Time Series: A First Course with Bootstrap Starter

Contents

Lesson 12-1: Sampling Distributions

- In time series (and statistics in general) we want to know the uncertainty in our estimates.
- First we study the sampling distributions of statistics.

Remark 12.1.4. Improving on the Normal Approximation

- A confidence interval for the mean requires us to know the *long-run variance*,

$$\sigma^2_\infty = \sum_{h=-\infty}^{\infty} \gamma(h).$$

- The 95% confidence interval based on the sample mean and the normal approximation is

$$\overline{X}_n \pm 1.96\sqrt{\frac{\widehat{\sigma^2_\infty}}{n}}.$$

- There are some issues with the accuracy of this interval:

1. The accuracy of the normal approximation in finite sample (non-normality of marginal distribution).
2. Serial correlation in the time series, since

$$\text{Var}[\sqrt{n}\overline{X}_n] \to \sigma^2_\infty,$$

   and the convergence can be slow.
3. Estimation of σ^2_∞ by some estimator $\widehat{\sigma^2_\infty}$ (this could be constructed from sample autocovariances).

- So we might consider directly approximating the distribution of $\overline{X}_n$ via *resampling* from $\{X_t\}$. This is the idea of the *bootstrap*.

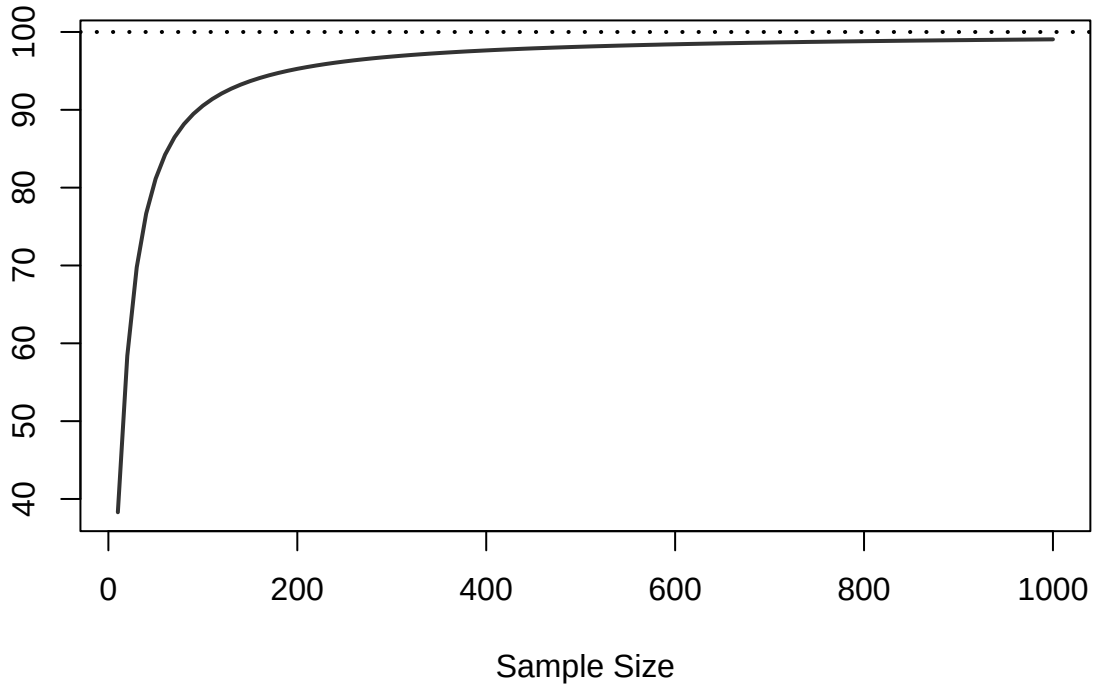
Example 12.1.6. Mean of a Gaussian AR(1).

- Suppose that $\{X_t\}$ is a stationary Gaussian AR(1) process, with parameter ϕ_1.
- So the spectral density is $f(\lambda) = \sigma^2 \left|1 - \phi_1 e^{-i\lambda}\right|^{-2}$.
- The asymptotic variance of $\sqrt{n}\overline{X}$ is the long-run variance: $f(0) = \sigma^2/(1-\phi_1)^2$.
- The actual variance of $\sqrt{n}\overline{X}$ is

$$\gamma(0) + 2\sum_{h=1}^{n-1}(1-h/n)\gamma(h) = \frac{\sigma^2}{1-\phi_1^2}\left(1 + \frac{2\phi_1}{1-\phi_1}\left[(1-\phi_1^n) - \frac{1-\phi_1^n(1+n(1-\phi_1))}{n(1-\phi_1)}\right]\right)$$

- For $\phi_1 > 0$, this quantity is smaller than the long-run variance.
- We illustrate for $\phi_1 = .9$ and $\sigma = 1$, so that $f(0) = 100$.

```
phi <- .9
n <- seq(10,1000,10)

longrun.var <- (1-phi^2)^(-1)*(1 + 2*phi*(1-phi)^(-1)*
    ((1-phi^n) - (1-phi^n*(1+n*(1-phi)))*(n*(1-phi))^(-1)))
longrun.var <- ts(longrun.var,start=10,frequency=.10)
plot(longrun.var,xlab="Sample Size",ylab="",lwd=2,col=grey(.2))
abline(h=1/(1-phi)^2,lty=3,lwd=2)
```



- Thus, using $f(0)$ instead of the true sampling variance will overestimate when n is small.

Lesson 12-2: Monte Carlo

- For this lesson, we suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. with common distribution G.

Fact 12.2.2. Parameters are Functionals of the Distribution

- Any parameter θ of a distribution can be expressed in terms of that distribution.
- Let G be the cumulative distribution function (cdf). Then we write θ as $\theta(G)$.

Example 12.2.5. Median Parameter

The median θ can be written as $G^{-1}(1/2)$, where G^{-1} denotes the quantile inverse. ($G^{-1}(p) = \inf\{x : G(x) \geq p\}$.)

Example 12.2.12. Monte Carlo Approximation to the Variance of a Statistic

- Suppose we want to know the variance of a statistic, $\eta = \text{Var}[\widehat{\theta}_n]$, but there is no nice analytic formula.
- Monte Carlo approach: generate multiple independent copies of $\widehat{\theta}_n$, and take the sample variance of these:

1. For a large integer M simulate:

$$X_1^{(1)}, X_2^{(1)}, \ldots, X_n^{(1)} \sim \text{i.i.d.} G$$
$$X_1^{(2)}, X_2^{(2)}, \ldots, X_n^{(2)} \sim \text{i.i.d.} G$$
$$\vdots$$
$$X_1^{(M)}, X_2^{(M)}, \ldots, X_n^{(M)} \sim \text{i.i.d.} G.$$

2. For $j = 1, \ldots, M$ compute $\widehat{\theta}_n^{(j)}$ from the pseudo-data $X_1^{(j)}, X_2^{(j)}, \ldots, X_n^{(j)}$.
3. Compute $\widehat{\mathbb{E}}[\widehat{\theta}_n] = M^{-1} \sum_{j=1}^{M} \widehat{\theta}_n^{(j)}$ and $\widehat{\eta} = M^{-1} \sum_{j=1}^{M} \left(\widehat{\theta}_n^{(j)} - \widehat{\mathbb{E}}[\widehat{\theta}_n]\right)^2$, which is our estimate of η.

