# LEC012 Facility Location

warehouse

find optimal location (medium long-term strategy decisions)

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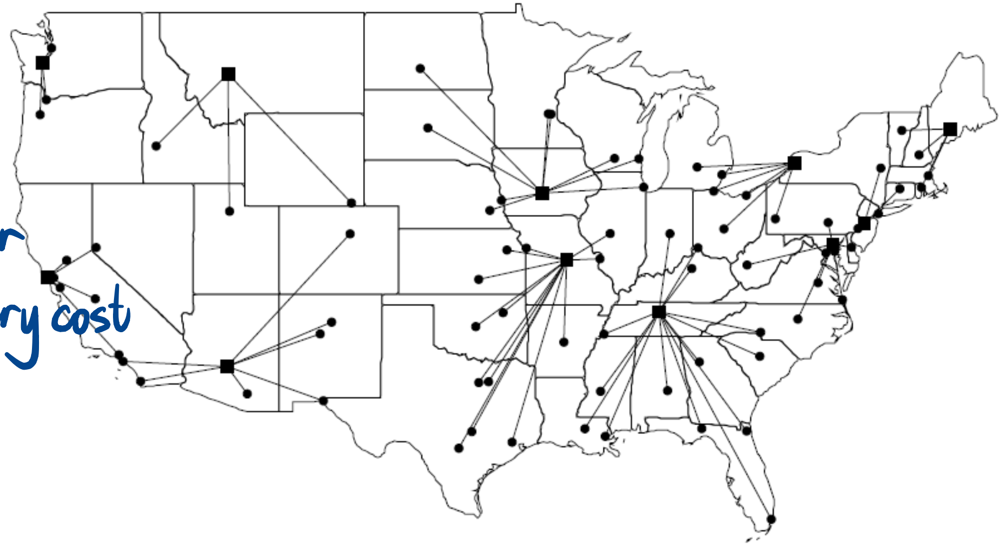
# Configurations trade-off

different strategy

- satisfy customers faster
- reduce last-mile delivery cost

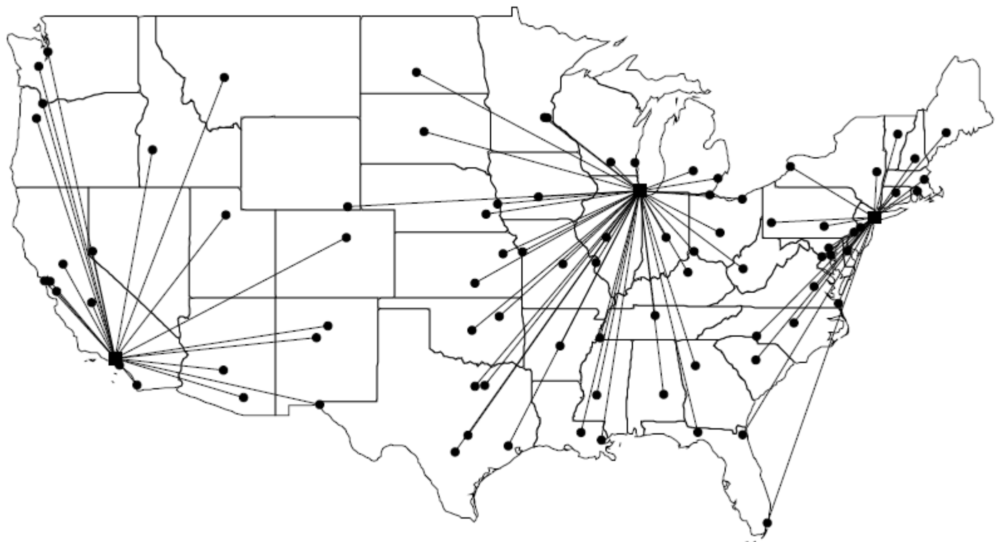
● demand sites

▪ facility sites



(a) Many facilities open.

- lower setup cost
- lower labour cost/ equipment cost



(b) Few facilities open.

# Formulation

map $I = ((a_1, b_1), (a_2, b_2), \ldots, )$

- Sets

demand sites

$I$ = set of customers

$J$ = set of potential facility locations

- Parameters

$d_i$

$h_i$ = annual demand of customer $i \in I$

$c_{ij}$ = cost to transport one unit of demand from facility $j \in J$ to customer $i \in I$

$f_j$ = fixed annual cost to open a facility at site $j \in J$

setup cost

- Decision Variables

$x_j$ = 1 if facility $j$ is opened, 0 otherwise binary

assignment/fullfillment

$y_{ij}$ = the fraction of customer $i$ 's demand that is served by facility $j$

# Transportation costs $c_{ij}$

- Euclidean distance

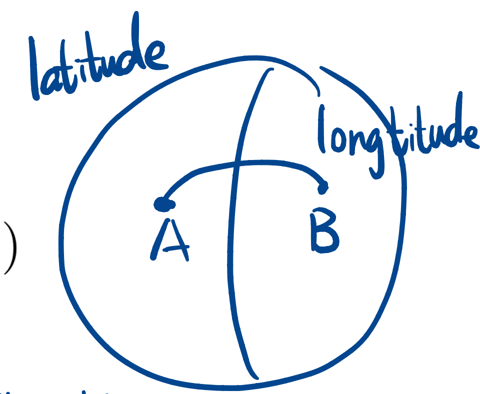
$$\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

