The Context Algorithm

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Outline

The online Bayes Algorithm

- Total loss of expert i

$$L_i^t = -\sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0$$

- Weight of expert i

$$w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s)$$

- Freedom to choose initial weights.
  $w_t^1 \geq 0$, $\sum_{i=1}^{n} w_i^1 = 1$
- Prediction of algorithm A

$$\mathbf{p}_A^t = \frac{\sum_{i=1}^{N} w_i^t \mathbf{p}_i^t}{\sum_{i=1}^{N} w_i^t}$$

Cumulative loss vs. Final total weight

Total weight: $W^t \doteq \sum_{i=1}^{N} w_i^t$

$$\frac{W^{t+1}}{W^t} = \frac{\sum_{i=1}^{N} w_i^t e^{\log p_i^t(c^t)}}{\sum_{i=1}^{N} w_i^t} = \frac{\sum_{i=1}^{N} w_i^t p_i^t(c^t)}{\sum_{i=1}^{N} w_i^t} = p_A^t(c^t)$$

$$-\log \frac{W^{t+1}}{W^t} = -\log p_A^t(c^t)$$

$$-\log W^{T+1} = -\log \frac{W^{T+1}}{W^1} = -\sum_{t=1}^{T} \log p_A^t(c^t) = L_A^T$$

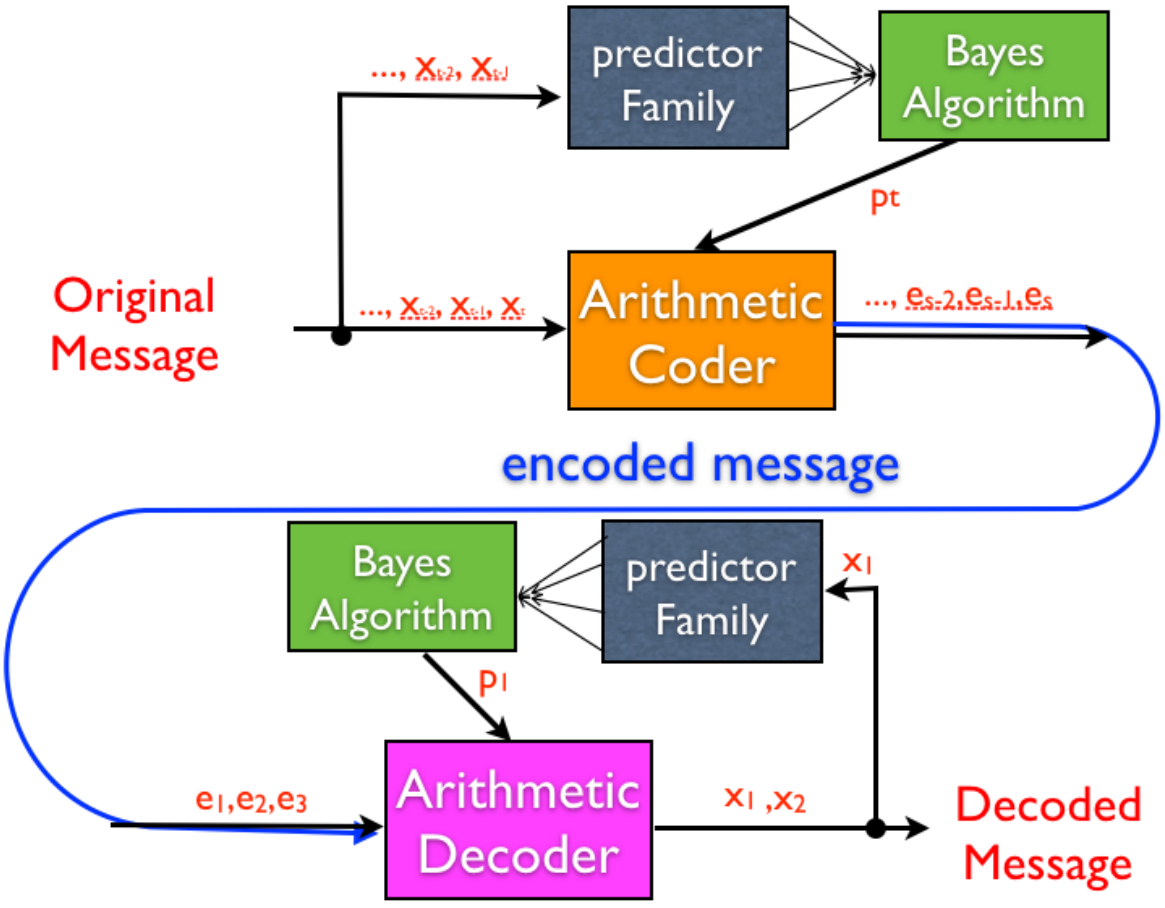
EQUALITY not bound!

