 0^-} \frac{1}{x} = -\infty \\ \lim_{x\to 0^+} \frac{1}{x} = \infty\end{array}\right\} \Rightarrow \lim_{x\to 0} \frac{1}{x} \text{ does not exist}$$

Recap: Each time that the "plug in" returns $\frac{1}{0}$ (or more in general $\frac{L}{0}$, with $L \neq 0$) we have to compute separately the left-hand and the right hand limit.

The value of the one-sided limits will be ∞ or $-\infty$ depending on the sign of the denominator.

We denote by 0^+ a very small (= close to 0) positive quantity and by 0^- a very small negative quantity and we have:

$$\frac{1}{0^+} = \infty \quad \text{and} \quad \frac{1}{0^-} = -\infty$$

More in general:

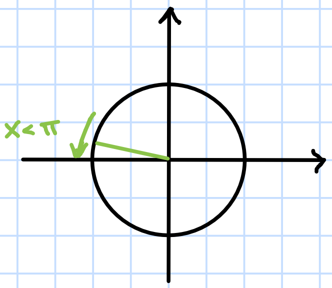
$\frac{L}{0^+} = \infty$ if $L > 0$	$\frac{L}{0^+} = -\infty$ if $L < 0$
$\frac{L}{0^-} = -\infty$ if $L < 0$	$\frac{L}{0^-} = +\infty$ if $L > 0$

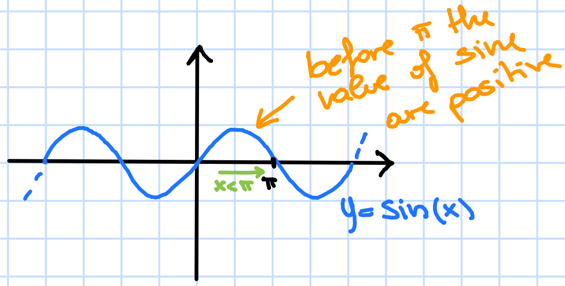
Example: $\lim\limits_{x \to \pi^-} \dfrac{\cos(x)}{\sin(x)} \underset{\substack{\uparrow \\ \text{plug in}}}{=} \dfrac{\cos(\pi)}{\sin(\pi)} = \text{“}\dfrac{-1}{0}\text{”}$

Hence the value of the limit will be ∞ or $-\infty$ depending on the sign of $\sin(x)$ when x approaches π from the left.

Let us consider the unit circle (or the graph of $\sin(x)$)

$\sin(x)$ is positive for angles close to π but less than π





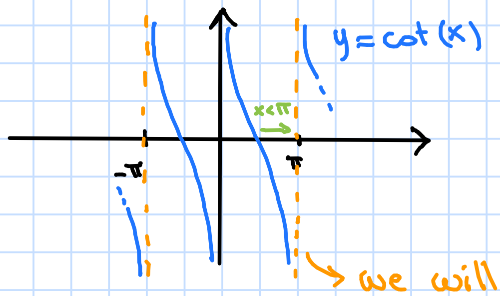
When $x \to \pi^-$ $(x<\pi)$ then $\sin(x) \to 0^+$ (sin x is close to 0 and positive).

Hence

$\lim\limits_{x \to \pi^-} \dfrac{\cos(x)}{\sin(x)} = \text{“}\dfrac{-1}{0^+}\text{”} = -\infty$

We can also plug in a value $<\pi$ for checkin the sign:
ex: $\dfrac{\cos(3.1)}{\sin(3.1)} \sim -24 < 0$

We could have achieved the same conclusion by remarking that $\dfrac{\cos(x)}{\sin(x)} = \cot(x)$ and by looking at the graph of the $\cot(x)$:



we will see that these orange vertical lines $(x=-\pi,\ x=\pi)$ are called vertical asymptotes.

ex: $\lim\limits_{x \to 2} \dfrac{x-3}{(x-2)^2} \underset{\substack{\uparrow \\ \text{plug in}}}{=} \text{“}\dfrac{-1}{0}\text{”}$

$\lim\limits_{x \to 2^+} \dfrac{x-3}{(x-2)^2} \underset{\substack{\uparrow \\ (x-2)^2 > 0}}{=} \text{“}\dfrac{-1}{0^+}\text{”} = -\infty$

(a square is always positive)

$$\lim_{x \to 2^-} \frac{x-3}{(x-2)^2} = \text{"}\frac{-1}{0^+}\text{"} = -\infty$$

$(x-2)^2 > 0$

Thus $\lim_{x \to 2} \frac{x-3}{(x-2)^2} = -\infty$.

