Research Statement

I study three-dimensional manifolds using combinatorial methods. More specifically, my research revolves around **veering triangulations**, a special class of ideal triangulations of three-manifolds that combinatorially encode **pseudo-Anosov flows**. These flows have been the focus of extensive research in the past due to their rich dynamics [BFM22, Fen16a, FB17], important connections with the geometry and topology of their underlying manifolds [Fen02, Fen16b, FM01, Mos91, Mos92], their foliations and laminations [Cal06a, Fen98, Fen07, Mos96], contact structures [EG02, FH13], and properties of their fundamental group [Cal06b, Fen12, PT72]. Introducing veering triangulations into the field has made it possible to prove some results that were previously out of reach.

In Section I, I discuss the connection between veering triangulations and pseudo-Anosov flows. Section II outlines my contributions to the field. These include results on **polynomial invariants of veering triangulations** (parts A, B, C, and G), **the relationship between veering triangulations and the Thurston norm** (part D), and **methods of constructing new veering triangulations out of old ones** (part E). I also discovered a **counterexample to a classification theorem of certain Anosov flows**, which was previously thought to be correct (part F).

In Section III, I describe my ongoing joint work with Henry Segerman on **drilling veering triangulations along their flow cycles** and its applications.

Section IV presents my future research plans. In part A, I discuss a project on **algebraic invariants of pseudo-Anosov flows that are not transverse to fibrations**. It is motivated by experimental data that I have already collected and analogous investigations in the fibered case. In part B, I describe my plan to **explore the combinatorics of veering triangulations that encode Bonatti-Langevin flows**, a well-known class of Anosov flows. The goal is to identify such triangulations in the Veering Census [GSS], which will be a step towards establishing a dictionary between veering triangulations from the census and the flows that they encode. In Section C, I propose to generalize my results, outlined in Sections II.D and II.E, concerning **pseudo-Anosov flows with equal cones of homology directions**. My focus will be on identifying which three-manifolds can admit such pairs of flows and exploring the connections between these flows.

I. Pseudo-Anosov flows and veering triangulations

A *flow* on a closed three-manifold N is a continuous map $\Psi : N \times \mathbb{R} \to N$ that satisfies $\Psi(x, 0) = x$ and $\Psi(\Psi(x, t), s) = \Psi(x, s + t)$ for all $x \in N$ and for all $s, t \in \mathbb{R}$. Given $x \in N$ the image $\gamma_x = \Psi(\{x\} \times \mathbb{R})$ is called an *orbit* of Ψ. We say that Ψ is *pseudo-Anosov* if N admits a pair of two-dimensional singular foliations which intersect transversely along the orbits of Ψ and satisfy a few additional conditions; see [Fen02, Definition 7.1]. These foliations are called the *stable* and the *unstable foliation* of Ψ. They may be *singular* along finitely many closed orbits of Ψ, called the *singular orbits* of Ψ. If Ψ does not have any singular orbits, we say that it is an *Anosov flow*. If Ψ admits an orbit which is dense in N, we say that Ψ is *transitive*. Such flows are very common; an unpublished work of Gabai-Mosher, outlined in [Mos96], implies that every closed hyperbolic three-manifold with a positive first Betti number admits a transitive pseudo-Anosov flow.

The first result connecting veering triangulations and pseudo-Anosov flows is due to Agol and Guéritaud. They showed that a triple (N, Ψ, Λ), where N is a closed three-manifold, Ψ is a transitive pseudo-Anosov flow on N, and Λ is a finite collection of closed orbits of Ψ satisfying a few technical conditions, determines a *veering triangulation* $\mathcal{V} = \mathcal{V}(N, \Psi, \Lambda)$ of $M = N - \Lambda$; see [LMT22, Section 4]. We will call Λ a *veering-enabling* set for Ψ. The triangulation $\mathcal{V}$ is an ideal triangulation additionally equipped with a *taut structure* and a *veering structure*. Roughly speaking, the former is a well-defined upwards direction that is consistent throughout the whole triangulation and encodes the direction of orbits of Ψ. The latter is given by a pair *branched surfaces* that are dual to the triangulation and encode (that is, *fully carry*) the stable and the unstable foliation of Ψ. Crucially, a veering-enabling set exists for any transitive pseudo-Anosov flow [Tsa22, Proposition 2.7]. Thus any transitive pseudo-Anosov flow can be encoded by some veering triangulation.