Simple Bound

- Use non-uniform initial weights $\sum_i w_i^1 = 1$
- Total Weight is at least the weight of the best expert.

$$
\begin{aligned}
L_A^T &= -\log W^{T+1} = -\log \sum_{i=1}^{N} w_i^{T+1} \\
&= -\log \sum_{i=1}^{N} w_i^1 e^{-L_i^T} \leq -\log \max_i \left(w_i^1 e^{-L_i^T} \right) \\
&= \min_i \left(L_i^T - \log w_i^1 \right)
\end{aligned}
$$

Universal Online coding

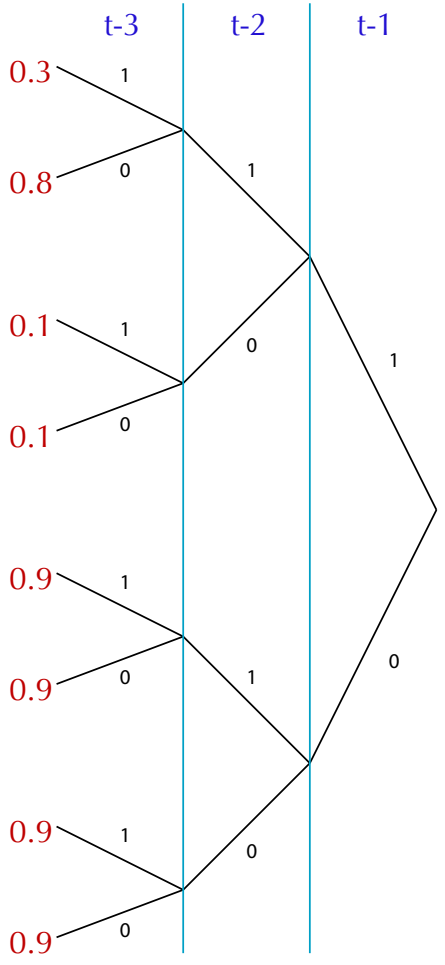


Combining large predictor families

- Log loss is **mixable** = each predictor in the family can use a Bayesian combination of a family of sub-predictors, with no additional loss.
- We talked about the KT preictor.
- Today we consider the much richer set of variable length markov models.
- The set of predictors is of exponential size, but the algorithm is efficient.

A fixed length Markov Model

- Observe a binary sequence.
- $x_1, \ldots, x_{t-1}$
- Predict next bit from past
- $P(x_t = 1 | x_{t-1}, x_{t-2}, \ldots, x_1)$
- Use only last k bits
- $P(x_t = 1 | x_{t-1}, \ldots, x_{t-k})$
- Markov model of order k



