Mathematics in the Real World
Math 107

Lectures 5 & 6: Logarithms and applications.

Continuous compounding and the number e. Recall that in case of the compound interest rate $r\%$ compounded n times per year, the amount accumulated after t years on the principal of A dollars is

$$A(t) = A\left(1 + \frac{r}{n \cdot 100}\right)^{nt}$$

What happens if the number of compounding periods per year is allowed to grow indefinitely? To simplify our computations, suppose that the principal $A = \$1$, $t = 1$ and $r = 100\%$. In this case, the formula reads

$$A(1) = \left(1 + \frac{1}{n}\right)^{n}.$$

Let us see how the result depends on n:

$$\begin{array}{ll} n = 1 & A(1) = (1+1)^1 = 2 \\ n = 2 & A(1) = (1+\frac{1}{2})^2 = \frac{9}{4} = 2.25 \\ n = 12 & A(1) = (1+\frac{1}{12})^{12} \simeq 2.613 \\ n = 365 & A(1) = (1+\frac{1}{365})^{365} \simeq 2.714 \\ n = 1000 & A(1) = (1+\frac{1}{1000})^{1000} \simeq 2.717 \end{array}$$

In fact, when n grows indefinitely, this sequence converges to a limit, namely to the number e:

$$\left(1 + \frac{1}{n}\right)^{n} \underset{n\to\infty}{\longrightarrow} e = 2.718281828459045\ldots$$

This number plays an important role in mathematics; for example, it appears as the base of the natural logarithm ln, which we will introduce below. One of the first definitions of e was given in the 17th century by the Swiss mathematician Jacob Bernoulli. He defined e as the annual growth factor with 100% compound interest rate *compounded continuously.*

If the compound interest rate is $r\%$, the amount accumulated after t years in case of continuous compounding is

$$A(t) = Ae^{\frac{rt}{100}}.$$

The effective annual rate in this case is determined by the formula

$$1 + \frac{r_{eff}}{100} = e^{\frac{r}{100}}.$$

Example 1. Let $A = 5000$, $r = 4\%$, compounded continuously.

(1) How much money will accumulate in $t = 5$ years?
(2) Find the effective annual rate.

Solution: Using the formula for continuous compounding, we get

$$A(5) = Ae^{\frac{4\cdot 5}{100}} = 5000e^{0.2} \simeq 6107.01.$$

For the effective annual rate, we have

$$1 + \frac{r_{eff}}{100} = e^{\frac{4}{100}} = e^{0.04} \simeq 1.04081.$$

Therefore, $r_{eff} \simeq 4.081\%$.

$\square$

Logarithms. To solve the compound interest equation for the number of years it takes to accumulate a certain amount, we need the logarithms. Here is a quick introduction. For positive numbers a and B, the logarithm $\log_a B$ is defined as the unique solution to the equation:

$$a^x = B \qquad \Longleftrightarrow \qquad x = \log_a B.$$

By definition, the exponential function a^x and the logarithmic function $\log_a x$ are inverse to each other. This means that if you apply one and then another, you get the same number you started with: for any positive a and x we have

$$a^{\log_a x} = x.$$

For any positive a and any real x we have

$$\log_a (a^x) = x.$$

The number a is the base of $\log_a$. We will denote

$$\log = \log_{10}, \quad \ln = \log_e,$$

where $e = 2.71828...$

For example, consider base 10 logarithm. Then $\log 1000 = 3$, since to obtain 1000, you need to raise 10 to the third power.

Here is a quick reminder of the *laws of logarithms*, the algebraic rules of dealing with the logarithms. Let a, B, C be positive numbers, and r a real number. The main properties of the logarithms are the following:

$$\log_a B + \log_a C = \log_a(BC), \qquad \log_a B - \log_a C = \log_a \left(\frac{B}{C}\right)$$

$$\log_a(B^r) = r \log_a B, \qquad \log_a B = \frac{\ln B}{\ln a},$$

$$\log_a 1 = 0, \qquad \log_a a = 1.$$

Example 2. In what follows, $\log = \log_{10}$, and $\ln = \log_e$. Compute each of the following expressions using the definition of the logarithm.

