Mathematics in the Real World
Math 107

Lectures 9 & 10: Finite geometric series and the mortgage.

Finite geometric series. A finite geometric series is a finite sum of consecutive powers of a fixed number:

$$(1 + r + r^2 + \cdots + r^n) = \sum_{k=0}^{n} r^k,$$

Here r is a real number and n is a natural number. The right hand side is a notation for the sum, where the index k takes all integer values between 0 and n. If r is 1, then the sum equals to $n + 1$. To compute this sum for $r \neq 1$, let us multiply it by $(1 - r)$:

$$(1 - r)(1 + r + r^2 + \cdots + r^n) = 1 - r + r - r^2 + r^2 - \ldots - r^n + r^n - r^{n+1}.$$

All the middle terms cancel, and we have:

$$(1 - r)(1 + r + r^2 + \cdots + r^n) = 1 - r^{n+1}.$$

Therefore, the original sum is given by the formula, that holds for all $r \neq 1$:

$$(1 + r + r^2 + \cdots + r^n) = \sum_{k=0}^{n} r^k = \frac{(1 - r^{n+1})}{1 - r}.$$

A more general form of the geometric series is

$$a + ar + ar^2 + \ldots + ar^n = a \cdot \sum_{k=0}^{n} r^k,$$

where a is another fixed real number. By taking the common factor out, we have:

$$a + ar + ar^2 + \ldots + ar^n = a(1 + r + r^2 + \ldots + r^n) = \frac{a(1 - r^{n+1})}{1 - r}.$$

This is the *finite geometric series formula* that holds for any real a, any natural n and a real $r \neq 1$.

Future value of money. The most prominent application of geometric series in daily life is in computation of mortgage payments. First we need to know how to compute the future value of money: when you take a loan or a mortgage, you will have to pay it back with all the interest accumulated by the time you pay it in full. In particular, we need to know the value of a dollar at a given point of time in the future (given a compounding scheme).

Example 1. Suppose a bank compounds quarterly, at an annual interest rate of 8%. How much is a dollar deposited in the bank today, worth in 2022?

The answer is computed using the usual compound interest formula: in ten years, the 2022-value of a dollar today is:

$$A(10) = 1 \cdot \left(1 + \frac{8}{4 \cdot 100}\right)^{4\times 10} = 1.02^{40} \simeq \$2.21.$$

Conversely, what is the *present value* of a dollar in 2022? To see this, if the value is A, then

$$1 = A\left(1 + \frac{8}{4 \cdot 100}\right)^{4\times 10} = A \cdot 1.02^{40} = 2.21A \quad \implies A = \frac{1}{2.21} \simeq \$0.45.$$

Practice problems: Given a dollar today, find its value under each of the following compounding schemes:

(1) In 2022; compounded quarterly at an annual rate of 12%. (*Answer:* \$3.26.)
(2) In 2002; compounded quarterly at an annual rate of 16%. (*Answer:* \$0.21.)
(3) In 2052; compounded quarterly at an annual rate of 10%. (*Answer:* \$51.98.)
(4) In 2021; compounded quarterly at an annual rate of 20%. (*Answer:* \$5.79.)

0.1. **Mortgage payments calculation.** Now we can calculate the mortgage payments. Here is the first example.

Example 2. If someone borrows a loan of $\$10,000$ using the above compounding scheme (8% compounded quarterly), what quarterly installment must they pay (starting the next quarter) for the next five years, to pay off the loan?

Solution: Suppose the quarterly installment is $\$A$. We then equate the 2017-value (in five years) of all payments made, with the 2017-value of the loan. The latter is easy to compute: five years means 20 quarters of compounding, so the 2017-value of the loan is:

$$A(5) = 10000 \cdot \left(1 + \frac{8}{4 \cdot 100}\right)^{4\times 5} = 10,000 \cdot 1.02^{20} = 14,859.474.$$

Computing the 2017-value of the payments is not too bad either: the first installment of A gets compounded 19 times, since it is paid one quarter from now. Thus, it is worth $A \cdot 1.02^{19}$ in 2017. The second is compounded 18 times, hence worth $A \cdot 1.02^{18}$, and so on. Thus, the 2017-value of our total payment is

$$A\left(1.02^{19} + 1.02^{18} + \cdots + 1\right),$$

where the last payment is made at the very end, and hence does not get compounded, and hence is worth $A = A \cdot 1$ in 2017.

