

NEW SEXTICS OF GENUS 6 AND 10 ATTAINING THE SERRE BOUND

ANNAMARIA IEZZI, MOTOKO QIU KAWAKITA, AND MARCO TIMPANELLA

ABSTRACT. We provide new examples of curves of genus 6 or 10 attaining the Serre bound. They all belong to the family of sextics introduced in [19] as a generalization of the Wiman sextics [36] and Edge sextics [9]. Our approach is based on a theorem by Kani and Rosen which allows, under certain assumptions, to fully decompose the Jacobian of the curve. With our investigation we are able to update several entries in <http://www.manypoints.org> ([35]).

1. INTRODUCTION

Let \mathbb{F}_q be a finite field with q elements, where q is a power of a prime p . A projective, absolutely irreducible, non-singular, algebraic curve C defined over \mathbb{F}_q is called \mathbb{F}_q -maximal if the number of its \mathbb{F}_q -rational points, denoted by $\#C(\mathbb{F}_q)$, attains the Hasse-Weil upper bound

$$\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q},$$

where g is the genus of the curve. A classical and well-studied example of maximal curve is the so called *Hermitian curve* \mathcal{H}_q of affine equation $x^q + x = y^{q+1}$. This curve is \mathbb{F}_{q^2} -maximal, and it has the largest possible genus for an \mathbb{F}_{q^2} -maximal curve [30]. By a result commonly referred to as the Kleiman–Serre covering result [21, 24], any curve defined over \mathbb{F}_q and \mathbb{F}_q -covered by an \mathbb{F}_q -maximal curve, is \mathbb{F}_q -maximal. This result provides a strong tool for constructing new examples of maximal curves from the known ones: indeed, if C is an \mathbb{F}_q -maximal curve, also the quotient curve C/G , where G is a finite subgroup of the automorphism group of C , is \mathbb{F}_q -maximal. Most of the known \mathbb{F}_{q^2} -maximal curves are obtained, following this approach, as quotient curves of the Hermitian curve \mathcal{H}_q , see for instance [7, 12, 29] and the references therein. In the research community, for a while, it was even speculated that all maximal curves could be obtained as quotient curves of the Hermitian curve. However, in 2009, Giulietti and Korchmáros [14] proved that this is false, as they constructed an \mathbb{F}_{q^6} -maximal curve not covered by the Hermitian curve when $q > 2$. Since then, a few other examples of maximal curves not covered by the Hermitian curve have been provided [4, 11, 33], but a complete solution of the classification problem for maximal curves seems to be out of reach, and looking for new examples is a very active line of research.

Clearly, \mathbb{F}_q -maximal curves can only exist when q is a square. In 1983, Serre provided a non-trivial improvement of the Hasse-Weil bound when q is not a square,

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namely

$$\#C(\mathbb{F}_q) \leq q + 1 + g\lfloor 2\sqrt{q} \rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function (see [31]). We refer to this bound as the *Serre bound*.

Curves attaining the Hasse-Weil or the Serre bound are interesting objects in their own right, but also for their applications in Coding Theory. Indeed, in [15], Goppa described a way to use algebraic curves to construct linear error correcting codes, the so called algebraic geometric codes (AG codes). As the relative Singleton defect of an AG code from a curve C is upper bounded by the ratio g/N , where g is the genus of C and N can be as large as the number of \mathbb{F}_q -rational points of C , it follows that curves with many rational points with respect to their genus are of great interest in Coding Theory. For this reason, maximal curves and curves attaining the Serre bound have been widely investigated in the last years, see for instance [2, 3, 6, 10, 23, 25, 26, 27, 34].

The aim of this paper is to provide new explicit examples of curves attaining the Hasse-Weil or the Serre bound. We stress the fact that while there is a wide literature about maximal curves, there are only a few examples of curves attaining the Serre bound and which are not maximal.

All the examples that we provide in this paper have genus 6 or 10 and belong to the following generalization of the families of the Wiman sextics [36] and Edge sextics [9] introduced in [19]

$$x^6 + y^6 + 1 + a(x^4y^2 + x^2 + y^4) + b(x^2y^4 + x^4 + y^2) + cx^2y^2 = 0,$$

with $a, b, c \in \mathbb{F}_q$. Our approach is based on a theorem by Kani and Rosen (see [17, Theorem B]), which provides a decomposition of the Jacobian of a curve under certain conditions on its automorphism group. This theorem has already been used to find examples of curves with many rational points with respect to their genus, see for instance [1, 18, 19, 22].

The paper is organized as follows. In Section 2 we recall the defining equations of the Wiman's and Edge's sextics and the statement of a theorem by Kani and Rosen. In Section 3 we prove a useful result to decompose completely the Jacobian of degree-6 hyperelliptic curve under certain assumptions (see Theorem 2). In Section 4 and 5 we use Theorem 2 to decompose the Jacobians of certain Wiman's and Edge's sextics, and we find new explicit examples of curves attaining the Hasse-Weil bound or the Serre bound. Our investigation allows to update several entries in <http://www.manypoints.org> ([35]). Finally, in Section 6, we prove that one of the maximal curves that we exhibit in the previous section is not Galois-covered by the Hermitian curve.

2. FAMILIES OF WIMAN'S AND EDGE'S SEXTICS

In 1896 Wiman introduced in [36] the family of sextics defined by the equation

$$W_{a,b}: x^6 + y^6 + 1 + a(x^4y^2 + x^2y^4 + x^4 + x^2 + y^4 + y^2) + bx^2y^2 = 0,$$

where $a, b \in \mathbb{C}$. Similarly, in 1981, Edge considered in [9] the family of sextics described by the equation

$$E_\alpha: x^6 + y^6 + 1 + (x^2 + y^2 + 1)(x^4 + y^4 + 1) - 12x^2y^2 + \alpha(y^2 - 1)(1 - x^2)(x^2 - y^2) = 0,$$

with $\alpha \in \mathbb{C}$. The geometry of the Wiman-Edge pencil is studied in [8].

