Vovk's algorithm
Mixable and unmixable loss functions

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Outline

Binary log-loss

- Prediction: $p \in [0,1]$ outcome $x \in \{0,1\}$
- Loss: $\lambda(p,x) = -x \log p - (1-x)\log(1-p)$
- N experts, expert i at time t outputs $q_i^t \in [0,1]$
- Cumulative loss of expert i at time t: $L_i^t = \sum_{s=1}^{t} \lambda(q_i^s, x^s)$
- Experts algorithm (Bayes Algorithm):
  - Assign weights: $w_i^t = \frac{1}{N}\exp(-L_i^{t-1})$
  - Master prediction: $q_M^t = \frac{\sum_{i=1}^{N} q_i^t w_i^t}{\sum_{i=1}^{N} w_i^t}$
- Regret Bound:

$$L_A^T \leq \min_i L_i^T + \ln N$$

Absolute loss

- Prediction: $p \in [0,1]$ outcome $x \in \{0,1\}$
- Loss: $\lambda(p,x) = |x-p|$
- N experts, expert i at time t outputs $q_i^t \in [0,1]$
- Cumulative loss of expert i at time t: $L_i^t = \sum_{s=1}^{t} \lambda(q_i^s, x^s)$
- Experts algorithm (Hedge):
  - Assign weights: $w_i^t = \frac{1}{N}\exp(-\eta L_i^{t-1})$
  - Master prediction: $q_M^t = \frac{\sum_{i=1}^{N} q_i^t w_i^t}{\sum_{i=1}^{N} w_i^t}$
- Regret Bound for known horizon.
  - Set η according to T: $\eta \approx \sqrt{\frac{2\ln N}{T}}$
  - Regret bound:

$$L_A \leq \min_i L_i + \sqrt{2T\ln N} + \ln N$$

Two other loss functions over $[0, 1]$

- Prediction: $p \in [0, 1]$ outcome $x \in [0, 1]$ (note x is not restricted to $\{0, 1\}$)
- Square loss (Breier Loss): $\lambda(p, x) = (p - x)^2$
- Hellinger Loss
  $\lambda(p, x) = \frac{1}{2}\left(\left(\sqrt{p} + \sqrt{x}\right)^2 + \left(\sqrt{1-p} + \sqrt{1-x}\right)^2\right)$

Vovk's general prediction game

Γ - prediction space. Ω - outcome space.
On each trial $t = 1, 2, \ldots$

1. Each expert $i \in \{1 \ldots N\}$ makes a prediction $\gamma_i^t \in \Gamma$
2. The learner, after observing $\langle \gamma_1^t \ldots \gamma_N^t \rangle$, makes its own prediction γ^t
3. Nature chooses an outcome $\omega^t \in \Omega$
4. Each expert incurs loss $\ell_t(i) = \lambda(\omega^t, \gamma_i^t)$
   The learner incurs loss $\ell_t(A) = \lambda(\omega^t, \gamma^t)$

Achievable loss bounds

- $L_A \doteq \sum_{t=1}^{T} \ell_t(A)$ - total loss of algorithm
- $L_i \doteq \sum_{t=1}^{T} \ell_t(i)$ - total loss of expert i
- **Goal:** find an algorithm which guarantees that

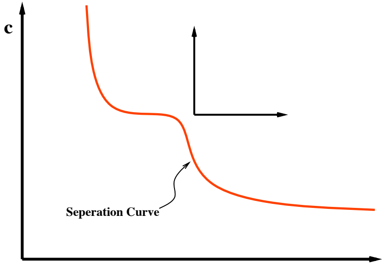
$$(a, c) \in [0, \infty), \quad L_A \leq aL_{\min} + c \ln N$$

  For any sequence of events.
- We say that the pair (a, c) is achievable.

The set of achievable bounds

- Fix loss function $\lambda : \Omega \times \Gamma \to [0, \infty)$
- The pair (a, c) is *achievable* if there exists *some* prediction algorithm such that for *any* $N > 0$, *any* set of N prediction sequences and *any* sequence of outcomes

$$L_A \leq a L_{\min} + c \ln N$$



Analysis for specific loss functions

- Outcomes: $\omega^1, \omega_2, \ldots \omega^t \in [0, 1]$
- Predictions: $\gamma^1, \gamma^2, \ldots \gamma^t \in [0, 1]$

Log loss (Entropy loss)

