Math 112a - Calculus of Functions of One Variable I

Final Exam

Solutions

(1) (a)

$$\lim_{x\to 9}\frac{x-4}{\sqrt{x}-2}=\frac{9-4}{\sqrt{9}-2}=5.$$

(b) Dividing both the numerator and the denominator by x^3 and recalling that for $x<0$, $\sqrt{x^6}=|x^3|=-x^3$, we get

$$\lim_{x\to-\infty}\frac{2x^3+3x+1}{\sqrt{1+x^3}}=\lim_{x\to-\infty}\frac{x^3(2+\frac{3}{x^2}+\frac{1}{x^3})}{-x^3\sqrt{\frac{1}{x^6}+1}}=-2.$$

(c) Using L'Hospital's rule, we have

$$\lim_{x\to 0}\frac{\tan(6x)}{\sin(2x)}=\lim_{x\to 0}\frac{6\cdot\sec^2(6x)}{2\cdot\cos(2x)}=\frac{6}{2}=3.$$

(d) Take ln of the function: $\ln(x^{\frac{1}{x}})=\frac{1}{x}\ln(x)$. Then the limit of the logarithm of the function is

$$\lim_{x\to\infty}\frac{1}{x}\ln(x)=\lim_{x\to\infty}\frac{\ln(x)}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0.$$

We used L'Hospital's rule in the last equality. Then the original limit is

$$\lim_{x\to\infty}x^{\frac{1}{x}}=e^0=1.$$

(2) (a) If $x=0$, we have the equation

$$y^3-2y^2=0 \quad\Longrightarrow\quad y^2(y-2)=0 \quad\Longrightarrow\quad y=0,\ y=2.$$

(b) Differentiating the equation implicitly, we get

$$3y^2\frac{dy}{dx}-4y\frac{dy}{dx}-2x+3x\frac{dy}{dx}+3y=0.$$

Solving this equation for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}=\frac{2x-3y}{3y^2-4y+3x}.$$

(c) Finally, at the point $(0,2)$ we have

$$\frac{dy}{dx}=\frac{-6}{12-8}=-\frac{3}{2}.$$

The equation of the tangent at $(0,2)$ is $y-2=-\frac{3}{2}(x-0)$, or equivalently $y=2-\frac{3}{2}x$.

(3) (a) Consider the function $f(x)=x+e^x$. We have $f(0)=e^0=1$, $f(-2)=-2+e^{-2}<-1<0$. The function is continuous everywhere. By the IVT, there is a point a between 0 and 2 where $f(a)=0$. Therefore, the equation $f(x)=0$ has at least one solution.

(b) The function $f(x)$ is differentiable everywhere. If there were two points a and b such that $a\neq b$ and $f(a)=f(b)=0$, then by the MVT (or by Rolle's theorem), there should be a point c between a and b such that $f'(c) = 0$. However, $f'(x) = 1 + e^x > 0$ for all real x, contradiction. Therefore there is exactly one solution to the equation $x + e^x = 0$.

(4) First find the most general expression for $f'(x)$ by taking the antiderivative:

$$f'(x) = \frac{1}{2}x^2 - \frac{1}{x} + C_1.$$

Since $f'(1) = \frac{1}{2} - 1 + C_1 = 0$, we have $C_1 = \frac{1}{2}$, and $f'(x) = \frac{1}{2}x^2 - \frac{1}{x} + \frac{1}{2}$. Then the most general expression for the function $f(x)$ is

$$f(x) = \frac{1}{6}x^3 - \ln(x) + \frac{1}{2}x + C_2,$$

where C_2 is any real number.