By considering Wiman's and Edge's sextics defined over a finite field rather than \mathbb{C} , in [18] and [20] it has been shown that they provide several examples of \mathbb{F}_{p^2} -maximal curves, or of curves attaining the Serre bound over \mathbb{F}_p and \mathbb{F}_{p^3} . This allowed to update several entries in <http://www.manypoints.org> ([35]).

By combining the defining equations of the Wiman's and Edge's sextics, a generalization of these two families was introduced and studied in [19]:

$$S: x^6 + y^6 + 1 + a(x^4y^2 + x^2 + y^4) + b(x^2y^4 + x^4 + y^2) + cx^2y^2 = 0.$$

It is readily seen that if $a = b$ then S is a Wiman's sextic, whereas if $a = (1 - \alpha)/2$, $b = (1 + \alpha)/2$ and $c = -6$ then S is an Edge's sextic.

In this paper we focus on $S_{a,b}$, obtained from S by setting $c = -3(a + b + 1)$, (see Section 4) and on $W_{a,b}$ (see Section 5).

The idea for studying the number of \mathbb{F}_q -rational points of $S_{a,b}$ and $W_{a,b}$ is to decompose their Jacobian varieties. For this we will make use of the following theorem by Kani and Rosen.

Theorem 1. [17, Theorem B] *Let C be a curve and let G be a finite subgroup of the automorphism group $\text{Aut}(C)$ such that $G = H_1 \cup \dots \cup H_n$, where the H_i 's are subgroups of G with $H_i \cap H_j = \{1_G\}$ for $i \neq j$. Then the following isogeny relation holds:*

$$\text{Jac}(C)^{n-1} \times \text{Jac}(C/G)^g \simeq \text{Jac}(C/H_1)^{h_1} \times \dots \times \text{Jac}(C/H_n)^{h_n},$$

where $g = |G|$ and $h_i = |H_i|$.

As we will see in Sections 4 and 5 we can always decompose $\text{Jac}(S_{a,b})$ and $\text{Jac}(W_{a,b})$ as a product of elliptic curves and Jacobians of hyperelliptic curves of degree 6. This is why in the next section, in order to fully decompose the Jacobians as a product of elliptic curves, we first study the decomposition of the Jacobian variety of a degree-6 hyperelliptic curve.

3. DECOMPOSING THE JACOBIAN OF CERTAIN CURVES OF GENUS TWO

In this section, we deal with the decomposition of the Jacobian variety $\text{Jac}(D)$ of a hyperelliptic curve

$$D: y^2 = f(x),$$

where $f(x)$ is a degree 6 polynomial defined over \mathbb{F}_q . In particular we prove that, under certain assumptions on $f(x)$, the Jacobian $\text{Jac}(D)$ splits completely as a product of two elliptic curves. The idea of the proof comes from Section 1.2 of [16], where the authors work over an algebraically closed field, rather than a finite field.

Theorem 2. *Assume that the polynomial $f(x)$ factors completely over the finite field \mathbb{F}_q , i.e.*

$$f(x) = c \cdot \prod_{i=1}^6 (x - a_i)$$

with $a_i \in \mathbb{F}_q$ for $i = 1, \dots, 6$, $a_i \neq a_j$ when $i \neq j$. Assume, moreover, that there exists a permutation of the roots such that

$$(a_2 - a_4)(a_1 - a_6)(a_3 - a_5) = (a_2 - a_6)(a_1 - a_5)(a_3 - a_4).$$

and set

$$\lambda = \frac{(a_1 - a_3)(a_2 - a_4)}{(a_2 - a_3)(a_1 - a_4)} \mu = \frac{(a_1 - a_3)(a_2 - a_5)}{(a_2 - a_3)(a_1 - a_5)},$$

$$\theta = c \cdot (a_2 - a_3)(a_1 - a_4)(a_1 - a_5)(a_1 - a_6).$$

Then, we have the following:

(a) *The hyperelliptic curve D is isomorphic to*

$$y^2 = \theta x(x-1)(x-\lambda)(x-\mu) \left(x - \frac{\lambda(1-\mu)}{1-\lambda} \right) \quad (1)$$

over the finite field \mathbb{F}_q .

(b) *Assume that there exists a square root of $\lambda(\lambda-\mu)$ in the finite field \mathbb{F}_q . Then the Jacobian of the hyperelliptic curve D decomposes over \mathbb{F}_q as*

$$\text{Jac}(D) \sim E_\sigma \times E_\tau,$$

where E_σ and E_τ are the elliptic curves defined by the equations:

$$s^2 = \frac{\theta(1-\mu)}{1-\lambda} t(t-1) \left(t - \frac{(1-\lambda)(\mu-2\lambda \pm 2(\lambda^2-\lambda\mu)^{1/2})}{\mu-1} \right).$$

Moreover, the number of rational points of D satisfies

$$\#D(\mathbb{F}_q) = \#E_\sigma(\mathbb{F}_q) + \#E_\tau(\mathbb{F}_q) - q - 1.$$

Proof. (a) Let us consider the map defined by

$$x \mapsto \frac{(a_1 - a_3)(a_1 - a_2) + x - a_1}{(x - a_1)(a_2 - a_3)}, \quad y \mapsto \frac{(a_1 - a_2)^2(a_1 - a_3)^2 y}{(a_2 - a_3)^2(x - a_1)^3}.$$

With this map, D is isomorphic to:

$$y^2 = \theta x(x-1)(x-\lambda)(x-\mu)(x-\nu),$$

$$\text{with } \nu = \frac{(a_1 - a_3)(a_2 - a_6)}{(a_2 - a_3)(a_1 - a_6)}.$$

Since $\nu = \frac{\lambda(1-\mu)}{1-\lambda}$ is equivalent to

$$(a_2 - a_4)(a_1 - a_6)(a_3 - a_5) = (a_2 - a_6)(a_1 - a_5)(a_3 - a_4),$$

we have (a).

