Calculus I - MAC 2311 - Section 001

Homework 1 - Solutions

Ex 1. (24 points) Compute the following limits and show all your work:

a) $$\lim_{x\to-\sqrt{2}} \frac{x^2}{x+1} \overset{\text{plug in}}{=} \frac{(-\sqrt{2})^2}{-\sqrt{2}+1} = \frac{2}{1-\sqrt{2}} \cdot \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{2+2\sqrt{2}}{1-2} = -2-2\sqrt{2}.$$

b) $$\lim_{t\to-1} \frac{t^2-1}{t^2+7t+6} = \lim_{t\to-1} \frac{(t+1)(t-1)}{(t+1)(t+6)} = \lim_{t\to-1} \frac{t-1}{t+6} \overset{\text{plug in}}{=} \frac{-1-1}{-1+6} = -\frac{2}{5}.$$

c) $$\lim_{x\to1} \frac{-\sqrt{x}+1}{2x-2} = \lim_{x\to1} \frac{1-\sqrt{x}}{2x-2} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to1} \frac{1-x}{2(x-1)(1+\sqrt{x})} =$$
$$= \lim_{x\to1} \frac{-(x-1)}{2(x-1)(1+\sqrt{x})} = \lim_{x\to1} \frac{-1}{2(1+\sqrt{x})} = -\frac{1}{4}.$$

d) $$\lim_{x\to\infty} \frac{2017x^{2017}+2017}{2018x^{2018}+2018} = \lim_{x\to\infty} \frac{x^{2017}\left(2017+\frac{2017}{x^{2017}}\right)}{x^{2018}\left(2018+\frac{2018}{x^{2018}}\right)} = \lim_{x\to\infty} \frac{2017+\frac{2017}{x^{2017}}}{x\left(2018+\frac{2018}{x^{2018}}\right)} =$$
$$= \text{“}\frac{2017+\frac{2017}{\infty}}{\infty\cdot\left(2018+\frac{2018}{\infty}\right)}\text{”} = \text{“}\frac{2017+0}{\infty\cdot(2018+0)}\text{”} = \text{“}\frac{2017}{\infty}\text{”} = 0.$$

e) $$\lim_{x\to-\infty} \frac{-3x^3+8x-1}{2x^3-x^2+4} = \lim_{x\to-\infty} \frac{x^3\left(-3+\frac{8}{x^2}-\frac{1}{x^3}\right)}{x^3\left(2-\frac{1}{x}+\frac{4}{x^3}\right)} = \lim_{x\to-\infty} \frac{-3+\frac{8}{x^2}-\frac{1}{x^3}}{2-\frac{1}{x}+\frac{4}{x^3}} =$$
$$= \text{“}\frac{-3+\frac{8}{\infty}-\frac{1}{-\infty}}{2-\frac{1}{-\infty}+\frac{4}{-\infty}}\text{”} = \frac{-3+0-0}{2-0+0} = -\frac{3}{2}.$$

f) $$\lim_{u\to-\infty} \frac{u^2+u+1}{-u+1} = \lim_{u\to-\infty} \frac{u^2\left(1+\frac{1}{u}+\frac{1}{u^2}\right)}{u\left(-1+\frac{1}{u}\right)} = \lim_{u\to-\infty} \frac{u\left(1+\frac{1}{u}+\frac{1}{u^2}\right)}{-1+\frac{1}{u}} =$$
$$= \text{“}\frac{-\infty\,(1+0+0)}{-1+0}\text{”} = \text{“}\frac{-\infty\cdot1}{-1}\text{”} = \text{“}\frac{-\infty}{-1}\text{”} = \infty.$$

