Security & Cryptography Class Notes

Anna Visman

Academic Year 2024-2025

Contents

1 Lecture 1

1.1 Security Overview

A computer system is said to be secure if it satisfies the following properties:

- **Confidentiality**: Unauthorized entities cannot access the system or its data
- **Integrity**: When you receive data, it is the right one
- **Availability**: The system or data is there when you need it

Remark 1. *The mere presence of these properties does not necessarily mean that the system is fully secure in practice.*

A secure system is reliable:

- Keep your personal data confidential
- Allow only authorised access or modifications to resources
- Ensure that any produced results are correct
- Give you correct and meaningful results whenever you want them

Terminology:

- **Assets**: Things we want to protect (hardware, software, data)
- **Vulnerabilities**: Weaknesses in a system that may be exploited in order to cause loss and harm
- **Threats**: A loss or harm that might befall a system (interception, interruption, modification, fabrication)
- **Attack**: An action which exploits a vulnerability to execute a threat
- **Control/Defence**: Removing/reducing a vulnerability. You control a vulnerability to prevent an attack and defend against a threat

Methods of Defence:

- Prevent it
- Deter it: make the attack harder or more expensive
- Deflect it: make yourself less attractive to attacker
- Detect it: notice that the attack is occurring
- Recover from it: mitigate the effects of the attack

Principle of Easiest Penetration: A system is only as secure as its weakest link. An attacker will go after whatever part of the system is easiest for them, not most convenient for you. In order to build secure systems, we need to learn how to think like an attacker!

1.2 Defense Overview

Software controls:

- Passwords and other forms of access control
- Operating systems separate users' actions from each other
- Virus scanner watch for malware
- Development controls enforce quality measures on the original source code
- Personal firewalls that run on your desktop

Hardware controls:

- Not usually protection of the hardware itself, but rather using separate hardware to protect the system as a whole
- Fingerprint readers
- Smart tokens
- Firewalls
- Intrusion detection systems

Physical Systems:

- Protection of the hardwell itself, as well as physical access to the console, storage media, etc.
- Locks
- Guards
- Off-site backups

Policies and Procedures:

- Non-technical means can be used to protect against some classes of attack (e.g. VPNs for accessing interal company network)
- Rules about choosing Passwords
- Training in best security practices

1.3 Cryptography Overview

Objectives of Cryptography:

- Protecting data privacy
- Authenticaion (message, data origin, entity)
- Non-repudiation: preventing the sender from later denying that they sent the message

Definition 1. *Kerkhoff's Principle: The adversary knows all details about a crypto system except the secret key.*

Definition 2. *Cipher: A method or algorithm used to transform readable data (called plaintext) into an unreadable format (called ciphertext) to protect its confidentiality.*

Encryption is the process of converting plaintext into ciphertext. Decryption is the reverse process. Encryption uses the key k, decryption uses the key k'. If k = k', the system is symmetric. If k ≠ k', the system is asymmetric. Decryption(Encryption(m)) = m.

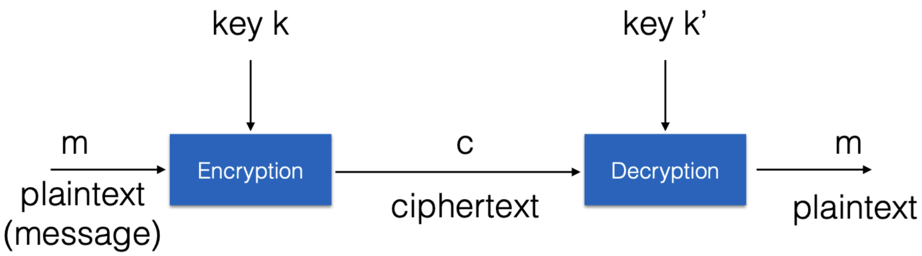


Figure 1: Encryption

Feature	**Private Key Encryption**	**Public Key Encryption**
Keys	Same key for encryption & decryption	Two keys: public and private
Speed	Faster	Slower
Key sharing	Must be kept secret	Only the private key is secret
Use cases	Encrypting large data, e.g., files	Secure key exchange, digital signatures

Table 1: Summary of Differences Between Private and Public Key Encryption

1.4 Topics Covered in Course

- Classical systems: simple ciphers, substitution, permutation, transposition, Caesar, Vigenere
- Information Theoretic Security
- Defining security: pseudorandomness, one-way functions, trapdoor functions
- Notions of security: perfect secrecy, semantic security, IND security
- Attacks on encryption schemes: objective, levels of computing power, amount of information available
- Types attacks: ciphertext-only, known plaintext, chosen plaintext, chosen ciphertext, adaptive
- Different types of adversaries: unbounded/polynomial computing power
- Security: unconditionally secure, computationally secure
- and more... see slides

2 Lecture 2

2.1 Number Theory

2.1.1 Modular Arithmetic

Definition 3. *A positive integer N is called the* modulus*. Two integers a and b are said to be congruent modulo N, written $a \equiv b \pmod{N}$, if N divides $b - a$.*

Examples:

$$18 \equiv 4 \pmod{7}, \quad -18 \equiv 3 \pmod{7}.$$

The set of integers modulo N is denoted by $\mathbb{Z}/N\mathbb{Z}$ or $\mathbb{Z}_N$:

$$\mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}, \quad \#(\mathbb{Z}/N\mathbb{Z}) = N.$$

Properties of Modular Arithmetic:

1. Addition is closed: $\forall a, b \in \mathbb{Z}/N\mathbb{Z} : a + b \in \mathbb{Z}/N\mathbb{Z}$.
2. Addition is associative: $\forall a, b, c \in \mathbb{Z}/N\mathbb{Z} : (a + b) + c = a + (b + c)$.
3. 0 is an additive identity: $\forall a \in \mathbb{Z}/N\mathbb{Z} : a + 0 = 0 + a = a$.
4. The additive inverse always exists: $\forall a \in \mathbb{Z}/N\mathbb{Z} : a + (N - a) = (N - a) + a = 0$.
5. Addition is commutative: $\forall a, b \in \mathbb{Z}/N\mathbb{Z} : a + b = b + a$.
6. Multiplication is closed: $\forall a, b \in \mathbb{Z}/N\mathbb{Z} : a \cdot b \in \mathbb{Z}/N\mathbb{Z}$.
7. Multiplication is associative: $\forall a, b, c \in \mathbb{Z}/N\mathbb{Z} : (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
8. 1 is a multiplicative identity: $\forall a \in \mathbb{Z}/N\mathbb{Z} : a \cdot 1 = 1 \cdot a = a$.
9. Multiplication and addition satisfy the distributive law: $\forall a, b, c \in \mathbb{Z}/N\mathbb{Z} : (a + b) \cdot c = a \cdot c + b \cdot c$.
10. Multiplication is commutative: $\forall a, b \in \mathbb{Z}/N\mathbb{Z} : a \cdot b = b \cdot a$.

2.1.2 Modular Exponentiation

Modular exponentiation is a technique used to efficiently compute expressions of the form $a^b \mod m$, especially for large b. The key idea is to repeatedly square the base a, reduce modulo m at each step, and combine results as needed.

Example: Compute $3^4 \mod 11$

1. Write the problem:
$$3^4 \mod 11$$
2. Break it into smaller steps using properties of modular arithmetic:
   (a) First, compute $3^2 \mod 11$:
   $$3^2 = 9 \quad \Rightarrow \quad 9 \mod 11 = 9$$
   (b) Then, square the result to get $3^4 \mod 11$:
   $$3^4 = (3^2)^2 = 9^2 = 81 \quad \Rightarrow \quad 81 \mod 11 = 4$$
3. Final result:
$$3^4 \mod 11 = 4$$

General Algorithm: Exponentiation by Squaring

1. If b is even:
$$a^b \mod m = \left(a^{b/2} \mod m\right)^2 \mod m$$

2. If b is odd:
$$a^b \mod m = \left(a \cdot a^{b-1} \mod m\right) \mod m$$

You can also simplify the problem by reducing the base modulo:

Definition 4. *For any a, b, n, if $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$ for any positive integer k.*

See an example of this in practice session 1 exercise 1e.

2.1.3 Groups and Rings

Definition 5. *A* group *is a set with an operation that is:*

- *Closed,*
- *Has an identity element,*
- *Associative, and*
- *Each element has an inverse.*

Definition 6. *A group is* abelian *if it is also commutative.*

Examples:

- The integers under addition $(\mathbb{Z}, +)$, where the identity is 0 and the inverse of x is $-x$.
- The nonzero rationals under multiplication $(\mathbb{Q}^*, \cdot)$, where the identity is 1 and the inverse of x is $1/x$.

Group types:

- Multiplicative group: operation is multiplication.
- Additive group: operation is addition.
- Cyclic abelian group: generated by a single element.

Definition 7. *An abelian group G is called* cyclic *if there exists an element in the group, called the* generator*, from which every other element in G can be obtained either by repeated application of the group operation to the generator, or by the use of the inverse operation.*

- *If the group operation is multiplication ($(G, \cdot)$), a generator g produces all elements by repeated multiplication or division: $h = g^x$, where h is an arbitrary element in the group.*
- *In modular arithmetic, g is a generator if $g^x \mod m$ produces all nonzero elements of the group as x varies.*

Example: The group $\mathbb{Z}_7^*$ (the multiplicative group of integers modulo 7) consists of the nonzero integers modulo 7 under multiplication. The elements of the group are:

$$\mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}.$$

An element $g \in \mathbb{Z}_7^*$ is a generator if the powers $g^x \mod 7$ (for $x = 1, 2, 3, \ldots, 6$) produce **all elements** of $\mathbb{Z}_7^*$ exactly once. Let's test whether 3 is a generator:

1. Compute the powers of 3 modulo 7:
$$\begin{gathered}
3^1 \mod 7 = 3, \\
3^2 \mod 7 = 9 \mod 7 = 2, \\
3^3 \mod 7 = 27 \mod 7 = 6, \\
3^4 \mod 7 = 81 \mod 7 = 4, \\
3^5 \mod 7 = 243 \mod 7 = 5, \\
3^6 \mod 7 = 729 \mod 7 = 1.
\end{gathered}$$

2. The results are:
$$\{3, 2, 6, 4, 5, 1\}.$$

Since this list contains all elements of $\mathbb{Z}_7^*$, 3 is a generator of $\mathbb{Z}_7^*$. Other generators of $\mathbb{Z}_7^*$ include 5. You can verify this by computing $5^x \mod 7$ for $x = 1, 2, \ldots, 6$.

Definition 8. *A* ring *is a set with two operations (+, ·) satisfying:*

- *The set is an abelian group under addition.*
- *Multiplication is associative and closed.*
- *Distributive laws hold.*

If multiplication is commutative, the ring is called *commutative.* Examples:

- Integers, real numbers, and complex numbers form infinite rings.
- $\mathbb{Z}/N\mathbb{Z}$ forms a finite ring.

2.1.4 Primes and Divisibility

Definition 9. *An integer a divides another integer b, denoted $a \mid b$, if $b = k \cdot a$ for some integer k.*

Definition 10. *A number p is* prime *if its only divisors are* 1 *and p.*

Examples of primes: $2, 3, 5, 7, 11, \ldots$.

Definition 11. *Greatest Common Divisor: $c = \gcd(a, b)$ if and only if c is the largest number that divides both a and b.*

Theorem 1. *Every positive integer can be written as a product of primes in a unique way.*

Definition 12. *Two integers a and b are coprime, relatively prime or mutually prime if the only positive integer that is a divisor of both of them is 1.*

Definition 13. *Euler's Totient Function: $\phi(p)$ is the number of integers less than p that are relatively prime to p.*

- *If N is a prime then $\phi(N) = N - 1$.*
- *If p and q are both prime and $p \neq q$, then $\phi(pq) = (p - 1)(q - 1)$*

$$\phi(N) = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right) = n\prod_{p|n}(1 - \frac{1}{p})$$

where $p_1, p_2, \ldots, p_k$ are the prime factors of N.

