

# Vovk's aggregating algorithm

## Mixable and unmixable loss functions

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Section 3.5 in “Prediction, Learning and Games”

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## Log Loss and Absolute loss

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The general prediction game

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Summary table

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4. Each expert incurs loss  $\ell_t(i) = \lambda(\omega^t, \gamma_i^t)$   
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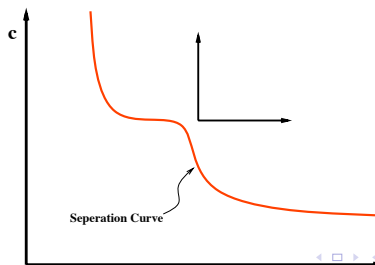
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- ▶ No triangle inequality  
 $\exists p_1, p_2, p_3 \lambda(p_1, p_3) > \lambda(p_1, p_2) + \lambda(p_2, p_3)$

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- ▶ Corresponds to regression.

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- ▶ Loss is bounded.
- ▶ Defines a metric.
- ▶  $\lambda_{\text{hel}}(p, q) \approx \lambda_{\text{ent}}(p, q)$  when  $p \approx q$  and  $p, q \in (0, 1)$

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- ▶ If  $P[\omega^t = 1] = q$ ,  $P[\omega^t = 0] = 1 - q$ , then the optimal prediction is

$$\gamma^t = \begin{cases} 1 & \text{if } q > 1/2, \\ 0 & \text{otherwise} \end{cases}$$

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- ▶ For the log loss the regret is  $O(\log N)$

## Structureless bounded loss

- ▶ Prediction is a distribution  $\gamma = \langle p_1, \dots, p_N \rangle$ ,  $p_i \geq 0$ ,  
 $\sum_{i=1}^N p_i = 1$
- ▶ Outcome is a loss vector  $\omega = \langle \omega_1, \dots, \omega_N \rangle$ ,  $0 \leq \omega_i \leq 1$
- ▶ Loss is the dot product:  $\lambda_{\text{dot}}(\omega, \gamma) = \gamma \cdot \omega$
- ▶ Corresponds to the hedging game.
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- ▶ **Which losses behave like entropy loss and which behave like hedge loss?**



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- ▶ There is **no universally optimal prediction**  
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- Vovk's result: **yes!** a good choice for  $\gamma_t$  always exists!

Vovk's algorithm is the the highest achiever [Vovk95]

The pair  $(a, c)$  is achieved by some algorithm if and only if it is achieved by Vovk's algorithm.

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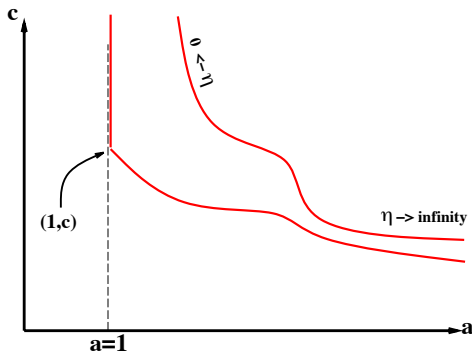
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- ▶ Equivalently - the image of the set  $\Gamma$  under the mapping  $F(\gamma) = \langle e^{-\eta \lambda(\omega, \gamma)} \rangle_{\omega \in \Omega}$  is concave.

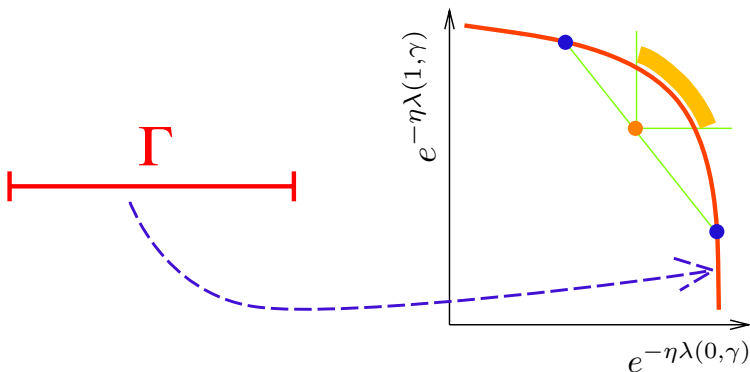
## convexity condition: Pictorially

- **Example:** Suppose  $\Omega = \{0, 1\}$ ,  $\Gamma = [0, 1]$ . then

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► **Example:** Suppose  $\Omega = \{0, 1\}$ ,  $\Gamma = [0, 1]$ . then

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- ▶ We are back to the online Bayes algorithm.

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- ▶ Which yields the bound

$$L_A \leq L_{\min} + 2 \ln N$$

## Summary of bounds for mixable losses

Loss Functions:	$c$ values: ( $\eta = 1/c$ )	
	$\text{pred}_{\text{wmean}}(v, x)$	$\text{pred}_{\text{Vovk}}(v, x)$
$L_{\text{sq}}(p, q)$	2	1/2
$L_{\text{ent}}(p, q)$	1	1
$L_{\text{hel}}(p, q)$	1	$1/\sqrt{2}$

Figure 2.  $(c, 1/c)$ -realizability:  $c$  values for loss and prediction