- Manhattan or rectilinear metric

$$|a_1 - a_2| + |b_1 - b_2|$$

- Great circle essential for long distance

$$r \arccos\left(\sin\alpha_1 \sin\alpha_2 + \cos\alpha_1 \cos\alpha_2 \cos(\Delta\beta)\right)$$




- GPS/MAP distance most useful one for any fullfillment(online)

solve LP for in an online fashion

# MILP Formulation

Decision variables:

$x_j = 1$ if facility $j$ is opened, 0 otherwise

$y_{ij} =$ the fraction of customer $i$ 's demand that is served by facility $j$

"facility location LP"

$$\text{minimize} \quad \underbrace{\sum_{j \in J} f_j x_j}_{\text{total setup cost}} + \underbrace{\sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij}}_{\text{transport/fulfillment cost}}$$

$$\text{subject to} \quad \sum_{j \in J} y_{ij} = 1, \qquad \forall i \in I$$ → Assignment Constraint

$$y_{ij} \leq x_j, \qquad \forall i \in I, \forall j \in J$$ → Linking Constraint

$$x_j \in \{0, 1\}, \qquad \forall j \in J$$

$$y_{ij} \geq 0, \qquad \forall i \in I, \forall j \in J$$

you can only serve demand from facility j if j is open

# Python + Gurobi Time!

# Lagrangian Relaxation

- Introduce a penalty term: $\sum_{i \in I} \lambda_i \left(1 - \sum_{j \in J} y_{ij}\right)$

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} + \sum_{i \in I} \lambda_i \left(1 - \sum_{j \in J} y_{ij}\right) \\
& = \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} \left(h_i c_{ij} - \lambda_i\right) y_{ij} + \sum_{i \in I} \lambda_i \\
\text{subject to} \quad & y_{ij} \le x_j, \qquad \forall i \in I, \forall j \in J \\
& x_j \in \{0, 1\}, \qquad \forall j \in J \\
& y_{ij} \ge 0, \qquad \forall i \in I, \forall j \in J
\end{aligned}
$$

# Lagrangian Relaxation

- It turns out that this relaxiation is easy to solve:

$$
\begin{array}{ll}
\text{minimize} & = \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} \left(h_i c_{ij} - \lambda_i\right) y_{ij} + \sum_{i \in I} \lambda_i \\
\text{subject to} & y_{ij} \le x_j, \qquad \forall i \in I, \forall j \in J \\
& x_j \in \{0,1\}, \qquad \forall j \in J \\
& y_{ij} \ge 0, \qquad \forall i \in I, \forall j \in J
\end{array}
$$

- Observe that if we open facility $j$, i.e., set $x_j = 1$

$$\beta_j = \sum_{i \in I} \min\left\{0, h_i c_{ij} - \lambda_i\right\} \longrightarrow \text{“benefit”}$$

- So we should open $j$ if and only if $\beta_j + f_j < 0$

# Lagrangian Relaxation

$$
\begin{array}{ll}
\text{minimize} & = \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} \left( h_i c_{ij} - \lambda_i \right) y_{ij} + \sum_{i \in I} \lambda_i \\
\text{subject to} & y_{ij} \leq x_j, \qquad \forall i \in I, \forall j \in J \\
& x_j \in \{0, 1\}, \qquad \forall j \in J \\
& y_{ij} \geq 0, \qquad \forall i \in I, \forall j \in J
\end{array}
$$

Solution:

$$
\bar{x}_j = \begin{cases} 1, & \text{if } \beta_j + f_j < 0 \\ 0, & \text{otherwise} \end{cases}
$$

$$
\bar{y}_{ij} = \begin{cases} 1, & \text{if } \bar{x}_j = 1 \text{ and } h_i c_{ij} - \lambda_i < 0 \\ 0, & \text{otherwise} \end{cases}
$$

Objective value:

$$
z_{\mathrm{LR}}(\lambda) = \sum_{j \in J} \min \{0, \beta_j + f_j\} + \sum_{i \in I} \lambda_i
$$

# Now Construct UB

- Use the “opened” facility in the relaxed solution
- Assign each customer to its nearest open facility

# Updating Multipliers

- Adjust the penalty multipliers:

If $\sum_{j \in J} y_{ij} = 0$, then $\lambda_i$ is too small; it should be increased.
If $\sum_{j \in J} y_{ij} > 1$, then $\lambda_i$ is too large; it should be decreased.
If $\sum_{j \in J} y_{ij} = 1$, then $\lambda_i$ is just right; it should not be changed.

- Then a natural updating rule becomes

$$\lambda_i^{t+1} = \lambda_i^t + \Delta^t \left( 1 - \sum_{j \in J} y_{ij} \right)$$

where the step-size is

$$\Delta^t = \frac{\left(\mathrm{UB} - z_{\mathrm{LR}}\left(\lambda^t\right)\right)}{\sum_{i \in I} \left(1 - \sum_{j \in J} y_{ij}\right)^2}$$