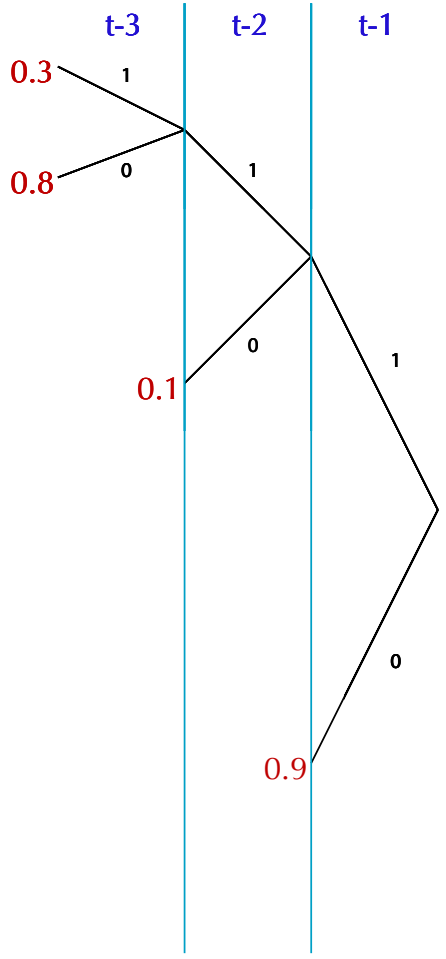
Learning a markov distribution

- Each tree leaf is associated with a binary sequence $y_1, \ldots, y_k$
- For each leaf keep two counters:
  - $a_{y_1,\ldots,y_k}$ = number of times $x_{t-1} = y_1, \ldots, x_{t-k} = y_k$ and $x_t = 0$
  - $b_{y_1,\ldots,y_k}$ = number of times $x_{t-1} = y_1, \ldots, x_{t-k} = y_k$ and $x_t = 1$
- Prediction (using Kritchevski Trofimov)

$$p(x_t = 1 | x_{t-1} = y_1, \ldots, x_{t-k} = y_k) = \frac{b_{y_1,\ldots,y_k} + 1/2}{a_{y_1,\ldots,y_k} + b_{y_1,\ldots,y_k} + 1}$$

- Total regret is at most $2^{k-1} \log T$

How variable length markov can reduce regret



- Reducing number of leaves from 8 to 4 means
- reducing regret from $4 \log T$ to $2 \log T$
- English example:
  B A R O Q U E
- When we have little data, we can get better prediction even if the children are not Exactly the same

Prefix trees / Tries

- In a prefix binary tree each node has either 0 or 2 children.
- A variable length markov model corresponds to a prefix tree.
- You can think of a prefix trees as different prunings of a maximal tree.
- We don't know a-priori which pruning to use!
- The number of prunings trees increases exponentially with the number of nodes in the maximal tree.
- We will use the Online Bayes to predict almost as well as the best prefix tree in hind-sight.
- First - simple but inefficient algorithm, Second - efficient algorithms.

Using online Bayes to learn the structure

- We assign to each tree an initial weight of 2^{-n} where n is the number of nodes in the pruned tree.
- We combine the predictions of the trees using online Bayes.
- The total regret would be $\frac{l}{2} \log T + n$ where l is the number of leaves in the prefix tree.
- This algorithm maintains a weight for each prefix tree.
- The number of prunings of a full tree of depth k is $O(2^{2^k})$ while maintaining all of the counts requires $O(2^k)$.

Efficient generation of prior

- Prior distribution is generated by a stochastic recursion.
- Start with root node (always exists)
- For each node flip a fair coin.
  - Heads Set node to be a leaf (0 children)
  - Tails Create 2 children nodes to the node.
- Defines a distribution over all prefix trees.
- Probability of a tree with n nodes is 2^{-n}

Efficient averaging over the prior (observations)

- Maintain a KT estimator at each node of the tree.
- Allocate counters only for nodes that have been visited.
- At iteration t only t counters need to be updated.
- Only k counters if depth of tree is bounded.
- Each node is visited on a subset of the iterations.
- Subset corresponding to node is contained in subset corresponding to node's parent.

Efficient averaging over the prior (procedure)

- This is not the method used in the original paper, it appears in a later paper by *Willems, Tjalkens and Ignatenko. Available on github.*

Definitions

- s is a bit sequence corresponding to a node in the tree. The chidren of this node are $0s$ and $1s$.
- The sequence of past bits up to time t is denoted x_1^{t-1}, the t'th bit is denoted X_t
- s determines a subsequence of x_1^{t-1}: the locations preceded by the reverse of s.
- $a_s(x_1^{t-1}), b_s(x_1^{t-1})$ count the number of 0's and 1's in the subsequence corresponding to s
- The KT estimate associated with node s.

$$P_e^s\left(X_t = 1 | x_1^{t-1}\right) = \frac{b_s(x_1^{t-1}) + 1/2}{a_s(x_1^{t-1}) + b_s(x_1^{t-1}) + 1}$$

Assigning probabilities to complete sequences

- Using the chain rule, we can use a prediction rule to assign probabilities to a complete sequence.

$$P(x_1 = y_1, \ldots, x_T = y_T) = p(x_1 = y_1)p(x_2 = y_2|x_1 = y_1)\ldots$$

- We can translate probabilities for complete sequences back into predictions.