(b) Using Equation (1), the following maps define three automorphisms of the hyperelliptic curve D :

$$\begin{aligned} \sigma: x &\mapsto \frac{\lambda(x-\mu)}{x-\lambda}, & y &\mapsto y \frac{\lambda^{3/2}(\lambda-\mu)^{3/2}}{(x-\lambda)^3}, \\ \iota: x &\mapsto x, & y &\mapsto -y, \end{aligned}$$

and $\tau = \sigma \cdot \iota$. Moreover, as $\lambda(\lambda-\mu)$ has a square root in \mathbb{F}_q , these automorphisms are defined over \mathbb{F}_q . Let E_σ and E_τ be the quotient curves $D/\langle \sigma \rangle$ and $D/\langle \tau \rangle$, respectively. By setting $t = x + \frac{\lambda(x-\mu)}{x-\lambda}$ and $s = y \frac{x - (\lambda \mp (\lambda^2 - \lambda\mu)^{1/2})}{(x-\lambda)^2}$, we have the following defining equations for E_σ and E_τ :

$$s^2 = \theta(t-\mu) \left(t - \frac{1-\lambda\mu}{1-\lambda} \right) \left(t - 2(\lambda \mp (\lambda^2 - \lambda\mu)^{1/2}) \right),$$

which are birationally equivalent to

$$s^2 = \frac{\theta(1-\mu)}{1-\lambda} t(t-1) \left(t - \frac{(1-\lambda)(\mu-2\lambda \pm 2(\lambda^2 - \lambda\mu)^{1/2})}{\mu-1} \right).$$

Hence, we have that $\text{Jac}(D) \sim E_\sigma \times E_\tau$ by Theorem B of [17].

Let us now prove that $\#D(\mathbb{F}_q) = \#E_\sigma(\mathbb{F}_q) + \#E_\tau(\mathbb{F}_q) - q - 1$. It is well known that $\#D(\mathbb{F}_q) = q + 1 - t$, where t is the trace of the Frobenius endomorphism acting

on a Tate module of $\text{Jac}(D)$. Since $\text{Jac}(D) \sim E_\sigma \times E_\tau$, then the Tate module of D is isomorphic to the direct sum of the Tate modules of E_σ and E_τ . Hence $t = t_1 + t_2$, where t_1 and t_2 are the traces of the Frobenius on the Tate modules of E_σ and E_τ respectively. The result follows by recalling that $t_1 = q + 1 - \#E_\sigma(\mathbb{F}_q)$ and $t_2 = q + 1 - \#E_\tau(\mathbb{F}_q)$. \square

4. NEW SEXTICS OF GENUS SIX ATTAINING THE SERRE BOUND

In this section we consider the family of sextics

$$S_{a,b}: x^6 + y^6 + 1 + a(x^4y^2 + x^2 + y^4) + b(x^2y^4 + x^4 + y^2) - 3(a + b + 1)x^2y^2 = 0.$$

We recall the following two results from [19].

Proposition 3. [19, Proposition 1] *Let k be a field of characteristic $p \geq 5$. Let $a, b \in k$ and let $D_{a,b}: y^2 = f_{a,b}(x)$ be the hyperelliptic curve with*

$$f_{a,b}(x) = -(x^3 + bx^2 + ax + 1)((a + b + 2)x^3 - (a + 2b + 3)x^2 + (b + 3)x - 1).$$

Then, the Jacobian variety of the sextic $S_{a,b}$ decomposes over k as

$$\text{Jac}(S_{a,b}) \sim \text{Jac}(D_{a,b}) \times \text{Jac}(D_{b,a})^2.$$

Note that if $a + b + 2$, $-4a^3 + a^2b^2 + 18ab - 4b^3 - 27$, and the resultant of $x^3 + bx^2 + ax + 1$ and $(a + b + 2)x^3 - (a + 2b + 3)x^2 + (b + 3)x - 1$ are not 0 then the genus of the hyperelliptic curve $D_{a,b}$ is 2, and therefore $S_{a,b}$ has genus 6.

Moreover, since $\#D_{b,a}(\mathbb{F}_q) = \#D_{a,b}(\mathbb{F}_q)$, we have the following corollary.

Corollary 4. [19, Corollary 2] *When $a, b \in \mathbb{F}_q$, then the number of \mathbb{F}_q -rational points of the sextic $S_{a,b}$ is*

$$\#S_{a,b}(\mathbb{F}_q) = 3\#D_{a,b}(\mathbb{F}_q) - 2q - 2.$$

By Corollary 4, in order to compute $\#S_{a,b}(\mathbb{F}_q)$, it is enough to compute $\#D_{a,b}(\mathbb{F}_q)$.

Corollary 5. *When the polynomial $f_{a,b}(x)$ satisfies the assumptions in Theorem 2, then the number of rational points of the sextic $S_{a,b}$ is*

$$\#S_{a,b}(\mathbb{F}_q) = 3\#E_\sigma(\mathbb{F}_q) + 3\#E_\tau(\mathbb{F}_q) - 5q - 5,$$

where E_σ and E_τ are defined as in Theorem 2(b).

Proof. The claim follows from Theorem 2 and Corollary 4. \square

Because of Corollary 5, we can determine the number of rational points of the sextic $S_{a,b}$ by computing the number of rational points of the two elliptic curves E_σ and E_τ . Using this fact and a MAGMA([5])-aided search, we were able to find new examples of curves of genus 6 attaining the Serre bound.

Example 6. The sextic

$$S_{444,469}: x^6 + y^6 + 1 + 444(x^4y^2 + x^2 + y^4) + 469(x^2y^4 + x^4 + y^2) + 1239x^2y^2 = 0$$

has 1760 rational points over the finite field \mathbb{F}_{1327} and therefore it attains the Serre bound. Its automorphism group has 12 elements.

