Mathematics in the Real World
Math 107

Lectures 7 & 8: Exponential growth and decay.

One of the commonly used tools in science is the mathematical model called *the exponential growth and decay.* If in a process a certain quantity doubles or halves at equal intervals of time, then this process can be modeled by an exponential function:

$$A(t) = A(0)e^{kt},$$

Where $A(t)$ is the quantity at the moment t, $A(0)$ is the initial quantity, and k is the growth or decay constant. It is also known as the relative growth rate, or the relative rate of decay. Note that if $k > 0$, then for $t > 0$ we have $A(t) > A(0)$ and we have a growth model; if $k < 0$, then $A(t) < A(0)$ and we observe a decay.

Population growth models. Let us start with an example involving *population growth.* Suppose that a population of a certain species doubles every T_d years (this parameter is called the *doubling time*, hence the notation). We have:

$t = 0$ years	$A(0)$ individuals
$t = T_d$ years	$2A(0)$ individuals
$t = 2T_d$ years	$4A(0)$ individuals
...	...
$t = n \cdot T_d$ years	$2^n A(0)$ individuals

In general,

$$A(n \cdot T_d) = 2^n A(0).$$

What if we want to know the population $A(t)$ at a moment of time t that is not a multiple of T_d? It is convenient to reformulate the process as exponential growth: $A(t) = A(0)e^{kt}$. Here $A(0)$ is the initial population, t is any moment later in time, and we only need to determine the relative growth rate k. At $t = T_d$ we have:

$$A(T_d) = 2A(0) = A(0)e^{kT_d} \quad \Longrightarrow \quad 2 = e^{kT_d} \quad \Longrightarrow kT_d = \ln 2,$$

and

$$k = \frac{\ln 2}{T_d}.$$

Note that the relative growth rate k depends on the unit of measurement of T_d. In the above example, T_d was given in years, and k shows the relative growth *per year.* If T_d was given in minutes, the obtained k would have been the relative growth of the population per minute. Finally, here is the formula that allows to determine the population $A(t)$ at any moment of time t, if the doubling time T_d is given:

$$A(t) = A(0)e^{kt}, \quad k = \frac{\ln 2}{T_d},$$

where t is the time measured in the same units as T_d, and $A(0)$ is the initial population. Conversely, if the relative growth rate k is given (say, per year), then the doubling time T_d (in years) is determined by the formula

$$T_d = \frac{\ln 2}{k}.$$

Example 1. A bacteria culture is observed to grow exponentially in time, with a relative growth rate $k = 0.0462$ per hour. If there are 2000 bacteria at the start of the experiment, we would like to

(1) find out how many hours it takes for the bacteria to reach 10000.
(2) compute the doubling time, i.e., how long it takes to reach 4000.

Solution: Since the growth model is exponential, we have the equation $A(t) = A_0e^{kt}$, where $A_0 = 2000$ and $k = 0.0462$.

To answer the first question, we write:

$$10000 = 2000e^{0.0462t} \quad \Rightarrow \quad e^{0.0462t} = 5.$$

Taking the natural logarithm of both sides and solving for t, we get the answer:

$$0.0462t = \ln(5) = 1.609 \quad \Rightarrow \quad t = \ln(5)/0.0462 = 34.86 \text{ hours}.$$

To answer the second question, we use the formula:

$$T_d = \frac{\ln 2}{k} = \frac{\ln 2}{0.0462} \simeq \frac{0.69315}{0.0462} \simeq 15.003,$$

which means that the doubling time is roughly 15 hours.

Example 2. The world population was 2560 millions in 1950 and 3040 millions in 1960. Assuming that the population grows exponentially, find the relative growth rate per year, and calculate the world population in 2010.

Solution: Set $t = 0$ in 1950. Then $A(0) = 2560 \cdot 10^6 = 2.56 \cdot 10^9$, and $A(10) = 3.04 \cdot 10^9$. According to the exponential growth model, we have

$$A(10) = A(0)e^{10k} \quad \Longrightarrow \quad 3.04 \cdot 10^9 = 2.56 \cdot 10^9 \cdot e^{10k} \quad \Longrightarrow \quad \frac{3.04}{2.56} = e^{10k}.$$

Applying the natural logarithm, we find

$$\ln\left(\frac{3.04}{2.56}\right) = 10k \quad \Longrightarrow \quad k = \frac{1}{10}\ln\left(\frac{3.04}{2.56}\right) \simeq 0.017185.$$

Therefore, the relative growth rate per year of the world population, as determined by the data of 1950 and 1960, is $k = 0.017185$.

