LEC008 Inventory Management III

VG441 SS2021

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Newsvendor Problem

random variable demand 分布

- Let D be random demand with $\mu = E[D]$ and $\sigma^2 = V[D]$
- Let c be unit cost, $r > c$ selling price, $s < c$ salvage value
- Question: what is the optimal ordering quantity S?

Newsvendor Problem

$(S-D)^+$

$= \max(S-D, 0)$

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revenue function

charging

$$\pi(S) = rE[\min(S, D)] + sE[(S-D)^+] - cS$$

sales

ordering cost

goal: $\max \pi(S)$

Transforming this objective into ...

$$\pi(S) = (r-c)\mu - g(S)$$

where $$g(S) = \underbrace{(c-s)}_{h} \underbrace{E[(S-D)^+]}_{\text{left over}} + \underbrace{(r-c)}_{p} \underbrace{E[(D-S)^+]}_{\text{shortage}}$$

newvender cost function

Newsvendor Problem

- Minimizing …

$\min_S g(S)$

convex in S

$$g(S) = hE[(S-D)^+] + pE[(D-S)^+]$$

- The optimal solution is called "newsvendor quantile":

$$\frac{dg(S)}{dS} = hF(S) + p(F(S)-1) = (h+p)F(S) - p = 0$$

quantile

$$S^* = F^{-1}\left(\frac{p}{h+p}\right)$$

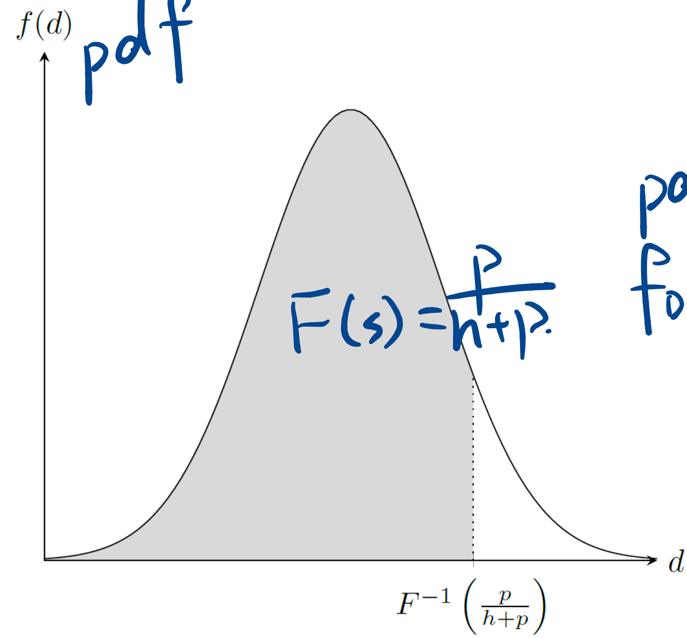
$$\frac{dE[(S-D)^+]}{dS}$$

why?

$$= dE[(S-D)\mathbb{I}(S\geq D) + 0\,\mathbb{I}(S<D)]$$

$$= E[\mathbb{I}(S\geq D)] = P[S\geq D] = F(S)$$

pdf



$F(s) = \frac{p}{h+p}$

D → pdf $f_D(\cdot)$, cdf $F_D(\cdot)$

Explicit Form for Normal Demand

- $D \sim N(50, 8^2), h = 0.18, p = 0.70$

optimal
order
quantity

$$S^* = F^{-1}\left(\frac{0.70}{0.18+0.70}\right) = F^{-1}(0.795) = 56.6$$

indication
$p \uparrow$ $S^* \uparrow$

- Derivation…

$$F\left(S^*\right) = \frac{p}{h+p}$$

$$\iff \quad \Phi\left(\frac{S^*-\mu}{\sigma}\right) = \frac{p}{h+p}$$

$$\iff \quad S^* = \mu + \sigma\Phi^{-1}\left(\frac{p}{h+p}\right)$$

- Concept of cycle stock and safety stock

optimal order
critical quantile

$$S^* = \mu + \underline{z_\alpha \sigma} \quad \text{where } \alpha = p/(h+p) \text{ and } z_\alpha = \Phi^{-1}(\alpha)$$

cycle stock safety stock
critical ratio

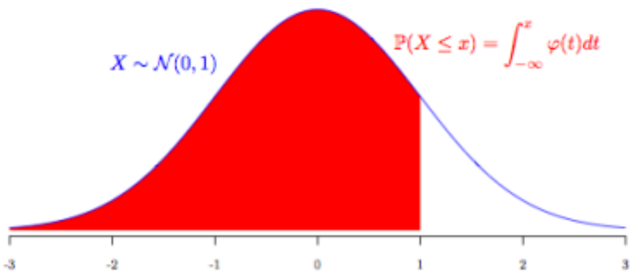
at least, order over the mean
buffer for uncertainty

focus on leadtime D $\Rightarrow$ long. S.C

if lead time Demand

σ: stdev of lead time D

Normal CDF Table



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Forecast and STDEV

- Don't use the stdev of demand

$$\hat{\sigma}_t = \sqrt{\frac{1}{N-1}\sum_{i=t-N}^{t-1}(d_t - \hat{\mu}_t)^2} \text{ where } \hat{\mu}_t = \frac{1}{N}\sum_{i=t-N}^{t-1} d_t$$