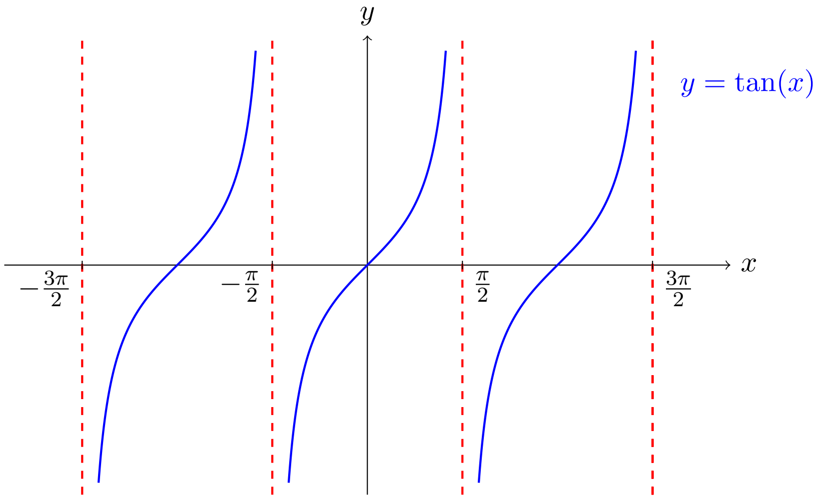
g) $$\lim_{\alpha\to0} \frac{\sin(8\alpha)}{2\alpha} = \lim_{\alpha\to0} \frac{\sin(8\alpha)}{2\alpha}\cdot\frac{4}{4} = \lim_{\alpha\to0} 4\cdot\frac{\sin(8\alpha)}{8\alpha} = 4\cdot\lim_{\alpha\to0}\frac{\sin(8\alpha)}{8\alpha} \overset{\lim_{x\to0}\frac{\sin x}{x}=1}{=} 4\cdot1 = 4.$$

h) $$\lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} \overset{\text{plug in}}{=} \text{“}\frac{1}{0}\text{”}.$$

This means the result of the limit will be ∞ or $-\infty$ and the sign will depend on the sing of the denominator. In this case, we have that when x is approaching $\frac{\pi}{2}$ from the left (i.e. $x<\frac{\pi}{2}$) then $\cos x>0$ (in order to convince yourself think about the unit circle or to the graph of the function cosine...). Thus:

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \text{“}\frac{1}{0^+}\text{”} = \infty.$$

You could also remark that $\frac{\sin x}{\cos x} = \tan x$ and, by using the graph of the tangent, get to the same conclusion that $\lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$.



i) $\lim\limits_{x\to 0} \dfrac{x-1}{x} \overset{\text{plug in}}{=} \text{“}\dfrac{1}{0}\text{”}$.

We will solve this limit by computing separately the left-hand and the right-hand limits:

$$\lim_{x\to 0^-} \frac{x-1}{x} = \text{“}\frac{0-1}{0^-}\text{”} = \text{“}\frac{-1}{0^-}\text{”} = \text{“}-1\cdot\frac{1}{0^-}\text{”} = \text{“}-1\cdot(-\infty)\text{”} = \infty.$$

$$\lim_{x\to 0^+} \frac{x-1}{x} = \text{“}\frac{0-1}{0^+}\text{”} = \text{“}\frac{-1}{0^+}\text{”} = \text{“}-1\cdot\frac{1}{0^+}\text{”} = \text{“}-1\cdot\infty\text{”} = -\infty.$$

Since $\lim\limits_{x\to 0^-} \dfrac{x-1}{x} \neq \lim\limits_{x\to 0^+} \dfrac{x-1}{x}$ then $\lim\limits_{x\to 0} \dfrac{x-1}{x}$ does not exist.

j) $\lim\limits_{x\to\infty} \dfrac{1}{x+\sqrt{3+x}} = \text{“}\dfrac{1}{\infty+\sqrt{3+\infty}}\text{”} = \text{“}\dfrac{1}{\infty+\sqrt{\infty}}\text{”} = \text{“}\dfrac{1}{\infty+\infty}\text{”} = \text{“}\dfrac{1}{\infty}\text{”} = 0.$

k) $\lim\limits_{x\to 1} f(x)$, where $\quad f(x) = \begin{cases} x^3-5x+7, & \text{when } x\le 1 \\ \sqrt{x+3}+1 & \text{when } x>1 \end{cases}$

Since we have to compute the limit of a piecewise function at its “breaking point”, we have first to compute separately the left-hand and the right-hand limits:

$$\lim_{x\to 1^-} f(x) \overset{x\le 1}{=} \lim_{x\to 1^-} x^3-5x+7 \overset{\text{plug in}}{=} 1-5+7 = 3.$$

$$\lim_{x\to 1^+} f(x) \overset{x\ge 1}{=} \lim_{x\to 1^-} \sqrt{x+3}+1 \overset{\text{plug in}}{=} \sqrt{1+3}+1 = 2+1 = 3.$$

Since $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = 3$ then $\lim\limits_{x\to 1} f(x) = 3$.