Exercise 12.6. Monte Carlo Approximation to the Variance of the Median

- We use Monte Carlo to approximate the variance of the sample median.
- Consider a sample of size $n = 100$ from an AR(1) process with mean 2, AR parameter $\phi_1 = .8$, and Student t inputs with 4 degrees of freedom.
- First we load the function to simulate an ARMA with Student t inputs.

```
armapq.simht <- function(n,burn,ar.coefs,ma.coefs,innovar,df,seed)
{
    p <- length(ar.coefs)
    q <- length(ma.coefs)
    set.seed(seed)
    if(df == Inf)
    {
        z <- rnorm(n+burn+p+q,sd=sqrt(innovar))
    } else
    {
        z <- sqrt(innovar)*rt(n+burn+p+q,df=df)
    }
    x <- filter(z,c(1,ma.coefs),method="convolution",sides=1)
    x <- x[(q+1):(q+n+burn+p)]
    y <- x[1:p]
    for(t in (p+1):(p+n+burn))
    {
        next.y <- sum(ar.coefs*y[(t-1):(t-p)]) + x[t]
        y <- c(y,next.y)
    }
    y <- y[(p+burn+1):(p+burn+n)]
    return(y)
}

n <- 100
phi1 <- .8
theta <- 2
monte <- 10000
med.mcs <- NULL
for(i in 1:monte)
{
    x.sim <- theta + armapq.simht(n,500,phi1,NULL,1,4,set.seed(i))
    med.mcs <- c(med.mcs,median(x.sim))
}
print(mean(med.mcs^2)-(mean(med.mcs))^2)
```

```
## [1] 0.44356
```

Lesson 13-3: The Plug-in Principle and the Bootstrap

- We describe the plug-in method of constructing estimators, which generalizes the method-of-moments.

Definition 12.3.1.

- If $X_1, \ldots, X_n$ are i.i.d. with common cdf G, then their empirical distribution function (edf) is

$$\widehat{G}(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{X_i \leq x\}}.$$

Paradigm 12.3.5. The Plug-In Principle

- The edf converges uniformly to the cdf, so we can estimate θ by *plugging in*:

$$\widehat{\theta} = \theta(\widehat{G}).$$

- This is called a *plug-in estimator*.
- It generalizes method-of-moments.

Example 12.3.8. Plug-In Estimator of the Median

The plug-in estimator for the median is

$$\widehat{\theta} = \widehat{G}^{-1}(1/2),$$

which is the sample median.

Paradigm 12.3.9. Classical Bootstrap for the Variance of a Statistic

- Consider a scenario where we want to know the variance of a statistic $\widehat{\theta}_n$ (computed from a sample of size n).
- Let $\eta = \text{Var}[\widehat{\theta}_n]$. Since n is fixed and the sample is i.i.d., $\eta = \eta(G)$.
- We could estimate η with the plug-in estimator.
- We would like to compute $\eta(\widehat{G})$, but maybe there is no formula!
- So we try to approximate it using the *bootstrap*.

1. For large M simulate

$$\begin{aligned} X_1^{*(1)}, \ldots, X_n^{*(1)} &\sim \text{i.i.d.} \widehat{G} \\ X_1^{*(2)}, \ldots, X_n^{*(2)} &\sim \text{i.i.d.} \widehat{G} \\ &\vdots \\ X_1^{*(M)}, \ldots, X_n^{*(M)} &\sim \text{i.i.d.} \widehat{G}. \end{aligned}$$

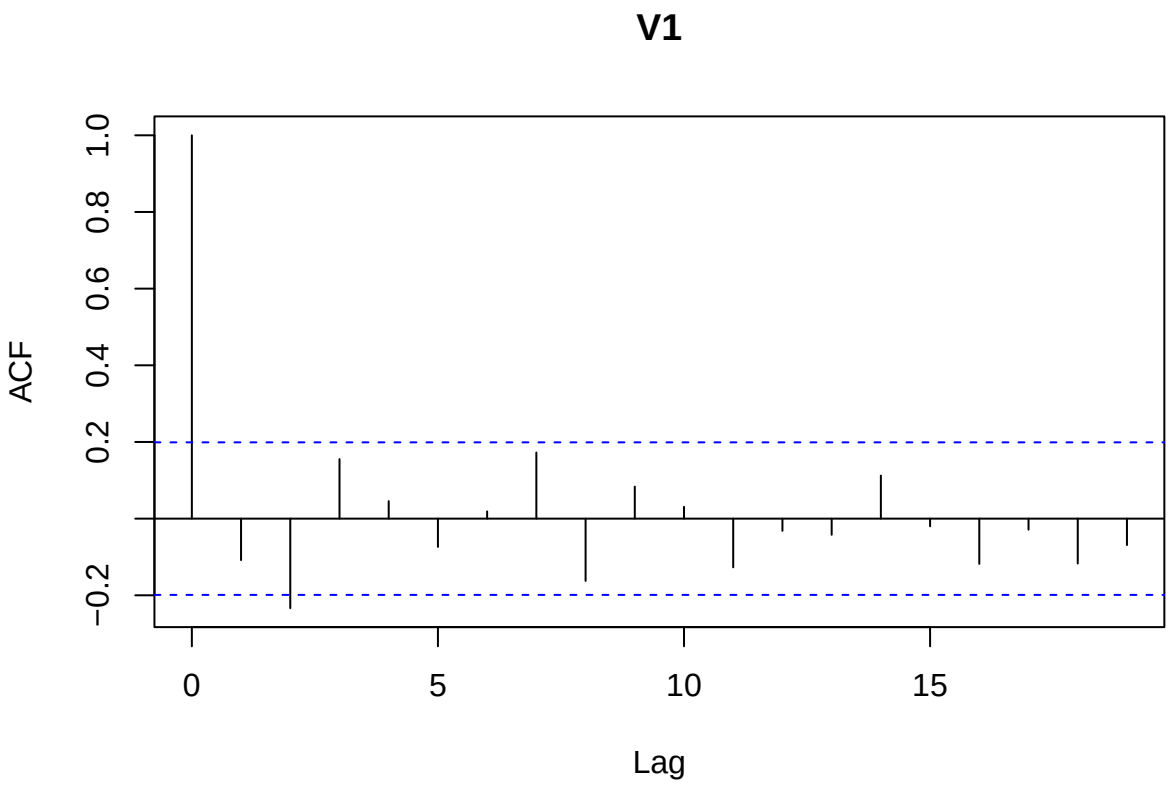
2. For $1 \leq j \leq M$ compute $\widehat{\theta}_n^{*(j)}$ from the *pseudo-sample* $X_1^{*(j)}, \ldots, X_n^{*(j)}$.
3. Our bootstrap estimator of η is

$$\frac{1}{M} \sum_{j=1}^{M} \left(\widehat{\theta}_n^{*(j)} - M^{-1} \sum_{k=1}^{M} \widehat{\theta}_n^{*(k)} \right)^2.$$

Example 12.3.10. Bootstrap for the Variance of U.S. Population Acceleration

- Consider the time series $\{Y_t\}$ of U.S. Population.
- One possible model is twice differencing, for which the series appears to be white noise.

```
pop <- read.table("USpop.dat")
pop <- ts(pop, start = 1901)
pop.diff <- diff(diff(pop))*10^(-3)
acf(pop.diff)
```



- Suppose that $X_t = (1-B)^2 Y_t$ is actually i.i.d.
- The sample mean of $\{X_t\}$ is 9.8227835 in units of millions.
- Suppose we want to estimate the variance of this sample mean using the bootstrap (of course we could use a formula as well) with $M = 10^5$.

```
n <- length(pop.diff)
pop.mean <- mean(pop.diff)
pop.edf <- sort(pop.diff)

monte.means <- NULL
Monte <- 100000
for(i in 1:Monte)
{
    monte.sample <- sample(pop.edf,size=n,replace=TRUE)
    monte.means <- c(monte.means,mean(monte.sample))
}
var.mean <- var(monte.means)
```

- The resulting estimate is 627.3034894 in units of millions.
- Note: results change each time notebook is rendered, because seed is not fixed!