We can apply Theorem 2 to the hyperelliptic curve $D_{444,469}: y^2 = f_{444,469}(x)$, where $f_{444,469}(x) = c(x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)(x - a_6)$, with $c = 412$, $a_1 = 548$, $a_2 = 541$, $a_3 = 289$, $a_4 = 364$, $a_5 = 344$, $a_6 = 28$. Note that the order of a_i for $i = 1, \dots, 6$ is important. With the notation of Theorem 2, we have $\lambda = 611$, $\mu = 656$ and $\theta = 696$. Hence, $\text{Jac}(D_{444,469}) \sim E_\sigma \times E_\tau$ where

$E_\sigma: s^2 = 247t(t-1)(t-811)$ and $E_\tau: s^2 = 247t(t-1)(t-1084)$. So $\#S_{444,469}(\mathbb{F}_q) = 3\#E_\sigma(\mathbb{F}_q) + 3\#E_\tau(\mathbb{F}_q) - 5q - 5$.

Example 7. The sextic

$$S_{0,7}: x^6 + y^6 + 1 + 7(x^2y^4 + x^4 + y^2) + 35x^2y^2 = 0$$

is maximal over the finite field \mathbb{F}_{59^2} . Its automorphism group has 12 elements.

We can apply Theorem 2 to the hyperelliptic curve $D_{0,7}: y^2 = f_{0,7}(x)$ with $f_{0,7}(x) = 50(x-39)(x-25)(x-8)(x-27)(x-23)(x-4)$. Therefore, with the notation of Theorem 2, we have $\lambda = 13$, $\mu = 5$ and $\theta = 33$. Hence $\text{Jac}(D_{0,7}) \sim E_\sigma \times E_\tau$ with $E_\sigma: s^2 = 11t(t-1)(t-30)$ and $E_\tau: s^2 = 11t(t-1)(t-37)$. So $\#S_{0,7}(\mathbb{F}_q) = 3\#E_\sigma(\mathbb{F}_q) + 3\#E_\tau(\mathbb{F}_q) - 5q - 5$.

Other examples for which $S_{a,b}$ is maximal over the finite field \mathbb{F}_{p^2} are obtained for $(p, a, b) \in \{(59, 0, 7), (71, 13, 23), (79, 9, 18), (83, 36, 52), (107, 53, 58), (139, 103, 115), (167, 7, 52), (179, 131, 154), (191, 61, 85), \dots\}$. We also observed that all the curves corresponding to the listed triplets have automorphism group of order 12.

Example 8. The sextic

$$S_{558,2522}: x^6 + y^6 + 1 + 558(x^4y^2 + x^2 + y^4) + 2522(x^2y^4 + x^4 + y^2) + 2425x^2y^2 = 0$$

attains the Serre bound over \mathbb{F}_{2917^3} . Its automorphism group has 12 elements.

We apply Theorem 2 to the hyperelliptic curve $D_{558,2522}: y^2 = f_{558,2522}(x)$ with $f_{558,2522}(x) = 2752(x-2694)(x-2107)(x-2027)(x-1853)(x-2095)(x-879)$. Therefore, $\lambda = 171$, $\mu = 932$ and $\theta = 893$. Hence, $\text{Jac}(D_{558,2522}) \sim E_\sigma \times E_\tau$ with $E_\sigma: s^2 = 2677t(t-1)(t-1194)$ and $E_\tau: s^2 = 2677t(t-1)(t-1495)$. So $\#S_{558,2522}(\mathbb{F}_q) = 3\#E_\sigma(\mathbb{F}_q) + 3\#E_\tau(\mathbb{F}_q) - 5q - 5$.

For $(p, a, b) \in \{(67, 6, 62), (101, 25, 59), (673, 40, 460), (677, 1, 76), (1153, 65, 957), (2113, 287, 438), (2311, 431, 1253), (2707, 143, 2372), (2909, 174, 1715), (2917, 558, 2522), (3361, 1062, 1788), \dots\}$, the sextic $S_{a,b}$ attains the Serre bound over the finite field \mathbb{F}_{p^3} . The automorphism group of the curves corresponding to the above listed examples has order 12, except when $(p, a, b) = (67, 6, 62)$, in which case it has order 60.

5. THE WIMAN SEXTICS OF GENUS TEN ATTAINING THE SERRE BOUND

We consider in this section the family of sextics

$$W_{a,b}: x^6 + y^6 + 1 + a(x^4y^2 + x^2y^4 + x^4 + x^2 + y^4 + y^2) + bx^2y^2 = 0.$$

In [18], the case $b = -6a - 3$ was investigated, and new examples of curves of genus 6 attaining the Serre bound were provided.

In [1], the Jacobian of the sextic $W_{0,b}$ over the finite field \mathbb{F}_{p^2} was completely decomposed, and this allowed to find new maximal curves of genus 10. Moreover, in [13], the Jacobian of $W_{0,b}$ was completely decomposed over any field k whose characteristic > 5 , bringing to find a new sextic of genus 10 attaining the Serre bound.

With a computer search on the family of genus 10 sextics $W_{a,b}$, we found new examples of curves with genus 10 and many rational points. We list such examples in Table 1 and we highlight how they improve the old entries from <http://www.manypoints.org> ([35]).

TABLE 1. Wiman sextics of genus 10 with many points

p	a	b	$\#W_{a,b}(\mathbb{F}_p)$	old entry	new entry
19	0	0	72	-100	72 - 100
23	4	8	84	-114	84 - 114
29	8	1	102	-130	102 - 130
31	0	12	108	-133	108 - 133
43	36	28	132	-170	132 - 170
47	19	0	144	-178	144 - 178
53	42	46	162	-191	162 - 191
59	5	21	168	-205	168 - 205
67	17	17	192	-222	192 - 222
79	0	30	216	-247	216 - 247
83	13	79	216	-256	216 - 256
89	37	43	246	-267	246 - 267
97	48	5	246	-284	246 - 284

In [18], we also proved the following result on the decomposition of the Jacobian of $W_{a,b}$ over a field k whose characteristic > 5 .