Euler's Totient function counts the number of positive up to a given integer N that are relatively prime to N.

2.1.5 Linear Congruences

Finding the solution to the linear congruence equation:

$$a \cdot x \equiv b \pmod{N}$$

We want to know how many solutions exist for x modulo N given the coefficients a, b, and the modulus N.

1. Compute the greatest common divisor (gcd) of a and N, denoted as $\gcd(a, N) = g$.
2. The following cases determine the number of solutions:

   (a) **If $g = 1$:**

   - When a and N are coprime ($\gcd(a, N) = 1$), the equation has **exactly one solution** modulo N. This is because a has a multiplicative inverse modulo N.

(b) **If $g > 1$ and $g \mid b$:**

- If $\gcd(a, N) = g > 1$ and g divides b, then there are **exactly g solutions** modulo N.
- These solutions can be determined by reducing the equation to a simpler congruence modulo N/g.

(c) **If $g > 1$ and $g \nmid b$:**

- If g does not divide b, then the equation has **no solution**. This is because b is not in the span of a modulo N.

Definition 14. *Multiplicative Inverse Modulo N: A number that, when multiplied by a given number a, gives a result of 1 modulo N. In other words, the multiplicative inverse of a modulo N is a number x such that:*

$$a \cdot x \equiv 1 \pmod N$$

- *The multiplicative inverse of a modulo N is denoted as a^{-1}.*
- *A multiplicative inverse of a modulo N exists only if a and N are coprime, i.e.,* $\gcd(a, N) = 1$.
- *If a and N are not coprime, it's impossible to find x such that* $a \cdot x \equiv 1 \pmod N$.
- *When N is a prime p, then for all non-zero values of $a \in \mathbb{Z}/p\mathbb{Z}$ we always obtain a unique solution to the equation* $a \cdot x \equiv 1 \pmod p$.

Inverse in this case means that the two numbers multiply to 1 modulo N. Think about regular numbers: the inverse of 2 is $\frac{1}{2}$ under multiplication, because $2 * \frac{1}{2} = 1$.

2.1.6 Fields

Definition 15. *A field is a set G with two operations $(G, \cdot, +)$. It satisfies the following properties:*

- *$(G, +)$ is an abelian group with identity element 0 (G is a commutative group under addition).*
- *$(G\backslash\{0\}, \cdot)$ is an abelian group ($G\backslash\{0\}$ is a commutative group under multiplicatio).*
- *Multiplication distributes over addition, i.e., $(G, \cdot, +)$ satisfies the distributive law.*

A field is like the "ideal playground" for numbers: You can add, subtract, multiply, and divide (except by 0). Both addition and multiplication behave nicely (associative, commutative, etc.). Examples of fields include familiar systems like real numbers and rational numbers. The key difference between rings and fields is that in a ring, division is not always possible. In a field, division (except by 0) is always possible, because every nonzero element has a multiplicative inverse.

$$\mathbb{Z}/N\mathbb{Z}$$

is a field if and only if N is prime (because then every nonzero element has a multiplicative inverse). Else, it is a ring.
Think of $\mathbb{Z}/N\mathbb{Z}$ as a "clock" with N hours. Once you pass $N - 1$, you wrap around back to 0. Arithmetic in $\mathbb{Z}/N\mathbb{Z}$ always "cycles" within the set $\{0, 1, \ldots, N - 1\}$.

$(\mathbb{Z}/N\mathbb{Z})^*$ is the set of all elements that are invertible (the set of elements that are coprime to N).

$$(\mathbb{Z}/N\mathbb{Z})^* = \{x \in \mathbb{Z}/N\mathbb{Z} : \gcd(x, N) = 1\}$$

The size of $(\mathbb{Z}/N\mathbb{Z})^*$ is given by Euler's Totient function: $\phi(N)$. If N is a prime p, then $(\mathbb{Z}/N\mathbb{Z})^* = \{1, ..., p - 1\}$.

2.1.7 Lagrange's Theorem

Lagrange's Theorem states that if $(G, \cdot)$ is a finite group with order (size) $n = \#G$, then for any element $a \in G$, the order of a (the smallest positive integer k such that $a^k = 1$) divides n. In particular, it follows that:

$$a^n = 1 \quad \text{for all } a \in G.$$

Application in Modular Arithmetic: In the context of modular arithmetic, consider the group of units $\mathbb{Z}/N\mathbb{Z}^*$ (the set of integers modulo N that are coprime to N, with multiplication as the group operation). If $x \in \mathbb{Z}/N\mathbb{Z}^*$, then the group has size $\phi(N)$, where $\phi(N)$ is Euler's totient function (the count of integers less than N that are coprime to N). Therefore:

$$x^{\phi(N)} \equiv 1 \pmod N.$$

2.1.8 Fermat's Little Theorem

Fermat's Little Theorem states that if p is a prime number and a is any integer, then:

$$a^p \equiv a \pmod{p}.$$

If a is not divisible by p, then this can be rewritten as:

$$a^{p-1} \equiv 1 \pmod{p}.$$

Explanation: This theorem tells us that raising a to the power of $p-1$ gives a remainder of 1 when divided by p, provided a and p are coprime. Fermat's Little Theorem is useful for simplifying modular exponentiation and serves as a foundation for more advanced results like Euler's theorem.

2.2 Basic Algorithms

2.2.1 Euclid's GCD Algorithm

The **Greatest Common Divisor** (GCD) of two integers a and b is the largest integer d such that d divides both a and b.

Key Idea: If we could factorize a and b, we could easily determine their GCD. For example, consider:

$$a = 2^4 \cdot 157 \cdot 4513^3, \quad b = 2^2 \cdot 157 \cdot 2269^3 \cdot 4513.$$

Here, the GCD is given by:

$$\gcd(a, b) = 2^2 \cdot 157 \cdot 4513 = 2{,}834{,}164.$$

However, computing prime factorizations is often impractical for large numbers. Instead, we use Euclid's Algorithm.

The Euclidean Algorithm is based on the principle:

$$\gcd(a, b) = \gcd(a \mod b, b).$$

The algorithm starts with two numbers a, b, where $a > b$. The remainder $r = a \mod b$ is computed. Then, a is repeatedly replaced with b (the smaller number), and b with r (the remainder). This process continues until the remainder is 0. The last non-zero remainder (b) is the GCD of a and b.

Steps:

1. Let $r_0 = a$ and $r_1 = b$.
2. Compute remainders $r_2, r_3, \ldots$ using:

$$r_{i+2} = r_i \mod r_{i+1}, \quad \text{where } r_{i+2} < r_{i+1}.$$

3. Stop when $r_{m+1} = 0$. The GCD is r_m.

Example: Compute $\gcd(21, 12)$:

$$\begin{aligned} \gcd(21, 12) &= \gcd(21 \mod 12, 12) = \gcd(9, 12), \\ \gcd(9, 12) &= \gcd(12 \mod 9, 9) = \gcd(3, 9), \\ \gcd(3, 9) &= \gcd(9 \mod 3, 3) = \gcd(0, 3). \end{aligned}$$

Thus, $\gcd(21, 12) = 3$.

2.2.2 The Extended Euclidean Algorithm

In addition to computing the GCD, the Extended Euclidean Algorithm finds integers x and y such that:

$$\gcd(a, b) = ax + by = r.$$

This is useful in many applications, such as finding modular inverses. For $\gcd(a, b) = d$ where $d = 1$, we can compute $ax + yN = 1$. Here **x** is the multiplicative inverse of a in modulo N. So, if $\gcd(a, N) = 1$, then $a^{-1} \mod N = x \mod N$.

Algorithm:

1. Start with $r_0 = a$, $r_1 = b$, $s_0 = 1$, $s_1 = 0$, $t_0 = 0$, $t_1 = 1$.
2. For each step, compute:
$$q_i = \left\lfloor \frac{r_{i-1}}{r_i} \right\rfloor, \quad r_{i+1} = r_{i-1} - q_i r_i,$$
$$s_{i+1} = s_{i-1} - q_i s_i, \quad t_{i+1} = t_{i-1} - q_i t_i.$$
3. Stop when $r_{i+1} = 0$. Then, $\gcd(a, b) = r_i$, and $x = s_i$, $y = t_i$.

Example: Compute $\gcd(36, 24)$ and coefficients x, y:

$$\text{Step 1: } q = \left\lfloor \frac{36}{24} \right\rfloor = 1, \quad r = 36 - 1 \cdot 24 = 12,$$
$$\text{Update: } x = 0 - 1 \cdot 1 = -1, \quad y = 1 - 1 \cdot 0 = 1,$$
$$\text{Step 2: } q = \left\lfloor \frac{24}{12} \right\rfloor = 2, \quad r = 24 - 2 \cdot 12 = 0,$$
$$\text{Update: } x = 1 - 2 \cdot (-1) = 3, \quad y = 0 - 2 \cdot 1 = -2.$$

Thus, $\gcd(36, 24) = 12$, with $x = -1$, $y = 1$.

2.2.3 The Chinese Remainder Theorem

Let $m_1, ..., m_r$ be pairwise relatively prime (i.e., $\gcd(m_i, m_j) = 1$ for all $i \neq j$). Let $x = a_i \mod m_i$ for all i. The CRT guarantees a unique solution given by:

$$x = \sum_{i=1}^{r} a_i M_i y_i \mod M$$

where

$$M_i = M / \; m_i$$

and

$$y_i = M_i^{-1} \mod m_i$$

y_i is the modular inverse of M_i modulo m_i (this can be computed using the Extended Euclidean Algorithm). The theorem is a way to solve a system of simultaneous modular congruences, finding a unique solution for a number that satisfies multiple modular equations, provided that the moduli are coprime/relatively prime.

$$x \equiv a_1 \mod m_1$$
$$x \equiv a_2 \mod m_2$$
$$...$$
$$x \equiv a_k \mod m_k$$

In that case, there exists a **unique solution** x modulo M, where M is the product of all the moduli:

$$M = m_1 \cdot m_2 \cdot \ldots \cdot m_k.$$

Example:

$$x \equiv 5 \mod 7$$
$$x \equiv 3 \mod 11$$
$$x \equiv 10 \mod 13$$

Then $M = 7 \cdot 11 \cdot 13 = 1001$. $M_1 = 1001/7 = 143$, $M_2 = 1001/11 = 91$, $M_3 = 1001/13 = 77$. $y_1 = 5, y_2 = 4, y_3 = 12$.

$$x = \sum_{i=1}^{r} a_i M_i y_i \mod M = 5 \cdot 143 \cdot 5 + 3 \cdot 91 \cdot 4 + 10 \cdot 77 \cdot 12 \mod 1001$$
$$= 3575 + 1092 + 9240 \mod 1001 = 13907 \mod 1001 = 894$$

add here why we need the CRT

2.2.4 Computing Legendre and Jacobi Symbols

Definition 16. *Let n be a positive integer. An integer a is called a* ***quadratic residue modulo*** n *if there exists an integer x such that:*

$$x^2 \equiv a \pmod{n}.$$

In other words, a is a quadratic residue modulo n if a is congruent to the square of some integer x modulo n.
If no such x exists, then a is called a ***quadratic non-residue modulo*** n.