Now, the time elapsed between 1950 and 2010 is 60 years. The exponential growth model gives

$$A(60) = A(0)e^{0.017185 \cdot 60} = 2.56 \cdot 10^9 \cdot e^{0.017185 \cdot 60} \simeq 7.18 \cdot 10^9.$$

In fact, the world population in 2010 was approximately 6.896 billion people. The difference between the actual and the predicted number can be a consequence of many reasons: first, we based our calculation on only two data points: that of 1950 and 1960. In fact, it would be wiser to find relative growth rates based on several pairs of data points, and take their average. Besides, new factors could have appeared that pushed the relative growth rate down, and are not reflected in the data yet. In any case, it is important to remember that any mathematical model is just a model; it gives correct answers only if all the assumption of the model are satisfied in the reality. The question of applicability should be decided in each particular case separately. In general, the more complicated the system, the harder it is to formulate an adequate model: for example, the exponential growth model describes reproduction of bacteria fairly adequately, but dealing with complex systems, such as human populations, many other factors should be taken into account. Still, simple models like exponential growth are useful since they quickly provide a rough estimate of the actual process.

Radioactive decay. Exponential function is used to describe the process of *radioactive decay*. In this case, the constant k is negative, which means that the quantity $A(t)$ given by the equation $A(t) = A(0)e^{kt}$ after time t is smaller than the initial quantity $A(0)$.

The method of *Carbon dating* is based on the exponential decay model for the carbon isotope C_{14}. It was discovered by an American physicist Willard Libby (University of Chicago) in 1949, and rewarded with a Nobel Prize in chemistry. Roughly, the idea is the following. Carbon has various isotopes, and the one with six protons and eight neutrons (C_{14}) is radioactive - and hence, decays according to the exponential model. While an animal or a plant is alive, it keeps replenishing its C_{14}-supply to balance with the atmospheric level, but as soon as it dies, the amount of C_{14} in its body starts to decrease at an exponential rate.

Thus, if $A(t)$ denotes the amount of C_{14} in a fossil t years after death, then

$$A(t) = A(0)e^{kt}.$$

Here, k is the *relative rate of decay*. However, in radioactive decay problems, the usual data supplied is the *half-life* $T_{1/2}$, which is the time taken to reduce by half the amount of radioactive substance (in this case, C_{14} in the fossil).

To relate the half-life $T_{1/2}$ and the relative rate of decay k we use the equation

$$A(T_{1/2}) = \frac{1}{2}A(0) = A(0)e^{kT_{1/2}}.$$

Eliminating $A(0)$ from both sides and solving for k in terms of $T_{1/2}$, we get (since $\ln(1/2) = -\ln 2$):

$$\ln\frac{1}{2} = kT_{1/2} \qquad \Longrightarrow \qquad k = \frac{\ln\frac{1}{2}}{T_{1/2}} = -\frac{\ln 2}{T_{1/2}}.$$

Since the behavior of atoms (when the energy is not too high or too low) is well understood, the exponential decay model gives remarkably precise results in case of a radioactive decay. The modern methods of radiocarbon dating use calibration curves, that take into account slight variations of the level of C_{14} in the atmosphere. The precision of the method depends on the age of the object. For example, medieval documents can be dated within a range of 30 years.

Example 3. (1) Carbon-14 has a half-life of 5730 years. Find its relative rate of decay.
(2) A parchment has about 74% as much of C_{14} as does the atmosphere. Estimate its age.

Solution:

(1)

$$k = -\frac{\ln 2}{T_{1/2}} = -\frac{\ln 2}{5730} \simeq -1.20968 \cdot 10^{-4} = -0.000120968.$$

(2)

$$A(t) = 0.74A(0) = A(0)e^{kt} = A(0)e^{-0.000120968t}.$$

Eliminating $A(0)$ from both sides and taking the natural logarithm, we get

$$\ln 0.74 = -0.000120968t \qquad \Rightarrow \qquad t \simeq 2489.$$

Therefore, the parchment is about 2500 years old.

The same model, and the same formulas describe the decay of other radioactive materials. The parameter that characterizes the process for a particular radioactive substance is the relative rate of decay k, or the half-life $T_{1/2}$.

Example 4. The mass of a sample of radioactive Radium-226 reduced from 100mg to 95.76mg in 100 years.

(1) Find the half-life of Radium-226.
(2) How long will it take for the mass of the sample to be reduced to 30mg?

