ML-AS0

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ASSIGMENT 0 - REVIEW

PART A - VECTOR OPERATIONS

$$p = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \qquad q = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \qquad r = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

1. $3p + 2q$

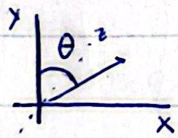
$$\begin{bmatrix} 6 \\ -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 22 \end{bmatrix}$$

2. $\hat{p}$

$$\|p\| = \sqrt{2^2 + (-1)^2 + (4)^2} = \sqrt{4+1+16} = \sqrt{21}$$

$$\hat{p} = \frac{p}{\|p\|} = \frac{1}{\sqrt{21}} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{21} \\ -1/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix}$$

3. $\|p\|$ and the angle with positive y-axis



$$y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\cos\theta = \frac{p \cdot \hat{y}}{\|p\| \cdot \|\hat{y}\|}$$

$$\cos\theta = \frac{2 \cdot 0 + (-1) \cdot 1 + 4 \cdot 0}{\sqrt{21} \cdot 1} = \frac{-1}{\sqrt{21}} \longrightarrow \theta = \arccos\left(\frac{-1}{21}\right)$$

$$\theta = 102.60°$$

4. Direction cosines of p

$$\cos p \longrightarrow \quad \cos\alpha = \frac{x}{\|p\|} \quad \cos\beta = \frac{y}{\|p\|} \quad \cos\gamma = \frac{z}{\|p\|}$$

$$\text{cosines of } p = \left(\frac{2}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right)$$

17.264

5. Angle between p and q

$$\cos\theta = \frac{p \cdot q}{\|p\| \cdot \|q\|} \longrightarrow \frac{\overset{-17}{2 \cdot 0 + 3 \cdot (-1) + 4 \cdot 5}}{\sqrt{21} \cdot \sqrt{3^2 + 5^2}} = 0{,}6362$$

$$\theta = \arccos(0{,}6362) = 50{,}49^\circ$$

6. $p \cdot q$ and $q \cdot p = 17$

7. $p \cdot q$ using the angle between p and q

$$p \cdot q = \cos\theta \cdot \|p\| \cdot \|q\| = \cos(50{,}49) \cdot \sqrt{21} \cdot \sqrt{34} = 17$$

8. Scalar projection

$$\text{proj}_p(q) = \frac{p \cdot q}{\|p\|} = \frac{17}{\sqrt{21}}$$

9. A vector that is perpendicular to p

$$p \cdot s = 0$$

$$2 \cdot x + (-1) \cdot y + 4 \cdot z = 0$$

$$s = [0; 4, -1]$$

10. $p \times q$ and $q \times p$ (cross product)

$$p \times q = \begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix} = [(-5) - 12]i - [10]j + [6]k$$

$$= \begin{bmatrix} -17 \\ -10 \\ 6 \end{bmatrix}$$

$$q \times p = \begin{vmatrix} i & j & k \\ 0 & 3 & 5 \\ 2 & -1 & 4 \end{vmatrix} = [12 + 5]i - [-10]j + [-6]k$$

$$= \begin{bmatrix} 17 \\ 10 \\ -6 \end{bmatrix}$$

11. Vector perpendicular to p and q

$2 \cdot x - y + 4z = 0$ $\qquad y = 1$

$3y + 5z = 0 \rightarrow y = \frac{-5}{3} z$

$2x - \frac{5}{3} z + 4z = 0 \rightarrow 2x - \frac{17z}{3} = 0$

$1 = \frac{-5}{3} z \rightarrow \boxed{z = \frac{-3}{5}}$ $\boxed{y = 1}$

$x = \frac{17}{6} z = \frac{17}{6} \cdot \left(-\frac{3}{5}\right) = \frac{17}{10}$

$v = \left(-\frac{17}{10}, 1, -\frac{3}{5}\right)$

or we could use the orthogonal to both p and q

$p \cdot (p \times q) = 0$ // $q \cdot (p \times q) = 0 \longrightarrow v = (-17, -10, 6)$

12. Linear dependency between p, q and r

$$|A| = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 3 & -2 \\ 4 & 5 & 2 \end{vmatrix} = (12-5) - (12-20) = 15 \neq 0$$

p, q, r are independent

13. $p^T q$ $\quad 1\times 3 \quad 3\times 1$

$$\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} = 17$$

$p q^T$ $\quad 3\times 1 \quad 1\times 3$

$$\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 10 \\ 0 & -3 & -5 \\ 0 & 12 & 20 \end{bmatrix}$$

PART B: Matrix Operations

$$X = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \qquad Y = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

