Name: ________________________

Math 112b Final Exam

May 4, 2010

Your signature: ________________________

Math 112b Section: __________

Instructions: There are ten problems, with the number of points for each problem indicated at the start of the problem. There is a total of 250 points. You are **not** allowed the use of a calculator, notes, or bluebooks. Also no cell phones, laptops, or other electronic devices are allowed during the test.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put your answers in the $\boxed{\textbf{Box}}$ provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.
- At the end of the test is attached the two Reference Pages from the text, which you may refer to and use during the test.

Problem 1	25 points	
Problem 2	25 points	
Problem 3	25 points	
Problem 4	25 points	
Problem 5	25 points	
Problem 6	25 points	
Problem 7	25 points	
Problem 8	25 points	
Problem 9	25 points	
Problem 10	25 points	
Total	250 points	

1. (25 points). Compute each of the following limits.

(a) $\lim_{x \to 9} \dfrac{x-4}{\sqrt{x}-2}$ Answer: $\boxed{}$

(b) $\lim_{x \to -\infty} \dfrac{2x^3+3x+1}{\sqrt{1+x^6}}$ Answer: $\boxed{}$

(c) $\displaystyle\lim_{x \to 0} \frac{\tan(6x)}{\sin(2x)}$ Answer: $\boxed{}$

(d) $\displaystyle\lim_{x \to \infty} x^{1/x}$. Answer: $\boxed{}$

2. (25 points). Consider the curve C described implicitly by the relationship

$$y^3 - 2y^2 - x^2 + 3xy = 0$$

(a) Show that $(0, 2)$ is on the curve, and find all other points on the curve whose x-coordinate is 0.

Answer: $\boxed{}$

(b) Compute the implicit derivative $\dfrac{dy}{dx}$ as a function of x and y.

Answer: $\boxed{}$

(c) Find the equation of the tangent line to C at the point $(0, 2)$.

Answer: $\boxed{}$

3. (25 points). Let $f(x) = x + e^x$.

(a) Prove that $f(x)$ has at least one real root (i.e. a solution to $f(x) = 0$).

(b) Prove that $f(x)$ has exactly one real root.

4. (25 points). Find the most general function $f(x)$ satisfying the conditions that

$$f''(x) = x + \frac{1}{x^2} \quad \text{and} \quad f'(1) = 0.$$

Answer: $\boxed{}$

5. (25 points). Compute the derivatives of the following functions:

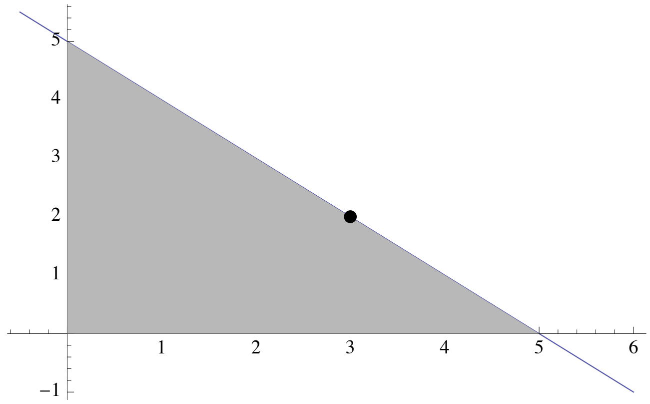
(a) $f(x) = \ln(\sin(e^x))$. Answer: $\boxed{}$

(b) $g(x) = \dfrac{x^2 + \sqrt{x}}{2x}$. Answer: $\boxed{}$

(c) $h(x) = \ln(x^{\sin x})$. Answer: $\boxed{}$

(d) $k(x) = 2^x + x^2$. Answer: $\boxed{}$

6. (25 points). A line with negative slope passing through the point $(3, 2)$ cuts off a triangular region in the first quadrant (see the illustration below for one example). Find the equation of the line for which the corresponding triangular region has minimal area. Your solution should justify that your answer is indeed a minimum.



Answer:

7. (25 points). Compute the following definite integrals, bearing in mind their geometric meaning.

(a) $\displaystyle\int_0^{\pi} \sin x \, dx.$ Answer: $\boxed{}$

(b) $\displaystyle\int_1^4 \left(3x^2 + \frac{1}{\sqrt{x}}\right) dx.$ Answer: $\boxed{}$

(c) $\int_0^4 |x-2|\,dx.$ Answer: $\boxed{}$

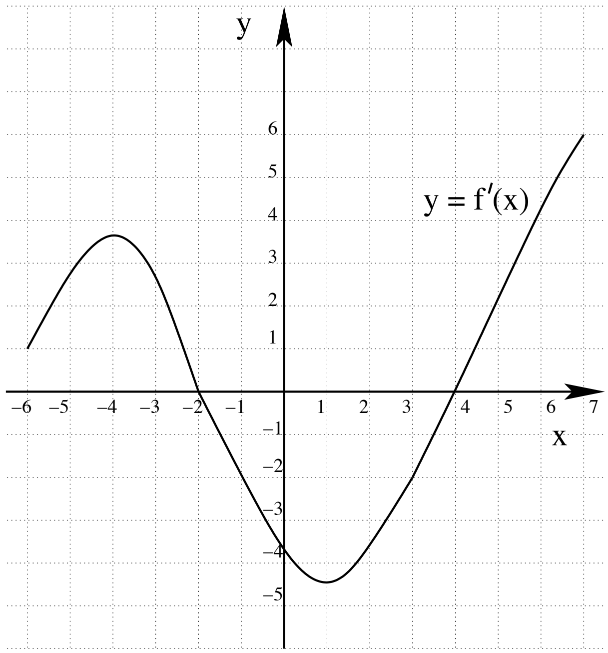
8. (25 points). Let $g(x)$ be the function defined by

$$g(x) = \int_1^{x^2+5} \ln t \, dt.$$

Compute its derivative $g'(x)$.

