DIFFERENTIATION FORMULAS (Sec. 2.3, 2.4)

Recall from the previous class that if f is a function, then the derivative of f is the function f' defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **DERIVATIVE OF A CONSTANT FUNCTION**

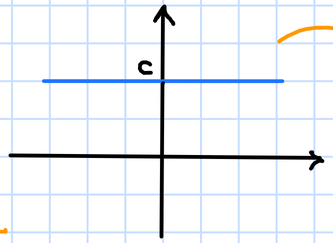
$f(x) = c$, where $c \in \mathbb{R}$.

$\Downarrow$

algebraically

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

geometrically



The graph of a constant function is a horizontal line, which is tangent to itself at each point and has slope zero.

Lagrange notation : $c \in \mathbb{R}$, $(c)' = 0$
Leibniz notation : $c \in \mathbb{R}$, $\frac{d}{dx}(c) = 0$

do not mix Lagrange and Leibniz notation

ex : $f(x) = \pi \Rightarrow f'(x) = 0$
$f(x) = e^{100} \Rightarrow f'(x) = 0$

- **DERIVATIVE OF A POWER FUNCTION**

* natural exponent

$f(x) = x^n$, $n \in \mathbb{N}$ *

* set of positive natural number $n > 0$

- $n = 1 \Rightarrow f(x) = x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}} = \lim_{h \to 0} 1 = 1$$

- $n = 2 \Rightarrow f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x$$

- $n=3 \Rightarrow f(x) = x^3$ recall $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}} = 3x^2$$

- ⋮

Recap:

$$f(x) = x^1 \Rightarrow f'(x) = 1 = 1\cdot x^{1-1}$$

$$f(x) = x^2 \Rightarrow f'(x) = 2x = 2x^{2-1}$$

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2 = 3x^{3-1}$$

You could compute the derivative of the general power function $f(x) = x^n$ by using the Binomial formula (see page 96 of the book).

We have:

POWER RULE

If n is a positive natural integer then

$$(x^n)' = n\,x^{n-1} \quad \text{(Lagrange notation)}$$

$$\frac{d}{dx} x^n = n\,x^{n-1} \quad \text{(Leibniz notation)}$$

ex: $f(x) = x^{2018} \Rightarrow f'(x) = 2018\,x^{2018-1} = 2018\,x^{2017}$

* negative exponent

$f(x) = x^k, \quad k \in \mathbb{Z}$ ← set of integers

- $k=-1 \Rightarrow f(x) = x^{-1} = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h} =$$

$$= \lim_{h\to 0}\frac{x-x-h}{h\cdot x(x+h)} = \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}\cdot x(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

Hence $f(x) = x^{-1} \Rightarrow f'(x) = -\frac{1}{x^2} = -x^{-2} = (-1)\cdot x^{-1-1}$

* rational exponent

$f(x) = x^{\frac{p}{q}}$, $\frac{p}{q} \in \mathbb{Q}$ ← set of rational numbers (fractions)

- $\frac{p}{q} = \frac{1}{2} \Rightarrow f(x) = x^{\frac{1}{2}} = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Hence $f(x) = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1}$

These examples show that the power rule can be generalized to a generic real exponent (even if we did not provide a formal proof, which would be too difficult)

POWER RULE (general version)

If α is a real number, then:

$$(x^\alpha)' = \alpha x^{\alpha-1} \quad \text{(Lagrange notation)}$$

$$\frac{d}{dx} x^\alpha = \alpha x^{\alpha-1} \quad \text{(Leibniz notation)}$$

Remark: The derivative of the constant funtion is a particular case of the power rule when $\alpha = 0$.

