ON THE CENTER OF THE SMALL QUANTUM GROUP

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ABSTRACT. Using the quantum Fourier transform $\mathcal{F}$ [LM], we describe the block decomposition and multiplicative structure of a subalgebra $\widetilde{\mathfrak{Z}} + \mathcal{F}(\widetilde{\mathfrak{Z}})$ of the center of the small quantum group $U_q^{fin}(\mathfrak{g})$ at a root of unity. It contains the known subalgebra $\widetilde{\mathfrak{Z}}$ [BG], which is isomorphic to the algebra of characters of finite dimensional $U_q^{fin}(\mathfrak{g})$-modules. We prove that the intersection $\widetilde{\mathfrak{Z}} \cap \mathcal{F}(\widetilde{\mathfrak{Z}})$ coincides with the annihilator of the radical of $\widetilde{\mathfrak{Z}}$. Applying representation-theoretical methods, we show that $\widetilde{\mathfrak{Z}}$ surjects onto the algebra of endomorphisms of certain indecomposable projective modules over $U_q^{fin}(\mathfrak{g})$. In particular this leads to the conclusion that the center of $U_q^{fin}(\mathfrak{g})$ coincides with $\widetilde{\mathfrak{Z}} + \mathcal{F}(\widetilde{\mathfrak{Z}})$ in the case $\mathfrak{g} = sl_2$.

1. INTRODUCTION AND NOTATION

Let $U_q^{fin}(\mathfrak{g})$ denote the small quantum group [Lus2] associated to a semisimple complex Lie algebra $\mathfrak{g}$ and an odd number l. We will assume l is greater than the Coxeter number of the root system and relatively prime to the determinant of the Cartan matrix of $\mathfrak{g}$.

In the present paper we combine Hopf-algebraic and representation-theoretical methods to obtain the block decomposition and multiplicative structure of a subalgebra in the center $\mathfrak{Z}$ of $U_q^{fin}(\mathfrak{g})$.

The starting point is the result in [BG], which describes a subalgebra $\widetilde{\mathfrak{Z}} \subset \mathfrak{Z}$ (Theorem 4.1). It can be obtained as the image of the complexification of the Grothendieck ring $\mathfrak{R}$ of the category of finite dimensional $U_q^{fin}(\mathfrak{g})$-modules under the homomorphism $\mathfrak{J}$ (§3), which was introduced in [Dr] for any quasitriangular Hopf algebra and can be viewed as an inverse of the quantum Harish-Chandra map. If q is not a root of unity, the image of $\mathfrak{J}$ coincides with the whole center of the quantum enveloping algebra $U_q(\mathfrak{g})$ (see e.g. [Bau]). In the case of the small quantum group, $\mathfrak{R}$ surjects onto a proper subalgebra of the center $\mathfrak{J}(\mathfrak{R}) \simeq \widetilde{\mathfrak{Z}}$, as was already observed in the $\mathfrak{sl}_2$ case in [Ker1].

On the other hand, viewing $U_q^{fin}(\mathfrak{g})$ as a finite dimensional Hopf algebra allows one to define an injective map $\phi^{-1} : \mathfrak{R} \to \mathfrak{Z}$ (§2), and obtain an ideal in the center $\mathfrak{Z}' = \phi^{-1}(\mathfrak{R})$. A combination of the two theories yields a map $\mathcal{F} \simeq \mathfrak{J} \circ \phi$ with the properties of a Fourier transform, which is a slight modification of a map introduced in [LM] as the basic involution in a projective action of the modular group on $U_q^{fin}(\mathfrak{g})$. In §5 we use the involutive property of $\mathcal{F}$ to prove the main structural result (Theorem 5.2):

$$\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}' \simeq \operatorname{Ann}(\operatorname{Rad} \widetilde{\mathfrak{Z}}).$$

Then we show that $\widetilde{\mathfrak{Z}} + \mathfrak{Z}' = \widetilde{\mathfrak{Z}} + \mathcal{F}(\widetilde{\mathfrak{Z}})$ is a subalgebra of the center, and describe its multiplicative structure and block decomposition (Theorem 5.5). It contains many elements which cannot be expressed in terms of any version of a Harish-Chandra map, including the two-sided integral of $U_q^{fin}(\mathfrak{g})$.

We would like to mention that the ideal $\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}'$ is an interesting object in itself. It has many recognizable properties: its basis is parametrized by the orbits of a certain group of affine reflections in P, it is invariant under a suitable version of the Fourier transform, and it appears naturally in the representation theory of a quantum group at a root of unity. This is reminiscent of the algebra of characters of the fusion category over $U_q^{res}(\mathfrak{g})$ (the Verlinde algebra) [AP], and it turns out that the analogy between the two structures goes even further. We study this structure in [La], and show that it can be considered a natural counterpart of the Verlinde algebra for the small quantum group.

In §6 we study the action of the center on projective $U_q^{fin}(\mathfrak{g})$-modules and prove that the subalgebra $\widetilde{\mathfrak{Z}}$ surjects onto the direct sum of algebras of endomorphisms of certain projective modules over $U_q^{fin}(\mathfrak{g})$. This result may be considered a quantum root of unity version of the category $\mathcal{O}$ statement [Soe1], where the center of the universal enveloping algebra $U(\mathfrak{g})$ surjects onto the algebra of endomorphisms of the projective cover of a simple $U(\mathfrak{g})$-module with anti-dominant highest weight. The surjection $\widetilde{\mathfrak{Z}} \to End_{U_q^{fin}(\mathfrak{g})}P(\lambda)$ holds for all the projective modules $P(\lambda)$ over $U_q^{fin}(\mathfrak{sl}_2)$, which allows us to prove that $\mathfrak{Z} = \widetilde{\mathfrak{Z}} + \mathfrak{Z}'$ in the $\mathfrak{sl}_2$ case. We discuss the cases $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_3$ in §7.

The constructed subalgebra is the smallest subspace of the center containing the obvious part $\widetilde{\mathfrak{Z}}$ and invariant under the quantum Fourier transform; it coincides with the whole center in the $\mathfrak{sl}_2$ case. We believe that its description will serve as a basepoint for understanding the structure of the center $\mathfrak{Z}$ of $U_q^{fin}(\mathfrak{g})$ in general.

1.1. **Definitions and notation.** To simplify the arguments we will assume that the root system R of $\mathfrak{g}$ is simply laced, though this restriction is not crucial. Let R_+ denote the positive roots, $\Pi \subset R_+$ the simple roots, Q the root lattice, P the weight lattice, and $r = \mathrm{rank}\mathfrak{g}$. Fix an odd number $l \geq h$, where h is the Coxeter number of the root system R. Starting from §3 we assume that $(l, \det a_{ij}) = 1$ for the Cartan matrix a_{ij} of R (see Lemma 3.1).

Let W be the Weyl group associated to the root system R. Its action on P is given by $s_\alpha(\mu) = \mu - \langle \mu, \alpha \rangle \alpha$ for any $\alpha \in R$, $\mu \in P$, where the form $\langle \cdot, \cdot \rangle$ is normalized so that $\langle \alpha, \alpha \rangle = 2$ for $\alpha \in R$. We will also use the shifted action $w \cdot \mu = w(\mu + \rho) - \rho$ for any $w \in W$, $\mu \in P$, where $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$. Let $\hat{W}_l$ denote the affine Weyl group generated by the reflections $s_{\alpha,k} \cdot \mu = s_\alpha \cdot \mu + kl\alpha$ for any $\alpha \in R_+$, $k \in \mathbf{Z}$ and $\mu \in P$. The natural and shifted actions of $\hat{W}_l$ are defined by $s_{\alpha,k}(\mu) = s_\alpha(\mu) + kl\alpha$ and $s_{\alpha,k} \cdot \mu = s_\alpha \cdot \mu + kl\alpha$ for any $\mu \in P$. The sets $\hat{X} = \{\lambda \in P_+ : \langle \lambda, \alpha \rangle \leq l \text{ for all } \alpha \in R_+\}$ and $\bar{X} = \{\lambda \in P : 0 \leq \langle \lambda + \rho, \alpha \rangle \leq l \text{ for all } \alpha \in R_+\}$ are the fundamental domains for the non-shifted and shifted affine Weyl group actions respectively.

Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum enveloping algebra associated to a semisimple simply-laced Lie algebra $\mathfrak{g}$ over $\mathbb{Q}(v)$, where v is a formal variable. It is generated by the elements $\{E_i, F_i, K_i^{\pm 1}\}_{i=1}^r$, with the standard set of relations (see e.g. [Lus1]). Denote by $U_v(\mathfrak{g})^{\mathbb{Z}}$ the $\mathbb{Z}[v, v^{-1}]$ subalgebra of $U_q(\mathfrak{g})$ generated by the

divided powers $E_i^{(k)} = \frac{E_i^k}{[k]!}$, $F_i^{(k)} = \frac{F_i^k}{[k]!}$ and K_i, K_i^{-1} for $i = 1 \ldots r$, $k \in \mathbb{N}, k \geq 1$, where $[k]! = \prod_{s=1}^{k} \frac{v^s - v^{-s}}{v - v^{-1}}$. Then the restricted quantum group $U_q^{res}(\mathfrak{g})$ is defined by $U_q^{res}(\mathfrak{g}) = U_v(\mathfrak{g})^{\mathbb{Z}} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(q)$, where v maps to $q \in \mathbb{C}$, which is set to be a primitive l-th root of unity. By [Lus1], we have for $1 \leq i \leq r$: $E_i^l = 0$, $F_i^l = 0$, $K_i^{2l} = 1$, and K_i^l is central in $U_q^{res}(\mathfrak{g})$.