So far we have always computed a limit when x approaches a number.

But we can also compute limits for x approaching ∞ or $-\infty$, i.e. when x is arbitrarily large (positive or negative).

Let us start from a very easy case: $\lim_{x \to \infty} \frac{1}{x}$

We have: "$\frac{1}{\infty}$" $= 0$.

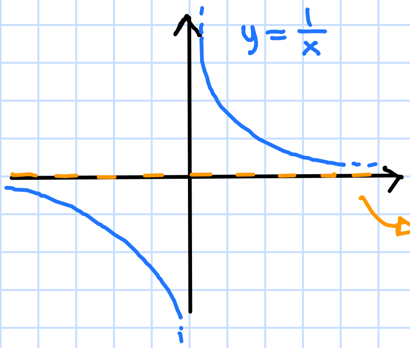
To convince yourself you can construct a table of values where x is very large ($1000 \cdot 1000{,}000$; etc...) or you can just imagine that you want to share fairly a PIZZA with many many people! How much pizza will everyone get?

when the number of people is very very big the quantity pizza available for everyone is pratically 0!

Conclusion: never share your pizza!

So we have $\lim_{x \to \infty} \frac{1}{x} = 0$ and we have also $\lim_{x \to -\infty} \frac{1}{x} = 0$

This is clear from the graph of $\frac{1}{x}$



we will see that the horizontal line $y=0$ is in this case called a horizontal asymptote

Note that when we compute the limit of a function at ∞ or $-\infty$ we can still plug in, by applying the following rules:

SUM

$L + \infty = \infty$

$L - \infty = -\infty$

$\infty + \infty = \infty$

$-\infty - \infty = -\infty$

$\infty - \infty =$ indeterminate form!
↓
we can not say anything!

PRODUCT

$L > 0, \quad L \cdot \infty = \infty$

$L < 0, \quad L \cdot \infty = -\infty$

$L > 0, \quad L \cdot (-\infty) = -\infty$

$L < 0, \quad L \cdot (-\infty) = -\infty$

$\infty \cdot \infty = \infty$

$\infty \cdot (-\infty) = -\infty$

$(-\infty) \cdot (-\infty) = \infty$

$+ \cdot + = +$
$+ \cdot - = -$
$- \cdot - = +$

$0 \cdot \infty, \quad 0 \cdot (-\infty) =$ indeterminate form!
↓
we can not say anything!

QUOTIENT

$\frac{L}{\pm\infty} = 0$

$L > 0, \quad \frac{\infty}{L} = \infty$

$L < 0, \quad \frac{\infty}{L} = -\infty$

$L > 0, \quad \frac{-\infty}{L} = -\infty$

$L < 0, \quad \frac{-\infty}{L} = +\infty$

$\frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{\infty}{-\infty}, \frac{-\infty}{-\infty} =$ indeterminate form!
↓
we can not say anything!

POWER / ROOT

n integer, $\quad \infty^n = \infty$

n even, $\quad (-\infty)^n = \infty$

n odd, $\quad (-\infty)^n = -\infty$

n integer, $\quad \sqrt[n]{\infty} = \infty$

n odd, $\quad \sqrt[n]{-\infty} = -\infty$

Example: $\lim_{x \to \infty} x^2 - x = $ "$\infty^2 - \infty$" $=$ "$\infty - \infty$": INDETERMINATE FORM

We can escape to the indeterminate form in the following way

$$\lim_{x \to \infty} x^2 - x = \lim_{x \to \infty} x(x-1) = \text{"}\infty(\infty - 1)\text{"} = \text{"}\infty \cdot \infty\text{"} = \infty$$

Limit at ∞ or $-\infty$ of a rational function

$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

For computing the limit of a rational function at $+\infty$ or $-\infty$ the technique is standard:

you have to factor the numerator and the denominator respectively by their higher power of x.