Example: For $n = 7$, the integers modulo 7 are $\{0, 1, 2, 3, 4, 5, 6\}$. Computing the squares of each integer modulo 7:

$$\begin{aligned}
0^2 &\equiv 0 \pmod{7},\\
1^2 &\equiv 1 \pmod{7},\\
2^2 &\equiv 4 \pmod{7},\\
3^2 &\equiv 2 \pmod{7},\\
4^2 &\equiv 2 \pmod{7},\\
5^2 &\equiv 4 \pmod{7},\\
6^2 &\equiv 1 \pmod{7}.
\end{aligned}$$

The quadratic residues modulo 7 are:

$$\{0, 1, 2, 4\}.$$

The quadratic non-residues modulo 7 are:

$$\{3, 5, 6\}.$$

Symmetry of squares: squaring numbers modulo n often produces repeated results due to symmetry in the group of residues. This means that different numbers can have the same square when considered modulo n. For each x, its symmetric counterpart $n - x$ produces the same square modulo n. The total number of unique quadratic residues is approximately half of n (or $\lfloor n/2 \rfloor + 1$ if 0 is included).

Definition 17. *Legendre Symbol: Let p be a prime number, and let a be an integer. The* ***Legendre symbol*** $\left(\frac{a}{p}\right)$ *is defined as follows:*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p},\\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p,\\ 0 & \text{if } p \mid a \text{ (i.e., if } a \equiv 0 \pmod{p}).\end{cases}$$

- $\left(\frac{a}{p}\right) = 1$ means there exists an integer x such that $x^2 \equiv a \pmod{p}$ (i.e., a is a quadratic residue modulo p).
- $\left(\frac{a}{p}\right) = -1$ means that no such integer x exists (i.e., a is a quadratic non-residue modulo p).
- $\left(\frac{a}{p}\right) = 0$ means that a is divisible by p (i.e., $a \equiv 0 \pmod{p}$).

In simpler terms, the Legendre symbol answers the question: **"Can a be written as the square of some number, when working modulo p?"**
To detect squares modulo a prime p, we define:

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}. \tag{1}$$

Additional formulae:

$$\left(\frac{a}{p}\right) = \left(\frac{a \pmod{p}}{p}\right), \tag{2}$$

$$\left(\frac{a \cdot b}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right), \tag{3}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}, \tag{4}$$

$$\left(\frac{a}{p}\right) = \begin{cases} -\left(\frac{p}{a}\right) & \text{if } p \equiv 3 \pmod 4, \\ \left(\frac{p}{a}\right) & \text{otherwise.} \end{cases} \tag{5}$$

Example: Compute the Legendre symbol $\left(\frac{15}{17}\right)$ to check if 15 is a quadratic residue modulo 17.

$$\begin{aligned} \left(\frac{15}{17}\right) &= \left(\frac{3}{17}\right) \cdot \left(\frac{5}{17}\right) && \text{by equation (3)} \\ &= \left(\frac{17}{3}\right) \cdot \left(\frac{17}{5}\right) && \text{by equation (5)} \\ &= \left(\frac{2}{3}\right) \cdot \left(\frac{2}{5}\right) && \text{by equation (2)} \\ &= (-1) \cdot (-1)^3 && \text{by equation (4)} \\ &= 1. \end{aligned}$$

So $\left(\frac{15}{17}\right) = 1$, and thus 15 is a quadratic residue modulo 17.

Instead of manually testing all possible values of x to see if $x^2 \equiv a \pmod p$, the Legendre symbol gives a quick, direct answer. This is especially helpful when working with large prime numbers.

- **Public-key cryptography** (like RSA) often relies on modular arithmetic and quadratic residues. The Legendre symbol helps in many cryptographic algorithms, such as those involving **Elliptic Curve Cryptography (ECC)**, **zero-knowledge proofs**, and **primality testing**.
- For example, in **the Diffie-Hellman key exchange**, one might need to check if certain numbers are quadratic residues in a modular group.

The Legendre symbol above is only defined when its denominator is a prime, but there is a generalization to composite denominators called the Jacobi symbol.

Definition 18. *For any integer a and odd integer n, the* ***Jacobi symbol*** *is defined as the product of the Legendre symbols corresponding to the* prime factors *of $n > 2$:*

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{e_1} \left(\frac{a}{p_2}\right)^{e_2} \cdots \left(\frac{a}{p_k}\right)^{e_k},$$

where

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$$

is the prime factorization of n.

It is defined as follows:

$$\left(\frac{a}{n}\right) = \begin{cases} 0 & \text{if } n \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } n \text{ and } a \not\equiv 0 \pmod n, \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } n. \end{cases}$$

Remark 2. *If a is square, then the Jacobi symbol will be 1. However, if the Jacobi symbol is 1, a* might *not be a square.*

2.3 Primality Tests

Prime numbers are needed almost always in every public key algorithm. How can you find prime numbers?

Theorem 2. ***The Prime Number Theorem:*** *The number of primes less than X can be given estimated with:*

$$\pi(X) \approx \frac{X}{\log(X)}$$

There are many prime numbers! The probability of a random value to be a prime is $\frac{1}{\log(p)}$. If we need a prime number with 100% certainty, we need a proof of primality.

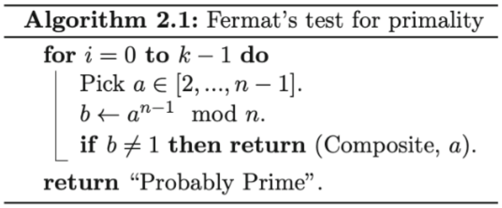
2.3.1 Fermat's Primality Test

Recall that

$$a^{\phi(N)} \equiv 1 \mod N$$

If N is a prime, this equality holds. However, if this equality holds, N is *not necessarily* prime. Probably prime: N is composite with a probability of $\frac{1}{2^k}$. k refers to the number of independent tests or iterations performed to check the primality of a number N. Each test involves choosing a random integer a and checking whether $a^{N-1} \equiv 1 \pmod{N}$.

Remark 3. *Carmichael numbers are composite numbers that pass the Fermat primality test for all possible values of a. They are rare but can be problematic in cryptographic applications. They always return* probably prime.

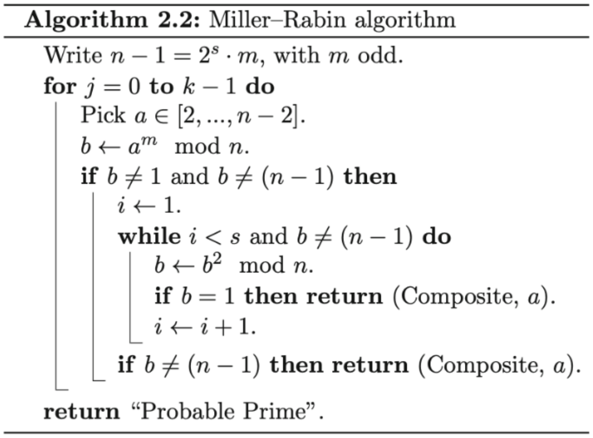


```
Algorithm 2.1: Fermat's test for primality
for $i = 0$ to $k - 1$ do
    Pick $a \in [2, ..., n-1]$.
    $b \leftarrow a^{n-1} \mod n$.
    if $b \neq 1$ then return (Composite, $a$).
return "Probably Prime".
```

Figure 2: Algorithm for Fermat Primality Test

2.3.2 Miller-Rabin Primality Test

The Miller-Rabin test is an improvement over the Fermat test. Unlike deterministic primality tests (which can definitively prove whether a number is prime), the Miller-Rabin test provides a result with high probability. If the test declares a number to be composite, then it is definitely not prime. However, if the test declares the number to be prime, there is still a small chance that it is actually composite (this is the "probabilistic" part). The Miller-Rabin test checks whether a number n passes certain conditions that hold for all prime numbers. It does this by examining the modular arithmetic properties of numbers related to n. If n passes these tests, it is likely prime. If it fails, n is definitely composite.



```
Algorithm 2.2: Miller–Rabin algorithm
Write $n - 1 = 2^s \cdot m$, with $m$ odd.
for $j = 0$ to $k - 1$ do
    Pick $a \in [2, ..., n-2]$.
    $b \leftarrow a^m \mod n$.
    if $b \neq 1$ and $b \neq (n-1)$ then
        $i \leftarrow 1$.
        while $i < s$ and $b \neq (n-1)$ do
            $b \leftarrow b^2 \mod n$.
            if $b = 1$ then return (Composite, $a$).
            $i \leftarrow i + 1$.
        if $b \neq (n-1)$ then return (Composite, $a$).
return "Probable Prime".
```

Figure 3: Algorithm for Miller-Rabin Test

2.4 Elliptic Curves

Definition 19. *An elliptic curve is an equation of the form* $F : y^2 = x^3 + ax + b \mod p$, *with constants a, b.*

- $p > 3$, *otherwise* $x^3 = x$
- *If P is on F, then also* $P + P, P + P + P, \ldots$ *are on F.*

Not all equations make good elliptic curves. They must satisfy a condition that ensures there are no sharp points or self-intersections. The curve must be *smooth* and *non-singular*. This means that the *discriminant must be nonzero*.

$$4a^3 + 27b^2 \neq 0$$

Point Addition:

- Addition of two points P and Q on the curve gives you another point R, which is also on the curve. To add $P = (x_1, y_1)$ and $Q = (x_2, y_2)$:

  1. Draw a straight line through P and Q. This line will generally intersect the curve at exactly one more point, say R'.
  2. Reflect R' across the x-axis to get $R = (x_3, y_3)$, the result of $P + Q$.

  The formulas to compute $R = (x_3, y_3)$ are:

  $$x_3 = \lambda^2 - x_1 - x_2$$

  $$y_3 = \lambda(x_1 - x_3) - y_1$$

  where λ (the slope of the line) is:

  $$\lambda = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\delta y}{\delta x}$$

- If you're adding P to itself (doubling), the line you draw is the tangent to the curve at P. The formulas for $R = 2P = (x_3, y_3)$ are:

  $$x_3 = \lambda^2 - 2x_1$$

  $$y_3 = \lambda(x_1 - x_3) - y_1$$

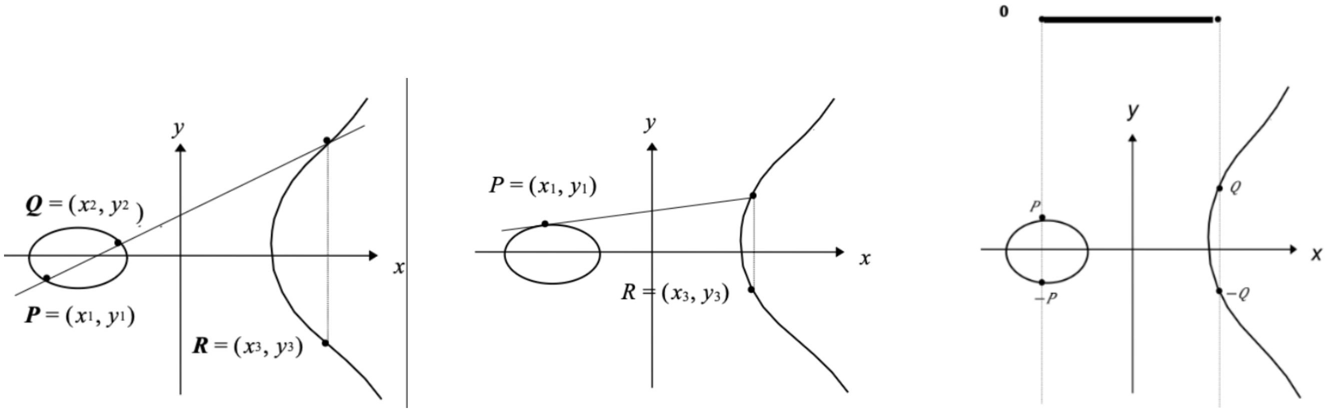
  Here, λ (the slope of the tangent) is:

  $$\lambda = \frac{3x_1^2 + a}{2y_1}$$

- If P and Q are vertical opposites (e.g., $P = (x, y)$ and $Q = (x, -y)$), the line through them is vertical, and their sum is the **point at infinity**, $\mathscr{O}$. Think of $\mathscr{O}$ as the "zero" for point addition.

