Mathematics in the Real World
Math 107

Lectures 11 & 12: Infinite geometric series. Fractals.

Infinite geometric series. An *infinite geometric series* is the following expression

$$(1 + r + r^2 + r^3 + \dots) = \sum_{k=0}^{\infty} r^k,$$

where the sum is of infinitely many consecutive powers of r. If $r > 1$, this sum is infinitely large. If $-1 < r < 1$, the sum is finite and is given by the simple formula:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

More generally, the formula

$$a(1 + r + r^2 + r^3 + \dots) = \frac{a}{1-r}$$

holds for any real a and a real r such that $-1 < r < 1$. This formula comes useful in many applications, as we will see below.

Repeating decimals. Infinite geometric series can be used to compute exact value of a number given as periodically repeating decimals. Consider the number $1.2222\dots$. To find its exact value, note that we have a *non-repeating part* - namely, 1 - and a repeating part - $0.2222\dots$. Let us compute the latter first. Simply add up the digits combining them with their place values:

$$\begin{aligned} 0.2222\dots &= (2/10) + (2/100) + (2/1000) + \dots = \frac{2}{10^1} + \frac{2}{10^2} + \dots + \frac{2}{10^n} + \dots \\ &= \frac{2}{10}\left(1 + \frac{1}{10^1} + \frac{1}{10^2} + \dots\right). \end{aligned}$$

In the brackets we have a geometric series with infinitely many terms, i.e.,

$$a(1 + r + r^2 + \dots),$$

where $a = 2/10$ and $r = 1/10$. Note that $-1 < r < 1$, so that we have a formula for the sum of this series. It equals $a/(1-r)$. In our example,

$$\frac{2}{10}\left(1 + \frac{1}{10^1} + \frac{1}{10^2} + \dots\right) = \frac{\frac{2}{10}}{1 - \frac{1}{10}} = \frac{\frac{2}{10}}{\frac{9}{10}} = \frac{2}{9}.$$

Thus, $0.2222\dots = 2/9$, so adding 1 to both sides, $1.2222\dots = 1 + (2/9) = 11/9$.

Practice problems: Find the exact value of each of the following:

(1) $0.7777\dots$. (*Answer:* 7/9.)
(2) $1.121212\dots$. (*Answer:* 37/33.)
(3) $1.2545454\dots$. (Hint: What is the non-repeating part and the repeating part of this decimal number?) (*Answer:* 69/55.)

Note that the inverse operation: converting $\frac{11}{9}$ into $1.2222\dots$ is done easily by any calculator; but no calculator knows how to get the *exact* value of a repeating decimal. No matter how many digits you write, you cannot obtain the exact value of an infinitely repeating periodic decimal. The exact value can be calculated as above using the geometric series.

Two trains and a fly. Two two trains move with a constant speed of 5 mph along the same straight track from the opposite directions. Initially the distance between the trains is 10 miles. At the same time, a fly starts flying from the first train to the second with a constant speed of 6 mph. It flies until it meets the second train, then turns around and flies back until it meets the first train, turns around, meets the second train again, and so on until the trains collide. Question: find the total distance covered by the fly before the trains collide.

We considered this question at the first lecture, when we discussed linear constant speed motion. Then we found a simple solution: since the trains take 1 hour to reach the meeting point, and the fly always moves with the same constant speed of 6 mph, the total distance covered by the fly is 6 miles.