Indeed $f(x) = c = c \cdot x^0 \Rightarrow f'(x) = 0 = 0 \cdot x^{0-1}$

ex: • $(\sqrt[3]{x^2})' = (x^{\frac{2}{3}})' = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3} \frac{1}{x^{\frac{1}{3}}} = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$.

rewrite as a power function

- DERIVATIVES OF TRIGONOMETRIC FUNCTIONS (SINE & COSINE)

ADDITION FORMULA FOR SINE
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

* $f(x) = \sin(x)$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h} =$$

$$= \lim_{h\to 0} \left[\sin(x)\frac{(\cos(h)-1)}{h} + \cos(x) \cdot \frac{\sin(h)}{h}\right] =$$

$$= \sin(x) \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{=0\,*} + \cos(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{=1 \text{ (special limit)}} =$$

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)$$

$$* \lim_{h\to 0} \frac{\cos(h)-1}{h} \cdot \frac{\cos(h)+1}{\cos(h)+1} = \lim_{h\to 0} \frac{\cos^2(h)-1}{h(\cos(h)+1)} =$$

$$= \lim_{h\to 0} \frac{-(1-\cos^2(h))}{h(\cos(h)+1)} = \lim_{h\to 0} \frac{-\sin^2(h)}{h(\cos(h)+1)} =$$

$\sin^2(h)+\cos^2(h)=1$

$\Rightarrow \sin^2(h) = 1-\cos^2(h)$

$$= \lim_{h\to 0} \left[-\frac{\sin(h)}{h} \cdot \frac{\sin(h)}{h} \cdot h \cdot \frac{1}{\cos(h)+1} \right] =$$

$$= -\underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{1} \cdot \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{1} \cdot \underbrace{\lim_{h\to 0} \frac{h}{\cos(h)+1}}_{0} = 1 \cdot 1 \cdot 0 = 0$$

Hence if $f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$

$*$ $f(x) = \cos(x) \Rightarrow f'(x) = -\sin(x)$

The proof is totally analogous to that of sine, but this time you have to use the formula of addition of cosine:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

DERIVATIVE OF SINE & COSINE

$(\sin(x))' = \cos(x)$; $(\cos(x))' = -\sin(x)$ (Lagrange)

$\frac{d}{dx}\sin(x) = \cos(x)$; $\frac{d}{dx}\cos(x) = -\sin(x)$ (Leibniz)

Now that we know the derivatives of the fundamental functions we need to study how the operation of differentiation behaves when we sum, multiply, divide, compose functions.

For example, can we easily compute the derivative of $\sqrt{x}+x$ by only knowing the derivatives of $\sqrt{x}$ and x?

- CONSTANT MULTIPLE RULE

Let $c \in \mathbb{R}$ (a constant), and f a differentiable function:

$$[cf(x)]' = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \to 0} c \frac{f(x+h) - f(x)}{h} =$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = cf'(x).$$

CONSTANT MULTIPLE RULE

If $c \in \mathbb{R}$ and f is a differentiable function, then:

$$[cf(x)]' = cf'(x) \qquad \text{(Lagrange)}$$

$$\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x) \qquad \text{(Leibniz)}$$

ex: • $(3\cos(x))' = 3(\cos(x))' = 3(-\sin(x)) = -3\sin(x)$

• $(888x^{888})' = 888 \cdot (x^{888})' = 888 \cdot 888 \cdot x^{887}$

- SUM AND DIFFERENCE RULE

Let f, g be two differentiable functions. Then:

$$[f(x) + g(x)]' = \lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h} =$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} =$$

$$\overset{\text{SUM LIMIT LAW}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} =$$

$$= f'(x) + g'(x)$$

Hence the derivative of the sum is the sum of the derivatives.