(5) (a)

$$f'(x) = \frac{1}{\sin(e^)} \cdot \cos(e^x) \cdot e^x.$$

(b)

$$g'(x) = \left(\frac{x}{2} + \frac{1}{2\sqrt{x}}\right)' = \frac{1}{2} - \frac{1}{4}x^{-\frac{3}{2}}.$$

(c)

$$h'(x) = (\ln(x^{\sin(x)}))' = (\sin(x) \cdot \ln(x))' = \cos(x) \cdot \ln(x) + \frac{\sin(x)}{x}.$$

(d)

$$k'(x) = 2^x \cdot \ln 2 + 2x.$$

(6) The equation of a line passing through the point $(3, 2)$ with slope $k < 0$ is $(y-2) = k(x-3)$. To find the sides of the triangle that the line cuts from the axes, set $x = 0$ and $y = 0$ in the equation:

$$y_0 = 2 - 3k, \quad x_0 = 3 - \frac{2}{k}.$$

The area we need to minimize is

$$A(k) = \frac{1}{2}(2-3k)(3-\frac{2}{k}) = \frac{1}{2}(6 - 9k - \frac{4}{k} + 6) = -\frac{9}{2}k - \frac{2}{k} + 6.$$

Taking the derivative with respect to k, and setting it equal to zero, we have

$$A'(k) = -\frac{9}{2} + \frac{2}{k^2} = 0 \quad \Longrightarrow \quad \frac{2}{k} = \frac{9}{2} \quad \Longrightarrow \quad k = \pm\frac{2}{3}.$$

Because $k < 0$, we have the only solution $k = -\frac{2}{3}$. This is indeed an absolute minimum, because the derivative $A'(k) = -\frac{9}{2} + \frac{2}{k^2}$ is negative for all $k < -\frac{2}{3}$, and positive for all $-\frac{2}{3} < k < 0$. The equation of the line is $y - 2 = -\frac{2}{3}(x-3)$, or equivalently, $y = -\frac{2}{3}x + 4$.

(7) (a)

$$\int_0^\pi \sin(x)dx = -\cos(x)\Big|_0^\pi = 1 + 1 = 2.$$

(b)

$$\int_1^4 \left(3x^2 + \frac{1}{\sqrt{x}}\right) dx = x^3 + 2\sqrt{x}\Big|_1^4 = 64 + 4 - 1 - 2 = 65.$$

(c) The function $y = |x-2|$ is given by $-(x-2)$ for $x < 2$, and $(x-2)$ for $x \geq 2$. Then the integral splits into two parts, on $[0, 2]$ and on $[2, 4]$. Noticing that the two parts are the areas of two equal triangles, we can compute only the second integral and double the result:

$$\int_0^4 |x-2|dx = 2\int_2^4 (x-2)dx = 2\left(\frac{1}{2}x^2 - 2x\Big|_2^4\right) = 2(8-8-2+4) = 4.$$

(8) Using the FTC and the chain rule, we get

$$\left(\int_1^{x^2+5} \ln t dt\right)' = \ln(x^2+5)\cdot(2x).$$

(9) (a) $x = -2$ is a local max because the derivative changes sign from positive to negative;
(b) $x = 4$ is a local min because the derivative changes sign from negative to positive;
(c) $x = -4, x = 1$ are the points where $f''(x) = 0$ because $f'(x)$ has horizontal tangent at these points;
(d) $f(x)$ is increasing for $x < -2$ and $x > 4$ because $f'(x)$ is positive there;
(e) $f(x)$ is concave down on $(-4, 1)$ because $f'(x)$ is decreasing and $f''(x) < 0$ there.

(10) The volume of the cone is $V = \frac{1}{3}\pi r^2 h$. Taking into account the equation $r = \frac{1}{2}h$, we get $V(h) = \frac{1}{12}\pi h^3$. Differentiating this equation with respect to time, we have

$$\frac{dV}{dt} = \frac{1}{12}\pi\cdot 3h^2\cdot\frac{dh}{dt}.$$

Plugging in the known values $\frac{dV}{dt} = 25\,\frac{\mathrm{ft}^3}{\mathrm{min}}$ and $h = 10\,\mathrm{ft}$, we get

$$25 = \frac{1}{4}\pi\cdot 100\frac{dh}{dt} \qquad\Longrightarrow\qquad \frac{dh}{dt} = \frac{1}{\pi}\,\frac{\mathrm{ft}}{\mathrm{min}}.$$

The height is growing at a rate of $\frac{1}{\pi}\,\frac{\mathrm{ft}}{\mathrm{min}}$.