Tracking a small set of Experts

Yoav Freund

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Based on “Tracking a Small Set of Experts by Mixing Past Posteriors” by Bousquet and Warmuth.

Vovk's meta-algorithm

- Fix an achievable pair (a, c) and set $\eta = a/c$
- 1.

$$W_i^t = \frac{1}{N} e^{-\eta L_i^t}$$

2. Choose γ_t so that, for all $\omega^t \in \Omega$:

$$\lambda(\omega^t, \gamma^t) - c \ln \sum_i W_i^t \leq -c \ln \left(\sum_i W_i^t e^{-\eta \lambda(\omega^t, \gamma_i^t)} \right)$$

- If choice of γ_t always exists, then the total loss satisfies:

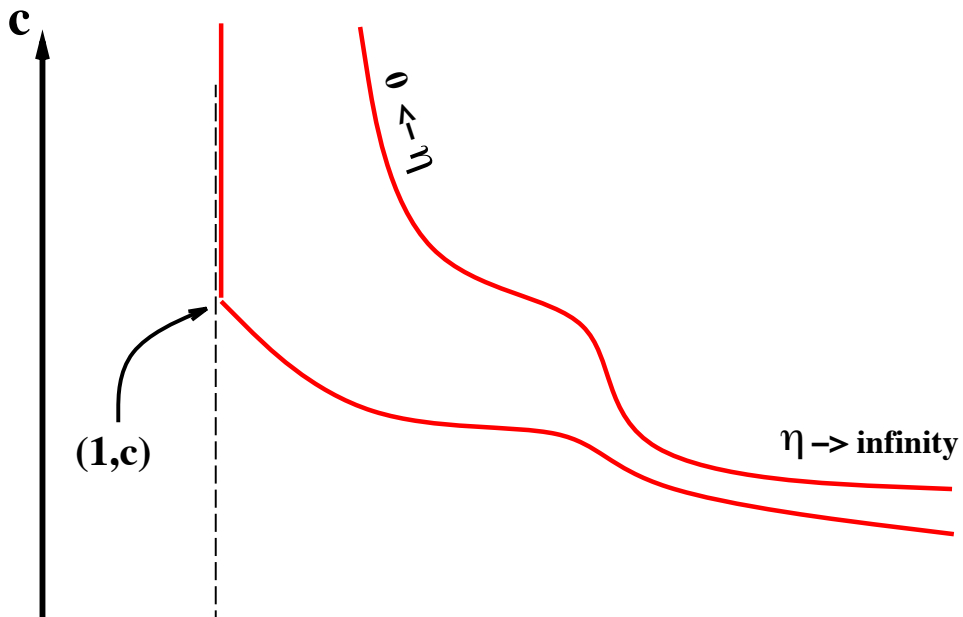
$$\sum_t \lambda(\omega^t, \gamma^t) \leq -c \ln \sum_i W_i^{T+1} \leq a L_{\min} + c \ln N$$

- Vovk's result: *yes!* a good choice for γ_t always exists!

The set of achievable bounds

- Fix loss function $\lambda : \Omega \times \Gamma \to [0, \infty)$
- The pair (a, c) is *achievable* if there exists *some* prediction algorithm such that for *any* $N > 0$, *any* set of N prediction sequences and *any* sequence of outcomes

$$L_A \leq aL_{\min} + c \ln N$$



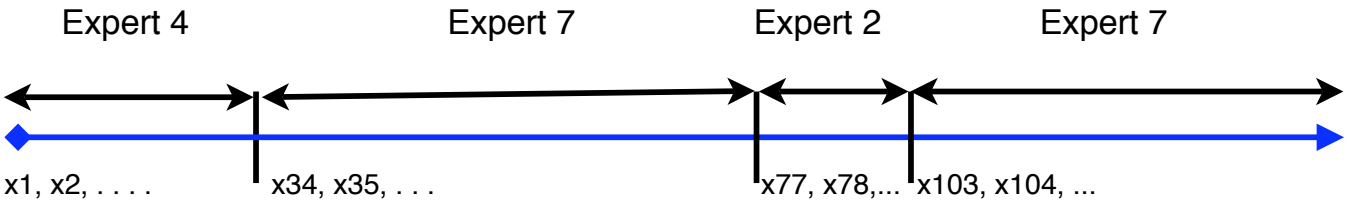
Definition of achievability for non-uniform prior

