LEC010 Maximum Flow

VG441 SS2021

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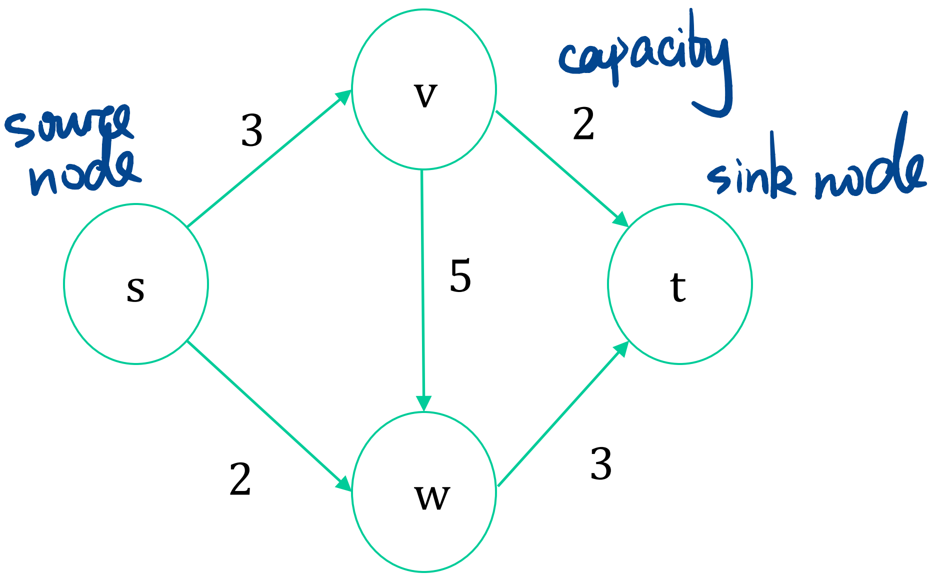
Max Flow



Applicants:

order fullfillment

process flexibility

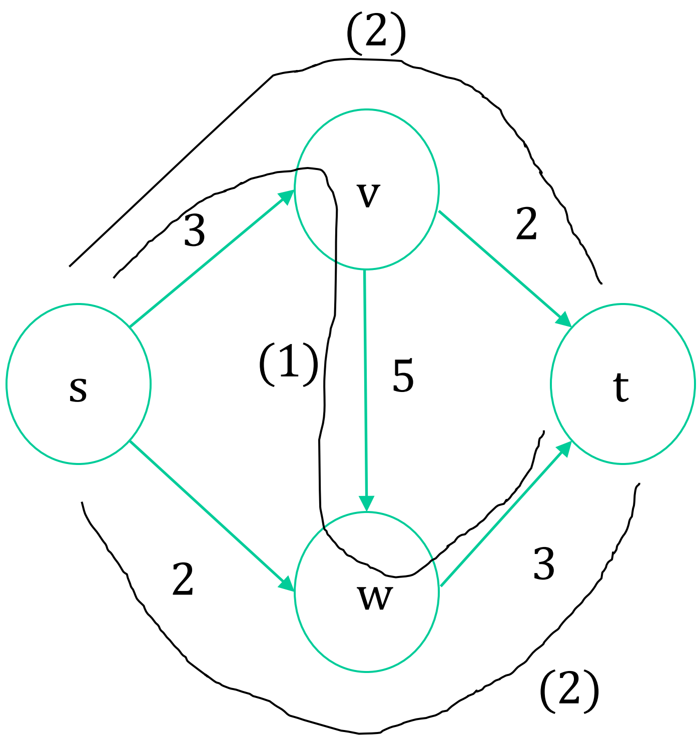
maximal matching

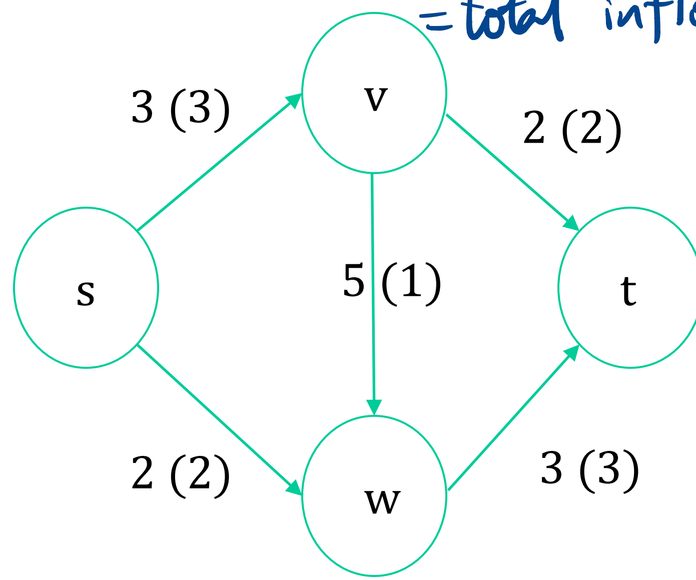
The number on each edge is capacity

Qn: Push as much flow as possible from s to t

Max Flow

opitmal soln s = total amount outflow to s
= total inflow to t





The number on each edge is capacity

Qn: Push as much flow as possible from *s* to *t*

Max Flow

- Input
  - a directed graph G, with vertices V and directed edges E
  - a source vertex $s \in V$ (no edges into s)
  - a sink vertex $t \in V$ (no edges out of t)
  - a nonnegative and integral capacity u_e for each edge $e \in E$
- Feasible solutions – flows
  - Nonnegativity constraints: $f_e \geq 0$ for every edge $e \in E$
  - Capacity constraints: $f_e \leq u_e$ for every edge $e \in E$
  - Conservation constraints: for every vertex v other than s and t
    amount of flow entering v = amount of flow exiting v
- Goal: maximize flow value = flow going out of S flow going into
  t

- Attempt #1:

A Naive Greedy Algorithm

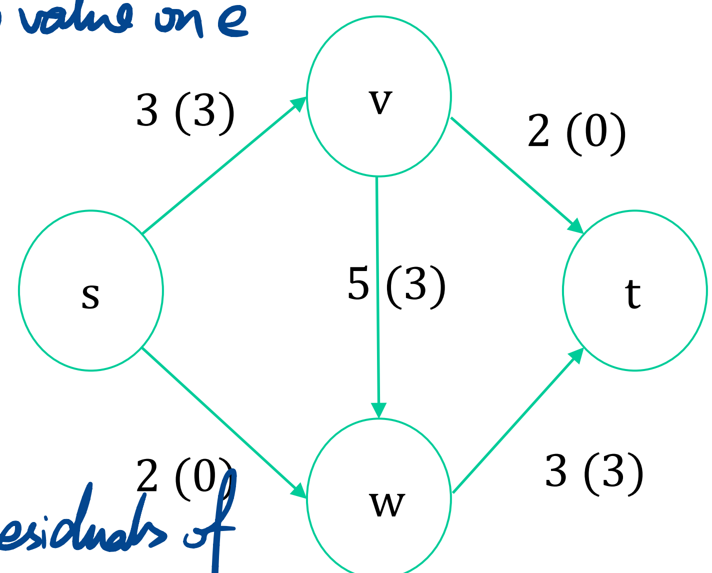
initialize $f_e = 0$ for all $e \in E$
repeat
  search for an $s-t$ path P such that $f_e < u_e$ for every $e \in P$
  // takes $O(|E|)$ time using BFS or DFS
  if no such path **then**
    halt with current flow $\{f_e\}_{e \in E}$
  else
    let $\Delta = \underbrace{\min_{e \in P} \overbrace{(u_e - f_e)}^{\text{room on } e}}_{\text{room on } P}$
    for all edges e of P do increase f_e by Δ

residual capacity

"bottle neck" along s-t path

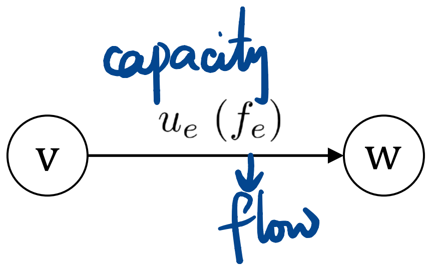
capone - current flow value on e

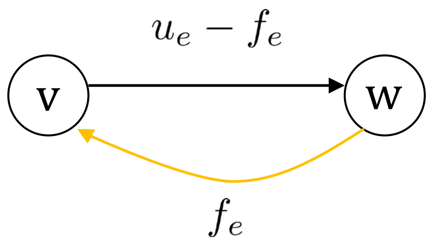
iteration #1 s→v→w→t Δ=3
send 3 units of flow along s→v→w→t
but s→w, v→w, v→t has residuals of capacity of 2 ⇒ 产生浪费