But now we can add up this expression using the geometric series formula! We have

$$A(1 + r + r^2 + \cdots + r^n) = \frac{A(1 - r^{n+1})}{1 - r}.$$

for every real number $r \neq 1$. Applying this to our case, $r = 1.02$ and $n = 19$, so the 2016-value of all our payments, is:

$$\frac{A(1 - 1.02^{20})}{1 - 1.02} = A \cdot (-0.4859/ - 0.02) = 24.295 \cdot A.$$

Finally, this equals the 2017-value of our loan, if we are to pay it off entirely. Thus,

$$A = 14859.474/24.295 \simeq \$611.63.$$

This is the quarterly installment that is needed to pay off the loan in five years. $\square$

Now let us try to derive a general strategy for solving the mortgage problems. Suppose you take a mortgage of M dollars. The bank uses a compounding scheme with $r\%$ annual interest rate, compounded n times per year. You have to pay the mortgage back in t years, by paying equal installments n times per year. Find the installment amount.

First, we compute the value of the mortgage in t years:

$$M\left(1 + \frac{r}{n \cdot 100}\right)^{n \cdot t} = Mp^{nt},$$

where we denoted the compounding factor $\left(1 + \frac{r}{n \cdot 100}\right)$ by p for convenience. Now, suppose each installment is A dollars. The first installment is paid *one compounding period after the mortgage is taken*, and therefore it gets compounded $(nt - 1)$ times. The next installment gets compounded $(nt - 2)$ times, and so on until the last installment that does not get compounded at all. We have the following sum, that is calculated using the geometric series formula:

$$A\left(1 + p + p^2 + \ldots + p^{nt-1}\right) = \frac{A(1 - p^{nt})}{1 - p}.$$

Then we equate the two amounts:

$$Mp^{nt} = \frac{A(1 - p^{nt})}{1 - p}.$$

Finally,

$$A = M \cdot \frac{p^{nt}(1 - p)}{(1 - p^{nt})}$$

In practice, it is convenient first to calculate the compounding factor $p = \left(1 + \frac{r}{n \cdot 100}\right)$, and then plug it in the formula. The compounding periods are usually months ($n = 12$) or quarters ($n = 4$).

The next question is a variation on a mortgage calculation.

Example 3. Suppose you want to establish a monthly deposit to a bank that compounds monthly at an interest rate 6%, so that in 15 years you can have \$100, 000. FInd the monthly payment.

Solution: In this case, the final amount in 15 years is already known: \$100, 000. Only the monthly payments get compounded. The compounding factor per month is

$$p = \left(1 + \frac{r}{n \cdot 100}\right) = \left(1 + \frac{6}{1200}\right) = 1.005.$$

Here is an important difference from the mortgage calculation: the first payment of A dollars is made *right now*, therefore it gets compounded $n \cdot t = 12 \cdot 15 = 180$ times, and not $n \cdot t - 1 = 179$ times. The next payment is compounded $nt - 1$ times, and so on, until the last payment, that is made a month before 15 years are over, and so *it gets compounded once.* Therefore, we have:

$$A\left(p + p^2 + p^3 + \ldots + p^{nt}\right).$$

Now, the sum in the brackets is not exactly the geometric series as it was defined on the first page, because it starts with p instead of 1. To apply the geometric series formula, we need to take out the common factor p. Then the sum in the brackets starts with 1 and the geometric series formula can be applied:

$$A\left(p + p^2 + p^3 + \ldots + p^{nt}\right) = A \cdot p \cdot \left(1 + p + p^2 + \ldots + p^{nt-1}\right) = \frac{A \cdot p \cdot (1 - p^{nt})}{1 - p}.$$

Plugging in the numbers $n = 12$, $t = 15$ and $p = 1.005$, and equating the amount to \$100, 000, we have

$$100,000 = A \cdot 1.005 \cdot \frac{1 - 1.005^{12 \cdot 15}}{1 - 1.005} \simeq A \cdot 1.005 \cdot \frac{-1.45409}{-0.005} \simeq 292.27A.$$

Therefore,

$$A = \frac{100,000}{292.27} \simeq \$342.15.$$

$\square$

To make the difference between the two cases clear, here is an example of a mortgage payment calculation for the same amount and under the same compounding scheme, but with different payment schedules: (1) if the first payment is made one compounding period from now, and (2), it is made right now.

Example 4. A person borrows \$500, 000 from a bank, to be repaid over the next 20 years in 80 quarterly installments. Suppose the bank compounds quarterly at an annual rate of 16%. Find the value of the quarterly installment, if the payments start

(1) a quarter from now;
(2) today.

(*Hint:* Think how many times are the installments compounded in both of these situations?)

Solution. The *crucial* element to solving mortgage-type questions is to understand that *each payment* is compounded a different number of times and contributes a different value to the total accumulated payment.