(1) $\log \frac{1}{100}$. *Solution:* This is the power you need to raise 10 to to get $\frac{1}{100}$. We have $10^{-2} = \frac{1}{100}$. Therefore, $\log \frac{1}{100} = -2$.
(2) $\log 10^{2t+4}$. *Answer:* $2t + 4$.
(3) log of one million. *Answer:* 6.
(4) $\log 1$. *Answer:* 0.
(5) Solve the equation using log: $10^x = 563$. *Answer:* $x = \log 563 \simeq 2.7505$.
(6) $\log 60 - \log 6$.
Solution: $\log 60 - \log 6 = \log(6 \cdot 10) - \log 6 = \log 6 + \log 10 - \log 6 = \log 10 = 1$.
(7) $\log(1+t) + \log(1-t) - \log(1-t^2)$.
Solution: $\log(1+t) + \log(1-t) - \log(1-t)(1+t) = \log(1+t) + \log(1-t) - (\log(1+t) + \log(1-t)) = 0$.
(8) $(\ln 9 + \ln 4)/\ln(6)$. *Solution:* $(\ln 9 + \ln 4)/\ln 6 = (\ln(3^2) + \ln(2^2))/\ln(3 \cdot 2) = (2\ln 3 + 2\ln 2)/(\ln 3 + \ln 2) = 2$.
(9) Solve using log and also using ln: $30 \cdot 2^t = 900$. Write down your answer to three places of decimal.
Solution: Dividing by 30, we get $2^t = 30$. Therefore, $t = \log 30/\log 2 = \ln 30/\ln 2 = 4.907$.
(10) Solve for k in: $50 \cdot 10^{5k} = 5$.
Solution: Dividing by 50, we get $10^{5k} = \frac{1}{10} = 10^{-1}$. Therefore, $5k = -1$ and $k = -1/5$.
(11) Solve the equation using log: $(\frac{1}{10})^{(3x-1)} = 38$.
Solution:

$$\left(\frac{1}{10}\right)^{3x-1} = ((10)^{-1})^{3x-1} = 10^{-3x+1} = 38.$$

The last equality follows from the property of exponents: $(A^a)^b = A^{ab}$. Now we can apply the logarithm (base 10):

$$-3x + 1 = \log 38 \quad \Rightarrow \quad x = \frac{1}{3}(1 - \log 38).$$

Time-related questions in compound interest models. Now we can use logarithms to answer the following type of questions: With the annual interest rate r% compounded n times per year, how long does it take for the amount to double (triple, grow by 50%, increase by 1000, etc.) We have

$$A(t) = A\left(1 + \frac{r}{n \cdot 100}\right)^{nt} = 2A$$

We need to find t. We can cancel the principal A from both sides of the equation and get

$$\left(1 + \frac{r}{n \cdot 100}\right)^{nt} = 2.$$

Then

$$(nt) \ln \left(1 + \frac{r}{n \cdot 100}\right) = \ln 2.$$

and

$$t = \frac{\ln 2}{n \cdot \ln \left(1 + \frac{r}{n \cdot 100}\right)}.$$

In case of continuous compounding, the same question can be asked: given an annual interest rate $r\%$, how long does it take for an investment to double. We need to solve the equation

$$Ae^{\frac{rt}{100}} = 2A$$

This leads to

$$e^{\frac{rt}{100}} = 2, \quad \Longrightarrow \frac{rt}{100} = \ln 2 \quad \Longrightarrow t = \frac{100}{r} \ln 2.$$

Example 3. How long does it take to for your investment to grow by 50%, if the annual interest rate is 6% compounded (a) quarterly, (b) continuously?
Answer: (a)

$$t = \frac{\ln 1.5}{4 \cdot \ln \left(1 + \frac{6}{4 \cdot 100}\right)} \simeq 6.81 \text{ years.}$$

(b)

$$t = \frac{100}{6} \ln 1.5 \simeq 6.75 \text{ years.}$$

Example 4. How long does it take for your investment to grow from $A = 10000$ to $A(t) = 12000$, if the annual interest rate is 4%, compounded quarterly?
Solution: We have $r = 4\%$, $n = 4$, $A = 10000$, $A(t) = 12000$.

$$A(t) = 12000 = 10000(1 + \frac{4}{4 \cdot 100})^{4t}.$$

Therefore,

$$12 = 10(1.01)^{4t} \quad \Rightarrow \quad \frac{6}{5} = (1.01)^{4t}.$$

Now is a good time to apply the logarithms. You can use a logarithm of any base, since the answer is independent of the base. We choose the natural logarithm because it is a one-touch operation in any calculator:

$$\ln (\frac{6}{5}) = \ln (1.01)^{4t} = 4t \cdot \ln 1.01 \quad \Rightarrow \quad t = \frac{\ln 1.2}{4 \ln 1.01}.$$

Finally, use your calculator:

$$t = \frac{\ln 1.2}{4 \ln 1.01} \simeq 4.6 \text{ years.}$$

Logarithmic scales. "Adders need logs to multiply". In other words, logs allow us to perform multiplication by doing addition instead:

$$\log_a(BC) = \log_a B + \log_a C.$$

As a consequence, the logarithms provide a tool to deal with data of very large range. If two positive numbers differ by a factor of 10, their logs (base 10) differ by one unit. For example, if the quantities $B = 100000$ and $C = 0.1$ need to be marked on the same scale, then picturing them together on the number line is problematic: the difference $B - C = 99999.9$ is too large compared to C. But on the logarithmic scale they become easy to visualize: instead of C, we mark $\log C = -1$, instead of B, $\log B = 5$. Then the difference between them is only 6 units.

