LEC011 Maximum Flow and Linear Program

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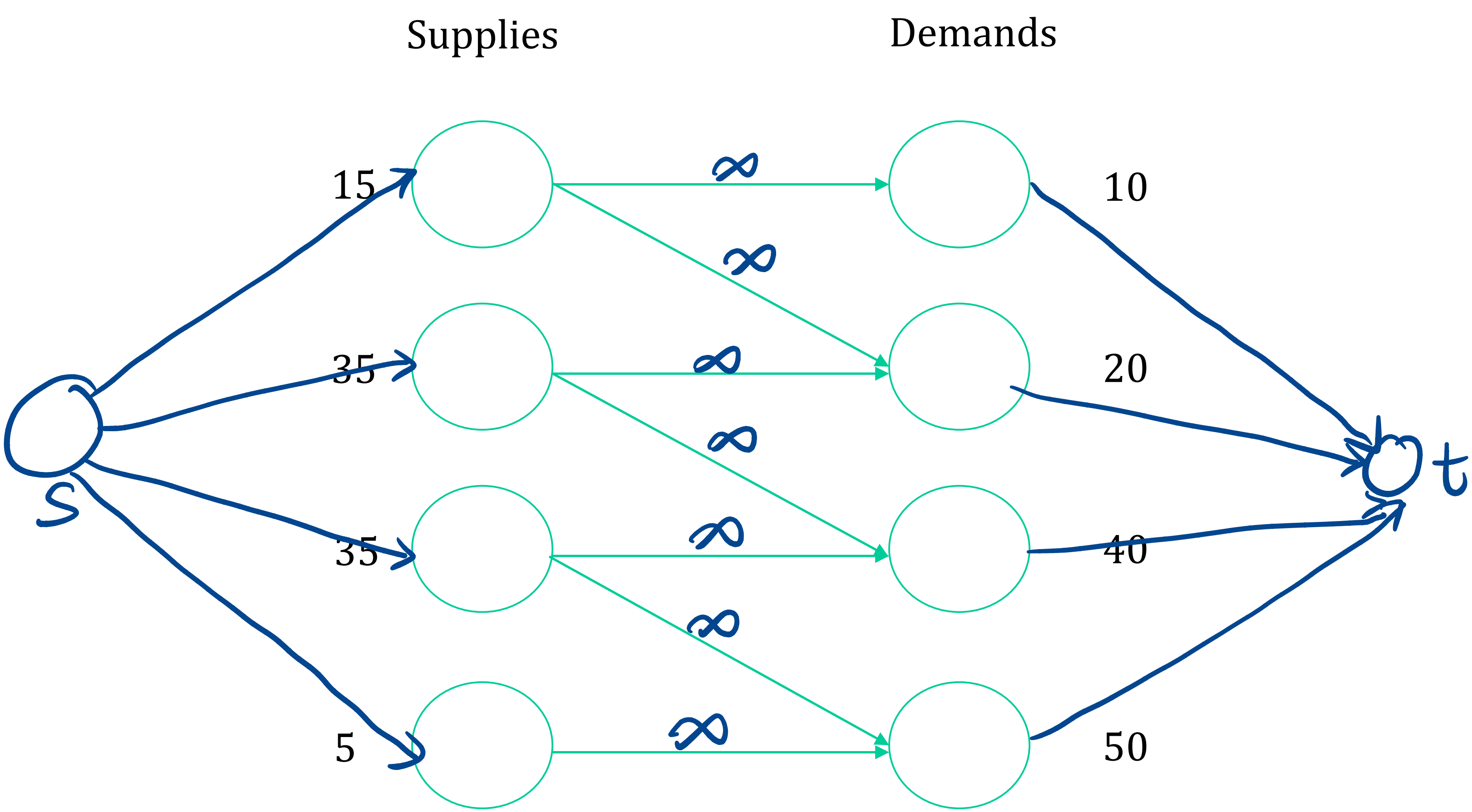
Max Flow Applications

matching problem



Linear Programming

- Comparison to systems of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$Ax = b$

