MATHEMATICS IN THE REAL WORLD

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Properties of exponents. Simple and compound interest.

In this chapter, we will study operations with powers. We explain what it means for a number to be raised to a positive, negative, integer or fractional power, and how to multiply or divide two fractional powers of a number. This technique is indispensable in modern-day banking and finance, and we will apply it to computation of compound interest. What does it mean to compound a thousand dollars monthly, at a given annual interest rate? In what follows, we explain these and related concepts.

MATH

Let us start with integer powers. We all know how to take the square of a number, or its cube: $1.6^2 = 1.6 \cdot 1.6 = 2.56$ and $2^3 = 8$. Similarly, if n is a positive integer, then the nth power of a (positive) number is

$$A^n = \underbrace{A \cdot A \cdot A \cdot \dots \cdot A}_{n}$$

(multiplied n times). These powers satisfy various properties: for example, $A^3 \cdot A^2 = A^5$ (multiply 3 times, then 2 times – i.e., $3+2$ times in all). Similarly, $(A^3)^2 = A^3 \cdot A^3 = A^6$.

What if we want to take *fractional powers*? Well, one of these is already very familiar: the power $\frac{1}{2}$ of a positive number is simply its square root. Similarly, the power $\frac{1}{3}$ is the cube root $\sqrt[3]{A}$, and so on.

It is also possible to take *negative powers*: $A^{-2} = 1/A^2$, for example. In general, given any positive A and any two positive integers n, m,

$$A^{-n} = 1/A^n, \qquad A^{n/m} = \sqrt[m]{A^n}.$$

In fact the same properties as above hold, whether or not the exponent is positive, negative, a power, or a root. In other words, suppose A and B are positive real numbers and c and d are *any* two real numbers (even irrational ones). Then the following properties hold:

$$A^c \cdot A^d = A^{c+d}, \qquad A^c/A^d = A^{c-d}, \qquad (A^c)^d = A^{cd},$$
$$A^c \cdot B^c = (AB)^c, \qquad A^c/B^c = (A/B)^c.$$

These simple rules are called the *laws/properties of exponents*. Note that they imply in particular that the zeroth power of any number is 1: $A^0 = A^{1-1} = A/A = 1$. The properties of exponents are extremely useful in simplifying expressions involving powers. For example,

$$2^3 \cdot \sqrt{2^5} = 2^3 \cdot (2^{1/2})^5 = 2^3 \cdot 2^{5/2} = 2^{3+(5/2)} = 2^{11/2}.$$

The first equality is the definition of a square root. In the second equality, we used the property above that $(A^c)^d = A^{cd}$, with $A = 2, c = 1/2, d = 5$. In the third equality, we used that $A^c \cdot A^d = A^{c+d}$, putting $A = 2, c = 3, d = 5/2$.

Here are some more examples.

EXAMPLES

Example 0.1. *Simplify* $2^{10} \cdot 5^5/20^4$.
Answer: Using the properties of exponents, we have:

$$\frac{2^{10} \cdot 5^5}{20^4} = \frac{(2^{2\cdot 5}) \cdot 5^5}{20^4} = \frac{(2^2)^5 \cdot 5^5}{20^4} = \frac{4^5 \cdot 5^5}{20^4} = \frac{(4 \cdot 5)^5}{20^4} = \frac{20^5}{20^4} = 20^{5-4} = 20.$$

□

It is very instructive to take each equality in the above calculation, and see why it is true – and whether you need to use one of the above properties of exponents to justify it.

The next example shows that the exponents need not be concrete real numbers – they can be "variables" as well.

Example 0.2. *Simplify* $9^b 4^b - 6^{2b}$. Here, b is a fixed (unknown) real number.
Answer: Using the properties of exponents, we compute:

$$9^b \cdot 4^b - 6^{2b} = (9 \cdot 4)^b - 6^{2b} = 36^b - 6^{2b} = (6^2)^b - 6^{2b} = 6^{2b} - 6^{2b} = 0.$$

□

Here is a third example. The general philosophy is to try and simplify as much as you can. Thus, do not be surprised if the overall answer is not such a simple expression as "20", or "0" (as in the above examples).