- Special properties:

  - **Commutative:** $P + Q = Q + P$
  - **Associative:** $(P + Q) + R = P + (Q + R)$
  - **Identity Element:** Adding the point at infinity $\mathscr{O}$ to any point P gives P (like adding zero).



(a) Point Addition

(b) Point Doubling

(c) Zero Point

Using elliptic curves lets us create very secure systems with shorter keys (really large prime numbers are not required), which means faster and more lightweight encryption.

Definition 20. ***Elliptic Curve Discrete Log Problem****: For a given integer m and a point P, it is easy to compute $Q = mP$. However, given P and Q, it is hard to compute m.*

Imagine a simple operation: start at one point on the curve, ”add” it to itself repeatedly, and you get another point. If I tell you the starting point and the number of additions, it’s easy to figure out the end point. But if I give you the end point and ask you to figure out the number of additions, that’s really hard. This is what makes elliptic curve cryptography (ECC) secure. For cryptographic validity, the curve must:

- Have a sufficiently large number of points (called the order) to ensure security.
- Avoid known vulnerabilities (e.g., weak curves susceptible to specific attacks like small subgroup attacks).

3 Lecture 3

3.1 The Syntax of Encryption

Some terms:

- M: message space
- C: ciphertext space
- K: key space
- Three algorithms
  - Key Generation: **Gen** is a *probabilistic* algorithm that outputs a key $k \in K$ chosen according to some distribution.
  - Encryption: **Enc** takes as input a key $k \in K$ and a message $m \in M$ and outputs a ciphertext $c \in C$.
  - Decryption: **Dec** takes as input a key $k \in K$ and a ciphertext $c \in C$ and outputs a message $m \in M$.

3.2 Classical Ciphers

3.2.1 Caesar Cipher

The Caesar cipher is one of the simplest and oldest encryption techniques, named after Julius Caesar, who used it to secure his communications. It works by shifting the letters of the alphabet by a fixed number of positions. For example, with a shift of 3, the letter "A" becomes "D", "B" becomes "E", and so on. To decrypt a message, the recipient simply shifts the letters back by the same number. Although it is easy to understand and implement, the Caesar cipher is considered insecure by modern standards, as there are only a limited number of possible shifts (25), making it vulnerable to brute force attacks.

3.2.2 Shift Cipher

A keyed variant of Caesar's cipher: the secret key can take value between 0 and 26. Each letter in the plaintext is shifted by a certain number of positions in the alphabet. The key difference is that the shift can vary, meaning the amount of movement of each letter can differ across the message. In a typical shift cipher, a number (the key) determines how far to shift each letter. Decryption requires shifting in the opposite direction by the same key. Like the Caesar cipher, the shift cipher is vulnerable to brute force attacks due to the limited number of possible shifts.

$$c = m + k \mod 26$$

Cryptanalysis of the shift cipher can be done using letter and bigram frequencies. This cipher is based on language, and language has structure. Thus it can be decrypted. This approach leverages the fact that in natural languages, certain letters or combinations of letters appear more frequently than others. Example approach:

- **Analyze the ciphertext**: Count the frequency of each letter in the encrypted message.
- **Compare with language statistics**: Match the observed frequencies with the known frequency distribution of letters in the target language (e.g., "E" is often the most frequent in English).
- **Make educated guesses**: Substitute frequently occurring ciphertext letters with likely plaintext letters.
- **Refine the substitution**: Use context, common words, and patterns to adjust and decode the message.

3.2.3 Statistical Distance

Used to evaluate the security of cryptographic systems by measuring how close an observed probability distribution (e.g., the distribution of ciphertexts) is to a desired distribution (usually uniform). Attacks can exploit patterns or non-uniformity in data. The goal is that a secure encryption scheme produces ciphertext that is statistically indistinguishable from random noise.

$$\delta[X, Y] = \frac{1}{2} \sum_{u \in V} |\Pr_{X \leftarrow D_1}[X = u] - \Pr_{Y \leftarrow D_2}[Y = u]|$$

where X, Y are random variables with distributions $D_1, D2$ respectively.

The statistical distance between the ciphertext distribution (produced by the encryption scheme) and the uniform distribution is measured. A smaller statistical distance implies stronger security, as it becomes harder for an attacker to distinguish between encrypted data and random noise.

3.2.4 Substitution Cipher

Each letter (or symbol) in the plaintext is replaced with another letter or symbol according to a specific rule or mapping. The mapping defines the "key" for the cipher, and the process is reversible if the recipient knows the key. The key space consists of all bijections, or permutations ($26! \approx 2^{88}$ for English alphabet). But using letter frequencies, the cipher can be easily solved.

Example:

- Plaintext: `HELLO`
- Cipher Key:

$$\text{A} \to \text{Q, B} \to \text{W, C} \to \text{E}, \dots, \text{Z} \to \text{P}$$

- Key mapping:
  - Plain alphabet: `ABCDEFGHIJKLMNOPQRSTUVWXYZ`
  - Cipher alphabet: `QWERTYUIOPASDFGHJKLZXCVBNM`
- Ciphertext: `XUBBE`

Types of Substitution Ciphers:

- *Mono-alphabetic*: Uses a single mapping for the entire message (e.g., Caesar cipher).
- *Poly-alphabetic*: Uses multiple mappings that change throughout the message for added complexity.

3.2.5 Vigenère Cipher

Also called a poly-alphabetic substitution cipher: encrypt each letter with a dfferent alphabet. Use a keyword to shift letters in the plaintext. Unlike the Caesar cipher, where each letter is shifted by a constant number, the Vigenère cipher uses a series of shifts based on the letters of a keyword. This makes it more secure than simple substitution ciphers, as the pattern of shifts changes across the message. Still, cryptanalysis is easy.

Definition 21. ***Kasiski Test:*** *Find the occurence of the repeated sequences and use* gcd *to determine the key length.*

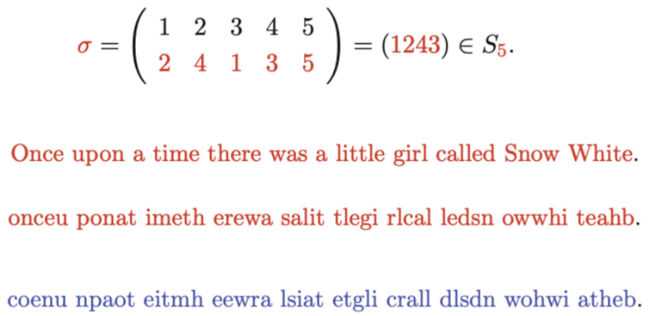
The plaintext is encrypted by shifting each letter by the number of positions specified by the corresponding letter in the keyword. If the keyword is shorter than the plaintext, it is repeated until it matches the length of the plaintext.

Example:

- Plaintext: `HELLO`
- Keyword: `KEY`
- Repeat the keyword to match the length of the plaintext: `KEYKE`
- Shift each letter in the plaintext by the corresponding letter in the keyword:
  - `H` (shift by `K`, which is 10) → `R`
  - `E` (shift by `E`, which is 4) → `I`
  - `L` (shift by `Y`, which is 24) → `J`
  - `L` (shift by `K`, which is 10) → `V`
  - `O` (shift by `E`, which is 4) → `S`
- Ciphertext: `RIJVS`

3.2.6 Permutation Cipher

The *positions* of the characters in the plaintext are rearranged according to a specific permutation (a predetermined key) rather than *replacing* characters with other symbols. The basic idea is that the order of the characters is shuffled, but no new symbols are introduced.



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} = (1243) \in S_5.$$

Once upon a time there was a little girl called Snow White.

onceu ponat imeth erewa salit tlegi rlcal ledsn owwhi teahb.

coenu npaot eitmh eewra lsiat etgli crall dlsdn wohwi atheb.

Figure 5: Permutation Cipher

The key is a sequence of numbers that indicates the new positions of each character in the plaintext. Here, $(1243) \in S_5$, meaning; position 1 goes to position 2, position 2 goes to position 4, and so on. The key space is $|S_n| = n!$.

Key take aways: all historical, classical systems are broken. They rely on substituion and permutation. Designing secure ciphers is hard.

4 Lecture 4

4.1 Information Theory

Information theory is the mathematical study of the quantification, storage, and communication of information. By Claude Shannon. A key measure in information theory is *entropy.* Entropy quantifies the amount of uncertainty involved in the value of a random variable or the outcome of a random process. For example, identifying the outcome of a fair coin flip (which has two equally likely outcomes) provides less information (lower entropy, less uncertainty) than identifying the outcome from a roll of a die (which has six equally likely outcomes).

4.2 Security Definitions

Definition 22. *Computationally Secure: it takes N operations using the* ***best known algorithm*** *to break a crytographic system and N is too large to be feasible.*

Definition 23. *Provably Secure: breaking the system is reduced to solving some well-studied hard problem.*

Definition 24. *Unconditional Secure/Perfectly Secure: the system is secure against an adversary with unlimited computational power.*

Key size is important. Advances in computer hardware and algorithms are important. In the future, it will be broken due to hardware or better algorithms.

4.3 Probability and Ciphers

Definition 25. *Let P denote the set of plaintexts, K the set of keys, and C denote the set of cipher texts. $p(P = m)$ is the probability that the plaintext is m. Then,*

$$p(C = c) = \sum_{k:c \in \mathbb{C}(k)} p(K = k) \cdot p(P = d_k(c))$$

This formula calculates the probability of a specific cipher text $C = c$ occuring. It sums over all keys k that could result in the ciphertext c , combining the probabilities of the key and the corresponding plaintext. It is fundamental in several cryptographic analyses:

- Ciphertext distribution: compute the distribution of the ciphertexts given the plaintext and key distributions. By ensuring $p(C = c)$ is uniform, cryptographers achieve perfect secrecy in schemes like the One-Time Pad.
- Evaluating security metrics: the uncertainty of C is maximized for a secure cipher, ensuring that an attacker cannot infer patterns.

4.4 Perfect Secrecy

Previously, the ciphertext revelas a lot of information about the plaintext. We want a system in which ciphertext does not reveal anything about the plaintext.

Definition 26. *Perfect secrecy: a cryptosystem has perfect secrecy if*

$$p(P = m|C = c) = p(P = m)$$

for all plain texts m and ciphertexts c.

Lemma 1. *Assume the cryptosystem is perfectly secure, then*

$$\#K \geq \#C \geq \#P$$

where $\#$ denotes the number of items in the corresponding set.

4.4.1 One-Time Pad

Theorem 3. *Shannon's Theorem: Let $(P, C, K, e_k(), d_k())$ denote a cryptosystem with $\#K = \#C = \#P$. Then the cryptosystem provides* perfect secrecy *if and only if:*

- *Every key is used with equal probability $1/\#K$*
- *For each $m \in P$ and $c \in C$, there is a unique key k such that $c = e_k(m)$*

Definition 27. *One-Time Pad a cryptographic system that provides perfect secrecy when implemented correctly. It is a symmetric encryption technique (shift cipher) where the sender and receiver use a shared secret key, known as the pad, that is as long as the plaintext message.*

$$\#K = \#P = \#C = 26^n$$

and $p(K = k) = 1/26^n$. Also known as the Vernam cipher. It is perfectly secure because:

- *The key is truly random*
- *The key is as along as the plaintext*
- *The ciphertext reveals no information about the plaintext. The ciphertext C is independent of the plaintext P without the key K, and observing the ciphertext does not reduce the uncertainty about the plaintext.*
- *The key is used only once (hence the name)*

Example: Suppose the plaintext P is "A", and the key K is a random letter. After encryption, the ciphertext C could be any letter, with equal probability. If an attacker intercepts $C = Z$, they cannot determine whether the plaintext P was "A", "B", or any other letter without the key. Every possible plaintext is equally likely.

