# Language Technology
# http://cs.lth.se/edan20/
## Chapter 5, part 2: Word Sequences

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# Word Sequences

Words have specific contexts of use.
Pairs of words like *strong* and *tea* or *powerful* and *computer* are not random associations.
Psychological linguistics tells us that it is difficult to make a difference between *writer* and *rider* without context
A listener will discard the improbable *rider of books* and prefer *writer of books*
A language model is the statistical estimate of a word sequence.
Originally developed for speech recognition
The language model component enables to predict the next word given a sequence of previous words: *the writer of books, novels, poetry*, etc. and not *the writer of hooks, nobles, poultry*, . . .

# *N*-Grams

The types are the distinct words of a text while the tokens are all the words or symbols.

The phrases from *Nineteen Eighty-Four*

> *War is peace*
> *Freedom is slavery*
> *Ignorance is strength*

have 9 tokens and 7 types.

Unigrams are single words

Bigrams are sequences of two words

Trigrams are sequences of three words

# Trigrams

| Word | Rank | More likely alternatives |
|---|---|---|
| *We* | 9 | *The This One Two A Three Please In* |
| *need* | 7 | *are will the would also do* |
| *to* | 1 | |
| *resolve* | 85 | *have know do...* |
| *all* | 9 | *the this these problems...* |
| *of* | 2 | *the* |
| *the* | 1 | |
| *important* | 657 | *document question first...* |
| *issues* | 14 | *thing point to...* |
| *within* | 74 | *to of and in that...* |
| *the* | 1 | |
| *next* | 2 | *company* |
| *two* | 5 | *page exhibit meeting day* |
| *days* | 5 | *weeks years pages months* |

# Counting Bigrams With Unix Tools

1. `tr -cs 'A-Za-z' '\n' < input_file > token_file`
   Tokenize the input and create a file with the unigrams.
2. `tail +2 < token_file > next_token_file`
   Create a second unigram file starting at the second word of the first tokenized file (+2).
3. `paste token_file next_token_file > bigrams`
   Merge the lines (the tokens) pairwise. Each line of `bigrams` contains the words at index $i$ and $i+1$ separated with a tabulation.
4. And we count the bigrams as in the previous script.

# Counting Bigrams in Python

```
bigrams = [tuple(words[inx:inx + 2])
           for inx in range(len(words) - 1)]
```

The rest of the count_bigrams function is nearly identical to count_unigrams. As input, it uses the same list of words:

```
def count_bigrams(words):
    bigrams = [tuple(words[inx:inx + 2])
               for inx in range(len(words) - 1)]
    frequencies = {}
    for bigram in bigrams:
        if bigram in frequencies:
            frequencies[bigram] += 1
        else:
            frequencies[bigram] = 1
    return frequencies
```

# Probabilistic Models of a Word Sequence

$$
\begin{aligned}
P(S) &= P(w_1, ..., w_n), \\
&= P(w_1)P(w_2|w_1)P(w_3|w_1, w_2)...P(w_n|w_1, ..., w_{n-1}), \\
&= \prod_{i=1}^{n} P(w_i|w_1, ..., w_{i-1}).
\end{aligned}
$$

The probability $P(\textit{It was a bright cold day in April})$ from *Nineteen Eighty-Four* corresponds to
*It* to begin the sentence, then *was* knowing that we have *It* before, then *a* knowing that we have *It was* before, and so on until the end of the sentence.

$$
\begin{aligned}
P(S) = & P(It) \times P(was|It) \times P(a|It, was) \times P(bright|It, was, a) \times ... \\
& \times P(April|It, was, a, bright, ..., in).
\end{aligned}
$$

# Approximations

Bigrams:

$$P(w_i|w_1, w_2, ..., w_{i-1}) \approx P(w_i|w_{i-1}),$$

Trigrams:

$$P(w_i|w_1, w_2, ..., w_{i-1}) \approx P(w_i|w_{i-2}, w_{i-1}).$$

Using a trigram language model, $P(S)$ is approximated as:

$$\begin{aligned} P(S) \quad \approx \quad & P(It) \times P(was|It) \times P(a|It, was) \times P(bright|was, a) \times ... \\ & \times P(April|day, in). \end{aligned}$$

# Maximum Likelihood Estimate

Bigrams:

$$P_{MLE}(w_i|w_{i-1}) = \frac{C(w_{i-1}, w_i)}{\sum\limits_{w} C(w_{i-1}, w)} = \frac{C(w_{i-1}, w_i)}{C(w_{i-1})}.$$

Trigrams:

$$P_{MLE}(w_i|w_{i-2}, w_{i-1}) = \frac{C(w_{i-2}, w_{i-1}, w_i)}{C(w_{i-2}, w_{i-1})}.$$

# Conditional Probabilities

A common mistake in computing the conditional probability $P(w_i|w_{i-1})$ is to use

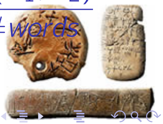
$$\frac{C(w_{i-1}, w_i)}{\#bigrams}.$$

This is not correct. This formula corresponds to $P(w_{i-1}, w_i)$.
The correct estimation is

$$P_{MLE}(w_i|w_{i-1}) = \frac{C(w_{i-1}, w_i)}{\sum_w C(w_{i-1}, w)} = \frac{C(w_{i-1}, w_i)}{C(w_{i-1})}.$$

Proof:

$$P(w_1, w_2) = P(w_1)P(w_2|w_1) = \frac{C(w_1)}{\#words} \times \frac{C(w_1, w_2)}{C(w_1)} = \frac{C(w_1, w_2)}{\#words}$$



# Training the Model

The model is trained on a part of the corpus: the **training set**

It is tested on a different part: the **test set**

The vocabulary can be derived from the corpus, for instance the 20,000 most frequent words, or from a lexicon

It can be closed or open

A closed vocabulary does not accept any new word

An open vocabulary maps the new words, either in the training or test sets, to a specific symbol, `<UNK>`