- Instead, use the stdev of forecast errors $S =$ point forecast $+ z_\alpha \cdot \sigma$

$$e_t = \text{forecast}_t - d_t$$

$$\hat{\sigma}_{e,t} = \sqrt{\frac{1}{N-1}\sum_{i=t-N}^{t-1}(e_t - \hat{e}_t)^2} \text{ where } \hat{e}_t = \frac{1}{N}\sum_{i=t-N}^{t-1} e_t$$

reason : 1) demand volatile, forecase model is good at predicting
⇒ residuals are small
2) demand less volatile → larger

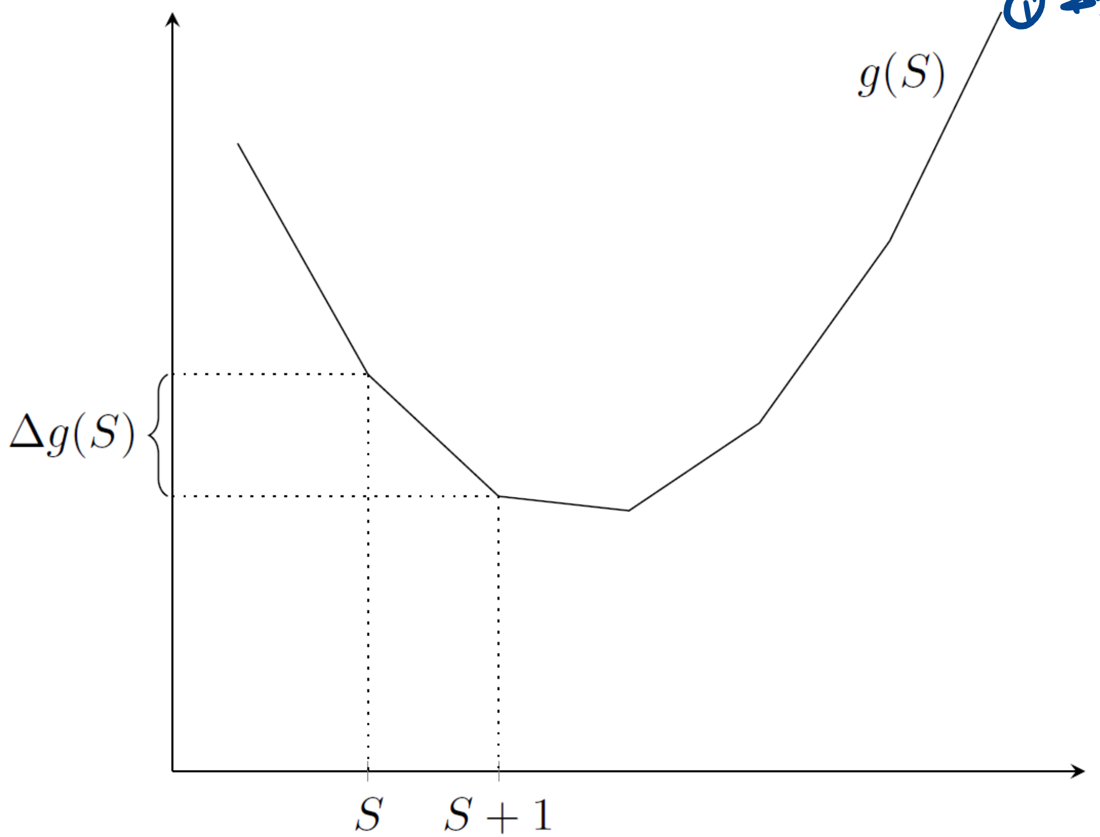
What about Discrete Demand?

Find the smallest S such that $F(S) \geq \dfrac{p}{h+p}$

fractional

critical ratio

① 判断是 convex



① $S = F^{-1}\left(\frac{p}{h+p}\right)$. however, S should be integer

Multiple-Period Newsvendor

dynamic programming

- $D_1, \dots, D_T$ i.i.d. demands over period $1, \dots, T$

- In each period $t = 1, \dots, T$
  - Raise the beginning inventory x to a new level y
  - Demand realizes and is satisfied to the max extent
  - Assess the cost (ordering, holding, backlogging)

- Objective: find the optimal ordering strategy

Multiple-Period Newsvendor ☆需调查.

value function (state) = the optimal cost over [t,T] given that the initial state = state$_0$