Now we will solve the same question using infinite geometric series. Let us find the distance the fly covers before it meets the second train for the first time. Since the train and the fly approach at a speed of $5 + 6 = 11$ mph, and at the beginning they are 10 miles apart, the time elapsed before they meet is $t = 10/11$ of an hour. In this time, the fly covers

$$6 \cdot \frac{10}{11} = \frac{60}{11} \text{ miles.}$$

This can be interpreted as $\frac{6}{11} \cdot 10$ miles, or $\frac{6}{11}$ of the distance between the fly and the second train. Now the fly turns around and approaches the first train. Since the trains move with the same speed, the picture is symmetric, so the fly will again cover $\frac{6}{11}$ of the distance to the first train. This distance is the difference between the distance covered by the fly in the first move and by the first train:

$$\frac{60}{11} - 5 \cdot \frac{10}{11} = \frac{10}{11} \text{ miles.}$$

Therefore, the distance covered by the fly in the second move is

$$\frac{6}{11} \cdot \frac{10}{11} = \frac{60}{11^2} \text{ miles.}$$

This process will be repeated to infinity, each time the distance covered by the fly will get divided by 11. So the total distance covered by the fly before the trains collide is

$$D = \frac{60}{11} + \frac{60}{11^2} + \frac{60}{11^3} + \ldots = \frac{60}{11}\left(1 + \frac{1}{11} + \frac{1}{11^2} + \ldots\right).$$

Applying the formula for the infinite geometric series, we get

$$D = \frac{60}{11}\frac{1}{1-\frac{1}{11}} = \frac{60}{11}\cdot\frac{11}{10} = \frac{60}{10} = 6 \text{ miles}.$$

This roundabout way to get the right answer serves two purposes: first, it shows that a finite distance can be presented as a sum of infinitely many ever decreasing pieces. Second, it comes with a great story.

The story has it that once this question was asked of the famous mathematician and computer scientist John von Neumann, who gave the correct answer in seconds. "Ah, you found the shortcut", the questioner said, "most people start summing up a geometric series!" "That's what I did", said von Neumann, "Is there a shortcut?"

Another famous story about dividing a distance into infinitely many pieces is the Zeno paradox of Achilles and the turtle. The turtle offers to run a race against Achilles, if only he can have a 100 ft head start. The speed of Achilles is assumed to be 10 times that of the turtle. While Achilles covers the 100 ft, the turtle advances by 10 ft. While Achilles runs that distance, the turtle goes 1 ft more, and so on to infinity. We trust the reader to formulate the mathematical model for this problem, and persuade themselves that Achilles will be able to overtake the turtle in finite time. To do this, you will have to sum up an infinite geometric series.

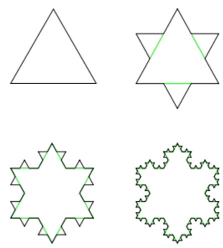
Geometric applications. Fractals. Geometric series can be used to compute areas of recursively defined self-similar geometric shapes, or fractals. One of the simplest visually engaging examples is the Koch snowflake. It is constructed as follows: To an equilateral triangle, add three smaller equilateral triangles on the middle third of each side. The obtained star has the boundary of 12 intervals. In the next step, an equilateral triangle is constructed on the middle third of each of these intervals, and so on to infinity.

The perimeter of the snowflake is infinite: at each step, the length of each side increases by 1/3, making the perimeter grow by a factor of 4/3 with each step.

Let us compute the area of the Koch snowflake. Clearly, the area is finite (the snowflake lies inside the escribed circle of the initial triangle). Suppose the area of the initial triangle is 1 (say, square inch). The area of the star after the first step (upper right on the picture) is

$$1 + 3\cdot\frac{1}{9} = 1 + \frac{1}{3},$$

Figure 1. The Koch snowflake - the first four steps.



since the area of each of the three added small triangles is 1/9 of the original one. At the next step, 12 more triangles are added, each with the area 1/9 of the small triangles, or $1/9^2$ of the initial triangle. The area grows by

$$\ldots + 12 \cdot \frac{1}{9^2} = \ldots + \frac{4}{3^3}$$

After that, at each next step 4 times as many triangles are added, because instead of each side gets replaced with 4 sides. Each added triangle has the area 1/9 of that of the triangles added at the previous step.