Let $U_q^{fin}(\mathfrak{g})$ be the finite dimensional subquotient of $U_q^{res}(\mathfrak{g})$ [Lus2], generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i=1}^{r}$ over $\mathbb{Q}(q)$, factorized over the two-sided ideal $\langle \{K_i^l - 1\}_{i=1}^{r} \rangle$. Then $U_q^{fin}(\mathfrak{g})$ is a Hopf algebra of dimension $l^{\dim \mathfrak{g}}$ over $\mathbb{Q}(q)$. We will use the same notation for the $\mathbb{C}$-algebras $U_q^{res}(\mathfrak{g})$ and $U_q^{fin}(\mathfrak{g})$, where the field $\mathbb{Q}(q)$ is extended to $\mathbb{C}$.

2. Hopf structure and the isomorphism of modules.

The properties of $U_q^{fin}(\mathfrak{g})$ as a finite dimensional Hopf algebra were studied in e.g. [Lyu], [Ker1]. In particular, it is unimodular, i.e. there exists a two-sided integral $\Lambda \in U_q^{fin}(\mathfrak{g})$ such that $a\Lambda = \Lambda a = \varepsilon(a)\Lambda$, for any $a \in U_q^{fin}(\mathfrak{g})$, where ε is the counit for $U_q^{fin}(\mathfrak{g})$, and Λ is unique up to a scalar multiple. The Hopf dual $U_q^{fin}(\mathfrak{g})^*$ of $U_q^{fin}(\mathfrak{g})$ is not unimodular; however the spaces of left and right integrals $\lambda_l, \lambda_r \in U_q^{fin}(\mathfrak{g})^*$ such that $p\lambda_l = p(1)\lambda_l$ and $\lambda_r p = p(1)\lambda_r$ for all $p \in U_q^{fin}(\mathfrak{g})^*$ are one-dimensional. For any $\mu = \sum_i n_i \alpha_i \in Q$ we set $K_\mu = \prod_i K_i^{n_i}$. The square of the antipode S^2 is an inner automorphism of $U_q^{fin}(\mathfrak{g})$ with $S^2(a) = K_{-2\rho} a K_{2\rho}$ for any $a \in U_q^{fin}(\mathfrak{g})$, where $K_{2\rho} = \prod_{\alpha \in R_+} K_\alpha$.

To formulate the next result we will need to turn $U_q^{fin}(\mathfrak{g})$ and $U_q^{fin}(\mathfrak{g})^*$ into left and right modules over themselves. Naturally $U_q^{fin}(\mathfrak{g})$ is a left module over itself via multiplication. Define a left $U_q^{fin}(\mathfrak{g})$-module structure on $U_q^{fin}(\mathfrak{g})^*$ by

$$a \rightharpoonup p(b) = p(S(a)b),$$

for any $a, b, \in U_q^{fin}(\mathfrak{g})$, $p \in U_q^{fin}(\mathfrak{g})^*$. $U_q^{fin}(\mathfrak{g})^*$ is a right module over itself via right multiplication. $U_q^{fin}(\mathfrak{g})$ can be given a right $U_q^{fin}(\mathfrak{g})^*$- module structure by setting

$$a \leftharpoonup p = \sum p(a_{(1)}) a_{(2)},$$

for any $a \in U_q^{fin}(\mathfrak{g})$, $p \in U_q^{fin}(\mathfrak{g})^*$, where we have used Sweedler's notation for comultiplication $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

The next theorem is a special case of a general result valid for finite dimensional Hopf algebras with an antipode.

Theorem 2.1. [Swe], §5.1; [Rad]. *There exists a linear map ϕ from $U_q^{fin}(\mathfrak{g})$ to $U_q^{fin}(\mathfrak{g})^*$, which is both an isomorphism of left $U_q^{fin}(\mathfrak{g})$-modules and right $U_q^{fin}(\mathfrak{g})^*$- modules, satisfying the following conditions:*

1. *$\phi(1) = \lambda_r$ is a right integral for $U_q^{fin}(\mathfrak{g})^*$ and $\phi^{-1}(\varepsilon) = \Lambda$ is an integral for $U_q^{fin}(\mathfrak{g})$;*

2. $\phi(a) = a \rightharpoonup \lambda_r$ for any $a \in U_q^{fin}(\mathfrak{g})$, and $\phi^{-1}(p) = \Lambda \leftharpoonup p$ for any $p \in U_q^{fin}(\mathfrak{g})^*$. Thus $U_q^{fin}(\mathfrak{g})^*$ is a free $U_q^{fin}(\mathfrak{g})$-module with basis λ_r, and $U_q^{fin}(\mathfrak{g})$ is a free $U_q^{fin}(\mathfrak{g})^*$-module with basis Λ with respect to the actions defined above.

With these conditions, ϕ is unique up to a scalar multiple.

Corollary 2.2. *The image of the center $\mathfrak{Z}$ of $U_q^{fin}(\mathfrak{g})$ under ϕ coincides with C_r, the set of all functionals $p \in U_q^{fin}(\mathfrak{g})^*$ such that $p(ab) = p(bS^{-2}(a))$. In particular, $\dim(\mathfrak{Z}) = \dim(C_r)$.*

Proof. The right integral λ_r is itself an element of C_r ([Rad], Thm.3). Since $S(\mathfrak{Z}) = \mathfrak{Z}$, the result of the left action of $\mathfrak{Z}$ on λ_r is again in C_r. Conversly, suppose $a \rightharpoonup \lambda_r \in C_r$ Then

$$\lambda_r(S(a)bc) = \lambda_r(cS^{-2}(S(a)b)) = \lambda_r(cS^{-1}(a)S^{-2}(b)),$$

which has to be equal to $\lambda_r(S(a)cS^{-2}(b)) = \lambda_r(cS^{-2}(b)S^{-1}(a))$ for all $b, c \in U_q^{fin}(\mathfrak{g})$, and this holds if and only if $a \in \mathfrak{Z}$. $\square$

Remark. The set C_r has a simple interpretation in terms of cocommutative elements of $U_q^{fin}(\mathfrak{g})^*$. Let C_0 be the set of all cocommutative elements of $U_q^{fin}(\mathfrak{g})^*$, i.e. $\mu(ab) = \mu(ba)$ for $\mu \in C_0$ and any $a, b \in U_q^{fin}(\mathfrak{g})$. Consider the functional $p \in U_q^{fin}(\mathfrak{g})^*$ defined by $p = K_{-2\rho} \rightharpoonup \mu$. Using the identity $S^2 = Ad(K_{-2\rho})$, it is easy to check that $p \in C_r$:

$$p(ab) = \mu(abK_{2\rho}) = \mu(bK_{2\rho}a) = \mu(bK_{2\rho}aK_{-2\rho}K_{2\rho}) = p(bS^{-2}(a)).$$

Similarly, for a functional $t \in C_l$ defined by the condition $t(ab) = t(bS^2(a))$, we have $t = K_{2\rho} \rightharpoonup \mu$ for some $\mu \in C_0$. Therefore, the sets C_r, C_l and C_0 differ only by the action of a semisimple grouplike element of $U_q^{fin}(\mathfrak{g})$, and since $\Delta(K_{2\rho}) = K_{2\rho} \otimes K_{2\rho}$, they are isomorphic as commutative algebras ([Dr], §3) and their dimensions coincide.

Let $\mathfrak{R}$ be the subspace of C_0 spanned by the traces of simple $U_q^{fin}(\mathfrak{g})$- modules. By [Lus2] they are parametrized by the l-restricted weights

$$(P/lP)_+ \equiv \{\lambda \in P : 0 \leq \langle \lambda, \alpha_i \rangle < l, \alpha_i \in \Pi\},$$

so the dimension of $\mathfrak{R}$ equals l^r. $\mathfrak{R}$ has an algebra structure induced from the tensor product in the category of finite dimensional $U_q^{fin}(\mathfrak{g})$- modules. It is a commutative subalgebra in $U_q^{fin}(\mathfrak{g})^*$. The integral form of $\mathfrak{R}$ is the Grothendieck ring of the category of finite dimensional $U_q^{fin}(\mathfrak{g})$-modules. By the above remark we can consider the isomorphic algebras of q- and q^{-1}-traces of $U_q^{fin}(\mathfrak{g})$- modules, defined as the usual traces of $U_q^{fin}(\mathfrak{g})$-modules, shifted by the action of $K_{2\rho}$ and $K_{-2\rho}$ respectively. They will be denoted by $\mathfrak{R}_l \subset C_l$ and $\mathfrak{R}_r \subset C_r$. We have $\mathfrak{R} \simeq \mathfrak{R}_r \simeq \mathfrak{R}_l$ as commutative algebras.

Let $\mathfrak{Z}' \subset \mathfrak{Z}$ denote the image of $\mathfrak{R}_r$ under the map ϕ^{-1}.

Proposition 2.3.

$$\mathfrak{Z}' \simeq \text{Ann}(\text{Rad } \mathfrak{Z}).$$

Proof. A central element z acts on a simple $U_q^{fin}(\mathfrak{g})$-module as multiplication by a scalar, which is nonzero whenever z is not nilpotent. Therefore, as a $\mathfrak{Z}$- module,

$\mathfrak{R}_r$ coincides with the maximal semisimple submodule of C_r, which is annihilated by the radical of $\mathfrak{Z}$.