Lesson 12-4: Model-based Bootstrap

- We want to extend the bootstrap idea to the case of time series data.
- We don't want to assume i.i.d. anymore, because time series data have serial dependence.

Paradigm 12.4.2. Bootstrapping an AR(1) Model

- Suppose $\{X_t\}$ is a stationary AR(1) process with i.i.d. inputs with cdf G:

$$X_t - \phi X_{t-1} = \epsilon_t,$$

  where $\epsilon_t \sim$ i.i.d.G.
- Suppose we want to estimate the cdf of $\widehat{\phi} - \phi$, where $\widehat{\phi}$ is the Yule-Walker estimator.
- So for any x, we want a bootstrap estimate of $\zeta = \mathbb{P}[\widehat{\phi} - \phi \leq x]$.
- We compute residuals

$$e_t = X_t - \widehat{\phi} X_{t-1},$$

  and use these as proxies for ϵ_t.
- Center the residuals by their sample mean, and let $\widehat{G}$ be their edf.

1. For large M simulate

$$\begin{aligned}
&\epsilon_1^{*(1)}, \ldots, \epsilon_n^{*(1)} \sim \text{i.i.d.} \widehat{G} \\
&\epsilon_1^{*(2)}, \ldots, \epsilon_n^{*(2)} \sim \text{i.i.d.} \widehat{G} \\
&\ldots \\
&\epsilon_1^{*(M)}, \ldots, \epsilon_n^{*(M)} \sim \text{i.i.d.} \widehat{G}.
\end{aligned}$$

2. For $1 \leq j \leq M$ construct

$$X_t^{*(j)} = \widehat{\phi} X_{t-1}^{*(j)} + \epsilon_t^{*(j)}$$

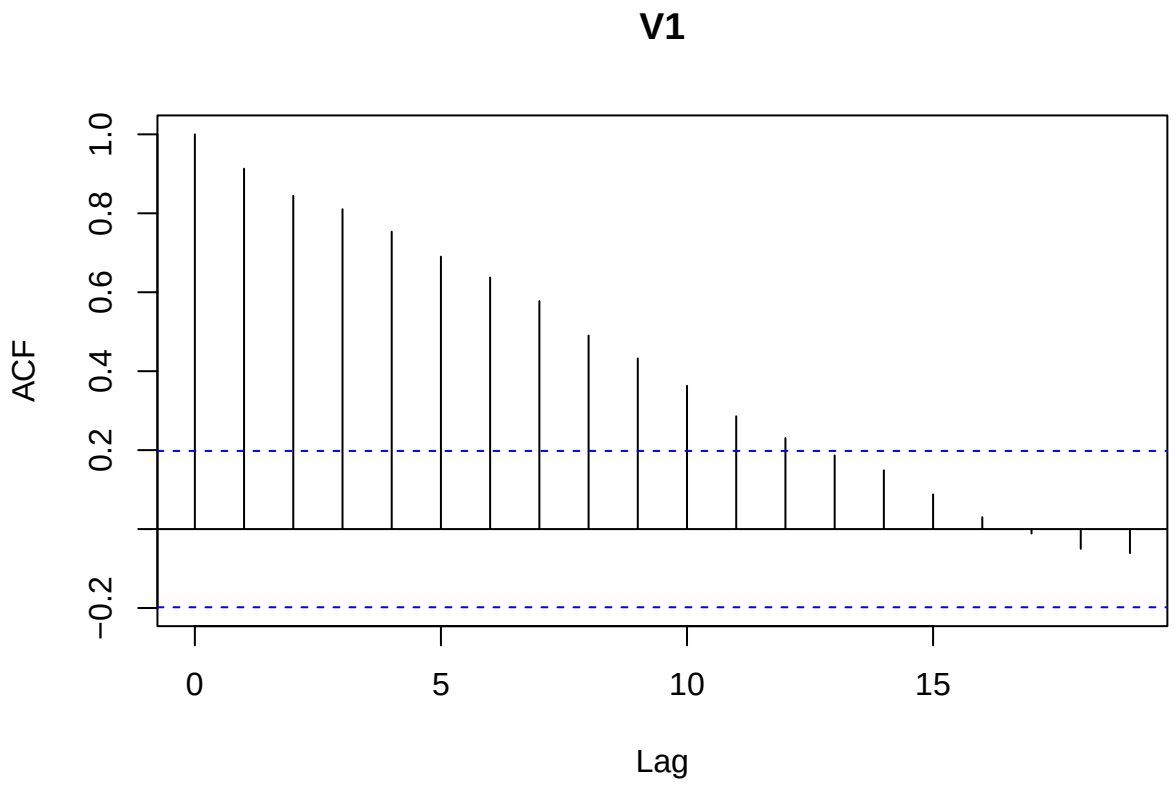
   for $1 \leq t \leq n$.
3. Compute $\widehat{\phi}^{*(j)}$ from the *pseudo-sample* $X_1^{*(j)}, \ldots, X_n^{*(j)}$.
4. Our bootstrap estimator of ζ is

$$\frac{1}{M} \sum_{j=1}^{M} 1_{\{\widehat{\phi}^{*(j)} - \widehat{\phi} \leq x\}}.$$

Example 12.4.3. Bootstrap for the AR(1) Coefficient of U.S. Population Growth

- For the U.S. Population time series $\{Y_t\}$, we can also consider fitting an AR(1) model to first differences.

```
pop <- read.table("USpop.dat")
pop <- ts(pop, start = 1901)
pop.diff <- diff(pop)*10^(-6)
acf(pop.diff)
```

