# THE DEFINITE INTEGRAL (Sec. 5.1 and 5.2)

We said at the beginning of the course that calculus has two major branches:

- differential calculus.
- integral calculus.

So far we have explored the branch of differential calculus that arised from the following two equivalent problems:

→ given the graph of a function find the tangent line to the graph at a given point.

→ given the position function of a moving object, find the instantaneous velocity at each time.

Recall that differential calculus is based on the notion of limit: indeed the slope of a tangent line is defined as the limit of slopes of secant lines. Same for the istantaneous velocity which is defined as the limit of average velocities.

We enter now the branch of INTEGRAL CALCULUS.

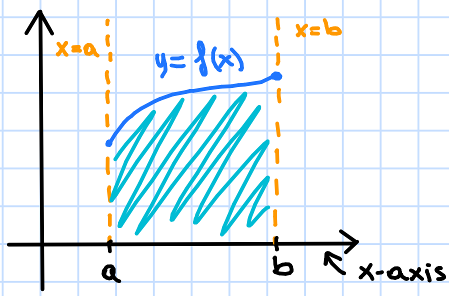
Let us consider the two following problems:

- the AREA PROBLEM: given the graph of a function, find the area "under the graph".
- the DISTANCE PROBLEM: given the instantaneous velocity of of an object, find the distance traveled by the object.

We will see that also in these problems we will use limits!

## The area problem

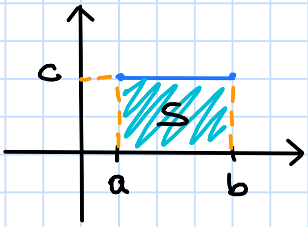
Let $f(x)$ be a function such that $f(x) \geq 0$ on $[a,b]$




**Problem:** Find the area between the lines $x=a$, $x=b$, the x-axis and the graph $y=f(x)$.

Let us consider first some easy cases:

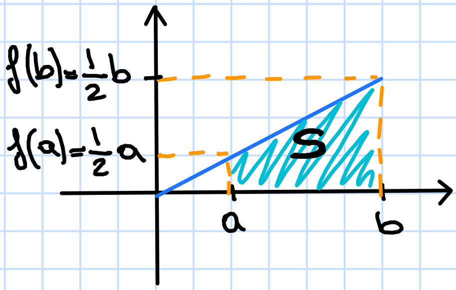
- $f(x) = c$, with $c > 0$ a constant.

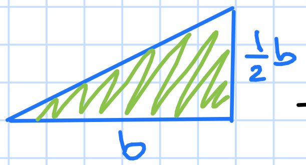


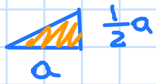

In this case the problem consists in computing the area of the rectangle shown above.
The rectangle has length $b-a$ and height $c$. Therefore its area is

$$\text{Area of } S = c\cdot(b-a)$$

- $f(x) = \frac{1}{2}\cdot x$




$S =$ 

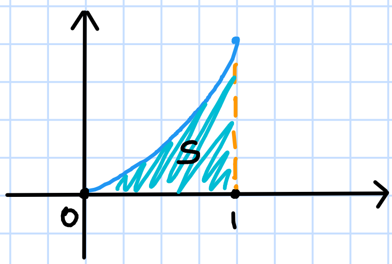
$-$ 

$\Rightarrow \text{Area of } S = \frac{1}{2}b\cdot\frac{1}{2}b - \frac{1}{2}a\cdot\frac{1}{2}a =$

$$= \frac{1}{4}b^2 - \frac{1}{4}a^2 = \frac{1}{4}(b^2-a^2).$$

In the previous two cases the computation is easy because the areas of the shapes under the graph can be obtained by using the formulas of the areas of a rectangle, triangle, etc...
This is not always the case.

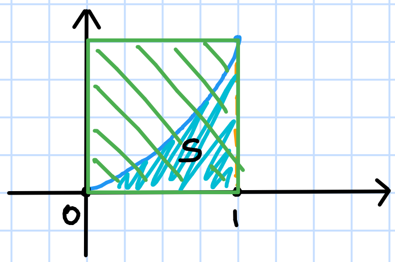
Let us consider, for example, the function $y = x^2$ on $[0,1]$ and let $S$ be the region under the parabola between $x=0$, $x=1$ and the x-axis:



In this case we do not obtain a shape for which the area can be easily calculated.

Nevertheless, we can try to approximate the area.