Going in the opposite direction, Agol and Tsang proved that a triple $(M, \mathcal{V}, s)$, where M is a cusped three-manifold, $\mathcal{V}$ is a veering triangulation of M, and s is a collection of Dehn filling slopes on the boundary components of M (satisfying very minor conditions) determines a transitive pseudo-Anosov flow $\Phi = \Phi(M, \mathcal{V}, s)$ on the Dehn-filled manifold $M(s)$ [AT, Theorem 5.1]. Moreover, if $\mathcal{V} = \mathcal{V}(N, \Psi, \Lambda)$ and $M(s) = N$ then the flows Ψ and Ψ are *orbit equivalent* [Tsa23, Theorem 2.4.1]. See also the work of Schleimer and Segerman [SSb, SSa]. The possibility to replace a transitive pseudo-Anosov flow with a veering triangulation, and vice versa, suggests that the combinatorics of veering triangulations can be used to prove important results about pseudo-Anosov flows.

Employing veering triangulations to study pseudo-Anosov flows has numerous advantages. First, since veering triangulations are finite combinatorial objects, they are easier to work with than the flows that they encode. Second, they can be rigorously classified and catalogued. This feature prompted the creation of the Veering Census [GSS] which contains the list of all veering triangulations with up to 16 tetrahedra. In some sense, this is the largest existing database of pseudo-Anosov flows. Third, a veering triangulation constructed via the Agol-Guéritaud construction from a pair (Ψ, Λ) is canonical for that pair. This makes veering triangulations superior to other combinatorial methods used to study pseudo-Anosov flows, such as Markov partitions and branched surfaces. Finally, thanks to veering triangulations one can study pseudo-Anosov flows experimentally, using the already existing software to study triangulations, such as Regina [BBP$^+$17] and SnapPy [CDW]. Schleimer and Segerman developed a Python package called Veering [PSS], which is specifically designed for studying taut and veering triangulations. I am a regular contributor to Veering; for example, I am the sole author of the `carried_surface` and `mutation` modules and have collaborated on several other modules, including `flow_cycles`, `taut_polynomial`, and `veering_polynomial`. These computational tools enable researchers to perform experiments, test hypotheses, identify veering triangulations with particular properties, and formulate new conjectures based on generated data.

II. Completed research projects

Below I outline the key findings from my research, listed chronologically.

A. Algorithms to compute the taut and veering polynomials of veering triangulations

Landry, Minsky, and Taylor introduced two polynomial invariants of veering triangulations, the *taut polynomial* and the *veering polynomial* [LMT23, Section 3]. These invariants carry information about the dynamical properties of the flow encoded by the triangulation; see [LMT23, Section 7] and [LMT22, Theorem 7.2]. In [Par24] I gave algorithms to compute both the taut and the veering polynomial of an arbitrary veering triangulation. I implemented these in collaboration with Schleimer and Segerman; they are now included in the Veering package.

B. Algorithms to compute the Teichmüller polynomials

The *Thurston norm* is a norm on the second homology group of a hyperbolic three-manifold. If the three-manifold is fibered over the circle, then the unit norm ball of its Thurston norm admits some number of *fibered faces*. They are characterized by the property that all primitive classes lying over their interior can be represented by fibers of fibrations over the circle [Thu86, Theorem 3].

