Online learning in repeated matrix games

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Based on “Adaptive Game Playing Using Multiplicative Weights” Freund and Schapire.

Outline

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- Game repeated many times.

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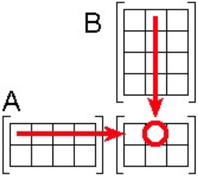
Mixed strategies in matrix notation



$$(A \times B)_{12} = \sum_{r=1}^{4} a_{1r} b_{r2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42}$$

- **Q** is a column vector. $\mathbf{P}^T$ is a row vector.

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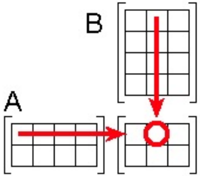

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- $\mathbf{Q}$ is a column vector. $\mathbf{P}^T$ is a row vector.
- $\mathbf{M}(\mathbf{P}, \mathbf{Q}) = \mathbf{P}^T\mathbf{M}\mathbf{Q} = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(i)\mathbf{M}(i,j)\mathbf{Q}(j)$

The minmax Theorem

When using pure strategies, second player has an advantage.

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John von Neumann, 1928.

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In words:

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- for mixed strategies, choosing second gives no advantage.
- There are min-max optimal mixed Strategies: $\mathbf{P}^*, \mathbf{Q}^*$
- $M(\mathbf{P}^*, \mathbf{Q}^*)$ is the value of the game.

Online Learning as matrix game

- Row = action

	$t = 1$	$t = 2$	...
*expert*1	0	1	...
*expert*2	0.2	0.1	...
*expert*3	0.5	0.2	...
...	...	...	...
Master	0.35	0.13	...

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- Row = action
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- Player chooses mixed strategy $\mathbf{P}_t$
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- Goal - minimize regret: $\sum_{t=1}^{T} \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) - \sum_{t=1}^{T} \mathbf{M}(\mathbf{P}^*, \mathbf{Q}_t)$

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Boosting as a matrix game (1)

- Row = example (x, y)

	h_1	h_2	$\dots$
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- Matrix entry for $(x, y), h_t$ is 0 if $h_t(x) = y$, 1 $h_t(x) \neq y$

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- The weighted majority vote over the weak rule is **always** correct.

Adaboost as a repeated matrix game

- Booster chooses distribution over examples = mixed strategy over rows $\mathbf{P}_t$

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- Booster chooses distribution over examples = mixed strategy over rows $\mathbf{P}_t$
- adversary chooses weak rule $\mathbf{Q}_t = \langle 0, \cdots, 0, 1, 0, \cdots, 0 \rangle$ the 1 is at position t
- **Goal 1:** produce a weighted majority rule that is highly accurate.
- **Goal 2:** Find a "hard" distribution over the training examples.

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- Our goal is to minimize regret.

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- follow the leader makes an error on each iteration.

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- Adding noise allows us to choose responses that are slightly worse than best response.
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- regret is $O(1/\sqrt{n})$ where n is number of actions.

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- $\eta > 0$ is the learning rate.

Generalized regret bound

- Regret relative to the best *pure strategy* i

$$\sum_{t=1}^{T} \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \leq \left(\frac{1}{1 - e^{-\eta}} \right) \min_i \left[\eta \sum_{t=1}^{T} \mathbf{M}(i, \mathbf{Q}_t) - \ln \mathbf{P}_1(i) \right]$$

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- regret with respect the the best *mixed strategy* $\mathbf{P}$:

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$$\mathrm{RE}\left(\mathbf{P} \parallel \mathbf{Q}\right) \doteq \sum_{i=1}^{n} \mathbf{P}(i) \ln \frac{\mathbf{P}(i)}{\mathbf{Q}(i)}$$

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- The sequence $\mathbf{P}_1, \ldots, \mathbf{P}_T$ produced by basic alg using $\eta > 0$ satisfies

$$\sum_{t=1}^{T} \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \leq \left(\frac{1}{1 - e^{-\eta}} \right) \min_{\mathbf{P}} \left[\eta \sum_{t=1}^{T} \mathbf{M}(\mathbf{P}, \mathbf{Q}_t) + \mathrm{RE}\left(\mathbf{P} \parallel \mathbf{P}_1 \right) \right]$$

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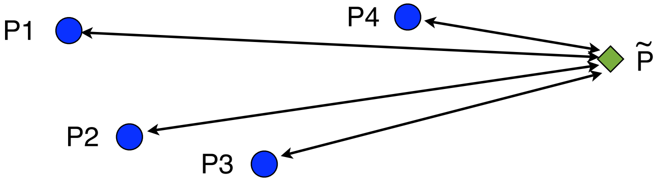
- Where

$$\Delta_{T,n} = \sqrt{\frac{2\ln n}{T}} + \frac{\ln n}{T} = O\left(\sqrt{\frac{\ln n}{T}}\right).$$

Visual intuition

- **Hedge**(η) : **If** $\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \gg \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}_t)$ **then:** distance between $\mathbf{P}_{t+1}$ and $\tilde{\mathbf{P}}$ smaller than distance between $\mathbf{P}_t$ and $\tilde{\mathbf{P}}$



