Mathematics in the Real World
Math 107

Lecture 1: Linear constant speed motion.

Questions on linear constant speed motion often appear in daily life. For example, how long does it take to get to a town 230 miles away, driving at an average speed of 60 miles per hour? We compute:

$$\begin{aligned}\text{time} \;=\; \tfrac{\text{distance}}{\text{speed}} = \tfrac{230}{60} &\simeq 3.833 \text{ hours} \\ &= 3h50min.\end{aligned}$$

□

Another question: suppose your friend is 30 miles ahead of you on a highway, moving forward with a constant speed of 50 miles per hour. Can you catch up with him before he gets to the next town, 120 miles ahead of you, without breaking the speed limit? Equivalently, assume you are moving at a constant speed of 65 miles per hour. How soon will you catch up with your friend: before or after he passes the town 120 miles ahead? The distance to the meeting point can be calculated in two ways: as the distance covered by you in t hours, and as the distance covered by your friend in t hours, plus 30 miles:

$$\text{distance} = 65t = 50t + 30.$$

From this we get $15t = 30$, and $t = 2$ hours. In 2 hours, you will be $2 \cdot 65 = 130$ miles ahead, so you cannot catch up with him before he passes the town at 120 miles. □

"The Seven Messengers". For a more elaborate example of a constant speed motion, we will analyze the situation described in the short story "The Seven Messengers" by Dino Buzzati.

The prince, who is the narrator of the story, sets out to explore his father's kingdom, in hope to find its boundaries. He and his knights start from the capital and move along a straight line (due south, or so they hope) at a constant speed of 40 leagues per day. After two days of travel, the prince sends his first messenger – Alessandro – back to the capital. The next six messengers, Bartolomeo, Caio, Domenico, Ettore, Federico, and Gregorio were sent back respectively after 3, 4, 5, 6, 7 and 8 days of travel. All messengers move one and a half times as fast as the prince, which corresponds to a constant speed of 60 leagues per day. Having reached the capital, each messenger immediately starts back along the same straight path to catch up with the prince. Thus, the seven messengers oscillate between the capital and the prince, while the prince is moving away from the capital with a constant speed. This information is sufficient to figure out the position of each of the messengers at any moment of time. For example, we can find the moments of time (in days since the start of the expedition) of consecutive returns of each messenger to the prince.

Let d be the number of days elapsed before a messenger (M) was first sent back. Let t denote the number of days M needs to catch up again with the prince (P). Then we have the equation:

$$\text{distance traveled by } M = 2\left(\begin{array}{c}\text{distance to the capital}\\ \text{when } M \text{ leaves}\end{array}\right)+\left(\begin{array}{c}\text{distance traveled by P}\\ \text{since } M \text{ left}\end{array}\right)$$

or, taking into account the given speeds of P and M,

$$60t = 2 \cdot 40d + 40t.$$

From this equation we find that $t = 4d$ is the time taken by M to catch up with P. By the time he catches up, $d + 4d = 5d$ days would have passed since the start of the expedition. Now we only need to plug in $d = 2$ for Alessandro, $d = 3$ for Bartolomeo, and so on, up to $d = 8$ for the last messenger, Gregorio, to obtain the times of their return to the prince. We find that they reunite with the prince after 10, 15, 20, 25, 30, 35 and 40 days, respectively.

A messenger is then immediately sent back to the capital, so for instance, Alessandro is sent back the second time after 10 days since the start of the expedition. By the same argument, we see that the second return of the messengers will occur after $5\cdot 5d = 25d$ since the start of the expedition. Next, they will return after $5\cdot 25d = 125d$, $5 \cdot 125d = 625d$, and so on. For example, Alessandro ($d = 2$) will return to the camp after $2 \cdot 5 = 10$, $2 \cdot 25 = 50$, $2 \cdot 125 = 250$, $2 \cdot 625 = 1250$, etc., days.

Here is the timetable (in days since the beginning of the journey) of the first five consecutive returns of the messengers to the prince:

	d	$5d$	$25d$	$125d$	$625d$
A	2	10	50	250	1250
B	3	15	75	375	1875
C	4	20	100	500	2500
D	5	25	125	625	3125
E	6	30	150	750	3750
F	7	35	175	875	4375
G	8	40	200	1000	5000

This table should contain enough information to check the numerical claims of the narrator. Here they are:

(1) *"...it was sufficient to multiply by* 5 *the days elapsed so far to know when the messenger would catch up with us".* This is exactly the formula we derived above: if a messenger leaves the prince after d days since the start of the expedition, he returns after $5d$ days.

