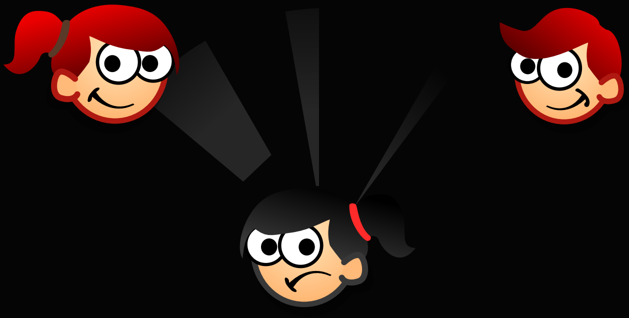
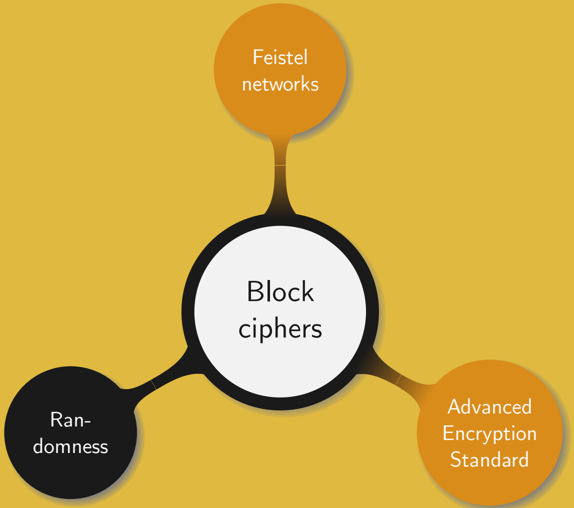


Introduction to Cryptography

2. Block ciphers

Manuel – Summer 2021

Chapter organisation



对称加密：采用单钥密码系统的加密方法，同一个密钥可以同时用作信息的加密和解密，这种加密方法称为对称加密，也称为单密钥加密。

A *block cipher* is composed of two functions, inverse of each other:

$$E : \{0,1\}^n \times \{0,1\}^k \to \{0,1\}^n \qquad D : \{0,1\}^n \times \{0,1\}^k \to \{0,1\}^n$$
$$(P, K) \mapsto C \qquad (C, K) \mapsto P$$

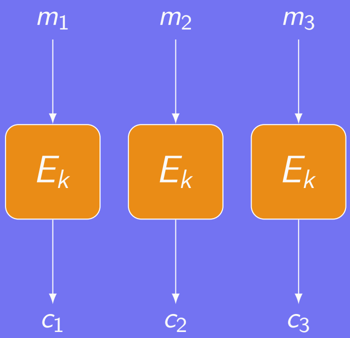
functions, inverse of each other

where n and k are the sizes of a block and the key, respectively.

Goal: given a key K, design an invertible function E whose output cannot be distinguished from a random permutation over $\{0,1\}^n$.

加密以后的数据不能太有规律性

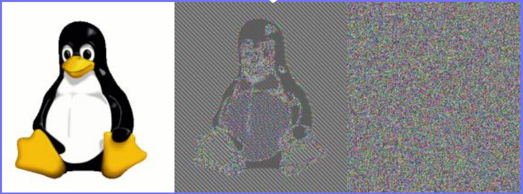
Mode ECB



Basic principle:

- Split the plaintext in blocks of size n
- Encrypt each block with a function E and a key K
- Electronic Code Block (ECB) mode

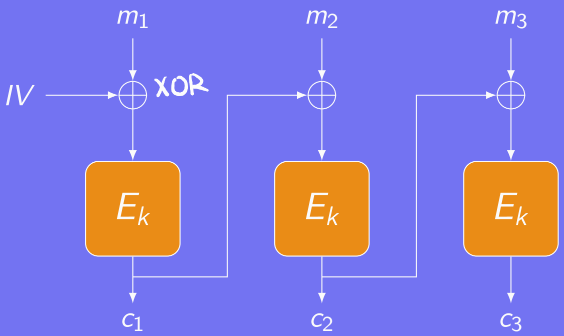
ECB RND



Limitation: what if a block is repeated several times over the message?

most common

Mode CBC

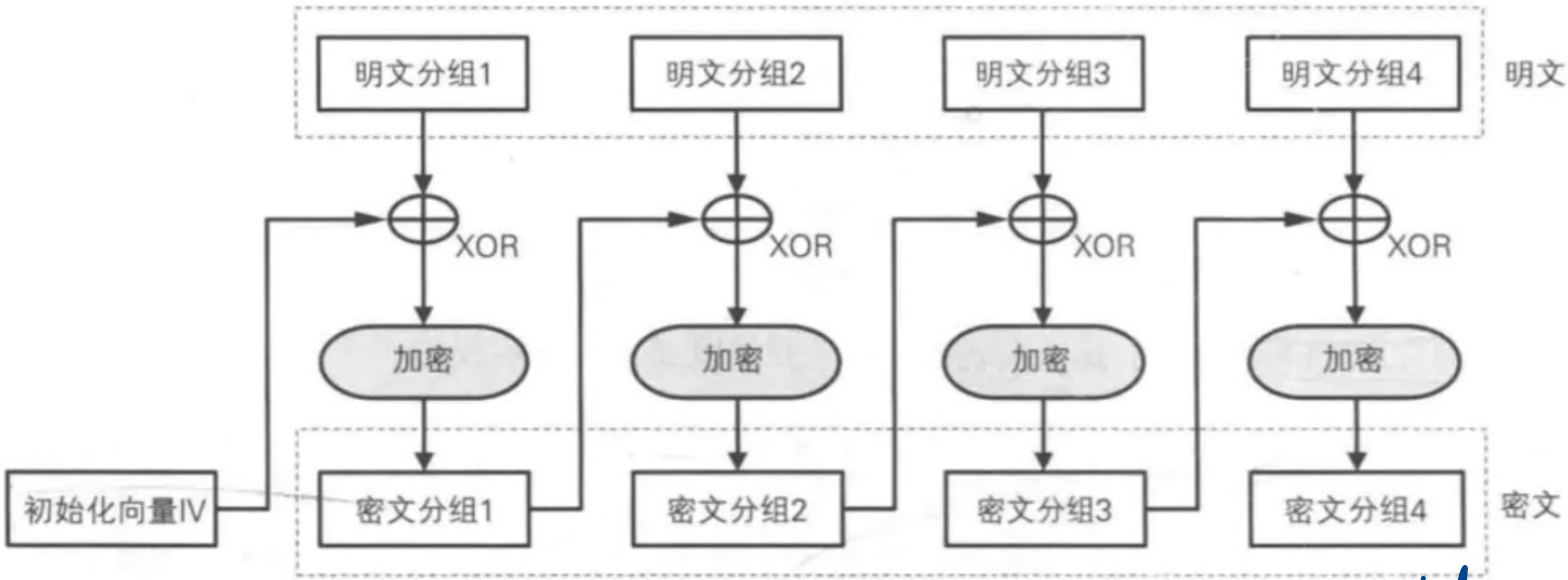


Basic principle:

- Cipher Block Chaining (CBC)
- Most commonly used mode
- Uses an Initialization Vector
- Can it be parallelized?