1. $X + 2Y$

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 8 & -2 & 4 \\ 6 & 0 & -6 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 4 \\ 5 & 3 & -2 \\ 5 & 6 & 0 \end{bmatrix}$$

2. XY

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -2 & +1 \\ 9 & 9 & -7 \\ 16 & -7 & -2 \end{bmatrix}$$

YX

$$\begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 5 & -8 \\ 15 & 9 & -6 \\ 3 & 9 & 6 \end{bmatrix}$$

3.

$$(XY)^T \begin{bmatrix} 11 & 9 & 16 \\ -2 & 9 & -7 \\ 1 & -7 & -2 \end{bmatrix}$$

$$Y^T X^T = (XY)^T$$

4. $|X|$ and $|Z|$

$$|X| = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} = (\cancel{-12} + \cancel{12}) - (16 + 2) = -18$$

$$|Z| = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix} = (16 - 1) - (12 + 10) = 17$$

5. None of them

$|X| \rightarrow 2 \cdot (-1) + 3 \cdot 1 + 0 = 1 \neq 0 \rightarrow$ not ortogonal

$Y \rightarrow 4 \cdot 3 + 2 \cdot (-3) \neq 0$

$Z \rightarrow 2 \cdot 1 + 1 \cdot 5 = 7 \neq 0$

6./7.

$$X = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$X^{-1} = \frac{1}{\det(X)} \cdot Adj(X)$$

$$C = \begin{bmatrix} -14 & 10 & -11 \\ 2 & -4 & -1 \\ 4 & -8 & 7 \end{bmatrix} \longrightarrow Adj(X) = C^T$$

$$X^{-1} = \frac{1}{18} \cdot \begin{bmatrix} -14 & 2 & 4 \\ 10 & -4 & -8 \\ -11 & -1 & 7 \end{bmatrix}$$

Y^{-1}

$$Y = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} +6 & -6 & 6 \\ 5 & 2 & -9 \\ 3 & 18 & 3 \end{bmatrix}$$

$$Y^{-1} = \frac{1}{42} \cdot \begin{bmatrix} 6 & 5 & 3 \\ -6 & 2 & 18 \\ 6 & -9 & 3 \end{bmatrix}$$

Z^{-1}

$$Z = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix} \qquad |Z| = -5 \qquad C = \begin{bmatrix} 3 & 13 & -11 \\ -1 & 7 & -2 \\ 4 & -11 & 8 \end{bmatrix}$$

$$Z^{-1} = \frac{1}{-5} \cdot \begin{bmatrix} 3 & -1 & 4 \\ 13 & 7 & -11 \\ -11 & -2 & 8 \end{bmatrix}$$

8. Xs

$$\underset{3\times3}{\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}} \cdot \underset{3\times1}{\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}}$$

9. Scalar projection of X onto s $\quad \|s\| = \sqrt{17}$

$$\boxed{scalar_s(r_i) = r_i \cdot \hat{s} = \frac{r_i \cdot s}{\|s\|} =}$$

$$r_1: \frac{1}{\sqrt{17}} \cdot (2\cdot(-1) + 1\cdot 4) = 2 = \frac{2}{\sqrt{17}} \Big\rangle \text{proj} \rightarrow \frac{1}{\sqrt{17}} \cdot (2, 13, 5)$$

$r_2: 13/\sqrt{17} \qquad r_3: 5/\sqrt{17}$

10. Vector projection of X onto s $\quad proj(r_i) = (r \cdot \hat{s}) \cdot \hat{s} = \frac{r_i \cdot s}{\|s\|^2} \cdot s =$

$r_1 = \frac{1}{17} \cdot [-2, 8, 0] \qquad r_3 = \frac{1}{17} \cdot [-5, 20, 0]$

$r_2 = \frac{1}{17} \cdot [-13, -52, 0]$

12. $Yt = s$

$Y^{-1} \cdot Y \cdot t = Y^{-1} s$

$t = Y^{-1} \cdot s$ $\quad 3 \times 3 \quad 3 \times 1$

$$t = \frac{1}{42} \cdot \begin{bmatrix} 6 & 5 & 3 \\ -6 & 2 & 18 \\ 6 & -9 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \\ -42 \end{bmatrix} =$$

$$t = \begin{bmatrix} 1/3 \\ 1/3 \\ -1 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

11. linear combination of X using s

$$c_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad c_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad c_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \quad s = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$(-1) \cdot c_1 + 4 \cdot c_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