$\min c^T x$

$Ax \leq b$

Linear Programming

- Ingredients of a Linear Program

1. Decision variables $x_1, \dots, x_n \in \mathbb{R}$
2. Linear constraints, each of the form

$$\sum_{j=1}^{n} a_{ij} x_j \quad \overbrace{(*)}^{=/\leq} \quad b_i$$

   where $(^*)$ could be $\leq$, $\geq$, or $=$

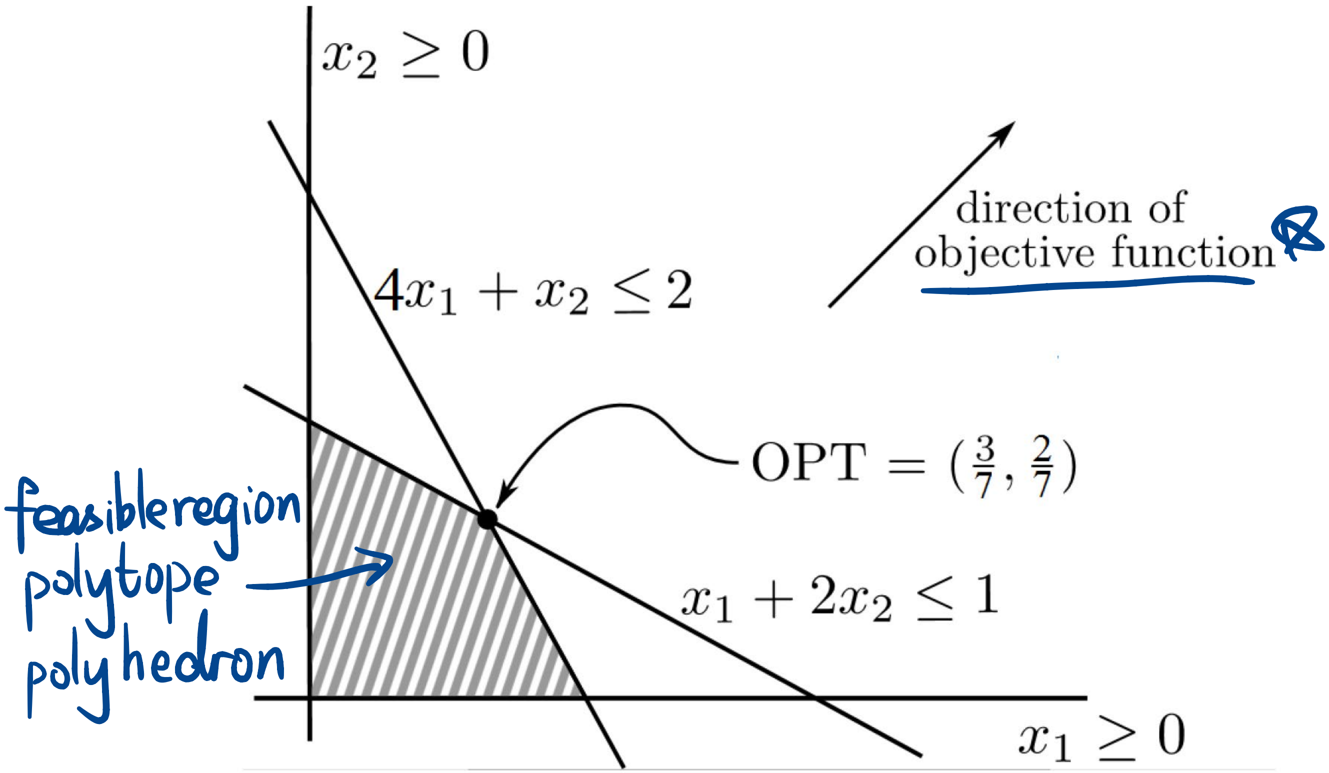
3. A linear objective function, of the form

$$\max \sum_{j=1}^{n} c_j x_j \qquad \text{or} \qquad \min \sum_{j=1}^{n} c_j x_j$$

A Simple Example

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



Python + Gurobi Time!

1. Excel Add-in Tools
2. small scale problem → Excel

LP - simple

jupyter

GUROBI
OPTIMIZATION

MaxFlow is a LP

- Decision variables

$$\{f_e\}_{e \in E}$$

- Constraints $(2m + n - 2)$

$$\underbrace{\sum_{e \in \delta^-(v)} f_e}_{\text{flow in}} - \underbrace{\sum_{e \in \delta^-(v)} f_e}_{\text{flow out}} = 0$$

$$f_e \leq u_e$$

$$f_e \geq 0$$

- Objectives

$$\max \textstyle\sum_{e \in \delta^+(s)} f_e$$

Generalization of MaxFlow → simple

⇓

F-F algorithm

(fast)

- Min-Cost MaxFlow

$$\min \sum_{e \in E} c_e f_e$$

maxflow ≥ θ

add additional constraints

- Easy to change the LP formulation!

