Mathematics in the Real World
Math 107

Lecture 13: Permutations and combinations. Introduction to probabilities.

Now we will discuss the basics of *probability theory*. We will start by understanding permutations and combinations.

Permutations. A *permutation* of n items is an arrangement of these items in a particular order. For example, a permutation of three colors, Blue, White and Red, is (WRB). Another permutation of the same items is (RWB). We will be interested in the *number* of possible permutations of n items. It is denoted by ${}_nP_n$. In case of three items (e.g, B, W and R), there are 6 possible permutations:

$$(BWR), (BRW), (RWB), (RBW), (WBR), (WRB),$$

so we have ${}_3P_3 = 6$. In general, when counting the number of permutations of n items, there are n choices for the first item, $(n-1)$ for the second, and so on, until the last item for which there is only one choice:

$${}_nP_n = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$$

where $n!$ is the notation for the product of consecutive whole numbers from 1 to n, which is read "n factorial".

How many different permutations are there of 3 items selected from a set of 5 items? Say, we want to count the number of permutations of three different colors out of possible five: Blue, White, Red, Yellow and Green. Then we have 5 choices for the first color, 4 choices for the second color and 3 choices for the last color:

$${}_5P_3 = 5 \cdot 4 \cdot 3 = 60.$$

We can write the same number using the factorial notation. Indeed, the number of permutations in this case is the product of all whole numbers from 3 to 5, which is the product of natural numbers up to 5 divided by the product of natural numbers up to 2:

$${}_5P_3 = 5 \cdot 4 \cdot 3 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} = 60.$$

In general, the number of permutations of k items out of n items is

$${}_nP_k = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

Now suppose we need to count the number of permutations of the letters in the word *bee*. Presumably, there should be ${}_3P_3 = 3! = 6$ permutations, but in fact we have only three:

$$(bee), (ebe), (eeb)$$

This happens because two of the letters are identical, so the number of permutations of 3 items has to be divided by 2 to account for the identical permutations of the kind (e_1be_2) and (e_2be_1). In general, the number of permutations of n items of which r are identical equals to

$$\frac{{}_nP_n}{{}_rP_r} = \frac{n!}{r!}.$$

If in addition there are p other identical items, then the number of distinct permutations will be

$$\frac{n!}{r!p!},$$

and so on.

Example 1. Cinderella has 6 dresses and the ball goes for 3 nights. How many ways are there for Cinderella to dress? Assume that only one dress can be worn per night and no dress can be worn on two different nights.

Solution: The number we are looking for is the number of permutations of 3 out of 6 items. It is given by

$${}_6P_3 = \frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$$

□

Example 2. How many distinct permutations can be made using the letters of the word *MASSACHUSETTS*?
Answer: $\frac{13!}{4!2!2!} = 64864800$.

Combinations. A *combination* of k items out of n items is a choice of a collection of k items out of n items where the order of the chosen items does not matter. Let us first count the number of combinations of 3 out of 3 items, denoted ${}_3C_3$. There are 6 possible permutations, but since the order does not matter, they constitute just one combination! Therefore, there is only 1 combination of 3 out of 3 items. The same is true for any number of items: the number of combinations of n out of n items is always 1:

$${}_nC_n = 1 \qquad \text{for all natural } \ n.$$

Now let us count the number of combinations of 3 out of 9 items, denoted by ${}_9C_3$. For example, you want to take 3 out of your favorite 9 books with you on a trip. What is the number of possible choices? First, let us count the number of permutations of 3 out of 9:

$${}_9P_3 = \frac{9!}{6!} = 9 \cdot 8 \cdot 7 = 504.$$

This number contains a collection of any three given books A, B and C six times: $(ABC), (ACB), (BAC), (BCA), (CAB), (CBA)$. For our purposes, they are indistinguishable. Therefore, to get the number of distinct collections of 3 books out of 9 books, we have to divide ${}_9P_3$ by the number of permutations of 3 items:

$${}_9C_3 = \frac{{}_9P_3}{{}_3P_3} = \frac{504}{6} = 84.$$

In general, to count the number of combinations ${}_nC_k$ of k out of n items, we have to divide the number of permutations of k out of n items ${}_nP_k$ by the number of permutations of k items ${}_kP_k$:

$${}_nC_k = \frac{{}_nP_k}{{}_kP_k} = \frac{n!}{(n-k)!k!}.$$

Example 3. How many ways do you have to choose 4 courses for a semester out of 16 available courses? (The order does not matter).