By restriction, ϕ is an isomorphism of $\mathfrak{Z}$-modules, and Theorem 2.1 provides an isomorphism of $\mathfrak{Z}$-module structures on C_r and $\mathfrak{Z}$, the latter given by multiplication. In particular, $\mathfrak{Z}' = \phi^{-1}(\mathfrak{R}_r)$ coincides with the annihilator of the radical (i.e. the socle) of $\mathfrak{Z}$. $\qquad\square$

3. QUASITRIANGULAR STRUCTURE IN $U_q^{fin}(\mathfrak{g})$.

By [Lyu], $U_q^{fin}(\mathfrak{g})$ is quasitriangular with the canonical element R satisfying

$$R\Delta(x)R^{-1} = \Delta^{op}(x)$$

for any $x \in U_q^{fin}(\mathfrak{g})$ and

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12},$$

where $\Delta^{op}(x) = \sum x_{(2)} \otimes x_{(1)}$, and the lower indices of R indicate the position of $R_{(1)}$ and $R_{(2)}$.

Let U^0 denote the subalgebra of $U_q^{fin}(\mathfrak{g})$ generated by $\{K_i^{\pm 1}\}_{i=1...r}$. Define a bimultiplicative symmetric form on U^0 by $\pi(K_\mu, K_\nu) = q^{(\mu|\nu)}$, where $(\mu|\nu) : Q \times Q \to \mathbb{C}$ is the bilinear pairing such that $(\alpha_i|\alpha_j) = a_{ij}$ for any $\alpha_i, \alpha_j \in \Pi$ and the Cartan matrix a_{ij} of $\mathfrak{g}$.

Lemma 3.1. *(cf. App. B, [Lyu]) Suppose that for the Cartan matrix a_{ij} of $\mathfrak{g}$, an odd number l satisfies the condition $(l, \det a_{ij}) = 1$. Then π is nondegenerate on U^0.*

Proof. Let $K_h \in Ann\pi$, which means that $\pi(K_h, K_\alpha) = 1$ for any $\alpha \in \Pi$. Then in general, $Ann\pi = \{K_\lambda, \lambda \in Q \cap lP\}$ [Lyu]. Write $h = \sum_{i=1}^r b_i\alpha_i$ for some coefficients $b_i \in \mathbb{Z}$, and

$$\pi(K_h, K_{\alpha_i}) = q^{(\sum b_j\alpha_j|\alpha_i)} = q^{\sum a_{ij}b_j} = 1.$$

Then $\sum a_{ij}b_j \equiv 0(\mathrm{mod}\,l)$, for any $i = 1, \dots, r$. Solving the system, we get $b_j = \frac{1}{\det a_{ij}} \sum A_{ij}k_i l$, where A_{ij} are the minors of a_{ij}, and $k_i \in \mathbb{Z}$. By assumption this implies $b_j \equiv 0(\mathrm{mod}\,l)$, therefore $h \in lQ$ and $Q \cap lP = lQ$. Clearly all $K_\alpha^l \in Ann\pi$, and hence $Ann\pi$ is generated by $\{K_\alpha^l\}_{\alpha \in \Pi}$, and π is nondegenerate on $U^0 = \{K_\mu, \mu \in Q/lQ\}$. $\qquad\square$

From now on we will assume that l is an odd integer greater than or equal to the Coxeter number of the root system of $\mathfrak{g}$, satisfying the conditions of Lemma 3.1. Then $U_q^{fin}(\mathfrak{g})$ is a quasitriangular Hopf algebra with the canonical element given by the formula [Lyu]:

$$R = \frac{1}{|Q/lQ|} \sum_{\mu,\nu \in Q/lQ} \pi(K_\mu, K_\nu)^{-1} K_\mu \otimes K_\nu \cdot \sum_{m_\alpha=0}^{l-1} \prod_\alpha \frac{q^{m_\alpha(m_\alpha-1)/2}(q-q^{-1})^{m_\alpha}}{[m_\alpha]!} F_\alpha^{m_\alpha} \otimes E_\alpha^{m_\alpha},$$

where α runs over all positive roots in a fixed order corresponding to the decomposition of the longest element of the Weyl group $w_0 \in W$.

$U_q^{fin}(\mathfrak{g})$ is also factorizable, meaning that the map $p \to m(p \otimes \mathrm{id})(R_{21}R_{12})$ from $U_q^{fin}(\mathfrak{g})^*$ to $U_q^{fin}(\mathfrak{g})$ is surjective. Here m denotes the multiplication in $U_q^{fin}(\mathfrak{g})$. By [Lyu], a small quantum group is factorizable if and only if for $\Gamma = Q/lQ$, $2\Gamma = \Gamma$,

where $2\Gamma = \{2x, x \in \Gamma\}$ is the group of squares. This always holds for $U_q^{fin}(\mathfrak{g})$ since l is odd.

Theorem 3.2. [Dr] *The map* $\mathfrak{J} : U_q^{fin}(\mathfrak{g})^* \to U_q^{fin}(\mathfrak{g})$ *defined by*

$$\mathfrak{J}(p) = m(p \otimes \mathrm{id})(R_{21}R_{12}),$$

restricted to C_r, *is an algebra homomorphism between* C_r *and* $\mathfrak{Z}$.

Corollary 3.3. *In the case of a factorizable finite dimensional Hopf algebra, the map*

$$\mathfrak{J} : C_r \to \mathfrak{Z}$$

is an isomorphism of commutative algebras.

Proof. Factorizability for a finite dimensional Hopf algebra means that $Ker(\mathfrak{J}) \subset U_q^{fin}(\mathfrak{g})^*$ is trivial. Since $\mathfrak{J}(C_r) \subset \mathfrak{Z}$ is a homomorphic image of C_r and $dim(\mathfrak{Z}) = dim(C_r)$ by Corollary 2.2, we get an isomorphism. $\qquad\square$

Denote the image of $\mathfrak{R}_r \subset C_r$ under $\mathfrak{J}$ by $\widetilde{\mathfrak{Z}} \subset \mathfrak{Z}$. By the Remark in §2, we have $\mathfrak{R}_r \simeq \mathfrak{R}$, the algebra of characters of finite dimensional $U_q^{fin}(\mathfrak{g})$- modules with multiplication induced by the tensor product. The algebraic structure of $\mathfrak{R}$ can be described using the representation theory of finite dimensional modules over $U_q^{res}(\mathfrak{g})$.

Let $\bar{\mathcal{C}}_f$ denote the category of finite dimensional modules over $U_q^{fin}(\mathfrak{g})$. For a module M in $\bar{\mathcal{C}}_f$ we will use the symbol chM for the formal character $chM = \sum_{\eta \in P}(\dim M_\eta)e^\eta \in \mathbb{C}[P]^W$, where we write e^η for the basis element in $\mathbb{C}[P]$ corresponding to $\eta \in P$. Here $\mathbb{C}[P]^W$ denotes the subspace of exponential invariants of the W-action in $\mathbb{C}[P]$.

Proposition 3.4.

$$\mathfrak{R} \simeq \mathbb{C}[P]^W \otimes_{\mathbb{C}[lP]^W} \mathbb{C},$$

where $\mathbb{C}[lP]^W \longrightarrow \mathbb{C}$ *is evaluation at* 1.

Proof. The finite dimensional simple modules of type **1** (i.e. all $\{K_i^l\}_{i=1...r}$ act on them by 1) over $U_q^{res}(\mathfrak{g})$ are parametrized by the dominant integral weights $\{L(\mu)\}_{\mu \in P_+}$. The algebra of characters, spanned by $\{chL(\mu) = \sum_{\nu \in P}\dim(L(\mu))_\nu e^\nu\}_{\mu \in P_+}$, is isomorphic to $\mathbb{C}[P]^W$. By Lusztig's tensor product theorem [Lus1],

$$L(\mu_0 + l\mu_1) \simeq L(\mu_0) \otimes L(l\mu_1),$$

where $\mu = \mu_0 + l\mu_1 \in P_+$ and $0 \le \langle \mu_0, \alpha_i \rangle \le l - 1$ for all $\alpha_i \in \Pi$. By restriction, simple $U_q^{res}(\mathfrak{g})$-modules of type **1** are modules over $U_q^{fin}(\mathfrak{g})$. The subquotient $U_q^{fin}(\mathfrak{g})$ of $U_q^{res}(\mathfrak{g})$ acts trivially on the simple modules $L(l\mu_1)$, whose weights are all in lP [APW2]. Therefore, the algebra of characters $\mathfrak{R}$ of $U_q^{fin}(\mathfrak{g})$-modules can be obtained from that of $U_q^{res}(\mathfrak{g})$ by setting the characters of simple modules with l- multiple highest weights equal to their dimensions. This leads to the formula above. $\qquad\square$

By the above, the subalgebra $\widetilde{\mathfrak{Z}} \equiv \mathfrak{J}(\mathfrak{R}_r) \subset \mathfrak{Z}$ is isomorphic to $\mathfrak{R}_r$.

4. The block decomposition of $\widetilde{\mathfrak{Z}}$.

The block decomposition of $\widetilde{\mathfrak{Z}}$ can be obtained using Theorem 4.5 of [BG]. To formulate the result, we need to define the action of the Weyl group on the restricted weight lattice $(P/lP)_+$.