$$p(x_t = 1|x_1 = y_1, \ldots, x_{t-1} = y_{t-1}) = \frac{p(x_1 = y_1, \ldots, x_{t-1} = y_{t-1}, x_t = 1)}{p(x_1 = y_1, \ldots, x_{t-1} = y_{t-1})}$$

Mixing Factors

- $P_w^s\left(X_t = 1 | x_1^{t-1}\right)$ The posterior average of the predictions associated with the descendants of the node s.
- Using the chain rule we can also define unconditional probabilities $P_w^s\left(x_1^{t-1}\right)$
- The mixing factors according to the Prior distribution is 0.5: stop, 0.5: continue
- After observing x_1^{t-1} the odds change. The updated odds are represented by $\beta^s(x_1^{t-1})$:
- $P_w^s\left(X_t = 1 | x_1^{t-1}\right)$ The posterior average of the predictions associated with the descendants of the node s.

Outline of algorithm

- **Forward**: Traverse the tree from root to leaf.
- **extend**: Add two children to the leaf. Initialized counts to 0,1.
- **Backward traversal**: Traverse back to root.
  For each node s
  - compute $P_e^s\left(X_t = 1|x_1^{t-1}\right)$ and $P_w^s\left(X_t = 1|x_1^{t-1}\right)$
  - update counts: a^s, b^s.
  - update β^s

Slides from Frans Willems

IX. Betas: Introduction

Consider an internal node s in the context tree $\mathcal{T}_{\mathcal{D}}$ and the corresponding *conditional* weighted probability $P_w^s(X_t = 1|x_1^{t-1})$. Assuming that $0s$ (and not $1s$) is a suffix of the context x_{1-D}^0, x_1^{t-1} of x_t, we obtain for this probability that

$$\begin{aligned} P_w^s(X_t = 1|x_1^{t-1}) &= \frac{P_e^s(x_1^{t-1}, X_t = 1) + P_w^{0s}(x_1^{t-1}, X_t = 1)P_w^{1s}(x_1^{t-1})}{P_e^s(x_1^{t-1}) + P_w^{0s}(x_1^{t-1})P_w^{1s}(x_1^{t-1})} \\ &= \frac{\beta^s(x_1^{t-1})P_e^s(X_t = 1|x_1^{t-1}) + P_w^{0s}(X_t = 1|x_1^{t-1})}{\beta^s(x_1^{t-1}) + 1} \end{aligned} \quad (1)$$

where

$$\beta^s(x_1^{t-1}) \triangleq \frac{P_e^s(x_1^{t-1})}{P_w^{0s}(x_1^{t-1})P_w^{1s}(x_1^{t-1})}. \quad (2)$$

If we start in the context-leaf and work our way down to the root, we finally find $P_w^{\lambda}(X_t = 1|x_1^{t-1})$.

Slides from Frans Willems

Implementation

Assume that in node s the counts $a_s(x_1^{t-1})$ and $b_s(x_1^{t-1})$ are stored, as well as $\beta^s(x_1^{t-1})$. We then get the following sequence of operations:

1. Node $0s$ delivers cond. wei. probability $P_w^{0s}(X_t = 1|x_1^{t-1})$ to node s.
2. Cond. est. probability $P_e^s(X_t = 1|x_1^{t-1})$ is determined as follows:

$$P_e^s(X_t = 1|x_1^{t-1}) = \frac{b_s(x_1^{t-1}) + 1/2}{a_s(x_1^{t-1}) + b_s(x_1^{t-1}) + 1}. \tag{3}$$

3. Now $P_w^s(X_t = 1|x_1^{t-1})$ can be computed as in (1).
4. The ratio $\beta^s(\cdot)$ is then updated with symbol x_t as follows:

$$\beta^s(x_1^{t-1}, x_t) = \beta^s(x_1^{t-1}) \cdot \frac{P_e^s(X_t = x_t|x_1^{t-1})}{P_w^{0s}(X_t = x_t|x_1^{t-1})}. \tag{4}$$

5. Finally, depending on the value x_t, either count $a_s(x_1^{t-1})$ or $b_s(x_1^{t-1})$ is incremented.

Binary Tree

Sequence: 0,0,1,1,0,1,0,1,0,1,1,0,1,1,1