- First of all it is easy to see that the region $S$ is contained in a square of side 1:

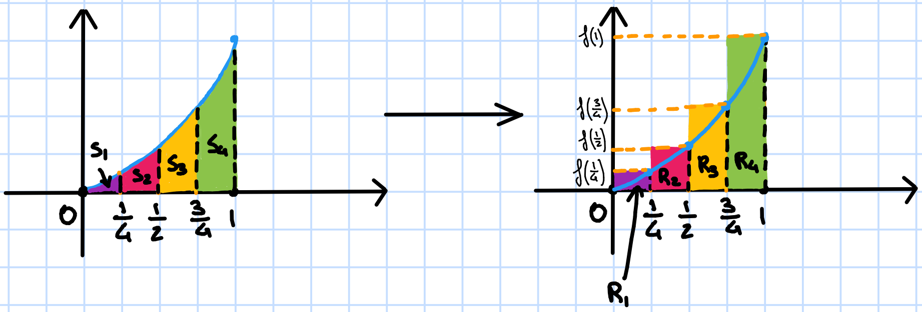



Therefore the area of $S$ is bounded by the area of the square:

$$\text{Area of } S \leq 1\cdot 1 = 1.$$

Of course this is not a good approximation and we can certainly do better.

- For example we can divide $S$ in several strips (4 in our example) and approximate each strip by a rectangle whose base is the same as the strip and whose height is the value of $f$ at the right edge of the strip:




We obtain:

$$\underset{\substack{\uparrow \\ \text{is contained in}}}{S \subseteq} R_1 \cup R_2 \cup R_3 \cup R_4 \Rightarrow \text{Area } S \leq \text{Area } R_1 + \text{Area } R_2 + \text{Area } R_3 + \text{Area } R_4 =$$

$$= \frac{1}{4}\cdot f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f(1) =$$

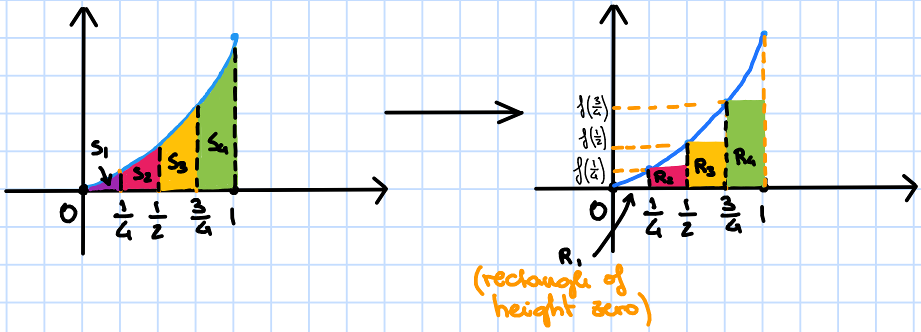
$$= \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 =$$

$$= \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{9}{16} + \frac{1}{4} =$$

$$= \frac{1+4+9+16}{64} = \frac{30}{64} = \frac{15}{32}$$

note that all the bases of the rectangles have same length $\frac{1}{4}$ (only the heights are different)

Instead of using the previous rectangles we could use the smaller rectangles whose heights are the values of $f$ at the left endpoints of the subintervals:




In this case we obtain:

$$S \supseteq R_1 \cup R_2 \cup R_3 \cup R_4 \Rightarrow \text{Area of } S \geq \text{Area } R_1 + \text{Area } R_2 + \text{Area } R_3 + \text{Area } R_4 =$$

contains

$$= \frac{1}{4} \cdot f(0) + \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right) =$$

$$= \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 =$$

$$= 0 + \frac{1}{64} + \frac{1}{16} + \frac{9}{64} =$$

$$= \frac{1+4+9}{64} = \frac{14}{64} = \frac{7}{32}$$

Then we have

$$\frac{7}{32} \leq \text{Area of } S \leq \frac{15}{32}.$$

- It is clear that if we split the region $S$ in more strips and we use more rectangles, then we get a better approximation of the area of $S$.

