# On the maximal number of points on singular curves over finite fields

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# Notation





- $\mathbb{F}_q$ the finite field with $q$ elements.
- With the word "curve" we will always refer to an absolutely irreducible projective algebraic curve.

# Smooth curves over finite fields





Let $X$ be a smooth curve over $\mathbb{F}_q$. We can associate to $X$ two nonnegative integers:

- $\sharp X(\mathbb{F}_q)$: the number of rational points on $X$ over $\mathbb{F}_q$;
- $g$: the genus of $X$.

The integers $q, \sharp X(\mathbb{F}_q)$ and $g$ satisfy the **Serre-Weil inequality**:

$$|\sharp X(\mathbb{F}_q) - (q+1)| \leq g[2\sqrt{q}]$$

Let us denote by

$$N_q(g)$$

the maximal number of rational points over $\mathbb{F}_q$ that a curve of genus $g$ can have. Clearly we have:

$$N_q(g) \leq q + 1 + g[2\sqrt{q}]$$

# ... and if $X$ is singular?



If now we remove the hypothesis of smoothness for $X$, can we still say something about $\sharp X(\mathbb{F}_q)$?

Yes, but we have to introduce another invariant for $X$,

**the arithmetic genus** $\pi$.

To define $\pi$ we have to recall some local properties of curves.

# Points and local rings

Let $X$ be a curve over $\mathbb{F}_q$ and let $\mathbb{F}_q(X)$ be the function field of $X$. Let $Q$ be a point on $X$ and let us define

$$\mathcal{O}_Q := \{f \in \mathbb{F}_q(X) \mid f \text{ is regular at } Q\}.$$

$\mathcal{O}_Q$ is a local ring with maximal ideal

$$\mathcal{M}_Q := \{f \in \mathcal{O}_Q \mid f \text{ vanishes at } Q\}$$

Moreover we have:

$$[\mathcal{O}_Q/\mathcal{M}_Q : \mathbb{F}_q] = \deg Q.$$

Fact: $\mathcal{O}_Q$ is integrally closed if and only if $Q$ is a nonsingular point.

$$\Downarrow$$

$X$ is smooth if and only if $\mathcal{O}_Q$ is integrally closed for every $Q$ on $X$.

# Normalization of a singular curve



Let $\tilde{X}$ be the **normalization** of $X$, i.e. the smooth curve together with a regular map

$$\nu : \tilde{X} \to X$$

such that $\nu$ is finite and birational.

In particular $X$ and $\tilde{X}$ have the same function field:

$$\mathbb{F}_q(X) = \mathbb{F}_q(\tilde{X}).$$

# Diagram

Let $Q$ be a point on $X$ and let $P_1, \ldots, P_n$ be the points on $\tilde{X}$ such that $\nu(P_i) = Q$ for all $i = 1, \ldots, n$.

$$
\begin{tikzcd}
 & \mathbb{F}_q(\tilde{X}) = \mathbb{F}_q(X) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathcal{O}_{P_1} \arrow[dr, dash] & \cdots \arrow[d, dash] & \mathcal{O}_{P_n} \arrow[dl, dash] \\
 & \overline{\mathcal{O}_Q} = \bigcap_{1 \leq i \leq n} \mathcal{O}_{P_n} \arrow[d, dash] & \\
 & \mathcal{O}_Q &
\end{tikzcd}
$$

$\overline{\mathcal{O}_Q}$ is the integral closure of $\mathcal{O}_Q$.

# The arithmetic genus

$\overline{\mathcal{O}_Q}/\mathcal{O}_Q$ is a finite dimensional $\mathbb{F}_q$-vectorial space. We set:

$$\delta_Q := \dim_{\mathbb{F}_q} \overline{\mathcal{O}_Q}/\mathcal{O}_Q$$

We can now define the **arithmetic genus** $\pi$ of a curve $X$ as the integer:

$$\pi := g + \sum_{Q \in \mathrm{Sing}\, X(\overline{\mathbb{F}_q})} \delta_Q,$$

where $g$ is the genus of the normalization $\tilde{X}$ of $X$ ($g$ is called the **geometric genus** of $X$).

- $\pi \geq g$;
- $\pi = g$ if and only if $X$ is a smooth curve;
- If $X$ is a plane curve of degree $d$, $\pi = \frac{(d-1)(d-2)}{2}$.