- $$\lambda_{\text{ent}}(\omega, \gamma) = \omega \ln \frac{\omega}{\gamma} + (1 - \omega) \ln \frac{1 - \omega}{1 - \gamma}$$
- When $q_t \in \{0, 1\}$ Cumulative log loss $=$ coding length ± 1
- If $P[\omega_t = 1] = q$, optimal prediction $\gamma^t = q$
- Unbounded loss.
- Not symmetric $\exists p, q \;\; \lambda(p, q) \neq \lambda(q, p)$.
- No triangle inequality
  $\exists p_1, p_2, p_3 \;\; \lambda(p_1, p_3) > \lambda(p_1, p_2) + \lambda(p_2, p_3)$

Square loss (Breier Loss)

- $$\lambda_{\text{sq}}(\omega, \gamma) = (\omega - \gamma)^2$$
- $P[\omega^t = 1] = q, \;\; P[\omega^t = 0] = 1 - q$,
  optimal prediction $\gamma^t = q$
- Bounded loss.
- Defines a metric (symmetric and triangle ineq.)
- Corresponds to regression.

Hellinger Loss

- $$\lambda_{\mathrm{hel}}(\omega, \gamma) = \frac{1}{2}\left((\sqrt{\omega} + \sqrt{\gamma})^2 + \left(\sqrt{1-\omega} + \sqrt{1-\gamma}\right)^2 \right)$$
- If $P[\omega^t = 1] = q, \;\; P[\omega^t = 0] = 1 - q$, optimal prediction $\gamma^t = q$
- Loss is bounded.
- Defines a metric.
- $\lambda_{\mathrm{hel}}(p, q) \approx \lambda_{\mathrm{ent}}(p, q)$ when $p \approx q$ and $p, q \in (0, 1)$

Absolute loss

- $$\lambda(\omega, \gamma) = |\omega - \gamma|$$
- Probability of making a mistake if predicting 0 or 1 using a biased coin
- If $P[\omega^t = 1] = q, \quad P[\omega^t = 0] = 1 - q$, then the optimal prediction is
$$\gamma^t = \begin{cases} 1 & \text{if } q > 1/2, \\ 0 & \text{otherwise} \end{cases}$$

Structureless bounded loss

- Prediction is a distribution $\gamma = \langle p_1, \ldots, p_N \rangle$, $p_i \geq 0$, $\sum_{i=1}^{N} p_i = 1$
- Outcome is a loss vector $\omega = \langle \omega_1, \ldots, \omega_N \rangle$, $0 \leq \omega_i \leq 1$
- Loss is the dot product: $\lambda_{\text{dot}}(\omega, \gamma) = \gamma \cdot \omega$
- Corresponds to the hedging game.
- For hedge loss the regret is $\Omega(\sqrt{T \log N})$.
- For the log loss the regret is $O(\log N)$
- **Which losses behave like entropy loss and which behave like hedge loss?**

Some technical requirements

- There should be a topology on the prediction set Γ such that
- Γ is compact.
- $\forall \omega \in \Omega$, the function $\gamma \to \lambda(\omega, \gamma)$ is continuous
- There is a universally reasonable prediction
  $\exists \gamma \in \Gamma, \forall \omega \in \Omega, \lambda(\omega, \gamma) < \infty$
- There is no universally optimal prediction
  $\neg \exists \gamma \in \Gamma, \forall \omega \in \Omega, \lambda(\omega, \gamma) = 0$

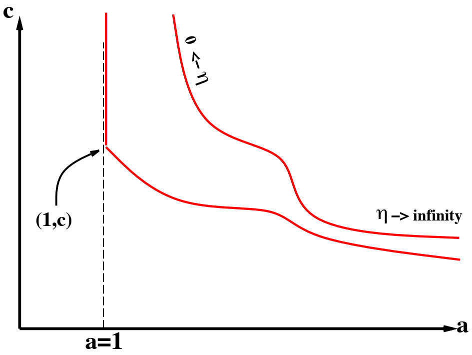
Vovk's meta-algorithm

- Fix an achievable pair (a, c) and set $\eta = a/c$
- 1.
$$W_i^t = \frac{1}{N}\, e^{-\eta L_i^t}$$
  2. Choose γ_t so that, for all $\omega^t \in \Omega$:
$$\lambda(\omega^t, \gamma^t) - c \ln \sum_i W_i^t \leq -c \ln \left(\sum_i W_i^t e^{-\eta\lambda(\omega^t, \gamma_i^t)} \right)$$
- If choice of γ_t always exists, then the total loss satisfies:
$$\sum_t \lambda(\omega^t, \gamma^t) \leq -c \ln \sum_i W_i^{T+1} \leq aL_{\min} + c \ln N$$
- Vovk's result: *yes!* a good choice for γ_t always exists!

Vovk's algorithm is the the highest achiever [Vovk95]

The pair (a, c) is achieved by some algorithm if and only if it is achieved by Vovk's algorithm.