Answer: $\boxed{}$

9. (25 points). In the figure below is shown the graph of $f'(x)$, the **DERIVATIVE** of a certain (unknown) function $f(x)$, on the interval $-6 \le x \le 7$. Based on this graph, answer the following questions about $f(x)$ and its derivatives. The numbers in all answers should be whole numbers.



(a) At what value(s) of x does $f(x)$ have a local maximum?

(b) At what value(s) of x does $f(x)$ have a local minimum?

(c) At what value(s) of x is $f''(x) = 0$?

(d) List the x-intervals on which $f(x)$ is increasing.

(e) List the x-intervals on which $f(x)$ is concave down.

10. (25 points). Sand is being dumped from a conveyor belt at a rate of 25 cubic feet per minute, and is piling up in the shape of a right circular cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 feet high? [Recall that the volume of a right circular cone with base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]

REFERENCE PAGES

ALGEBRA

ARITHMETIC OPERATIONS

$$a(b+c) = ab + ac \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

EXPONENTS AND RADICALS

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

FACTORING SPECIAL POLYNOMIALS

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

BINOMIAL THEOREM

$$(x+y)^2 = x^2 + 2xy + y^2 \qquad (x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\binom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot \cdots \cdot k}$

QUADRATIC FORMULA

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

INEQUALITIES AND ABSOLUTE VALUE

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

$|x| > a$ means $x > a$ or $x < -a$

GEOMETRY

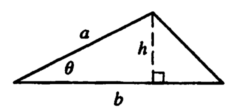
GEOMETRIC FORMULAS

Formulas for area A, circumference C, and volume V:

Triangle

$A = \frac{1}{2}bh$

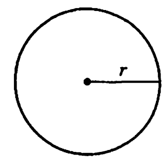
$= \frac{1}{2}ab\sin\theta$



Circle

$A = \pi r^2$

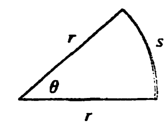
$C = 2\pi r$



Sector of Circle

$A = \frac{1}{2}r^2\theta$

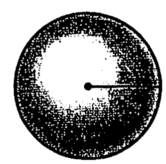
$s = r\theta$ (θ in radians)



Sphere

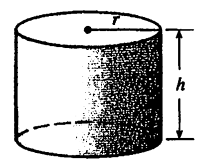
$V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$



Cylinder

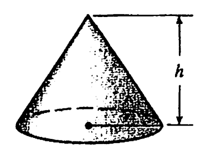
$V = \pi r^2 h$



Cone

$V = \frac{1}{3}\pi r^2 h$

$A = \pi r\sqrt{r^2 + h^2}$



DISTANCE AND MIDPOINT FORMULAS

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

LINES

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

CIRCLES

Equation of the circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

TRIGONOMETRY

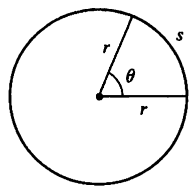
ANGLE MEASUREMENT

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad 1$ rad $= \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

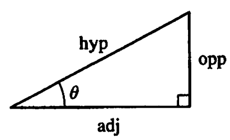


RIGHT ANGLE TRIGONOMETRY

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\text{adj}}{\text{opp}}$

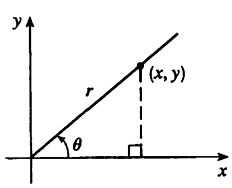


TRIGONOMETRIC FUNCTIONS

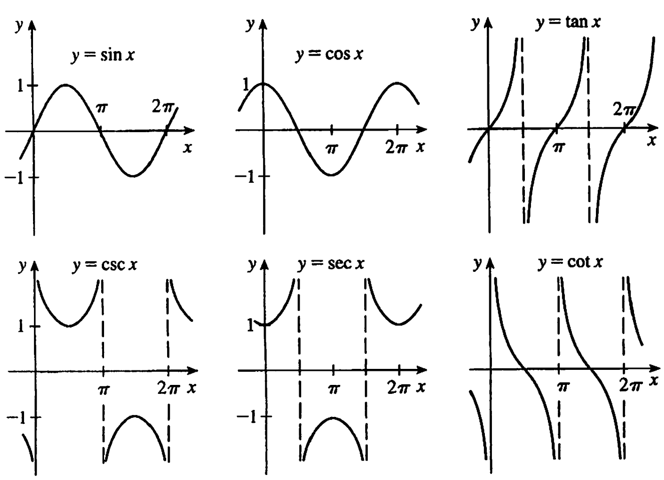
$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$



GRAPHS OF TRIGONOMETRIC FUNCTIONS



TRIGONOMETRIC FUNCTIONS OF IMPORTANT ANGLES

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

FUNDAMENTAL IDENTITIES

$\csc\theta = \dfrac{1}{\sin\theta} \qquad \sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} \qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$

$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta \qquad \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

THE LAW OF SINES

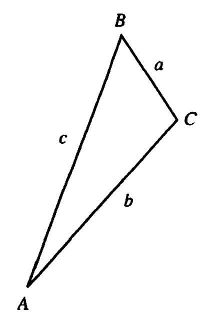
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

THE LAW OF COSINES

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$



ADDITION AND SUBTRACTION FORMULAS

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

DOUBLE-ANGLE FORMULAS

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

HALF-ANGLE FORMULAS

$\sin^2 x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$