McMullen showed that to each fibered face $\mathtt{F}$ of a three-manifold N one can associate a polynomial invariant $\Theta_{\mathtt{F}}$, called the *Teichmüller polynomial* of $\mathtt{F}$. Its main feature is that it carries information about the *stretch factors* of the monodromies of all fibrations lying over $\mathtt{F}$ [McM00, Theorem 5.1]; see also Part IV.A below. On the other hand, $\mathtt{F}$ determines a pseudo-Anosov flow Ψ on N [Fri82b, Theorem 7], and thus a veering triangulation $\mathcal{V}$ on the complement of the singular orbits $\mathrm{Sing}(\Psi)$ of Ψ [Ago11, Section 4]. Landry, Minsky, and Taylor proved that the Teichmüller polynomial of $\mathtt{F}$ is equal to the image of the taut polynomial of $\mathcal{V}$ under the homomorphism induced by the inclusion $N - \mathrm{Sing}(\Psi) \hookrightarrow N$ [LMT23, Theorem 7.2]. In [Par24, Section 8] I used this result and the work described in Part A to devise and implement an algorithm to compute the Teichmüller polynomial of any fibered face of the Thurston norm ball.

C. A proof that the taut polynomial of a veering triangulation is a twisted Alexander polynomial of the underlying manifold

The original definition of the taut polynomial relies heavily on the combinatorics of veering triangulations. However, in [Par23c, Theorem 5.7] I proved that the taut polynomial of a veering triangulation $\mathcal{V}$ of M is just a special case of a *twisted Alexander polynomial* of M, where the twisting depends only on a certain *orientation homomorphism* $\omega : \pi_1(M) \to \{-1, 1\}$ associated to $\mathcal{V}$.

In the fibered setup, I used [Par23c, Theorem 5.7] to find equations relating the Teichmüller polynomial of a fibered face of the Thurston norm ball and the (untwisted) Alexander polynomial of the underlying manifold [Par23c, Theorem 6.3]. This generalized a previous result of McMullen [McM00, Theorem 7.1]. Furthermore, [Par23c, Theorem 5.7] implies that for any three-manifold M there are only finitely many potential candidates for the taut polynomial of a veering triangulation of M, one for each homomorphism $\pi_1(M) \to \{-1, 1\}$. This observation is particularly noteworthy in the context of the *finiteness conjecture for veering triangulations*, which asserts that any hyperbolic three-manifold can admit only finitely many distinct veering triangulations. Finally, [Par23c, Theorem 5.7] implies that the taut polynomial can be computed using *Fox calculus*. This method drastically speeds up the computation by reducing the dimensions of the matrices involved.

D. Non-fibered faces represented by two topologically inequivalent pseudo-Anosov flows

A connection between pseudo-Anosov flows and the Thurston norm was discovered by Fried and Mosher in the 90's [Fri82b, Mos92]. In particular, Mosher introduced the notion for a face of the Thurston norm ball to be *dynamically represented* by a pseudo-Anosov flow. Roughly speaking, this means that the homology classes of surfaces that are almost transverse to the flow span the cone on the face. While many questions about dynamical representation were answered by Mosher himself, it was still unknown if one face can be represented by two topologically inequivalent flows.

Veering triangulations turned out to be an excellent tool to tackle this problem. I discovered many pairs of distinct veering triangulations that *combinatorially represent* the same face of the Thurston norm ball and described them in Section 4 of [Par23b]. In Section 5 of [Par23b], relying on the work of Landry [Lan22, Theorem A] and Agol-Tsang [AT, Theorem 5.1], I proved that some of these veering triangulations give rise to pairs of topologically inequivalent pseudo-Anosov flows that dynamically represent the same face of the Thurston norm ball. This is an example of a result concerning pseudo-Anosov flows that would be very hard to obtain without the use of veering triangulations.

E. Combinatorial mutations of taut triangulations

Taut triangulations form a class of ideal triangulations of three-manifolds that contains veering triangulations as a proper subclass. In [Par23b, Section 3] I introduced a new operation that one can perform on a taut triangulation, called a *combinatorial mutation*. I found sufficient and necessary conditions on the *mutant triangulation* to admit a taut structure [Par23b, Proposition 3.16] and analyzed the homeomorphism type of its underlying manifold [Par23b, Theorem 3.10]. I further refined a combinatorial mutation into a *veering mutation*, which can be performed on a veering triangulation to produce another veering triangulation. The main motivation for these constructions came from investigating veering triangulations representing the same face of the Thurston norm ball that I mentioned in part D. I showed that many of them (but not all) differ exactly by a veering mutation [Par23b, Fact 4.2].