This is why logarithmic scales are used in many applications with wide range of data, for example to measure the relative brightness of stars, intensity of earthquakes (the Richter scale), sound intensity (the decibel scale), liquid acidity (the PH scale).

Noise levels. The decibel scale measures sound intensity with respect to the reference level, the *threshold of hearing* (TOH). Given a sound intensity P (in kilowatts), the decibel measure of the noise level is given by

$$L_{dB} = 10 \log\left(\frac{P}{P_0}\right),$$

where P_0 is the TOH intensity in kilowatts. The decibel scale is a convenient description of human perception of noise. For example, the kilowatt level of whisper is about 100 times the kilowatt level of TOH: $P_{\text{whisper}} = 100P_0$. Then the decibel level of whisper is

$$L_{\text{whisper}} = 10 \log\left(\frac{P_{\text{whisper}}}{P_0}\right) = 10 \log 100 = 10 \cdot 2 = 20\ dB.$$

Other examples include

TOH	$0\ dB$,
normal voice	$60\ dB$,
street trafic	$70\ dB$
nightclub	$100\ dB$.

Conversely, if we know the decibel level of a sound, we can find its kilowatt intensity with respect to P_0:

$$L = 10 \log\left(\frac{P}{P_0}\right) \quad \Longrightarrow \frac{L}{10} = \log\left(\frac{P}{P_0}\right) \quad \Longrightarrow \frac{P}{P_0} = 10^{\frac{L}{10}}.$$

Example 5. Given that the noise level at a nightclub is approximately 100 dB and that of the normal voice about 60 dB, find the kilowatt ratio between the two noise levels. *Solution:* Let $L_n = 100\ dB$ and $L_v = 60\ dB$ be the decibel noise levels of nightclub music and normal voice, respectively, and P_n and P_v their kilowatt intensities. Then

$$\frac{P_n}{P_0} = 10^{\frac{L_n}{10}} = 10^{10}; \qquad \frac{P_v}{P_0} = 10^{\frac{L_v}{10}} = 10^{6}.$$

Therefore,

$$\frac{P_n}{P_v} = \frac{10^{10}}{10^6} = 10^4.$$

The nightclub noise is 10000 times as intense as normal voice.

Apparent magnitude of stars. A logarithmic scale of relative *apparent magnitude* of stars originates from ancient Greece; in particular, dividing the visible stars into six levels of magnitudes (now 0th to 5th) according to their brightness was practiced by Ptolemy. The Greeks also postulated that the brightest visible stars (of magnitude 0) are 100 times as bright as the faintest visible stars (of magnitude 5). Given these conditions, we can calculate the base of a logarithmic scale suitable to describe the relative magnitude of the stars. 5 steps on this scale should correspond to the brightness ratio of $\frac{1}{100}$:

$$5 = \log_X \frac{1}{100} \quad \Longrightarrow X^5 = \frac{1}{100} \quad \Longrightarrow X = \sqrt[5]{\frac{1}{100}} = \frac{1}{\sqrt[5]{100}} = 10^{-\frac{2}{5}}.$$

Indeed, according to the modern scale, the apparent magnitude of a star is defined as

$$m = \log_{10^{-\frac{2}{5}}} \left(\frac{F}{F_0}\right),$$

where F is the observed flux (brightness) of the star and F_0 the flux (brightness) of Vega, which is taken as the reference point. To make the computations easier, we can use the definition of the logarithm to reformulate the scale in terms of logarithms base 10:

$$10^{-\frac{2}{5}m} = \frac{F}{F_0} \quad \Leftrightarrow \quad -\frac{2}{5}m = \log\left(\frac{F}{F_0}\right) \quad \Leftrightarrow \quad m = -\frac{5}{2}\log\left(\frac{F}{F_0}\right).$$

This scale can also be used to calculate apparent magnitude of stars visible only in telescope, and brighter objects like the moon and the sun. Because of the minus sign in front of the log, this is an example of a reverse logarithmic scale: the brighter the star, the smaller its magnitude. Very bright objects have negative apparent magnitude.