# Probability of a Sentence: Unigrams

<s> *A good deal of the literature of the past was, indeed, already being transformed in this way* </s>

| $w_i$ | $C(w_i)$ | #words | $P_{MLE}(w_i)$ |
|---|---:|---:|---:|
| <s> | 7072 | – | |
| *a* | 2482 | 108140 | 0.023 |
| *good* | 53 | 108140 | 0.00049 |
| *deal* | 5 | 108140 | $4.62\ 10^{-5}$ |
| *of* | 3310 | 108140 | 0.031 |
| *the* | 6248 | 108140 | 0.058 |
| *literature* | 7 | 108140 | $6.47\ 10^{-5}$ |
| *of* | 3310 | 108140 | 0.031 |
| *the* | 6248 | 108140 | 0.058 |
| *past* | 99 | 108140 | 0.00092 |
| *was* | 2211 | 108140 | 0.020 |
| *indeed* | 17 | 108140 | 0.00016 |
| *already* | 64 | 108140 | 0.00059 |
| *being* | 80 | 108140 | 0.00074 |
| *transformed* | 1 | 108140 | $9.25\ 10^{-6}$ |
| *in* | 1759 | 108140 | 0.016 |
| *this* | 264 | 108140 | 0.0024 |
| *way* | 122 | 108140 | 0.0011 |
| </s> | 7072 | 108140 | 0.065 |

# Probability of a Sentence: Bigrams

*<s> A good deal of the literature of the past was, indeed, already being transformed in this way </s>*

| $w_{i-1}, w_i$ | $C(w_{i-1}, w_i)$ | $C(w_{i-1})$ | $P_{MLE}(w_i \mid w_{i-1})$ |
|---|---:|---:|---:|
| <s> *a* | 133 | 7072 | 0.019 |
| *a good* | 14 | 2482 | 0.006 |
| *good deal* | 0 | 53 | 0.0 |
| *deal of* | 1 | 5 | 0.2 |
| *of the* | 742 | 3310 | 0.224 |
| *the literature* | 1 | 6248 | 0.0002 |
| *literature of* | 3 | 7 | 0.429 |
| *of the* | 742 | 3310 | 0.224 |
| *the past* | 70 | 6248 | 0.011 |
| *past was* | 4 | 99 | 0.040 |
| *was indeed* | 0 | 2211 | 0.0 |
| *indeed already* | 0 | 17 | 0.0 |
| *already being* | 0 | 64 | 0.0 |
| *being transformed* | 0 | 80 | 0.0 |
| *transformed in* | 0 | 1 | 0.0 |
| *in this* | 14 | 1759 | 0.008 |
| *this way* | 3 | 264 | 0.011 |
| *way* </s> | 18 | 122 | 0.148 |

# Sparse Data

Given a vocabulary of 20,000 types, the potential number of bigrams is $20,000^2 = 400,000,000$
With trigrams $20,000^3 = 8,000,000,000,000$
Methods:

- Laplace: add one to all counts
- Linear interpolation:

$$
\begin{aligned}
P_{DelInterpolation}(w_n|w_{n-2}, w_{n-1}) = & \lambda_1 P_{MLE}(w_n|w_{n-2}w_{n-1}) + \\
& \lambda_2 P_{MLE}(w_n|w_{n-1}) + \lambda_3 P_{MLE}(w_n)
\end{aligned}
$$

- Good-Turing: The discount factor is variable and depends on the number of times a n-gram has occurred in the corpus.
- Back-off

# Laplace's Rule

$$P_{Laplace}(w_{i+1}|w_i) = \frac{C(w_i, w_{i+1}) + 1}{\sum_w (C(w_i, w) + 1)} = \frac{C(w_i, w_{i+1}) + 1}{C(w_i) + Card(V)},$$

| $w_i, w_{i+1}$ | $C(w_i, w_{i+1}) + 1$ | $C(w_i) + Card(V)$ | $P_{Lap}(w_{i+1}\|w_i)$ |
|---|---|---|---|
| <s> *a* | 133 + 1 | 7072 + 8635 | 0.0085 |
| *a good* | 14 + 1 | 2482 + 8635 | 0.0013 |
| *good deal* | 0 + 1 | 53 + 8635 | 0.00012 |
| *deal of* | 1 + 1 | 5 + 8635 | 0.00023 |
| *of the* | 742 + 1 | 3310 + 8635 | 0.062 |
| *the literature* | 1 + 1 | 6248 + 8635 | 0.00013 |
| *literature of* | 3 + 1 | 7 + 8635 | 0.00046 |
| *of the* | 742 + 1 | 3310 + 8635 | 0.062 |
| *the past* | 70 + 1 | 6248 + 8635 | 0.0048 |
| *past was* | 4 + 1 | 99 + 8635 | 0.00057 |
| *was indeed* | 0 + 1 | 2211 + 8635 | 0.000092 |
| *indeed already* | 0 + 1 | 17 + 8635 | 0.00012 |
| *already being* | 0 + 1 | 64 + 8635 | 0.00011 |
| *being transformed* | 0 + 1 | 80 + 8635 | 0.00011 |
| *transformed in* | 0 + 1 | 1 + 8635 | 0.00012 |
| *in this* | 14 + 1 | 1759 + 8635 | 0.0014 |
| *this way* | 3 + 1 | 264 + 8635 | 0.00045 |
| *way* </s> | 18 + 1 | 122 + 8635 | 0.0022 |

# Good–Turing

Laplace's rule shifts an enormous mass of probability to very unlikely bigrams. Good–Turing's estimation is more effective

Let's denote $N_c$ the number of n-grams that occurred exactly $c$ times in the corpus.

$N_0$ is the number of unseen n-grams, $N_1$ the number of n-grams seen once, $N_2$ the number of n-grams seen twice The frequency of n-grams occurring $c$ times is re-estimated as:

$$c* = (c+1)\frac{E(N_{c+1})}{E(N_c)},$$

Unseen n-grams: $c* = \dfrac{N_1}{N_0}$ and $N$-grams seen once: $c* = \dfrac{2N_2}{N_1}$.

# Good-Turing for *Nineteen eighty-four*

*Nineteen eighty-four* contains 37,365 unique bigrams and 5,820 bigram seen twice.