l) $\lim\limits_{\alpha\to\frac{\pi}{2}} \dfrac{\sqrt{1-\cos(\alpha)}-\sqrt{1+\cos(\alpha)}}{\cos(\alpha)} =$

$$= \lim_{\alpha \to \frac{\pi}{2}} \frac{\sqrt{1-\cos(\alpha)} - \sqrt{1+\cos(\alpha)}}{\cos(\alpha)} \cdot \frac{\sqrt{1-\cos(\alpha)} + \sqrt{1+\cos(\alpha)}}{\sqrt{1-\cos(\alpha)} + \sqrt{1+\cos(\alpha)}} =$$

$$= \lim_{\alpha \to \frac{\pi}{2}} \frac{(1-\cos(\alpha)) - (1+\cos(\alpha))}{\cos(\alpha)\left(\sqrt{1-\cos(\alpha)} + \sqrt{1+\cos(\alpha)}\right)} =$$

$$= \lim_{\alpha \to \frac{\pi}{2}} \frac{-2\cos(\alpha)}{\cos(\alpha)\left(\sqrt{1-\cos(\alpha)} + \sqrt{1+\cos(\alpha)}\right)} = \lim_{\alpha \to \frac{\pi}{2}} \frac{-2}{\sqrt{1-\cos(\alpha)} + \sqrt{1+\cos(\alpha)}} \overset{\text{plug in}}{=}$$

$$\overset{\text{plug in}}{=} \frac{-2}{\sqrt{1-\cos\left(\frac{\pi}{2}\right)} + \sqrt{1+\cos\left(\frac{\pi}{2}\right)}} = -\frac{2}{\sqrt{1-0} + \sqrt{1+0}} = -\frac{2}{2} = -1.$$

Ex 2. (20 points) Sketch the graph of a function f which satisfies simultaneously the following conditions:

a) $\lim\limits_{x\to\infty} f(x) = -2$,

b) The line $y = 3$ is a horizontal asymptote,

c) $f(3) = -3$,

d) The line $x = -1$ is a vertical asymptote,

e) $\lim\limits_{x\to -1^+} f(x) = \infty$,

f) $\lim\limits_{x\to -1^-} f(x) = 1$,

g) $x = -1$ is a solution for the equation $f(x) = 1$,

h) f has a removable discontinuity at $x = -3$.

Solution:

Let us translate some of these conditions geometrically.

a) $\lim\limits_{x\to\infty} f(x) = -2$: this means that the line $y = -2$ is a horizontal asymptote for the graph of the function f.

b) The line $y = 3$ is a horizontal asymptote: this means that $\lim\limits_{x\to\infty} f(x) = 3$ or $\lim\limits_{x\to -\infty} f(x) = 3$. Since we know already from a) that $\lim\limits_{x\to\infty} f(x) = -2$ (and the limit is unique) then we get $\lim\limits_{x\to -\infty} f(x) = 3$.

c) $f(3) = -3$: the graph of the function passes through the point $(3, -3)$.

d) The line $x = -1$ is a vertical asymptote.

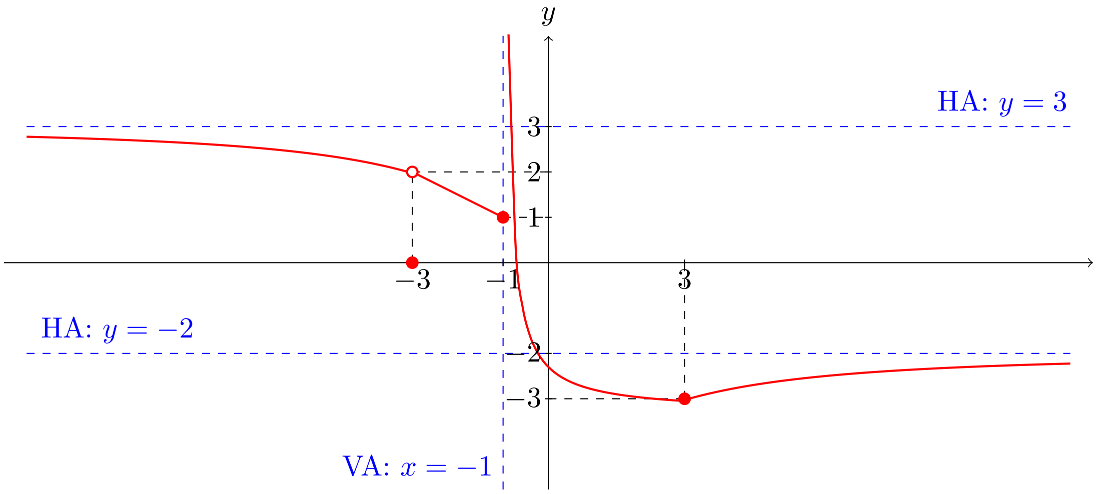
e) $\lim\limits_{x\to -1^+} f(x) = \infty$,

g) $\lim\limits_{x\to -1^-} f(x) = 1$,

f) $x = -1$ is a solution for the equation $f(x) = 1$: this means that $f(-1) = 1$, i.e. the graph of the function passes through the point $(-1, 1)$.