LP优点：customize

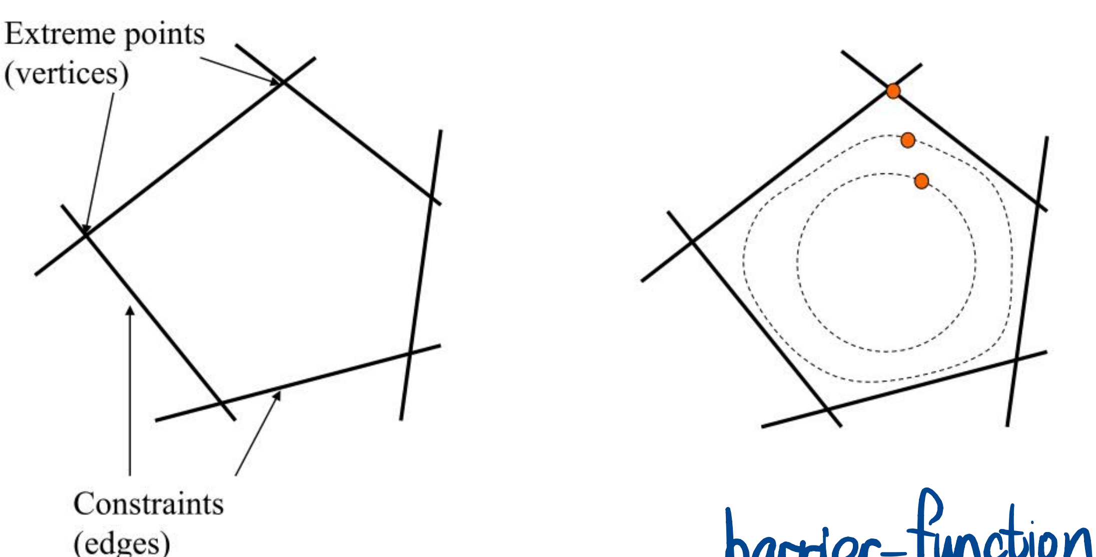
Solution Approaches(LP)

- Simplex methods
- Interior point methods

start of every vertex, check neighbours, stop when no neighbour is better (只找得到local optimization)

polytope : feasible region ^Grobi

Interior point methods



barrier-function.

Simplex: search from vertex to vertex along the edges

Interior-point methods: go through the inside of the feasible space

CVX(develop by stanford) solve this by interior

Primal LP ⇔ Dual LP LP Duality 二重性

- Question: given a feasible solution, how can we know whether it is optimal or close to optimal? can we actually get the upper bound of solution

$$\boxed{\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & 4x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \leq 4x_1 + x_2 \leq \underbrace{2}_{\text{upper bound}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \leq x_1 + 2x_2 \leq \underbrace{1}_{\text{upper bound}}$$

how can I make sure that I have the best upper bound

Can we get an even better upper bound?

$$x_1 + x_2 \leq \frac{1}{7}\underbrace{(4x_1 + x_2)}_{\leq 2 \text{ by } (2)} + \frac{3}{7}\underbrace{(x_1 + 2x_2)}_{\leq 1 \text{ by } (3)} \leq \frac{1}{7} \cdot 2 + \frac{3}{7} \cdot 1 = \frac{5}{7}$$

better upper bound

$x_1^* = \frac{3}{7}$ $x_2^* = \frac{2}{7}$ $obj^* = \frac{5}{7}$

Deriving the Dual LP

Compact form

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

$$\begin{array}{llcl} \max & \sum_{j=1}^{n} c_j x_j & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{1j} x_j \leq b_1 & \longrightarrow & y_1 \geq 0 \\ & \sum_{j=1}^{n} a_{2j} x_j \leq b_2 & \longrightarrow & y_2 \geq 0 \\ & \vdots \leq \vdots & & \\ & \sum_{j=1}^{n} a_{mj} x_j \leq b_m & \longrightarrow & y_m \geq 0 \\ & x_1, \ldots, x_n \geq 0 & & \end{array}$$

"Multipliers"

The idea is to "dominates the objective coefficients":

$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j$$

More compactly,

Find $\mathbf{y} \geq 0$ such that $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$

mixing all the constraints

Deriving the Dual LP

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}$$

Find $\mathbf{y} \ge 0$ such that $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$

clever thing to do:

$$\boxed{\begin{array}{l} \min\ y^t b. \\ A^t y \ge c \\ y \ge 0 \end{array}}$$

dual LP

$$\begin{aligned} \underbrace{\sum_{j=1}^{n} c_j x_j}_{\text{x's obj fn}} &\le \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} y_i a_{ij}\right)} x_j \\ &= \sum_{i=1}^{m} y_i \cdot \left(\sum_{j=1}^{n} a_{ij} x_j\right) \\ &\le \underbrace{\sum_{i=1}^{m} y_i b_i}_{\text{upper bound}} \end{aligned}$$

dominant c

$\le b_i \ \forall i$

把第1项弄得更general一点。

More compactly, $\mathbf{c}^T\mathbf{x} \le \left(\mathbf{A}^T\mathbf{y}\right)^T\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \le \mathbf{y}^t\mathbf{b}$