Its vocabulary of 8,635 words generates $86352^2$ = 74,563,225 bigrams whose 74,513,701 are unseen.

New counts:

- Unseen bigrams: $\dfrac{37,365}{74,513,701} = 0.0005.$
- Unique bigrams: $2 \times \dfrac{5820}{37,365} = 0.31.$
- Etc.

| **Freq. of occ.** | $N_c$ | $c*$ | **Freq. of occ.** | $N_c$ | $c*$ |
|---|---|---|---|---|---|
| 0 | 74,513,701 | 0.0005 | 5 | 719 | 3.91 |
| 1 | 37,365 | 0.31 | 6 | 468 | 4.94 |
| 2 | 5,820 | 1.09 | 7 | 330 | 6.06 |
| 3 | 2,111 | 2.02 | 8 | 250 | 6.44 |
| 4 | 1,067 | 3.37 | 9 | 179 | 8.93 |

# Backoff

If there is no bigram, then use unigrams:

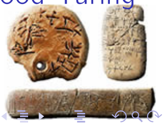
$$P_{\text{Backoff}}(w_i|w_{i-1}) = \begin{cases} \tilde{P}(w_i|w_{i-1}), & \text{if } C(w_{i-1}, w_i) \neq 0, \\ \alpha P(w_i), & \text{otherwise.} \end{cases}$$

$$P_{\text{Backoff}}(w_i|w_{i-1}) = \begin{cases} P_{\text{MLE}}(w_i|w_{i-1}) = \dfrac{C(w_{i-1}, w_i)}{C(w_{i-1})}, & \text{if } C(w_{i-1}, w_i) \neq 0, \\ P_{\text{MLE}}(w_i) = \dfrac{C(w_i)}{\#\text{words}}, & \text{otherwise.} \end{cases}$$

# Backoff: Example

| $w_{i-1}, w_i$ | $C(w_{i-1}, w_i)$ | | $C(w_i)$ | $P_{Backoff}(w_i \mid w_{i-1})$ |
|---|---|---|---|---|
| <s> | | | 7072 | — |
| <s> *a* | 133 | | 2482 | 0.019 |
| *a good* | 14 | | 53 | 0.006 |
| *good deal* | 0 | backoff | 5 | $4.62\ 10^{-5}$ |
| *deal of* | 1 | | 3310 | 0.2 |
| *of the* | 742 | | 6248 | 0.224 |
| *the literature* | 1 | | 7 | 0.00016 |
| *literature of* | 3 | | 3310 | 0.429 |
| *of the* | 742 | | 6248 | 0.224 |
| *the past* | 70 | | 99 | 0.011 |
| *past was* | 4 | | 2211 | 0.040 |
| *was indeed* | 0 | backoff | 17 | 0.00016 |
| *indeed already* | 0 | backoff | 64 | 0.00059 |
| *already being* | 0 | backoff | 80 | 0.00074 |
| *being transformed* | 0 | backoff | 1 | $9.25\ 10^{-6}$ |
| *transformed in* | 0 | backoff | 1759 | 0.016 |
| *in this* | 14 | | 264 | 0.008 |
| *this way* | 3 | | 122 | 0.011 |
| *way* </s> | 18 | | 7072 | 0.148 |

The figures we obtain are not probabilities. We can use the Good-Turing technique to discount the bigrams and then scale the unigram probabilities. This is the Katz backoff.



# Quality of a Language Model (I)

The quality of a language model corresponds to its accuracy in predicting word sequences: $P(w_1, ..., w_n)$: The higher, the better.
We derive the model (the statistics) from a training set and evaluate this quality on a long unseen sequence sequence: The test set.
With the *n*-gram approximations, we have:

$$\begin{array}{rcll} P(w_1, ..., w_n) & = & \prod_{i=1}^{n} P(w_i) & \text{Unigrams} \\ P(w_1, ..., w_n) & = & P(w_1) \prod_{i=2}^{n} P(w_i | w_{i-1}) & \text{Bigrams} \\ P(w_1, ..., w_n) & = & P(w_1) P(w_2 | w_1) \prod_{i=3}^{n} P(w_i | w_{i-2}, w_{i-1}) & \text{Trigrams} \end{array}$$

etc.

# Quality of a Language Model (II)

The probability value will depend on the length of the sequence. We take the geometric mean instead to standardize across different lengths:

$$\sqrt[n]{\prod_{i=1}^{n} P(w_i)} \quad \text{Unigrams}$$

$$\sqrt[n]{P(w_1)\prod_{i=2}^{n} P(w_i|w_{i-1})} \quad \text{Bigrams}$$

...

In practice, we use the log to compute the per word probability of a word sequence, the entropy rate:

$$H(L) = -\frac{1}{n}\log_2 P(w_1, ..., w_n).$$

Here the lower, the better

The figures are usually presented with the perplexity metric:

$$PP(p,m) = 2^{H(L)}.$$

# Mathematical Background

Entropy rate: $H_{rate} = -\frac{1}{n} \sum_{w_1,\ldots,w_n \in L} p(w_1,\ldots,w_n) \log_2 p(w_1,\ldots,w_n)$,

Cross entropy:

$$H(p,m) = -\frac{1}{n} \sum_{w_1,\ldots,w_n \in L} p(w_1,\ldots,w_n) \log_2 m(w_1,\ldots,w_n).$$

We have:

$$\begin{aligned} H(p,m) &= \lim_{n\to\infty} -\frac{1}{n} \sum_{w_1,\ldots,w_n \in L} p(w_1,\ldots,w_n) \log_2 m(w_1,\ldots,w_n), \\ &= \lim_{n\to\infty} -\frac{1}{n} \log_2 m(w_1,\ldots,w_n). \end{aligned}$$

We compute the cross entropy on the complete word sequence of a test set, governed by $p$, using a bigram or trigram model, $m$, from a training set.