F. A counterexample to the classification theorem of Anosov flows on BL-manifolds

Identifying the two boundary components of the orientable circle bundle over a two-holed $\mathbb{R}P^2$ via some homeomorphism A yields a closed three-manifold N_A. Among dynamicists, such manifolds are often called *BL-manifolds* due to the fact that on one such manifold Bonatti and Langevin constructed the first known example of an Anosov flow which is transverse a torus, but is not a suspension of an Anosov homeomorphism [BL94]. Barbot generalized their result by showing that whenever N_A is not a circle bundle it admits an Anosov flow with such properties [Bar98, Theorem A]. He called these flows *BL-flows*. Furthermore, he claimed that a BL-flow on a BL-manifold is unique up to topological equivalence [Bar98, Theorem B]. I found a counterexample to that statement and described it in Remark 5.5 of [Par23b]. The idea is that one can construct a BL-flow on both N_A and N_{-A}, these manifolds are homeomorphic, yet no homeomorphism between them sends the orbits of one flow to the orbits of the other.

G. Infinitely many distinct veering triangulations with the same taut polynomial

Since Landry-Minsky-Taylor defined the taut polynomial, it was an open question whether it can be the same for infinitely many distinct veering triangulations. In [Par23a] I answered this question positively, by constructing an infinite sequence of veering triangulations, with the number of tetrahedra tending to infinity, whose taut polynomials are all zero.

III. In-progress research projects

One substantial research project that I am currently involved in concerns **drilling veering triangulations along their flow cycles**. This is joint work with Henry Segerman.

A construction of Agol-Guéritaud produces a veering triangulation $\mathcal{V}$ from a transitive pseudo-Anosov flow Ψ and a veering-enabling set Λ. If γ is a closed orbit of Ψ that is not in Λ, we may construct another veering triangulation $\mathcal{V}'$ encoding the dynamics of Ψ by applying the Agol-Guéritaud construction to Ψ and $\Lambda' = \Lambda \cup \{\gamma\}$. We call the triangulation $\mathcal{V}'$ a *veering parent* of $\mathcal{V}$. While it is completely straightforward to describe the difference between the pairs (Ψ, Λ) and (Ψ, Λ'), the difference in the combinatorics of $\mathcal{V}$ and $\mathcal{V}'$ is much more elusive.

Landry, Minsky, and Taylor proved that γ can be represented by a primitive cycle c in a certain graph associated to $\mathcal{V}$, called the *flow graph* of $\mathcal{V}$. Together with Henry Segerman we devised and implemented an algorithm which constructs $\mathcal{V}'$ using just $\mathcal{V}$ and c. The Agol-Guéritaud construction cannot be applied here directly, because it relies on an infinite object — namely, the orbit space of the flow — while we can work only with finite-type data. Our approach is based on approximating a compact subset of the orbit space and appropriately modifying the Agol-Guéritaud construction.

This project will lead to a new preprint in the coming months, and the accompanying code implementation will be added to the Veering package. We plan to use this code for the following purposes.

1. **Find a *layered veering parent* for each non-layered veering triangulation in the Veering Census.**

   The distinction between *layered* and *non-layered* veering triangulations is based on whether the triangulation represents a fibered face of the Thurston norm ball, or not [LMT23, Theorem 5.15]. It follows from a result of Brunella [Bru95, Theorem 1] that every non-layered veering triangulation admits a layered veering parent. We seek experimental data on layered parents of non-layered veering triangulations to gain an insight on how to improve the existing results concerning *Birkhoff sections* for pseudo-Anosov flows [Tsa22, Theorem 6.3].