Proposition 9. [18, Proposition 10] *The Jacobian of the Wiman sextic $W_{a,b}$ over a field k satisfies the following isogeny relation:*

$$\text{Jac}(W_{a,b}) \sim V_1^3 \times V_2 \times \text{Jac}(V_3)^3,$$

where V_1, V_2 and V_3 are defined by

$$V_1: y^2 = ((3a - b - 3)x - a + 3)(1 + (a - 3)x(1 - x)),$$

$$V_2: x^3 + y^3 + 1 + a(x^2y + xy^2 + x^2 + x + y^2 + y) + bxy = 0,$$

$$V_3: y^2 = -((a + 1)x^3 + (2a + b)x^2 + 4ax + 4)(x^3 + ax^2 + ax + 1).$$

In the cases where Theorem 2 applies to the hyperelliptic curve V_3 of genus 2, we can completely decompose the Jacobian of V_3 , and hence we can completely decompose the Jacobian of $W_{a,b}$.

With a computer search we were able to find the following examples.

Example 10. The sextic

$$W_{5,17}: x^6 + y^6 + 1 + 5(x^4y^2 + x^2y^4 + x^4 + x^2 + y^4 + y^2) + 17x^2y^2 = 0$$

has 990 rational points over \mathbb{F}_{23^2} and it is a maximal curve of genus 10. Its automorphism group has 24 elements. In Section 6, we will prove that this maximal curve is not a quotient of the Hermitian curve \mathcal{H}_{23} defined over \mathbb{F}_{23^2} .

From Proposition 9, we have that $\text{Jac}(W_{5,17}) \sim V_1^3 \times V_2 \times \text{Jac}(V_3)^3$. By applying Theorem 2 to $V_3: y^2 = 17(x - 22)(x - 14)(x - 13)(x - 11)(x - 6)(x - 5)$, we obtain that $\text{Jac}(V_3) \sim E_1 \times E_2$ where

$$E_1: s^2 = 20t(t - 1)(t - 21), \quad E_2: s^2 = 20t(t - 1)(t - 22).$$

Since V_1, V_2 are birational equivalent to

$$E_3: s^2 = t(t - 1)(t - 3), \quad E_4: s^2 = 6t(t - 1)(t - 22).$$

respectively, we have that

$$\text{Jac}(W_{5,17}) \sim E_1^3 \times E_2^3 \times E_3^3 \times E_4.$$

For $(p, a, b) \in \{(167, 27, 40), (191, 49, 131), (239, 119, 216), (263, 51, 123), (431, 257, 322), (503, 33, 274), (599, 352, 358), (719, 254, 557), (887, 388, 609), \dots\}$, the Wiman sextic $W_{a,b}$ is maximal over the finite field \mathbb{F}_{p^2} . The automorphism group of all the explicit examples listed here has 24 elements.

Example 11. The sextic

$$W_{7,120} : x^6 + y^6 + 1 + 7(x^4y^2 + x^2y^4 + x^4 + x^2 + y^4 + y^2) + 120x^2y^2 = 0$$

has 7242678 rational points over \mathbb{F}_{193^3} . It has genus 10 and it attains the Serre bound. Its automorphism group has 24 elements.

From Proposition 9, we have that $\text{Jac}(W_{7,120}) \sim V_1^3 \times V_2 \times \text{Jac}(V_3)^3$. Applying Theorem 2 to $V_3 : y^2 = 185(x - 192)(x - 122)(x - 101)(x - 110)(x - 89)(x - 86)$, we have that $\text{Jac}(V_3) \sim E_1 \times E_2$ where

$$E_1 : s^2 = 38t(t - 1)(t - 56), \quad E_2 : s^2 = 38t(t - 1)(t - 7).$$

Since V_1, V_2 are birational equivalent to

$$E_3 : s^2 = 14t(t - 1)(t - 164), \quad E_4 : s^2 = 88t(t - 1)(t - 173)$$

respectively, we have that

$$\text{Jac}(W_{7,120}) \sim E_1^3 \times E_2^3 \times E_3^3 \times E_4.$$

For $(p, a, b) = (2909, 2271, 2350), (4349, 1169, 4282)$ the sextic $W_{a,b}$ attains the Serre bound over the finite field \mathbb{F}_{p^3} . Both automorphism groups have 24 elements.

6. $W_{5,17}$ IS NOT GALOIS-COVERED BY \mathcal{H}_{23} OVER \mathbb{F}_{23^2}

In the previous section we proved that the sextic $W_{5,17}$ is an \mathbb{F}_{23^2} -maximal curve of genus 10. In this section, we prove that this maximal curve is not a quotient curve of the Hermitian curve \mathcal{H}_{23} defined over \mathbb{F}_{23^2} .

Recall that every subgroup G of the automorphism group of \mathcal{H}_q , which is isomorphic to $\text{PGU}(3, q)$, produces a quotient curve \mathcal{H}_q/G , and the cover $\mathcal{H}_q \rightarrow \mathcal{H}_q/G$ is a Galois cover defined over \mathbb{F}_{q^2} . The degree of the different divisor Δ of this covering is given by the Riemann-Hurwitz formula [32, Theorem 3.4.13],

$$\Delta = (2g(\mathcal{H}_q) - 2) - |G|(2g(\mathcal{H}_q/G) - 2). \quad (2)$$

On the other hand,

$$\Delta = \sum_{\sigma \in G \setminus \{id\}} i(\sigma),$$

where $i(\sigma) \geq 0$ is given by the Hilbert's different formula [32, Theorem 3.8.7], namely

$$i(\sigma) = \sum_{P \in \mathcal{H}_q(\bar{\mathbb{F}}_q)} v_P(\sigma(t) - t), \quad (3)$$

where t is a local parameter at P .