# Other Statistical Formulas

- Mutual information (The strength of an association):

$$I(w_i, w_j) = \log_2 \frac{P(w_i, w_j)}{P(w_i)P(w_j)} \approx \log_2 \frac{N \cdot C(w_i, w_j)}{C(w_i)C(w_j)}.$$

- T-score (The confidence of an association):

$$\begin{aligned} t(w_i, w_j) &= \frac{mean(P(w_i, w_j)) - mean(P(w_i))mean(P(w_j))}{\sqrt{\sigma^2(P(w_i, w_j)) + \sigma^2(P(w_i)P(w_j))}}, \\ &\approx \frac{C(w_i, w_j) - \frac{1}{N}C(w_i)C(w_j)}{\sqrt{C(w_i, w_j)}}. \end{aligned}$$

# T-Scores with Word *set*

| **Word** | **Frequency** | **Bigram set + word** | ***t*-score** |
|---|---:|---:|---:|
| *up* | 134,882 | 5512 | 67.980 |
| *a* | 1,228,514 | 7296 | 35.839 |
| *to* | 1,375,856 | 7688 | 33.592 |
| *off* | 52,036 | 888 | 23.780 |
| *out* | 12,3831 | 1252 | 23.320 |

Source: Bank of English

# Mutual Information with Word *surgery*

| **Word** | **Frequency** | **Bigram word + surgery** | **Mutual info** |
|---|---:|:---:|---:|
| *arthroscopic* | 3 | 3 | 11.822 |
| *pioneeing* | 3 | 3 | 11.822 |
| *reconstructive* | 14 | 11 | 11.474 |
| *refractive* | 6 | 4 | 11.237 |
| *rhinoplasty* | 5 | 3 | 11.085 |

Source: Bank of English

# Mutual Information in Python

```
def mutual_info(words, freq_unigrams, freq_bigrams):
    mi = {}
    factor = len(words) * len(words) / (len(words) - 1)
    for bigram in freq_bigrams:
        mi[bigram] = (
            math.log(factor * freq_bigrams[bigram] /
                     (freq_unigrams[bigram[0]] *
                      freq_unigrams[bigram[1]]), 2))
    return mi
```

# T-Scores in Python

```
def t_scores(words, freq_unigrams, freq_bigrams):
    ts = {}
    for bigram in freq_bigrams:
        ts[bigram] = ((freq_bigrams[bigram] -
                      freq_unigrams[bigram[0]] *
                      freq_unigrams[bigram[1]] /
                      len(words)) /
                      math.sqrt(freq_bigrams[bigram]))
    return ts
```

# Dimension Reduction

One-hot encoding of TFIDF encoding can produce very long vectors:
Imagine a vocabulary one one million words per language with 100 languages.

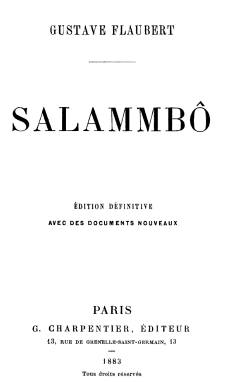
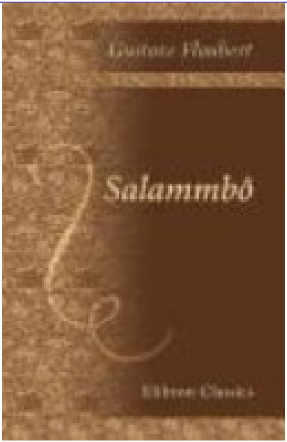
A solution is to produce dense vectors also called word embeddings using a dimension reduction

This reduction is very close to principal component analysis or singular value decomposition

It can be automatically obtained through training or initialized with pretrained vectors

# A Data Set: *Salammbô*

A corpus is a collection – a body – of texts.

| French original | English translation |
|---|---|
|  |  |

GUSTAVE FLAUBERT

SALAMMBÔ

ÉDITION DÉFINITIVE

AVEC DES DOCUMENTS NOUVEAUX

PARIS

G. CHARPENTIER, ÉDITEUR

13, RUE DE GRENELLE-SAINT-GERMAIN, 13

—

1883

Tous droits réservés

# Supervised Learning

## Letter counts from *Salammbô*

| Chapter | French | | English | |
|---|---|---|---|---|
| | # characters | # *A* | # characters | # *A* |
| Chapter 1 | 36,961 | 2,503 | 35,680 | 2,217 |
| Chapter 2 | 43,621 | 2,992 | 42,514 | 2,761 |
| Chapter 3 | 15,694 | 1,042 | 15,162 | 990 |
| Chapter 4 | 36,231 | 2,487 | 35,298 | 2,274 |
| Chapter 5 | 29,945 | 2,014 | 29,800 | 1,865 |
| Chapter 6 | 40,588 | 2,805 | 40,255 | 2,606 |
| Chapter 7 | 75,255 | 5,062 | 74,532 | 4,805 |
| Chapter 8 | 37,709 | 2,643 | 37,464 | 2,396 |
| Chapter 9 | 30,899 | 2,126 | 31,030 | 1,993 |
| Chapter 10 | 25,486 | 1,784 | 24,843 | 1,627 |
| Chapter 11 | 37,497 | 2,641 | 36,172 | 2,375 |
| Chapter 12 | 40,398 | 2,766 | 39,552 | 2,560 |
| Chapter 13 | 74,105 | 5,047 | 72,545 | 4,597 |
| Chapter 14 | 76,725 | 5,312 | 75,352 | 4,871 |
| Chapter 15 | 18,317 | 1,215 | 18,031 | 1,119 |