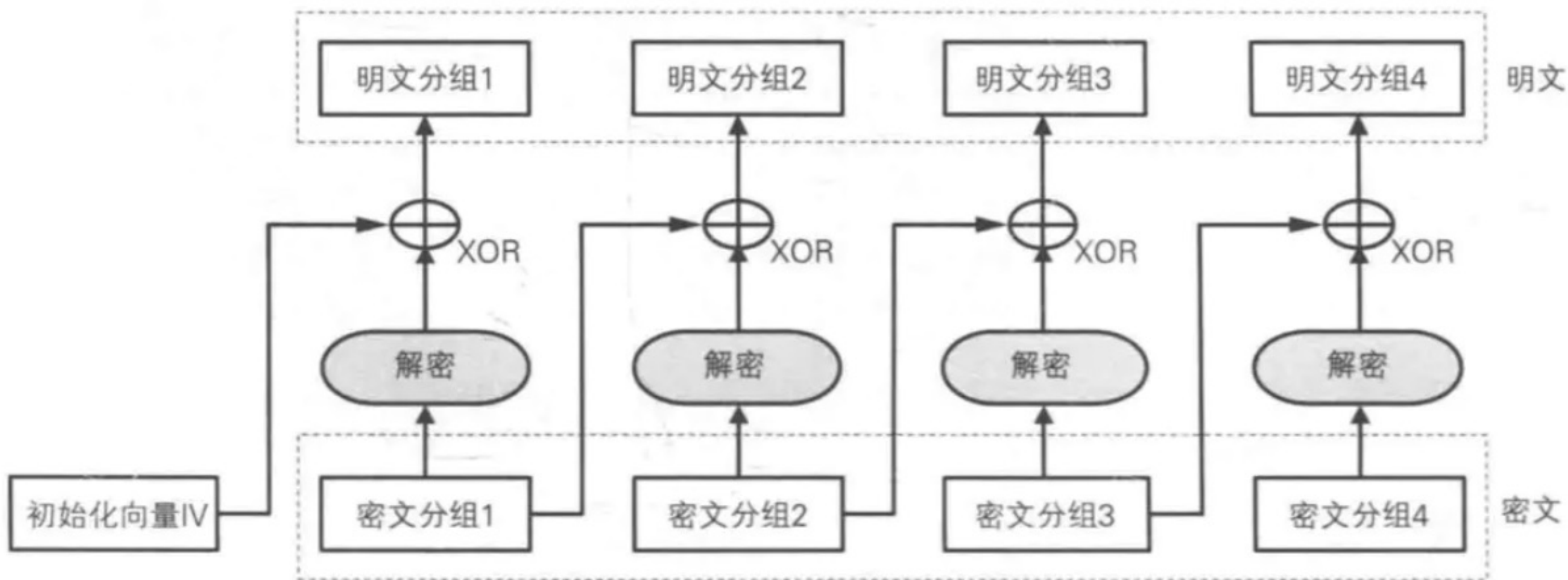
decrytion could be parallel

CBC模式的加密



parallel

CBC模式的解密

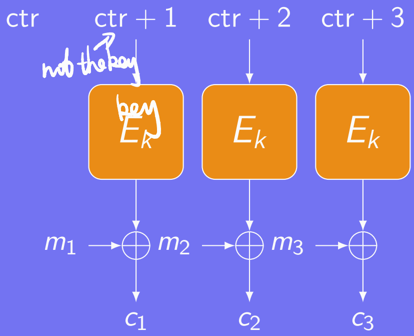


CTR:计数器模式

Mode CTR
flow

ctr: 任意一个保证长时间不产生重复输出的函数

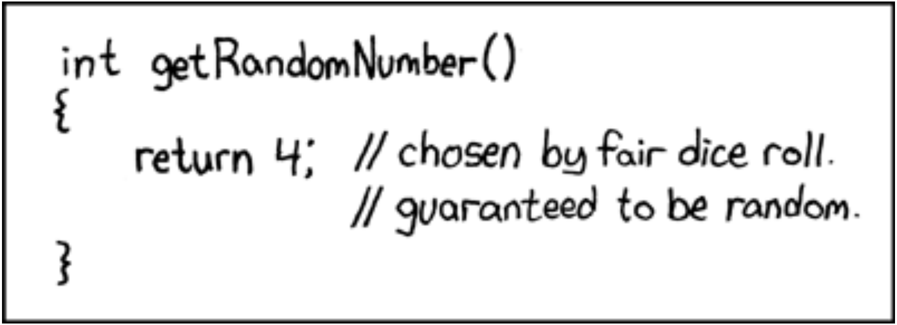
all the blocks are not secure



Basic principle:

- CTR stands for counter
- The counter acts like an IV
- The E_K function randomizes the counter
- Can be run in parallel
  and decrypt in parallel

What is randomness?



用K(s)表示

Definition (Kolmogorov randomness)

Let x be a string.

- We say that x is *random* if and only if it is not larger than any program that can produce it in any language.
- The *entropy* of x is the minimum number of bits necessary to describe x.

Remark. A random string of length k cannot be compressed in any way, therefore it has entropy k

Example. 3434 3434 3434 3434 3434 3434 3434 3434
ed71 b38f 4316 6907 a8ea 75d3 c141 735f

Generating true randomness is not simple:

- Toss a coin
- Measure physical phenomena that are expected to be random
- In case of a lack of entropy the output is blocked

Example. The thermal noise from a semiconductor resistor
A nuclear decay radiation source measured by a Geiger counter

Random function from the C standard:

```c
/*  Linear congruential generator */
static unsigned long next = 1;

/* RAND_MAX assumed to be 32767 */
int rand(void) {
  next = next * 1103515245 + 12345;
  return((unsigned)(next/65536) % 32768);
}

void srand(unsigned int seed) {
  next = seed;
}
```

Pseudo-random bits generation – BBS generator

A secure method from Blum, Blum and Shub:

1. Generate two large primes p and q, both being 3 mod 4
2. Set $n = pq$
3. Choose a random integer x coprime to n
4. Define
$$\begin{cases} x_0 & \equiv & x^2 \bmod n \\ x_{i+1} & \equiv & x_i^2 \bmod n \end{cases}$$
5. At each iteration select the least significant bit of x_i

最后一位bit

Why is BBS secure?

Can bits generated using BBS be predicted?

Problem (Quadratic Residuosity (QR))

Let $n = pq$ be the product of two primes. Given an integer y, is it a square mod n, i.e. is there an x such that $x^2 \equiv y \bmod n$?

This loose formulation will be refined in the next chapter (3.32).

Strategy:

- Prove that the QR problem is hard
- If this is hard the previous bit cannot be predicted

  cannot predict
- A sequence a pseudo-random bits generated by BBS cannot be compressed

① 证明第一步

In order to prove that the QR problem is hard we first recall and prove few results from number theory. The goal is to prove that solving the QR problem is as hard as factoring. That is, knowing how to solve one implies knowing how to solve the other one.

p不整除 a

Theorem (Fermat's little theorem) 费马小定理

Let $p \in \mathbb{N}$ and $a \in \mathbb{Z}$. If p is prime and $p \nmid a$, then

$$a^{p-1} \equiv 1 \bmod p.$$

More generally, for any prime $p \in \mathbb{N}$ and $a \in \mathbb{Z}$,

$$a^p \equiv a \bmod p.$$

Lemma

If $p \equiv 3 \bmod 4$ is prime, then the equation $x^2 \equiv -1 \bmod p$ has no solution.