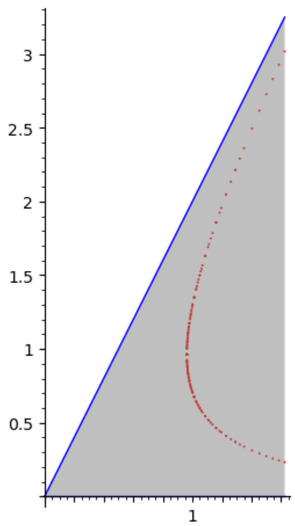
2. **Study the properties of the *veering ancestry graph.***

   The *veering ancestry graph* is the graph $\mathcal{A}$ whose vertices correspond to veering triangulations, and in which a veering triangulation $\mathcal{V}_1$ is connected to a veering triangulation $\mathcal{V}_2$ by a directed edge if and only if $\mathcal{V}_2$ can be obtained from $\mathcal{V}_1$ by drilling it along a single flow cycle. We want to study the properties of $\mathcal{A}$, and in particular describe its connected components. We hope that this project will shed some light on a conjecture of Ghys which states that all transitive Anosov flows with transversely orientable invariant foliations are *almost equivalent* — that is, topologically equivalent after drilling out finitely many of their closed orbits; see [Tsa24, Section 1.4].

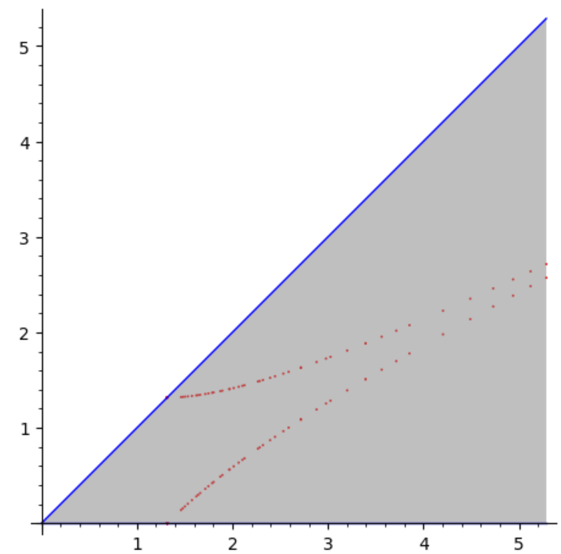
IV. Research plans

A. Non-fibered faces and the taut polynomial

Let $\mathtt{F}$ be a fibered face of the Thurston norm ball of a three-manifold M. The Teichmüller polynomial $\Theta_{\mathtt{F}}$ of $\mathtt{F}$ is an element of the integral group ring $\mathbb{Z}[H]$, where $H = H_1(M;\mathbb{Z})/\text{torsion}$. Thus it can be represented as a sum $\sum_{h \in H} a_h \cdot h$, where $a_h \in \mathbb{Z}$ is nonzero for only finitely many $h \in H$. McMullen proved that if η is a fibered class from the interior of the cone on $\mathtt{F}$, then the stretch factor of the monodromy of the fibration whose fiber represents η is the largest real root of $\Theta_{\mathtt{F}}^{\eta}(z) = \sum_{h \in H} a_h z^{\langle \eta, h \rangle}$, the *specialization* of $\Theta_{\mathtt{F}}$ at η [McM00, Theorem 5.1]. The behavior of stretch factors of distinct fibrations lying over $\mathtt{F}$ is well-understood; see [Fri82a, Theorems E and F] and [McM00, Corollary 5.4] for details, and Figure 1(a) for an illustration.



(a) Cone on a fibered face.



(b) Cone on a non-fibered face.

Figure 1: A plot of $\log|\lambda| \cdot \eta$, where η is a carried class such that the largest in the absolute value real root λ of the specialization of the taut polynomial at η satisfies $|\lambda| > 1$.

The taut polynomial of a veering triangulation generalizes the Teichmüller polynomial in the sense that if $\mathcal{V}$ is layered then its taut polynomial $\Theta_{\mathcal{V}}$ is equal to Teichmüller polynomial of the fibered face represented by $\mathcal{V}$ [LMT23, Theorem 7.1]. When $\mathcal{V}$ combinatorially represents a non-fibered face $\mathtt{F}$, its taut polynomial $\Theta_{\mathcal{V}}$ is not, strictly speaking, an invariant of $\mathtt{F}$, because $\mathtt{F}$ can be represented by different veering triangulations with different taut polynomials [Par23b, Fact 6.1]. Nonetheless, it remains interesting to explore the meaning of the roots of $\Theta_{\mathcal{V}}^{\eta}$ for $\eta \in \mathbb{R}_+ \cdot \mathtt{F}$.