Data set: `https://github.com/pnugues/ilppp/tree/master/programs/ch04/salammbo`

# Principal Component Analysis

We will use a small dataset to explain principal component analysis: The characters in *Salammbô*

| Ch. | 'a' | 'b' | 'c' | 'd' | 'e' | 'f' | 'g' | 'h' | 'i' | 'j' | 'k' | 'l' | 'm' | 'n' | 'o' | 'p' | 'q' | 'r' | 's' | 't' | 'u' | 'v' | 'w' | 'x' | 'y' | 'z' | 'à' | 'â' |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 01_fr | 2503 | 365 | 857 | 1151 | 4312 | 264 | 349 | 295 | 1945 | 65 | 4 | 1946 | 726 | 1896 | 1372 | 789 | 248 | 1948 | 2996 | 1938 | 1792 | 414 | 0 | 129 | 94 | 20 | 128 | 36 |
| 02_fr | 2992 | 391 | 1006 | 1388 | 4993 | 319 | 360 | 350 | 2345 | 81 | 6 | 2128 | 823 | 2308 | 1560 | 977 | 281 | 2376 | 3454 | 2411 | 2069 | 499 | 0 | 175 | 89 | 23 | 136 | 50 |
| 03_fr | 1042 | 152 | 326 | 489 | 1785 | 136 | 122 | 126 | 784 | 41 | 7 | 816 | 397 | 778 | 612 | 315 | 102 | 792 | 1174 | 856 | 707 | 147 | 0 | 42 | 31 | 7 | 39 | 9 |
| 04_fr | 2487 | 303 | 864 | 1137 | 4158 | 314 | 331 | 287 | 2028 | 57 | 3 | 1796 | 722 | 1958 | 1318 | 773 | 274 | 2000 | 2792 | 2031 | 1734 | 422 | 0 | 138 | 81 | 27 | 110 | 43 |
| 05_fr | 2014 | 268 | 645 | 949 | 3394 | 223 | 215 | 242 | 1617 | 67 | 3 | 1513 | 651 | 1547 | 1053 | 672 | 166 | 1601 | 2192 | 1736 | 1396 | 315 | 1 | 83 | 67 | 18 | 90 | 67 |
| 06_fr | 2805 | 368 | 910 | 1266 | 4535 | 332 | 384 | 378 | 2219 | 97 | 3 | 1900 | 841 | 2179 | 1569 | 868 | 285 | 2205 | 3065 | 2293 | 1895 | 453 | 0 | 151 | 80 | 39 | 131 | 42 |
| 07_fr | 5062 | 706 | 1770 | 2398 | 8512 | 623 | 622 | 620 | 4018 | 126 | 19 | 3726 | 1596 | 3851 | 2823 | 1532 | 468 | 4015 | 5634 | 4116 | 3518 | 844 | 0 | 272 | 148 | 71 | 246 | 50 |
| 08_fr | 2643 | 325 | 869 | 1085 | 4229 | 307 | 317 | 359 | 2102 | 85 | 4 | 1857 | 811 | 2041 | 1367 | 833 | 239 | 2132 | 2814 | 2134 | 1788 | 437 | 0 | 135 | 64 | 30 | 130 | 43 |
| 09_fr | 2126 | 289 | 771 | 920 | 3599 | 278 | 289 | 279 | 1805 | 52 | 6 | 1499 | 619 | 1711 | 1130 | 651 | 187 | 1719 | 2404 | 1763 | 1448 | 348 | 0 | 119 | 58 | 20 | 90 | 24 |
| 10_fr | 1784 | 249 | 546 | 805 | 3002 | 179 | 202 | 215 | 1319 | 60 | 5 | 1462 | 598 | 1246 | 922 | 557 | 172 | 1242 | 1769 | 1423 | 1191 | 270 | 0 | 65 | 61 | 11 | 73 | 18 |
| 11_fr | 2641 | 381 | 817 | 1078 | 4306 | 263 | 277 | 330 | 1985 | 114 | 0 | 1886 | 900 | 1966 | 1356 | 763 | 230 | 1912 | 2564 | 2218 | 1737 | 425 | 0 | 114 | 61 | 25 | 101 | 40 |
| 12_fr | 2766 | 373 | 935 | 1237 | 4618 | 329 | 350 | 349 | 2273 | 65 | 2 | 1955 | 812 | 2285 | 1419 | 865 | 272 | 2276 | 3131 | 2274 | 1923 | 455 | 0 | 149 | 98 | 37 | 129 | 33 |
| 13_fr | 5047 | 725 | 1730 | 2273 | 8678 | 648 | 566 | 642 | 3940 | 140 | 22 | 3746 | 1597 | 3984 | 2736 | 1550 | 425 | 4081 | 5599 | 4387 | 3480 | 767 | 0 | 288 | 119 | 41 | 209 | 55 |
| 14_fr | 5312 | 689 | 1754 | 2149 | 8870 | 628 | 630 | 673 | 4278 | 143 | 2 | 3780 | 1610 | 4255 | 2713 | 1599 | 512 | 4271 | 5770 | 4467 | 3697 | 914 | 0 | 283 | 145 | 41 | 224 | 75 |