Example 0.3. *Simplify* $2^a 5^b/10^{a+b}$, where a, b are real numbers.
Answer: Using the properties of exponents,

$$\begin{aligned} \frac{2^a \cdot 5^b}{10^{a+b}} = \frac{2^a \cdot 5^b}{10^a \cdot 10^b} = \frac{2^a}{10^a} \cdot \frac{5^b}{10^b} &= (2/10)^a \cdot (5/10)^b = (1/5)^a \cdot (1/2)^b \\ &= (5^{-1})^a \cdot (2^{-1})^b = 2^{-b}5^{-a}. \end{aligned}$$

□

Example 0.4. *Solving equations involving exponents:* Solve the following equations for the unknown real variable:

(1) $5(1 + x)^3 = 40$.
Solution: First divide both sides by 5 to get: $(1 + x)^3 = 8$. Now take the cube root of both sides. The left-hand side can now be computed using the properties of exponents, to equal:

$$((1 + x)^3)^{1/3} = (1 + x)^{3\cdot 1/3} = 1 + x,$$

while the right-side becomes $8^{1/3} = 2$. So we get: $1 + x = 2$, or $x = 1$.

(2) $(3+x)^5 = 32(2x+1)^5$.
Solution: Take the fifth root of both sides - i.e., raise them to the 1/5th power. Using the properties of exponents (do the work!), we can see that

$$(3+x) = (2^5 \cdot (2x+1)^5)^{1/5} = 2 \cdot (2x+1) = 4x+2.$$

This is a linear equation, so moving all terms involving x to the right-hand side, we get:

$$3-2 = 4x - x = 3x.$$

Finally, $x = 1/3$. □

Practice problems. Simplify each of the following expressions using the properties of exponents above:

(1) $12^6/6^{12}$. (*Answer:* $1/3^6 = (1/3)^6$.)
(2) $(25^3 5^t)^{6-t}$ is what power of 5? (*Answer:* 5^{36-t^2}.)
(3) $\sqrt[4]{2}\sqrt{2}\sqrt[4]{8}$. (*Answer:* $2^{\frac{1}{4}} \cdot 2^{\frac{1}{2}} \cdot 8^{\frac{1}{4}} = 2^{\frac{1}{4}+\frac{1}{2}+\frac{3}{4}} = 2^{\frac{3}{2}} = 2\sqrt{2}$.)
(4) $\dfrac{(\sqrt{RS})^7}{R^3S^3}$, where R, S are positive numbers. (*Answer:* $\sqrt{RS}$.)
(5) $\dfrac{\sqrt[6]{27} \cdot \sqrt[6]{16}}{\sqrt{6}\sqrt[6]{2}}$.
(*Answer:* $\dfrac{\sqrt[6]{27} \cdot \sqrt[6]{16}}{\sqrt{6}\sqrt[6]{2}} = \dfrac{3^{\frac{3}{6}} \cdot 2^{\frac{4}{6}}}{2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot 2^{\frac{1}{6}}} = 3^{\frac{1}{2}-\frac{1}{2}} \cdot 2^{\frac{2}{3}-\frac{1}{2}-\frac{1}{6}} = 3^0 \cdot 2^0 = 1$.)
(6) $3 \cdot 6^{2x} \cdot 5^{x+1} - 5 \cdot 3^{3x+1} \cdot 4^x$. (*Answer:* $15 \cdot 6^{2x}(5^x - 3^x)$.)
(7) Solve for β: $(2\beta^2+1)^5 = 243$. (*Answer:* $\beta^2 = 1$, so $\beta = \pm 1$.)
(8) Using your calculator (or even google), find all positive numbers r that satisfy:

$$\left(1 + \frac{r}{400}\right)^{20} = 1.4859.$$

Round your answer off to three decimal places. (Hint: The syntax for raising a number to the 1/20th power, say in google search, is: 5^(1/20).) (*Answer:* $r = 8$.)

Computations such as the last one, are very important and commonly show up in studying compound interest, as we will see in Application 2 below.

APPLICATION 1

Computations with exponents are necessary in modeling quickly developing processes, such as chain reactions and avalanches.

Example 0.5. High in the mountains, a comparatively small initial impact (a snowball) can cause a larger snowfall and eventually lead to a huge avalanche.

Suppose that each second, the volume of the sliding snow grows by a factor of 1/3. What is the volume of the snow in the avalanche after 10 seconds? after half a minute?