4.5 Entropy

Due to the key distribution problem (the key must be as long as the message), perfect secrecy is not practical. Instead, we need a cryptosystem in which **one key can be used many times**, and **a small key can encrypt a long message**. Such a system is not perfectly secure, but it should be computationally secure. We need to measure the amount of information first: Shannon's entropy.

Definition 28. *Shannon's Entropy: Let X be a random variable which takes a finite set of values x_i, with $1 \leq 1 \leq n$, and has a probability distribution $p(x)$. We use the convention that if $p_i = 0$ then $p_i \log_2(p_i) = 0$. The entropy of X is defined as:*

$$H(X) = -\sum_{i=1}^{n} p_i \cdot \log_2 p_i$$

Properties:

- *$H(X) \geq 0$*
- *$H(X) = 0$ if $p_i = 1$ and $p_j = 0$ for $i \neq j$*
- *if $p_i = 1/n$ for all i, then $H(X) = \log_2(n)$*

Example: For a specific question X: "Will you go out with me?", the answer is Yes or No. If you always say No, the amount of information, $H(X) = 0$. If you always say Yes, $H(X) = 0$. You know the result. If you say Yes and No with equal probability, $H(X) = 1$. When you get the answer, no matter what it is, you learn a lot. Here, $H()$ is the entropy, and is independent of the length of X.

4.6 Joint Entropy, Conditional Entropy and Mutual Information

Used to measure the uncertainty and interdependence between random variables.

Definition 29. *The* ***joint entropy*** *of two random variables X and Y measures the total uncertainty in the pair (X, Y), considering them together. The total amount of information contained in* one *observation of both X and Y.*

$$H(X, Y) = -\sum_{x \in X} \sum_{y \in Y} P(x, y) \log_2 P(x, y)$$

where $P(x, y)$ is the joint probability *of $X = x$ and $Y = y$.*

$$H(X, Y) \leq H(X) + (HY)$$

Definition 30. *The* ***conditional entropy*** *of X given Y, $H(X|Y)$, measures uncertainty in X after knowing Y. It quantifies how much information about X is still unknown once Y is known.*
The entropy of X given an observation of $Y = y$:

$$H(X \mid Y = y) = -\sum_{x} p(X = x \mid Y = y) \cdot \log_2 p(X = x \mid Y = y).$$

Then,

$$H(M \mid C) = \sum_{c} p(C = c) \cdot H(M \mid C = c) = -\sum_{m} \sum_{c} p(C = c) \cdot p(M = m \mid C = c) \cdot \log_2 p(M = m \mid C = c).$$

Definition 31. *Conditional and joint entropy are connected as follows:*

$$H(X, Y) = H(Y) + H(X|Y)$$

$$H(X|Y) \leq H(X)$$

Definition 32. ***Mutual information*** *quantifies the amount of shared information between X and Y. It measures how much knowing X reduces the uncertainty about Y (or vice versa). The expected amount of information that Y gives about X (or X about Y).*

$$I(X; Y) = H(X) + H(Y) - H(X, Y) = H(Y) - H(Y \mid X) = H(X) - H(X \mid Y)$$

Mutual information represents the reduction in uncertainty about one variable due to knowledge of the other. If X and Y are independent, $I(X; Y)$ because knowing one provides no information about the other.

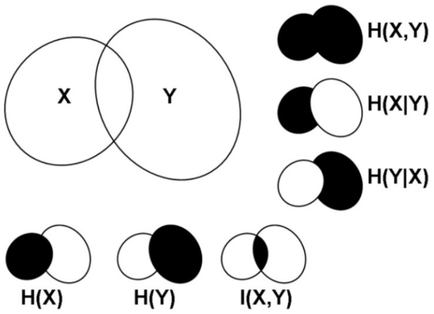
Example Suppose X and Y represent the outcomes of rolling two six-sided dice:

- $H(X)$: Entropy of die X (e.g., how random die X's outcomes are).
- $H(X, Y)$: Joint entropy, the uncertainty of the combined outcome of X and Y.
- $H(Y \mid X)$: Given a roll of X, the remaining uncertainty in Y.
- $I(X; Y)$: Shared information (likely 0 if the dice rolls are independent).

4.6.1 Application to Ciphers

- $H(P|K, C) = 0$: If you have the cipher text and the key, you can decrypt and obtain the plaintext. There is no surprise, hence 0.
- $H(C|P, K) = 0$: When you have the key and the plaintext to can encrypt and obtain the ciphertext. True for deterministic cryptosystems. Not for modern ones that use randomness (the same plaintext can be encrypted with the same key but correspond to a different ciphertext).
- $H(K, C) = H(K) + H(P)$
- Then, $H(K|C) = H(K, C) - H(C) = H(K) + H(P) - H(C)$. Given the ciphertext, can we obtain information of the key?

Graphical Representation



4.7 Spurious Keys and Unicity Distance

Definition 33. *Spurious keys: the remaining possible but incorrect keys.*

Tactic of an attacker is to reduce the number of spurious keys to zero. Then the one correct key remains. Plaintext are not random: natural languages... Random text has an entropy of $H_L = 4.7$ bits (5 bits per letter). *continue...*

4.7.1 Entropy of a Natural Language

Definition 34. *The entropy of a natural language measures the average* amount of information *conveyed per character, word, or sentence in that language, accounting for redundancy and patterns in its structure. It is defined by*

$$H_L = \lim_{n \to \infty} \frac{H(P^n)}{n}$$

where n is the length of letters, and P^n is the n-grams.

For English (estimation): $1.0 \leq H_L \leq 1.5$ bits per character. This means you need 5 bits of data to represent only 1.5 bits of information. Natural languages are highly redundant (e.g., letter and word patterns), making them less random than completely independent symbols. The low entropy of natural languages makes cryptographic attacks easier because it introduces predictability and redundancy (useless information) in the plaintext, which attackers can exploit. To mitigate this, cryptographic systems should:

- Use strong encryption methods that destroy plaintext patterns (e.g., AES or OTP).
- Introduce randomness (e.g., padding).
- Avoid predictable plaintext in sensitive communications.

4.7.2 Redundancy

Definition 35. *Redundancy is information that is expressed more than once. The repetition or predictability of information, where certain elements (like letters, words, or phrases) appear more frequently or follow patterns. It is defined as:*

$$R_L = 1 - \frac{H_L}{\log_2 \#\mathbb{P}}$$

where $\#\mathbb{P}$ is the message space.

Example: For English,

$$R_L \approx 1 - \frac{1.25}{\log_2 26} = 0.75$$

This means 75 percent redundancy, 75 % of the information in a message is predictable or can be removed without losing meaning. You can zip your files 3 quarters.

Definition 36. *The* ***average number of spurious keys*** *is:*

$$\bar{s}_n \geq \frac{\#\mathbb{K}}{\#\mathbb{P}^{n \cdot R_L}} - 1$$

An attacker wants this number to be zero.

We can make this number zero making the number of keys equal to the number of n-grams times the redundancy. This leads to unicity distance.

4.7.3 Unicity Distance and Ciphers

Definition 37. ***Unicity distance*** *is the average number of cipher texts for which the expected number of spurious keys becomes 0.*

$$n_0 \approx \frac{\log_2 \#\mathbb{K}}{R_L \cdot \log_2 \#\mathbb{P}}$$

In the context of a substition cipher, the unicity distance is the length of the ciphertext required for an attacker to break the cipher with high probability (i.e., to uniquely determine the key). It depends on the number of possible keys and the redundancy of the language used. Given $\#P = 26, \#K = 26!, R_L = 0.75$,

$$n_0 \approx \frac{88.4}{0.75 \cdot 4.7} \approx 25$$

To successfully break the cipher (i.e., determine the correct key), an attacker needs to analyze at least 25 characters of ciphertext. This is because, with this many characters, the redundancy and patterns in the language (due to its predictable letter frequencies) will provide enough information for the attacker to determine the key. This highlights the cipher's vulnerability: there are 26! possible keys! The smaller the unicity distance, the easier it is to break the cipher with limited ciphertext.

For a modern stream cipher $\#P = 2$ (binary symbol 0 or 1), $\#K = 2^l, R_L = 0.75$, then

$$n_0 \approx \frac{l}{0.75} = \frac{4 \cdot l}{3}$$

Here, the unicity distance tells us how many ciphertext bits are needed to break the streamcipher. The distance is proportional to the key length l. Longer key lengths lead to larger unicity distances, making it harder to break the cipher.

With no redundancy, $R_L = 0$. Then:

$$n_0 \approx \frac{l}{0} = \infty$$

There are no predictable patterns in the plaintext. Perfect randomness would imply that every symbol in the plaintext carries the maximum amount of information, with no repeated or predictable patterns.In such a case, the language would have maximum entropy, and there would be no chance of making guesses about the plaintext from patterns in the ciphertext. In practice: no natural language has zero redundancy.

5 Lecture 5

5.0.1 What does it mean to be secure?

Modern cryptography is focused on three key aspects:

1. **Definitions:** Concrete mathematical definition of what it means for a particular cryptographic mechanism to be secure.
2. **Schemes:** Design the schemes which will (hopefully) meet the security definitions.
3. **Proofs:** Check whether the design meets the security definitions (provable/reductionist security).

For instance with factorization, given a large number, it is very difficult to factorize. An algorithm to do this efficiently in polynomial time would break all schemes that rely on factorization.

5.1 Security Games

Security games are mathematical frameworks used in cryptography to formally evaluate the security of a computational problem or cryptographic system. They involve an interaction between two entities:

- **The challenger**, who provides an input (e.g., a computational problem).
- **The adversary** (A), who attempts to solve the problem or break the cryptographic system under specific rules and constraints.

The goal is to quantify the probability of the adversary's success within a defined resource limit (like time or computational power).

5.1.1 The FACTOR Problem

In this example, the security game focuses on the **Factorization Problem**, where the adversary tries to factorize a large number N, generated as the product of two large primes p and q.

Steps in the Security Game:

1. Setup by the Challenger:
   - Two large primes p and q are randomly chosen, each with a bit size of $v/2$.
   - The product $N = p \cdot q$ is computed and given to the adversary A.
2. Goal of the Adversary:
   - The adversary A "wins" the game if they can find two integers p' and q' such that:

$$\begin{aligned} p' \cdot q' &= N, \\ p', q' &\neq p, q. \end{aligned}$$

3. Time Constraint:
   - The adversary has a limited amount of computational time t to solve the problem.

5.1.2 Measuring the Adversary's Advantage

The advantage of the adversary A in this game is defined as:

$$\mathrm{Adv}_v^X(A, t) = \Pr\big[A \text{ wins the game } X \text{ for } v = \log_2 N \text{ in time less than } t\big].$$

Here:

- v is the bit-length of N, representing the problem's complexity.
- t is the allowed time for A.

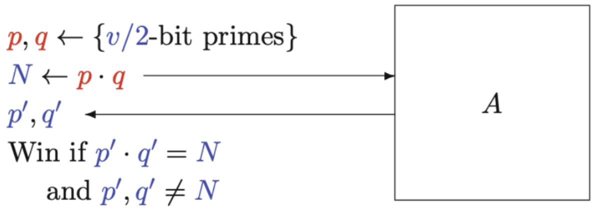


Figure 6: Security game to define the FACTOR problem

- Pr is the probability that A successfully factors N under these conditions.

$$\mathrm{Adv}_v^X(A) = 2 \cdot \left| \Pr\left[A \text{ wins } X \text{ for } v = \log_2 N\right] - \frac{1}{2} \right|$$

A always wins: 1. A always guesses: 0.