Proof. Suppose such an x exists. Then raising it to the power of $(p-1)/2$ and applying Fermat's little theorem (2.13) yields

$4k+3-1/2$.

$$(x^2)^{\frac{p-1}{2}} \equiv x^{p-1} \equiv 1 \bmod p.$$

On the other hand $p \equiv 3 \bmod 4$, implies $(p-1)/2$ odd and

$$(-1)^{\frac{p-1}{2}} \equiv -1 \bmod p. \lightning$$

$\square$

Proposition

Let $p \equiv 3 \bmod 4$ be a prime, y be an integer and $x \equiv y^{\frac{p+1}{4}} \bmod p$.

- If y has a square root mod p, then its square roots are $\pm x \bmod p$
- If y has no square root mod p, then the square roots of $-y$ are $\pm x \bmod p$

$-y$一定有平方根吗

Proof. The case $y \equiv 0 \bmod p$ being trivial, we assume $y \not\equiv 0 \bmod p$. Applying Fermat's little theorem (2.13) we get

$$x^4 \equiv y^{p+1} \equiv y^2 y^{p-1} \equiv y^2 \bmod p. \tag{2.1}$$

Proof (continued). Since p is prime all the non zero elements have a multiplicative inverse (prop. 1.33). Therefore rewriting eq. (2.1) into

$$(x^2 - y)(x^2 + y) \equiv 0 \bmod p,$$

implies $x^2 \equiv \pm y \bmod p$. Hence at least one of y and $-y$ is a square mod p.

↓ y和-y不能同时 square

Suppose that both y and $-y$ are square mod p, i.e. there exist a and b such that $y \equiv a^2 \bmod p$ and $-y \equiv b^2 \bmod p$.

Then $\left(b^{-1}a\right)^2 \equiv -1 \bmod p$, that is -1 is a square mod p, contradicting lem. 2.14↯.

Hence exactly one of y and $-y$ has square roots $\pm x \bmod p$.

□

Keeping in mind the initial goal of studying the BBS generator where the squares are computed mod $n = pq$, with both p and q congruent to 3 modulo 4, we recall the following result.

Theorem (Chinese Remainder Theorem (CRT))

Let $m_1, \dots, m_k \in \mathbb{N}\setminus\{0\}$ be pairwise relatively prime and $a_1, \dots, a_k \in \mathbb{Z}$. Then the system of congruences

$$\begin{cases} x & \equiv & a_1 \bmod m_1, \\ x & \equiv & a_2 \bmod m_2, \\ & \vdots & \\ x & \equiv & a_n \bmod m_k. \end{cases}$$

has a unique solution modulo $m = m_1 m_2 \dots m_k$.

CRT的具体算法

若需寻找一个数 $\begin{cases} x \equiv a_1 \bmod m_1 \\ x \equiv a_2 \bmod m_2 \\ x \equiv a_3 \bmod m_3 \end{cases}$

则拆成3个问题.

寻找 $m_2 m_3 k \equiv a_1 \bmod m_1$

$m_1 m_2 k = a_3 \bmod m_3$

$m_1 m_3 k = a_2 \bmod m_2$

$\Rightarrow$ 把这三个加起来 $\Rightarrow$ final answer

Example. Find x such that $x^2 \equiv 71 \bmod 77$.

As $77 = 7 \times 11$, the congruency can be rewritten

$$\begin{cases} x^2 \equiv 71 \equiv 1 \bmod 7 \\ x^2 \equiv 71 \equiv 5 \bmod 11. \end{cases}$$

As both 7 and 11 and 3 mod 4, from prop. 2.15 we derive

$$\begin{cases} x \equiv \pm 1 \bmod 7 \\ x \equiv \pm 4 \bmod 11. \end{cases}$$

how come

Finally, by applying the CRT (2.17) the four solutions can be recombined modulo 77 such as to get

$$x \equiv \pm 15, \ \pm 29 \bmod 77.$$

In the previous example we used the factorisation of n in order to calculate the square root of x modulo n. We now show that if we know the square root then we can factorize n.

Proposition

Let n be a product of two unknown primes p and q, both being 3 mod 4. Let $x \equiv \pm a, \ \pm b \bmod n$ be the four solutions to $x^2 \equiv y \bmod n$. Then $\gcd(a-b, n)$ is a non-trivial factor of n.

Proof. From the construction of a and b, we know that $a \equiv b \bmod p$ and $a \equiv -b \bmod q$ (or the other way around). Therefore $p|(a-b)$ while $q \nmid (a-b)$, which means that $\gcd(a-b, n) = p$. $\square$

Remarks on the BBS generator

We showed that:

- Solving the factorization problem allows to solve the QR problem
- Solving the QR problem gives the factorization of the modulus

The previous reasoning is:

- Not a formal security reduction
- Enough to "informally" consider BBS as a secure pseudo-random number generator

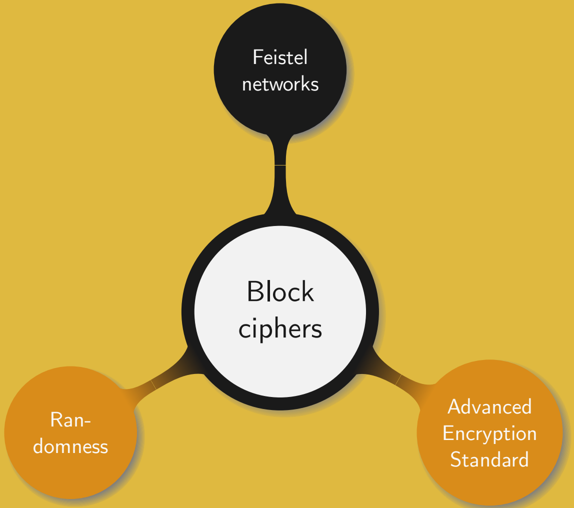
A few informal definitions:

随机预言

- A *random oracle* is a “black box” that returns a truly uniform random output on an input. Submitting the same input more than once leads to the same output.
- A *pseudorandom function* is a function that emulates a random oracle

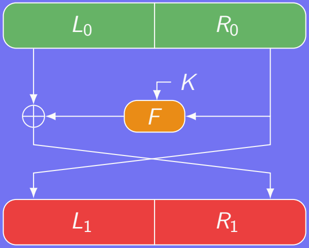
- A pseudorandom function that cannot be distinguished from a random permutation is called *pseudo random permutation*
- A *blockcipher* is a pseudorandom permutation
- A *one way function* is a function easy to evaluate but hard to invert

Chapter organisation



Basic Feistel network

We want to build a random bijection over 2n bits



- Size of a block: $2n$ bits
- Split the block into two blocks of n bits each
- $F : \{0,1\}^n \times \{0,1\}^k \to \{0,1\}^n$

We define the function

$$\begin{aligned} \Psi_F : \{0,1\}^{2n} &\longrightarrow \{0,1\}^{2n} \\ [L, R] &\longmapsto [R, L \oplus F(R, K)] \end{aligned}$$

Proposition

For any function F, Ψ_F is a bijection and $\Psi_F^{-1} = \sigma \circ \Psi_F \circ \sigma$, with

$$\begin{aligned} \sigma : \{0,1\}^{2n} &\longrightarrow \{0,1\}^{2n} \\ [L, R] &\longmapsto [R, L] \end{aligned}$$

对σ的说明

Proof. By definition of Ψ_F, $\Psi_F\left([L_0, R_0]\right) = [R_0, L_0 \oplus F(R_0, K)] = [L_1, R_1]$.