| 15_fr | 1215 | 173 | 402 | 582 | 2195 | 150 | 134 | 148 | 969 | 27 | 6 | 950 | 387 | 906 | 697 | 417 | 103 | 985 | 1395 | 1037 | 893 | 206 | 0 | 63 | 36 | 3 | 48 | 20 |
| 01_en | 2217 | 451 | 729 | 1316 | 3967 | 596 | 662 | 2060 | 1823 | 22 | 200 | 1204 | 656 | 1851 | 1897 | 525 | 19 | 1764 | 1942 | 2547 | 704 | 258 | 653 | 29 | 401 | 18 | 0 | 0 |
| 02_en | 2761 | 551 | 777 | 1548 | 4543 | 685 | 769 | 2530 | 2163 | 13 | 284 | 1319 | 829 | 2218 | 2237 | 606 | 21 | 2019 | 2411 | 3083 | 861 | 295 | 769 | 37 | 475 | 31 | 0 | 0 |
| 03_en | 990 | 183 | 271 | 557 | 1570 | 279 | 253 | 875 | 783 | 4 | 82 | 520 | 333 | 816 | 828 | 194 | 13 | 711 | 864 | 1048 | 298 | 94 | 254 | 8 | 145 | 15 | 0 | 0 |
| 04_en | 2274 | 454 | 736 | 1315 | 3814 | 595 | 559 | 1978 | 1835 | 22 | 198 | 1073 | 690 | 1771 | 1865 | 514 | 33 | 1726 | 1918 | 2704 | 745 | 245 | 663 | 60 | 467 | 19 | 0 | 0 |
| 05_en | 1865 | 400 | 553 | 1135 | 3210 | 515 | 525 | 1693 | 1482 | 7 | 153 | 949 | 571 | 1468 | 1586 | 517 | 17 | 1357 | 1646 | 2178 | 663 | 194 | 568 | 26 | 330 | 33 | 0 | 0 |
| 06_en | 2606 | 518 | 797 | 1509 | 4237 | 687 | 669 | 2254 | 2097 | 26 | 216 | 1239 | 763 | 2174 | 2231 | 613 | 25 | 1931 | 2192 | 2955 | 899 | 277 | 733 | 49 | 464 | 37 | 0 | 0 |
| 07_en | 4805 | 913 | 1521 | 2681 | 7834 | 1366 | 1163 | 4379 | 3838 | 42 | 416 | 2434 | 1461 | 3816 | 4091 | 1040 | 39 | 3674 | 4060 | 5369 | 1552 | 465 | 1332 | 74 | 843 | 52 | 0 | 0 |
| 08_en | 2396 | 431 | 702 | 1416 | 4014 | 621 | 624 | 2171 | 2011 | 24 | 216 | 1152 | 748 | 2085 | 1947 | 527 | 33 | 1915 | 1966 | 2765 | 789 | 266 | 695 | 65 | 379 | 24 | 0 | 0 |
| 09_en | 1993 | 408 | 653 | 1096 | 3373 | 575 | 517 | 1766 | 1648 | 16 | 146 | 861 | 629 | 1728 | 1698 | 442 | 20 | 1561 | 1626 | 2442 | 683 | 208 | 560 | 25 | 328 | 18 | 0 | 0 |
| 10_en | 1627 | 359 | 451 | 933 | 2690 | 477 | 409 | 1475 | 1196 | 7 | 131 | 789 | 506 | 1266 | 1369 | 325 | 23 | 1211 | 1344 | 1759 | 502 | 181 | 410 | 31 | 255 | 20 | 0 | 0 |
| 11_en | 2375 | 437 | 643 | 1364 | 3790 | 610 | 644 | 2217 | 1830 | 16 | 217 | 1122 | 799 | 1833 | 1948 | 486 | 23 | 1720 | 1945 | 2424 | 767 | 246 | 632 | 20 | 457 | 39 | 0 | 0 |
| 12_en | 2560 | 489 | 757 | 1566 | 4331 | 677 | 650 | 2348 | 2033 | 28 | 234 | 1102 | 746 | 2125 | 2105 | 581 | 32 | 1939 | 2152 | 3046 | 750 | 278 | 721 | 35 | 418 | 40 | 0 | 0 |
| 13_en | 4597 | 987 | 1462 | 2689 | 7963 | 1254 | 1201 | 4278 | 3634 | 39 | 432 | 2281 | 1493 | 3774 | 3911 | 1099 | 49 | 3577 | 3894 | 5540 | 1379 | 437 | 1374 | 77 | 673 | 49 | 0 | 0 |
| 14_en | 4871 | 948 | 1439 | 2799 | 8179 | 1335 | 1140 | 4534 | 3829 | 36 | 427 | 2218 | 1534 | 4053 | 3989 | 1019 | 36 | 3689 | 3946 | 5858 | 1490 | 539 | 1377 | 90 | 856 | 49 | 0 | 0 |
| 15_en | 1119 | 229 | 335 | 683 | 1994 | 323 | 281 | 1108 | 912 | 9 | 112 | 579 | 351 | 924 | 1004 | 305 | 9 | 863 | 997 | 1330 | 310 | 108 | 330 | 14 | 150 | 9 | 0 | 0 |