5.2 Pseudo-Random Functions

Definition 38. *A pseudorandom function $F_k(x)$ is a deterministic function that takes an input x and produces an output that is computationally indistinguishable from a truly random function, given only oracle access to the function. No efficient adversary can distinguish between $F_k(x)$ and a truly random function, without knowing the secret key k.*

Example in Security Game: The game below does not follow Kerkhoff's principle. The adversary always wins, because everything is known. There is no key/security parameter. The adversary can easily check if $F(x) = y$.

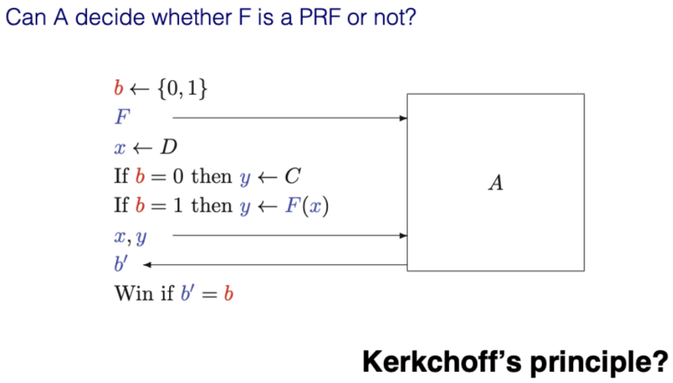

Can A decide whether F is a PRF or not?
$b \leftarrow \{0,1\}$
F
$x \leftarrow D$
If $b = 0$ then $y \leftarrow C$
If $b = 1$ then $y \leftarrow F(x)$
x, y
b'
Win if $b' = b$
A
Kerkchoff's principle?

Figure 7: Security game to define the PRF problem. Here, the adversary always wins!

Changing the SG to include a key parameter k: we are not having one function F, but several functions chosen based on a secret key: $F_k(x)$. The adversary does not know the key. The adversary now has access to an oracle:

Definition 39. *An **oracle** is an abstract, theoretical concept representing a "black box" that provides answers to specific queries according to a defined rule or function. The term is often used to model adversaries' access to resources or information in a controlled manner during security analyses.*

- *The oracle operates according to predefined rules or a specific function (e.g., encryption, decryption, signature generation, or random sampling).*
- *The adversary can query the oracle to obtain information or perform operations, but cannot directly access the oracle's internal workings.*
- *For example, an encryption oracle takes a plaintext as input and outputs the corresponding ciphertext, while a decryption oracle does the reverse.*

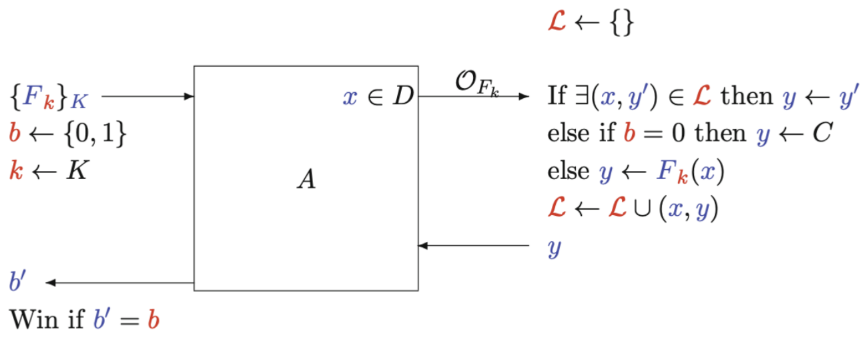

$\mathcal{L} \leftarrow \{\}$
$\{F_k\}_K$
$b \leftarrow \{0,1\}$
$k \leftarrow K$
$x \in D$
$\mathcal{O}_{F_k}$
A
If $\exists(x, y') \in \mathcal{L}$ then $y \leftarrow y'$
else if $b = 0$ then $y \leftarrow C$
else $y \leftarrow F_k(x)$
$\mathcal{L} \leftarrow \mathcal{L} \cup (x, y)$
y
b'
Win if $b' = b$

Figure 8: The final security game for a PRF

Example: Oracle in a Security Game For a PRF function security game, the oracle $\mathscr{O}_k(x)$ behaves as follows:

- With secret key k, the oracle implements $\mathscr{O}_{F_k}$, where F_k is a pseudorandom function.
- Alternatively, the oracle can implement $\mathscr{O}_R$, where R is a truly random function.
- The oracle should not give two different answers for the same input x. Hence, it has a memory $\mathscr{L}$.

The adversary queries the oracle with different x values and observes the outputs. The adversary's goal is to distinguish whether the oracle is implementing a PRF or truly random function, without knowing the secret key k. In figure 8, this is indicated by the bit b. The adversary wins if they can correctly guess b.

5.2.1 Adversary's Advantage

The adversary's *advantage* measures how well it can distinguish the oracle's behavior from random guessing. This is formally defined as:

$$\text{Adv}^{\text{PRF}}_{\{F_k\}_K}(A) = 2 \cdot \left| \Pr\left[A^{O_{F_k}} \text{ wins}\right] - \frac{1}{2} \right|$$

Key Components:

- $\Pr\left[A^{O_{F_k}} \text{ wins}\right]$: The probability that A correctly guesses the bit b.
- $\frac{1}{2}$: The probability of guessing b purely at random.

Interpretation:

- If A cannot distinguish $F_k(x)$ from a random function, its success probability is $\frac{1}{2}$, meaning $\text{Adv}^{\text{PRF}}_{\{F_k\}_K}(A) = 0$.
- If A has some advantage in distinguishing $F_k(x)$ from random, the value Adv increases, indicating the weakness of the PRF.

5.2.2 Why is this important?

This security game is critical for evaluating whether a function $F_k(x)$ behaves indistinguishably from a truly random function. In cryptographic systems:

- Strong PRFs ensure adversaries can't predict outputs or distinguish the function, which is vital for the security of encryption, key derivation, and other cryptographic primitives.
- A PRF that is distinguishable from random may leak information or fail to provide the desired security guarantees.

5.3 Trapdoor Functions

Definition 40. *A **one-way function** is a function that is easy to compute in one direction but* hard to invert.

- ***Easy to compute:*** *Given an input x, it is computationally efficient to calculate $f(x)$.*
- ***Hard to invert:*** *Given $f(x)$, it is computationally infeasible to find x (or any x' such that $f(x') = f(x)$) without additional information.*

Definition 41. *A **trapdoor** function is a one-way function that is easy to compute in one direction but hard to invert unless a secret "trapdoor" is known. An extra piece of information helps to invert the function, e.g. RSA.*

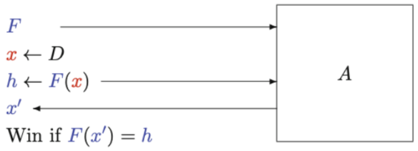


Figure 9: Security game for a one-way function

5.4 Public Key Cryptography

- **Key distribution problem** in symmetric (private key) systems.
- A **key pair** is needed:
  - **Private key** for decryption (only the owner can decrypt).
  - **Public key** for encryption (everyone can encrypt).
- Public and private keys are linked in a mathematical way:
  - Obtaining the private key from the public key is **NOT easy**.
  - Obtaining the public key from the private key is **easy**.

The security of an encryption scheme (both symmetric and asymmetric) is dependent on:

- The goal of the adversary
- The types of attacks allowed
- The computational model

5.5 Basic Notions of Security

Notation and valid encryption:

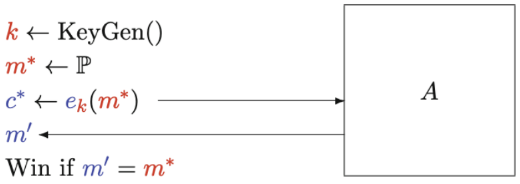
$$\forall k \in \mathbb{K}, \forall m \in \mathbb{P}, d_k(e_k(m)) = m$$

What does it mean for a symmetric encryption scheme to be secure? Adversary should not learn the underlying (plaintext) message.

5.5.1 OW-PASS attack

One-wayness under passive attack.

- Adversary's Capability: The adversary only observes the encryption of a plaintext message but cannot interact with the encryption mechanism in any way.
- Goal: The adversary is given the ciphertext $c = \text{Enc}(m)$ and attempts to recover the plaintext m.
- Assumptions:
  - The adversary has no control over the plaintexts being encrypted.
  - Security is evaluated against passive observers who can only eavesdrop.
- Use Case: Suitable for scenarios where the attacker does not have active access to the encryption process.



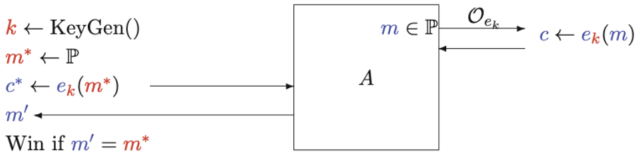
$k \leftarrow \text{KeyGen}()$
$m^* \leftarrow \mathbb{P}$
$c^* \leftarrow e_k(m^*)$
m'
Win if $m' = m^*$
A

Figure 10: Security game for symmetric key OW-PASS

5.5.2 OW-CPA attack

OW-PASS is very limiting for the adversary. She should have an encryption oracle. One-Wayness under Chosen Plaintext Attack:

- Adversary's Capability: The adversary can interact with the encryption mechanism by choosing plaintexts and obtaining their corresponding ciphertexts.
- Goal: The adversary, after observing ciphertexts for chosen plaintexts, attempts to recover the plaintext of a given ciphertext.
- Assumptions:
  - The adversary can actively query the encryption oracle with plaintexts of their choice.
  - This security definition is stronger than OW-PA because it assumes a more powerful adversary.
- Use Case: Applies to scenarios where the attacker has partial or controlled access to the encryption process, such as in chosen plaintext attacks on public-key cryptography.



$k \leftarrow \text{KeyGen}()$
$m^* \leftarrow \mathbb{P}$
$c^* \leftarrow e_k(m^*)$
m'
Win if $m' = m^*$
A
$m \in \mathbb{P}$
$\mathcal{O}_{e_k}$
$c \leftarrow e_k(m)$

Figure 11: Security game for symmetric key OW-CPA

5.5.3 OW-CCA attack

The adversary should be able to decrypt as too (limited number). Thus, One-wayness under Chosen Ciphertext Attack:

- Adversary's Capability: The adversary is allowed to interact with both the encryption and decryption oracles. Specifically, they can:
  - Query the decryption oracle for the decryption of any ciphertext c, except for the target ciphertext c^*.
  - Query the encryption oracle to obtain the ciphertext for any plaintext m of their choice.
- Goal: The adversary attempts to recover the plaintext m^* from a given target ciphertext $c^* = \text{Enc}(m^*)$.
- Assumptions:
  - The adversary has powerful capabilities to both query encryption for chosen plaintexts and query decryption for chosen ciphertexts.
  - The decryption oracle cannot be used directly on the target ciphertext c^* to prevent trivial success.
- Use Case:

- OW-CCA security is crucial for applications requiring strong guarantees against active attackers who can manipulate ciphertexts.
  - It is relevant in scenarios such as securing communications where adversaries can intercept, modify, and replay ciphertexts.

- Key Features:
  - Adversary's Advantage: OW-CCA tests the robustness of a cryptographic scheme against the most powerful adversaries who can both observe and manipulate encrypted communications.
  - Stronger Security: OW-CCA provides stronger guarantees compared to OW-PA and OW-CPA as it accounts for adversaries who have decryption capabilities.