This will be more clear on some examples:

if $P(x) = a_n x^n + \dots + a_0$ with $a_n \neq 0$ then $\deg P = n$ (ex: $\deg(x^7+2x+1)=7$)

ex 1: $\deg(P) = \deg(Q)$

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{x^2\left(\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}\right)}{x^2\left(\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}\right)} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\cancel{x^2}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} =$$

$$= \text{"}\frac{3 - \frac{1}{\infty} - \frac{2}{\infty}}{5 + \frac{4}{\infty} + \frac{1}{\infty}}\text{"} = \frac{3-0-0}{5+0+0} = \frac{3}{5}$$

recall $\frac{L}{\infty} = 0$

note that this is the ratio of the leading coefficients of P and Q.

ex 2: $\deg(P) < \deg(Q)$

$$\lim_{x\to\infty} \frac{x^2+1}{4x^3+5x-4} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1 + \frac{1}{x^2}\right)}{\cancel{x^3}\left(4 + \frac{5}{x^2} - \frac{4}{x^3}\right)} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{x\left(4+\frac{5}{x^2}-\frac{4}{x^3}\right)} =$$

$$= \text{"}\frac{1+\frac{1}{\infty}}{\infty\left(4+\frac{5}{\infty}-\frac{4}{\infty}\right)}\text{"} = \text{"}\frac{1+0}{\infty\cdot(4+0-0)}\text{"} = \text{"}\frac{1}{\infty\cdot 4}\text{"} = \frac{1}{\infty} = 0$$

ex 2: $\deg(P) > \deg(Q)$

$$\lim_{x\to-\infty} \frac{x^3-1}{-x^2+3} = \lim_{x\to-\infty} \frac{x^{\cancel{3}}\left(1-\frac{1}{x^3}\right)}{\cancel{x^2}\left(-1+\frac{3}{x^2}\right)} = \lim_{x\to-\infty} \frac{x\left(1-\frac{1}{x^3}\right)}{-1+\frac{3}{x^2}} =$$

$$= \text{"}\frac{-\infty\left(1-\frac{1}{-\infty}\right)}{-1+\frac{3}{\infty}}\text{"} = \text{"}\frac{-\infty\cdot(1-0)}{-1+0}\text{"} = \text{"}\frac{-\infty\cdot 1}{-1}\text{"} = \text{"}\frac{-\infty}{-1}\text{"} = \infty$$

More in general we have:

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \begin{cases} 0 & \text{if } \deg P < \deg Q \\ \text{ratio of the leading coefficients} & \text{if } \deg P = \deg Q \\ \infty \text{ or } -\infty & \text{if } \deg P > \deg Q \end{cases}$$

Indeed we have the following theorem:

Th: If $a_n, b_m \neq 0$ then $\displaystyle\lim_{x \to \pm\infty} \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$.

you can find the proof at the end of this PDF.

Asymptotes

Def: The line $x=a$ is a vertical asymptote of the curve $y=f(x)$ if $x=a$ is an infinite discontinuity, i.e. if one of the following is true

$$\lim_{x \to a} f(x) = \infty \text{ or } -\infty$$

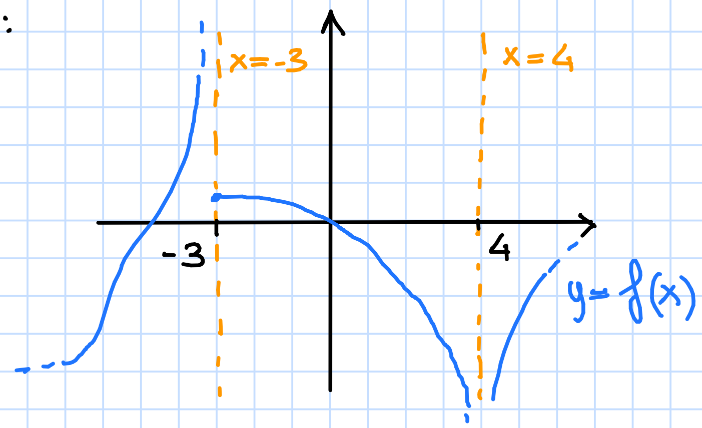
or

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty$$

or

$$\lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

ex:

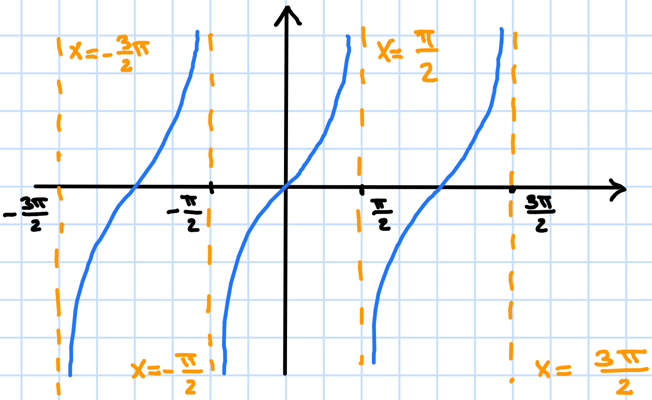


$x=-3$ and $x=4$ are two vertical asymptotes for f. Indeed:

$$\lim_{x \to -3^-} f(x) = \infty$$

$$\lim_{x \to 4} f(x) = -\infty$$

Remark: • A function can have infinitely many asymptotes. For instance the function $\tan(x)$ has a vertical asymptote at $x = \frac{\pi}{2} + k\pi$, for every integer k.