Note that the shifted action of the affine Weyl group $\hat{W}_l$ preserves the $(lP - \rho)$-lattice in P:

$$s_{\alpha,k} \cdot (l\mu - \rho) = s_\alpha(l\mu) + kl\alpha - \rho \in (lP - \rho).$$

Therefore, the shifted action of $\hat{W}_l$ on $(P/lP)_+$ can be defined by

$$w \bullet \lambda = w \cdot \lambda \bmod (lP)$$

for any $\lambda \in (P/lP)_+$, $w \in \hat{W}_l$. By definition, the result of this action belongs again to $(P/lP)_+$. The affine Weyl group $\hat{W}_l$ is isomorphic to $W \ltimes lQ$, where the elements of lQ generate the translations. Since $lQ \subset lP$, it is enough to consider the $\bullet$-action of just the elements $w \in W$ of the finite Weyl group.

To find the fundamental domain of the $W\bullet$ action, consider the affine group $W_l^P = W \ltimes lP$, where the translations are generated by the lP lattice. Let Ω be the subgroup of W_l^P stabilizing the alcove $\bar{X}$ with respect to the shifted action. Recall that with the assumption on the odd number l given by Lemma 3.1, we have $lP \cap Q = lQ$. Then by [Hum], §4.5, $\hat{W}_l \cap \Omega = 1$, so in fact W_l^P is the semidirect product of $\hat{W}_l$ and Ω, and $\Omega \simeq W_l^P/\hat{W}_l$. The order of Ω is equal to $|W_l^P/\hat{W}_l| = |P/Q|$, the index of connection. With the above assumption on l, the order of the open alcove $|X|$ is divisible by $|P/Q|$.

Denote by $\bar{\mathcal{X}}$ the set of representatives of the orbits of the shifted action of Ω in the alcove $\bar{X}$. Then $\bar{\mathcal{X}}$ is the set of representatives of the orbits of the $W\bullet$ action in $(P/lP)_+$. Let $\mathcal{X} \subset \bar{\mathcal{X}}$ denote the subset of regular weights in $\bar{\mathcal{X}}$, such that their stabilizer in W with respect to the $W\bullet$ action is trivial. Then $|\mathcal{X}| = |X|/|P/Q|$.

Comparing to the results in [APW2], one can deduce that the set $\bar{\mathcal{X}}$ enumerates the blocks of the category of finite dimensional modules over $U_q^{fin}(\mathfrak{g})$ (see [La]).

Similarly one can define the natural action of $\hat{W}_l$ on $(P/lP)_+$ by

$$w \circ \lambda = w(\lambda) \bmod (lP)$$

for any $\lambda \in (P/lP)_+$, $w \in \hat{W}_l$. It is easy to see that $W\bullet$ and $W\circ$ have the same orbit structure in $(P/lP)_+$.

We will also need the restriction of the action of $\hat{W}_l$ to the finite root lattice Q/lQ. The natural (non-shifted) action of $\hat{W}_l$ on $Q \subset P$ can be defined by restriction from P, since the action preserves the root lattice:

$$s_{\alpha,k}(\beta) = s_\alpha(\beta) + kl\alpha$$

for any $\alpha, \beta \in Q$, $k \in \mathbb{Z}$. Then the $\circ$-action of $\hat{W}_l$ on Q/lQ is defined by

$$w \circ \beta = w(\beta) \bmod lQ$$

for any $\beta \in Q/lQ$. As before, it is enough to consider the $\circ$- action on Q/lQ of elements of the finite Weyl group W.

Now we can state the following corollary of Theorem 4.5 in [BG].

Theorem 4.1. *Let $\bar{\mathcal{X}}$ denote the set of representatives of the orbits of the $W\bullet$-action in $(P/lP)_+$, and for any $\mu \in \bar{\mathcal{X}}$ let W_μ be the stabilizer subgroup of μ in W for this action. Then*

$$\mathfrak{R} \simeq \oplus_{\mu \in \bar{\mathcal{X}}} \mathbb{C}[P]^{W_\mu} \otimes_{\mathbb{C}[P]^W} \mathbb{C},$$

where $\mathbb{C}[P]^W \longrightarrow \mathbb{C}$ *is evaluation at* 1.

Proof. Since R is assumed to be simply laced, we will always identify the root lattice Q with $\check{Q}$, the $\mathbb{Z}$-span of the coroots $\check{\alpha}, \alpha \in R$. Denote by H^* the real vector space spanned by the simple roots, and let H be its dual space with the basis $\{h_i\}_{i=1}^r$ such that $\alpha_i(h) = \kappa(h_i, h)$ for any $h \in H$. Here $\kappa(\cdot, \cdot)$ is the restriction of the Killing form in $\mathfrak{g}$. Then Q^*, the $\mathbb{Z}$-span of $\{h_i\}_{i=1}^r$, is a lattice in H dual to Q. Let $T = \{e^{2\pi i h}, h \in H \bmod Q^*\}$ denote the maximal torus of the simply connected group G with $Lie(G) = \mathfrak{g}$. Then the elements of $\mathbb{C}[P]$ can be considered as characters on T by setting $e^\mu(e^{2\pi i h}) = e^{2\pi i \mu(h)} \in \mathbb{C}$, for any $t = e^{2\pi i h} \in T$.

Recall that $\mathfrak{R} \simeq \mathbb{C}[P]^W \otimes_{\mathbb{C}[lP]^W} \mathbb{C}$. Then we are in the setting of Theorem 4.5 in [BG] (in our case $\chi_s = 1$ in the notations of [BG]), and we have

$$\mathfrak{R} \simeq \oplus_{\{t \in T : t^l = 1\}/W} \mathbb{C}[P]^{W(t)} \otimes_{\mathbb{C}[P]^W} \mathbb{C},$$

where the map $\mathbb{C}[P]^W \to \mathbb{C}$ is given by the evaluation at 1. The notation $\{t \in T : t^l = 1\}/W$ stands for the set of representatives of the W orbits in $\{t \in T : t^l = 1\}$, which has a natural W action. The subgroup $W(t) \subset W$ is generated by the reflections $\{s_\alpha, \alpha \in R \,|\, e^\alpha(t) = 1\}$.

We can rewrite the result parametrizing the blocks by the $W\circ$-orbits in the restricted root lattice. Indeed, the set $\{t \in T : t^l = 1\}$ contains all $t = e^{2\pi i h}$ where $lh \in Q^*$. Therefore, it is parametrized by $h \in Q^*/lQ^*$, which under duality corresponds to $\beta \in Q/lQ$, with the orbits of the W-action in $\{t \in T : t^l = 1\}$ translated to the orbits of the $W\circ$-action in Q/lQ. Thus the maximal ideals of $\mathfrak{R}$ can be parametrized by $(Q/lQ)/W\circ$, the representatives of $W\circ$-orbits in the restricted root lattice. The stabilizer subgroup $W(t)$ is generated by those $s_\alpha, \alpha \in R$ such that $e^\alpha(t) = e^{2\pi i \alpha(h)} = 1$, or $\alpha(h) \in \mathbb{Z}$ for $h \in Q^*/lQ^*$. In the dual picture this corresponds to those $s_\alpha, \alpha \in R$, such that $\langle \alpha, \beta \rangle \equiv 0 (mod l)$ for $\beta \in Q/lQ$. These are exactly the reflections $s_\alpha \in W$ that stabilize the element $\beta \in Q/lQ$ with respect to the $W\circ$-action. Denote by $W_\beta \subset W$ the subgroup generated by such reflections for a fixed $\beta \in (Q/lQ)/W\circ$. Then we get

$$\mathfrak{R} \simeq \otimes_{\beta \in (Q/lQ)/W\circ} \mathbb{C}[P]^{W_\beta} \otimes_{\mathbb{C}[P]^W} \mathbb{C}.$$

However, if we want to consider the subalgebra $\widetilde{\mathfrak{Z}} \subset \mathfrak{Z}$, which is isomorphic to $\mathfrak{R}$, then it will be more natural to parametrize its blocks by weights in $\bar{\mathcal{X}}$. This agrees with the decomposition of $U_q^{fin}(\mathfrak{g})$ as a module over itself, where the block corresponding to an element $\mu \in \bar{\mathcal{X}}$ contains only the composition factors of highest weights in the $W\bullet$-orbit of μ. The set $(P/lP)_+$ under the $W\bullet$-action has the same orbit structure as Q/lQ under the $W\circ$-action. Indeed, with the conditions on l formulated in Lemma 3.1, for any $\beta \in Q/lQ$ there exists a unique $\lambda \in (P/lP)_+$ such that $\langle \beta, \alpha \rangle = \langle \lambda, \alpha \rangle \bmod(l), \alpha \in \Pi$. This provides a one-to-one correspondence between the orbits of the $W\circ$ actions in Q/lQ and $(P/lP)_+$. Clearly the $W\bullet$-action on $(P/lP)_+$ has the same orbit structure. Therefore we can translate the parametrization by $(Q/lQ)/W\circ$ to $(P/lP)/W\bullet \simeq \bar{\mathcal{X}}$. In this interpretation, the subgroup W_μ is generated by the reflections in W (not necessarily simple) which stabilize the weight $\mu \in \bar{\mathcal{X}}$ with respect to the $W\bullet$-action in $(P/lP)_+$, and we obtain the statement of the theorem. $\square$

Remark. Using the isomorphism $\mathfrak{J} : \mathfrak{R}_r \to \widetilde{\mathfrak{Z}}$ and Theorem 4.1, we can write $\widetilde{\mathfrak{Z}} \simeq \oplus_{\mu \in \bar{\mathcal{X}}} \widetilde{\mathfrak{Z}}_\mu$, where $\widetilde{\mathfrak{Z}}_\mu \equiv \mathbb{C}[P]^{W_\mu} \otimes_{\mathbb{C}[P]^W} \mathbb{C}$ for any $\mu \in \bar{\mathcal{X}}$. Each block $\widetilde{\mathfrak{Z}}_\mu$ is a local

algebra by §4.3 in [BG]. It is also Frobenius, since $\oplus_{\mu \in \bar{\mathcal{X}}} \widetilde{\mathfrak{Z}}_\mu$ can be realized as the center of a matrix algebra $\oplus_{\mu \in \bar{\mathcal{X}}} \mathrm{Mat}_{l^{|R_+|}}(\widetilde{\mathfrak{Z}}_\mu)$ [BG]. Therefore, for each $\mu \in \bar{\mathcal{X}}$, the socle of $\widetilde{\mathfrak{Z}}_\mu$ is one-dimensional.