13. $Zt = s$

$$t = Z^{-1} \cdot s = \frac{1}{-5} \cdot \begin{pmatrix} 3 & -1 & 4 \\ 13 & 7 & -11 \\ -11 & -2 & 8 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -7 \\ 15 \\ 3 \end{pmatrix} \cdot \frac{1}{5}$$

PART D

$f(x) = 2x^2 - 1 \qquad g(x) = 3x^2 + 4 \qquad h(x,y) = x^2 + y^2 + xy$

1. $f'(x) = 4x \qquad f''(x) = 4$

2. $\frac{\partial h}{\partial x} = 2x + y \qquad \frac{\partial h}{\partial y} = 2y + x$

3. $\nabla h(x,y) = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) = (2x + y, 2y + x)$

4. $\frac{d}{dx} f(g(x))$

CHAIN RULE

$= \frac{d}{dx}\left[2\cdot(3x^2+4)^2 - 1\right] = 4\cdot(3x^2+4)\cdot 6x = \boxed{24x\cdot(3x^2+4)}$

NO CHAIN RULE

$2\cdot(3x^2+4)^2 - 1 = 2\cdot(9x^4 + 24x^2 + 16) - 1$

$\frac{\partial}{\partial x}(18x^4 + 42x^2 + 32 - 1) = 4\cdot 18x^3 + 2\cdot 42x$

$= 72x^3 + 96x$

PART C

$M = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \qquad N = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

1. Eigenvalues and eingenvectors of M

$\det(M - \lambda I) = \begin{bmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = (3-\lambda)\cdot(4-\lambda) + 2$

$= \lambda^2 - 7\lambda + 14 = 0$

$\lambda = \frac{7 \pm \sqrt{49-56}}{2} = \frac{7 \pm \sqrt{-7}}{2} = \frac{7}{2} \pm \frac{i\sqrt{7}}{2}$

$\begin{bmatrix} 3-\lambda_1 & 2 \\ -1 & 4-\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \longrightarrow v_1 = \begin{bmatrix} 1 \\ \frac{1+i\sqrt{7}}{4} \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ \frac{1-i\sqrt{7}}{4} \end{bmatrix}$

2. Dot product of eigenvectors of M

$$v_1 \cdot v_2 = \begin{bmatrix} 1 \\ \frac{1+i\sqrt{7}}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{1-i\sqrt{7}}{2} \end{bmatrix} = 1 \cdot 1 + \frac{(1+i\sqrt{7}) \cdot (1-i\sqrt{7})}{4}$$

$$= \frac{1+i\sqrt{7} \cdot 1-i\sqrt{7}}{4} = \frac{1+7}{4} = 2$$

$$= 1+2 = 3$$

3. Dot product of eigenvectors of N

$$\det(N - \lambda I) = \begin{vmatrix} 5-\lambda & -3 \\ -3 & 6-\lambda \end{vmatrix} = (5-\lambda)\cdot(6-\lambda) - 9 = 0$$

$$30 - 6\lambda - 5\lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 - 11\lambda + 21 = 0$$

$$\lambda_1 = \frac{11+\sqrt{37}}{2} \quad \lambda_2 = \frac{11-\sqrt{37}}{2}$$

$$\begin{bmatrix} 5-\lambda_1 & -3 \\ -3 & 6-\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$(5-\lambda_1)x - 3y = 0 \rightarrow y = \frac{5-\lambda_1}{3}x$$

$$v_1 = \begin{bmatrix} 1 \\ \frac{5-\lambda_1}{3} \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ \frac{5-\lambda_2}{3} \end{bmatrix}$$

$$v_1 \cdot v_2 = \begin{bmatrix} 1 \\ \frac{5-\lambda_1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{5-\lambda_2}{3} \end{bmatrix} = 0$$

4.

Since the matrix N is symetric the eigenvalues are orthogonal.

5. $Pt = 0$ trivial

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = 0$$

$2t_1 + 4t_2 = 0$

$4t_1 + 8t_2 = 0 - t_1 + 4t_2$

$t_1 = 0 \quad t_2 = 0$

6. Non trivial

$t_1 + 2t_2 = 0 \rightarrow t_1 = -2t_2 \quad t_1 = -2$

$-2t_2 + 4t_2 = 0 = 2t_2 \rightarrow t_2 = 1$

$t = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad t = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$|N| = 0$

7. $Mt = 0$

$$\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = 0$$

$3t_1 + 2t_2 = 0$

$-t_1 + 4t_2 = 0$

Only sol $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$t_1 = 4t_2 \rightarrow 12t_2 + 2t_2 = 0$

$14t_2 = 0$

$t_2 = 0$

It only has one solution because $|M| = 14 \neq 0$ thus is invertible and the null space contains only the zero vector