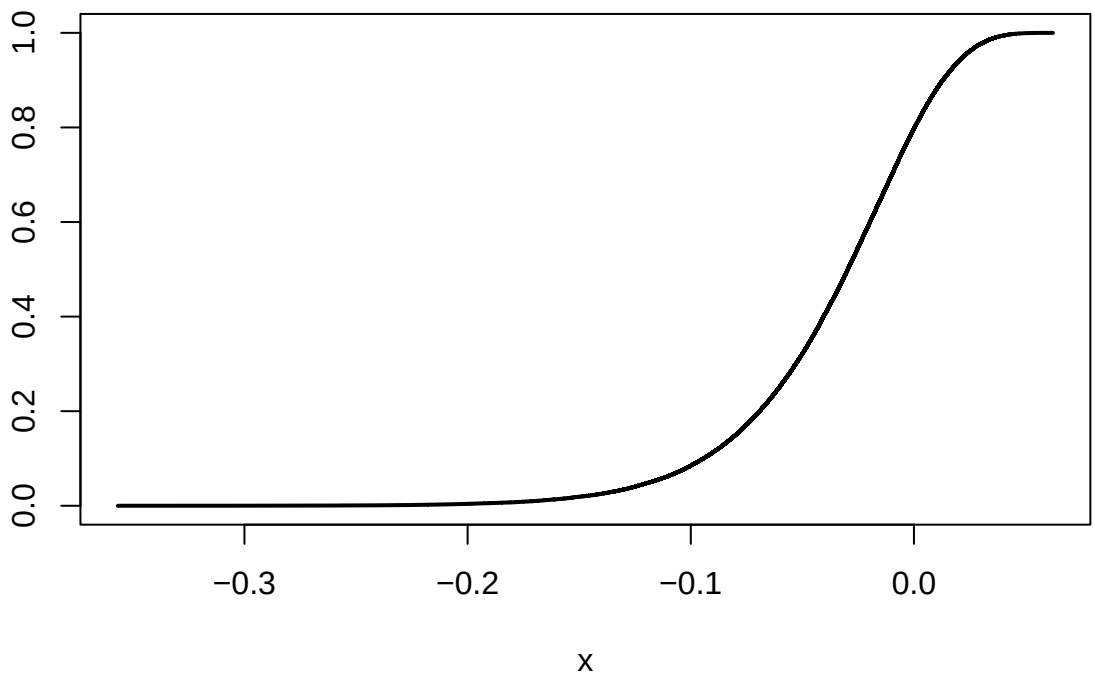


- Let $X_t = (1-B)Y_t$, and consider the AR(1) model for $\{X_t\}$.
- Suppose we want to estimate the cdf of $\widehat{\phi} - \phi$ using the bootstrap with $M = 10^5$.

```
n <- length(pop.diff)
kappa.hat <- pacf(pop.diff,lag=n-1,plot=FALSE)$acf[,,1]
pop.ar1 <- kappa.hat[1]
pop.resids <- pop.diff[2:n] - pop.ar1*pop.diff[1:(n-1)]
pop.resids <- pop.resids - mean(pop.resids)
pop.edf <- sort(pop.resids)

monte.roots <- NULL
Monte <- 100000
for(i in 1:Monte)
{
    monte.resids <- sample(pop.edf,size=n,replace=TRUE)
    init.value <- sample(pop.diff,size=1)
    monte.sample <- filter(monte.resids,pop.ar1,method="recursive",init=init.value)
    monte.root <- pacf(monte.sample,lag=n-1,plot=FALSE)$acf[,,1][1] - pop.ar1
    monte.roots <- c(monte.roots,monte.root)
}
# hist(monte.roots)
interval <- c(sort(monte.roots)[floor(.025*Monte)],sort(monte.roots)[floor(.975*Monte)])
```

- The AR(1) coefficient is estimated to be 0.9130219.
- The 95% confidence interval based on the bootstrap is [0.8838385,1.0534122].
- We plot the bootstrap edf.

```
plot(sort(monte.roots),seq(1,Monte)/Monte,type="l",xlab="x",ylab="",lwd=2)
```



Paradigm 12.4.7. Bootstrap and the Model-Free Principle

- Using a transformation (instead of a model) that produces i.i.d. residuals from the data process is called the *model-free principle*.
- So to do a time series bootstrap, we should seek such a transformation, bootstrap the residuals, reconstruct the process, and evaluate the statistic on the pseudo-samples.
- Suppose there exists an invertible transformation Π such that $\underline{\epsilon} = \Pi(\underline{X})$ is a vector of i.i.d. components, where $\underline{X} = [X_1, \ldots, X_n]'$.
- Let G denote the cdf of ϵ_t.

- Suppose we have a statistic $\widehat{\theta}_n$ and we want the cdf $\zeta = \mathbb{P}[\widehat{\theta}_n - \theta \leq x]$.
- Compute the residuals, and estimate G via the residual edf $\widehat{G}$.
- Then the *Model-free bootstrap* is:

1. For large M simulate

$$\begin{aligned}
&\epsilon_1^{*(1)}, \ldots, \epsilon_n^{*(1)} \sim \text{i.i.d.} \widehat{G} \\
&\epsilon_1^{*(2)}, \ldots, \epsilon_n^{*(2)} \sim \text{i.i.d.} \widehat{G} \\
&\ldots \\
&\epsilon_1^{*(M)}, \ldots, \epsilon_n^{*(M)} \sim \text{i.i.d.} \widehat{G}.
\end{aligned}$$

2. For $1 \leq j \leq M$ construct

$$\underline{X}^{*(j)} = \Pi^{-1}[\underline{\epsilon}^{*(j)}].$$

3. Compute $\widehat{\theta}_n^{*(j)}$ from the *pseudo-sample* $\underline{X}^{*(j)}$.
4. Our bootstrap estimator of ζ is

$$\frac{1}{M}\sum_{j=1}^{M} 1_{\{\widehat{\theta}_n^{*(j)}-\widehat{\theta}\leq x\}}.$$

Lesson 12-5: Sieve Bootstraps

- We now investigate two transformations Π: the AR sieve and the MA sieve.

Paradigm 12.5.2. Sieves

- Consider the case that the transformation Π involves infinitely many parameters.
- So we consider a sequence of transformations $\Pi_1, \Pi_2, \ldots$, where Π_j has j parameters.
- We suppose these transformations to be nested. This means that Π_j is obtained from Π_{j+1} by restricting the $j+1$th parameter to some constant value (such as zero).
- Also we suppose that taking the limit of these transformations gives Π.
- Such a collection is called a *sieve*.
- The *method of sieves* is to apply Π_m to the sample $X_1, \ldots, X_n$, with m chosen large enough that $\Pi_m \approx \Pi$, while also m is small enough that we can estimate all the parameters.
- If we get additional data (n increases), then we would also increase m.

Paradigm 12.5.4. Autoregressive Sieve and the AR Sieve Bootstrap

- Suppose we have an AR(∞) process:

$$\Xi(B)X_t = \epsilon_t \sim \text{i.i.d.}G.$$

- We take Π_p to be an AR(p) model. This is called the *AR sieve*.
- Notation:

$$\phi^{(p)}(B) = 1 - \sum_{j=1}^{p} \phi_j^{(p)} B^j.$$

- So as p increases, all the coefficients can change (and we get more coefficients, too).
- The order p can be linked to sample size n by a formula. Or p can be determined empirically, as a statistic of the sample.
- Once this Π_p is determined, we can do an AR(p) bootstrap (generalizing the $p = 1$ case considered in previous notebook).
- Because p grows with n, this is called the *AR sieve bootstrap*.

Example 12.5.6. Lag 12 Autocorrelation of Gasoline Sales

- Consider the seasonally adjusted gasoline sales data.
- Apply logs and differences, obtaining a linear process (this is an assumption).