Equivalently,
$$\begin{cases} R_0 = L_1 \\ L_0 = R_1 \oplus F(L_1, K). \end{cases}$$

Moreover
$$\begin{aligned} \sigma \circ \Psi_F \circ \sigma([L_1, R_1]) &= \sigma \circ \Psi_F \circ \sigma([R_0, L_0 \oplus F(R_0, K)]) \\ &= \sigma\left(\Psi_F([L_0 \oplus F(R_0, K), R_0])\right) \\ &= \sigma\left(R_0, L_0 \oplus F(R_0, K) \oplus F(R_0, K)\right) \\ &= [L_0, R_0]. \end{aligned}$$

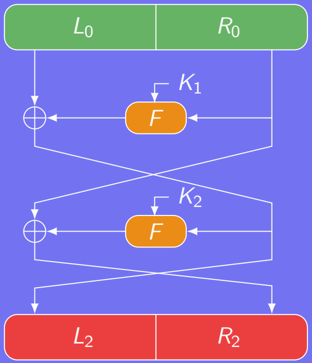
$\square$

Setting up two black boxes, a random oracle and a Feistel network, the goal for an attacher is to distinguish them.

Number of messages necessary to reach the goal:

Rounds	KPA	CPA	CPCA
1	1	1	1
2	$\mathcal{O}\left(\sqrt{2^n}\right)$	2	2
3	$\mathcal{O}\left(\sqrt{2^n}\right)$	$\mathcal{O}\left(\sqrt{2^n}\right)$	3
4	$\mathcal{O}\left(2^n\right)$	$\mathcal{O}\left(\sqrt{2^n}\right)$	$\mathcal{O}\left(\sqrt{2^n}\right)$

Two rounds – CPA



Attack strategy:

- For simplicity we denote $F(X, K)$ by $F_k(X)$ and $\Psi_{F_{k_2}} \circ \Psi_{F_{K_1}}$ by $\Psi^2_{F_{K_1}, F_{K_2}}$
- $\Psi^2_{F_{K_1}, F_{K_2}}\left([L_0, R_0]\right) = [L_2, R_2]$ with $L_2 = L_0 \oplus F_{K_1}(R_0)$ and $R_2 = R_0 \oplus F_{K_2}(L_2)$
- The inverse of $\Psi^2_{F_{K_1}, F_{K_2}}$ is

$$\begin{aligned} \Psi^{-2}_{F_{K_1}, F_{K_2}} &= \Psi^{-1}_{F_{K_1}} \circ \Psi^{-1}_{F_{K_2}} \\ &= \sigma \circ \Psi_{F_{K_1}} \circ \sigma \circ \sigma \circ \Psi_{F_{K_2}} \circ \sigma \\ &= \sigma \circ \Psi^2_{F_{K_2}, F_{K_1}} \circ \sigma \end{aligned}$$

What if we use $m_1 = [m_{1_L}, m_{1_R}]$ and $m_2 = [m_{2_L}, m_{2_R}]$ such that

$$\begin{cases} m_{1_L} \neq m_{2_L} \\ m_{1_R} = m_{2_R} \end{cases}$$

Basic KPA strategy:

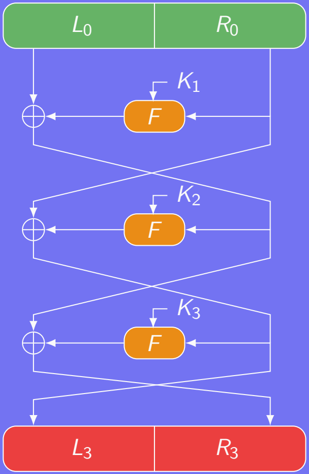
1. Find a collision over the m_{i_R}, $1 \leq i \leq 2^n$
2. If a collision is found for m_j and m_l check if

$$m_{j_{L_2}} \oplus m_{l_{L_2}} = m_{j_{L_0}} \oplus m_{l_{L_0}}$$

By the birthday paradox (slide 4.11):

- Number of plaintext-ciphertext pairs needed: $\mathcal{O}(\sqrt{2^n})$
- No better than $\mathcal{O}(\sqrt{2^n})$:
  - Collision on $m_{i_{L_2}} = m_{i_{L_0}} \oplus F_{K_1}(m_{i R_0})$ for two messages
  - The variables $m_{i_{L_2}}$, $m_{i_{L_0}}$, and $m_{i_{R_0}}$ are fixed
  - It only depends on F_{K_1}, which can take 2^n different values
  - From l messages $\frac{l(l-1)}{2}$ pairs can be constructed
  - Probability of collision: $\approx \frac{l(l-1)}{2 \cdot 2^n}$

Three rounds – CPCA



Attack strategy:

- $\Psi^3_{F_{k_1},F_{k_2},F_{k_3}}([L_0, R_0]) = [L_3, R_3]$ with
$$\begin{cases} L_3 = R_0 \oplus F_{K_2}(L_2) \\ R_3 = L_2 \oplus F_{K_3}(L_3) \end{cases}$$
and $L_2 = L_0 \oplus F_{K_1}(R_0)$
- Notice for a pair of messages (m_a, m_b)
$$\begin{aligned} m_{a_{R_0}} &= m_{b_{R_0}} \Leftrightarrow m_{a_{L_2}} \oplus m_{b_{L_2}} = m_{a_{L_0}} \oplus m_{b_{L_0}} \\ m_{a_{L_2}} &= m_{b_{L_2}} \Leftrightarrow m_{a_{L_3}} \oplus m_{b_{L_3}} = m_{a_{R_0}} \oplus m_{b_{R_0}} \\ m_{a_{L_3}} &= m_{b_{L_3}} \Leftrightarrow m_{a_{R_3}} \oplus m_{b_{R_3}} = m_{a_{L_2}} \oplus m_{b_{L_2}} \end{aligned} \tag{2.2}$$

Three rounds – CPCA

Attack strategy:

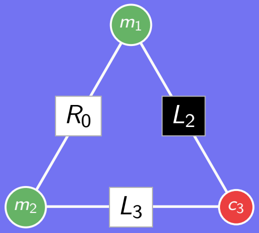
- Choose two plaintexts m_1 and m_2 such that $m_{2_{R_0}} = m_{1_{R_0}}$
- Choose a ciphertext c_3 such that $m_{3_{L_3}} = m_{2_{L_3}}$
- From the third equation of (2.2)

$$m_{3_{R_3}} \oplus m_{2_{R_3}} = m_{3_{L_2}} \oplus m_{2_{L_2}}$$

- Using the first equation of (2.2) enforce $m_{3_{L_2}}$ to be $m_{1_{L_2}}$

$$m_{3_{R_3}} = m_{2_{R_3}} \oplus m_{1_{L_0}} \oplus m_{2_{L_0}}$$

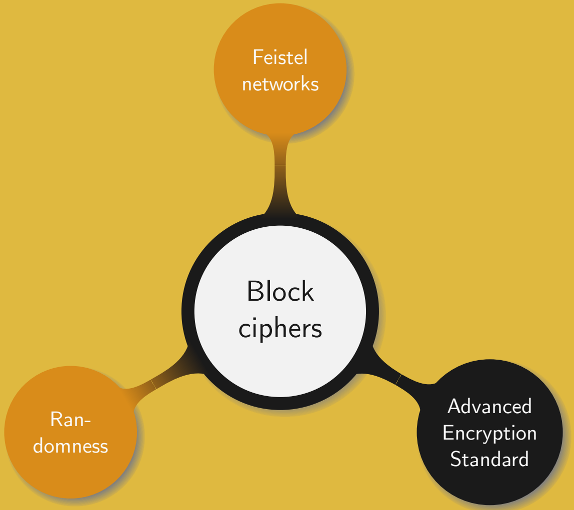
- How to conclude?