By analyzing the geometric properties of the elements $\sigma \in \text{PGU}(3, q)$, it turns out that there are only a few possibilities for $i(\sigma)$. This is stated in the next results on how an element of a given order in $\text{PGU}(3, q)$ acts on the set of \mathbb{F}_{q^2} -rational points of \mathcal{H}_q , denoted by $\mathcal{H}_q(\mathbb{F}_{q^2})$. We recall that a linear collineation σ of $\text{PG}(2, \mathbb{K})$ is a (P, ℓ) -*perspectivity*, if σ preserves each line through the point P (the *center* of σ), and fixes each point on the line ℓ (the *axis* of σ). A (P, ℓ) -perspectivity is either an *elation* or a *homology* according as $P \in \ell$ or $P \notin \ell$.

Lemma 12. ([28, Lemma 2.2]) *For a nontrivial element $\sigma \in \text{PGU}(3, q)$, one of the following cases holds.*

- (A) $\text{ord}(\sigma) \mid (q+1)$ and σ is a homology whose center P is a point of \mathcal{H}_q and whose axis ℓ is a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$ such that (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- (B) $\text{ord}(\sigma)$ is coprime to p and σ fixes the vertices P_1, P_2, P_3 of a non-degenerate triangle T .
 - (B1) The points P_1, P_2, P_3 are \mathbb{F}_{q^6} -rational, $P_1, P_2, P_3 \notin \mathcal{H}_q$ and the triangle T is self-polar with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$. Also, $\text{ord}(\sigma) \mid (q+1)$.
 - (B2) The points P_1, P_2, P_3 are \mathbb{F}_{q^6} -rational, $P_1 \notin \mathcal{H}_q$, $P_2, P_3 \in \mathcal{H}_q$. Also, $\text{ord}(\sigma) \mid (q^2 - 1)$ and $\text{ord}(\sigma) \nmid (q+1)$.
 - (B3) The points P_1, P_2, P_3 have coordinates in $\mathbb{F}_{q^6} \setminus \mathbb{F}_{q^2}$, $P_1, P_2, P_3 \in \mathcal{H}_q$. Also, $\text{ord}(\sigma) \mid (q^2 - q + 1)$.
- (C) $\text{ord}(\sigma) = p$ and σ is an elation whose center P is a point of \mathcal{H}_q and whose axis ℓ is a tangent of $\mathcal{H}_q(\mathbb{F}_{q^2})$; here (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- (D) $\text{ord}(\sigma) = p$ with $p \neq 2$, or $\text{ord}(\sigma) = 4$ and $p = 2$. In this case σ fixes an \mathbb{F}_{q^6} -rational point P , with $P \in \mathcal{H}_q$, and a line ℓ which is a tangent of $\mathcal{H}_q(\mathbb{F}_{q^2})$; here (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- (E) $p \mid \text{ord}(\sigma)$, $p^2 \nmid \text{ord}(\sigma)$, and $\text{ord}(\sigma) \neq p$. In this case σ fixes two \mathbb{F}_{q^6} -rational points P, Q , with $P \in \mathcal{H}_q$, $Q \notin \mathcal{H}_q$.

In the rest of the section, a nontrivial element of $\text{PGU}(3, q)$ is said to be of type (A), (B), (B1), (B2), (B3), (C), (D), or (E), as given in Lemma 12.

Lemma 13. ([28, Theorem 2.7]) *For a nontrivial element $\sigma \in \text{PGU}(3, q)$ one of the following cases occurs.*

- (1) If $\text{ord}(\sigma) = 2$ and $2 \mid (q+1)$, then $i(\sigma) = q+1$.
- (2) If $\text{ord}(\sigma) = 3$, $3 \mid (q+1)$ and σ is of type (B3), then $i(\sigma) = 3$.
- (3) If $\text{ord}(\sigma) \neq 2$, $\text{ord}(\sigma) \mid (q+1)$ and σ is of type (A), then $i(\sigma) = q+1$.
- (4) If $\text{ord}(\sigma) \neq 2$, $\text{ord}(\sigma) \mid (q+1)$ and σ is of type (B1), then $i(\sigma) = 0$.
- (5) If $\text{ord}(\sigma) \mid (q^2 - 1)$ and $\text{ord}(\sigma) \nmid (q+1)$, then σ is of type (B2) and $i(\sigma) = 2$.
- (6) If $\text{ord}(\sigma) \neq 3$ and $\text{ord}(\sigma) \mid (q^2 - q + 1)$, then σ is of type (B3) and $i(\sigma) = 3$.
- (7) If $p = 2$ and $\text{ord}(\sigma) = 4$, then σ is of type (D) and $i(\sigma) = 2$.
- (8) If $\text{ord}(\sigma) = p$, $p \neq 2$ and σ is of type (D), then $i(\sigma) = 2$.
- (9) If $\text{ord}(\sigma) = p$ and σ is of type (C), then $i(\sigma) = q+2$.
- (10) If $\text{ord}(\sigma) \neq p$, $p \mid \text{ord}(\sigma)$ and $\text{ord}(\sigma) \neq 4$, then σ is of type (E) and $i(\sigma) = 1$.

Finally, the following useful corollary can be deduced from the proof of Theorem 5 in [14].

Proposition 14. *Let \mathcal{Y} be the quotient curve \mathcal{H}_q/G , where G is a subgroup of $\text{PGU}(3, q)$. Then*

$$\frac{|\mathcal{H}_q(\mathbb{F}_{q^2})|}{|\mathcal{Y}(\mathbb{F}_{q^2})|} \leq |G| \leq \frac{2g(\mathcal{H}_q) - 2}{2g(\mathcal{Y}) - 2}.$$

Proposition 15. $W_{5,17}$ is not Galois-covered by \mathcal{H}_{23} over \mathbb{F}_{23^2} .