The total area is given by the infinite geometric series:

$$1 + \frac{1}{3} + \frac{4}{3^3} + \frac{4^2}{3^5} + \frac{4^3}{3^7} \ldots = 1 + \frac{1}{3}\left(1 + \frac{4}{9} + \frac{4^2}{9^2} + \frac{4^3}{9^3} + \ldots\right).$$

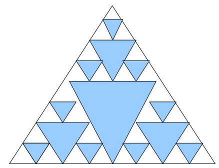
The geometric series in the brackets can be summed using the general formula with $a = \frac{1}{3}$ and $r = \frac{4}{9}$:

$$1 + \frac{1}{3}\left(1 + \frac{4}{9} + \frac{4^2}{9^2} + \frac{4^3}{9^3} + \ldots\right) = 1 + \frac{1}{3} \cdot \frac{1}{1 - \frac{4}{9}} = 1 + \frac{1}{3} \cdot \frac{9}{5} = 1 + \frac{3}{5} = \frac{8}{5}.$$

Therefore, the area of the Koch snowflake is 8/5 of the area of the initial equilateral triangle. Even though the object is formed by infinitely many tiny shapes put together, its area is finite and can be computed using the geometric series.

Here is another example. The picture below shows the Sierpinski triangle - an equilateral triangle with several shaded inverted equilateral triangles inside. The first shaded triangle is constructed by joining the middles of the sides of the original triangle. Next, the same operation is performed with the three white triangles to the left, to the right and above the shaded triangle, and so on to infinity (the picture shows only three consecutive steps, but it is obvious that this process can be continued indefinitely).

Figure 2. The Sierpinski triangle, the first three steps.



Suppose that the area of the original white triangle is 1. What is the total area of all shaded triangles? The first shaded triangle has the area of 1/4 since it is one of the four equal triangles forming the original one. Next, three triangles of area 1/16 are added. Next, 9 triangles of area 1/64 are added, and so on. We have the geometric series:

$$\frac{1}{4}+\frac{3}{16}+\frac{9}{64}+\dots.$$

One quarter is the common factor in all terms:

$$\frac{1}{4}\left(1+\frac{3}{4}+\frac{3^2}{4^2}+\frac{3^3}{4^3}+\dots\right)=\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k.$$

This is an infinite geometric series with $r=\frac{3}{4}$. Since $r<1$, the sum is finite and is given by the formula $\frac{1}{1-r}$. So we have:

$$\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k=\frac{1}{4}\cdot\frac{1}{1-\frac{3}{4}}=\frac{1}{4}\cdot\frac{1}{\frac{1}{4}}=\frac{1}{4}\cdot\frac{4}{1}=1.$$

After infinitely many steps, the shaded triangles will fill all of the white triangle! This means that we can get the total area of the shaded triangles as close as we like to the area of the white triangle by repeating the process sufficiently many times.

Fractal dimension. What is the dimension of a fractal, for example, of the Sierpinski triangle?

First, let us try to formalize the idea of dimension of a line segment or a square. Everyone knows that a line is one-dimensional and a flat square is two-dimensional. In these cases, linear algebra provides a way of defining the dimension as the number of *linearly independent* directions on the object. This definition is not suitable for a fractal.

In fact, we can think of a line segment and a square as of self-similar objects, just like fractals. If we divide a line segment into 2 equal pieces, each piece is similar to the original segment (meaning that it can be obtained by shrinking of the original object). Each of the smaller pieces can be stretched back to become the original line segment, if we multiply its length by 2. Therefore, the *magnification factor* of line segments of half the original length is 2. In the same way, we can divide a line segment into any number of equal parts, with magnification factor equal to the number of parts.

In case of a square, the picture is different. We can divide the square into 4 equal smaller squares, with sides half the sides of the original square. In this case, the magnification factor is 2. If we divide the square into, say, 25 smaller squares, then the magnification factor is 5.

We observe a clear difference between the line segment and the square: in the first case, the number of the self-similar parts is equal to the magnification factor, while in the second case, it is the square of the magnification factor. It turns out that this difference is exactly the difference between one-dimensional and two-dimensional objects.