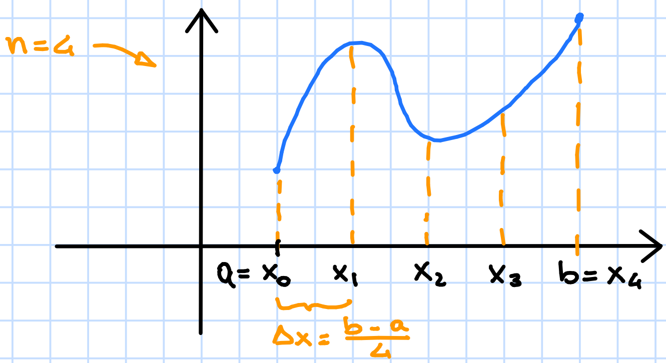
This method for getting an approximation of the area of the region under the graph can be applied to any positive function and generalizes as described here below.

Let $f(x)$ be a function such that $f(x) \geq 0$ for all $x$ in $[a,b]$.

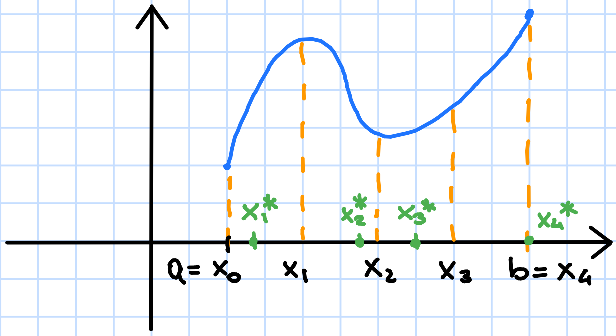
We split $[a,b]$ into $n$ equal (same lenght) subintervals:

$$[x_0, x_1], \quad [x_1, x_2], \quad \ldots, \quad [x_{n-1}, x_n], \quad \text{with } x_0 = a \text{ and } x_n = b$$

Each interval $[x_{i-1}, x_i]$ has lenght $\Delta x = \frac{b-a}{n}$.



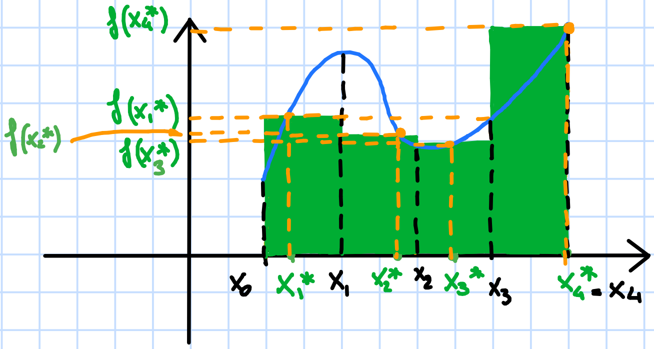
In each subinterval $[x_{i-1}, x_i]$ I choose a number $x_i^*$.
We call $x_i^*$ a sample point.




I compute the sum:

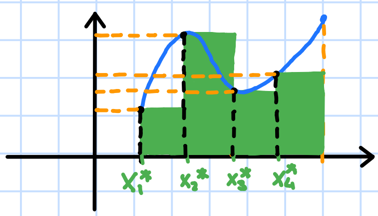
$$f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \dots + f(x_n^*)\,\Delta x$$

which geometrically corresponds to the sum of the areas of the rectangles of base $\Delta x$ and height $f(x_i^*)$:

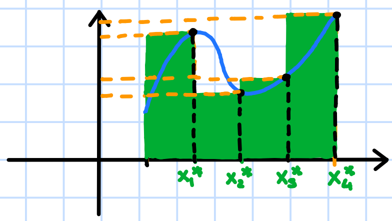



This sum is called RIEMANN SUM:

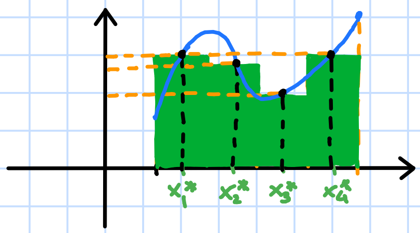
- If all the sample points are the left endpoints of the intervals $[x_{i-1}, x_i]$, i.e. $x_i^* = x_{i-1}$, then it is called left Riemann sum.




- If all the sample points are the right endpoints of the intervals $[x_{i-1}, x_i]$, i.e. $x_i^* = x_i$, then it is called right Riemann sum.



- If all the sample points are the midpoints of the intervals $[x_{i-1}, x_i]$, i.e. $x_i^* = \frac{x_i + x_{i-1}}{2}$ then it is called midpoint Riemann sum.