h) f has a removable discontinuity at $x=-3$: this means that $\lim_{x\to -3} f(x) = L$ exists (and is a number) and either f is undefined at $x=-3$ or $f(-3) \neq L$. In the example below we have $\lim_{x\to -3} f(x) = 2$ and $f(-3) = 0$.

Of course there exist infinitely many examples of functions satisfying simultaneously all the previous conditions. An example is given by the function whose graph is the following:



Ex 3. (20 points) Let a and b be two constants (= two real numbers) and f be the function:

$$f(x) = \begin{cases} x^2 - 3x + a, & \text{when } x < -1 \\ 2\cos(\pi x), & \text{when } -1 \leq x \leq 2 \\ \frac{-2x+2b^2}{x}, & \text{when } x > 2. \end{cases}$$

a) Compute $f(-1)$, $\lim_{x\to(-1)^-} f(x)$, $\lim_{x\to(-1)^+} f(x)$, $f(2)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$.

b) Find the values of a and b that make f continuous everywhere.

Solution:

We remark that $f(x)$ is a piecewise function whose branches are respectively defined on the intervals $(-\infty,-1)$, $[-1,2]$ and $(2,\infty)$.

a)
- When $x=-1$ then $f(x) = 2\cos(\pi x)$, hence:
  $f(-1) = 2\cos(\pi\cdot(-1)) = 2\cos(-\pi) = -2$.
- When $x<-1$ then $f(x) = x^2-3x+a$, hence:
  $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 - 3x + a = (-1)^2 - 3\cdot(-1) + a = 4+a$.
- When $x>-1$ then $f(x) = 2\cos(\pi x)$, hence:

$$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} 2\cos(\pi x) = 2\cos(\pi\cdot(-1)) = 2\cos(-\pi) = -2.$$

- When $x = 2$ then $f(x) = 2\cos(\pi x)$, hence:
  $f(2) = 2\cos(2\pi) = 2.$
- When $x < 2$ then $f(x) = 2\cos(\pi x)$, hence:
  $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2\cos(\pi x) = 2\cos(2\pi) = 2.$
- When $x > 2$ then $f(x) = \frac{-2x+2b^2}{x}$, hence:
  $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{-2x+2b^2}{x} = \frac{-2\cdot 2+2b^2}{2} = \frac{-4+2b^2}{2} = -2+b^2.$$

b) First we remark that the function f is continuos on $(-\infty,-1)$ (because x^2-3x+a is a polynomial), on $(-1,2)$ (because $2\cos(\pi x)$ is continuous) and on $(2,\infty)$ (because the only discontinuity of the rational function $\frac{-2x+2b^2}{x}$ is $x=0$ which is outside the interval $(2,\infty)$). Thus, the function f is continuous everywhere if and only if it is continuous simultaneously at $x=-1$ and $x=2$ (its breaking points).

Now:

- f is continuous at 1 $\Leftrightarrow \lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(1) \Leftrightarrow -2 = 4+a \Leftrightarrow a = -6.$
- f is continuous at 2 $\Leftrightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) \Leftrightarrow -2+b^2 = 2 \Leftrightarrow b^2 = 4 \Leftrightarrow b = 2$ or $b = -2$.

Therefore f is continuous simultaneously at $x=-1$ and $x=2$ if and only if ($a=-6$ and $b=2$) or ($a=-6$ and $b=-2$)

Ex 4. (20 points)

a) It is the Sunday before the test. A calculus student, following the suggestion of his instructor, decides to go hiking on the highest mountain in Florida in order to understand the Intermediate Value Theorem in a more concrete situation.
Let $h(t)$ be the function that at each time t (in hours) represents the height of the student above sea level (in feet). If

$$h(t) = -t^2 + 5t + 1,$$

prove that there is a time between 0 and 3 hours at which the student is 6 feet above sea level.

b) Compute the instantaneous rate of change of $h(t)$ at $t=1$, that is $h'(1)$, by using the definition of derivative.