Data Encryption Standard (DES):

- 1974: IBM uses Feistel networks to create LUCIFER
- 1975: LUCIFER is sent to NSA for review and modifications
- 1977: renamed DES and becomes the official encryption standard
- 2002: DES is not secure anymore and is replaced by AES

Chapter organisation



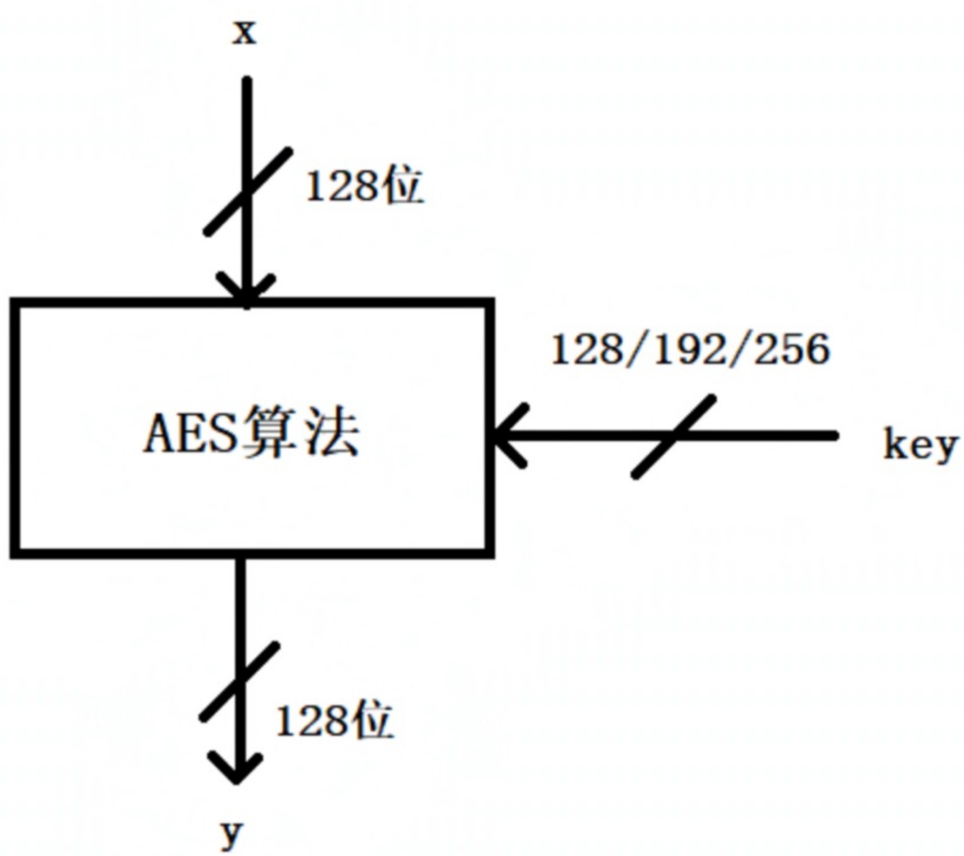
Advanced Encryption Standard (AES):

- 1997: call for candidates to replace DES
- Requirements:
  - Possible key sizes: 128, 192 and 256 bits
  - Input block size: 128 bits
  - Work on various hardware (e.g. 8-bit processors)
  - Speed
- Five finalists: MARS, RC6, Rijndael, Serpent, and Twofish
- 2001: Rijndael is chosen to become AES

AES – General description

Brief outline of AES:

- 10 rounds for a 128-bit key (12 and 14 for 192 and 256-bit)
- A round is formed of layers
  - *SubBytes*: substitution operation
  - *ShiftRows*: linear mixing step on the rows
  - *MixColumns*: linear mixing on the columns 伽瓦罗域运算
  - *AddRoundKey*: apply a round key derived from the main key


x
128位
AES算法
128/192/256
key
128位
y

AES – Encryption

Plaintext

AddRoundKey

SubBytes

ShiftRows

MixColumns

AddRoundKey

Rounds 1 to 9

SubBytes

ShiftRows

AddRoundKey

Round 10

Ciphertext

AES setup:

8 bits

- The 128 bits are grouped into 16 bytes
- Each byte is composed of 8 bits:

$$a_{0,0}, a_{1,0}, a_{2,0}, a_{3,0}, a_{1,1}, \cdots, a_{3,3}$$

- Bits are arranged in a 4×4 matrix:

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

So far we worked with the set $S = \{0, \cdots, n-1\}$ using modular congruences (def. 1.19). In the proof of prop. 2.15 we noted that when n is prime all the non-zero elements of S are invertible.

Example.

1. The set $S = \{0, \cdots, 4\}$ has five elements, and since five is prime all the non-zero elements are invertible. Indeed,

$$1 \cdot 1 \equiv 1 \bmod 5, \;\; 2 \cdot 3 = 6 \equiv 1 \bmod 5, \text{ and } 4^2 = 16 \equiv 1 \bmod 5.$$

2. The set $S = \{0, \cdots, 5\}$ has six elements, and as six is not prime some non-zero elements are not invertible. In fact since

$$2 \cdot 3 = 6 \equiv 0 \bmod 6,$$

we conclude that 2 and 3 are not invertible mod 6.

Loosely speaking a set where the addition and multiplication operations are defined and such that every non-zero element is invertible for the multiplication is called a *field*.

只讨论有限元素的域

When a field has a finite number of elements it is called *finite field*. For each prime p and positive integer n there exists a finite field with p^n elements, often denoted $\mathrm{GF}(p^n)$ or $\mathbb{F}_{p^n}$ (GF standing for Galois Field).

Remark. The set $S = \{0, \cdots, 8\}$ has $9 = 3^2$ elements and is not a field since 3 is not invertible. Therefore the question remaining to answer is "how to construct a finite field with nine elements", or more generally with p^n elements.

只有 $m=p^n$, p为素数, 阶为m的域才存在

p: 特征

Polynomials over finite fields

Similarly to how polynomials are defined over common fields such as the real numbers, they can also be defined over finite fields. The main difference relies on their coefficients which take their values in the base field.

In a field, a polynomial which cannot be written as the product of two polynomials of lower degree is said to be *irreducible*.
Example.