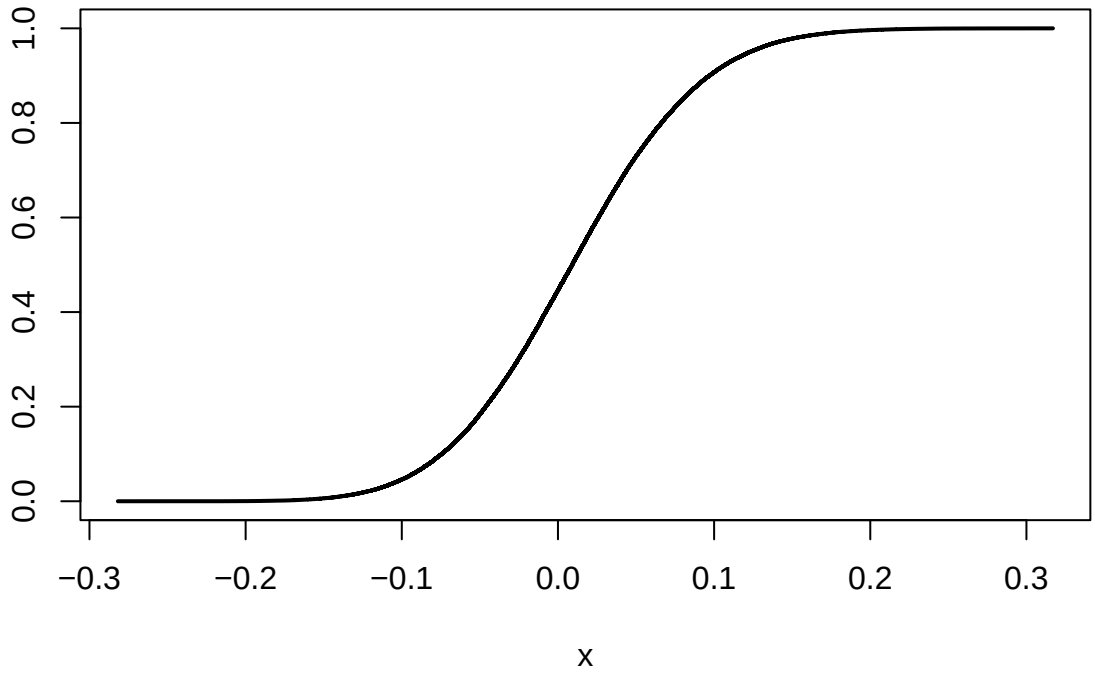
```
gassa <- read.table("GasSA_2-11-13.dat")
gassa <- ts(log(gassa),start=1992,frequency=12)
gas.diff <- diff(gassa)
n <- length(gas.diff)
```

- We want to estimate $\rho(12)$, and get the cdf of $\widehat{\rho}(12) - \rho(12)$.
- We use an AR sieve bootstrap with $p = 12$ (based on analysis of the PACF plot) for this sample size. There are $M = 10^5$ replications, and the pseudo-samples are constructed using a burnin of 500.

```
rho.hat <- acf(gas.diff,lag=n-1,type="correlation",plot=FALSE)$acf[,,1]
gas.acf12 <- rho.hat[13]
p.order <- 12
phi.ar <- solve(toeplitz(rho.hat[1:p.order])) %*% rho.hat[2:(p.order+1)]

gas.resids <- gas.diff[(p.order+1):n]
for(i in 1:p.order) { gas.resids <- gas.resids - phi.ar[i]*gas.diff[(p.order+1-i):(n-i)] }
gas.resids <- gas.resids - mean(gas.resids)
gas.edf <- sort(gas.resids)

monte.roots <- NULL
burnin <- 500
Monte <- 100000
for(i in 1:Monte)
{
    monte.resids <- sample(gas.edf,size=n+burnin,replace=TRUE)
    init.value <- rep(0,p.order)
    monte.sample <- filter(monte.resids,phi.ar,method="recursive",init=init.value)[(burnin+1):(burnin+n)
    monte.root <- acf(monte.sample,lag=n-1,plot=FALSE)$acf[,,1][13] - gas.acf12
    monte.roots <- c(monte.roots,monte.root)
}
# hist(monte.roots)
interval <- c(sort(monte.roots)[floor(.025*Monte)],sort(monte.roots)[floor(.975*Monte)])
```

- The lag 12 autocorrelation is estimated to be -0.239469.
- The 95% confidence interval based on the bootstrap is [-0.3850592,-0.1226657].
- We plot the bootstrap edf.

```
plot(sort(monte.roots),seq(1,Monte)/Monte,type="l",xlab="x",ylab="",lwd=2)
```



Paradigm 12.5.8. Linear Process Bootstrap

- Consider a stationary process $\{Y_t\}$ with mean μ and acvf $\gamma(h)$.
- For a sample of size n, we have Γ_n is the Toeplitz covariance matrix of the sample. Recall that $\Gamma_n = \{\gamma(j-k)\}$.
- We can taper the sample autocovariance estimators in order to estimate the whole matrix:

$$\breve{\gamma}(h) = \Lambda(h/d)\widehat{\gamma}(h).$$

- Here Λ is a *taper*, which is a symmetric function on $[-1, 1]$ with non-negative values, which down-weights $\widehat{\gamma}$ when $|h|$ is large.
- Also d is the *bandwidth*, which is chosen by the user, and typically satisfies $d/n \to 0$.
- For example, Λ can be a trapezoid function.
- Then we construct $\breve{\Gamma}_n$ by inserting $\breve{\gamma}(h)$ for $\gamma(h)$, and ensuring the matrix is positive-definite.
- There is a Cholesky decomposition of the matrix, of the form

$$\breve{\Gamma} = L\,D\,L',$$

  where L is unit lower-triangular and D is diagonal with positive entries.
- Then we can transform the data to residuals by first subtracting the sample mean, and then multiplying the sample vector by $D^{-1/2}L^{-1}$.
- To the resulting residuals we apply the i.i.d. bootstrap; this whole procedure is called the *Linear process bootstrap*.
- When using a trapezoidal taper, $\breve{\gamma}(h) = 0$ if $|h| > d$, so we can think of these autocovariance estimates as corresponding to an MA(d) process.
- If we use some taper such that $\breve{\gamma}(h)$ truncates at $h = q$, the structure resembles that of an MA(q) process, and the resulting procedure is called the *MA sieve bootstrap*.

Example 12.5.11. Lag 1 Autocovariance of Non-Defense Capitalization

- Consider the Non-Defense Capitalization time series.
- After differencing, we wish to estimate $\gamma(1)$ and compute the cdf $\zeta = \mathbb{P}[\widehat{\gamma}(1) - \gamma(1) \leq x]$.