The amount borrowed from the bank is \$500,000 today - so its 2032-value (in 20 years, at the annual rate of 16%, compounded quarterly) is:

$$500,000 \times \left(1 + \frac{16/4}{100}\right)^{20\times 4} = 500000 \cdot 1.04^{80} = 11,524,899.60.$$

This is to equal the amount paid in installments (and compounded over 20 years). We start with question (1) and suppose each installment is \$$A$. There are 80 total payments. We can denote them A_1, A_2, up to A_{80}. Now let us compute the 2032-value of each installment. The first is A_1, which is paid a quarter from today, is compounded 79 times. Hence, $A_1 = A(1.04)^{79}$.The second payment of A is compounded 78 times, so $A_2 = A(1.04)^{78}$, and so on. The last (80th) payment is not compounded at all, so $A_{80} = A$.

Now it remains to add them up, and equate to the above amount. We have:

$$11,524,899.6 = A\left(1.04^{79} + 1.04^{78} + \cdots + 1.04 + 1\right).$$

This adds up to

$$\frac{1 - 1.04^{80}}{1 - 1.04}.$$

So we have

$$11,524,899.60 = A \cdot \frac{1 - 1.04^{80}}{1 - 1.04}.$$

And finally,

$$A = \$\ 20,907.04.$$

Now solve (2). Denote each installment by B. There are again 80 installments. The first B_1, which is paid today, *is compounded* 80*, not* 79 *times*. Hence, $B_1 = B(1.04)^{80}$. The second payment of B is compounded 79 times, so $B_2 = B(1.04)^{79}$, and so on. The last (80th) payment is made a quarter before the closing of the mortgage, and hence *it is compounded once*, so $B_{80} = B(1.04)$. As before, we need to add them up and equate with the value of the loan in 20 years. We have:

$$11,524,899.6 = B\left(1.04^{80} + 1.04^{79} + \cdots + (1.04)^2 + 1.04\right).$$

Here it is important to remember, that the formula for finite geometric series *only works when the starting term is* 1. In this case, the starting term is 1.04, so we *can not apply the formula* to the equation as it stands!

Instead, take 1.04 out as a common factor from every term, and write:

$$11,524,899.6 = B \cdot 1.04\left(1.04^{79} + 1.04^{78} + \cdots + 1.04^{1} + 1\right).$$

Now we have a geometric series to which the formula can be applied. Hence, we get

$$11,524,899.6 = B \cdot 1.04 \cdot \frac{1 - 1.04^{80}}{1 - 1.04}.$$

So this sum results in the exponent 80, *and not 81* - but now there is an extra factor of 1.04 outside, which was not there in part (1). From here, you can solve for B:

$$B = \$\ 20,102.92.$$

$\square$

Grains on the chessboard. According to legend, the reward requested by the inventor of the game of chess was to be calculated as follows:
1 grain of rice on the 1st square of the chessboard, plus
2 grains on the 2nd square, plus
4 grains on the 3rd square, plus
8 grains on the 4th square, and so on,
doubling the number of grains on each next square until all 64 squares are filled.

It quickly turned out that the amount of rice requested was well beyond the king of India's means. We have the following geometric series:

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^{63}.$$

According to the geometric series formula, this sums up to

$$\frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1 \simeq 1.85 \cdot 10^{19} \text{ grains.}$$

Assuming that the volume of one grain of rice is 0.03 cm^3, or $3 \cdot 10^{-8}\text{m}^3$, we get the total volume of the rice

$$V_{\text{rice}} \simeq 1.84 \cdot 10^19 \cdot 3 \cdot 10^{-8} \simeq 5 \cdot 10^{11} \text{ m}^3.$$

This amount of rice would cover the total area of India approximately half foot deep.

Example 5. In a race competition, there are 80 participants. The prizes are awarded according to the following rule: the first prize is \$1000; the second prize is 3/4 of the first prize, the third prize 3/4 of the second prize, and so on, till the 80th prize for the last arriving athlete. How much money do the organizers need to have for the prize fund? (Assume that no two athletes will arrive at the same time.)

Solution: The prizes are distributed as follows:

$$\begin{array}{ll} 1st & 1000 \\ 2nd & \frac{3}{4}\cdot 1000 \\ 3rd & \left(\frac{3}{4}\right)^2 \cdot 1000 \\ \dots & \dots \\ 80th & \left(\frac{3}{4}\right)^{79} \cdot 1000. \end{array}$$

To find the total amount of all prizes, we write

$$1000 + 1000 \cdot \frac{3}{4} + 1000 \cdot \left(\frac{3}{4}\right)^2 + \ldots + 1000 \cdot \left(\frac{3}{4}\right)^{79} =$$

$$= 1000 \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \ldots + \left(\frac{3}{4}\right)^{79}\right) =$$

$$= 1000 \cdot \frac{1 - \left(\frac{3}{4}\right)^{80}}{1 - \frac{3}{4}} \simeq 1000 \cdot \frac{1 - 10^{-10}}{\frac{1}{4}} \simeq 4000.$$

Therefore, the organizers will need to have a total of \$4000 for the prize fund.

$\square$