Thesis 10

Anna Lucia Lamacchia
Statistics

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Wiener Process and Geometric Brownian Motion (GBM)

1 Wiener Process

1.1 Meaning

The Wiener Process, also known as Brownian Motion, is a continuous-time stochastic process that models the random and unpredictable movement of particles. Named after mathematician Norbert Wiener, this process is characterized by its continuous paths and has profound applications in various fields, including physics, finance, and biology. The fundamental idea is that the position of a particle undergoing Brownian Motion evolves randomly, with no specific trend or direction, resulting in a trajectory that appears erratic and unpredictable.

1.2 Derivation

1. **Start at the Origin**: The process begins at the origin

   $$W(0) = 0$$

   This ensures that the particle starts from a fixed point, usually interpreted as the starting position of the random walk

2. **Independent Normally Distributed Increments**: For any $t_1 < t_2$, the increments are independent and normally distributed with mean zero and variance proportional to the length of the time interval $t_2 - t_1$. Mathematically

   $$W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$$

This property reflects the random and continuous nature of the process. The independence of increments implies that future movements are not influenced by past movements, and the normal distribution captures the inherent randomness of the process.

2 Geometric Brownian Motion (GBM)

2.1 Meaning

Geometric Brownian Motion (GBM) is a continuous-time stochastic process that models the exponential growth or decay of a variable. It is commonly used in financial contexts to describe the behavior of asset prices, capturing the compounding effect seen in various natural processes and financial markets. GBM is particularly suitable for modeling phenomena where percentage changes are of primary interest.

2.2 Derivation

The mathematical formulation of GBM involves a stochastic differential equation (SDE). Let $S(t)$ represent the variable of interest (e.g., asset price) at time t, and $W(t)$ be a standard Wiener process (Brownian Motion). The GBM is described by the SDE:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where:

- μ is the drift, representing the average growth rate.
- σ is the volatility, measuring the degree of random fluctuations.
- $dW(t)$ is the Wiener process increment.

The solution to this SDE, in integral form, is given by:

$$S(t) = S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$$

This solution indicates that the logarithm of the variable $(\log(S(t)))$ follows a standard Brownian Motion with drift $\left(\mu - \frac{\sigma^2}{2}\right)t$ and volatility σ, and exponentiating this Brownian Motion gives the variable $S(t)$ following GBM.

3 Simulation in Python

In this simulation, the Wiener process (left plot) and Geometric Brownian Motion (right plot) are generated using random increments. The resulting paths demonstrate the typical continuous and random behavior of these processes, respectively.

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
T = 1  # Total time
N = 1000  # Number of time steps
mu = 0.1  # Drift
sigma = 0.2  # Volatility

# Time vector
t = np.linspace(0, T, N)  # Ensure N matches the number of points in Brownian motion

# Brownian motion simulation
dW = np.sqrt(T/N) * np.random.normal(size=N)
W = np.cumsum(dW)

# Geometric Brownian Motion simulation
S = np.exp((mu - 0.5 * sigma**2) * t + sigma * W)

# Plot Brownian motion and GBM
plt.figure(figsize=(12, 6))
plt.subplot(1, 2, 1)
plt.plot(t, W, label='Brownian Motion')
plt.xlabel('Time')
plt.ylabel('Brownian Motion Value')
plt.title('Brownian Motion Simulation')
plt.legend()

plt.subplot(1, 2, 2)
plt.plot(t, S, label='Geometric Brownian Motion')
plt.xlabel('Time')
plt.ylabel('GBM Value')
plt.title('Geometric Brownian Motion Simulation')
plt.legend()

plt.tight_layout()
plt.show()
```

Figure 1: Code

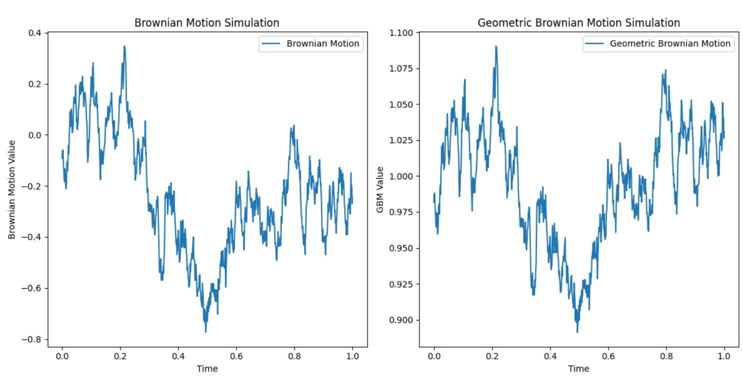

Figure 2: Chart