Solution: The number we are looking for is the number of combinations of 4 out of 16:

$${}_{16}C_4 = \frac{16!}{12!4!} = \frac{16 \cdot 15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = 4 \cdot 5 \cdot 7 \cdot 13 = 1820.$$

□

Example 4. Compute ${}_5C_4$, ${}_5C_1$, ${}_5C_3$, ${}_5P_2$, ${}_5P_3$.
Answer:

$${}_5C_4 = 5, \quad {}_5C_1 = 5, \quad {}_5C_3 = 10, \quad {}_5P_2 = 20, \quad {}_5P_3 = 60.$$

Can you think of examples where counting the number of choices results in these numbers? Remember, that permutations count the choices if the order of the items matters, and combinations if the order does not matter.

The next two examples illustrate the difference between permutations and combinations.

Example 5. How many three-scoop bowls of ice-cream with three different flavors can be made out of 15 possible flavors?
Solution: Clearly, in this case the order does not matter: a bowl of strawberry, vanilla and chocolate is the same as a bowl of chocolate, strawberry and vanilla. The answer is given by the number of combinations of 3 out of 15,

$${}_{15}C_3 = \frac{15!}{12!3!} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 5 \cdot 7 \cdot 13 = 455.$$

□

Example 6. How many three-scoop bowls of ice-cream with two different flavors can be made out of possible 15 flavors?
Solution: Here we have to pick 2 flavors out of 15, but the order of the two matters: a bowl with two scoops of vanilla and one scoop of strawberry is different from the bowl of one scoop of vanilla and two scoops of strawberry. Say, we compose the bowl of two scoops of the first chosen flavor and one scoop of the second chosen flavor.

Then the number of possible choices equals to the number of permutations of 2 out of 15:

$$ {}_{15}P_2 = \frac{15!}{13!} = 15 \cdot 14 = 210. $$

$\square$

0.1. **Combining permutations and combinations.** Some situations require a more creative application of the notions of permutations and combinations. Here is a killer example.

Example 7. Suppose you are the head of R&D of a dessert factory and you want to design new flavors for your desserts. The company has bought a 5-nozzle machine that can drop 5 different flavors and mix them together. Each nozzle is designed to release 1 drop of the flavor. You have 15 flavors available and you want to run evaluation tests on all possible flavor combinations that can be achieved with 5 nozzles. For example, one of the mixtures can contain 3 drops of vanilla, 1 drop of apple flavor and 1 drop of cinnamon. Another mixture can contain 5 drops of 5 different flavors. You need to estimate the cost of testing all possible flavor combinations. To do this, you need to know how many flavor combinations can be obtained using this machine.

Solution: Here the number of flavors used in one mixture can be anything from 1 to 5. It is assumed that you always have to use each nozzle, so each mixture contains 5 drops of flavors, some of which might repeated. We will consider the question case by case, according to the number of flavors used.

(1). How many different 5-drop mixtures can be made with just one flavor out of 15? This is easy: just pick a flavor and take 5 drops of it. The number of choices of 1 out of 15 is given by either ${}_{15}P_1$ or ${}_{15}C_1$, which are equal (why?):

$$ {}_{15}P_1 = {}_{15}C_1 = 15. $$

(2) Now suppose the mixture has two flavors out of 15. For example, if we choose strawberry and vanilla, the resulting mixtures might be:

1 drop of strawberry	4 drops of vanilla
2 drops of strawberry	3 drops of vanilla
3 drops of strawberry	2 drops of vanilla
4 drops of strawberry	1 drops of vanilla

Here the order of chosen flavors matters: 4 S and 1 V is not the same as 4 V and one 1 S. In addition, there are two ways to divide 5 into two unequal parts: 4 and 1, and 2 and 3. Therefore, we need to compute the number of permutations of 2 out of 15 and multiply the result by 2:

$$ 2 \cdot {}_{15}P_2 = 2 \cdot \frac{15!}{13!} = 2 \cdot 15 \cdot 14 = 420. $$

(3) Next, suppose the mixture has 3 different flavors. Then either it has 3 drops of the first flavor, and 1 drop of each of the second and the third, or 1 drop of the first flavor, and 2 drops each of the second and the third. In both cases, we need to choose one distinguished flavor, and two undistinguishable flavors out of 15 flavors available. We will count the ways to do it, and then multiply by 2 to account for the two different splits $(3, 1, 1)$ and $(1, 2, 2)$.

