

Tracking the best Expert

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HW 3

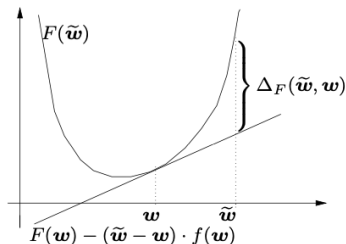
- ▶ Let $x \in \{0, 1\}$ and let $y \in [0, 1]$.
- ▶ Show that log loss $x \log y + (1 - x) \log(1 - y)$ is mixable.
- ▶ Show that square loss $(x - y)^2$ is mixable.
- ▶ Show that absolute loss $|x - y|$ is not mixable.
- ▶ HW 3 is due on Feb 18.
- ▶ No class on Tues 11 (ALT)
- ▶ There will be no mid-term exam.

Bregman Divergences [Br,CL,Cs]

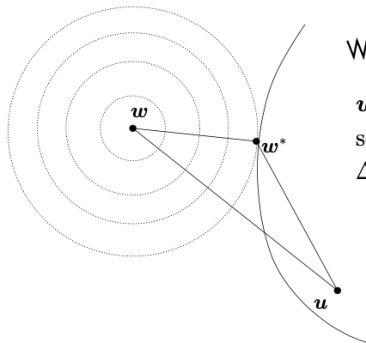
For **any** differentiable convex function F

$$\Delta_F(\tilde{w}, w) = F(\tilde{w}) - F(w) - (\tilde{w} - w) \cdot \underbrace{\nabla_w F(w)}_{f(w)}$$

$$= F(\tilde{w}) - \begin{array}{l} \text{supporting hyperplane} \\ \text{through } (w, F(w)) \end{array}$$



A Pythagorean Theorem [Br,Cs,A,HW]

 \mathcal{W}

w^* is **projection** of w onto convex set \mathcal{W} w.r.t. Bregman divergence Δ_F :

$$w^* = \operatorname{argmin}_{u \in \mathcal{W}} \Delta_F(u, w)$$

Theorem:

$$\Delta_F(u, w) \geq \Delta_F(u, w^*) + \Delta_F(w^*, w)$$

Unnormalized Relative entropy

- ▶ prediction, outcome \mathbf{p}, \mathbf{q} are n dimensional vectors with non-negative coordinates.
- ▶ Loss is RE extended to non-negative vectors:

$$\text{RE}(\mathbf{p} \parallel \mathbf{q}) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} - \sum_{i=1}^n (q_i - p_i)$$

Coincides with RE when $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$

- ▶ Unnormalized RE is the Bregman divergence corresponding to the unnormalized entropy:

$$F(\mathbf{p}) = \sum_{i=1}^n p_i \log p_i - \sum_{i=1}^n p_i$$

Inequalities for Unnormalized Relative entropy

► No triangle inequality

$$\exists \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \quad \text{RE}(\mathbf{p}_1 \parallel \mathbf{p}_3) > \text{RE}(\mathbf{p}_1 \parallel \mathbf{p}_2) + \text{RE}(\mathbf{p}_2 \parallel \mathbf{p}_3)$$

► Generalized Pythagorean inequality For any closed convex set S and any point $\mathbf{p}_1 \notin S$, define the projection of \mathbf{p}_1 on S to be $\mathbf{p}_2 = \operatorname{argmin}_{\mathbf{u} \in S} \text{RE}(\mathbf{p}_1 \parallel \mathbf{u})$, then:

$$\forall \mathbf{p}_3 \in S; \quad \text{RE}(\mathbf{p}_1 \parallel \mathbf{p}_3) \geq \text{RE}(\mathbf{p}_1 \parallel \mathbf{p}_2) + \text{RE}(\mathbf{p}_2 \parallel \mathbf{p}_3)$$

half squared euclidean distance

- ▶ prediction, outcome $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$



$$\lambda_{\text{sq}}(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \|\mathbf{u} - \mathbf{v}\|^2 = \frac{1}{2} \sum_{i=1}^n (u_i - v_i)^2$$

- ▶ Bregman divergence with respect to the square euclidean norm

$$\|\mathbf{v}\|_2$$

- ▶ Triangle inequality does not hold.
- ▶ **Pythagoras inequality** : For any closed convex set S and any point $\mathbf{v}_1 \notin S$, define the projection of \mathbf{v}_1 on S to be $\mathbf{v}_2 = \operatorname{argmin}_{\mathbf{u} \in S} \|\mathbf{v}_1 - \mathbf{u}\|^2$, then:

$$\forall \mathbf{v}_3 \in S; \quad \|\mathbf{v}_1 - \mathbf{v}_3\|^2 \geq \|\mathbf{v}_1 - \mathbf{v}_2\|^2 + \|\mathbf{v}_2 - \mathbf{v}_3\|^2$$

Bregman divergence regularization

- ▶ Idea: Set \mathbf{w}_{t+1} to be \mathbf{u} that minimizes:

$$\Delta_F(\mathbf{w}_t, \mathbf{u}) + \alpha \ell_t(\mathbf{u})$$

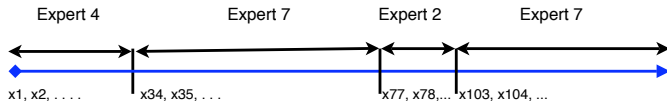
- ▶ In general, hard to compute the minimum.
- ▶ Efficient approximation **Mirror Descent**. Will be covered later.