If we imagine all possible choices of sample points, we can think of taking the limit of all possible Riemann sums as n becomes large:

$$\lim_{n \to \infty} \left( f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x \right)$$

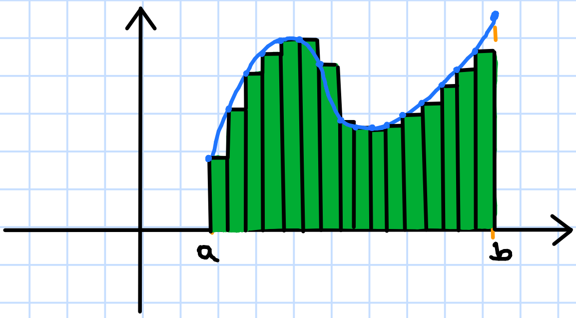
Note that, as n becomes large, the length of the subintervals

$$\Delta x = \frac{b-a}{n}$$

approaches zero and the rectangles become thinner and thinner.

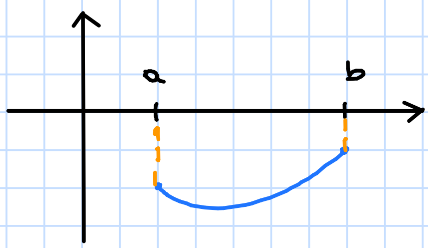
If the limit exists, then it is equal to the area of the region S that lies under the graph of f.

In other words, the area of S is the limit of the sum of the areas of approximating rectangles:



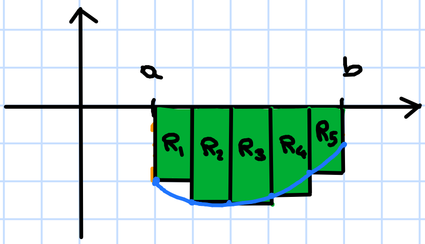

Remark : So far we considered functions that are positive over $[a,b]$.

Assume now that $f(x) \leq 0$ over $[a,b]$:




Then, if we compute a Riemann sum we

will get a negative number, since $f(x_i^*) < 0$ for all $i$:




This number represents the opposite of the sum of the areas of the rectangles.

We have the following definition:

**Def**: If $f$ is a function defined on $[a,b]$ the **definite integral** of $f$ from $a$ to $b$ is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left( f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x \right)$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^*$ are points belonging to the intervals $[x_{i-1}, x_i]$ provided that this limit exists.

If it does exist, we say that $f$ is integrable on $[a,b]$

**Notation**: The symbol $\int$ was introduced by Leibniz.

It is an elongated S which wants to emphasize that an integral is a limit of Sums.

limits of integration:
b = upper limit
a = lower limit

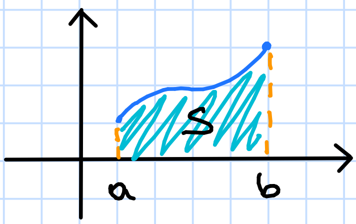
$$\int_a^b f(x)\,dx$$

the independent variable is $x$

integrand

**Remarks**:

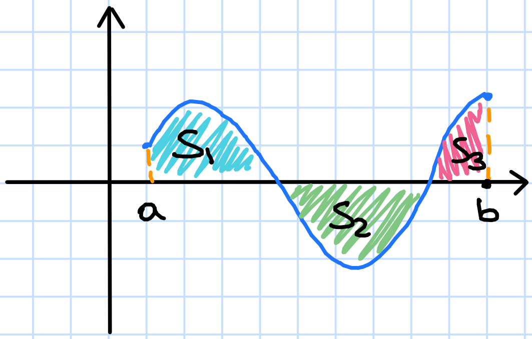
- If $f$ is positive and integrable over $[a,b]$, then $\int_a^b f(x)\,dx$ is the area of the region $S$ that lies under the graph of $f$.




$$\text{Area of } S = \int_a^b f(x)\,dx.$$

Note that, since $\int_a^b f(x)\,dx$ represents an area, it is a number (and not a function).

- If $f$ is integrable and takes on both positive and negative values, then we saw that the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis. When we take the limit we get:




$$\Rightarrow \int_a^b f(x)\,dx = \text{Area } S_1 \underbrace{- \text{Area } S_2}_{\text{negative because the region } S_2 \text{ lies under the x-axis}} + \text{Area } S_3$$

- $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(v)\,dv \ldots$

Are there any conditions for $f$ in order to be integrable? The following theorem answers to this question:

<u>Theorem</u>: If $f$ is continuous on $[a,b]$ or if $f$ has only a finite number of jump discontinuity then $f$ is integrable on $[a,b]$, i.e. the definite integral $\int_a^b f(x)\,dx$ exists.