The separation curve is $\left\{ \left(a(\eta), \frac{a(\eta)}{\eta} \right) \middle| \eta \in [0, \infty] \right\}$



Mixable Loss Functions

- A Loss function is mixable if a pair of the form $(1, c),\ c < \infty$ is achievable.

$$L_A \leq L_{\min} + c \ln N$$

- Vovk's algorithm with $\eta = 1/c$ achieves this bound.
- $\lambda_{\text{ent}}, \lambda_{\text{sq}}, \lambda_{\text{hel}}$ are mixable
- $\lambda_{\text{abs}}, \lambda_{\text{dot}}$ are not mixable

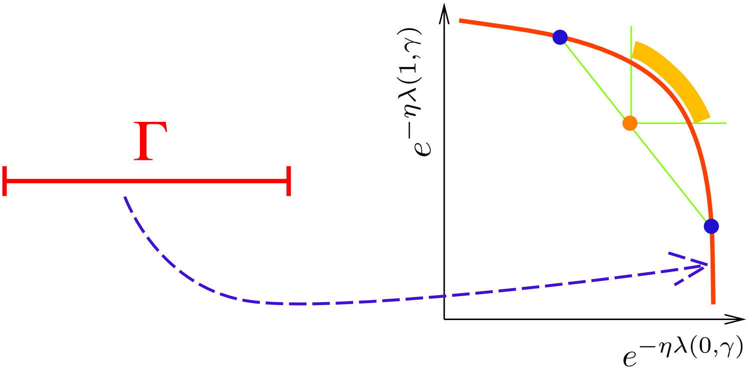
The convexity condition

- requirement for loss to be $(1, 1/\eta)$ mixable
- $\forall \langle (\gamma_1, W_1), \ldots, (\gamma_N, W_N) \rangle$
  $\exists \gamma \in \Gamma$
  $\forall \omega \in \Omega$:
  $$\lambda(\omega, \gamma) - \frac{1}{\eta} \ln \sum_i W_i \leq -\frac{1}{\eta} \ln \left(\sum_i W_i e^{-\eta \lambda(\omega, \gamma_i)} \right)$$
- Can be re-written as:
  $$e^{-\eta \lambda(\omega, \gamma)} \geq \sum_i \left(\frac{W_i}{\sum_j W_j} \right) e^{-\eta \lambda(\omega, \gamma_i)}$$
- Equivalently - the image of the set Γ under the mapping $F(\gamma) = \left\langle e^{-\eta \lambda(\omega, \gamma)} \right\rangle_{\omega \in \Omega}$ is concave.

convexity condition: Pictorially

- **Example:** Suppose $\Omega = \{0, 1\}$, $\Gamma = [0, 1]$. then

$$F(\gamma) = \left\langle e^{-\eta\lambda(0,\gamma)}, e^{-\eta\lambda(1,\gamma)} \right\rangle$$



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Vovk Algorithm for log loss

- The log loss is mixable with $\eta = 1$
- The image of $[0, 1]$ through $F(\gamma) = \left\langle e^{-\eta\lambda(0,\gamma)}, e^{-\eta\lambda(1,\gamma)} \right\rangle$ is a straight line segment.
- The only satisfactory prediction is

$$\gamma = \frac{\sum_i W_i \gamma_i}{\sum_i W_i}$$

- We are back to the online Bayes algorithm.

Vovk algorithm for square loss

- The square loss is mixable with $\eta = 2$.
- Prediction must satisfy

$$1 - \sqrt{-\frac{1}{2}\ln\sum_i V_i^t e^{-2(1-p_i^t)^2}} \leq p^t \leq \sqrt{-\frac{1}{2}\ln\sum_i V_i^t e^{-2(p_i^t)^2}}$$

  where $V_i^t = \frac{W_i^t}{\sum_s W_i^s}$.

- $$L_A \leq L_{\min} + \frac{1}{2}\ln N$$

Simple prediction for square loss

- We can use the prediction

$$\gamma = \frac{\sum_i W_i \gamma_i}{\sum_i W_i}$$

- But in that case we must use $\eta = 1/2$ when updating the weights.
- Which yields the bound

$$L_A \leq L_{\min} + 2 \ln N$$

Summary of bounds for mixable losses

Loss Functions:	c values: $(\eta = 1/c)$	
	$\mathbf{pred}_{\text{wmean}}(v, x)$	$\mathbf{pred}_{\text{Vovk}}(v,$
$L_{\text{sq}}(p, q)$	2	$1/2$
$L_{\text{ent}}(p, q)$	1	1
$L_{\text{hel}}(p, q)$	1	$1/\sqrt{2}$

Figure 2. $(c, 1/c)$-realizability: c values for loss and predictio