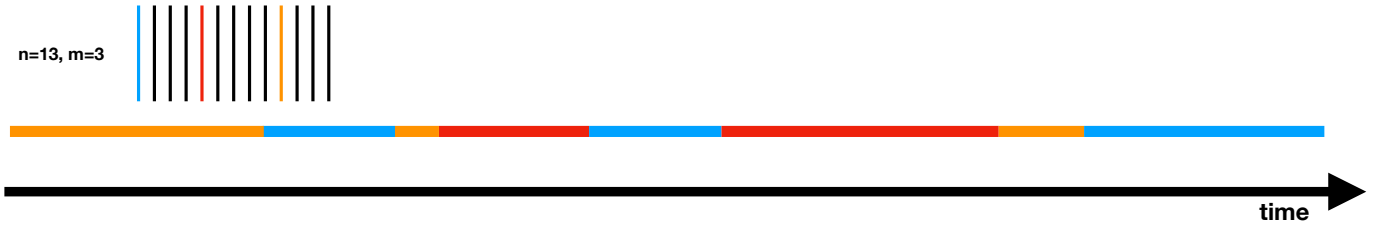
Definition 1 *(Haussler et al., 1998, Vovk, 1998) Let* $c, \eta > 0$. *A loss function* L *and prediction function* `pred` *are* (c, η)*-realizable if, for any weight vector* $\boldsymbol{v} \in \mathcal{P}_n$, *prediction vector* $\boldsymbol{x}$ *and outcome* y,

$$L(y, \texttt{pred}(\boldsymbol{v}, \boldsymbol{x})) \leq -c \ln \sum_{i=1}^{n} v_i e^{-\eta L(y, x_i)} \quad . \tag{1}$$

- Specifies c, η instead of a, c and $\eta = a/c$
- Mixable: $a = 1$ or equivalently $\eta = 1/c$

Switching in a small set

- Switching experts: n experts. Algorithm's total loss compared to total loss of best expert sequence with k switches.
- Regret bound **Regret** $\leq ck \log n + c \log \frac{T-1}{k}$
- 

- Switching in a small set: Same, but comparator restricted to a subset of $m \ll n$ experts.
- 

Short term vs. long term

- In the short term - track the sequence of switching experts. Regret of $k \log n$ per switch.
- In the long term - Identify the set of $m \ll n$ experts and switch only among them. Regret of $c \log m$ per switch.
- Practical implication: when the set of models is small, transitions are tracked more quickly.

An inefficient algorithm for switching experts

- Fix:
  - l - sequence length
  - k - number of switches
  - n - number of experts
- Consider one partition-expert per sequence of switching experts.
- No. of partition-experts : $\binom{l}{k-1} n(n-1)^k = O\left(n^{k+1}\left(\frac{el}{k}\right)^k\right)$
- The regret for a mixable loss with constant c is $c\left((k+1)\log n + k\log\frac{l}{k} + k\right)$

An inefficient algorithm for switching in a small set

- Define a meta-expert for each subset S of $\{1, \ldots, n\}$ of size m.
- There are $\binom{n}{m}$ meta-experts.
- Each meta-expert considers switches among the experts in S
- A standard exponential weights algorithm is used to combine all meta-experts.

Regret bound for inefficient algorithm

- For inefficient switching experts:
  $R \approx ck \log n + ck \log \frac{l}{k}$
- For inefficient switching in a small set:
  $R \approx ck \log m + ck \log \frac{l}{k} + cn \log \frac{n}{m}$
- Remember two part coding for log loss: encode the model and then the data given the model. The length of the description of the model is the regret.

Efficient algorithm

Parameters: $0 < \eta, c$ and $0 \leq \alpha \leq 1$
Initialization: Initialize the weight vector to $\boldsymbol{v}_1 = \frac{1}{n}\mathbf{1}$ and denote $\boldsymbol{v}_0^m = \frac{1}{n}\mathbf{1}$
FOR $t = 1$ **TO** T **DO**

- **Prediction:** After receiving the vector of experts' predictions $\boldsymbol{x}_t$, predict with

$$\hat{y}_t = \texttt{pred}(\boldsymbol{v}_t, \boldsymbol{x}_t) \ .$$

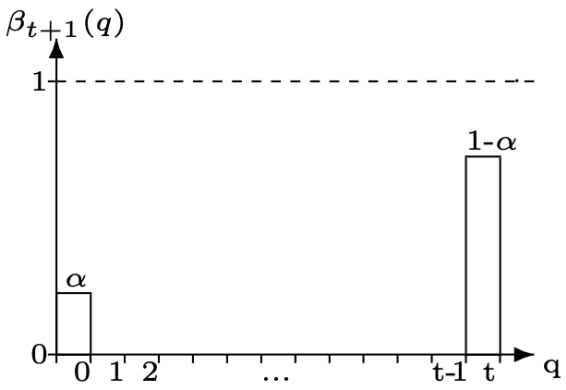
- **Loss Update:** After receiving the outcome y_t, compute for $1 \leq i \leq n$,

$$v_{t,i}^m = \frac{v_{t,i} e^{-\eta L_{t,i}}}{\sum_{j=1}^n v_{t,j} e^{-\eta L_{t,j}}}, \qquad \text{where } L_{t,i} = L(y_t, x_{t,i}) \ .$$