Denote the initial volume by V_0. Then after 1 second, we have $V_0 + \frac{1}{3}V_0 = \frac{4}{3}V_0$, after 2 seconds $\frac{4}{3} \cdot \frac{4}{3}V_0$, and so on. After 10 seconds, the volume is

$$V_{10} = \left(\frac{4}{3}\right)^{10} V_0 \simeq 17.76\, V_0.$$

After half a minute, we have

$$V_{30} = \left(\frac{4}{3}\right)^{30} V_0 \simeq 5600\, V_0.$$

For example, if the initial volume V_0 was 1 ft^3 (7.5 gallons), then after 10 seconds we have 133 gallons, and after 30 seconds, almost 42000 gallons of sliding snow.

Now suppose we know that the volume of snow in an avalanche grows in such a way that after 30 seconds, it is 100 times (1000 times) the initial volume. How much does the volume grow per second?

Denote by r the factor of volume growth per second. Then we have an equation:

$$(1+r)^{30} = 100.$$

This implies

$$1 + r = \sqrt[30]{100} \simeq 1.1659,$$

and $r \simeq 0.1659$. In the second case,

$$(1+r)^{30} = 1000, \;\Rightarrow\; 1 + r = \sqrt[30]{1000} \simeq 1.2589,$$

and $r \simeq 0.2589$.

□

Another avalanche type system is provided by a rumor spread.

Example 0.6. Suppose a person obtains some valuable trading information. He probably will not share it massively, but he will tip his closest 5 friends. Next day, they do the same, and so on. How many more people will know the information each day during the next week? We calculate:

day 1	1 person
day 2	5 more people
...	...
day 7	$5^6 = 15625$ more people

In case of a trading tip, at some point in this process saturation will be reached, where all interested people already know the news, and others don't care. Saturation is the reason of failure of financial pyramids, which require exponential increases in participants to sustain them.

□

Finally, here is an example of a different kind, where proficiency with powers comes useful.

Example 0.7. Suppose that a store allows to apply consecutively three discounts:
10% membership discount,
10% seasonal sale discount, and
10% promotional discount.
Another store with the same merchandise offers flat 28% discount on all sales. Which deal is better?

Here the main point is that the three discounts in the first store are applied consecutively. The first discount reduces the price of an item P to $0.9P$, then the second discount reduces it further to $0.9 \cdot 0.9P = (0.9)^2 P$, and the last discount results in a final price of $0.9 \cdot (0.9)^2 P = (0.9)^3 P$. Therefore, a customer would pay $(0.9)^3 P = 0.729P$ in the first store, and only $0.72P$ in the second store. The second deal is better.

□

APPLICATION 2

Raising positive numbers to powers is necessary in finance, in particular in computing *compound interest*. If you take a loan of a thousand dollars from a bank, or deposit a thousand dollars in it, then, after a couple of years, you will either owe or have earned *more* than that amount. The original amount of money borrowed or deposited is the *principal* and the extra amount is called the *interest*.

Simple interest. Let us start with *simple interest*, which is, of course, quite simple. If you deposit a thousand dollars in a bank that has a 5% annual simple interest rate, then the deposited money accumulates interest, fifty dollars per year (which is five percent of \$1,000). Thus, after one year you have \$1,050, after two years \$1,100, and so on.

Here is the "general formula" for simple interest: if we deposit an initial sum of money A in a bank that offers an annual simple interest rate of $r\%$, then each year the principal accumulates the same amount of interest, $A \cdot r/100$. Thus, the interest that we would have accumulated on A after t years is: $A \cdot r \cdot t/100$. The total amount after t years is given by:

$$A(t) = A\left(1 + \frac{rt}{100}\right).$$

Example 0.8. A few years ago, I deposited \$2,000 in a bank with a simple interest scheme. The money in my account today is \$2,500.

(1) If I deposited the money ten years ago, what is the simple interest rate in the bank?
(2) If the rate of simple interest is 5%, then how long ago did I deposit the money in the bank?

Solution: In both calculations, we set $A = 2000$ and $A(t) = 2500$.

(1) If the money was deposited ten years ago, then $t = 10$ and r is unknown, so compute:

$$2500 = 2000\left(1 + \frac{r \cdot 10}{100}\right).$$

Solving for r, $500 = 20 \cdot r \cdot 10$, whence $r = 2.5\%$.