## Properties of the definite integral

Let $f$ and $g$ be two integrable functions. We have:

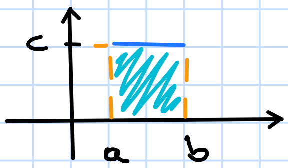
(1) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

→ Indeed in the Riemann sum $\Delta x$ changes from $\dfrac{b-a}{n}$ to $\dfrac{a-b}{n}$.

(2) $\displaystyle\int_a^a f(x)\,dx$

→ if $a=b$ then $\Delta x = 0$.

(3) $\int_a^b c \, dx = c(b-a)$, where $c$ is any constant.

→ 


$\int_a^b c\, dx$ is the area of a rectangle of base $b-a$ and height $c$.

(4) $\int_a^b f(x) + g(x) \, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

$\int_a^b f(x) - g(x) \, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

→ $\int_a^b f(x) + g(x)\, dx = \lim_{n\to\infty} [f(x_1^*) + g(x_1^*)]\Delta x + \cdots + [f(x_n^*) + g(x_n^*)]\Delta x =$

$= \lim_{n\to\infty} f(x_1^*)\Delta x + \cdots f(x_n^*)\Delta x + g(x_1^*)\Delta x + \cdots + g(x_n^*)\Delta x =$

sum rule of limits → $= \lim_{n\to\infty} [f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x] + \lim_{n\to\infty} [g(x_1^*)\Delta x + \cdots + g(x_n^*)\Delta x] =$

$= \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

↑ f,g integrable

(5) $\int_a^b c f(x)\, dx = c \int_a^b f(x)\, dx$

→ $\int_a^b c f(x)\, dx = \lim_{n\to\infty} c f(x_1^*)\Delta x + \cdots + c f(x_n^*)\Delta x =$

$= \lim_{n\to\infty} c [f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x] =$

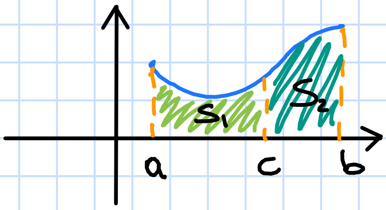
limit law → $= c \lim_{n\to\infty} [f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x] =$

$= c \int_a^b f(x)\, dx.$

↑ f integrable

(6) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

→ If $a < c < b$ and $f(x) \geq 0$ this property has an easy geometric interpretation:




$$\Rightarrow \underbrace{\int_a^b f(x)\,dx}_{\text{Area } S_1 \cup S_2} = \underbrace{\int_a^c f(x)\,dx}_{\text{Area } S_1} + \underbrace{\int_c^b f(x)\,dx}_{\text{Area } S_2}$$

(7) • If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x) \geq 0$.

• If $f(x) \leq 0$ on $[a,b]$, then $\int_a^b f(x) \leq 0$.

(8) If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

→ If $f(x) \geq g(x)$ then $f(x) - g(x) \geq 0 \overset{(7)}{\Longrightarrow}$

$\overset{(7)}{\Longrightarrow} \int_a^b f(x) - g(x)\,dx \geq 0 \overset{(4)}{\Longrightarrow} \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$

$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(9) If $m \leq f(x) \leq M$ on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a)$$

→ This is a consequence of (8). Indeed if $m \leq f(x) \leq M$ then:

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$\Downarrow (3)$$

$$m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a)$$

## EXERCISES

(1) Approximate $\int_0^2 e^x - 2\, dx$ using a left Riemann sum with $n = 4$.