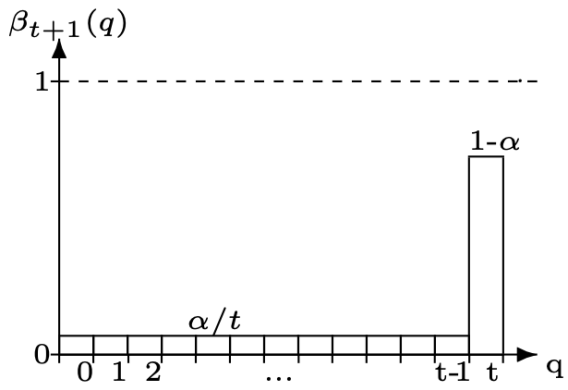
- **Mixing Update:** Choose non-negative mixture coefficients $\beta_{t+1}(q)$ $(q = 0, \ldots, t)$ such that $\sum_{q=0}^t \beta_{t+1}(q) = 1$ and compute

$$\boldsymbol{v}_{t+1} = \sum_{q=0}^{t} \beta_{t+1}(q) \boldsymbol{v}_q^m \ .$$

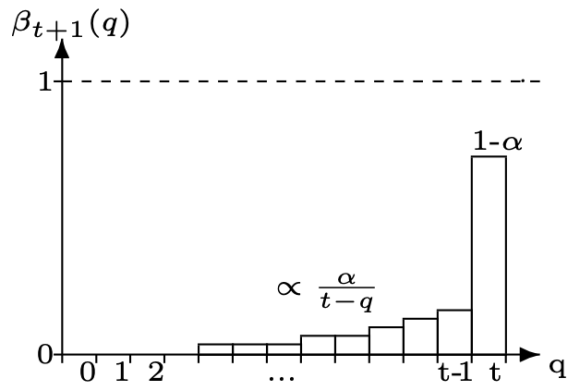
Mixing in past weights



FS to Start Vector



FS to Uniform Past



FS to Decaying Past

Bound For Uniform Past

Bound for the Fixed Share to Uniform Past Mixing Scheme. Consider a mixing scheme that equally penalizes all vectors in the past: $\beta_{t+1}(q) = \alpha \frac{1}{t}$ $(q = 0..t-1)$.

Corollary 8 *For the Mixing Algorithm A with the Fixed Share to Uniform Past mixing scheme and for any sequence of T comparison vectors $\boldsymbol{u}_t$ with k shifts from a pool of m convex combinations, we have*

$$L_{1..T,A} \leq \sum_{t=1}^{T} \boldsymbol{L}_t \cdot \boldsymbol{u}_t + cm \ln n + ck \ln \frac{1}{\alpha} + c(T-k-1) \ln \frac{1}{1-\alpha} + ck \ln(T-1) \ .$$

Bound For Decaying Past

Bound for the Fixed Share to Decaying Past Mixing Scheme. We now show that an improvement of the above corollary is possible by choosing $\beta_{t+1}(q) = \alpha \frac{1}{(t-q)^\gamma Z_t}$ for $0 \le q \le t-1$, with $Z_t = \sum_{q=0}^{t-1} \frac{1}{(t-q)^\gamma}$.

Corollary 9 *For the Mixing Algorithm A with the Fixed Share to Decaying Past mixing scheme with $\gamma = 1$ and for any sequence of T comparison vectors $\boldsymbol{u}_t$ with k shifts from a pool of m convex combinations, we have*

$$L_{1..T,A} \le \sum_{t=1}^{T} \boldsymbol{L}_t \cdot \boldsymbol{u}_t + cm \ln n + ck \ln \frac{1}{\alpha} + c(T-k-1) \ln \frac{1}{1-\alpha} + ck \ln \frac{(T-1)(m-1)}{k} + ck \ln \ln(eT) \ .$$

Proof The proof follows the proof of Corollary 8 and is given in Appendix B. ■

Experiments

Switch to paper (BousquetW02.pdf)