Table: Character counts per chapter, where the fr and en suffixes designate the language, either French or English

Each chapter is modeled by a vector of characters.

# Character Counts

| | French | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 |
| a | 2503 | 2992 | 1042 | 2487 | 2014 | 2805 | 5062 | 2643 | 2126 | 1784 | 2641 | 2766 | 5047 | 5312 | 1215 |
| b | 365 | 391 | 152 | 303 | 268 | 368 | 706 | 325 | 289 | 249 | 381 | 373 | 725 | 689 | 173 |
| c | 857 | 1006 | 326 | 864 | 645 | 910 | 1770 | 869 | 771 | 546 | 817 | 935 | 1730 | 1754 | 402 |
| d | 1151 | 1388 | 489 | 1137 | 949 | 1266 | 2398 | 1085 | 920 | 805 | 1078 | 1237 | 2273 | 2149 | 582 |
| e | 4312 | 4993 | 1785 | 4158 | 3394 | 4535 | 8512 | 4229 | 3599 | 3002 | 4306 | 4618 | 8678 | 8870 | 2195 |
| f | 264 | 319 | 136 | 314 | 223 | 332 | 623 | 307 | 278 | 179 | 263 | 329 | 648 | 628 | 150 |
| g | 349 | 360 | 122 | 331 | 215 | 384 | 622 | 317 | 289 | 202 | 277 | 350 | 566 | 630 | 134 |
| h | 295 | 350 | 126 | 287 | 242 | 378 | 620 | 359 | 279 | 215 | 330 | 349 | 642 | 673 | 148 |
| i | 1945 | 2345 | 784 | 2028 | 1617 | 2219 | 4018 | 2102 | 1805 | 1319 | 1985 | 2273 | 3940 | 4278 | 969 |
| j | 65 | 81 | 41 | 57 | 67 | 97 | 126 | 85 | 52 | 60 | 114 | 65 | 140 | 143 | 27 |
| k | 4 | 6 | 7 | 3 | 3 | 3 | 19 | 4 | 6 | 5 | 0 | 2 | 22 | 2 | 6 |
| l | 1946 | 2128 | 816 | 1796 | 1513 | 1900 | 3726 | 1857 | 1499 | 1462 | 1886 | 1955 | 3746 | 3780 | 950 |
| m | 726 | 823 | 397 | 722 | 651 | 841 | 1596 | 811 | 619 | 598 | 900 | 812 | 1597 | 1610 | 387 |
| n | 1896 | 2308 | 778 | 1958 | 1547 | 2179 | 3851 | 2041 | 1711 | 1246 | 1966 | 2285 | 3984 | 4255 | 906 |
| o | 1372 | 1560 | 612 | 1318 | 1053 | 1569 | 2823 | 1367 | 1130 | 922 | 1356 | 1419 | 2736 | 2713 | 697 |
| p | 789 | 977 | 315 | 773 | 672 | 868 | 1532 | 833 | 651 | 557 | 763 | 865 | 1550 | 1599 | 417 |
| q | 248 | 281 | 102 | 274 | 166 | 285 | 468 | 239 | 187 | 172 | 230 | 272 | 425 | 512 | 103 |
| r | 1948 | 2376 | 792 | 2000 | 1601 | 2205 | 4015 | 2132 | 1719 | 1242 | 1912 | 2276 | 4081 | 4271 | 985 |
| s | 2996 | 3454 | 1174 | 2792 | 2192 | 3065 | 5634 | 2814 | 2404 | 1769 | 2564 | 3131 | 5599 | 5770 | 1395 |
| t | 1938 | 2411 | 856 | 2031 | 1736 | 2293 | 4116 | 2134 | 1763 | 1423 | 2218 | 2274 | 4387 | 4467 | 1037 |
| u | 1792 | 2069 | 707 | 1734 | 1396 | 1895 | 3518 | 1788 | 1448 | 1191 | 1737 | 1923 | 3480 | 3697 | 893 |
| v | 414 | 499 | 147 | 422 | 315 | 453 | 844 | 437 | 348 | 270 | 425 | 455 | 767 | 914 | 206 |
| w | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| x | 129 | 175 | 42 | 138 | 83 | 151 | 272 | 135 | 119 | 65 | 114 | 149 | 288 | 283 | 63 |
| y | 94 | 89 | 31 | 81 | 67 | 80 | 148 | 64 | 58 | 61 | 61 | 98 | 119 | 145 | 36 |
| z | 20 | 23 | 7 | 27 | 18 | 39 | 71 | 30 | 20 | 11 | 25 | 37 | 41 | 41 | 3 |
| à | 128 | 136 | 39 | 110 | 90 | 131 | 246 | 130 | 90 | 73 | 101 | 129 | 209 | 224 | 48 |
| â | 36 | 50 | 9 | 43 | 67 | 42 | 50 | 43 | 24 | 18 | 40 | 33 | 55 | 75 | 20 |
| æ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 3 | 0 | 2 |
| ç | 35 | 28 | 10 | 22 | 24 | 30 | 46 | 34 | 16 | 16 | 34 | 23 | 61 | 56 | 17 |
| è | 102 | 147 | 49 | 138 | 112 | 122 | 232 | 119 | 99 | 68 | 108 | 151 | 237 | 260 | 58 |
| é | 423 | 513 | 194 | 424 | 367 | 548 | 966 | 502 | 370 | 304 | 438 | 480 | 940 | 1019 | 221 |
| ê | 43 | 68 | 24 | 36 | 44 | 57 | 96 | 54 | 43 | 53 | 68 | 60 | 126 | 94 | 32 |
| ë | 1 | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| î | 17 | 20 | 12 | 15 | 11 | 15 | 42 | 11 | 8 | 15 | 26 | 13 | 32 | 28 | 12 |
| ï | 2 | 0 | 0 | 2 | 8 | 12 | 9 | 1 | 2 | 5 | 15 | 3 | 5 | 2 | 0 |
| ô | 20 | 20 | 27 | 15 | 23 | 15 | 41 | 14 | 13 | 38 | 50 | 15 | 37 | 45 | 24 |
| ù | 14 | 9 | 4 | 6 | 18 | 14 | 30 | 6 | 5 | 3 | 7 | 11 | 24 | 21 | 7 |
| û | 7 | 9 | 7 | 4 | 15 | 15 | 38 | 8 | 15 | 10 | 9 | 14 | 30 | 21 | 11 |
| œ | 5 | 5 | 2 | 8 | 7 | 9 | 9 | 5 | 3 | 5 | 7 | 0 | 13 | 12 | 6 |