Proposition 4.2. *The elements of $\widetilde{\mathfrak{Z}}$ distinguish the blocks of $U_q^{fin}(\mathfrak{g})$. In other words, the system $\{1_\mu \in \widetilde{\mathfrak{Z}}_\mu\}_{\mu \in \bar{\mathcal{X}}}$ of idempotents in the algebra $\widetilde{\mathfrak{Z}}$ is a complete set of central simple idempotents in $U_q^{fin}(\mathfrak{g})$, and $\sum_{\mu \in \bar{\mathcal{X}}} 1_\mu = 1 \in U_q^{fin}(\mathfrak{g})$.*

Proof. The algebra of functionals C_r is mapped homomorphically onto $\mathfrak{Z}$ by the isomorphism $\mathfrak{J}$. A functional in $C_r \setminus \mathfrak{R}_r$ is not a trace, and hence is zero when evaluated on any element $K_\mu \in U^0$. Any such functional is mapped to a nilpotent central element by $\mathfrak{J}$. Therefore, since $\mathfrak{J}$ is a bijection, all idempotents in $\mathfrak{Z}$ have their pre-images in $\mathfrak{R}_r$, or equivalently all central idempotents belong to the subalgebra $\widetilde{\mathfrak{Z}} \subset \mathfrak{Z}$.

As a module over itself by left multiplication, $U_q^{fin}(\mathfrak{g})$ decomposes into a direct sum of two-sided ideals $U_q^{fin}(\mathfrak{g}) \simeq \oplus_{\mu \in \bar{\mathcal{X}}} A_\mu$, corresponding to the blocks of the category of its finite dimensional modules. This induces a decomposition of the center $\mathfrak{Z} = \oplus_{\mu \in \bar{\mathcal{X}}} \mathfrak{Z}_\mu$. By Theorem 4.1, $\widetilde{\mathfrak{Z}} = \oplus_{\mu \in \bar{\mathcal{X}}} \widetilde{\mathfrak{Z}}_\mu$ with each $\widetilde{\mathfrak{Z}}_\mu \subset \mathfrak{Z}_\mu$ being a local algebra with the idempotent 1_μ. Since all idempotents are in $\widetilde{\mathfrak{Z}}$, the element 1_μ is the idempotent in $\mathfrak{Z}_\mu$, and $\oplus_{\mu \in \bar{\mathcal{X}}} 1_\mu = 1 \in \mathfrak{Z}$. In particular, each $\mathfrak{Z}_\mu$ is a local algebra, $\mathfrak{Z}_\mu = 1_\mu + \mathrm{Rad}(\mathfrak{Z}_\mu)$. $\qquad\square$

5. Intersection and sum of two central subalgebras.

Since $\mathfrak{Z}'$ is an ideal of $\mathfrak{Z}$, the sum $\widetilde{\mathfrak{Z}} + \mathfrak{Z}'$ is a subalgebra in the center. To describe its multiplicative structure, we need to find the intersection $\mathfrak{Z}' \cap \widetilde{\mathfrak{Z}}$. This requires a combination of the two mappings used to define $\mathfrak{Z}'$ and $\widetilde{\mathfrak{Z}}$.

Define the bijective mappings $\mathcal{F} : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ and $\mathcal{F}' : C_r \longrightarrow C_r$ by

$$\mathcal{F}(a) \equiv \mathfrak{J} \circ \phi(a) = m((\lambda_r \hookleftarrow S(a)) \otimes \mathrm{id})(R_{21}R_{12})),$$

$$\mathcal{F}'(f) \equiv \phi \circ \mathfrak{J}(f) = \lambda_r \hookleftarrow S(m((f \otimes \mathrm{id})(R_{21}R_{12}))).$$

For the involutive properties discussed below, $\mathcal{F}$ and $\mathcal{F}'$ will be called the quantum Fourier transforms. This differs slightly from the original definition in [LM], where the Fourier transforms are defined on $U_q^{fin}(\mathfrak{g})$ and $U_q^{fin}(\mathfrak{g})^*$ respectively, but we will need here only their restrictions to $\mathfrak{Z}$ and C_r.

Theorem 5.1. 1. *For any $a \in \mathfrak{Z}$, $\mathcal{F}^2(a) = S^{-1}(a)$;*
2. *Let η denote the isomorphism of commutative algebras $\eta : C_0 \to C_r$, $\eta(f) = f \hookleftarrow K_{2\rho}$, where $f(\hookleftarrow K_{2\rho})(x) = f(K_{2\rho}x)$ for any $x \in U_q^{fin}(\mathfrak{g})$. Then for any $f \in C_0$, we have $\eta^{-1} \circ \mathcal{F}'^2 \circ \eta(f) = f \circ S$.*

Proof. (1) is equivalent to the analogous statement in [LM]:
Set $\mathfrak{S}_-(a) \equiv m(\mathrm{id} \otimes \lambda_l)(R_{12}^{-1}(1 \otimes a)R_{21}^{-1})$, $a \in \mathfrak{Z}$, then $\mathfrak{S}_-^2 = S^{-1}$.
Let $R = \sum a_i \otimes b_i$. For a central element a write

$$\mathcal{F}(a) = m(\lambda_r \otimes \mathrm{id})(S(a) \otimes 1)(R_{21}R_{12}) =$$

$$= m(\lambda_r \otimes \mathrm{id})(S \otimes \mathrm{id})((S^{-1} \otimes \mathrm{id})(\sum b_i a_j \otimes a_i b_j)(a \otimes 1)) =$$

$$= m(\lambda_r \circ S \otimes \mathrm{id})((\sum S^{-1}(a_j)S^{-1}(b_i) \otimes a_i b_j)(a \otimes 1)) =$$

$$= m(\lambda_l \otimes \mathrm{id})((\sum S^{-1}(b_i)S(a_j) \otimes a_i b_j)(a \otimes 1)) =$$

$$= m(\lambda_l \otimes \mathrm{id})((R_{21}^{-1}R_{12}^{-1})(a \otimes 1)) = m(\mathrm{id} \otimes \lambda_l)((R_{12}^{-1}R_{21}^{-1})(1 \otimes a)) = \mathfrak{S}_-(a).$$

Here we have used the fact that $\lambda_r \circ S$ is a left integral, the property of left-invariant functionals $\lambda_l(xy) = \lambda_l(yS^2(x))$ and the identities $(S \otimes \mathrm{id})(R) = (\mathrm{id} \otimes S^{-1})(R) = R^{-1}$ [Dr].

(2) follows from (1). For $f_r \in C_r$ write

$$\mathcal{F}'^2(f_r) = \mathfrak{J}^{-1} \circ \mathcal{F}^2 \circ \mathfrak{J}(f_r) = \mathfrak{J}^{-1} \circ S^{-1} \circ \mathfrak{J}(f_r) =$$

$$= \mathfrak{J}^{-1}\{m(f_r \otimes S^{-1})(R_{21}R_{12})\} = \mathfrak{J}^{-1}\{m(f_r \circ S \otimes \mathrm{id})(S^{-1} \otimes S^{-1})(R_{21}R_{12})\} =$$

$$= \mathfrak{J}^{-1}\{m(f_r \circ S \otimes \mathrm{id})(R_{12}R_{21})\} = \mathfrak{J}^{-1}\{m(\mathrm{id} \otimes f_r \circ S)(R_{21}R_{12})\},$$

where we have used $(S^{-1} \otimes S^{-1})(R) = R$. For $f_r \in C_r$, $f_r \circ S \in C_l$. Consider the algebra isomorphism from C_l to C_r, given by the action of $K_{4\rho}$. Then by [Dr] for any $g \in C_l$, $(\mathrm{id} \otimes g)(R_{21}R_{12}) = ((g \leftharpoonup K_{4\rho}) \otimes \mathrm{id})(R_{21}R_{12})$. Therefore we have

$$\mathcal{F}'^2(f) = \mathfrak{J}^{-1}\{m((f_r \circ S) \leftharpoonup K_{4\rho}) \otimes \mathrm{id})(R_{21}R_{12}) = (f_r \circ S) \leftharpoonup K_{4\rho}.$$

Now let $f_r = \eta(f)$, $f \in C_0$. Then

$$\eta^{-1} \circ \mathcal{F}'^2 \circ \eta(f) = ((f \leftharpoonup K_{2\rho}) \circ S) \leftharpoonup K_{2\rho} = f \circ S,$$

as required.   □

In particular, Theorem 5.1 implies that the square of the Fourier transform, conjugated by η, maps the character of a simple $U_q^{fin}(\mathfrak{g})$-module to the character of its dual:

$$(\eta^{-1} \circ \mathcal{F}'^2 \circ \eta)(\mathrm{ch}L(\mu)) = \mathrm{ch}(L(\mu)^*).$$