(2) *After* 50 *days, the interval between the arrivals of the messengers had increased to* 25 *days.* This is confirmed by the third column of the table.

(3) *After* 6 *months, the interval had increased to at least* 4 *months.* We need to translate 6 months into days. There is an ambiguity: we don't know which months they are, resulting in a number of days between 181 (January - June) and 184 (July - December). But in any case, the interval increases to at least 4 months (125 days) a little later - after 250 days, or over 8 months of travel. Since the narrator does not claim the change occurred immediately after 6 months have passed, the claim is legitimate.

(4) *After* 4 *years, the interval had increased to at least* 20 *months.* Here we have to convert 4 years into days. Taking into account one leap year, we have $3 \cdot 365 + 366 = 1461$ days. 20 months contain a full year and 8 month, resulting in approximately $365 + 4 \cdot 31 + 4 \cdot 30 = 609$ days. In fact, from the last column of the table we deduce that after 1250 days of travel, the interval increased to 625 days, which is consistent with the statement.

(5) *Domenico returns exactly* 8 *years,* 6 *months and* 15 *days after the beginning of the expedition.* Looking at the fourth line of the table, we easily find a suitable time of return for Domenico: it happens 3125 days since the start of the expedition, which we have to compare with the claimed 8 years, 6 months and 15 days. Taking into account two leap years, and considering the possible range in the number of days in 6 months, 181 to 184, we get $6 \cdot 365 + 2 \cdot 366 + 181 + 15 = 3118$, or $6 \cdot 365 + 2 \cdot 366 + 184 + 15 = 3121$. In any case, the claimed interval is 4 to 7 days shorter than we expected. This discrepancy requires an explanation, which we will return to later.

(6) *The previous return of Domenico occurred almost* 7 *years ago.* According to our table, the previous return of Domenico occurred 625 days since the start of the expedition, and we compute: $3125 - 625 = 2500$, and $2500/365 \simeq 6.849$ years, or almost 7 years.

(7) *The next return of Domenico will occur after* 34 *years.* We compute: the next return of Domenico will happen after $5 \cdot 3125 - 3125 = 4 \cdot 3125 = 12500$ days, and $12500/365 \simeq 34.246$ years, or after 34 years.

(8) *Between the present return of Domenico, and the return of the next messenger, Ettore, a year and* 8 *months will pass.* We look at the last column in the table to find the interval between the next return of Ettore and the present return of Domenico: $3750 - 3125 = 625$. We have already computed a year and 8 months to contain approximately 609 days, so the statement is true.

Therefore, we have confirmed all statements of the narrator, except (5). Clearly, the author was well aware and careful about the mathematics involved in the story. For example, in claim (1), he confirms our main formula, that determines the dates of the consecutive returns of the messengers. This suggests that the few missing days were intentionally subtracted by the author. Domenico catches up with the expedition a little sooner than expected: just a few days sooner. In fact, the narrator knows that this might happen at the next return of Domenico: "In thirty-four years (sooner in fact, much sooner), Domenico will unexpectedly perceive the fires of my camp and ask why I have made so little progress in the meantime". It is tempting to conjecture that the progress of the narrator has already slowed down, without him noticing that. The prince and his knights now cover, probably, a little less than 40 leagues per day, and this has resulted in his messenger catching up with him sooner than expected. The melancholy of the story is amplified by the fact that the narrator realizes that his old age will inevitably diminish his powers, and eventually interrupt his project, but he cannot see that his decline has already started, and is already affecting his progress.

Scale of distances. How far can one ride a horse on Earth, without changing direction? In other words, what is the longest distance on land along one direction? Here are some examples:

- East Coast - West Coast distance in the US is approximately $3,000$ miles, or $4,828$km.
- Longest continuous distance on land along a longitude : $7,590$km (Northen Russia to Southern Thailand, $99°$ East).
- Longest continuous distance on land along a latitude: $10,726$km (Western France to Eastern China, $48°$ North.)
- Longest distance on land along any great circle: $13,573$km (Liberia to China).