For non-layered veering triangulations from the Veering Census [GSS], $\Theta_{\mathcal{V}}^{\eta}$ typically does not have any real roots. However, there are some mysterious examples for which there is a carried class η such that $\Theta_{\mathcal{V}}^{\eta}$ has a real root with absolute value greater than one. Furthermore, in this case such a class is not unique over the represented face, and the real roots associated to all such classes arrange into a non-random pattern; see Figure 1(b) for an example. One can observe that the behavior is noticeably different than in the fibered case. In particular, the blow up appears in the middle of the cone, instead of at its boundary. In light of this, I want to pursue the following project.

Project 1. Suppose that a veering triangulation $\mathcal{V}$ represents a non-fibered face $\mathtt{F}$ in $H_2(M, \partial M; \mathbb{R})$. Find sufficient conditions on $\eta \in \mathbb{R}_+ \cdot \mathtt{F}$ for the specialized taut polynomial $\Theta_{\mathcal{V}}^{\eta}$ to have a real root with absolute value greater than one. Explore what dynamical properties of the underlying flow (relative to a Thurston norm minimizing surface representing η) are responsible for this phenomenon.

The main objective is to get a better understanding of algebraic invariants of pseudo-Anosov flows that are not transverse to fibrations. This work will build upon and extend the well-established results concerning the fibered case and the Teichmüller polynomial.

B. Veering triangulations encoding BL-flows

This project concerns BL-flows described in Part II.F. It follows from [Tsa22, Proposition 2.7] that for every BL-flow Ψ on a BL-manifold N_A there is an orbit ℓ of Ψ such that $\{\ell\}$ is a veering-enabling set for Ψ. Applying the Agol-Guéritaud construction to $(\Psi, \{\ell\})$ yields a one-vertex veering triangulation encoding Ψ. I used Regina [BBP+17] to find many veering triangulations in the Veering Census [GSS] that encode BL-flows, but my list is unlikely to be complete. More generally, we know very little about which pseudo-Anosov flows are encoded in the Veering Census which undermines the claim that it can serve as a census of pseudo-Anosov flows. For this reason, I plan to pursue the following project.

Project 2. Let Ψ be a BL-flow on a BL-manifold N_A. Fix a closed orbit ℓ of Ψ such that $\{\ell\}$ is a veering-enabling set for Ψ. Let $\mathcal{V}_A(\ell)$ be the Agol-Guéritaud veering triangulation of $N_A - \ell$ which encodes Ψ. Find a formula relating the pair (A, ℓ) and the number of tetrahedra of $\mathcal{V}_A(\ell)$. More generally, describe the combinatorics of $\mathcal{V}_A(\ell)$.

As a starting point, I plan to analyze the veering triangulations that I already know encode BL-manifolds, focusing on how their combinatorics depends on the deleted orbit and the gluing homeomorphism A. Completing this study will serve as a step towards establishing a dictionary between veering triangulations from the Veering Census and pseudo-Anosov flows that they encode.

C. Pseudo-Anosov flows with equal cones of homology directions

In [Par23b, Theorem 5.2] I proved that there are non-fibered faces of the Thurston norm ball that are dynamically represented by multiple topologically inequivalent pseudo-Anosov flows. The *cones of homology directions* of these flows, spanned by the homology classes of their closed orbits, are equal. For a certain class of non-hyperbolic three-manifolds that admit a pair of Anosov flows with equal cones of homology directions, I explained how the two flows are connected and how one can be obtained from the other [Par23b, Fact 4.2]. I plan to investigate this problem in more generality.

Project 3.

- Find sufficient and necessary conditions for a closed three-manifold to admit multiple topologically inequivalent pseudo-Anosov flows with equal cones of homology directions.
- Discover topological and dynamical properties that orbit inequivalent pseudo-Anosov flows with equal cones of homology directions must share.
- Identify all possible operations that can be performed on a pseudo-Anosov flow to produce another pseudo-Anosov flow on the same manifold and with the same cone of homology directions.