1. In $\mathbb{F}_2[X]$, $X^2 + \overset{X}{\underline{3X}} + 1$ and $X^2 + X + 1$ are equal.
2. In $\mathbb{F}_5[X]$, $X^3 + X + 3 = (X + 4)(X^2 + X + 2)$ is not irreducible.
3. In $\mathbb{F}_{17}[X]$, $X^3 + X + 3$ is irreducible.

Theorem

Let $P(X)$ be an irreducible polynomial of degree n in $\mathbb{F}_p[X]$, and F be the set of all the polynomials of degree less than n. Then F is a finite field with p^n elements.

mod p

Proof. Assuming addition and multiplication are properly defined we need to prove that F has p^n elements and that all but 0 are invertible.

It is simple to see that F has p^n elements since each of the n monomials (from degree 0 to $n-1$) can take p different values (from 0 to $p-1$).

Proof (continued). Let $A(X)$, $B(X)$ and $C(X)$ be three distinct non-zero polynomials such that

$$A(X)B(X) \equiv A(X)C(X) \bmod P(X).$$

This implies $A(X)\left(B(X) - C(X)\right) \equiv 0 \bmod P(X)$, which is not possible since $P(X)$ is irreducible.

Hence multiplying a polynomial $A(X)$ by all the non-zero elements of F results in covering all the non-zero polynomials of F, meaning that there is a polynomial $B(X)$ such that

$$A(X)B(X) \equiv 1 \bmod P(X).$$

$\square$

Finite fields in the AES

另类表述方式

In Rijndael $\mathbb{F}_{2^8}$ is used:

- $P(X) = X^8 + X^4 + X^3 + X + 1$ is the irreducible over $\mathbb{F}_2[X]$
- Each element of $\mathbb{F}_{2^8}$ is a polynomial of the form
$$a_7X^7 + a_6X^6 + a_5X^5 + a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0$$
- The polynomial is described as a byte $a_7a_6a_5a_4a_3a_2a_1a_0$
- The sum of two polynomials is the XOR of their bit representation
- Multiplying a polynomial $Q(X)$ by X:
  1. Shift left the byte representation of $Q(X)$ and append a 0
  2. If the first bit is 0 stop and otherwise XOR with $P(X)$
- Multiplying $Q(X)$ by $R(X)$:
  1. Split $R(X)$ into the monomials $M_i(X)$, $i \leq \deg R(X)$
  2. For $M_i(X)$ applying the multiplication by X $\deg M_i(X)$ times
  3. Add all the results using XOR

Example. Let $Q(X) = X^7 + X^4 + X + 1$ and $R(X) = X^2 + 1$. Determine the product $Q(X)R(X)$ in $\mathbb{F}_{2^8}[X]$.

1. Regular strategy: multiply and reduce mod $P(X)$
   - $Q(X)R(X) = X^9 + X^7 + X^6 + X^4 + X^3 + X^2 + X + 1$
   - Since $P(X) = 0$, $X^9 = X^5 + X^4 + X^2 + X$ and
   $$Q(X)R(X) \equiv X^7 + X^6 + X^5 + X^3 + 1 \bmod P(X)$$
2. Represent polynomials as bytes and apply XOR operations:

Write $Q(X) = 10010011$ and decompose $R(X)$ as $X \cdot X + 1$

- $Q(X) \cdot X = 100100110 \oplus 100011011 = 000111101$
- $(Q(X) \cdot X) \cdot X = 001111010$
- $(Q(X) \cdot X) \cdot X + Q(X) = 01111010 \oplus 10010011 = 11101001$
- $Q(X)R(X) \equiv X^7 + X^6 + X^5 + X^3 + 1 \bmod P(X)$

在扩展域$GF(2^8)$中，计算$(x^5+x^2+x^1)\otimes(x^7+x^4+x^3+x^2+x^1)$的结果，

其中使用 $\otimes$ 表示扩展域中多项式乘法，不可约多项式为：$P(x)=x^8+x^4+x^3+x^1+1$

$$A(x)=x^5+x^2+x^1$$

$$B(x)=x^7+x^4+x^3+x^2+x^1$$

$$
\begin{aligned}
A(x)\otimes B(x) &= (x^5+x^2+x^1)\otimes(x^7+x^4+x^3+x^2+x^1)\\
&= (x^{12}+\underline{x^9+x^9}+\underline{x^8+x^8}+x^7+\underline{x^6+x^6}+\underline{x^5+x^5}+\underline{x^4+x^4}+\underline{x^3+x^3}+x^2)\\
&\quad mod(x^8+x^4+x^3+x^1+1)\\
&= (x^{12}+x^7+x^2)mod(x^8+x^4+x^3+x^1+1)
\end{aligned}
$$

$$
\begin{array}{r|l}
 & x^4 \ +1 \\
\hline
x^8+x^4+x^3+x^1+1 & x^{12}+x^7+x^2 \\
 & x^{12}+x^8+x^7+x^5+x^4 \\
\hline
 & \quad x^8+x^5+x^4+x^2 \\
 & \quad x^8+x^4+x^3+x^1+1 \\
\hline
\text{余数:} & \qquad x^5+x^3+x^2+x^1+1
\end{array}
$$

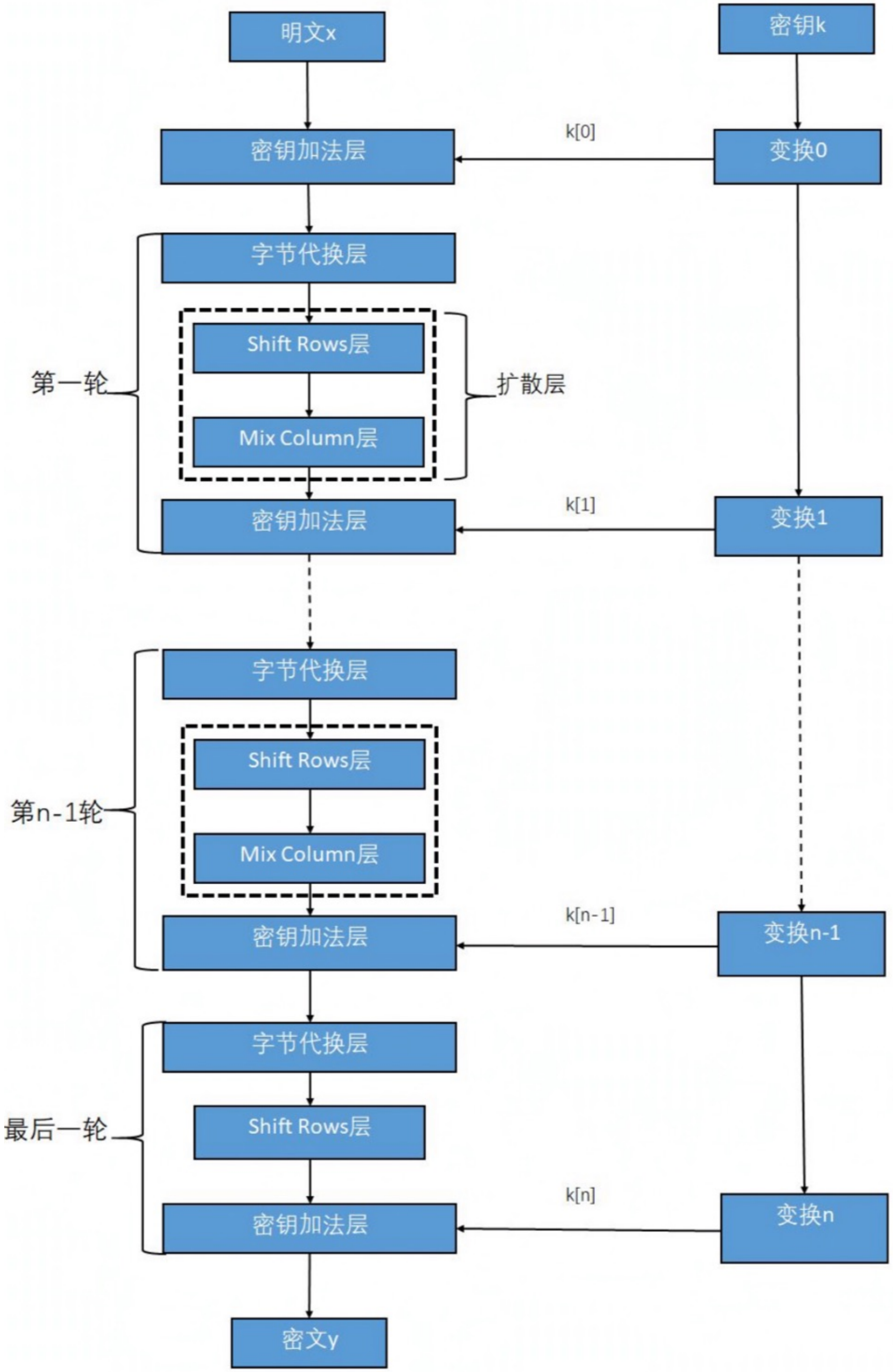
最终结果： $A(x)\otimes B(x)=\ x^5+x^3+x^2+x^1+1$