| | English | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 |
| a | 2217 | 2761 | 990 | 2274 | 1865 | 2606 | 4805 | 2396 | 1993 | 1627 | 2375 | 2560 |
| b | 451 | 551 | 183 | 454 | 400 | 518 | 913 | 431 | 408 | 359 | 437 | 489 |
| c | 729 | 777 | 271 | 736 | 553 | 797 | 1521 | 702 | 653 | 451 | 643 | 757 |
| d | 1316 | 1548 | 557 | 1315 | 1135 | 1509 | 2681 | 1416 | 1096 | 933 | 1364 | 1566 |
| e | 3967 | 4543 | 1570 | 3814 | 3210 | 4237 | 7834 | 4014 | 3373 | 2690 | 3790 | 4331 |
| f | 596 | 685 | 279 | 595 | 515 | 687 | 1366 | 621 | 575 | 477 | 610 | 677 |
| g | 662 | 769 | 253 | 559 | 525 | 669 | 1163 | 624 | 517 | 409 | 644 | 650 |
| h | 2060 | 2530 | 875 | 1978 | 1693 | 2254 | 4379 | 2171 | 1766 | 1475 | 2217 | 2348 |
| i | 1823 | 2163 | 783 | 1835 | 1482 | 2097 | 3838 | 2011 | 1648 | 1196 | 1830 | 2033 |
| j | 22 | 13 | 4 | 22 | 7 | 26 | 42 | 24 | 16 | 7 | 16 | 28 |
| k | 200 | 284 | 82 | 198 | 153 | 216 | 416 | 216 | 146 | 131 | 217 | 234 |
| l | 1204 | 1319 | 520 | 1073 | 949 | 1239 | 2434 | 1152 | 861 | 789 | 1122 | 1102 |
| m | 656 | 829 | 333 | 690 | 571 | 763 | 1461 | 748 | 629 | 506 | 799 | 746 |
| n | 1851 | 2218 | 816 | 1771 | 1468 | 2174 | 3816 | 2085 | 1728 | 1266 | 1833 | 2125 |
| o | 1897 | 2237 | 828 | 1865 | 1586 | 2231 | 4091 | 1947 | 1698 | 1369 | 1948 | 2105 |
| p | 525 | 606 | 194 | 514 | 517 | 613 | 1040 | 527 | 442 | 325 | 486 | 581 |
| q | 19 | 21 | 13 | 33 | 17 | 25 | 39 | 33 | 20 | 23 | 23 | 32 |
| r | 1764 | 2019 | 711 | 1726 | 1357 | 1931 | 3674 | 1915 | 1561 | 1211 | 1720 | 1939 |
| s | 1942 | 2411 | 864 | 1918 | 1646 | 2192 | 4060 | 1966 | 1626 | 1344 | 1945 | 2152 |
| t | 2547 | 3083 | 1048 | 2704 | 2178 | 2955 | 5369 | 2765 | 2442 | 1759 | 2424 | 3046 |
| u | 704 | 861 | 298 | 745 | 663 | 899 | 1552 | 789 | 683 | 502 | 767 | 750 |
| v | 258 | 295 | 94 | 245 | 194 | 277 | 465 | 266 | 208 | 181 | 246 | 278 |
| w | 653 | 769 | 254 | 663 | 568 | 733 | 1332 | 695 | 560 | 410 | 632 | 721 |
| x | 29 | 37 | 8 | 60 | 26 | 49 | 74 | 65 | 25 | 31 | 20 | 35 |
| y | 401 | 475 | 145 | 467 | 330 | 464 | 843 | 379 | 328 | 255 | 457 | 418 |
| z | 18 | 31 | 15 | 19 | 33 | 37 | 52 | 24 | 18 | 20 | 39 | 40 |
| à | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| â | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| æ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ç | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| è | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| é | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ê | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ë | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| î | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ï | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ô | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ù | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| û | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| œ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table: Character counts per chapter in French, left part, and English, right part

Each characters is modeled by a vector of chapters.

# Singular Value Decomposition

There are as many as 40 characters: the 26 unaccented letters from *a* to *z* and the 14 French accented letters

**Singular value decomposition** (SVD) reduces these dimensions, while keeping the resulting vectors semantically close

$\mathbf{X}$ is the $m \times n$ matrix of the letter counts per chapter, in our case, $m = 30$ and $n = 40$.

We can rewrite $\mathbf{X}$ as:

$$\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^\intercal,$$

where $\mathbf{U}$ is a matrix of dimensions $m \times m$, $\Sigma$, a diagonal matrix of dimensions $m \times n$, and $\mathbf{V}$, a matrix of dimensions $n \times n$.

The diagonal terms of $\Sigma$ are called the **singular values** and are traditionally arranged by decreasing value.

We keep the highest values and set the rest to zero.

# Code Example

Jupyter Notebook 3.1-SVD

# Word Embeddings

We can extend singular value decomposition from characters to words.
The rows will represent the words in the corpus, and the columns, documents,
We can replace documents by a context of a few words to the left and to the right of the focus word: $w_i$.
A context $C_j$ is then defined by a window of $2K$ words centered on the word:

$$w_{i-K}, w_{i-K+1}, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{i+K-1}, w_{i+K},$$

where the context representation uses a bag of words.
We can even reduce the context to a single word to the left or to the right of $w_i$ and use bigrams.

# Word Embeddings

We store the word-context pairs $(w_i, C_j)$ in a matrix.
Each matrix element measures the association strength between word $w_i$ and context $C_j$, for instance mutual information.
Mutual information, often called pointwise mutual information (the strength of an association) is defined as:

$$I(w_i, w_j) = \log_2 \frac{P(w_i, w_j)}{P(w_i)P(w_j)} \approx \log_2 \frac{N \cdot C(w_i, w_j)}{C(w_i)C(w_j)}.$$

| D#\Words | $C_1$ | $C_2$ | $C_3$ | ... | $C_n$ |
|---|---|---|---|---|---|
| $w_1$ | $MI(w_1, C_1)$ | $MI(w_1, C_2)$ | $MI(w_1, C_3)$ | ... | $MI(w_1, C_n)$ |
| $w_2$ | $MI(w_2, C_1)$ | $MI(w_2, C_2)$ | $MI(w_2, C_3)$ | ... | $MI(w_2, C_n)$ |
| $w_3$ | $MI(w_3, C_1)$ | $MI(w_3, C_2)$ | $MI(w_3, C_3)$ | ... | $MI(w_3, C_n)$ |
| ... | ... | ... | ... | ... | ... |
| $w_m$ | $MI(w_m, C_1)$ | $MI(w_m, C_2)$ | $MI(w_m, C_3)$ | ... | $MI(w_m, C_n)$ |

# Word Embeddings

We compute the word embeddings with a singular value decomposition, where we truncate the matrix $\mathbf{U}\Sigma$ to 50, 100, 300, or 500 dimensions.
The word embeddings are the rows of this matrix.
We usually measure the similarity between two embeddings $\mathbf{u}$ and $\mathbf{v}$ with the cosine similarity:

$$\cos(\mathbf{u},\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\cdot||\mathbf{v}||},$$

ranging from -1 (most dissimilar) to 1 (most similar) or with the cosine distance ranging from 0 (closest) to 2 (most distant):

$$1 - \cos(\mathbf{u},\mathbf{v}) = 1 - \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\cdot||\mathbf{v}||}.$$

# Popular Word Embeddings

Embeddings from large corpora are obtained with iterative techniques
Some popular embedding algorithms with open source programs:

- word2vec: https://github.com/tmikolov/word2vec
- GloVe: Global Vectors for Word Representation https://nlp.stanford.edu/projects/glove/
- ELMo: https://allennlp.org/elmo
- fastText: https://fasttext.cc/

To derive word embeddings, you will have to apply these programs on a very large corpus
Embeddings for many languages are also publicly available. You just download them
gensim is a Python library to create word embeddings from a corpus.
https://radimrehurek.com/gensim/index.html