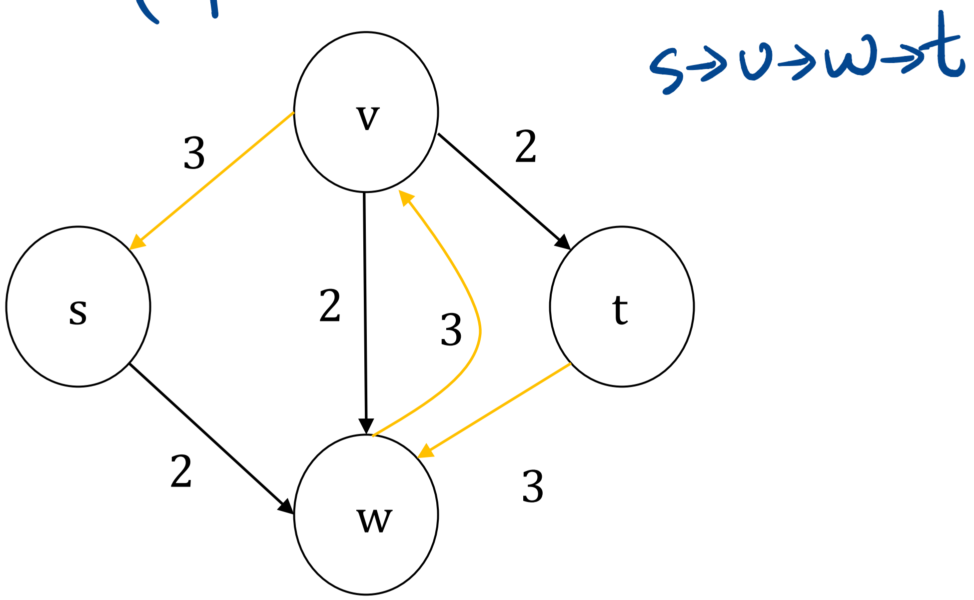
Max Flow

- Attempt #2: Allow "undo" operations





- Residual network (G_f)



Max Flow

most famous algorithm in network optimization

Ford-Fulkerson Algorithm

initialize $f_e = 0$ for all $e \in E$

repeat

search for an $s - t$ path P in the current residual graph G_f such that every edge of P has positive residual capacity

// takes $O(|E|)$ time using BFS or DFS

if no such path **then**

halt with current flow $\{f_e\}_{e \in E}$

else

let $\Delta = \min_{e \in P}$ (e' s residual capacity in G_f)

