Math 112a - Calculus of Functions of One Variable I

Final Exam

12/14/2011

NAME:

Section:

Please answer all TEN questions.
Show all your work and justify clearly where appropriate.
Cite any theorem or result that you use.
No books, notes, calculators, computers, or cellphones are allowed.

Good luck!

1	2	3	4	5	6	7	8	9	10	Total (160)

(1) (18 pts.) Find the following limits, or show they do not exist.

(a)

$$\lim_{x\to -1} \frac{x^2+3x+2}{x^2-x-2}$$

(b) Below $\tan^{-1}$ stands for the inverse function of tan:

$$\lim_{x\to\infty} \frac{\tan^{-1}(x)}{\ln(1+\frac{1}{x})}$$

(c)

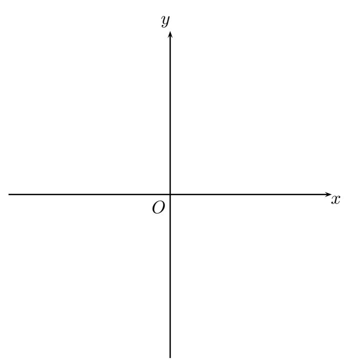
$$\lim_{x\to\infty} \frac{e^{2x}+6e^x+5}{4e^{2x}-4e^x+1}$$

(d)

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$$

(2) (18 pts.) Consider the function $f(x) = \frac{1}{\sqrt{1+x^2}}$.

(a) Find the domain, range and symmetries of $f(x)$.
(b) Find the asymptotes of $f(x)$.
(c) Find the intervals on which $f(x)$ is increasing and decreasing.
(d) Find the local maxima and minima of $f(x)$
(e) Find the intervals in which $f(x)$ is concave up and concave down and determine the points of inflection.
(f) Sketch the graph of $f(x)$ in the provided coordinate system.



(3) (16 pts.) Compute the derivatives of the following functions

(a)

$$f(x) = 3^{\ln x}$$

(b)

$$g(x) = \sqrt{1+\sqrt{x}}$$

(c) Below $\cos^{-1}$ stands for the inverse function of cos:

$$h(x) = \cos^{-1}(x^2) + \cos(x^2)$$

(d)

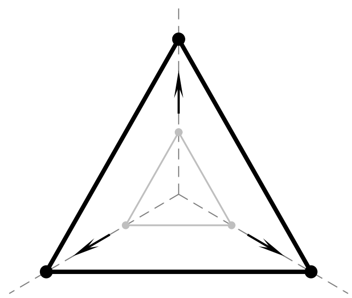
$$k(x) = (\ln x)^{\ln x}$$

(4) (12 pts.) Find the tangent line to

$$4(x+y)^2 + (y-x)^2 = 32$$

at the point $(x, y) = (3, -1)$.

(5) (18 pts.) The vertices of an equilateral triangle are being pulled away from the center of the triangle so that the distance between each vertex and the center of the triangle is growing at a constant rate of 1 cm per second. How fast is the area of the triangle growing when the distance between each vertex and the center of the triangle is 20 cm?



(6) (12 pts.) Find $h(x)$ if $h'(1) = \frac{1}{e}$, $h(1) = -\frac{1}{e}$, and

$$h''(x) = -\frac{1}{e^x} + 15\sqrt{x},$$

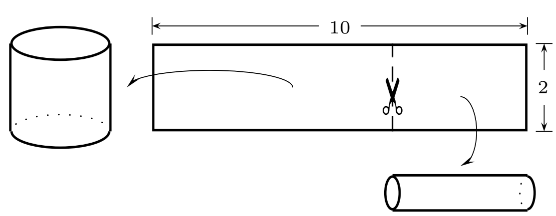
where $x > 0$.

(7) (20 pts.) A rectangular sheet of metal 2 feet wide and 10 feet long is to be cut into two smaller rectangles that are rolled into hollow cylinders (see picture). One cylinder is rolled upright, so that it has height 2 feet, and the other cylinder is rolled sideways so that it has circumference 2 feet. How should the sheet of metal be cut so that the total volume of the cylinders is

(a) a minimum,

(b) a maximum ?

(Recall that the volume of a cylinder of height h and radius r is given by $V = \pi h r^2$)



(8) (18 pts.) The function $f(x)$ is twice differentiable everywhere and $f''(x) < 0$ for all real x. In addition, $f(2) = f(7) = 4$.

(a) Suppose that $f(5) \leq 4$. Can the derivative $f'(x)$ be positive for all x in $(2, 5)$? Justify your answer.

(b) Suppose that $f(5) \leq 4$. Can the derivative $f'(x)$ be negative for all x in $(5, 7)$? Justify your answer.

(c) Use the Mean Value Theorem to show that if (a) and (b) hold, then there is a point c in $(2,7)$ such that $f''(c) \geq 0$. Conclude that $f(5) > 4$.

(9) (18 pts.) Compute the following definite integrals

(a)

$$\int_1^3 \frac{\sqrt{x} - x^2}{x^{\frac{3}{2}}} dx$$

(b)

$$\int_0^{\frac{1}{2}} \frac{2dy}{\sqrt{1-y^2}}$$

(c)

$$\int_{-4}^{4} \left(-1 - \sqrt{16 - x^2}\right) dx.$$

(10) (10 pts.) Find $g'(x)$ if

$$g(x) = \int_1^{3x} \frac{3}{1+s^2} ds, \qquad x \geq 0.$$