Work on this project will bring new insights on the variety of pseudo-Anosov flows co-exisiting on the same three-manifold, a topic that has been of interest to dynamicists and topologists since the 90's; see [Mos92, Bar98, BFM22].

References

[Ago11] Ian Agol. Ideal triangulations of pseudo-Anosov mapping tori. In W. Li, L. Bartolini, J. Johnson, F. Luo, R. Myers, and J. H. Rubinstein, editors, *Topology and Geometry in Dimension Three: Triangulations, Invariants, and Geometric Structures*, volume 560 of *Contemporary Mathematics*, pages 1–19. American Mathematical Society, 2011.

[AT] Ian Agol and Chi Cheuk Tsang. Dynamics of veering triangulations: infinitesimal components of their flow graphs and applications. To appear in *Algebraic & Geometric Topology*. `arXiv:2201.02706v1 [math.GT]`.

[Bar98] Thierry Barbot. Generalizations of the Bonatti-Langevin example of Anosov flow and their classification up to topological equivalence. *Communications in Analysis and Geometry*, 6(4):749 – 798, 1998.

[BBP+17] Benjamin A. Burton, Ryan Budney, William Pettersson, et al. Regina: Software for low-dimensional topology. `http://regina-normal.github.io/`, 1999–2017.

[BFM22] Thomas Barthelmé, Steven Frankel, and Kathryn Mann. Orbit equivalences of pseudo-Anosov flows. `arXiv:2211.10505 [math.DS]`, 2022.

[BL94] Christian Bonatti and Remi Langevin. Un exemple de flot d'Anosov transitif transverse à un tore et non conjugué à une suspension. *Ergodic Theory and Dynamical Systems*, 14:633 – 643, 1994. https://doi.org/10.1017/S0143385700008099.

[Bru95] Marco Brunella. Surfaces of section for expansive flows on three-manifolds. *J. Math. Soc. Japan*, 47(3):491–501, 1995.

[Cal06a] Danny Calegari. Promoting essential laminations. *Inventiones mathematicae*, 166:583 – 643, 2006.

[Cal06b] Danny Calegari. Universal circles for quasigeodesic flows. *Geometry & Topology*, 10:2271 – 2298, 2006.

[CDW] Marc Culler, Nathan Dunfield, and Jeffrey R. Weeks. SnapPy. A computer program for studying the geometry and topology of 3-manifolds. `http://snappy.computop.org/`.

[EG02] John Etnyre and Robert Ghrist. Tight contact structures and Anosov flows. *Topology and Its Applications*, 124(2):211 – 219, 2002.

[FB17] Sérgio R. Fenley and Thomas Barthelmé. Counting periodic orbits of Anosov flows in free homotopy classes. *Commentarii Mathematici Helvetici*, 92:641 – 714, 2017.

[Fen98] Sérgio R. Fenley. The structure of branching in Anosov flows of 3-manifolds. *Commentarii Mathematici Helvetici*, 73:259 – 297, 1998.

[Fen02] Sérgio R. Fenley. Foliations, topology and geometry of 3-manifolds: $\mathbb{R}$-covered foliations and transverse pseudo-Anosov flows. *Comment. Math. Helv.*, 77:415–490, 2002.

[Fen07] Sérgio R. Fenley. Laminar free hyperbolic 3-manifolds. *Commentarii Mathematici Helvetici*, 82:247 – 321, 2007.

[Fen12] Sérgio R. Fenley. Laminar free hyperbolic 3-manifolds. *Geometry & Topology*, 16:1 – 110, 2012.

[Fen16a] Sérgio R. Fenley. Diversified homotopic behavior of closed orbits of some $\mathbb{R}$-covered Anosov flows. *Ergodic Theory and Dynamical Systems*, 36(3):767 – 780, 2016.

[Fen16b] Sérgio R. Fenley. Quasigeodesic pseudo-Anosov flows in hyperbolic 3-manifolds and connections with large scale geometry. *Advances in Mathematics*, 303:192–278, 2016.

[FH13] Patrick Foulon and Boris Hasselblatt. Contact Anosov flows on hyperbolic 3–manifolds. *Geometry & Topology*, 17(2):1225— 1252, 2013.