For example, to find the apparent magnitude of Sirius, given that its observed flux is 3.63 times that of Vega, we calculate

$$m_{\text{Sirius}} = -\frac{5}{2}\log\left(\frac{F_{\text{Sirius}}}{F_0}\right) = -\frac{5}{2}\log 3.63 \simeq -1.4.$$

Conversely, given that the apparent magnitude of the faintest stars visible in an urban neighborhood is 4, we can find their relative brightness with respect to Vega:

$$F = 10^{-\frac{2}{5}m}F_0 = 10^{-\frac{8}{5}}F_0 \simeq 0.025F_0.$$

These stars are about 2.5% as bright as Vega.

Example 6. The apparent magnitude of the Moon is $m_M = -12.74$, and the apparent magnitude of Venus at its maximum brightness is $m_V = -4.89$. What is the ratio of brightness between the Moon and Venus?
Solution: We need to find $\frac{F_M}{F_V}$, where F_M and F_V are the brightness of the Moon and Venus. We have

$$m_M - m_V = -\frac{5}{2}\left(\log\left(\frac{F_M}{F_0}\right) - \log\left(\frac{F_V}{F_0}\right)\right) = -\frac{5}{2}\log\left(\frac{\frac{F_M}{F_0}}{\frac{F_V}{F_0}}\right) =$$

$$= -\frac{5}{2}\log\left(\frac{F_M}{F_V}\right) = -12.74 + 4.89 = -7.85.$$

Then

$$\log\left(\frac{F_M}{F_V}\right) = 3.14 \qquad \Longrightarrow \frac{F_M}{F_V} = 10^{3.14} \simeq 1380.$$

The Moon is 1380 times as bright as Venus.

Chromatic scale in music. European music is based on a seven pitch octave scale with ratios between the frequencies given by quotients of small integers. For example, the octave interval has the ratio of frequencies $2:1$; perfect fifth $3:2$; perfect fourth $4:3$; major third $5:4$, minor third $6:5$. It is believed that our preference for such intervals originates from their presence in nature, which can be explained by the physics of natural sounds - the simplest drums and whistles produce exactly this kind of intervals.

But the 7-pitch scale had a serious drawback: it didn't behave well under translations. For example, if the ratio between the first and the fifth pitch is $3:2$, and between the first and the third $5:4$, then the ratio between the third and the fifth pitch is $3/2 : 5/4 = 6:5$. Therefore, the third and the fifth pitch form a minor third, while the first and the third pitch - a major third, a different interval, even though there is exactly one skipped pitch in between in both cases. In other words, with the 7-pitch octave, it was impossible to translate a song a few pitches up or down; this required building a new instrument. This happened because the 7-pitch scale, as it is defined, is not equidistant: the ratios between each two *consecutive* frequencies are different from pitch to pitch.

This difficulty was especially problematic in organ music, where each pipe is set to a particular pitch. A solution was found in the 17th century: a new scale was proposed, that satisfied the following requirements:

(1) The scale is logarithmically equidistant: the ratios between each two consecutive pitches are equal. This assures translation invariance: any song can be translated a few pitches up or down, with the relative intervals staying exactly the same.

(2) Among the pitches in the new scale, the old seven ones can be identified, such that their relative frequencies are as close as possible to the original 7-pitch octave.
(3) The number of pitches per octave can be increased, but it should not be too large.

The resulting scale, the *chromatic scale*, is now used in most instruments, including the organ and the piano. It contains 12 pitches per octave with the ratio of frequencies between each two consecutive pitches equal to

$$\sqrt[12]{2} = 2^{1/12}.$$

This is a uniform logarithmic scale with step $\frac{1}{12}\ln 2$. The ratio of frequencies between the kth and the lth pitch is

$$\frac{f_k}{f_l} = (\sqrt[12]{2})^{k-l}.$$

Thus, the octave contains 12 equal intervals, and the ratio between the first and the thirteenth pitch is exactly $2 = (\sqrt[12]{2})^{12}$. Other intervals suffered, but not substantially. The new scale was popularized by J.-S. Bach in his *Well-Tempered Clavier*, a collection of pieces in each of the 12 new keys defined by the 12 new *semitones*. The old seven pitches in the chromatic scale are: 1st, 3rd, 5th, 6th, 8th, 10th and 12th.

Example 7. Find the ratio of frequencies between the 6th and the 1st pitch of the chromatic scale. Which of the perfect intervals does it approximate?
Solution: We compute:

$$(\sqrt[12]{2})^5 = 1.3348.$$

The closest perfect interval is $4 : 3 = 1.333\ldots$. The error of approximation is pretty small: $\frac{13348-1.3333}{1.3333} \simeq 0.0011$, or 0.11%.

Mathematicians calculated that next best approximation to the classical seven pitch scale contains 19 equidistant pitches. Imagine what a 19-semitone keyboard would look like!