A choice of 1 distinguished flavor can be made in ${}_{15}P_1 = 15$ ways. From the remaining 14 flavors, we need to choose 2 whose order does not matter:

$${}_{14}C_2 = \frac{14!}{12!2!} = \frac{14 \cdot 13}{2 \cdot 1} = 7 \cdot 13 = 91.$$

Multiplying them together, we get $15 \cdot 91 = 1365$.

There is another way to count the number of choices of one distinguished and 2 undistinguishable items out of 15. First we choose 3 ordered items out of 15. The number of choices is given by ${}_{15}P_3$:

$${}_{15}P_3 = \frac{15!}{12!} = 15 \cdot 14 \cdot 13.$$

Then we need to divide by the number of permutation of 2 items, to account for the fact that 2 of our chosen items are indistinguishable. The total number of choices then is

$$\frac{{}_{15}P_3}{{}_{2}P_2} = \frac{15 \cdot 14 \cdot 13}{2 \cdot 1} = 15 \cdot 7 \cdot 13 = 1365,$$

same as before.

Finally, we need to multiply this number by 2, to account for the $(3, 1, 1)$ and $(1, 2, 2)$ splits, and we get

$$1365 \cdot 2 = 2730.$$

(4) Now suppose the mixture has 4 flavors out of 15. This can only happen if one of the 4 flavors is used twice. So, we need to choose 1 distinguished flavor, of which will be used in 2 nozzles, and 3 more indistinguishable flavors. The first operation can be done in ${}_{15}P_1 = 15$. Of the remaining 14 flavors, we need to choose 3 whose order does not matter:

$${}_{14}C_3 = \frac{14 \cdot 13 \cdot 12}{3 \cdot 2 \cdot 1} = 14 \cdot 13 \cdot 2 = 364.$$

Therefore, the total number mixtures with 4 flavors is

$$15 \cdot 364 = 5460.$$

Similarly to the case of 3 flavors, there is another way to obtain the same answer (how?).

(5) Now suppose the bowl has to have 5 different flavors in it. The order does not matter, and the number of possibilities is given by the number of combinations of 5

out of 15:

$$_{15}C_5 = \frac{15!}{10!5!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 13 \cdot 3 \cdot 11 = 3003.$$

Finally, we have to sum up the numbers obtained in (1) through (5):

$$15 + 420 + 2730 + 5460 + 3003 = 11628$$

possible flavor combinations can be obtained using the 5-nozzle machine with 15 flavors available.

□

Probabilities. Now we come to the probability theory proper. We denote by Ω the set of all outcomes of an experiment. For example, if you randomly draw one card from a deck, then Ω is the set of all outcomes: from the 2 of clubs to the ace of spades.

We would like to compute the odds, or chances - or probability - of an event. An *event* is a set of outcomes; in other words, a subset of Ω. For example, one event could be $E_1 = \{$the drawn card is red$\}$. Another could be: $E_2 = \{$the drawn card is an ace$\}$. In the first case, the set of outcomes corresponding to the event E_1 is all hearts and all diamonds. In the second case, E_2 consists of four outcomes: the four aces.

What is the probability of any event E? It is simply $P(E) = \#E/\#\Omega$, the proportion of outcomes that lie in E. Thus, $P(E_1) = 26/52 = 1/2$, and $P(E_2) = 4/52 = 1/13$.

Let us see how permutations and combinations can be used to compute probabilities.

Example 8. A club has 5 men and 7 women members. 3 members are selected at random to attend a conference. What is the probability of selecting 3 women?
Solution: The number of ways to select 3 people out of 12 people is given by the number of combinations of 3 out of 12 (order doesn't matter):

$$_{12}C_3 = \frac{12!}{9!3!} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2} = 220.$$

The number of ways to select 3 women out of 7 women is $_7C_3$:

$$_7C_3 = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35.$$

Therefore, the probability is

$$P(3 \text{ women}) = \frac{_7C_3}{_{12}C_3} = \frac{35}{220} = \frac{7}{44}.$$

□

Example 9. What is the probability of drawing 3 spades in a row from a standard deck of 52 cards?

Solution: The number of ways to draw 3 cards out of 52 (order does not matter) is ${}_{52}C_3$. The number of ways to draw 3 cards out of 13 spades in the deck (order does not matter) is ${}_{13}C_3$. Therefore, the probability of drawing three spades in a row is

$$P = \frac{{}_{13}C_3}{{}_{52}C_3} = \frac{13 \cdot 12 \cdot 11 \cdot 3!}{52 \cdot 51 \cdot 50 \cdot 3!} = \frac{11}{17 \cdot 50} = \frac{11}{850}.$$

$\square$