# Bounds for singular curves

In 1996, Aubry and Perret give the following result on singular curves:

$$|\sharp\tilde{X}(\mathbb{F}_q) - \sharp X(\mathbb{F}_q)| \leq \pi - g,$$

from which they obtain directly the equivalent of Serre-Weil bound for singular curves:

$$|\sharp X(\mathbb{F}_q) - (q+1)| \leq g[2\sqrt{q}] + \pi - g.$$

# The quantity $N_q(g, \pi)$





We define an analogous quantity of $N_q(g)$ for singular curves:

**Definition**

For $q$ a power of a prime, $g$ and $\pi$ non negative integers such that $\pi \geq g$, let us define the quantity

$$N_q(g, \pi)$$

as the maximal number of rational points over $\mathbb{F}_q$ that a curve defined over $\mathbb{F}_q$ of geometric genus $g$ and arithmetic genus $\pi$ can have.

Obviously we have

$$N_q(g, g) = N_q(g),$$

$$N_q(g, \pi) \leq N_q(g) + \pi - g$$

# Fukasawa, Homma and Kim's curve

In 2011, Fukasawa, Homma and Kim consider and study the rational plane curve $B$ over $\mathbb{F}_q$ defined by the image of

$$\begin{array}{cccc} \Phi : & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (s,t) & \mapsto & (s^{q+1}, s^q t + s t^q, t^{q+1}) \end{array}$$

Properties of $B$:

1. $B$ is a rational curve of degree $q+1 \Rightarrow g = 0, \pi = \frac{q^2-q}{2}$;
2. For $P \in \mathbb{P}^1$, $\Phi(P) \in \mathrm{Sing}(B)$ if and only if $P \in \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q) \Rightarrow B$ has $\frac{q^2-q}{2}$ ordinary double points.
3. $\sharp B(\mathbb{F}_q) = q + 1 + \frac{q^2-q}{2} \Rightarrow$ $\underline{B \text{ attains the Aubry-Perret bound}}$!!

$$\Downarrow$$

$$N(0, \frac{q^2-q}{2}) = N_q(0) + \frac{q^2-q}{2}$$

# Question



Does there exist other different values of $g$ and $\pi$ for which

$$N_q(g, \pi) = N_q(g) + \pi - g\,?$$

To try to answer this question we need to find some way to construct singular curves with prescribed geometric genus $g$ and arithmetic genus $\pi$ and "many" rational points.

# Singular curves with many points

**Theorem**

*Let $X$ be a smooth curve of genus $g$ defined over $\mathbb{F}_q$. Let $\pi$ be an integer of the form*

$$\pi = g + a_2 + 2a_3 + 3a_4 + \cdots + (n-1)a_n$$

*with $0 \leq a_i \leq B_i(X)$, where $B_i(X)$ is the number of closed points of degree $i$ on the curve $X$. Then there exists a (singular) curve $X'$ over $\mathbb{F}_q$ of arithmetic genus $\pi$ such that $X$ is the normalization of $X'$ (so that $X'$ has geometric genus $g$) and*

$$\sharp X'(\mathbb{F}_q) = \sharp X(\mathbb{F}_q) + a_2 + a_3 + a_4 + \cdots + a_n.$$

# Sketch of the proof





Without loss of generality we can limit ourselves to the affine case; the general case will follow directly by covering $X$ by affine opens.

Let us take on the curve $X$:

- $a_2$ closed points of degree 2 : $S_2 = \{Q_1^{(2)}, Q_2^{(2)}, \ldots, Q_{a_2}^{(2)}\}$;
- $a_3$ closed points of degree 3 : $S_3 = \{Q_1^{(3)}, Q_2^{(3)}, \ldots, Q_{a_3}^{(3)}\}$;

  $\vdots$
- $a_n$ closed points of degree $n$ : $S_n = \{Q_1^{(n)}, Q_2^{(n)}, \ldots, Q_{a_n}^{(n)}\}$;

$\Downarrow$

$S := S_2 \cup S_3 \cup \cdots \cup S_n$.

# Sketch of the proof

Let $\mathcal{O}$ be the sheaf of local rings of $X$. Starting from $\mathcal{O}$ we are going now to define a new sheaf of local rings in the following way:

- for every $Q \in X - S$ we put $\mathcal{O}'_Q := \mathcal{O}_Q$.
- for every $Q \in S$ we set $\mathcal{O}'_Q := \mathbb{F}_q + \mathcal{M}_Q$;

The set of $\mathcal{O}'_Q$, for $Q \in X$, form a subsheaf $\mathcal{O}'$ of $\mathcal{O}$.

