Practice Final Solution

April 28, 2014

1

(a)

$$\lim_{x\to -1} \frac{x^2+3x+2}{x^2-x-2} = \lim_{x\to -1} \frac{x+2}{x-2} = \frac{-1}{3}.$$

(b) Since $\tan^{-1}(\infty) = \frac{\pi}{2}$ and $\ln(1+\frac{1}{\infty}) = 0$, we have

$$\lim_{x\to\infty} \frac{\tan^{-1}(x)}{\ln(1+\frac{1}{x})} = \infty.$$

(c) Divide top and bottom by e^{2x}, we get

$$\lim_{x\to\infty} \frac{e^{2x}+6e^x+5}{4e^{2x}-4e^x+1} = \lim_{x\to\infty} \frac{1+6e^{-x}+5e^{-2x}}{4-4e^{-x}+e^{-2x}} = \frac{1}{4}.$$

(d) Notice that

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{\sin x \cdot x}.$$

Using L'Hospital's rule twice, we get

$$\lim_{x\to 0} \frac{x-\sin x}{\sin x \cdot x} = \lim_{x\to 0} \frac{1-\cos x}{\cos x \cdot x + \sin x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - \sin x \cdot x} = 0.$$

2

(a) domain: $\mathbb{R}$, range: $0 < a < 1$, even function.

(b) horizontal asymptotes: $y = 0$.

(c) f is increasing on the interval $(-\infty, 0)$ and decreasing on the interval $(0, \infty)$.

(d) f has local maximum at $x = 0$.

(e) f is concave up when $x < \frac{-1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$, and f is concave down when $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. The points of inflection are $x = \pm\frac{1}{\sqrt{2}}$.

3

(a)

$$f'(x) = \ln(3) \cdot 3^{\ln x} \cdot \frac{1}{x}.$$

(b) Using chain rule, we get

$$g'(x) = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x(1+\sqrt{x})}}.$$

(c) Again by chain rule, we have

$$h'(x) = \frac{-2x}{\sqrt{1-x^4}} - 2x\sin(x^2).$$

(d) We use the fact that

$$k(x) = e^{\ln(x)\cdot\ln\ln(x)}.$$

Therefore, by chain rule, we have

$$k'(x) = (\ln x)^{\ln x} \cdot \left(\frac{\ln\ln(x)}{x} + \frac{1}{x} \right).$$

4

Differentiating both side by x, by chain rule, we get

$$8(x+y)(1+y') + 2(y-x)(y'-1) = 0.$$

Plug in $(x,y) = (3,-1)$ and solve for y'. We then obtain $y' = -3$, and the equation of the tangent line $y+1 = -3(x-3)$, or equivalently $y = -3x+8$.

5

Denote the distance between each vertex and the center by y. Hence, we have $\frac{dy}{dt} = 1\,\text{cm/sec}$. Moreover, the area A of the triangle is $\frac{3\sqrt{3}y^2}{4}$. Thus, by chain rule, we have

$$A' = \frac{3\sqrt{3}}{4} \cdot 2yy'.$$

At $y = 20$cm, area A changes at the rate of

$$A' = 30\sqrt{3}\,\text{cm}^3/\text{sec}.$$

6

Finding the anti-derivative, we get

$$f(x) = -e^{-x} + 4x^{5/2} + ax + b.$$

Now, $f'(1) = 1/e$ implies that $a = -10$. Finally, by $f(1) = -1/e$, we have $b = 6$.

7

Assume that we cut the sheet at $(10 - x)$ feet (for rolled uptight) and x feet. Then by the volume formula, the one that is rolled uptight has volume $\frac{(10-x)^2}{2\pi}$. On the other hand, the one that is rolled sideway has volume $\frac{x}{\pi}$. Consequently, we hope to minimize/maximize the total volume $\frac{(10-x)^2}{2\pi} + \frac{x}{\pi} = \frac{x^2-18x+100}{2\pi}$. Setting the first derivative to 0, we find that the function has a minimum at $x = 9$. The minimum volume is $V(9) = \frac{1}{2\pi} + \frac{9}{\pi}\,\mathrm{ft}^3$. To find the maximum, compare the values at the endpoints: $V(0) = \frac{100}{2\pi}\,\mathrm{ft}^3$, $V(10) = \frac{10}{\pi}\,\mathrm{ft}^3$. Therefore the maximum volume is $\frac{100}{2\pi}\,\mathrm{ft}^3$ when $x = 0\,\mathrm{ft}$.

8

(a) If $f(5) \leq 4$, then by the MVT there is a point a in the interval $(2, 5)$ such that $f'(a) = \frac{f(5)-f(2)}{5-2} < 0$, so f' cannot be positive for all x in $(2, 5)$.

(b) If $f(5) \leq 4$, then by the MVT there is a point b in the interval $(5, 7)$ such that $f'(b) = \frac{f(7)-f(5)}{7-5} > 0$, so f' cannot be negative for all x in $(5, 7)$.

(c) Suppose now that f' is non-positive for some $x \in (2, 5)$, and f' is non-negative for some $x \in (5, 7)$. Denote these points as $f'(a) \leq 0$, and $f'(b) \geq 0$. By mean value theorem, there exists some $c \in (a, b)$ such that $f''(c) = \frac{f'(b)-f'(a)}{b-a} \geq 0$. This contradicts the original assumption. Hence, $f(5) > 4$.

9

(a) We have

$$\int_1^3 \frac{\sqrt{x} - x^2}{x^{3/2}} dx = \int_1^3 x^{-1} - x^{1/2} dx = \ln x - \frac{2}{3}x^{3/2}\Big|_1^3 = \ln 3 - 2\sqrt{3} + \frac{2}{3}.$$

(b)

$$\int_0^{\frac{1}{2}} \frac{2dy}{\sqrt{1-y^2}} = 2\sin^{-1}(y)\Big|_0^{\frac{1}{2}} = \frac{\pi}{3}.$$

(c)

$$\int_{-4}^{4} \left(-1 - \sqrt{16 - x^2}\right) dx = -8 - \int_{-4}^{4} \sqrt{16 - x^2} dx.$$

The second integral is exactly the area of a semicircle with radius 4, which is 8π. Therefore,

$$\int_{-4}^{4} \left(-1 - \sqrt{16 - x^2}\right) dx = -8 - 8\pi.$$

10

Set $f(x) = \int_1^x \frac{3}{1+s^2} ds$, and $h(x) = 3^x$. One can see that

$$g(x) = f(h(x)).$$

Thus, by chain rule and FTC, we have

$$g'(x) = f'(h(x)) \cdot h'(x) = \frac{3}{1 + 3^{2x}} \cdot \ln 3 \cdot 3^x = \frac{\ln 3 \cdot 3^{x+1}}{1 + 3^{2x}}.$$