Mirror Descent

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Material follows Chapter 11 of “Prediction Learning and Games”
Sections 11.{1,2,3}

Outline

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Linear Pattern Recognition

Potential Based Gradient descent

Duality

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- Instance: $(\mathbf{x}_t, y_t) \in \mathbb{R}^d \times \mathbb{R}$

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- Regret: $\mathbf{R}_t(\mathbf{u}) = \sum_{i=1}^{t} [\ell(\mathbf{w}_t \cdot \mathbf{x}_t, y_t) - \ell(\mathbf{u} \cdot \mathbf{x}_t, y_t)]$

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- Experts that correspond to exponential distributions - we can use conjugate priors. (recall: biased coins).
- We need a new trick to compute $\mathbf{w}_t = \nabla\Phi(\mathbf{R}_t)$ efficiently.

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- U and V are dual vector spaces, with dual norms.

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- L_2 is self-dual.

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$$F^*(\mathbf{u}) = \sup_{\mathbf{v} \in A} (\mathbf{u} \cdot \mathbf{v} - F(\mathbf{v}))$$

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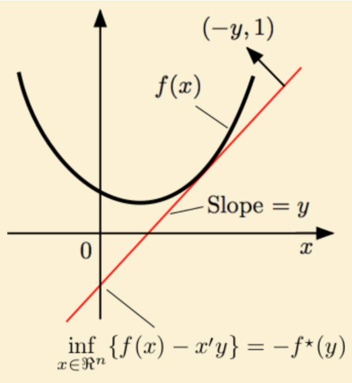

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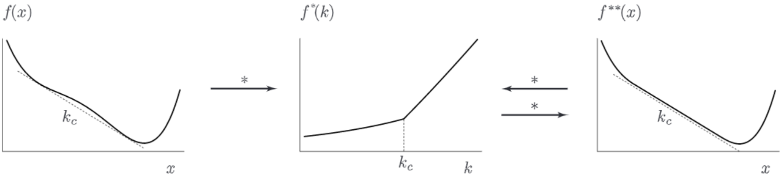

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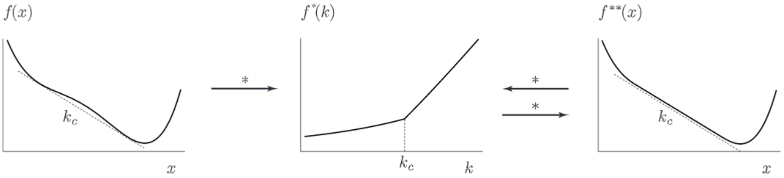

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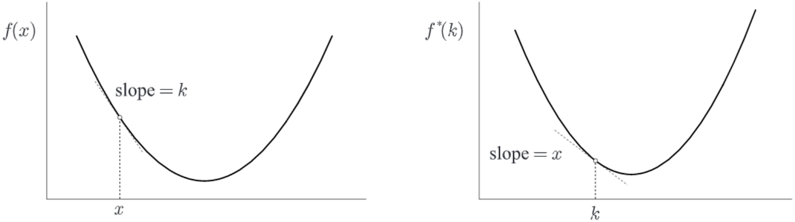

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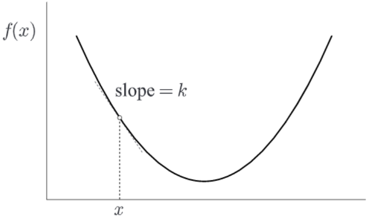
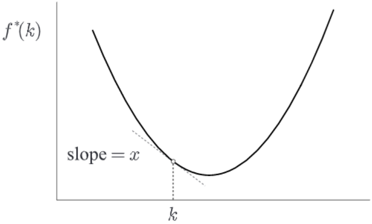
- In general:

$$\nabla F^* = (\nabla F)^{-1}$$





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- Note $(\nabla F)^{-1} = \nabla F^*$

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- $\mathbf{r}_t$ regret for single step.
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- re-written using Duality:

$$\nabla\Phi^*(\mathbf{w}_t) = \nabla\Phi(\mathbf{w}_{t-1}) + \mathbf{r}_t$$

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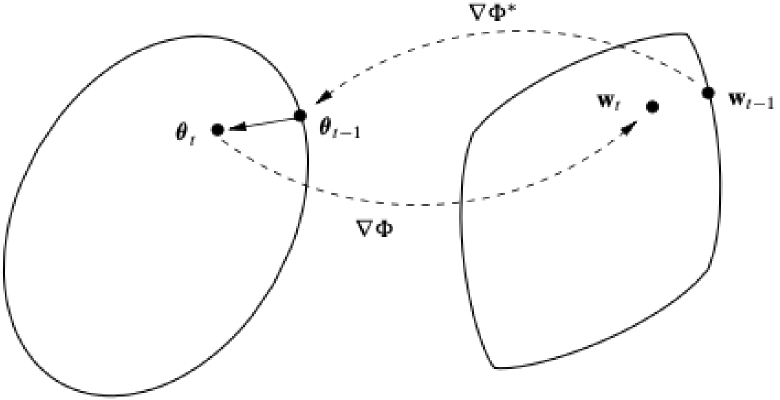
- As $\nabla \Phi$ is the inverse of $\nabla \Phi^*$ we get

$$\mathbf{w}_t = \nabla \Phi(\nabla \Phi^*(\mathbf{w}_{t-1}) - \lambda \nabla \ell_t(\mathbf{w}_{t-1}))$$

A picture of mirror descent

$$\mathbf{w}_t = \nabla\Phi(\nabla\Phi^*(\mathbf{w}_{t-1}) - \lambda\nabla\ell_t(\mathbf{w}_{t-1}))$$



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- Regret: $\mathbf{R}_t(\mathbf{u}) = L_{A,t} - L_t(\mathbf{u})$
- Theorem: For all example sequences $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_T, y_T)$, any initial vector $\mathbf{w}_0 \in \mathbb{R}^d$. all $\lambda > 0$ and all $\mathbf{u} \in \mathbb{R}^d$:

$$\mathbf{R}_T(\mathbf{u}) \leq \frac{1}{\lambda} D_{\Phi^*}(\mathbf{u}, \mathbf{w}_0) - \frac{1}{\lambda} \sum_{t=1}^{T} D_{\Phi^*}(\mathbf{w}_{t-1}, \mathbf{w}_t)$$

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- Loss Bound: $L_{A,T} \leq \frac{L_T(\mathbf{u})}{1-\epsilon} + \frac{X_\infty^2 \ln d}{2\epsilon(1-\epsilon)}$