// augment the flow f using the path P

$f_e = f_e + \Delta$ is e is black

for all edges e of G whose corresponding forward edge is in P **do**

increase f_e by Δ

for all edges e of G whose corresponding reverse edge is in P **do**

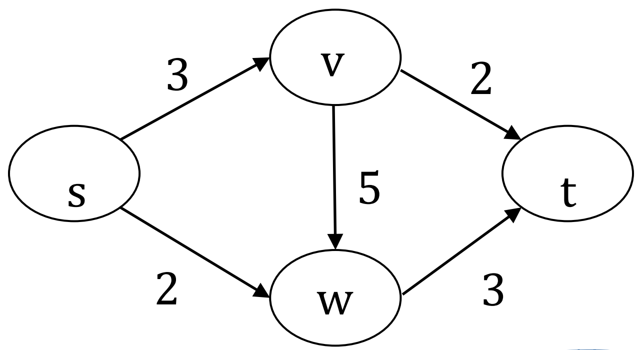
decrease f_e by Δ

$f_e = f_e - \Delta$ if e is orange

Max Flow

- Run Ford-Fulkerson on the simple example

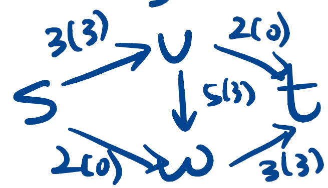
G



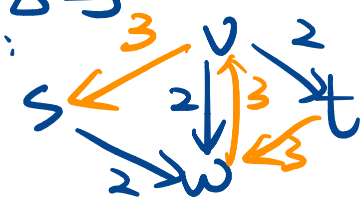
iteration#1 the residual diagram = G

$s \to v \to w \to t \quad \Delta = 3$

$\Rightarrow$



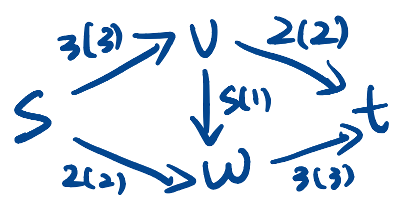
Gf:



iteration #2

$s \to w \to v \to t \quad \Delta = 2$

$\Rightarrow$

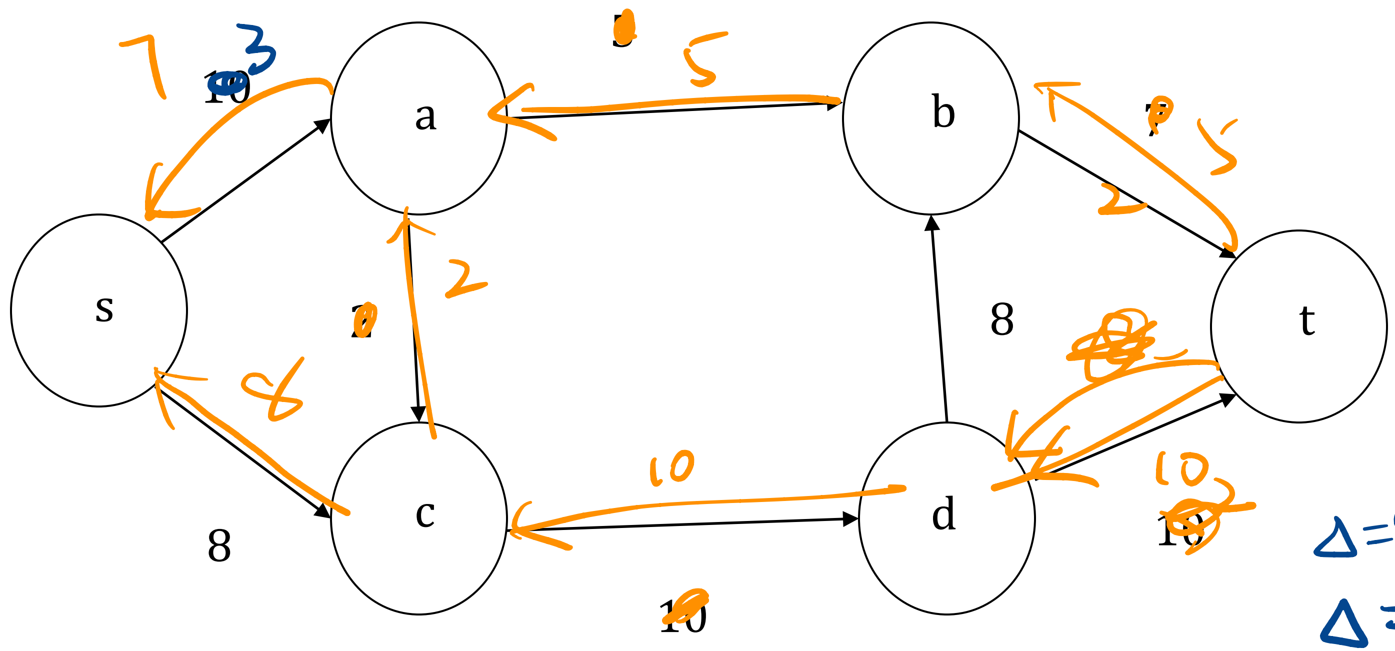


Gf:

iteration #3

no path

Exercise



$\Delta = 8$

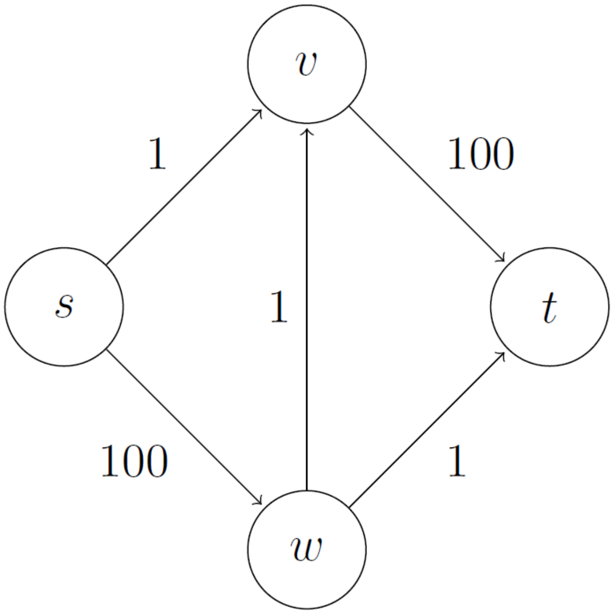
$\Delta = 5$

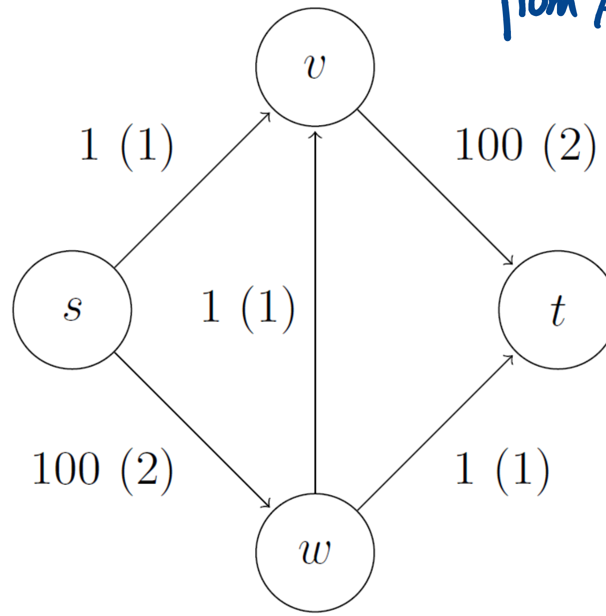
$\Delta = 2$

How do we know we are done?

1. whether F-F terminates in finite iters
2. whether F-F outputs the optimal solutions

total capacity from A to B





Two-Step Paradigm:

{s,w} {v,t}

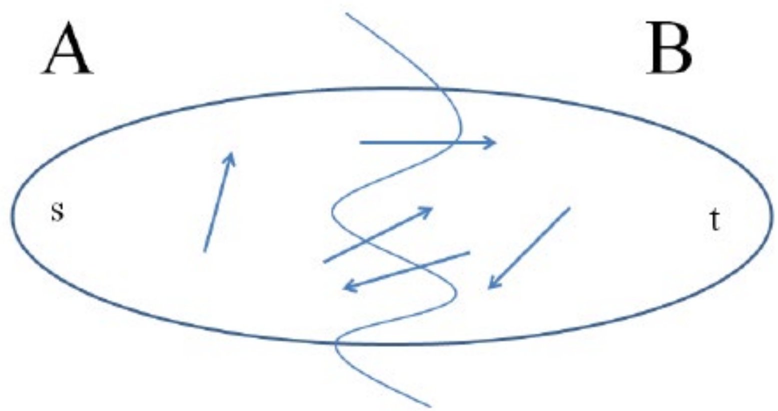
- Identify "optimality condition" find partition [A,B]
- Design an algorithm that terminates w/ the optimality condition satisfied

(s,t) cuts

- "Dual" flows

Definition An (s, t) -cut of a graph $G = (V, E)$ is a partition of V into sets A, B with $s \in A$ and $t \in B$

$\delta^+(A)$
$=$ all the outgoing arcs (edges) from A to B
$\delta^-(A)$
$=$ all the incoming arcs into A from B



The capacity of an (s, t) -cut (A, B) is defined as

$$\sum_{e \in \delta^+(A)} u_e$$

Σ: focused on outgoing arcs

serves as UB of flow value

Equivalence of (1) (2) (3)