**Solution**

We split the interval $[\overset{a}{0}, \overset{b}{2}]$ into 4 subintervals of same length $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$:

$$\left[0, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, 1\right], \left[1, \tfrac{3}{2}\right], \left[\tfrac{3}{2}, 2\right].$$

For each subinterval the sample point is given by the left endpoint of the interval:

$$\left[0, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, 1\right], \left[1, \tfrac{3}{2}\right], \left[\tfrac{3}{2}, 2\right].$$

$x_1^* = 0$ $\quad x_2^* = \frac{1}{2}$ $\quad x_3^* = 1$ $\quad x_4^* = \frac{3}{2}$

We compute the corresponding Riemann sum:

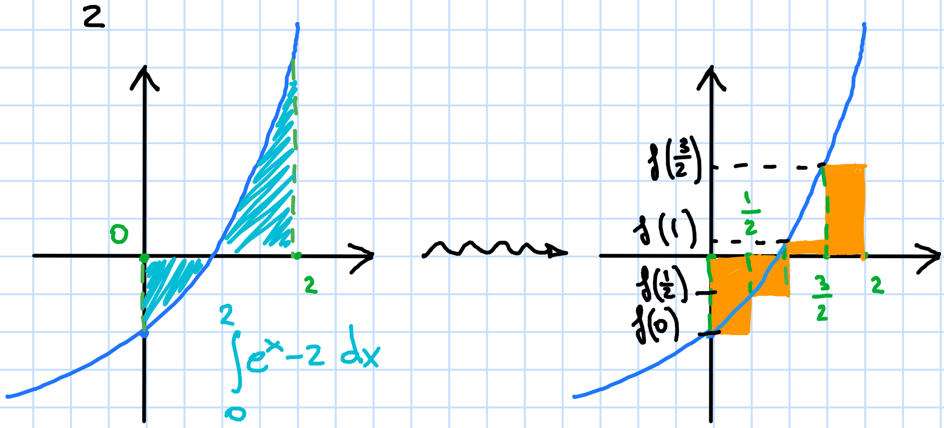
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x.$$

We get:

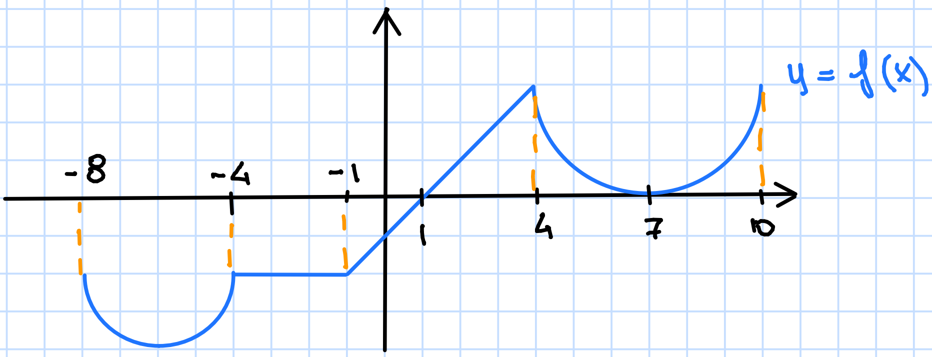
$$(e^0 - 2)\cdot\frac{1}{2} + \left(e^{\frac{1}{2}} - 2\right)\cdot\frac{1}{2} + \left(e^1 - 2\right)\cdot\frac{1}{2} + \left(e^{\frac{3}{2}} - 2\right)\cdot\frac{1}{2} =$$

$$= \frac{1}{2} - 1 + \frac{\sqrt{e}}{2} - 1 + \frac{e}{2} - 1 + \frac{\sqrt{e^3}}{2} - 1 =$$

$$= \frac{1 + \sqrt{e} + e + e\sqrt{e}}{2} - 4 \approx 0.924$$



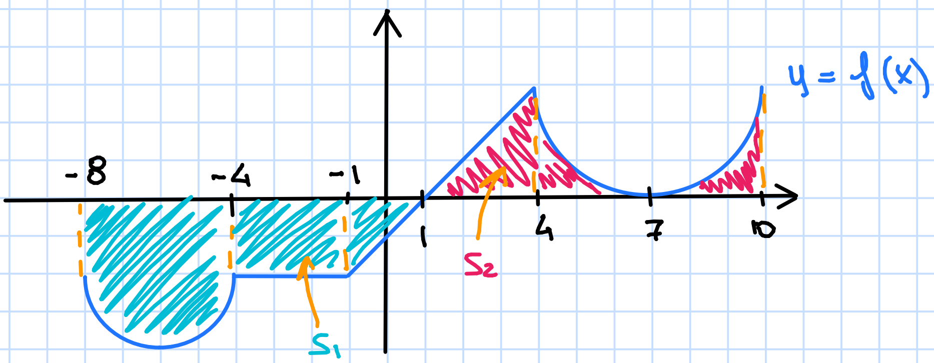
(2) Let $f$ be the function whose graph is the following