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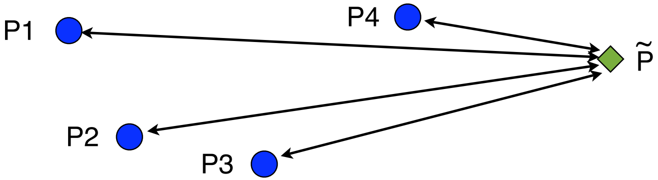
- $\mathrm{RE}\left(\tilde{\mathbf{P}} \parallel \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \parallel \mathbf{P}_t\right) \leq \eta \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}_t) - (1 - e^{-\eta})\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t)$



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\min_{\mathbf{P}} \max_{\mathbf{Q}} \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{Q} &\leq \max_{\mathbf{Q}} \overline{\mathbf{P}}^{\mathrm{T}}\mathbf{M}\mathbf{Q} \\
&= \max_{\mathbf{Q}} \frac{1}{T}\sum_{t=1}^{T} \mathbf{P}_t{}^{\mathrm{T}}\mathbf{M}\mathbf{Q} \quad \text{by definition of } \overline{\mathbf{P}}
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&= \max_{\mathbf{Q}} \frac{1}{T}\sum_{t=1}^{T} {\mathbf{P}_t}^{\mathrm{T}}\mathbf{M}\mathbf{Q} \quad \text{by definition of } \overline{\mathbf{P}} \\
&\leq \frac{1}{T}\sum_{t=1}^{T} \max_{\mathbf{Q}} {\mathbf{P}_t}^{\mathrm{T}}\mathbf{M}\mathbf{Q}
\end{aligned}$$

Proving minmax Theorem using online learning (2)

$$= \frac{1}{T}\sum_{t=1}^{T}\mathbf{P}_t^{\mathrm{T}}\mathbf{M}\mathbf{Q}_t \qquad \text{by definition of } \mathbf{Q}_t$$

Proving minmax Theorem using online learning (2)

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\begin{aligned}
&= \frac{1}{T}\sum_{t=1}^{T}\mathbf{P}_t{}^{\mathrm{T}}\mathbf{M}\mathbf{Q}_t && \text{by definition of } \mathbf{Q}_t \\
&\leq \min_{\mathbf{P}} \frac{1}{T}\sum_{t=1}^{T}\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{Q}_t + \Delta_{T,n} && \text{by the Corollary}
\end{aligned}
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but $\Delta_{T,n}$ can be set arbitrarily small.

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- Without loss of generality $\mathbf{Q}_t$ is a pure strategy (prob. 1 on a single action).
- Let $\overline{\mathbf{P}} \doteq \frac{1}{T}\sum_{t=1}^{T} \mathbf{P}_t$, $\overline{\mathbf{Q}} \doteq \frac{1}{T}\sum_{t=1}^{T} \mathbf{Q}_t$

Using average distributions

- Von Neumann Min/Max Thm:
  $v \doteq \min_{\mathbf{P}} \max_{\mathbf{Q}} \mathbf{M}(\mathbf{P}, \mathbf{Q}) = \max_{\mathbf{Q}} \min_{\mathbf{P}} \mathbf{M}(\mathbf{P}, \mathbf{Q})$

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- Fixing T and letting $\eta = \ln \left(1 + \sqrt{\frac{2 \ln n}{T}}\right)$
- Two immediate corrolaries of the proof of the min/max Thm:

$$\max_{\mathbf{Q}} \mathbf{M}(\overline{\mathbf{P}}, \mathbf{Q}) \leq v + \Delta_{T,n}. \min_{\mathbf{P}} \mathbf{M}(\mathbf{P}, \overline{\mathbf{Q}}) \geq v - \Delta_{T,n}$$

Using the final row distribution vMW

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- **Good Enough:** If $\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \leq u$ the row player does nothing $\mathbf{P}_{t+1} = \mathbf{P}_t$
- **Learn:** If $\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) > u$ set

$$\eta = \ln \frac{(1-u)\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t)}{u(1-\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t))} .$$

Bound for vMW

- Let $\tilde{\mathbf{P}}$ be any mixed strategy for the rows such that $\max_{\mathbf{Q}} \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}) \leq u$

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- Let $\tilde{\mathbf{P}}$ be any mixed strategy for the rows such that $\max_{\mathbf{Q}} \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}) \leq u$
- Then on any iteration of algorithm vMW in which $\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \geq u$ the relative entropy between $\tilde{\mathbf{P}}$ and $\mathbf{P}_{t+1}$ satisfies

$$\mathrm{RE}\left(\tilde{\mathbf{P}} \parallel \mathbf{P}_{t+1}\right) \leq \mathrm{RE}\left(\tilde{\mathbf{P}} \parallel \mathbf{P}_{t}\right) - \mathrm{RE}\left(u \parallel \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t)\right) .$$