Proof. Assume, by the way of contradiction, that $W_{5,17}$ is Galois-covered by \mathcal{H}_{23} over \mathbb{F}_{23^2} . So, there exists $G \leq \text{PGU}(3, 23)$ such that $W_{5,17}$ is the quotient curve

\mathcal{H}_{23}/G . First, by Proposition 14, it follows that

$$12 < \frac{|\mathcal{H}_{23}(\mathbb{F}_{23^2})|}{|W_{5,17}(\mathbb{F}_{23^2})|} \leq |G| \leq \frac{2g(\mathcal{H}_{23}) - 2}{2g(W_{5,17}) - 2} = 28.$$

Furthermore, as $|G|$ must divide $|\mathrm{PGU}(3, 23)| = 2^7 \cdot 3^3 \cdot 11 \cdot 13^2 \cdot 23^3$, we have that $|G| \in \{13, 16, 18, 22, 23, 24, 26, 27\}$. Observe that, by the Riemann-Hurwitz formula, the degree of the different divisor Δ can be computed as

$$\Delta = (2g(\mathcal{H}_{23}) - 2) - |G|(2g(W_{5,17}) - 2). \quad (4)$$

To conclude the proof, we will rule out each of these possibilities for $|G|$.

- $|G| = 13$. In this case G contains exactly 12 elements of order 13, and hence Lemma 13 yields $\Delta = 12 \cdot 3 = 36$, a contradiction with Equation (4).
- $|G| = 16$. A MAGMA aided computation shows that there are 9 subgroups (up to conjugation) of order 16 in $\mathrm{PGU}(3, 23)$, namely $G_i = S[i]$ where $S := \mathrm{Subgroups}(\mathrm{PGU}(3, 23) : \mathrm{OrderEqual} := 16)$ and $i \in \{1, \dots, 9\}$. Let N_i be the normalizer of G_i in $\mathrm{PGU}(3, 23)$ and Q_i be the factor group N_i/G_i . Then, from Galois theory, Q_i is a subgroup of $\mathrm{Aut}(W_{5,7}) \cong S_4$. By direct check with MAGMA, the order of Q_i is not a divisor of 24 for $i \in \{1, \dots, 5\}$, a contradiction. In the remaining cases we have $|Q_6| = |Q_7| = 24$ and $|Q_8| = |Q_9| = 12$. However, all such groups are abelian, a contradiction with the structure of S_4 .
- $|G| = 18$. By Lemma 13, G does not contain any non-trivial elements of orders 9 and 18. Therefore, by Sylow theorems, it contains exactly 8 elements of order 3, and Lemma 13 yields

$$180 = \Delta = 24 \cdot i_2 + 8 \cdot k + (17 - 8 - i_2) \cdot l,$$

where $i_2 \in \{1, 3, 9\}$ is the number of involutions in G , $k \in \{0, 3, 24\}$ and $l \in \{0, 24\}$. This is a contradiction as 180 is not divisible by 8.

- $|G| = 22$. In this case G is isomorphic to either the cyclic or the dihedral group of order 22. Now, Lemma 13 yields

$$\Delta = 24 + 10 \cdot 2 + 10 \cdot 2,$$

in the former case, and

$$\Delta = 24 \cdot 11 + 10 \cdot 2,$$

in the latter case. In both cases this is a contradiction with $\Delta = 108$.

- $|G| = 23$. In this case G contains 22 elements of order 23, whence Lemma 13 yields

$$\Delta = 22 \cdot i,$$

where $i \in \{2, 25\}$. This is a contradiction with $\Delta = 90$.

- $|G| = 24$. First observe that if G contains more than 3 involutions, Lemma 13 yields $\Delta > 24 \cdot 3$, a contradiction with $\Delta = 72$. Checking with MAGMA, there are 20 subgroups (up to conjugation) of order 24 in $\mathrm{PGU}(3, 23)$ with at most 3 involutions, namely $G_i = S[i]$ where $S := \mathrm{Subgroups}(\mathrm{PGU}(3, 23) : \mathrm{OrderEqual} := 24)$; and $i \in \{1, \dots, 18\} \cup \{22, 23\}$. For $i \in \{1, \dots, 18\} \cup \{22, 23\}$, let N_i be the normalizer of G_i in $\mathrm{PGU}(3, 23)$ and Q_i be the factor group N_i/G_i . Then, if $i \in \{1, \dots, 10\}$, the order of Q_i is not a divisor of 24, a contradiction with Q_i being a subgroup of S_4 . On the other hand,

in the remaining cases Q_i contains an abelian subgroup of order 12, a contradiction with the structure of S_4 .

- $|G| = 26$. By Sylow theorems G contains exactly 12 elements of order 13. Therefore Lemma 13 yields

$$\Delta \geq 12 \cdot 3 + 24 \cdot i,$$

where $i \geq 1$ is the number of involutions in G . This is a contradiction with $\Delta = 36$.

- $|G| = 27$. Checking with MAGMA, there is a unique (up to conjugation) subgroup of order 27 in $\text{PGU}(3, 23)$, which is generated by the following automorphisms of \mathcal{H}_{23} :

$$\{[x : y : z] \rightarrow [a_1x : a_2y : a_3z] \mid a_i^3 = 1, i = 1, 2, 3\}$$

and

$$[x : y : z] \rightarrow [z : x : y].$$

A MAGMA computation shows that the quotient curve of \mathcal{H}_{23} by this group has genus 7, a contradiction with $W_{5,17}$ having genus 10. \square