Compute $\displaystyle\int_{-8}^{10} f(x)\,dx$

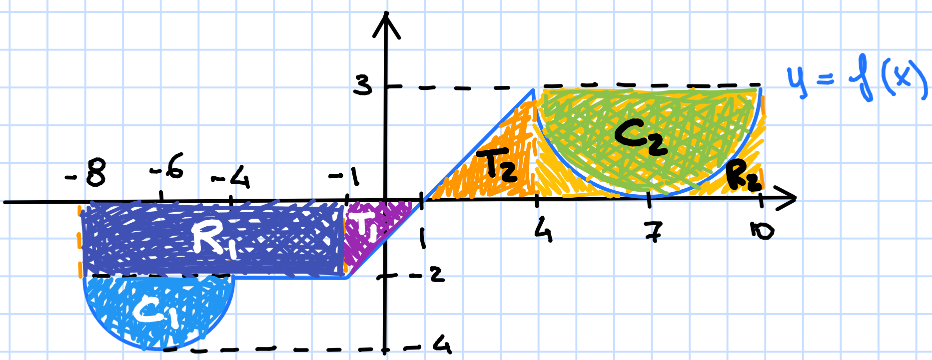
## Solution

If in the previous graph we highlight the region below the x-axis and the region above the x-axis.




We have $\displaystyle\int_{-8}^{10} f(x)\,dx = \text{Area } S_2 - \text{Area } S_1$

Moreover we split $S_1$ and $S_2$ in subshapes for which we have a formula for computing the area.




We have:

$S_1 = R_1 \cup C_1 \cup T_1 \Rightarrow \text{Area } S_1 = \text{Area } R_1 + \text{Area } C_1 + \text{Area } T_1$

$S_2 = (R_2 - C_2) \cup T_2 \Rightarrow \text{Area } S_2 = \text{Area } R_2 - \text{Area } C_2 + \text{Area } T_2$

$\text{Area } R_1 = [-1-(-8)]\cdot 2 = 7\cdot 2 = 14$

$\text{Area } T_1 = \frac{1}{2}(1-(-1))\cdot 2 = \frac{1}{2}\cdot 2 \cdot 2 = 2$

$\text{Area } C_1 = \pi\cdot(-4-(-6))^2 = \pi \cdot 2^2 = 4\pi$

$\text{Area } R_2 = (10-4)\cdot 3 = 18$

$\text{Area } T_2 = \frac{1}{2}(4-1)\cdot 3 = \frac{9}{2}$

$\text{Area } C_2 = \pi\,(10-7)^2 = 9\pi$

$$\Rightarrow \begin{cases} \text{Area } S_1 = \text{Area } R_1 + \text{Area } C_1 + \text{Area } T_1 = 14 + 4\pi + 2 = 16 + 4\pi \\ \text{Area } S_2 = \text{Area } R_2 - \text{Area } C_2 + \text{Area } T_2 = 18 - 9\pi + \frac{9}{2} = \frac{45}{2} - 9\pi \end{cases}$$

In conclusion $\int_{-8}^{10} f(x)\,dx = \text{Area } S_2 - \text{Area } S_1 = \frac{45}{2} - 9\pi - (16+4\pi) =$

$$= \frac{45}{2} - 16 - 9\pi - 4\pi = \frac{13}{2} - 13\pi = -34.34.$$

---

(3) Compute $\int_{3}^{-2} f(x)\,dx$ if $\int_{-2}^{5} f(x) = 2$ and $\int_{5}^{3} f(x) = -1$

<u>Solution</u>

If $a=-2$, $b=3$, $c=5$ we have:

$$\int_{\underset{a}{-2}}^{\overset{b}{3}} f(x)\,dx + \int_{\underset{b}{3}}^{\overset{c}{5}} f(x)\,dx = \int_{\underset{a}{-2}}^{\overset{c}{5}} f(x)\,dx$$

flip and change sign

$$\Downarrow$$

$$-\int_{3}^{-2} f(x)\,dx - \int_{5}^{3} f(x)\,dx = 2$$

$$\Downarrow$$

$$-\int_{3}^{-2} f(x)\,dx - (-1) = 2$$

$$\Downarrow$$

$$\int_{3}^{-2} f(x)\,dx = 1-2 = -1.$$