[FM01] Sérgio R. Fenley and Lee Mosher. Quasigeodesic flows in hyperbolic 3-manifolds. *Topology*, 40(3):503–537, 2001.

[Fri82a] David Fried. Flow equivalence, hyperbolic systems and a new zeta function for flows. *Commentarii mathematici Helvetici*, 57:237 – 259, 1982.

[Fri82b] David Fried. The geometry of cross sections to flows. *Topology*, 21(4):353 – 371, 1982.

[GSS] Andreas Giannopolous, Saul Schleimer, and Henry Segerman. A census of veering structures. `https://math.okstate.edu/people/segerman/veering.html`.

[Lan22] Michael Landry. Veering triangulations and the Thurston norm: Homology to isotopy. *Advances in Mathematics*, 396, 2022.

[LMT22] Michael Landry, Yair N. Minsky, and Samuel J. Taylor. Flows, growth rates, and the veering polynomial. *Ergodic Theory and Dynamical Systems*, 43(9):3026 – 3107, 2022. `arXiv:2107.04066 [math.GT]`.

[LMT23] Michael Landry, Yair N. Minsky, and Samuel J. Taylor. A polynomial invariant for veering triangulations. *J. Eur. Math. Soc.*, DOI 10.4171/JEMS/1368, 2023. `arXiv:2008.04836 [math.GT]`.

[McM00] Curtis T. McMullen. Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations. *Ann. Scient. Éc. Norm. Sup.*, 33(4):519–560, 2000.

[Mos91] Lee Mosher. Surfaces and branched surfaces transverse to pseudo-Anosov flows on 3-manifolds. *J. Differential Geometry*, 34:1–36, 1991.

[Mos92] Lee Mosher. Dynamical systems and the homology norm of a 3-manifold II. *Inventiones mathematicae*, 107:243 – 281, 1992.

[Mos96] Lee Mosher. Laminations and flows transverse to finite depth foliations. Preprint, 1996.

[Par23a] Anna Parlak. Arbitrarily large veering triangulations with a vanishing taut polynomial. To appear in *Groups, Geometry, and Dynamics*. `arXiv:2309.01752 [math.GT]`, 2023.

[Par23b] Anna Parlak. Mutations and faces of the Thurston norm ball dynamically represented by multiple distinct flows. To appear in *Geometry&Topology*. `arXiv:2303.17665 [math.GT]`, 2023.

[Par23c] Anna Parlak. The taut polynomial and the Alexander polynomial. *Journal of Topology*, 16(2):720–756, 2023. `https://doi.org/10.1112/topo.12302`.

[Par24] Anna Parlak. Computation of the taut, the veering and the Teichmüller polynomials. *Experimental Mathematics*, 33(1):1–26, 2024.

[PSS] Anna Parlak, Saul Schleimer, and Henry Segerman. GitHub Veering repository. Regina-Python and sage code for working with transverse taut and veering ideal triangulations. `https://github.com/henryseg/Veering`.

[PT72] J. F. Plante and W. P. Thurston. Anosov flows and the fundamental group. *Topology*, 11:147–150, 1972.

[SSa] Saul Schleimer and Henry Segerman. From veering triangulations to dynamic pairs. `arXiv:2305.08799 [math.GT]`.

[SSb] Saul Schleimer and Henry Segerman. From veering triangulations to link spaces and back again. `arXiv:1911.00006 [math.GT]`.

[Thu86] William P. Thurston. A norm for the homology of 3-manifolds. *Memoirs of the American Mathematical Society*, 59(339):100–130, 1986.

[Tsa22] Chi Cheuk Tsang. Constructing Birkhoff sections for pseudo-Anosov flows with controlled complexity. `arXiv:2206.09586v1 [math.DS]`, June 2022.

[Tsa23] Chi Cheuk Tsang. *Veering triangulations and pseudo-Anosov flows*. PhD thesis, University of California, Berkeley, https://escholarship.org/uc/item/26h9h6dp, 2023.

[Tsa24] Chi Cheuk Tsang. Examples of Anosov flows with genus one Birkhoff sections. `arXiv:2402.00229 [math.DS]`, 2024.