Solution:

a) Mathematically we can rewrite the problem of the exercise in the following way:

If $h(t) = -t^2+5t+1$, *show that there exists a number* c *in* $(0,3)$ *such that* $h(c) = 6$.

Recall:

Theorem *(Intermediate Value Theorem). Let f be a continuous function on a closed interval $[a,b]$, with $f(a) \neq f(b)$. Then for every number N between $f(a)$ and $f(b)$ there exists c in (a,b) such that $f(c) = N$.*

Let us apply the Intermediate Value Theorem to our exercise in 4 steps:

♣ **Set the function and the closed interval**

Let us consider the function $h(t) = -t^2 + 5t + 1$ on the closed interval $[0,3]$.

♣ **Point out that the function is continuous on the closed interval**

The function h is continuous everywhere (and in particular on $[0,3]$) since it is a polynomial.

♣ **Compute the value of the function at the endpoints of the interval**

We have:

$$h(0) = -0 + 5 \cdot 0 + 1 = 1 \qquad \text{and} \qquad h(1) = -3^2 + 5 \cdot 3 + 1 = 7.$$

♣ **Conclusion**

Now 6 is a number between 1 and 7 ($1 < 6 < 7$), therefore by the Intermediate Value Theorem, there exists a number c in $(0,3)$ such that $h(c) = 6$.
In our original problem this number c represents the time at which the calculus student is 6 feet above sea level.

b) Recall the definition of the derivative of a function $f(x)$ at a point a:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We use the previous definition for computing the instantaneous rate of change of $h(t)$ at $t = 1$, that is $h'(1)$:

$$\begin{aligned} h'(1) = \lim_{t \to 1} \frac{h(t) - h(1)}{t - 1} &= \lim_{t \to 1} \frac{-t^2 + 5t + 1 - (-1 + 5 + 1)}{t - 1} = \\ &= \lim_{t \to 1} \frac{-t^2 + 5t - 4}{t - 1} = \\ &= \lim_{t \to 1} \frac{-(t^2 - 5t + 4)}{t - 1} = \\ &= \lim_{t \to 1} \frac{-(t-4)(t-1)}{t - 1} = \\ &= \lim_{t \to 1} \frac{-(t-4)}{1} = \frac{-(1-4)}{1} = 3. \end{aligned}$$

Ex 5. (20 points) Which statements are True/False? Justify your answers.

a) A function can have at most 2 horizontal asymptotes.

True. Indeed $y = L$ is a horizontal asymptote if and only if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$. Hence, the maximum number of horizontal asymptotes that a function can have is two, and this situation occurs when $\lim_{x\to\infty} f(x) = L_1$ and $\lim_{x\to-\infty} f(x) = L_2$, with $L_1 \neq L_2$.

b) If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function and a is a number such that $Q(a) = 0$ then $x = a$ is a vertical asymptote for f.

False. Indeed a number a such that $Q(a) = 0$ can also be a removable discontinuity for f (and in this case it does not correspond to a vertical asymptote). Consider as an example the following rational function:

$$f(x) = \frac{x(x+1)}{x}.$$

The number $x = 0$ makes the denominator equal zero, but

$$\lim_{x\to 0} \frac{x(x+1)}{x} = \lim_{x\to 0} x + 1 = 1.$$

Hence $x = 0$ is a removable (and not infinite) discontinuity.

c) If $s(t)$ is a position function and $s(3) = 0$, then the velocity at $t = 3$ is zero.

False. Indeed, if $s(t)$ is a position function, the instantaneous velocity at a time t is given by the slope of the tangent line to the graph of $s(t)$ at the point $(t, s(t))$ (and not by the values of $s(t)$).
If we consider the position function $s(t) = t - 3$ then $s(3) = 0$, but $v(3) = 1 \neq 0$ (1 is indeed the slope of the line $s = t - 3$).

d) If $-|x-1| \leq f(x) \leq |x-1|$ near 1, then $\lim_{x\to 1} f(x) = 0$.

True. Indeed $\lim_{x\to 1} -|x-1| = \lim_{x\to 1} |x-1| = 0$. Then, since $-|x-1| \leq f(x) \leq |x-1|$, by the Squeeze Theorem one has also $\lim_{x\to 1} f(x) = 0$.