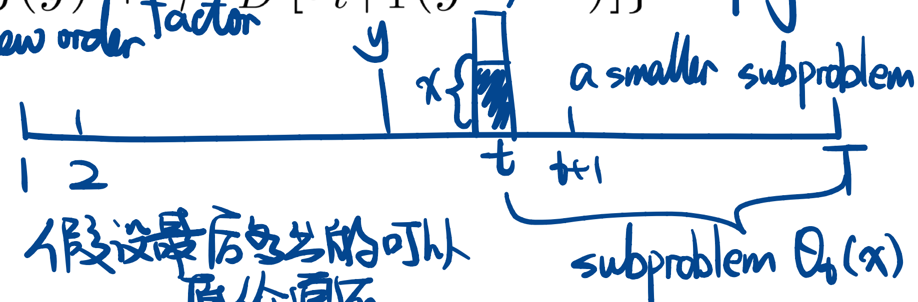
- Let $\theta_t(x)$ be the optimal expect cost for periods $[t, T]$ if we begin with inventory x in period t

Bellman equation

discount factor   future cost   new order $q = y - x$

$$\theta_t(x) = \min_{y \geq x} \{ \underbrace{c(y-x)}_{\text{ordering cost}} + g(y) + \gamma \mathbb{E}_D [\theta_{t+1}(y - D)] \}$$

new order



- Boundary condition:

假设最后多出的可以原价退还

$$\theta_{T+1}(x) = -cx$$

$x_t = f(x_t) = x_t - D$ state transition

assess the cost in period t (immediate cost)

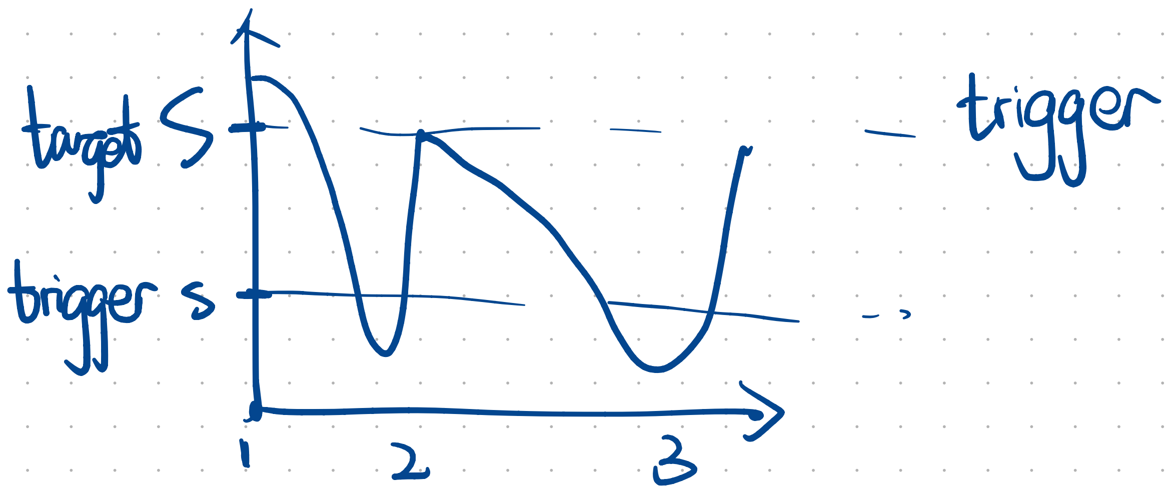
- The optimal strategy is the called base-stock policy:

$$S^* = F^{-1}\left(\frac{p - (1-\gamma)c}{h + p}\right)$$ 是一个常数

$g(y) = E_D[h(y-D)^+ + p(D-y)^+]$

holding   backlogging

(s, S) policy → when your holding cost is fixed



Optimality of Base-Stock

$$\theta_{T+1}(x) = -cx$$

$$\theta_t(x) = \min_{y \geq x} \{H_t(y) - cx\}$$

where $H_t(y) = cy + g(y) + \gamma \mathbb{E}_D \left[\theta_{t+1}(y - D)\right]$

Claim:

If $\theta_{t+1}(x)$ is convex, then:
(a) $H_t(y)$ is convex.
(b) A base-stock policy is optimal in period t
and any minimizer of $H_t(y)$ is an optimal base-stock level.
(c) $\theta_t(x)$ is convex.

By assumption, $\theta_{T+1}(x)$ is convex. Therefore, by Claim, a base-stock policy is optimal in period T. Moreover, $\theta_T(x)$ is convex by Claim. This implies that a base-stock policy is optimal in period $T-1$ and that $\theta_{T-1}(x)$ is convex. Continuing this logic, a base-stock policy is optimal in every period.

Multiple-Period Newsvendor

$$\underbrace{\theta_t(x)}_{g(x)} = \min_{y \geq x} \left\{ \underbrace{H_t(y) - cx}_{f(x,y)} \right\}$$

Claim:

If $f(x, y)$ is jointly convex in x and y, and C is a convex set, then the function $g(x) = \min_{y \in C} f(x, y)$ is convex in x.