Now we can see the difference of two functions as the sum of the first one with the opposite of the second one:

$$[f(x) - g(x)]' = [f(x) + \underbrace{(-g(x))}_{\text{opposite function}}]' \underset{\text{sum rule}}{=} f'(x) + (-g(x))' =$$

$$\underset{\text{constant rule}}{=} f'(x) + (-1) \cdot g'(x) = f'(x) - g'(x)$$

SUM AND DIFFERENCE RULE

If f and g are differentiable functions, then:

SUM

$$[f(x)+g(x)]' = f'(x)+g'(x) \qquad \text{(Lagrange)}$$

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \qquad \text{(Leibniz)}$$

DIFFERENCE

$$[f(x)-g(x)]' = f'(x)-g'(x) \qquad \text{(Lagrange)}$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) \qquad \text{(Leibniz)}$$

ex:

- $(\sqrt{x}+x)' = (\sqrt{x})' + (x)' = \frac{1}{2\sqrt{x}} + 1$
- $(x^5+4x^3-2x+3)' = (x^5)' + (4x^3)' - (2x)' + (3)' =$
  $= 5x^4 + 4(x^3)' - 2(x)' + 0 =$
  $= 5x^4 + 4\cdot 3x^2 - 2\cdot 1 + 0 =$
  $= 5x^4 + 12x^2 - 2.$
- $(4x^2 - 7\cos(x))' = (4x^2)' - (7\cos(x))' =$
  $= 4(x^2)' - 7\cdot(\cos(x))' =$
  $= 8x - 7(-\sin(x)) =$
  $= 8x + 7\sin(x)$

- PRODUCT RULE

Let $f(x)$ and $g(x)$ two differentiable functions. Then:

$$[f(x)g(x)]' = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$$

I am not changing anything if I add and substract the same quantity (it is like I am adding zero)

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-f(x)g(x+h)+f(x)g(x+h)}^{=0}-f(x)g(x)}{h} =$$

$$= \lim_{h\to 0}\left[g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h}\right] =$$

$$= \lim_{h\to 0} g(x+h)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot\frac{g(x+h)-g(x)}{h} =$$

$$= g(x)f'(x) + f(x)g'(x)$$

⚠ Warning: The derivative of a product of functions is not the product of the derivatives!!

PRODUCT RULE

If f and g are differentiable functions, then:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \quad \text{(Lagrange)}$$

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right] \quad \text{(Leibniz)}$$

ex:

- $$(\underbrace{x}_{f(x)} \cdot \underbrace{\sin(x)}_{g(x)})' = \underbrace{(x)'}_{f'(x)} \cdot \underbrace{\sin(x)}_{g(x)} + \underbrace{x}_{f(x)} \cdot \underbrace{(\sin(x))'}_{g'(x)} = 1 \cdot \sin(x) + x\cos(x)$$

- $$(\cos^2(x))' = (\cos(x) \cdot \cos(x))' = \cos(x)\,(\cos(x))' + (\cos(x))'\cos(x) =$$
$$= \cos(x)\,(-\sin(x)) + (-\sin(x))\cos x =$$
$$= -\sin(x)\cos(x) - \sin(x)\cos x = -2\sin(x)\cos(x)$$

we will see that there is a simpler way (called chain rule) for computing derivatives like this one.

- ## QUOTIENT RULE

Let f and g two differentiable functions. Then

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x+h)\,g(x)} =$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x+h)\,g(x)} =$$

$$= \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h\,g(x+h)\,g(x)} - \lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h\,g(x+h)\,g(x)} =$$

$$= g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0}\frac{1}{g(x+h)g(x)} - f(x)\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\lim_{h\to 0}\frac{1}{g(x+h)g(x)} =$$

$$= g(x)f'(x) \cdot \frac{1}{g^2(x)} - f(x)g'(x) \cdot \frac{1}{g^2(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

QUOTIENT RULE

If f and g are differentiable functions, then:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)} \qquad \text{(Lagrange)}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\left(\frac{d}{dx}f(x)\right)g(x) - f(x)\left(\frac{d}{dx}g(x)\right)}{g^2(x)} \qquad \text{(Leibniz)}$$

⚠ Warning: The order in which the terms appear in the numerator is important, since now we have a difference (and not a sum as in the product).

→ THE DERIVATIVE OF THE TANGENT

Among the trigonometric functions we did not compute the derivative of the tangent.