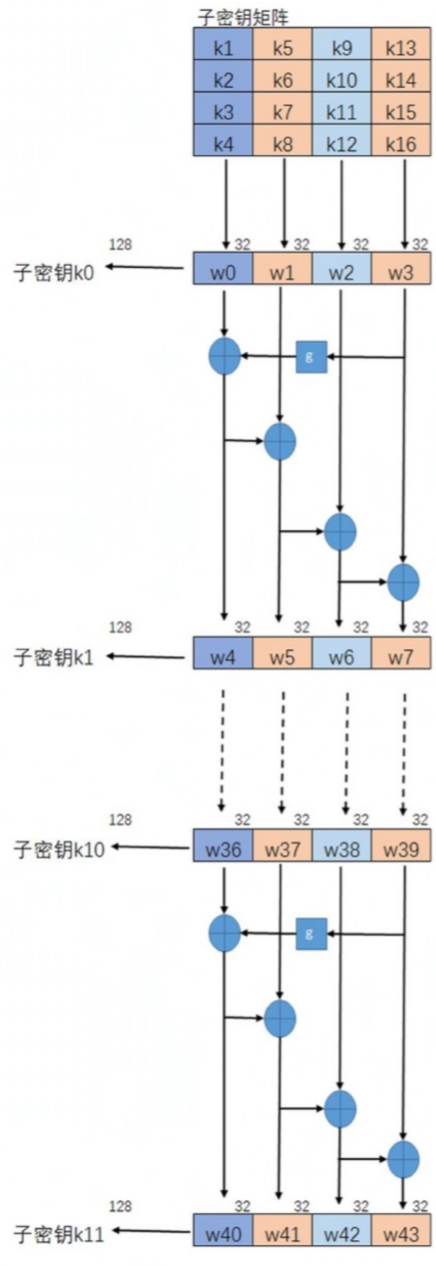
输入的字节顺序:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

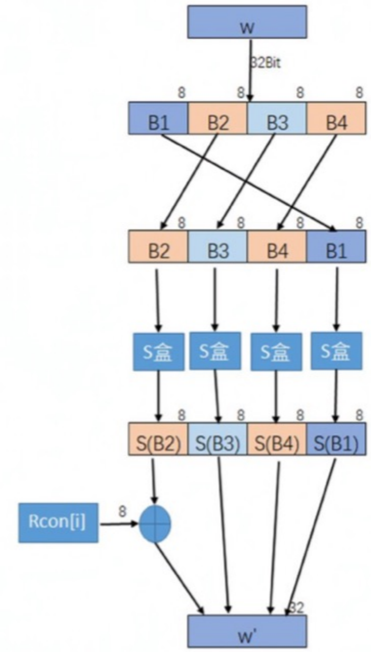
规定的字节排列方式:

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16





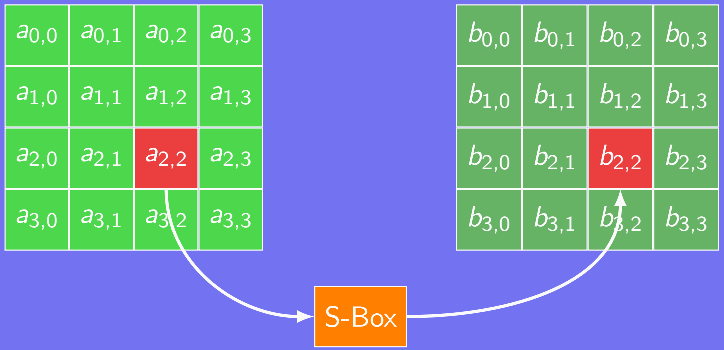
密钥生成大致流程



G函数处理流程

00 01 01 10

The SubBytes layer



For each byte in the matrix:

- Split it into two 4-bit numbers a and b
- Find byte c in the S-Box table at row a and column b
- Replace the original byte by c

S-Box

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	99	124	119	123	242	107	111	197	48	1	103	43	254	215	171	118
1	202	130	201	125	250	89	71	240	173	212	162	175	156	164	114	192
2	183	253	147	38	54	63	247	204	52	165	229	241	113	216	49	21
3	4	199	35	195	24	150	5	154	7	18	128	226	235	39	178	117
4	9	131	44	26	27	110	90	160	82	59	214	179	41	227	47	132
5	83	209	0	237	32	252	177	91	106	203	190	57	74	76	88	207
6	208	239	170	251	67	77	51	133	69	249	2	127	80	60	159	168
7	81	163	64	143	146	157	56	245	188	182	218	33	16	255	243	210
8	205	12	19	236	95	151	68	23	196	167	126	61	100	93	25	115
9	96	129	79	220	34	42	144	136	70	238	184	20	222	94	11	219
10	224	50	58	10	73	6	36	92	194	211	172	98	145	149	228	121
11	231	200	55	109	141	213	78	169	108	86	244	234	101	122	174	8
12	186	120	37	46	28	166	180	198	232	221	116	31	75	189	139	138
13	112	62	181	102	72	3	246	14	97	53	87	185	134	193	29	158
14	225	248	152	17	105	217	142	148	155	30	135	233	206	85	40	223
15	140	161	137	13	191	230	66	104	65	153	45	15	176	84	187	22

Generating the S-Box

Simple construction:

- For a in $\mathbb{F}_{2^8}^*$ compute its inverse $b = a^{-1}$ or set $b = 0$ if $a = 0$
- Represent b as a column vector $B = (b_0, \cdots, b_7)$
- Compute

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix}$$

- The entry located at row $(a_7 \cdots a_4)_2$ and column $(a_3 \cdots a_0)_2$ of the S-Box is $(c_7 \cdots c_0)_2$

Example. Find the S-Box entry corresponding to the byte 11001011?