Theorem 5.2.

$$\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}' = \mathrm{Ann}\,(\mathrm{Rad}\,\widetilde{\mathfrak{Z}}).$$

The proof is based on the following two lemmas

Lemma 5.3. *The character of the Steinberg module $St = L((l-1)\rho)$ annihilates the radical of $\mathfrak{R}$.*

Proof. Consider the block decomposition of the center $\mathfrak{Z} = \oplus_{\mu \in \bar{\mathcal{X}}}\mathfrak{Z}_\mu$. The block $\mathfrak{Z}_{(l-1)\rho} = \mathfrak{Z}_{St}$ consists of elements which act nontrivially only on indecomposable modules with highest weights in the •-orbit of $(l-1)\rho$. Since this weight is stabilized by the • action, and the Steinberg module is simple, projective and injective in the category of finite dimensional modules over $U_q^{fin}(\mathfrak{g})$ [APW2], the block $\mathfrak{Z}_{St}$ is one-dimensional. By the structure of $\mathfrak{Z}'$ (Proposition 2.3) and $\widetilde{\mathfrak{Z}}$ (Theorem 4.1), we have $\widetilde{\mathfrak{Z}}_{St} = \mathfrak{Z}'_{St} = \mathfrak{Z}_{St}$. Let $z_{St} = \phi^{-1}(ch_{q^{-1}}St) \in \mathfrak{Z}_{St}$. Then $z_{St} \in \widetilde{\mathfrak{Z}} \cap \mathfrak{Z}'$.

Write the sequence of mappings

$$\mathfrak{R}_r \xrightarrow{\mathfrak{J}} \widetilde{\mathfrak{Z}} \xrightarrow{\phi} \phi(\widetilde{\mathfrak{Z}}) \xrightarrow{\mathfrak{J}} \mathfrak{J} \circ \phi(\widetilde{\mathfrak{Z}}) \xrightarrow{\phi} (\phi \circ \mathfrak{J})^2(\mathfrak{R}_r) = \mathfrak{R}_r.$$

The first arrow is the definition of $\mathfrak{J}$, and the last equality follows by Theorem 5.1, since $\mathcal{F}' = \phi \circ \mathfrak{J}$. Then we have $\phi^{-1}(\mathcal{F}'^2(\mathfrak{R}_r)) = \phi^{-1}(\mathfrak{R}_r) = \mathfrak{Z}'$ by the definition of ϕ, and hence $\mathfrak{J} \circ \phi(\widetilde{\mathfrak{Z}}) = \mathcal{F}(\widetilde{\mathfrak{Z}}) = \mathfrak{Z}'$. Also, $\mathcal{F}(\mathfrak{Z}') = \mathfrak{J} \circ \phi(\mathfrak{Z}') = \mathfrak{J}(\mathfrak{R}_r) = \widetilde{\mathfrak{Z}}$.

Now since $z_{St} \in \widetilde{\mathfrak{Z}} \cap \mathfrak{Z}'$, we have $\mathcal{F}(z_{St}) \in \mathcal{F}(\widetilde{\mathfrak{Z}}) \cap \mathcal{F}(\mathfrak{Z}') = \mathfrak{Z}' \cap \widetilde{\mathfrak{Z}}$. The ideal $\mathfrak{Z}'$ annihilates the radical of $\mathfrak{Z}$ by Proposition 2.3, and therefore its intersection with $\widetilde{\mathfrak{Z}}$ belongs to the annihilator of the radical of $\widetilde{\mathfrak{Z}}$. We obtain $\mathcal{F}(z_{St}) = \mathfrak{J} \circ \phi(z_{St}) \in \mathrm{Ann}(\mathrm{Rad}\,\widetilde{\mathfrak{Z}})$. The map $\mathfrak{J}$ is an algebra isomorphism between $\mathfrak{R}_r$ and $\widetilde{\mathfrak{Z}}$, and

therefore $\phi(z_{St}) \in \mathrm{Ann}\,(\mathrm{Rad}\mathfrak{R}_r)$. By the definition of z_{St}, $ch_{q^{-1}}St = \phi(z_{St})$, and hence this element annihilates the radical of $\mathfrak{R}_r$, or equivalently, $chSt$ annihilates the radical of $\mathfrak{R}$. Here we used the isomorphism of algebras $\mathfrak{R} \to \mathfrak{R}_r$, given by the action of $K_{-2\rho}$ (§2). $\qquad\square$

Lemma 5.4. *The square of the Steinberg character spans the square of the annihilator of the radical of $\mathfrak{R}$.*

Proof. Recall that $\mathfrak{R}$ is a direct sum of local Frobenius algebras. Its socle (annihilator of the radical) is spanned by the socles of each of the $|\bar{\mathcal{X}}|$ blocks. Only one of these blocks is semisimple, the one corresponding to the orbit of the Steinberg weight $(l-1)\rho$. The socle of this block is an idempotent, while the socles of all other blocks are nilpotent of second degree. Therefore, the square of the socle of $\mathfrak{R}$ is one-dimensional. Since $chSt \in \mathrm{Soc}(\mathfrak{R})$ by Lemma 5.3, $(chSt)^2 \in (\mathrm{Soc}(\mathfrak{R}))^2$ and it is nonzero as a character of a direct sum of modules. Therefore, $(chSt)^2$ spans $(\mathrm{Soc}(\mathfrak{R}))^2$. $\qquad\square$

Proof. of the Theorem.

Obviously $\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}' \subset \mathrm{Ann}\,(\mathrm{Rad}\,\widetilde{\mathfrak{Z}}) \equiv \mathrm{Soc}\,\widetilde{\mathfrak{Z}}$. We will show that $\mathrm{Ann}\,(\mathrm{Rad}\,\widetilde{\mathfrak{Z}}) \subset \mathfrak{Z}'$.

For each $\mu \in \bar{\mathcal{X}}$, denote by z_μ an element of the one- dimensional socle of the block $\widetilde{\mathfrak{Z}}_\mu$. We want to show that each z_μ belongs to $\mathfrak{Z}'$, or equivalently that $\phi(z_\mu)$ is in $\mathfrak{R}_r$. We will use the lemmas above, recalling that $\mathfrak{R} \simeq \mathfrak{R}_r$, the isomorphism given by the action of $K_{2\rho}$ which maps characters of $U_q^{fin}(\mathfrak{g})$-modules to q^{-1}-characters.

As before let $z_{(l-1)\rho} \equiv z_{St} = \phi^{-1}(ch_{q^{-1}}St)$. Then the element z_{St} spans the square of $\mathrm{Soc}\,\widetilde{\mathfrak{Z}}$, and therefore, by Lemma 5.4 the isomorphism $\mathfrak{J}$ maps $ch_{q^{-1}}St^2$ to the subspace spanned by z_{St}. By Lemma 5.3, $ch_{q^{-1}}St$ is mapped by $\mathfrak{J}$ to an element $\sum_{\nu \in \bar{\mathcal{X}}} a_\nu z_\nu \in \mathrm{Soc}\,\widetilde{\mathfrak{Z}}$ for some coefficients $a_\nu \in \mathbb{C}$. Then up to nonzero scalar multiples one can write the following sequence of mappings:

$$ch_{q^{-1}}St^2 \xrightarrow{\mathfrak{J}} z_{St} \xrightarrow{\phi} ch_{q^{-1}}St \xrightarrow{\mathfrak{J}}$$

$$\xrightarrow{\mathfrak{J}} \sum_{\nu \in \bar{\mathcal{X}}} a_\nu z_\nu \xrightarrow{\phi} \sum_{\nu \in \bar{\mathcal{X}}} a_\nu p(\nu) = ch_{q^{-1}}St^2.$$

Here set $p(\nu) \equiv \phi(z_\nu)$. The last equality is a consequence of Theorem 5.1 since $chSt$ and its square are invariant under the antipode. Since the square of $\mathrm{Soc}(\mathfrak{R}_r)$ is one-dimensional, $ch_{q^{-1}}St^2$ coincides up to a scalar multiple with the product $ch_{q^{-1}}St^2 \cdot f$ for any $f \in \mathfrak{R}_r$. Therefore by weight considerations, $ch_{q^{-1}}St^2$ decomposes into a sum containing elements in all blocks of the category, and hence all a_ν in $\sum_{\nu \in \bar{\mathcal{X}}} a_\nu p(\nu) = ch_{q^{-1}}St^2$ are nonzero. Act on both sides of the last equality by an idempotent of one block 1_μ. Then we get $a_\mu p_\mu = pr_\mu(ch_{q^{-1}}St)^2$, the projection of the q^{-1}-character of St^2 to the block $(\mathfrak{R}_r)_\mu$, which is spanned by the q^{-1}-characters of simple modules with highest weights in the $W\bullet$ orbit of μ. This means that each $p(\mu)$ belongs to $\mathfrak{R}_r$. This completes the proof of the Theorem. $\quad\square$

Remark. The subalgebra $\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}'$ is an ideal in $\mathfrak{Z}$ closed under the quantum Fourier transform, which suggests that it has an interesting representation theoretical interpretation. In fact, one can show that it is isomorphic to the ideal of characters of the subcategory of projective modules over $U_q^{fin}(\mathfrak{g})$. The properties of $\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}'$ are studied in detail in [La].