- Max-Flow-Min-Cut Theorem "F-F theorem"
  ↳ most important in combinatorics

(1) f is a maximum flow of G

(2) there is an (s,t)-cut (A,B) s.t. the value of f equals the capacity of (A,B)

(3) there is no s–t path (with positive residual capacity) in the residual G_f

(2) => (1)

(2) there is an (s,t)-cut (A,B) s.t. the value of f equals the capacity of (A,B)

implies

(1) f is a maximum flow of G

Claim:

for every flow f and every (s,t)-cut (A,B) value of $f \leq$ capacity of (A,B)

$$\text{value of } f = \underbrace{\sum_{e\in\delta^+(s)} f_e}_{\text{flow out of } s} = \sum_{e\in\delta^+(s)} f_e - \underbrace{\sum_{e\in\delta^-(s)} f_e}_{\text{vacuous sum}}$$

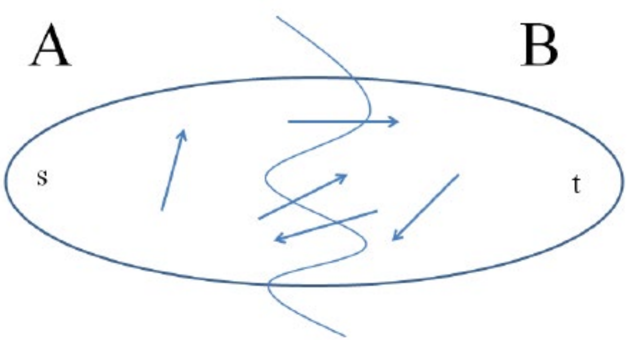
and

for $v \neq s,t$

$$\underbrace{\sum_{e\in\delta^+(v)} f_e}_{\text{flow out of } v} - \underbrace{\sum_{e\in\delta^-(v)} f_e}_{\text{flow into of } v} = 0$$

flow conservation

$$\begin{aligned}
\text{value of } f &= \sum_{v\in A}\left(\sum_{e\in\delta^+(v)} f_e - \sum_{e\in\delta^-(v)} f_e\right) \\
&= \sum_{e\in\delta^+(A)} \underbrace{f_e}_{\leq u_e} - \sum_{e\in\delta^-(A)} \underbrace{f_e}_{\geq 0} \\
&\leq \sum_{e\in\delta^+(A)} u_e \\
&= \text{capacity of } (A,B)
\end{aligned}$$



(1)=>(3)

(1) f is a maximum flow of G

implies

(3) there is no s–t path (with positive residual capacity) in the residual G_f

contrapositive

$\sim(3) \Rightarrow \sim(1)$

$\exists$ s-t path in G_f f is not optimal

push $\Delta \geq 1$

flow from $s \to t$

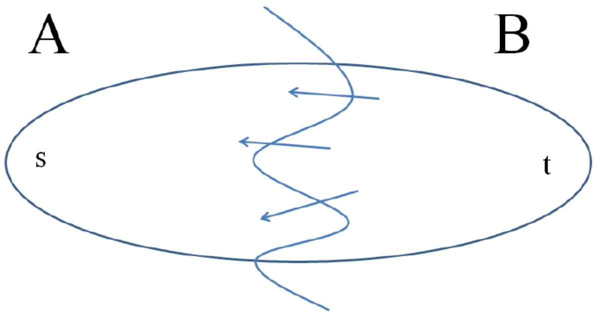
(3)=>(2)

(3) there is no s–t path (with positive residual capacity) in the residual G_f

implies

(2) there is an (s, t) -cut (A, B) s.t. the value of f equals the capacity of (A, B)

$A = \{v \in V : \text{ there is an } s \rightsquigarrow v \text{ path in } G_f\}$



Run BFS from s until stuck

(1) $\forall e \in \delta^+(A)$, $U_e - f_e = 0$ (no forward edges)
(2) $\forall e \in \delta^-(A)$, $f_e = 0$ (no "flow-inducded" backward edges)

$$\text{value of } f = \sum_{e \in \delta^+(A)} f_e - \sum_{e \in \delta^-(A)} f_e = \sum_{e \in \delta^+(A)} u_e = \text{cap}(A, B)$$