Recall that it is possible to define algebraically the tangent of an angle as the quotient of the sine and cosine of that angle:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

This implies, in particular, that the tangent is not defined when $\cos(x)=0$, i.e. when $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ (i.e. k is an integer).

We will compute the derivative of $\tan(x)$ by using the quotient rule:

$$(\tan(x))' = \frac{(\sin(x))'\cos(x) - \sin(x)\cdot(\cos(x))'}{\cos^2(x)} = \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} =$$

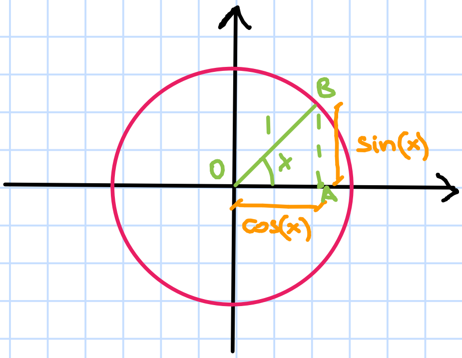
$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \begin{cases} \text{PATH 1} \\ \text{PATH 2} \end{cases}$$

In simplifying the last expression we can take two different paths:

PATH 1

We will use the Pythagorean trigonometric identity:

$$\sin^2(x) + \cos^2(x) = 1$$



The triangle $\hat{OAB}$ is a right triangle where:

lenght hypotenuse $= 1$
lenght opposite leg $= \sin(x)$
lenght adjacent leg $= \cos(x)$

$\downarrow$ Pythagorean theorem

$\sin^2(x) + \cos^2(x) = 1$

$$\underset{\sin^2(x)+\cos^2(x)=1}{=} \frac{1}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)^2 = \sec^2(x)$$

PATH 2

$$\underset{\text{split the fraction}}{=} \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = 1 + \left(\frac{\sin(x)}{\cos(x)}\right)^2 = 1 + \tan^2(x)$$

Hence we have $\left(\tan(x)\right)' = \sec^2(x)$ or $\left(\tan(x)\right)' = 1 + \tan^2(x)$

(Since $\sec^2(x)$ and $1+\tan^2(x)$ are derivatives of the same function we have also the identity:

$$\sec^2(x) = 1 + \tan^2(x) \quad \text{for all } x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z})$$

ex: $$\left(\frac{x-3}{x^2+1}\right)' = \frac{(x-3)'(x^2+1) - (x-3)(x^2+1)'}{(x^2+1)^2} = \frac{1\cdot(x^2+1) - (x-3)\cdot 2x}{(x^2+1)^2} =$$

when you compute the derivative of a qotient, first of all draw the line and put in the denominator the square of the denominator of the initial function.

$$= \frac{x^2+1-2x^2+6x}{(x^2+1)^2} = \frac{-x^2+6x+1}{(x^2+1)^2}$$

In the next page we recap all the differential rules that we got so far.

DIFFERENTIATION RULES

- CONSTANT RULE : $c \in \mathbb{R}, \quad (c)' = 0$
- POWER RULE : $\alpha \in \mathbb{R}, \quad (x^{\alpha})' = \alpha x^{\alpha-1}$
- TRIGONOMETRIC DERIVATIVES : $(\sin(x))' = \cos(x)$

  $(\cos(x))' = -\sin(x)$

  $(\tan(x))' = \sec^2(x)$ or $1+\tan^2(x)$

Let $c \in \mathbb{R}$ and f, g two differentiable functions:

- CONSTANT MULTIPLE RULE : $(c f(x))' = c f'(x)$
- SUM RULE : $(f(x) + g(x))' = f'(x) + g'(x)$
- DIFFERENCE RULE : $(f(x) - g(x))' = f'(x) - g'(x)$
- PRODUCT RULE : $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- QUOTIENT RULE : $\dfrac{f(x)}{g(x)} = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$