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REFERENCES

- [1] D. Bartoli, M. Giulietti, M. Kawakita, and M. Montanucci. New examples of maximal curves with low genus. *Finite Fields Appl.*, 68:101744, 32, 2020.
- [2] D. Bartoli, M. Montanucci, and G. Zini. AG codes and AG quantum codes from the GGS curve. *Des. Codes Cryptogr.*, 86(10):2315–2344, 2018.
- [3] D. Bartoli, M. Montanucci, and G. Zini. Multi point AG codes on the GK maximal curve. *Des. Codes Cryptogr.*, 86(1):161–177, 2018.
- [4] P. Beelen and M. Montanucci. A new family of maximal curves. *J. Lond. Math. Soc. (2)*, 98(3):573–592, 2018.
- [5] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. volume 24, pages 235–265. 1997. Computational algebra and number theory (London, 1993).
- [6] A. S. Castellanos and G. C. Tizziotti. Two-point AG codes on the GK maximal curves. *IEEE Trans. Inform. Theory*, 62(2):681–686, 2016.
- [7] A. Cossidente, G. Korchmáros, and F. Torres. Curves of large genus covered by the Hermitian curve. *Comm. Algebra*, 28(10):4707–4728, 2000.
- [8] I. Dolgachev, B. Farb, and E. Looijenga. Geometry of the Wiman-Edge pencil, I: algebro-geometric aspects. *Eur. J. Math.*, 4(3):879–930, 2018.
- [9] W. L. Edge. A pencil of four-nodal plane sextics. *Math. Proc. Cambridge Philos. Soc.*, 89(3):413–421, 1981.
- [10] S. Fanali and M. Giulietti. One-point AG codes on the GK maximal curves. *IEEE Trans. Inform. Theory*, 56(1):202–210, 2010.
- [11] A. Garcia, C. Güneri, and H. Stichtenoth. A generalization of the Giulietti-Korchmáros maximal curve. *Adv. Geom.*, 10(3):427–434, 2010.
- [12] A. Garcia, H. Stichtenoth, and C.-P. Xing. On subfields of the Hermitian function field. *Compositio Math.*, 120(2):137–170, 2000.
- [13] M. Giulietti, M. Kawakita, S. Lia, and M. Montanucci. Three sextics attaining the serre bound over the same finite field. 2020.

- [14] M. Giulietti and G. Korchmáros. A new family of maximal curves over a finite field. *Math. Ann.*, 343(1):229–245, 2009.
- [15] V. D. Goppa. Codes on algebraic curves. *Dokl. Akad. Nauk SSSR*, 259(6):1289–1290, 1981.
- [16] T. Ibukiyama, T. Katsura, and F. Oort. Supersingular curves of genus two and class numbers. *Compositio Math.*, 57(2):127–152, 1986.
- [17] E. Kani and M. Rosen. Idempotent relations and factors of Jacobians. *Math. Ann.*, 284(2):307–327, 1989.
- [18] M. Q. Kawakita. Wiman’s and Edge’s sextic attaining Serre’s bound II. In *Algorithmic arithmetic, geometry, and coding theory*, volume 637 of *Contemp. Math.*, pages 191–203. Amer. Math. Soc., Providence, RI, 2015.
- [19] M. Q. Kawakita. Certain sextics with many rational points. *Adv. Math. Commun.*, 11(2):289–292, 2017.
- [20] M. Q. Kawakita. Wiman’s and Edge’s sextics attaining Serre’s bound. *Eur. J. Math.*, 4(1):330–334, 2018.
- [21] S. L. Kleiman. Algebraic cycles and the Weil conjectures. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 359–386. North-Holland, Amsterdam, 1968.
- [22] G. Korchmáros, S. Lia, and M. Timpanella. A generalization of Bring’s curve in any characteristic. Preprint, 2022. arXiv:2112.10886.
- [23] G. Korchmáros, G. P. Nagy, and M. Timpanella. Codes and gap sequences of Hermitian curves. *IEEE Trans. Inform. Theory*, 66(6):3547–3554, 2020.
- [24] G. Lachaud. Sommes d’Eisenstein et nombre de certaines courbes algébriques sur les corps finis. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):729–732, 1987.
- [25] L. Landi and L. Vicino. Two-point AG codes from the Beelen-Montanucci maximal curve. *Finite Fields Appl.*, 80:Paper No. 102009, 17, 2022.
- [26] S. Lia and M. Timpanella. AG codes from \mathbb{F}_{q^7} -rational points of the GK maximal curve. *Applicable Algebra in Engineering, Communications and Computing*, 2021.
- [27] M. Montanucci, M. Timpanella, and G. Zini. AG codes and AG quantum codes from cyclic extensions of the Suzuki and Ree curves. *J. Geom.*, 109(1):Paper No. 23, 18, 2018.
- [28] M. Montanucci and G. Zini. Some Ree and Suzuki curves are not Galois covered by the Hermitian curve. *Finite Fields Appl.*, 48:175–195, 2017.
- [29] M. Montanucci and G. Zini. On the spectrum of genera of quotients of the Hermitian curve. *Comm. Algebra*, 46(11):4739–4776, 2018.
- [30] H.-G. Rück and H. Stichtenoth. A characterization of Hermitian function fields over finite fields. *J. Reine Angew. Math.*, 457:185–188, 1994.
- [31] J.-P. Serre. Sur le nombre des points rationnels d’une courbe algébrique sur un corps fini. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(9):397–402, 1983.
- [32] H. Stichtenoth. *Algebraic function fields and codes*, volume 254 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, second edition, 2009.
- [33] S. Tafazolian, A. Teherán-Herrera, and F. Torres. Further examples of maximal curves which cannot be covered by the Hermitian curve. *J. Pure Appl. Algebra*, 220(3):1122–1132, 2016.
- [34] G. Tizzotti and A. S. Castellanos. Weierstrass semigroup and pure gaps at several points on the GK curve. *Bull. Braz. Math. Soc. (N.S.)*, 49(2):419–429, 2018.
- [35] G. van der Geer, E. W. Howe, K. E. Lauter, and C. Ritzenthaler. Tables of curves with many points. 2009. Retrieved [2023].
- [36] A. Wiman. Ueber eine einfache Gruppe von 360 ebenen Collineationen. *Math. Ann.*, 47(4):531–556, 1896.

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI “RENATO CACCIOPOLI”, UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II, VIA CINTIA, MONTE S. ANGELO - I-80126 NAPOLI, ITALY
 Email address: annamaria.iezzi@unina.it

DIVISION OF MATHEMATICS, SHIGA UNIVERSITY OF MEDICAL SCIENCE, SETA TSUKINOWA-CHO, OTSU, SHIGA, 520-2192 JAPAN
 Email address: kawakita@belle.shiga-med.ac.jp

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DEGLI STUDI DI PERUGIA - UNIVERSITY OF PERUGIA VIA VANVITELLI, 1 - 06123 PERUGIA, ITALY
 Email address: marco.timpanella@unipg.it