Now we combine the results of §2, §4 and §5 to formulate the main theorem on the structure of a subalgebra in the center of $U_q^{fin}(\mathfrak{g})$:

Theorem 5.5. *The center $\mathfrak{Z}$ of the small quantum group $U_q^{fin}(\mathfrak{g})$ contains a subalgebra $\widetilde{\mathfrak{Z}} + \mathfrak{Z}'$, which is invariant under the quantum Fourier transform, of dimension*

$$\dim(\widetilde{\mathfrak{Z}} + \mathfrak{Z}') = \sum_{\mu \in \bar{\mathcal{X}}} (2[W : W_\mu] - 1) = 2l^r - |\bar{\mathcal{X}}|,$$

where $\bar{\mathcal{X}} \subset (P/lP)_+$ enumerates the blocks of the category of finite dimensional $U_q^{fin}(\mathfrak{g})$-modules. The intersection $\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}'$ is an ideal in $\mathfrak{Z}$ isomorphic to $\mathrm{Soc}\,\widetilde{\mathfrak{Z}}$. It contains one element in each block of $\mathfrak{Z}$, has dimension $\dim(\widetilde{\mathfrak{Z}} \cap \mathfrak{Z}') = |\bar{\mathcal{X}}|$, and is also invariant under the quantum Fourier transform. The ideal $\mathfrak{Z}' \subset \mathfrak{Z}$ is isomorphic to $\mathrm{Soc}\,\mathfrak{Z}$. The subalgebra $\widetilde{\mathfrak{Z}} \subset \mathfrak{Z}$ is isomorphic to the algebra $\mathfrak{R}$ of characters of finite dimensional $U_q^{fin}(\mathfrak{g})$-modules. The dimensions of $\widetilde{\mathfrak{Z}}$ and $\mathfrak{Z}'$ coincide and are equal to $\sum_{\mu \in \bar{\mathcal{X}}}[W : W_\mu] = l^r$.

Remark. The constructed subalgebra $\widetilde{\mathfrak{Z}} + \mathfrak{Z}'$ provides a supply of central elements, while the mappings $\mathcal{F}$, ϕ can serve as tools to describe them. For example, it is clear that the two-sided integral Λ of $U_q^{fin}(\mathfrak{g})$, which is not an element of $\widetilde{\mathfrak{Z}}$, is the Fourier transform of the unit,

$$\Lambda = \mathcal{F}(1).$$

Another possible application originates from the theory of quantum topological invariants of 3-manifolds [RT], [Hen] [Lyu], [Ker2]. It is known that a quantum group at a root of unity gives rise to a family of quantum surgical invariants, computed by assigning an element of the algebra to each component of the knot, using the quasitriangular structure to unknot the interlacings, and then evaluating the result against a functional in the dual quantum group. The invariant of the Reshetikhin-Turaev type is generated by a functional μ_{RT} which is a linear combination of q^{-1}-characters of certain simple modules [RT]. At the same time, the right integral λ_r of a quazitriangular finite dimensional Hopf algebra yields the Kauffman-Radford invariant [KR]. It was pointed out in [Hen] that all surgical invariants associated to a finite dimensional quantum group can be obtained from the Kauffman-Radford invariant by the left action of certain central elements, in particular $\mu_{RT} = z_{RT} \rightharpoonup \lambda_r$. In our case it means that $z_{RT} = \phi^{-1}(\mu_{RT})$, where $\mu_{RT} \in \mathfrak{R}_r$. Then Proposition 2.3 implies that $z_{RT} \in \mathfrak{Z}' = \mathrm{Soc}\,\mathfrak{Z}$ and z_{RT} annihilates the radical of $\mathfrak{Z}$ (cf. in [Ker2]: $z_{RT}^2 = 0$). A more careful consideration based on the properties of μ_{RT} shows that z_{RT} does not belong to the obvious subalgebra $\widetilde{\mathfrak{Z}}$ of the center. Although we consider here the case $U_q^{fin}(\mathfrak{g})$, the above arguments can be carried out in a more general situation, e.g. for a finite dimensional quantum group specialized at an even root of unity, which was the original setting for the Reshetikhin-Turaev invariant. One can deduce that the two invariants are always related by the left action of an element which is not contained in the obvious part of the center and annihilates its radical.

6. The action of the center in projective modules

In this section we derive a consequence of Theorem 5.2, which describes the action of certain central elements by the endomorphisms of projective modules. We will use freely the known facts from the representation theory of $U_q^{res}(\mathfrak{g})$ and

$U_q^{fin}(\mathfrak{g})$ in the proof below; a detailed exposition can be found in [Lus2], [And], [APW1], [APW2].

Theorem 6.1. *For an indecomposable projective $U_q^{fin}(\mathfrak{g})$- module $P((l-1)\rho+\mu)$, which is a restriction of the projective $U_q^{res}(\mathfrak{g})$-module of highest weight $(l-1)\rho+\mu$ with $\mu \in \tilde{X} = \{\nu \in P_+ : \langle \nu, \alpha \rangle < l \text{ for all } \alpha \in R_+\}$, there exists a unique surjective homomorphism from $\tilde{\mathfrak{Z}}$ to $\mathrm{End}_{U_q^{fin}} P((l-1)\rho+\mu)$, which is an isomorphism on the block $\tilde{\mathfrak{Z}}_\mu$, and zero on all other blocks.*

Proof. The restriction of $P((l-1)\rho+\mu)$ to $U_q^{fin}(\mathfrak{g})$ is a projective indecomposable module and the projective cover of $L((l-1)\rho+w_0(\mu))$ [APW2]. By the definition of the block $\tilde{\mathfrak{Z}}_\mu$, its elements act as endomorphisms of a projective module with composition factors in the corresponding $W\bullet$-orbit of μ, while elements of the other blocks $\mathfrak{Z}_{\nu \neq \mu}$ act on it by zero. Let $z_\mu \in \mathrm{Soc}\,\tilde{\mathfrak{Z}}_\mu$. By Lemma 5.3, $chSt \cdot \mathfrak{R} \subset$ Ann(Rad $\mathfrak{R}$). For any $\nu \in (P/lP)_+$, it is possible to find $f \in \mathfrak{R}$ such that $chSt \cdot f$ contains $chL(\nu)$ with a nonzero coefficient in its decomposition with respect to the basis of simple characters. Therefore, any character of a simple module appears in some combination in the linear span of Ann(Rad $\mathfrak{R}$). This means that z_μ acts nontrivially in each projective cover of a simple module with the highest weight in the $W\bullet$-orbit of μ, mapping it to its simple socle.

Since $\tilde{\mathfrak{Z}}$ is Frobenius, for any $z_1 \in \tilde{\mathfrak{Z}}_\mu$ there exists $z_2 \in \tilde{\mathfrak{Z}}_\mu$ such that $z_1 z_2 = z_\mu$. Then the kernel of the mapping of $\tilde{\mathfrak{Z}}_\mu$ to $\mathrm{End}_{U_q^{fin}(\mathfrak{g})} P((l-1)\rho+\mu)$ is trivial. Now compare the dimensions.

First note that $\mathrm{End}_{U_q^{res}} P((l-1)\rho+\mu) = [W : W_\mu]$, where W_μ is the stabilizer of μ with respect to the $W\bullet$ action. This follows from the structure of filtrations of projective modules over $U_q^{res}(\mathfrak{g})$. Each projective module has a filtration by Weyl modules $W(\nu)$, which are the universal highest weight modules for $U_q^{res}(\mathfrak{g})$ [APW1]. Then we have the reciprocity relation for the projective cover of a simple module $[P((l-1)\rho+\mu) : W(w \cdot \mu)] = [W(w \cdot \mu) : L((l-1)\rho+w_0(\mu))]$ ([AJS], §4.15). The Weyl filtration for a projective module in the $\tilde{X} + (l-1)\rho$ alcove has the following form:

$$chP((l-1)\rho+\mu) = \sum_{w \in W/W_\mu} chW((l-1)\rho+w(\mu)).$$

The last formula of course can be deduced from the general result on the characters of tilting modules in [Soe2] (since all projective modules over $U_q^{res}(\mathfrak{g})$ are tilting, [APW2]). It also follows by direct computation, using the fact that any projective module appears as a direct summand of a tensor product $St \otimes E$ for some finite dimensional module E ([APW1]).

We need to check that there are no extra endomorphisms of the restriction of $P((l-1)\rho+\mu)$ over $U_q^{fin}(\mathfrak{g})$. Suppose there are additional composition factors in $P((l-1)\rho+\mu)$ isomorphic to its maximal semisimple quotient $L((l-1)\rho+w_0(\mu))$. Then by the tensor product decomposition for simple modules ([Lus1]), a simple $U_q^{res}(\mathfrak{g})$-module $L((l-1)\rho+w_0(\mu)+l\nu)$ has to appear in the filtration of $P((l-1)\rho+\mu)$ for some $\nu \in P_+$. By the strong linkage principle (Thm. 8.1, [APW1]), this means that the weight $(l-1)\rho+w_0(\mu)+l\nu$ should be less than or equal to at least one of the weights $\{(l-1)\rho+w(\mu)\}_{w \in W}$ of the Weyl composition factors in $P((l-1)\rho+\mu)$, and belong to the same $\hat{W}_l\cdot$-orbit. For $\mu = \rho$, we should have then

$$(l-1)\rho+w(\rho) - ((l-1)\rho-\rho+l\nu) = w(\rho)+\rho-l\nu \in P_+,$$

which is false for any $w \in W$ and any dominant $\nu \neq 0$. Therefore, there are no composition factors of the form $L((l-1)\rho + w_0(\rho))$ in any of the $W((l-1)\rho + w(\rho))$. The statement for other weights $\mu \in \tilde{X}$ such that $(l-1)\rho + \mu$ is regular, is derived using the translation principle [APW1] which states that the structure of the filtration of a Weyl module is the same for all highest weights in the same alcove. The proof for the lowest walls of $(l-1)\rho + \tilde{X}$ is obtained similarly, replacing ρ by a suitable fundamental weight ω_i in the argument above. This means that $\dim(\mathrm{End}_{U_q^{fin}} P((l-1)\rho + \mu)) = \dim(\mathrm{End}_{U_q^{res}} P((l-1)\rho + \mu))$ for any $\mu \in \tilde{X}$.