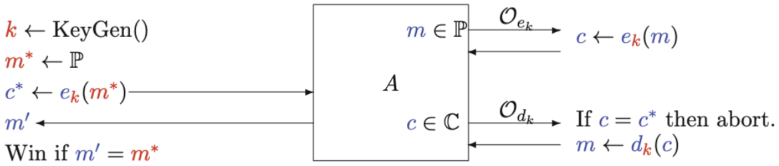


Figure 12: Security game for symmetric key OW-CCA

5.6 Modern Notions of Security

Before, the attacks assumed that you obtain the whole key. In real life, the adversary can also partially break the system (one bit of the key). The adversary should not be allowed to obtain **any** information about the plaintext!

Perfect secrecy is not practical, because every plaintext needs to have the same probability. To do so, the key needs to be as long as the message.

Definition 42. *Semantic security: like perfect security but the adversary has polynomially bounded computing power. The adversary cannot extract partial knowledge.*

$$g : M \to \{0, 1\}$$

$$Pr[g(m) = 1] = Pr[g(m) = 0] = \frac{1}{2}$$

$$Adv_{\Pi}^{SEM}(S) = 2 \cdot \left| Pr[S(c) = g(d_k(c))] - \frac{1}{2} \right|$$

Semantic security ensures that an adversary cannot learn any meaningful information about a plaintext from its ciphertext, even if they have some prior knowledge about the plaintext. Formally, a cryptosystem is semantically secure if, given a ciphertext, the adversary cannot distinguish between the encryptions of two chosen plaintexts.

Semantic security is *difficult to show*! Polynomial security (IND) is easier to show. If a system is IND secure, it is also semantically secure.

Definition 43. *IND Security: an encryption scheme is* IND-secure *if the ciphertexts of two plaintexts are indistinguishable to an adversary, ensuring that the adversary gains no advantage from observing ciphertexts, even with access to oracles (depending on the attack model).*

IND Security ensures that an adversary cannot distinguish between the ciphertexts of two chosen plaintexts, even if they select the plaintexts themselves. This is a fundamental property of secure encryption. IND Security requires the use of *probabilistic* encryption for public-key encryption schemes to ensure that ciphertexts for the same plaintext look different every time they are encrypted. This randomness prevents adversaries from exploiting patterns in ciphertexts to distinguish between plaintexts.

5.6.1 IND Security Games

The IND framework is defined under different adversarial capabilities:

IND-CPA (Indistinguishability under Chosen Plaintext Attack):

- The adversary can choose two plaintexts m_0 and m_1.
- A random bit b is chosen, and the ciphertext of m_b is given to the adversary.
- The adversary must guess b.
- Success probability $> \frac{1}{2}$ implies a weakness in the scheme.

IND-CCA Indistinguishability under Chosen Ciphertext Attack):

- Similar to IND-CPA, but the adversary is also allowed to query a decryption oracle.
- The adversary cannot query the decryption oracle for the target ciphertext.
- This provides a stronger security guarantee than IND-CPA.

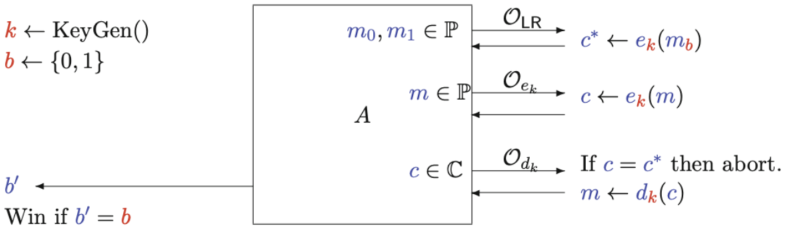


Figure 13: Security game for symmetric key IND-CCA

If the cryptosystem is deterministic, you can fool the oracles. In figure 13, $\mathscr{O}_{LR}$ will only encrypt one of m_1, m_2. You use the other message in the second step $\mathscr{O}_{e_k}$, and check if you receive the same ciphertext as in step 1. The decryption oracle $\mathscr{O}_{d_k}$ can be fooled if there is a relationship between the plain text and the cypertext. This is possible in homomorphic encryption. For secure communication, homomorphism is not appreciated. But we have it because in public cryptosystems we rely on mathematically difficult problems.

Definition 44. *An encryption algorithm is secure if it is semantically secure against a CPA attack.*

Definition 45. *An encryption algorithm is secure if it is IND-CCA secure.*

Theorem 4. *A system which is IND-PASS secure must be semanticallt secure against passive adversaries.*

$$\prod \text{ is IND-CCA} \Rightarrow \prod \text{ is IND-CPA} \Rightarrow \prod \text{ is IND-PASS}$$
$$\prod \text{ is IND-XXX} \Rightarrow \prod \text{ is OW-XXX}$$

5.7 Other Notions of Security

- Many Time Security: How many times can we use the LR oracle?
- Real-or-Random: Oracle encrypts either the real message or a random message of the same size.
- Lunchtime Attacks: (CCA1) Adversary has decryption oracle during the find stage for some time.
- Nonce-Based Encryption: Deterministic algorithms are not IND-CPA secure. Therefore, nonce (number used once) should be used.
- Data Encapsulation Mechanism: Symmetric system but key is used once.
- Non-malleability: Given a ciphertext of an unknown plaintext, the adversary can compute a new ciphertext for a *related* plaintext.
- Plaintext-aware: One needs to know the plaintext to encrypt.

5.7.1 Malleability

Malleability in cryptography refers to a property of some encryption schemes where an adversary, given a ciphertext C, can modify it to produce another ciphertext C' such that C' decrypts to a related plaintext M', where M' is a function of the original plaintext M.
In a *malleable encryption scheme*, the attacker does not need to know M to create C'. This is considered a vulnerability because the adversary can manipulate encrypted data without breaking the encryption itself. Why is malleability a problem?

1. **Breaking data integrity:** Encryption schemes are often expected to protect both confidentiality and integrity of the plaintext. Malleability breaks this assumption, as it allows attackers to modify data without detection.

2. **Practical attacks:** Malleability can be exploited in protocols that rely on encrypted data. For instance, in electronic payments, malleable encryption could allow an attacker to modify the transaction amount in a secure message.

Example: Suppose we are using a simple additive encryption scheme:

$$C = M + k$$

where k is the encryption key, M is the plaintext, and C is the ciphertext.
An attacker, without knowing k, can modify C by adding some value d:

$$C' = C + d$$

When C' is decrypted:

$$M' = C' - k = (M + k + d) - k = M + d$$

Thus, the attacker has changed the plaintext M to $M' = M + d$ without needing to know the key k.

Mitigating Malleability:

1. **Authenticated encryption:** Combine encryption with mechanisms to verify data integrity, such as using a MAC (Message Authentication Code) or digital signatures.

2. **Non-malleable encryption schemes:** Use encryption schemes that inherently prevent modification of ciphertexts to produce related plaintexts. For example, many modern schemes like AES-GCM or RSA-OAEP are designed to be non-malleable.

5.8 Random Oracle Model

Finish later

6 Lecture 6

Randomness is extremely important in cryptography, and the same random values should never be used twice. It is hard to achieve, often we use pseudo-random number generators (PRNGs) to generate randomness.

Example of Importance of Randomness: In a perfect secrecy scheme, the key is as long as the message. If the same key is reused even once, the messages can be decrypted. XORing their ciphertexts effectively removes the key, leaving a direct relationship between the two plaintexts.

For a message m and key k, the ciphertext is $c = m \oplus k$, where $\oplus$ is the XOR-operation. For two messages encrypted with the same key:

$$c_1 = m_1 \oplus k$$

$$c_2 = m_2 \oplus k$$

Now, if we XOR the ciphertexts:

$$c_1 \oplus c_2 = (m_1 \oplus k) \oplus (m_2 \oplus k)$$

Using the property of XOR that $a \oplus a = 0$ and $a \oplus b \oplus a = b$, we can deduce:

$$c_1 \oplus c_2 = m_1 \oplus m_2 \oplus k \oplus k$$

$$c_1 \oplus c_2 = m_1 \oplus m_2$$

Thus, the XOR of the two ciphertexts directly reveals the XOR of the two plaintexts. The "distance" between two messages m_1 and m_2 represented by their XOR operation $m_1 \oplus m_2$, reveals the bitwise differences between them. This is often inpreted as the Hamming distance (the number of positions at which two strings of the same length differ) between the two messages. The XOR distance not only represents differences but can be exploited to reconstruct messages in insecure systems where the same key is reused (for instance, when the adversary knows one of the plaintexts already and can reconstruct the other).

6.1 Stream Ciphers

Definition 46. *Stream ciphers are based on zeroes and ones. They encrypt bits (bit by bit) instead of blocks of plaintext. Due to bit operations, they are very fast. Easy to implement on hardware and software.*

Consider the following stream cipher:

$$c = m \oplus F_k(0)$$

Where $F_k(0)$ is a function that generates a keystream based on the key k. The keystream is XORed with the plaintext to produce the ciphertext. The keystream is generated by a pseudo-random number generator (PRNG) that takes the key as input.

If the key was the same length of the message it would be perfect, but creating such a long key would be very inefficient (only done for government to government communication). However, we can use a shorter key to mimic this behaviour. The short key is used to choose the output of a PRF (the key is the seed of the PRNG). It will mimic pseudo-randomness long enough (until it repeats itself). The cipher will be passive secure if the PRF is random enough.

6.2 Linear Feedback Shift Registers (LFSRs)

We can use LFSRs for generating a binary stream.

Definition 47. *A* ***Linear Feedback Shift Register*** *(LFSR) is a sequential shift register that generates a sequence of bits based on a linear feedback function. It consists of:*

- *A sequence of single-bit registers (0 or 1)*
- *Feedback function: a linear function (usually XOR) that determines how bits are fed back into the register.*

- *Tap positions in the register used in the feedback function.*

How it works:

1. Initialization: The register is initialized with a seed value (a non-zero starting state).
2. Bit Shifting: The register shifts all bits to the right (or left), discarding the last bit.
3. Feedback Calculation: A new bit is computed based on the XOR of the bits in the "tapped" positions.
4. Repeat: The new bit is inserted into the first position of the register, and the process repeats to generate the next bit in the sequence.

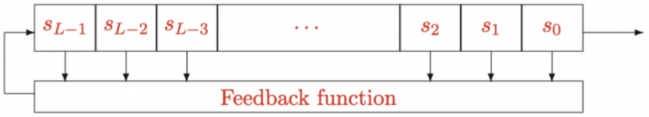


Figure 14: Feedback shift register

The initial state is very important, as it determines the entire sequence (whole sequence of zeroes always results in zeroes). The feedback function and length of the registers are also important.

6.2.1 Properties of LFSRs

- Periodicity: LFSRs are periodic; they generate a sequence that eventually repeats. The maximum period is $2^n - 1$, where n is the register size, achieved when the feedback taps correspond to a primitive polynomial.
- Deterministic: If the initial state (seed) and feedback function are known, the sequence is fully predictable.
- Efficiency: LFSRs are computationally efficient, using only shift and XOR operations.

6.2.2 Feedback Functions

Feedback functions in shift registers are used to compute the new bit that is fed back into the register. The key difference between linear and nonlinear feedback functions lies in how they combine the bits from the register.

Feature	**Linear Feedback**	**Nonlinear Feedback**
Operations	XOR and other linear operations	Nonlinear operations (AND, OR, S-boxes, etc.)
Predictability	Easier to predict with known state	Harder to predict, more secure
Efficiency	Computationally efficient	More computationally intensive
Periodicity	Periodic, with a maximum of $2^n - 1$	Can be non-periodic or have a very long period
Applications	Pseudo-random generators, CRC	Secure encryption, cryptographic systems

Table 2: Comparison of Linear and Nonlinear Feedback Functions

We would like to have a non-linear feedback function, but these are difficult to design. Difficult to estimate whether it is a good non-linear function. When you hit 0 the whole value chain becomes 0, and the cycle is broken.