Tracking Linear Experts

- ▶ **Usually:** compare algorithm's total loss to total loss of the best expert.
- ▶ **drifting experts:** Compare with a sequence of experts that change over time.
- ▶ The amount of change is measured using total bregman divergence.
- ▶ Regret depends on $\sum_t \Delta_F(\mathbf{u}_t - \mathbf{1}, \mathbf{u}_t)$
- ▶ **The Projection Update** After computing the unconstrained update, project the \mathbf{w}_{t+1} onto a convex set.
- ▶ Does not allow the algorithm to over-commit to an extreme vector from which it is hard to recover.

Switching experts setup

- ▶ **Usually:** compare algorithm's total loss to total loss of the best expert.
- ▶ **Switching experts:** compare algorithm's total loss to total loss of **best expert sequence** with **k switches**.
- ▶



An inefficient algorithm

- ▶ Fix:
 - ▶ l - sequence length
 - ▶ k - number of switches
 - ▶ n - number of experts
- ▶ Consider one **partition-expert** per sequence of switching experts.
- ▶ No. of **partition-experts** : $\binom{l}{k-1} n(n-1)^k = O\left(n^{k+1} \left(\frac{el}{k}\right)^k\right)$
- ▶ The log-loss regret is at most $(k+1) \log n + k \log \frac{l}{k} + k$
- ▶ Requires maintaining $O\left(n^{k+1} \left(\frac{el}{k}\right)^k\right)$ weights.

generalization to mixable losses

- ▶ In this lecture we assume loss function is **mixable**.
- ▶ There is an exponential weights algorithm with learning rate η that achieves (in the non-switching case) a bound

$$L_A \leq \min_i L_i + \frac{1}{\eta} \log n$$

- ▶ Then using the **partition-expert** algorithm for the switching-experts case we get a bound on the regret $\frac{1}{\eta} ((k+1) \log n + k \log \frac{l}{k} + k)$

Weight sharing algorithms

- ▶ Update weights in two stages: loss update then share update.
- ▶ Prediction uses the normalized s weights $w_{t,i}^s / \sum_j w_{t,j}^s$
- ▶ **Loss update** is the same as always, but defines intermediate m weights:

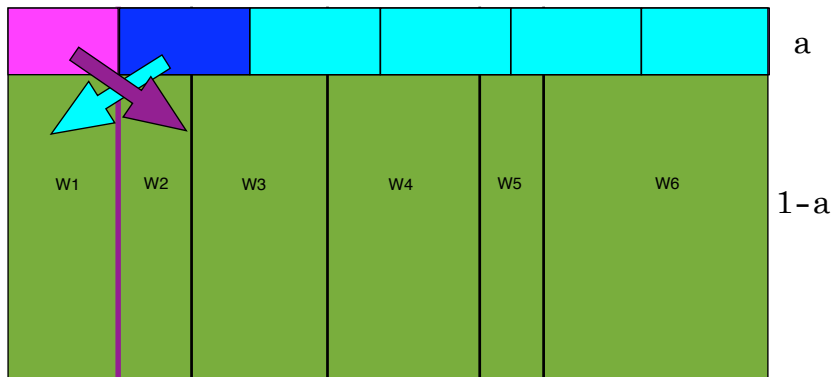
$$w_{t,i}^m = w_{t,i}^s e^{-\eta L(y_t, x_{t,i})}$$

- ▶ **Share update**: redistribute the weights
- ▶ **Fixed-share**:

$$pool = \alpha \sum_{i=1}^n w_{t,i}^m$$

$$w_{t+1,i}^s = (1 - \alpha) w_{t,i}^m + \frac{1}{n - 1} (pool - \alpha w_{t,i}^m)$$

The fixed-share algorithm



Proving a bound on the fixed-share

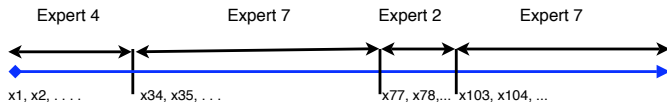
- ▶ The relation between algorithm loss and total weight does not change because share update does not change the total weight.
- ▶ Thus we still have

$$L_A \leq \frac{1}{\eta} \sum_{i=1}^n w_{l+1,i}^s$$

- ▶ The harder question is how to lower bound $\sum_{i=1}^n w_{l+1,i}^s$

Lower bounding the final total weight

- Fix some switching experts sequence:



- “follow” the weight of the chosen expert i_t .
- The loss update reduces the weight by a factor of $e^{-\eta \ell_{t,i_t}}$.
- The share update reduces the weight by a factor larger than:
 - $1 - \alpha$ on iterations with no switch.
 - $\frac{\alpha}{n-1}$ on iterations where a switch occurs.