In particular for every $Q \in S$ we have:

- $\mathcal{O}'_Q$ is local with maximal ideal $\mathcal{M}_Q$ and

$$\left[\mathcal{O}'_Q/\mathcal{M}_Q : \mathbb{F}_q\right] = 1;$$

- $\mathcal{O}_Q$ is the integral closure of $\mathcal{O}'_Q$;
- $\mathcal{O}_Q/\mathcal{O}'_Q$ is an $\mathbb{F}_q$-vectorial space of dimension $\deg Q - 1$.

# Sketch of the proof



Let us denote

$$A' := \bigcap_{Q \in X} \mathcal{O}'_Q.$$

$A'$ is a $\mathbb{F}_q$-algebra of finite type corresponding to an affine irreducible curve $X'$ defined over $\mathbb{F}_q$.

By construction we obtain that:

- $\tilde{X'}$=X so that $X'$ has geometric genus $g$;
- $\sharp X'(\mathbb{F}_q) = \sharp X(\mathbb{F}_q) + |S| = \sharp X(\mathbb{F}_q) + a_2 + a_3 + \cdots + a_n$;
- $X'$ has arithmetic genus $\pi = g + a_2 + 2a_3 + 3a_4 + \cdots + (n-1)a_n$.

# Remarks



- Unfortunately this construction is not explicit;
- this construction corresponds to a glueing of points on the curve obtained from $X$ by extension of the base field to its algebraic closure.

# Diagram

$$
\begin{tikzcd}
 & & & & \overline{\mathbb{F}_q}(X) \arrow[dlll, dash] \arrow[dll, dash] \arrow[dl, dash] \arrow[dd, dash] \\
 & \mathcal{O}_{P_1} \arrow[dr, dash] & \cdots \arrow[d, dash] & \mathcal{O}_{P_d} \arrow[dl, dash] & \\
 & & \bigcap_{1\le i\le d} \mathcal{O}_{P_i} \arrow[dll, dash] \arrow[d, "\cap \mathbb{F}_q(X)"] & & \mathbb{F}_q(X) \arrow[dll, dash] \\
\overline{\mathbb{F}_q} + r\left(\bigcap_{1\le i\le d} \mathcal{O}_{P_i}\right) \arrow[d, "\cap \mathbb{F}_q(X)"] & & \mathcal{O}_Q \arrow[dll, dash] & & \\
\mathcal{O}'_Q := \mathbb{F}_q + \mathcal{M}_Q & & & &
\end{tikzcd}
$$

# The case of rational curves





Let start from $X = \mathbb{P}^1$, the projective line, over a finite field $\mathbb{F}_q$.
As

$$B_2(\mathbb{P}^1) = \frac{q^2 - q}{2},$$

we have:

**Proposition**

*For any $\pi \leq \frac{q^2-q}{2}$, there exists a (singular) rational curve $X'$ over $\mathbb{F}_q$ of arithmetic genus $\pi$ that attains the Aubry-Perret bound, i.e.*

$$\sharp X(\mathbb{F}_q) = q + 1 + \pi.$$

*In other terms we have*

$$N_q(0, \pi) = N_q(0) + \pi = q + 1 + \pi.$$

# Maximal curves



### Definition

A (not necessarily smooth) curve $X$ defined over $\mathbb{F}_q$ is called maximal if

$$\sharp X(\mathbb{F}_q) = q + 1 + g[2\sqrt{q}] + \pi - g.$$

### Proposition

*If $X$ is a maximal curve defined over $\mathbb{F}_q$ with $q$ a square, of geometric genus $g$ and arithmetic genus $\pi$, then:*

$$2g(\sqrt{q} + q - 1) + 2\pi \leq q^2 - q.$$

In particular, for a maximal rational curve (and for $q$ not necessarily square), this proposition implies:

$$\pi \leq \frac{q^2 - q}{2}$$

**Proposition**

*We have*

$$N_q(0, \pi) = q + 1 + \pi$$

*if and only if* $\pi \leq \frac{q^2 - q}{2}$.

With this proposition we completely answer the question when $g = 0$.


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