• The vertical asymptotes of a rational function have to be found in correspondence of the values that make the denominator 0.

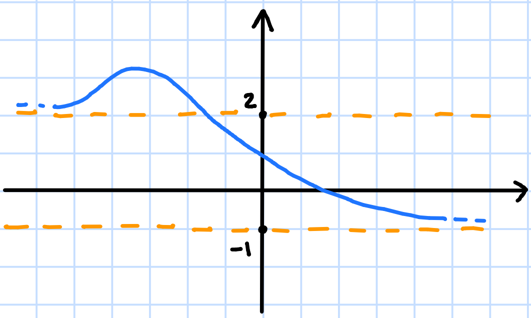
⚠ Warning: Not all the values that make the denominator 0 correspond to vertical asymptotes.

Indeed if $\frac{P(x)}{Q(x)}$ is a rational function and $Q(a)=0$, then $x=a$ can be an infinite or a removable discontinuity, and if a is a removable discontinuity then $x=a$ is not a vertical asymptote.

(An example of this fact is provided later).

Def: The line $y=L$ is a horizontal asymptote of the curve $y=f(x)$ if:

either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.



$y=2$ and $y=-1$ are two horizontal asymptotes since

$\lim_{x\to-\infty} f(x) = 2$ and $\lim_{x\to\infty} f(x) = -1$

Note that a function can cross its horizontal asymptotes.

Remark: • A function f can have at most two different horizontal asymptotes, one at ∞ and one at $-\infty$.

In particular f has exactly two different horizontal asymptotes if $\lim_{x\to\infty} f(x) = L_1$ and $\lim_{x\to-\infty} f(x) = L_2$, with $L_1 \neq L_2$.

• A constant function $f(x) = c$ has a horizontal asymptote of equation $y = c$.

Typical exercise about asymptotes

Write the equations of the vertical and horizontal asymptotes of the following rational function:

$$f(x) = \frac{x^2+6x+9}{x^2+2x-3}.$$

Solution

• HORIZONTAL ASYMPTOTE(S) → Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+6x+9}{x^2+2x-3} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1+\frac{6}{x}+\frac{9}{x^2}\right)}{\cancel{x^2}\left(1+\frac{2}{x}-\frac{3}{x^2}\right)} = \frac{1}{1} = 1$$

In the same way it is possible to show that $\lim_{x\to-\infty} f(x) = 1$,

Then $y = 1$ is the only horizontal asymptote for f.

recall that for a horizontal line it is the y-coordinate to be constant.

• VERTICAL ASYMPTOTE(S) → Find the value(s) that make the denominator 0 and compute the limit when x approaches those values.

$$\text{denominator} = 0 \iff x^2+2x-3=0 \iff (x-1)(x+3)=0$$
$$\iff x=1 \text{ or } x=-3.$$

Our candidates to be infinite discontinuities are $x=1$ and $x=-3$.

We have:

• $\lim\limits_{x \to 1^+} \dfrac{x^2+6x+9}{x^2+2x-3} = \lim\limits_{x \to 1^+} \dfrac{(x+3)^{\cancel{2}}}{(x-1)\cancel{(x+3)}} = \text{“}\dfrac{1+3}{0^+}\text{”} = \text{“}\dfrac{4}{0^+}\text{”} = +\infty \Rightarrow$

$x>1 \Leftrightarrow x-1>0$

$\Rightarrow$ 1 is an infinite discontinuity and $x=1$ a vertical asymptote.

• $\lim\limits_{x \to -3} \dfrac{x^2+6x+9}{x^2+2x-3} = \lim\limits_{x \to -3} \dfrac{(x+3)^{\cancel{2}}}{(x-1)\cancel{(x+3)}} = \dfrac{-3+3}{-3-1} \to \dfrac{0}{-4} = 0$

$\Rightarrow$ -3 is a removable discontinuity and does not correspond to a vertical asymptote.