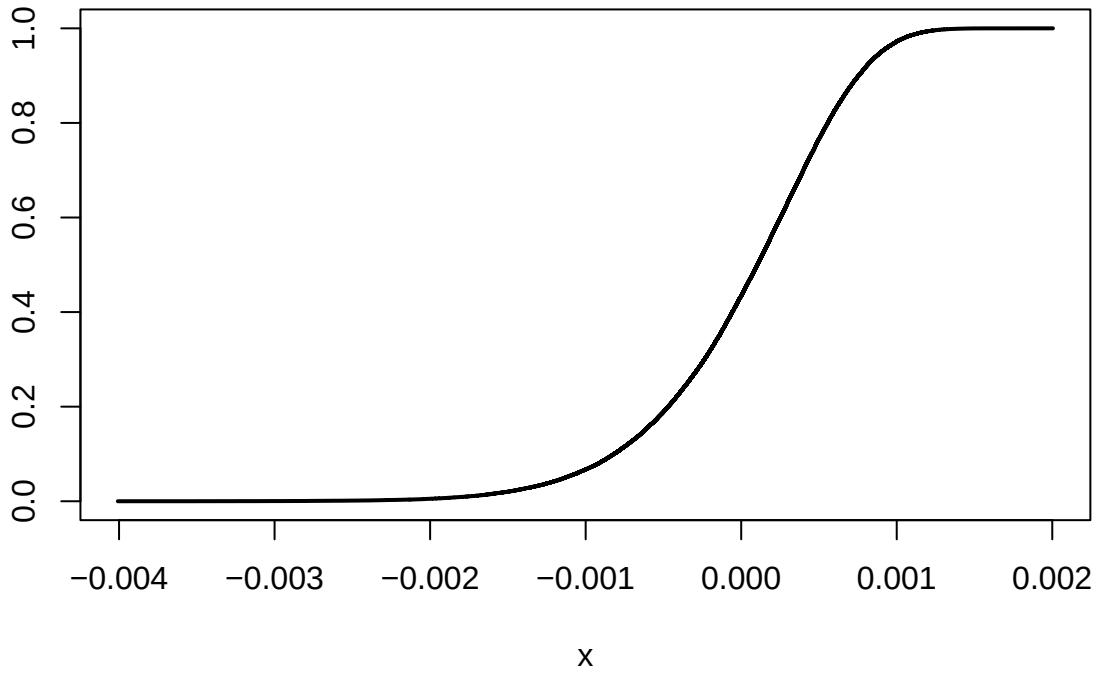
```
ndc <- read.table("Nondefcap.dat")
ndc <- ts(ndc[,2],start=c(1992,3),frequency=12,names= "NewOrders")
ndc.diff <- diff(ndc)
n <- length(ndc.diff)
```

- We use an MA sieve bootstrap with the truncation taper and $q = 10$ for this sample size. There are $M = 10^5$ replications.

```
gamma.hat <- acf(ndc.diff,lag=n-1,type="covariance",plot=FALSE)$acf[,,1]
ndc.acf1 <- gamma.hat[2]
q.order <- 10
gamma.mat <- toeplitz(c(gamma.hat[1:(q.order+1)],rep(0,n-(q.order+1))))

gamma.chol <- t(chol(gamma.mat))
ndc.resids <- solve(gamma.chol,ndc.diff)
ndc.resids <- ndc.resids - mean(ndc.resids)
ndc.edf <- sort(ndc.resids)

monte.roots <- NULL
Monte <- 100000
for(i in 1:Monte)
{
    monte.resids <- sample(ndc.edf,size=n,replace=TRUE)
    monte.sample <- gamma.chol %*% monte.resids
    monte.root <- acf(monte.sample,lag=n-1,plot=FALSE,type="covariance")$acf[,,1][2] - ndc.acf1
    monte.roots <- c(monte.roots,monte.root)
}
# hist(monte.roots)
interval <- c(sort(monte.roots)[floor(.025*Monte)],sort(monte.roots)[floor(.975*Monte)])
```

- The lag 1 autocovariance is estimated to be -0.002489.
- The 95% confidence interval based on the bootstrap is [-0.003506,-0.0010687].
- We plot the bootstrap edf.

```
plot(sort(monte.roots),seq(1,Monte)/Monte,type="l",xlab="x",ylab="",lwd=2)
```



Lesson 12-6: Time Frequency Toggle Bootstrap

- We study a frequency domain sieve.

Paradigm 12.6.1. Spectral Sieve

- The DFT $\underline{\widetilde{X}}$ is obtained from the sample $\underline{X}$ via multiplication by Q:

$$\underline{\widetilde{X}} = Q\,\underline{X}.$$

- The DFT has asymptotic covariance given by diagonal matrix Λ; the diagonal entries are the spectral density evaluated at Fourier frequencies.
- So $\Lambda^{-1/2}\underline{\widetilde{X}}$ is asymptotically uncorrelated with common variance 1.
- Then

$$\underline{\epsilon} = Q\,\Lambda^{-1/2}\,Q^*\,\underline{X}$$

  is an entropy-increasing transformation of the sample, and is an asymptotically uncorrelated sequence.
- We can base a bootstrap on $\underline{\epsilon}$; this is called a *spectral sieve*, because it involves an approximation to the true spectral density, which improves as the grid of Fourier frequencies becomes finer.
- To implement, we need an estimate of the spectral density.

Example 12.6.3. Wolfer Sunspot Spectral Sieve

- Consider the Wolfer sunspot time series.
- We estimate the DFT.

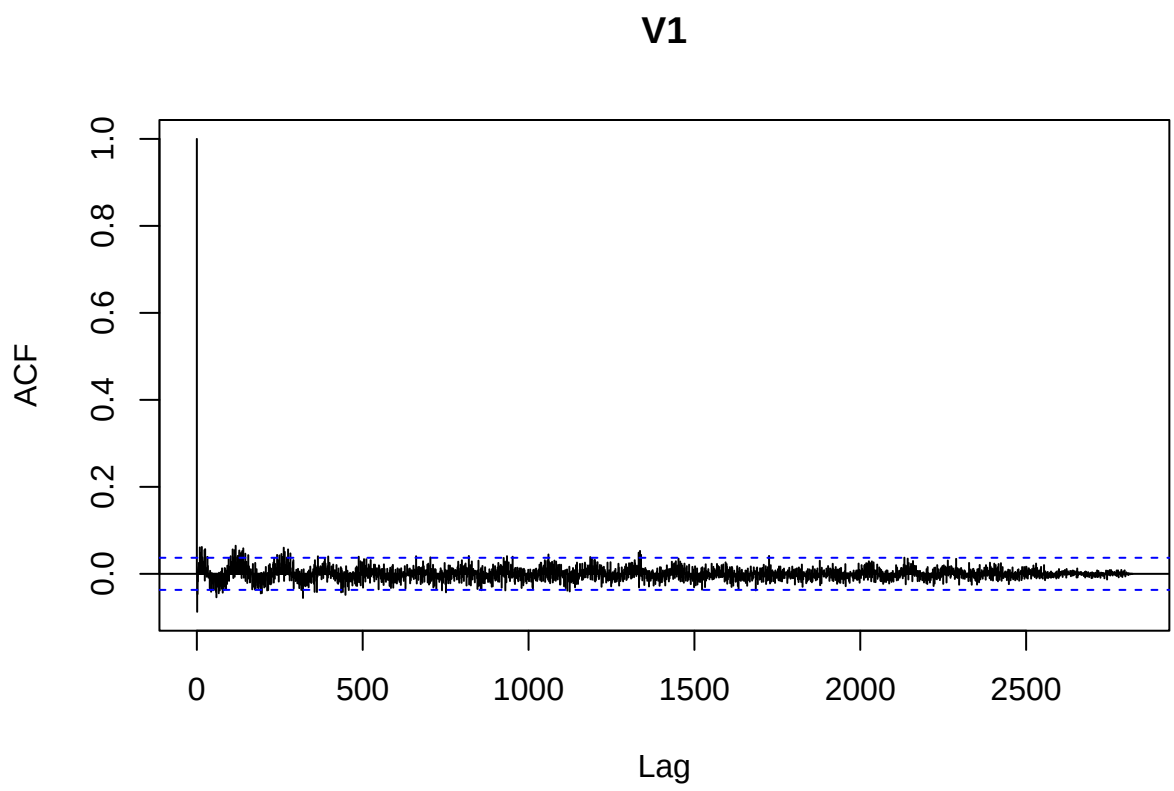
```
wolfer <- read.table("wolfer.dat")
wolfer <- ts(wolfer,start=1749,frequency=12)
n <- length(wolfer)
gamma.hat <- acf(wolfer,lag=n-1,type="covariance",plot=FALSE)$acf[,,1]

lambda <- seq(-floor(n/2)+1,floor(n/2))*2*pi/n
Q.mat <- exp(1i*seq(-floor(n/2)+1,floor(n/2)) %x% t(lambda))/sqrt(n)
wolfer.dft <- Conj(Q.mat) %*% (wolfer-mean(wolfer))
```