(2) In this part, $r = 5\%$ and t is unknown, so:

$$2500 = 2000\left(1 + \frac{t \cdot 5}{100}\right).$$

Solving for t, $500 = 20 \cdot t \cdot 5$, whence $t = 5$ years. □

Compound interest. This is the model actually used by most banks today. The difference between simple and compound interest is that in case of the simple interest, *only the principal* earns interest. In case of the compound interest, *both the principal and the accumulated interest* earn interest. For example, suppose I deposit a thousand dollars in a bank that offers a 4% annual rate of compound interest. Then in the first year, I accumulate \$40 interest. In the second year, the entire amount of \$1,040 is earning interest, so that at the end of the second year 40% of \$1,040, or \$41.60 is added. Thus, the amount after two years is \$1,081.60, and so on.

Let us write down a general formula for this. If the deposit amount is A dollars, and the annual compound interest rate is $r\%$, then after t years, we have:

$$A(t) = A \underbrace{\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)\cdots\left(1 + \frac{r}{100}\right)}_{t \text{ times}} = A\left(1 + \frac{r}{100}\right)^t.$$

Here we have a formula that involves a variable power t.

Some banks compound more often than others. Thus, if the compounding is quarterly as opposed to annually, then we would multiply four times as many factors for each year – one for each quarter – but the corresponding rate of interest would itself be quartered. In other words, the formula for the amount accumulating after t years would be:

$$A(t) \quad = \quad A\left(1 + \frac{r}{400}\right)^{4t} \qquad \text{(quarterly)}$$

Example 0.9. Suppose I borrow \$10,000 from a bank, and my loan accumulates interest at an annual rate of 6%, compounded quarterly. How much do I owe the bank after three years?

Solution: Use the formula for quarterly compounding, with $A = 10000$, $r = 6$, $n = 4$, and $t = 3$. Thus,

$$A(3) = 10000\left(1 + \frac{6}{400}\right)^{4\cdot 3} = 10000(1 + 0.015)^{12} = 10000(1.015)^{12} \simeq 11956.18.$$

Thus, the loan is worth \$11,956.18 after three years. □

Similarly, if a bank compounds monthly, the number of factors per year would grow to 12, but the rate of interest would get divided by 12:

$$A(t) \quad = \quad A\left(1 + \frac{r}{1200}\right)^{12t} \quad \text{(monthly)}$$

In general, suppose there are n compounding periods per year, and the principal of A dollars is to accrue interest at an *annual* compound interest rate of $r\%$. Then the total amount after t years is

$$A(t) = A\left(1 + \frac{r}{n \cdot 100}\right)^{nt}.$$

Strategy for problem solving. In questions involving compound interest, start with the equation for compound interest (monthly, quarterly, n periods per year). Now plug in the numbers that you are given and identify the unknown quantity. Then solve for the unknown quantity, as is done in examples below. To make your answer meaningful, you should indicate the units of measurement. For instance, if the amount of money is unknown, the answer should be given in a unit of currency (most often \$). If the interest rate is unknown, the answer should be given in percent.

When solving the equation, it is a good idea first to isolate the unknown quantity on one side, writing it in terms of the known quantities, and only then use your calculator to compute the answer. This way, even if the numerical answer is wrong, you can get partial credit for the correct expression, and for doing most of the work to solve the problem. If you have to round your answer, then you should do so only at the *end* of your computations, not in the middle. Here is why: compare 1.01^{20} rounded to one decimal place (you get 1.2) with 1.01 rounded to one decimal place *and then* raised to the 20th power (you get $1.0^{20} = 1.0$).

Example 0.10. How much should be deposited today in an account that earns interest at an annual rate of 6%, compounded monthly, so that it will accumulate to \$20,000 in 5 years?
Solution: We have $r = 6$, $n = 12$, $t = 5$, and $A(t) = 20000$. We need to find A.

$$A(t) = 20000 = A\left(1 + \frac{6}{1200}\right)^{12\cdot 5} = A(1 + 0.005)^{60}.$$

Solving for A, we get:

$$A = \frac{20000}{(1.005)^{60}}.$$

Now use your calculator.

$$A \simeq 14827.44.$$

Thus, the amount that should be deposited today is \$14,827.44. $\square$

Example 0.11. Suppose I borrow \$10,000 from a bank that compounds quarterly. Determine the annual compound interest rate (to four decimal places) if after one year I owe \$10,824.32.
Solution: Suppose the annual compound interest rate is r%. Using the formula for interest with quarterly compounding,

$$A(t) = 10824.32 = 10000\left(1 + \frac{r}{400}\right)^{4\cdot 1} = 10000\left(1 + \frac{r}{400}\right)^{4}.$$