Solution:

(1) We will first find k, and then use the formula $T_{1/2} = -\frac{\ln 2}{k}$ to find the half-life. To find k, in the formula $A(t) = A(0)e^{kt}$ we plug $A(t) = 95.76$, $A(0) = 100$, $t = 100$ years. Then

$$95.76 = 100e^{100k} \quad \Longrightarrow \quad 0.9576 = e^{100k} \quad \Longrightarrow \quad \ln 0.9576 = 100k,$$

and

$$k \simeq -0.0004332.$$

Then the half-life of Radium-226 is

$$T_{1/2} = -\frac{\ln 2}{-0.0004332} \simeq 1600 \text{ years}.$$

(2) From the same formula with unknown t, we get

$$A(t) = 100e^{-0.0004332t} = 30.$$

Dividing by 100 and applying the natural logarithm, we have

$$e^{-0.0004332t} = 0.3 \quad \Longrightarrow \quad -0.0004332 \cdot t = \ln 0.3 \quad \Longrightarrow \quad t = -\frac{\ln 0.3}{-0.0004332} \simeq 2779.$$

Therefore, it would take approximately 2779 years.

The Voynich manuscript. The Beinecke Library at Yale holds one of the most mysterious medieval manuscripts in the world. It is a vellum manuscript containing 240 pages with text written in ink a flowing cursive script, and hundreds of color illustration. Most probably, the book was intended as a scientific or magical treatise. Here are some of the characteristics of the manuscript that earned it the reputation of an ultimate enigma in the world of ancient texts:

(1) Title, author, exact date and place of creation are unknown.
(2) Script and language of the text are unknown. Similarly to many European languages, the text seems to be composed of 20 to 30 "letters" grouped into "words"; the word distribution follows natural patters. However, unlike in most European languages, there are almost no one- or two-letter words, some words are repeated up to three times in a row, and some differ by a single letter. Despite the efforts of many professional cryptographers, including modern counter-intelligence code breakers, and hundreds of amateurs, the text remains undeciphered. The hypotheses range from "an eastern asian language written in an invented script" to "a very elaborate cipher (code) involving multiple alphabets" to the claim that the text, mostly meaningless, contains information hidden in minute details, like tiny pen strokes, that can be interpreted as an ancient Greek shorthand. Some evidence can be cited in favor, and some against each of the hypothesis. High definition images of all pages of the manuscript are available online from the Beinecke library website for anyone wishing to try their luck with the text.

(3) Judging from the illustrations, the text can be divided into six chapters: botanical, astronomical, biological or medical, cosmological, pharmaceutical, and "a list of recipes". Color illustrations are plentiful, but mysterious. For instance, none of the botanical pictures correspond to a known plant.
(4) Origins of the manuscript are mysterious. Allegedly, it first surfaced in Prague in the 16th century, where it was in the possession of an English astrologer John Dee. The first well-documented mention dates from 1912, when a Polish-American book dealer Wilfrid Voynich bought it from a Jesuit college near Rome in Italy. Subsequently, the book acquired his name. Eventually it was purchased from Voynich's heirs and presented to the Beinecke library.

In this situation, any scientifically confirmed information can be a big help. And recently, one piece of the puzzle was set in place: the approximate date of manufacturing of the vellum on which the manuscript is written. In 2009, a group of researchers from the University of Arizona, led by assistant research scientist Greg Hodgins, performed a radiocarbon analysis of the vellum of four different pages of the manuscript. All four pieces dated between 1404 and 1438. Because different pages dated within the same narrow range, it is reasonable to assume that this should be the time when the manuscript was created. This puts the creation of the manuscript in early 15th century, about 100 years earlier than was assumed before; in particular it almost certainly rules out John Dee (1527-1608), as well as the English philosopher Roger Bacon (1214-1294) as possible authors. The dating of the vellum also allows to discard 19th or 20th century hoax hypothesis. It is very unlikely that someone who lived centuries later would be able to procure a substantial quantity (requiring dozens of calf hides) of high quality ancient vellum. A chemical analysis of the paint used in the illustrations, performed in 2009 by German scientists, revealed no traces of industrial manufacture and confirmed that the paints could have been produced by medieval methods around the same time.

An obvious next step would be a radiocarbon analysis of the ink used in the manuscript. Then if the ink turned out to be about as old as the vellum, this would lock the date of the creation of the manuscript in early 15th century. Unfortunately, the amount of ink needed for the analysis would require to destroy several pages, which is probably too high a price to pay for an answer. Another possible next move would be to try and determine the DNA of some of the cattle whose hide was used. Comparing it with the DNA of modern cattle, one could determine the region where the animals most probably lived, and hence narrow down geographical regions where the manuscript could have been created.

As of today, the Voynich manuscript mystery remains unsolved: stored under the catalog number MS 408 in the Medieval and Renaissance Manuscript section in Beinecke, the book is waiting for its secrets to be revealed. Below is the image of the page 14*v* of the botanical section of the manuscript.

FIGURE 1. The Voynich Manuscript, page 14v.