Therefore, we get

$$\dim(\mathrm{End}_{U_q^{fin}} P((l-1)\rho + \mu)) = [W : W_\mu] = \dim \widetilde{\mathfrak{Z}}_\mu,$$

and the two algebras are isomorphic. $\square$

Corollary 6.2. *For $\mathfrak{g} = \mathfrak{sl}_2$, the subalgebra $\widetilde{\mathfrak{Z}} + \mathfrak{Z}'$ coincides with the whole center of $U_q^{fin}(\mathfrak{g})$.*

Proof. We know that the subalgebra $\widetilde{\mathfrak{Z}}$ contains all idempotents of $\mathfrak{Z}$ (Proposition 4.2). We will estimate from above the dimension of the radical of the center, computing the number of inequivalent nilpotent edomorphisms of all projective modules.

Fix $\mu \in (P/lP)_+$. If $\mu = (l-1)\rho$, then it is fixed by the $W\bullet$ action and the corresponding block contains the single projective module St which is simple and has no nilpotent endomorphisms. Suppose that μ is regular, then its $W\bullet$ orbit contains two weights. Denote the corresponding indecomposable projective modules by P_1 and P_2. In the case $\mathfrak{g} = \mathfrak{sl}_2$, all indecomposable projective modules have highest weights in $(l-1)\rho + \tilde{X}$. Therefore, they all satisfy the condition of Theorem 6.1. Then since $\dim \widetilde{\mathfrak{Z}}_\mu = 2$ for any regular μ by Theorem 4.1, we have $\dim(\mathrm{End} P_1) = \dim(\mathrm{End} P_2) = 2$, and each algebra of endomorphisms contains a nilpotent element and an identity. Therefore, the center of $\mathrm{End}(P_1 \oplus P_2)$ contains at most two nilpotent elements. On the other hand, the dimension of the regular block $\mathfrak{Z}'_\mu$ of $\mathfrak{Z}'$ is equal to $|W| = 2$, which means that all nilpotent central elements are contained in $\mathfrak{Z}'$, and hence $\mathfrak{Z} = \widetilde{\mathfrak{Z}} + \mathfrak{Z}'$. $\square$

Remark. The above argument fails in higher rank. By ([AJS], §19) the restriction to $U_q^{fin}(\mathfrak{g})$ of a projective $U_q^{res}(\mathfrak{g})$-module with a regular highest weight outside of $(l-1)\rho + \tilde{X}$, has more than $|W|$ linearly independent endomorphisms.

7. EXAMPLES.

7.1. **The case $\mathfrak{g} = \mathfrak{sl}_2$.** The center $\mathfrak{Z}$ decomposes as a sum of ideals parametrized by the set $\bar{\mathcal{X}} = \{0, \ldots (\frac{l-3}{2}), (l-1)\}$. A regular orbit contains two weights $\{i, (l-2-i)\}_{i=0}^{\frac{l-3}{2}}$. The only singular orbit contains the Steinberg weight $(l-1)$. Therefore,

$$\mathfrak{Z} = \oplus_{i=0}^{\frac{l-3}{2}} \mathfrak{Z}_i \oplus \mathfrak{Z}_{l-1}.$$

The subalgebra $\widetilde{\mathfrak{Z}} = \mathfrak{J}(\mathfrak{R}_r) \subset \mathfrak{Z}$, is isomorphic to the algebra $\mathfrak{R}$,

$$\mathfrak{R} \simeq \mathbb{C}[x]^W \otimes_{\mathbb{C}[x^l]^W} \mathbb{C} \simeq \mathbb{C}[x + x^{-1}]/\langle x^l + x^{-l} - 2 \rangle,$$

which is spanned by the characters of simple modules $chL(i) \equiv \xi(i)$ for $i = 0, \ldots, (l-1)$.

Theorem 4.1 specializes to

$$\widetilde{\mathfrak{Z}} = \oplus_{i=0}^{\frac{l-3}{2}} \widetilde{\mathfrak{Z}}_i \oplus \widetilde{\mathfrak{Z}}_{l-1} = \oplus_{i=0}^{\frac{l-3}{2}} \mathbb{C}[x] \otimes_{\mathbb{C}[x+x^{-1}]} \mathbb{C} \oplus \mathbb{C}.$$

This means that each regular block $\mathfrak{Z}_i$ contains a two-dimensional subalgebra isomorphic to $\mathbb{C}[(x-1)]/\langle(x-1)^2\rangle$, and $\mathfrak{Z}_{l-1} = \widetilde{\mathfrak{Z}}_{l-1} = \mathbb{C}$.

By Proposition 2.3, the socle of $\mathfrak{Z}$ coincides with the inverse image of the Grothendieck ring under the isomorphism ϕ, which preserves the block structure. Therefore, each regular $\mathfrak{Z}_i$ contains two linearly independent elements of the socle, corresponding to the two simple characters in the orbit: $\phi^{-1}(\xi(i))$ and $\phi^{-1}(\xi(l-2-i))$; their product and both squares are zero, since the socle of a regular block contains no idempotents. The singular block $\mathfrak{Z}_{l-1}$ consists of one idempotent element $\phi^{-1}(\xi(l-1))$.

By Theorem 4.1, the subspace $\mathrm{Soc}\,\widetilde{\mathfrak{Z}}_i$ is one-dimensional for each block. By Lemma 5.3, its image in $\mathfrak{R}$ contains the characters of tensor products of the Steinberg module with finite dimensional modules. For $\mathfrak{g} = \mathfrak{sl}_2$, $chSt\cdot\mathfrak{R}$ is spanned by the characters of the tilting modules $chT(2l-2-i) = chT(l+i) = 2(\xi(i)+\xi(l-2-i))$, and therefore $\mathrm{Soc}\,\mathfrak{R}$ is spanned by

$$\{(\xi(i) + \xi(l-2-i)), \xi(l-1)\}_{i=0}^{\frac{l-3}{2}}.$$

Its dimension is $\frac{l+1}{2}$.

By Corollary 6.2, the whole center $\mathfrak{Z}$ coincides with the sum of the two subalgebras $\widetilde{\mathfrak{Z}} + \mathfrak{Z}'$ in this case. Therefore, we have

$$\mathfrak{Z} \simeq \widetilde{\mathfrak{Z}} + \mathfrak{Z}' \simeq \oplus_{i=0\ldots\frac{l-3}{2}, i=l-1}(\widetilde{\mathfrak{Z}} + \mathfrak{Z}')_i,$$

where $(\widetilde{\mathfrak{Z}} + \mathfrak{Z}')_i \simeq \{1_i, t_i, s_i\}$ with 1_i acting as the unit of the block, and $t_i s_i = t_i^2 = s_i^2 = 0$ for $i = 0, \ldots \frac{l-3}{2}$, and $(\widetilde{\mathfrak{Z}} + \mathfrak{Z}')_{l-1} \simeq \{1_{l-1}\}$. This coincides with the description of the center obtained in [Ker1] in this particular case.

7.2. **The case $\mathfrak{g} = \mathfrak{sl}_3$.** The set $\bar{\mathcal{X}}$ consists of a one-dimensional orbit corresponding to the Steinberg module; $(l-1)$ 3-dimensional orbits corresponding to the weights stabilized by one reflection in $W\bullet$; and $\frac{(l-2)(l-1)}{6}$ 6-dimensional orbits corresponding to $W\bullet$-regular weights.

Check the dimension:

$$1 + 3(l-1) + 6\frac{(l-2)(l-1)}{6} = l^2 = \dim\mathfrak{R}.$$

Theorem 4.1 specializes to

$$\widetilde{\mathfrak{Z}} \simeq \bigoplus_{i=1}^{\frac{(l-1)(l-2)}{6}} \mathbb{C}[x,y,z]/I_1 \oplus \bigoplus_{i=1}^{l-1} \mathbb{C}[x]/I_2 \oplus \mathbb{C},$$

where $I_1 = \langle x+y+z, xy+yz+xz, xyz \rangle$, and $I_2 = \langle x^3 \rangle$.

The algebra $\mathfrak{Z}'$ has the same block decomposition as $\widetilde{\mathfrak{Z}}$ and the same dimension of each block. Denote by C_n the n-dimensional vector space with zero multiplication. Then

$$\mathfrak{Z}' \simeq \oplus_{i=1}^{\frac{(l-1)(l-2)}{6}} C_6 \oplus \oplus_{i=1}^{l-1} C_3 \oplus \mathbb{C}.$$

The multiplication between $\widetilde{\mathfrak{Z}}$ and $\mathfrak{Z}'$ is determined by the condition that $\mathfrak{Z}' \simeq \mathrm{Soc}\,\mathfrak{Z}$.

The dimension of $\widetilde{\mathfrak{Z}} + \mathfrak{Z}'$ is equal to

$$1 + 5(l-1) + 11\frac{(l-1)(l-2)}{6},$$

which is an integer since $(l, 3) = 1$ by the assumption on l.

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