So we can conclude that the dimension is the exponent of the magnification factor, needed to get the number of self-similar pieces into which the shape may be broken. In formulas, we have

$$\dim = \frac{\log(\text{number of pieces})}{\log(\text{magnification factor})}.$$

In our examples, the line segment L and the square S, we have

$$\dim L = \frac{\log 2}{\log 2} = 1, \quad \dim S = \frac{\log 4}{\log 2} = \frac{\log 25}{\log 5} = 2.$$

This definition of dimension also works for fractals! Consider the Sierpinski triangle (ST). It contains 3 self-similar triangles, whose sides are $1/2$ of the sides of the original triangle. Therefore, the number of self-similar pieces is 3, and the magnification factor is 2. For the dimension, we find

$$\dim ST = \frac{\log 3}{\log 2} \simeq 1.58.$$

Let us compute the fractal dimension of the Koch snowflake (KS). At each step of the construction, we add 4 times as many triangles, each triangle with sides $1/3$ of those added in the previous step. Therefore, the number of pieces is 4 and the magnification factor 3. The fractal dimension of the Koch snowflake is

$$\dim KS = \frac{\log 4}{\log 3} \simeq 1.26.$$

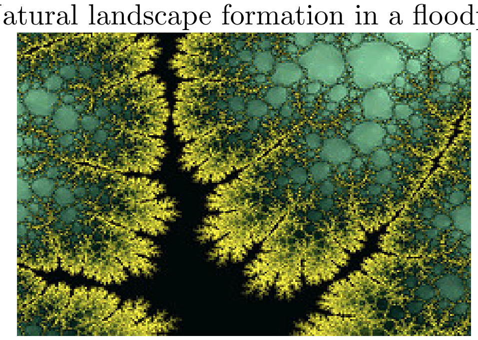
It is intuitively clear that the Koch snowflake is almost a curve, while the Sierpinski triangle is almost a solid triangle – and the fractal dimension reflects this fact: 1.58 for ST is closer to 2, while 1.26 for KS is closer to 1. Fractal dimension is used to measure the complexity of a pattern, and also its space-filling capacity.

Many real-world phenomena show (at least statistically) some self-similar behavior, and the fractal dimension has been a useful tool in image analysis, in particular in diagnostic medicine and neuroscience.

An application of fractal geometry in archeology. Methods of fractal geometry were recently applied by a group of scientists to demonstrate the existence at some point in the past of human activity on a large scale in an area of Egypt, known as the necropolis of Dahshur.

The royal necropolis is situated in a desert not far from the Nile valley. Big rivers are surrounded by branching networks of channels that play the major role in the formation of landscape topography, if this formation is defined only by natural factors. The channels carve fractal patterns in the land, as shown in the picture below. The royal necropolis of Dahshur is situated in a desert about 20 miles south of Cairo and close to Nile's floodplain. If the landscape was formed naturally, it should have followed the fractal pattern similar to the one show in the picture. Mathematically, this means that there should be a strong correlation (clear dependence) between the fractal dimensions of the channel network and of the surface topography in the area. Indeed, this is the case in a large area surrounding Dahshur.

FIGURE 3. Natural landscape formation in a floodplain of a river.



But this is not the case in the vicinity of the pyramid district of Dahshur. A group of German scientists calculated that the correlation between the fractal dimensions of the channel network and the surface topography is low or insignificant in a surprisingly large area - at least 2.5 square miles. This indicates that the creation of the landscape was disrupted at one point by a human impact of a large scale. The scientists conclude that the whole area once experienced a major construction or landscape architecture project, which took place most probably during the reign of Sneferu (Old Kingdom, around 2600 BC). Traditional methods of archaeologic research did not bring much evidence of human activity of this scale, and so the discovery is entirely due to the mathematical methods of fractal geometry, applied in archaeology.

Figure 4. The necropolis of Dahshur with the Red Pyramid.