To solve for r, we need to divide both sides by $10,000$ and then take the fourth root – i.e., raise both sides to the 1/4th power. (Do you see why taking the fourth root is the same as taking the square root of the square root? Use properties of exponents.) Thus,

$$(1.082432)^{\frac{1}{4}} = \left(1 + \frac{r}{400}\right)^{4\cdot(1/4)} = 1 + \frac{r}{400}.$$

In other words, $1 + (r/400) = 1.01999996$. Solving for r, $r = 7.999984 \simeq 8\%$ is the annual compound interest rate (to four decimal places). $\square$

Example 0.12. A bank uses a compound interest model, so that a deposited amount doubles in 18 years. How much should be deposited today in the bank, to obtain \$50,000 in 10 years?
Solution: We need to find the amount of money X such that

$$50000 = X\left(1+\frac{r}{n\cdot 100}\right)^{10n} \quad\Longrightarrow\quad X = \frac{50000}{(1+\frac{r}{n\cdot 100})^{10n}}.$$

However, neither the interest rate r, nor the period of compounding n is given. All we have is the equation:

$$2A = A\left(1+\frac{r}{n\cdot 100}\right)^{18n}$$

for any deposit amount A. Dividing by A and taking the 18th power root, we derive

$$\left(1+\frac{r}{n\cdot 100}\right)^{18n} = 2 \quad\Longrightarrow\quad \left(1+\frac{r}{n\cdot 100}\right)^{n} = 2^{\frac{1}{18}}.$$

This is all we need to solve the problem. Using the properties of exponents, we have:

$$X = \frac{50000}{(1+\frac{r}{n\cdot 100})^{10n}} = \frac{50000}{((1+\frac{r}{n\cdot 100})^{n})^{10}} = \frac{50000}{(2^{\frac{1}{18}})^{10}}.$$

Now, $(2^{\frac{1}{18}})^{10} = 2^{\frac{10}{18}} = 2^{\frac{5}{9}}$. Finally,

$$X = \frac{50000}{2^{\frac{5}{9}}} \simeq 34019.75.$$

The deposit has to be \$34,019.75. □
Remark: The important fact to understand here is that the number of years enters the compound interest formula as an exponent. Therefore, if a deposit doubles in 18 years, we can tell right away that in one year, *any* deposit will grow by a factor of $2^{\frac{1}{18}} = \sqrt[18]{2}$. Then the growth factor over 10 years is computed by raising $2^{\frac{1}{18}}$ to the 10th power, to obtain $2^{\frac{10}{18}} = 2^{\frac{5}{9}}$.

Effective annual rate. In dealing with banks, you may sometimes encounter the expression *effective annual rate* of compounding. Suppose I borrow \$10,000 from a bank with an annual rate of 8%, compounded quarterly. After one year, if the money was compounded only annually, then my loan would be \$10,800. But because the compounding is done more frequently, my loan actually increases. See Example 0.11 above for the calculations: the loan is now \$10,824.32 – some twenty-four dollars more!

The effective annual rate essentially measures this discrepancy. It is defined as the simple interest rate that produces the same amount of money at the end of one year, as the stated compound annual rate, compounded n times per year (quarterly, monthly, ...). Thus, if r% is the compound annual rate, and it is compounded n times per year, then the effective annual rate r_{eff} is computed according to the formula

$$1+\frac{r_{\text{eff}}}{100} = \left(1+\frac{r}{n\cdot 100}\right)^{n}.$$

For instance, in the above example, the loan after one year is \$10,824.32. Hence the effective annual rate is 8.2432%. (Note that the annual rate was given to be 8%.)

Example 0.13. Find the effective annual rate for the compound annual rate of 6% compounded monthly. Round off your answer to three decimal places.
Solution: Using $r = 6$ and $n = 12$, we compute:

$$1 + \frac{r_{\text{eff}}}{100} = \left(1 + \frac{6}{12 \cdot 100}\right)^{12} = (1 + 0.005)^{12} \simeq 1.0616778.$$

Solving this simple equation, we get $r_{\text{eff}} \simeq 6.168\%$. □

The effective annual rate is the way to determine which of two (or more) competing banking models yields greater returns/interest over the same length of time. Simply compute which effective annual interest rate is greater!