Algebraic Curves over Finite Fields

Homework 1

The first homework assignment consists in **5 problems** of your choice among those ones listed here below. It is **due on Monday, October 8**.

Note: This homework assignment is in constant evolution... Problems will be added as the semester goes on, but once an exercise is posted, it will not change. Discussion of the homework problems with me, or collaboration (in a reasonable degree) with your classmates, is encouraged, but you have to provide a note on which problems you had assistance.

Ex 1. Let $k = \mathbb{C}$ and let X be the (affine) conic described by the polynomial

$$f(x, y) = x^2 + y^2 - 1 \in k[x, y].$$

(a) Find the rational parametrization

$$\begin{array}{ccc} k & \longrightarrow & X \\ t & \longmapsto & (\varphi(t), \psi(t)) \end{array}$$

obtained via the construction given in class by using the point $(-1, 0) \in X$.

(b) Interpret geometrically the parametrization found in (a) in order to prove the trigonometric identities:

$$\sin(\theta) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}, \quad \cos(\theta) = \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}, \quad \text{for all } \theta \in (-\pi, \pi) \subset \mathbb{R}.$$

(c) Find the general form for the solution in $\mathbb{Q}^2$ of the equation $x^2 + y^2 - 1 = 0$.

(d) A *Pythagorean triple* is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. Use (c) in order to prove the *Euclid's formula* that generates a Pythagorean triple for each $m, n \in \mathbb{N}$ with $m > n > 0$:

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$

Ex 2. Consider the ideal $I = (x^3 + y^3 - 1) \subset \mathbb{Q}[x, y]$.

(a) Show that I is a prime ideal of $\mathbb{Q}[x, y]$.

(b) Show that $V(I)$ is a reducible algebraic set of $\mathbb{A}^2(\mathbb{Q})$.

(c) Explain why this proves that in the Hilbert's Nullstellensatz the hypothesis that k is algebraically closed can not be removed.

Ex 3. Let k be an algebraically closed field and consider the algebraic set $V = V(y - F(x)) \subset \mathbb{A}^2(k)$, where $F(x) \in k[x]$.

(a) Show that V is an affine variety.

(b) Show that V is isomorphic to $\mathbb{A}^1(k)$ by exhibiting an example of two morphisms $\varphi : V \to \mathbb{A}^1(k)$ and $\psi : \mathbb{A}^1(k) \to V$ such that $\psi \circ \varphi = \mathrm{id}_V$ and $\varphi \circ \psi = \mathrm{id}_{\mathbb{A}^1(k)}$.

(c) Explain why $k[V] \cong k[x]$ (where $k[x]$ is the ring of polynomials in one variable) and exhibit a ring isomorphism that fixes k (i.e. an isomorphism of k-algebras) between $k[V]$ and $k[x]$.