Bound for arbitrary α

- Combining we lower bound the final weight of the last expert in the sequence

$$w_{l+1, e_k}^s \geq \frac{1}{n} e^{-\eta L_*} (1 - \alpha)^{l-k-1} \left(\frac{\alpha}{n-1} \right)^k$$

Where L_* is the cumulative loss of the switching sequence of experts.

- Combining the upper and lower bounds we get that for any sequence

$$L_A \leq L_* + \frac{1}{\eta} \left(\ln n + (l - k - 1) \ln \frac{1}{1 - \alpha} + k \left(\ln \frac{1}{\alpha} + \ln(n - 1) \right) \right)$$

Tuning α

- ▶ let k^* be the best number of switches (in hind sight) and $\alpha^* = k^*/I$
- ▶ Suppose we use $\alpha \approx \alpha^*$ then the bound that we get is

$$L_A \leq L_* + \frac{1}{\eta}((k+1) \ln n + (I-1)(H(\alpha^*) + D_{\text{KL}}(\alpha^*||\alpha)))$$

Where

$$H(\alpha^*) = -\alpha^* \ln \alpha^* - (1 - \alpha^*) \ln(1 - \alpha^*)$$

$$D_{\text{KL}}(\alpha^*||\alpha) = \alpha^* \ln \frac{\alpha^*}{\alpha} + (1 - \alpha^*) \ln \frac{1 - \alpha^*}{1 - \alpha}$$

- ▶ This is very close to the loss of the computationally inefficient algorithm.
- ▶ For the log loss case this is essentially optimal.
- ▶ Not so for square loss!

What can we hope to improve?

- ▶ In the fixed-share algorithm, the weight of a suboptimal expert never decreases below α/n .
- ▶ The algorithm does not concentrate only on the best expert, even if the last switch is in the distant past.
- ▶ The regret depends on the length of the sequence.

The idea of variable-share

- ▶ Let the fraction of the total weight given to the best expert get arbitrarily close to **1**.
- ▶ we can get a regret bound that depends only on the number of switches, not on the length of the sequence.
- ▶ Requires that the loss be bounded.
- ▶ Works for **square** loss, but not for **log** loss!

Variable-share

$$pool = \sum_{i=1}^n \left(1 - (1 - \alpha)^{\ell_{t,i}}\right) w_{t,i}^m$$

$$w_{t+1,i}^s = (1 - \alpha)^{\ell_{t,i}} w_{t,i}^m + \frac{1}{n-1} \left(pool - (1 - (1 - \alpha)^{\ell_{t,i}}) w_{t,i}^m \right)$$

If $\ell_{t,i} = 0$, then expert i does not contribute to the pool.
Expert can get fraction of the total weight arbitrarily close to 1.
Shares the weight quickly if $\ell_{t,i} > 0$

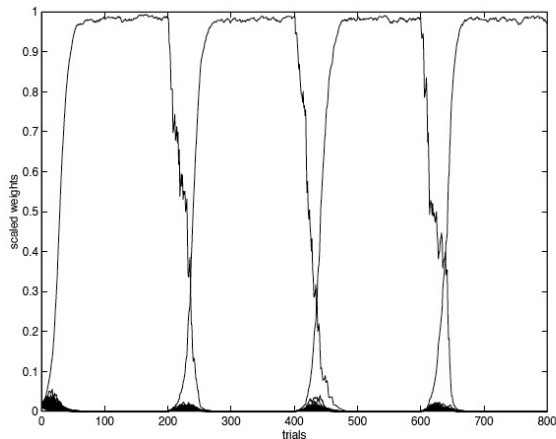
Bound for variable share



$$\frac{1}{\eta} \ln n + \left(1 + \frac{1}{(1-\alpha)\eta}\right) L_* + k \left(1 + \frac{1}{\eta} \left(\ln n - 1 + \ln \frac{1}{\alpha} + \ln \frac{1}{1-\alpha}\right)\right)$$

- α should be tuned so that it is (close to) $\frac{k}{2k+L_*}$

An experiment using variable share



Switching within a small subset

- ▶ Suppose the best switching sequence is repeatedly switching among a small subset of the experts $n' \ll n$
- ▶ In the context of speech recognition - the speaker repeatedly uses a small number of phonemes.
- ▶ If we know the subset, we can pay $\ln n'$ per switch rather than $\ln n$
- ▶ Can track switches much more closely.
- ▶ Easy to describe an inefficient algorithm (consider all $\binom{n}{n'}$ subsets.)
- ▶ Switching to Slides from Manfred Warmuth.