The byte 11001011 stands for $a(X) = X^7 + X^6 + X^3 + X + 1$, and we observe that

$$\begin{aligned} a(X) \cdot X^2 &= X^9 + X^8 + X^5 + X^3 + X^2 \\ &\equiv X^8 + X^4 + X^3 + X \bmod P(X) \\ &\equiv 1 \bmod P(X). \end{aligned}$$

Therefore we calculate

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

从下往上数

and finally conclude that the entry at row 12 and column 11 is 31.

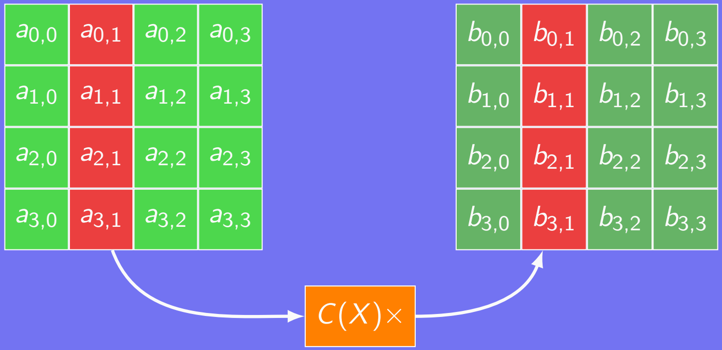
The ShiftRows layer

$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$
$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$
$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
$a_{3,0}$	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$

$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$
$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,0}$
$a_{2,2}$	$a_{2,3}$	$a_{2,0}$	$a_{2,1}$
$a_{3,3}$	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$

Cyclically shift to the left row i by offset i, $0 \leq i \leq 3$

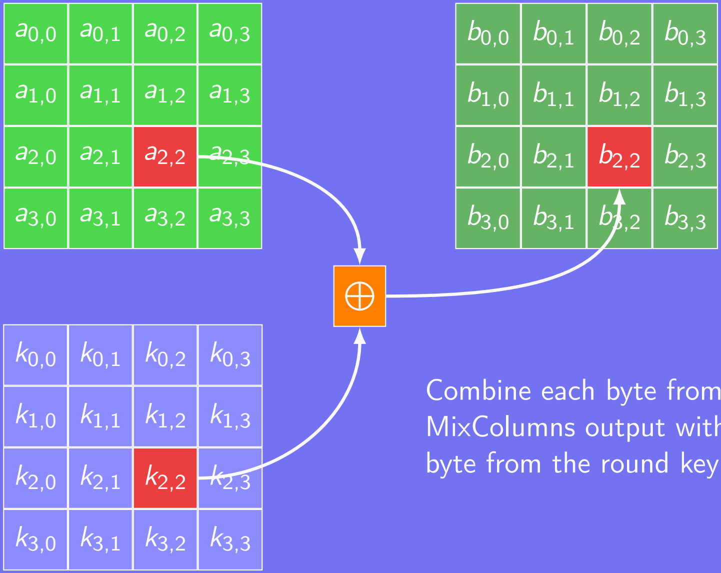
The MixColumns layer



Left multiply the output of ShiftRows by the matrix

$$C(X) = \begin{pmatrix} 00000010 & 00000011 & 00000001 & 00000001 \\ 00000001 & 00000010 & 00000011 & 00000001 \\ 00000001 & 00000001 & 00000010 & 00000011 \\ 00000011 & 00000001 & 00000001 & 00000010 \end{pmatrix}$$

The AddRoundKey layer



Combine each byte from the MixColumns output with a byte from the round key

Generating a round key

The original 128 bits key is arranged into a 4×4 matrix $K(X)$

Label the first four columns $K(0), \cdots, K(3)$ and add forty more:

- $K(i) = K(i-4) \oplus K(i-1)$, for $i \not\equiv 0 \bmod 4$
- $K(i) = K(i-4) \oplus T(K(i-1))$, for $i \equiv 0 \bmod 4$

The transformation $T(K(i-1))$ is defined over the column i:

- Compute $r(i) = 00000010^{\frac{i-4}{4}}$ 1
- Cyclically top shift the elements of the column by 1
- Apply the SubBytes layer (2.42) to each byte of the column and get the column vector (a, b, c, d) 00010110
- Finally return the column vector

$$T(K(i-1)) = (a \oplus r(i), b, c, d)$$

The i-th round key is given by the columns $K(4i), \cdots, K(4i+3)$

Example. $K(i)$ being simple to generate for $i \not\equiv 0 \bmod 4$, we focus on the case $i \equiv 0 \bmod 4$. For instance if $i = 40$ and $K(39)$ is the column vector $(10001100, 00001100, 11000110, 11110011)$, then

- Cyclical top shit: $(00001100, 11000110, 11110011, 10001100)$
- SubBytes transformation:

$$\begin{array}{ccccccc} 00001100 & \to & 11111110, & & 11000110 & \to & 10110100 \\ 11110011 & \to & 00001101, & & 10001100 & \to & 01100100 \end{array}$$

- $r(40) = X^9 \equiv X^5 + X^4 + X^2 + X \bmod P(X) = 00110110$
- Get the final column vector $T(K(39))$

$$\begin{aligned} T(K(39)) &= (11111110 \oplus 00110110, 10110100, 00001101, 01100100) \\ &= (11001000, 10110100, 00001101, 01100100) \end{aligned}$$

- Finally define $K(40)$ as $K(36) \oplus T(K(39))$

Decryption

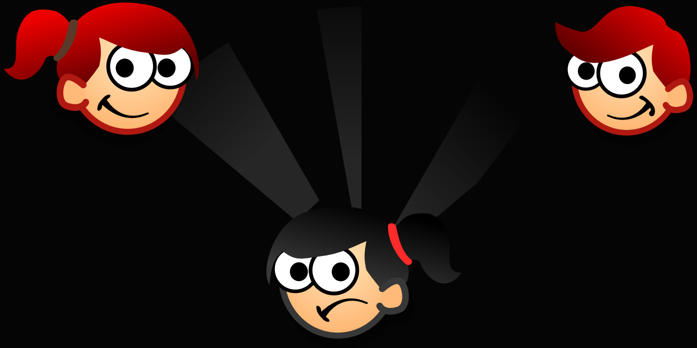
The decryption process is simple:

- Perform all the operations in reverse order
- Replace the SubBytes, ShitRows and MixColumns operations by their inverse

Remark. It is possible to construct an inverse cipher performing decryption by applying a sequence of inverse operations in the same order as it is done for encryption

Key points

- What does it mean to be random?
- Recall Fermat's little theorem
- Recall the CRT
- How is a Feistel network organised?
- Describe AES



Thank you!

References I

2.4 https://upload.wikimedia.org/wikipedia/commons/5/56/Tux.jpg

2.4 https://upload.wikimedia.org/wikipedia/commons/f/f0/Tux_ecb.jpg

2.4 https://upload.wikimedia.org/wikipedia/commons/a/a0/Tux_secure.jpg

2.7 https://www.xkcd.com/221/

Inverse of a mod b