Deriving the Dual LP

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}$$

weak duality

OPT(P) ≤ OPT(D)

Find $\mathbf{y} \ge 0$ such that $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$

$$\mathbf{c}^T\mathbf{x} \le \left(\mathbf{A}^T\mathbf{y}\right)^T\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \le \mathbf{y}^t\mathbf{b}$$

$$\begin{array}{ll} \min & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{array}$$

strong duality:

OPT(P) = OPT(D)

proof is non-trivial:

hyperplane separation

Farkas' Lemma

A Simple Example

Primal problem

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$

$b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Dual Problem

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & 2y_1 + y_2 \\ \text{s.t.} \quad & 4y_1 + y_2 \geq 1 \\ & y_1 + 2y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

$x_1^* = 3/7, x_2^* = 2/7$

$y_1^* = 1/7, y_2^* = 3/7$

Optimal objectives are the same!

Weak and Strong Duality

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array} \qquad \begin{array}{ll} \min & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{array}$$

We have

OPT(P) ≤ OPT(D)

- Weak Duality $\underline{\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}}$ for any feasible solutions $\mathbf{x}$ and $\mathbf{y}$

OPT(P) = OPT(D)

- Strong Dualiy $\underline{\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*}$ at optimality

Proof = Separating hyperplance theorem + Farkas's Lemma

More General Form

Primal	Dual
variables $x_1, \ldots, x_n$	n constraints
m constraints	variables $y_1, \ldots, y_m$
objective function $\mathbf{c}$	right-hand side $\mathbf{c}$
right-hand side $\mathbf{b}$	objective function $\mathbf{b}$
$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
constraint matrix $\mathbf{A}$	constraint matrix $\mathbf{A}^T$
i-th constraint is "$\leq$"	$y_i \geq 0$
i-th constraint is "$\geq$"	$y_i \leq 0$
i-th constraint is "$=$"	$y_i \in \mathbb{R}$
$x_j \geq 0$	j-th constraint is "$\geq$"
$x_j \leq 0$	j-th constraint is "$\leq$"
$x_j \in \mathbb{R}$	j-th constraint is "$=$"

constraint $\Longleftrightarrow$ variable

variable $\Longleftrightarrow$ constraints

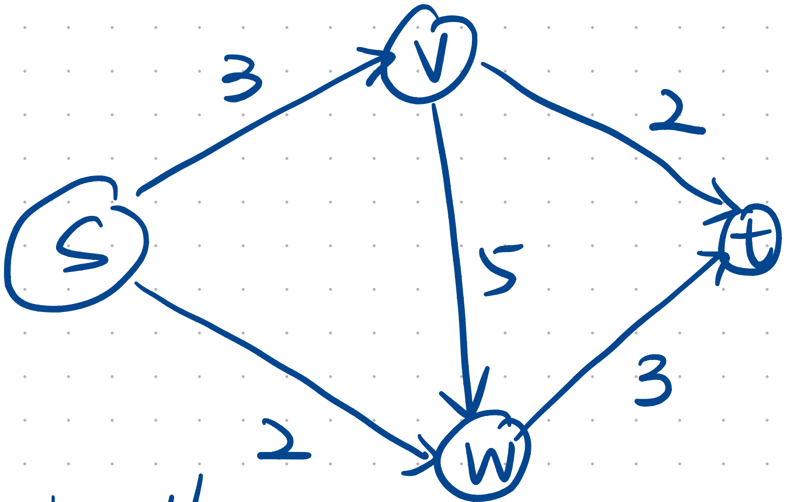
Back to MaxFlow

- Let the decision variable be flow on an s-t path

Primal

$$\begin{aligned} \max \quad & \sum_{P \in \mathcal{P}} f_P \\ \text{s.t.} \quad & \underbrace{\sum_{P \in \mathcal{P}: e \in P} f_P}_{\text{total flow on } e} \leq u_e \ \text{ for all } e \in E \\ & f_P \geq 0 \text{ for all } P \in \mathcal{P} \end{aligned}$$

Question: how to write its dual and what is its interpretation?



s-t path:

$P_1: s \to v \to t$ $\quad f_{P_1} = 2$

$P_2: s \to v \to w \to t$ $\quad f_{P_2} = 1$

$P_3: s \to w \to t$ $\quad f_{P_3} = 2$

$$\max \sum_{p \in P} f_p$$

$$\text{st. } \sum_{p \in P : e \in P} f_p \le u_e, \forall e \in E$$

$$f_p \geq 0 \ \forall p \in \mathcal{P}$$

$\downarrow$ convert into matrix notation

$$\max \mathbb{1}^T f$$

$$Af \le u$$

$$f \geq 0$$