- We apply a tapered spectral estimator with Bartlett taper to estimate f.

```
d <- 3*floor(n^{1/3})
wolfer.spec <- cos(0*lambda)*gamma.hat[1]
for(h in 1:(n-1))
{
    wolfer.spec <- wolfer.spec + 2*(max(1-h/d,0))*cos(h*lambda)*gamma.hat[h+1]
}
wolfer.spec <- ts(wolfer.spec,start=0,frequency=n)
```

- Then we construct the residuals, and check that they are a white noise.
- The decorrelation is imperfect, but is sufficiently low statistically.

```
wolfer.sieved <- wolfer.dft/sqrt(wolfer.spec)
wolfer.resids <- Re(Q.mat %*% wolfer.sieved)
gamma.resids <- acf(wolfer.resids,lag=n-1,type="correlation")$acf[,,1]
```



Paradigm 12.6.8. Time Frequency Toggle Bootstrap

- We construct a bootstrap based on the spectral sieve.
- Let $\widehat{f}$ be a spectral density estimate, and $\widehat{\Lambda}$ is diagonal with entries given by evaluating at the Fourier frequencies.
- The *frequency domain residuals* are defined as $\widehat{\Lambda}^{-1/2}\underline{\widetilde{X}}$; these are asymptotically i.i.d. and Gaussian, by theory from Chapter 9.
- So we can generate bootstrap copies of the frequency domain residuals, transform to time domain by applying Q, and evaluate our statistic.
- This is called the *time frequency toggle* (TFT) bootstrap.

Example 12.6.11. Lag 12 Autocorrelation of Gasoline Sales via TFT

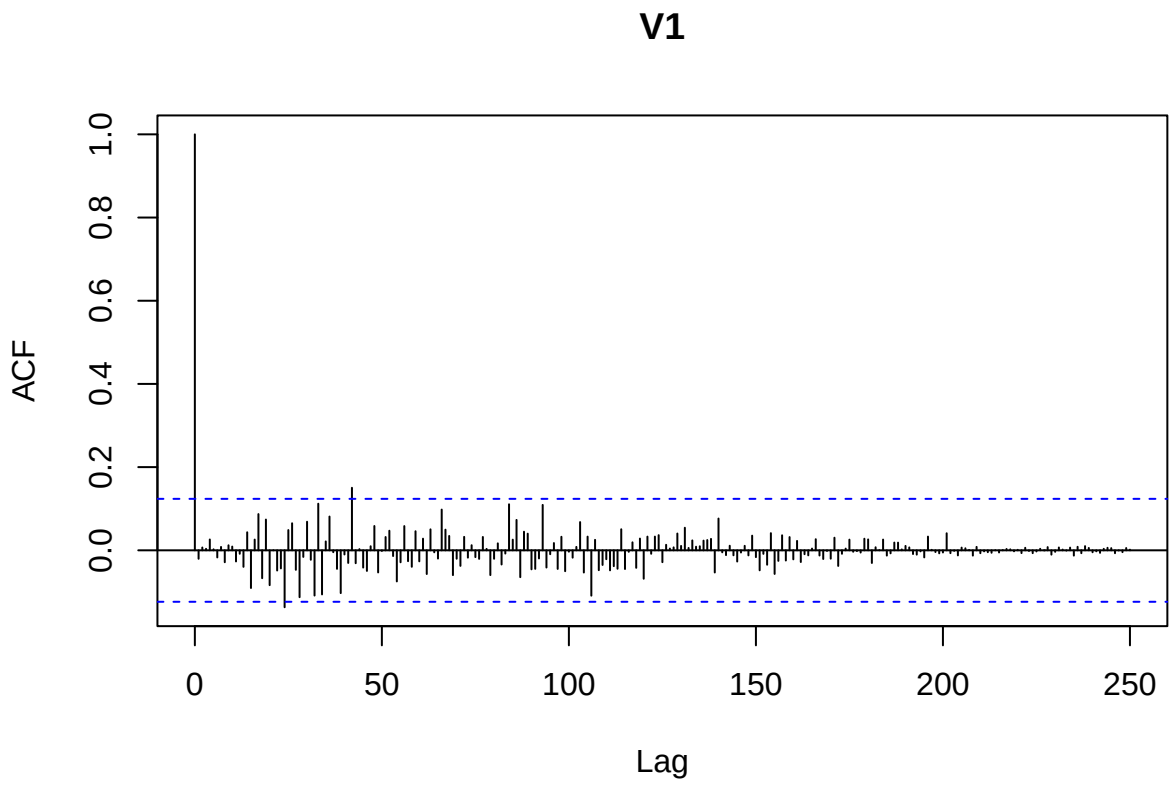
- Consider the Gasoline sales time series data.
- We want to estimate the lag 12 autocorrelation, and measure the uncertainty using the TFT bootstrap.

```
gassa <- read.table("GasSA_2-11-13.dat")
gassa <- ts(log(gassa),start=1992,frequency=12)
gas.diff <- diff(gassa)
n <- length(gas.diff)
gamma.hat <- acf(gas.diff,lag=n-1,type="covariance",plot=FALSE)$acf[,,1]
rho.hat <- acf(gas.diff,lag=n-1,type="correlation",plot=FALSE)$acf[,,1]
gas.acf12 <- rho.hat[13]
print(gas.acf12)
```

```
## [1] -0.239469
```

- We compute the DFT, and construct a spectral estimator based on a fitted AR(12).

```
lambda <- seq(-floor(n/2),floor(n/2))*2*pi/n
Q.mat <- exp(1i*seq(-floor(n/2),floor(n/2)) %x% t(lambda))/sqrt(n)
gas.dft <- Conj(Q.mat) %*% (gas.diff-mean(gas.diff))

p.order <- 12
phi.ar <- solve(toeplitz(gamma.hat[1:p.order])) %*% gamma.hat[2:(p.order+1)]
sig2.ar <- gamma.hat[1] - sum(phi.ar*gamma.hat[2:(p.order+1)])
gas.spec <- rep(1,n)
for(j in 1:p.order) { gas.spec <- gas.spec - phi.ar[j]*exp(-1i*j*lambda) }
gas.spec <- sig2.ar*Mod(gas.spec)^{-2}
gas.sieved <- gas.dft/sqrt(gas.spec)
#plot(ts(Re(gas.sieved)))
#ceps.sieve <- Q.mat %*% Lambda.mat %*% Conj(Q.mat)
gas.resids <- Re(Q.mat %*% gas.sieved)
gas.tft <- gas.sieved[1:(floor(n/2)+1)]
gamma.resids <- acf(gas.resids,lag=n-1,type="correlation")$acf[,,1]
```