To find the distance covered so far by the prince and his knights (supposedly, they always move due south), we need to multiply 40 leagues by 3120 (approximately) days of travel. A league is an ancient measure of distance that varies from country to country, but is approximately equal to the distance a person can walk in an hour. We suppose that Dino Buzzati, being an Italian writer, would use the Roman league, equal to 1.4 miles, or 2.2225km. In this case, the distance traveled by the prince at the time of the narration is $2.2225 \cdot 40 \cdot 3120 = 277,368$km. No distance on land along one direction on Earth is that long, which suggests either a fantastic setting, or that the expedition, despite the hopes of the narrator, is not moving along a straight path and might never be able to reach its goal.

International date line. We can propose another hypothesis to explain the discrepancy between the calculated time of Domenico's return (3125 days) and the time reported by the narrator (between 3118 and 3121 days).

Recall the story of the first world circumnavigation headed by Ferdinand Magellan. He started to sail his 5 ships due west from the coast of Spain on September 20, 1519. Almost 3 years later, on September 6, 1522, the 18 survivors of his original crew (Magellan himself was killed in March 1521 in Philippines) and his only surviving ship, Victoria, returned to Spain. But the ship's log had the date as September 5, 1522. The ship's log was recorded with utmost care and accuracy, in particular, the leap year 1520 was taken into account. The mystery of a missing day was discussed by the leading scientsts of the time, among them the Venetian astronomer Gasparo

Contarini, who suggested the right explanation: moving westward, and going a whole circle around the Earth, you gain one day; moving eastward, you lose it. It wasn't until the 19th century when the International Date Line was established in the sense and position it has now: an imaginary line between the north and south poles at approximately 180° East separating Russia and Asia from the Americas, and one calendar day on Earth from the next. Now a person crossing the International Date Line traveling eastbound has to subtract a day; when traveling westbound, add a day.

If we suppose for a moment that the expedition was moving west, or mostly west, instead of south, the missing days can be explained by the same effect. A renaissance setting of Buzzati's story suggests that the narrator (just like Magellan's crew) might not have known about the necessity to add a day for each complete circle when traveling westward on an Earth-like planet. Let us estimate the distances. The radius of Earth $R_\oplus$ is between 6353 and 6384km (larger at the equator). Therefore, the circumference at the equator is approximately $2\pi R_\oplus \simeq 40086$km. By the time of the narration, the expedition of the prince has covered $277,368$km. Dividing this distance by the circumference of the equator, we get $277,368/40086 \simeq 6.9$. Therefore, 6 days (more if he was moving along a higher latitude, less if he was deviating from the straight westward direction) might have to be added to the number of days in the narrator's log to obtain the actual number of days elapsed in the capital, or for a messenger who moves east and west, back and forth. Even if this explanation was not intended by the author, one has to remember that the medieval or renaissance world was full of mysteries, and the date discrepancy between round the world travelers and stationary observers was one of the most remarkable among them. Its consideration in the context is consistent with the enigmatic atmosphere of the story.

Messengers in the real world. Where in the real world do we have a situation of an object moving away from a fixed observer, and messengers oscillating between them? A space probe such as Voyager is an example. This is an instrument, usually a radio telescope, propelled by a rocket into the deep outer space to send back to Earth information of remote planets and stars. The messengers are the light particles that carry information between the space ship and the command center on Earth. There are many more than seven of them, and their speed is the speed of light: $300,000 = 3 \cdot 10^5$km per second. The distances are much bigger, too: the approximate size of the Solar System is 4.5 billion km (4.5×10^9km). As the probe moves further away from us, the messages we get from it become more and more outdated: for example, a snapshot sent to Earth from the orbit of Mars takes about 240 seconds, or 4 minutes to arrive, while a message sent from the edge of the Solar System reflects the situation that was observed by the probe about 14667 seconds (4 hours) ago.

In fact, since the universe is expanding, any star is an object moving away from us, and its visible light is a messenger from it. For comparison, the approximate diameter of our Galaxy is $100,000$ light years, or $9.4 \cdot 10^{17}$km. The approximate diameter of the universe is 93 billion light years, or $8.8 \cdot 10^{23}$km. There are of course the gravitational effects, which make the object trajectories curve, and the relativistic effects, which make time flow differently for them and for us, but we still receive their messengers, their beams of light, that tell us what they looked like thousands of years ago, and we hope they might be able to see us as well.