Conclusion : f has a horizontal asymptote at $y=1$ and a vertical asymptote at $x=1$.

EXERCISE

Sketch the graph of a function f which satisfies simultaneously the following conditions:

$\lim\limits_{x \to -\infty} f(x) = -2$, $\lim\limits_{x \to 1^-} f(x) = \infty$, $f(1) = 2$,

$\lim\limits_{x \to 1^+} f(x) = 2$, $\lim\limits_{x \to \infty} f(x) = 1$.

Solution

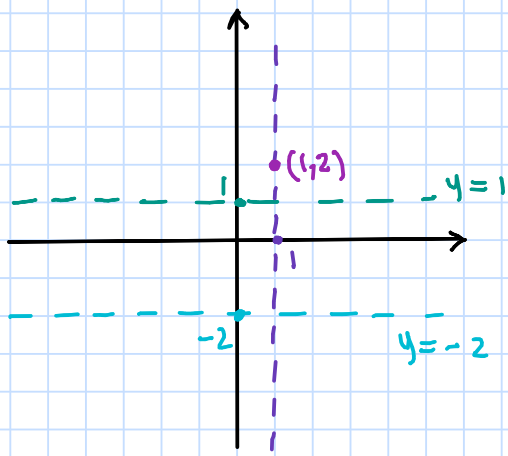
① Recognize and draw the horizontal / vertical asymptotes of the function and the points through which the graph passes.

$\lim\limits_{x \to -\infty} f(x) = -2 \Rightarrow y=-2$ is a horiz. asymp.

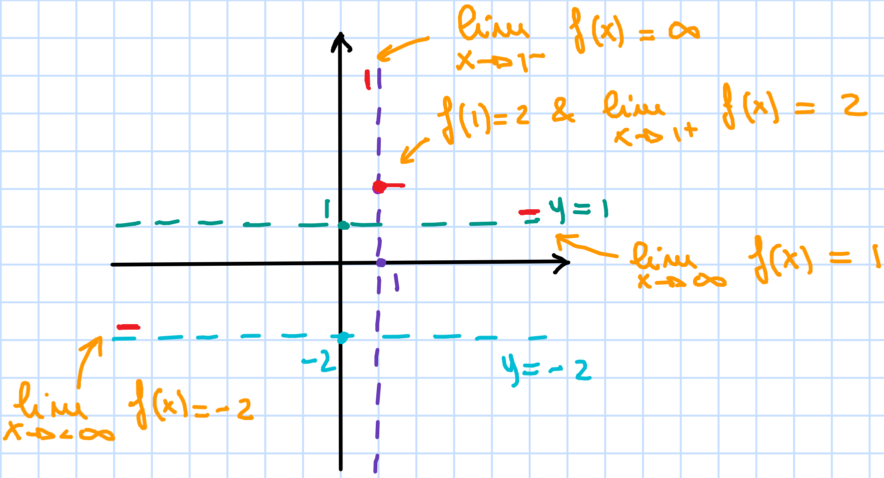
$\lim\limits_{x \to \infty} f(x) = 1 \Rightarrow y=1$ is a horiz. asymp.

$\lim\limits_{x \to 1^-} f(x) = \infty \Rightarrow x=1$ is a vertical asymp.

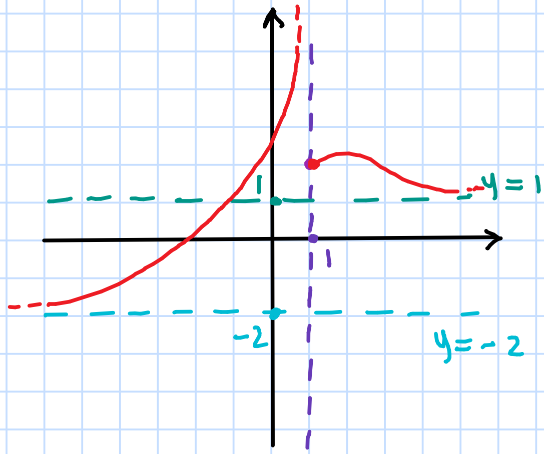
$f(1) = 2 \Rightarrow (1,2)$ is a point of the graph



② Write all the conditions as little marks — on your graph.



③ Connect the marks together (and make sure that your graph passes the vertical line test)



④ Check that your graph is one of the correct answers to the problem by verifying that it satisfies all the required conditions.