Thesis 11

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Statistics

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The Functional CLT (Donsker's Invariance Principle)

1 Meaning

Donsker's Invariance Principle is a key result in probability theory, specifically within the realm of the Functional Central Limit Theorem (CLT). It extends the traditional CLT from convergence in distribution of random variables to the convergence in distribution of stochastic processes or functionals. Donsker's Invariance Principle asserts that properly normalized empirical distribution functions of a sequence of independent and identically distributed random variables converge in distribution to a limiting process known as a Brownian Bridge. This has profound implications in various fields, including statistics and mathematical finance.

2 Proof of Functional CTL

Let $X_1, X_2, \ldots, X_n, \ldots$ be independent and identically distributed (i.i.d.) random variables with $E[X_i] = 0$ and $Var(X_i) = \sigma^2$. Define $S_n = X_1 + X_2 + \ldots + X_n$, and consider the standardized random processes $Z_n(t) = \frac{S_{[nt]}}{\sqrt{n}}$ for $0 \leq t \leq 1$.

Theorem Donsker's Invariance Principle. Donsker's Invariance Principle states that the finite-dimensional distributions of $\{Z_n(t), 0 \leq t \leq 1\}$ converge weakly to those of a standard Brownian Bridge as $n \to \infty$.

Proof.

2.1 Step 1: Tightness

Let $\epsilon > 0$. By Prokhorov's theorem, it suffices to show that for any $\epsilon > 0$, there exists a compact set K_ϵ such that $P(\sup_{0\leq t\leq 1} |Z_n(t)| > \epsilon) < \epsilon$ for all n. This can be accomplished using moment inequalities and the fact that $Var(S_n) = n\sigma^2$.

2.2 Step 2: Weak Convergence

Next, show weak convergence of the finite-dimensional distributions of $\{Z_n(t), 0 \leq t \leq 1\}$. For any $0 \leq t_1 < t_2 < \ldots < t_k \leq 1$, demonstrate that the characteristic function of $\{Z_n(t_1), Z_n(t_2), \ldots, Z_n(t_k)\}$ converges to that of a standard Brownian Bridge. The characteristic function of $Z_n(t)$ is given by:

$$\phi_{Z_n(t)}(u) = E[e^{iuZ_n(t)}]$$

By the central limit theorem, as $n \to \infty$, the characteristic function converges to the characteristic function of a standard normal distribution.

2.3 Step 3: Skorokhod Representation

By the Skorokhod Representation Theorem, find a probability space and random variables $\tilde{Z}_n(t)$ defined on this space such that $\tilde{Z}_n(t) \stackrel{d}{=} Z_n(t)$ and almost surely $\tilde{Z}_n(t) \to Z(t)$ for all t, where $Z(t)$ is a standard Brownian Bridge. This completes the proof of Donsker's Invariance Principle, demonstrating the tightness, weak convergence, and the existence of the limiting process in distribution. $\square$

3 Simulation in Python

In this simulation, we generate five i.i.d. random processes and compute the corresponding empirical processes. The plot demonstrates the convergence of these empirical processes to a Brownian bridge, as indicated by the characteristic random-walk-like behavior.

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
n = 1000  # Number of observations
t_values = np.linspace(0, 1, n)  # Time values

# Simulate i.i.d. random processes
X = np.random.normal(size=(5, n))

# Compute empirical processes
G = np.cumsum(X, axis=1) / np.sqrt(n)

# Plot the empirical processes
plt.figure(figsize=(10, 6))
plt.plot(t_values, G.T, alpha=0.7)
plt.title("Empirical Processes - Donsker's Invariance Principle")
plt.xlabel('Time')
plt.ylabel('Empirical Process Value')
plt.show()
```

Figure 1: Code

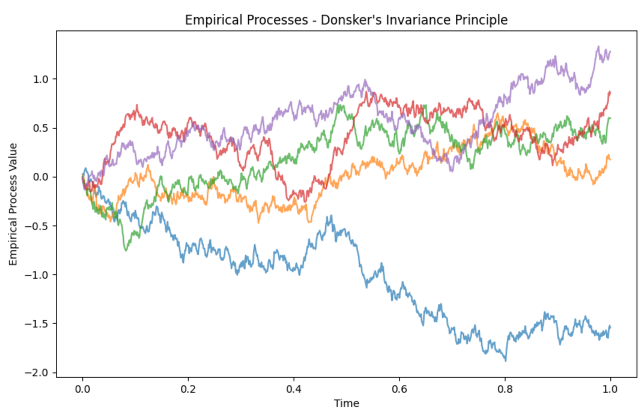

Figure 2: Chart