6.2.3 Zero State in Feedback Functions:

If an LFSR (or any shift register using feedback functions) reaches the value **zero**, the system will typically "lock" into this state and remain there indefinitely. This is because:

1. Feedback Function Output: In a linear feedback system, the XOR of all zeros is zero, meaning that once all bits are zero, the feedback function will continue producing zeros.
2. State Transition: The state of the LFSR will not change, and the register will stay stuck at zero, generating an output sequence that consists entirely of zeros.

Implications:

- **In LFSRs:**
  - The sequence degenerates and loses all pseudo-randomness, becoming a constant stream of zeros.
  - This reduces the period of the LFSR to just 1, which is undesirable, especially in cryptographic or pseudo-random number generation applications.
- **In Cryptographic Systems:**
  - If the feedback function hits zero during an encryption or random number generation process, it can render the output insecure.
  - Attackers can easily exploit the fact that the system produces predictable, non-random output (all zeros).

Avoiding the Zero State:

1. Seed Initialization: The initial state (seed) of the register is carefully chosen to ensure it is non-zero.
2. Primitive Polynomials: For LFSRs, the taps (feedback positions) are chosen based on primitive polynomials. This ensures the register cycles through all possible non-zero states ($2^n - 1$) before repeating.
3. Nonlinear Feedback Functions: Nonlinear systems often include mechanisms to avoid degenerative states like zero, introducing additional complexity to prevent such behavior.
4. External Checks: Some systems include checks to ensure that a zero state is never reached, restarting the LFSR with a valid non-zero state if necessary.

6.2.4 Mathematical Expression

Cell is tapped or not (content of the cell contributes to the feedback function yes or no):

$$[c_1, c_2, \ldots, c_L]$$

Initial state of the registers:

$$[s_{L-1}, \ldots, s_1, s_0]$$

Output:

$$[s_0, s_1, s_2, s_3, \ldots, s_{L-1}, s_L, s_{L+1}, \ldots]$$

Then, for $j \geq L$:

$$s_j = c_1 \cdot s_{j-1} \oplus c_2 \cdot s_{j-2} \oplus \cdots \oplus c_L \cdot s_{j-L}.$$

However, we need to see a repetition after a certain period, N, such that:

$$s_{N+i} = s_i.$$

N can be maximum $2^L - 1$.

The **connection polynomial** $C(X)$ describes how the bits stored in the LFSR are combined to produce the feedback bit, which determines the next state of the LFSR. It is defined as:

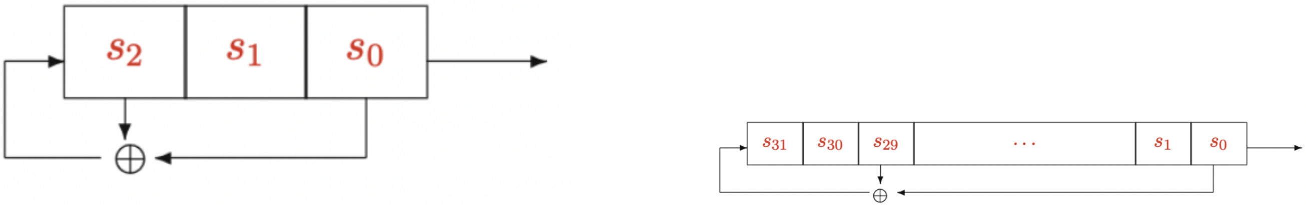
$$C(X) = 1 + c_1 \cdot X + c_2 \cdot X^2 + \cdots + c_L \cdot X^L \in \mathbb{F}_2[X]$$

where $\mathbb{F}_2[X]$ is the set of polynomials over the binary field (coefficients are either 0 or 1). Proper choice of $C(X)$ can result in a maximal period of $2^L - 1$.

The **characteristic polynomial** $G(X)$ describes the LFSR's output sequence and the polynomial that can be used to generate the entire sequence of output bits. It is a function of the feedback logic defined by the connection polynomial.

$$G(X) = X^L \cdot C(1/X)$$

Example:



(a) $X^3 + X + 1$

(b) $X^{32} + X^3 + X + 1$

Figure 15: Linear Feedback Shift Register Examples

a) This is a small LFSR with length $L = 3$, described by the connection polynomial $C(X) = X^3 + X + 1$. There are 3 registers: s_0, s_1, s_2. The feedback function involves: X^3 (feedback from the last register), X (feedback from the first register), and a constant 1. At each step, the contents of the tapped cells (s_2, s_0) are XOR-ed to produce a new bit.

b) This is a much larger LFSR with length $L = 32$, described by the connection polynomial $C(X) = X^{32} + X^3 + X + 1$. There are 32 registers. The feedback function involves the 32nd (last) register, the 3rd register, the 1st register, and a constant 1. The characteristic polynomial $G(X)$ is used to generate the entire output sequence. *I dont get the exp values for this one??*

Note: The connection polynomial determines which registers contribute to the feedback. But is not a direct description of the output stream itself. Instead, it describes the feedback mechanism of the LFSR.

How do we know that the feedback function is a good one, such that we achieve the maximum length before the stream cipher repeats itself?

6.2.5 Primitive Polynomial

Definition 48. *A* primitive polynomial *is a polynomial that generates a maximal-length sequence in a finite field, particularly in the context of binary fields $\mathbb{F}_2$ (i.e., polynomials whose coefficients are either 0 or 1). Specifically, for a polynomial to be primitive, it must satisfy the following conditions:*

- *The polynomial must be* ***irreducible*** *over $\mathbb{F}_2$ (i.e., it cannot be factored into the product of lower-degree polynomials with coefficients in $\mathbb{F}_2$).*
- *The polynomial must generate a sequence that cycles through all the nonzero elements of the finite field $\mathbb{F}_{2^L}$, where L is the degree of the polynomial.*

In simpler terms, a primitive polynomial produces a sequence of numbers that goes through all possible states except the zero state (when viewed as an LFSR) before repeating.

When used in LFSRs, primitive polynomials ensure that the generated sequence has a *maximum length* before it repeats. The length of the sequence is $2^L - 1$, where L is the degree of the polynomial.

Theorem 5. *If N is the period, then the characteristic polynomial $f(x)$ is a factor of $1 - X^N$.*

$$C(X) = 1 + c_1 \cdot X + c_2 \cdot X^2 + \cdots + c_L \cdot X^L \in \mathbb{F}_2 [X]$$

Conditions for $C(X)$:

- The highest coefficient $c_L = 0$: the polynomial is singular. This means that $C(X)$ is not a full degree of L and cannot properly generate sequences of maximal length in an LFSR.
- The highest coefficient $c_L = 1$: the polynomial is non-singular. $C(X)$ is a full degree of L and there is a periodicity in the sequences generated by the polynomial. The behaviour of the sequence depends on whether $C(X)$ is irreducible or not.

$C(X)$ is irreducible if it cannot be factored into smaller polynomials over $\mathbb{F}_2 [X]$ (other than itself and 1). Then, the period of the sequence generated by the LFSR is the *smallest* N such that $C(X)$ divides $1 + X^N$. N will satisfy $N|(2^L - 1)$.

Example: 3 registers, maximum length is 7:

$$m = 3, p = 2^3 - 1 = 7$$

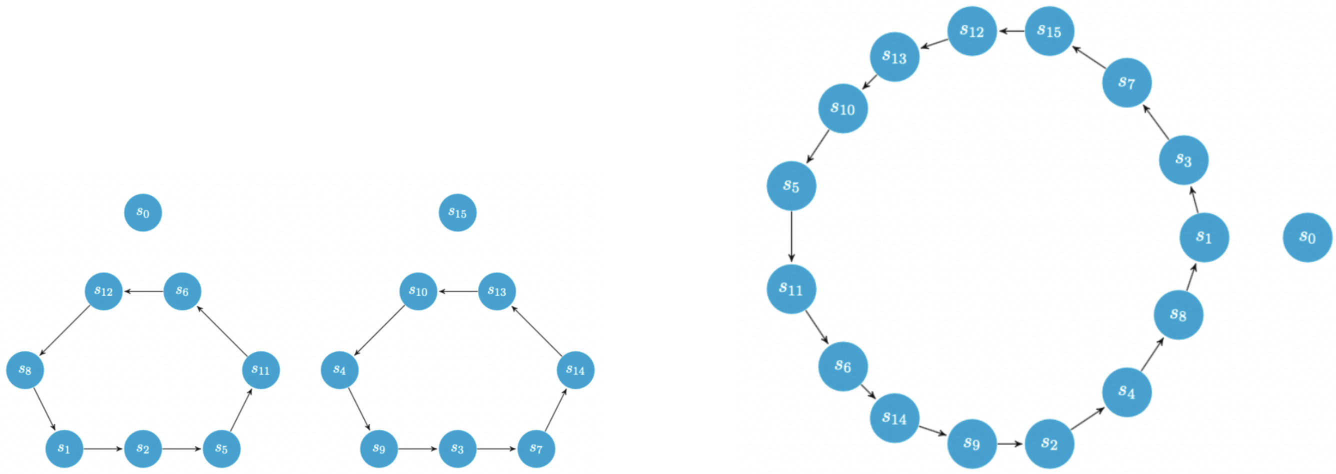
So we need to find the factors of:

$$1 - x^y = (1 - x)(1 + x + x^3)(1 + x^2 + x^3)$$

The divisors of this polynomial (irreducible polynomials) will have a period of 7.

6.3 Period

The presence of disjoint cycles in an LFSR, such as in figure 16, has implications for randomness:



(a) $X^4 + X^3 + X^2 + 1$

(b) $X^4 + X + 1$

Figure 16: State transitions of 4-bit LFSRs with connection polynomials

- The output sequence splits into two disjoint cycles of length 7 instead of a single cycle of length $2^L - 1$ (15 for $L = 4$).
- Reduced Period: The sequence repeats earlier, reducing the overall period and uniformity.
- Loss of Randomness: Missing states lead to gaps, making the sequence predictable and non-random.
- Cause: The connection polynomial is *not primitive*, preventing the LFSR from achieving a maximum-length sequence.

The right polynomial results in a big cycle with all states included, except for the zero state, as expected.

6.4 Security of LFSRs

With L registers and $2L$ known output bits, the connection polynomial can be revealed. It is a linear system. If you have sufficient equations, you can solve the linear system.

- First L bits reveal the s values
- We need to learn L unknowns: c values

$$s_j = \sum_{i=1}^{L} c_i \cdot s_{j-i} \pmod 2$$

Therefore, LFSRs are not secure!
Linear complexity notes here

6.5 Combining LFSRs

LFSR-based stream ciphers are very fast ciphers, suitable for implementation in hardware, to encrypt real-time data such as voice or video. But they need to be augmented with a method to produce a form of non-linear output. Create fast and secure non-linear behaviour by combining multiple LFSRs into a non-linear combination function:

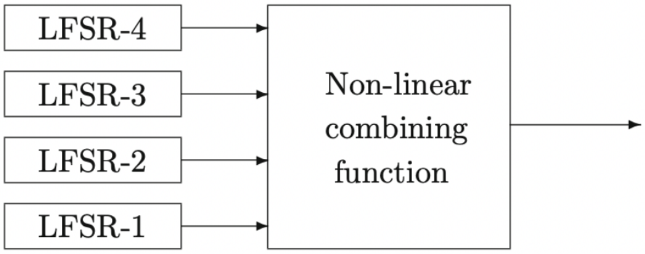


Figure 17: Combining LFSRs

6.5.1 Non-linear Combiners

Examples covered in lecture are:

- Alternating-step generator
- Shrinking generator
- A5/1
- Trivium
- RC4

How much do we need to know about these?