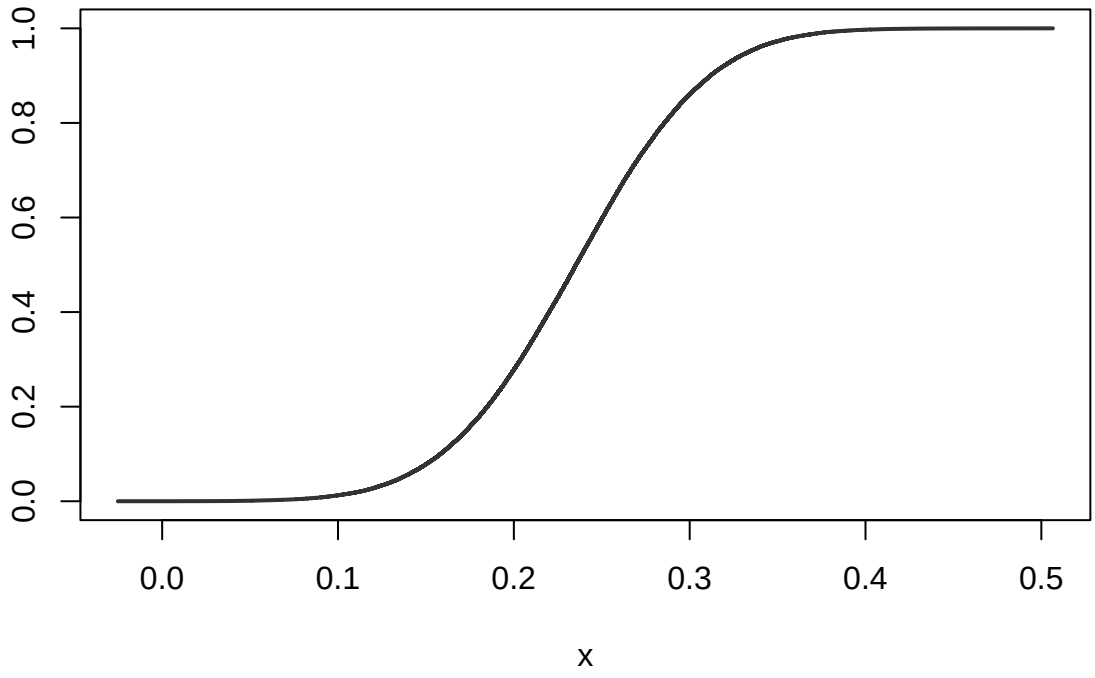


- Next, we do the bootstrapping on the frequency domain residuals.

```
gas.tft <- gas.tft - mean(gas.tft)
gas.edf.re <- sort(Re(gas.tft))
gas.edf.im <- sort(Im(gas.tft))

monte.roots <- NULL
Monte <- 100000
for(i in 1:Monte)
{
    monte.resids <- sample(gas.edf.re,size=(floor(n/2)+1),replace=TRUE) +
                1i*sample(gas.edf.im,size=(floor(n/2)+1),replace=TRUE)
    monte.resids[(floor(n/2)+1)] <- Re(monte.resids[(floor(n/2)+1)])
    monte.resids <- c(monte.resids,Conj(rev(monte.resids)[-1]))
    monte.sample <- Q.mat %*% monte.resids*sqrt(gas.spec)
    monte.sample <- Re(monte.sample)
    monte.root <- acf(monte.sample,lag=n-1,plot=FALSE,type="correlation")$acf[,,1][13] - gas.acf12
    monte.roots <- c(monte.roots,monte.root)
}
print(c(sort(monte.roots)[floor(.025*Monte)],sort(monte.roots)[floor(.975*Monte)]))
```

```
## [1] 0.1176180 0.3520214
```

```
plot(sort(monte.roots),seq(1,Monte)/Monte,type="l",xlab="x",ylab="",lwd=2,col=grey(.2))
```



Lesson 12-7: Subsampling

- An alternative to the bootstrap is to replicate properties of the sample by examining *subsamples*.

Paradigm 12.7.1. Roots and Subsampling

- Suppose that $\{X_t\}$ is strictly stationary.
- We also suppose that the process is m-dependent, which says that serial dependence vanishes between variables that are more than lag m apart.

- Suppose $\widehat{\theta}_n$ is an estimator of θ based on the sample $X_1, \ldots, X_n$.
- Suppose that τ_n is a rate of convergence for the estimator, such that

$$\tau_n(\widehat{\theta}_n - \theta) \Rightarrow S,$$

  where S is some random variable with cdf J.

- The cdf of our centered statistic is

$$J_n(x) = \mathbb{P}[\tau_n(\widehat{\theta}_n - \theta) \leq x].$$

- So $J_n(x) \to J(x)$ as $n \to \infty$.
- We can compute the statistic on smaller sub-spans of the full sample, and the corresponding cdf will also converge.

Paradigm 12.7.2. Blocking Schemes

- We can divide $X_1, \ldots, X_n$ into smaller blocks.
- Here we focus on overlapping blocks of size b, where $b < n$.
- There are $Q = n - b + 1$ such blocks, or *subsamples*. For $1 \leq i \leq Q$

$$X_i, \ldots, X_{i+b-1}$$

  is the ith subsample.
- Adjacent blocks have $b - 1$ values in common.

Paradigm 12.7.5. Subsampling Methodology

- Consider overlapping blocks, and evaluate the statistic on each:

$$\widehat{\theta}_{b,i} = \widehat{\theta}(X_i, \ldots, X_{i+b-1}).$$

- The centered statistic is then

$$Z_{b,i} = \tau_b(\widehat{\theta}_{b,i} - \theta).$$

  Note the rate is τ_b, not τ_n.
- As $b \to \infty$, the cdf of $Z_{b,i}$ tends to J, for each i.
- Fixing b, each $Z_{b,i}$ has the same distribution, and they are dependent random variables. The idea is to take their edf to estimate $J_n(x)$.
- However, θ in $Z_{b,i}$ is unknown, so we replace it by $\widehat{\theta}_n$ based on the whole sample, which converges at a faster rate.

$$\widehat{Z}_{b,i} = \tau_b(\widehat{\theta}_{b,i} - \widehat{\theta}).$$

- The *classical subsampling estimator* is then defined as

$$L_{n,b}(x) = \frac{1}{Q} \sum_{i=1}^{Q} 1_{\{\widehat{Z}_{b,i} \leq x\}}.$$

- This is an edf. Its quantiles are the order statistics of $\widehat{Z}_{b,i}$.
- Neighboring blocks share lots of time series observations, and the corresponding $\widehat{Z}_{b,i}$ random variables will be more highly correlated.
- By the m-dependence assumption, blocks that are sufficiently separated (far apart from one another) will be uncorrelated.
- Assuming that $\tau_b/\tau_n \to 0$, the result is a consistent estimator of $J(x)$, and the difference $L_{n,b}(x) - J_n(x)$ tends to zero in probability.

Example 12.7.13. Subsampling Inference for Lag 1 Autocovariance of Non-Defense Capitalization

- Consider the Non-Defense Capitalization time series.
- After differencing, we wish to estimate $\gamma(1)$ and compute the cdf $\zeta = \mathbb{P}[\widehat{\gamma}(1) - \gamma(1) \leq x]$.

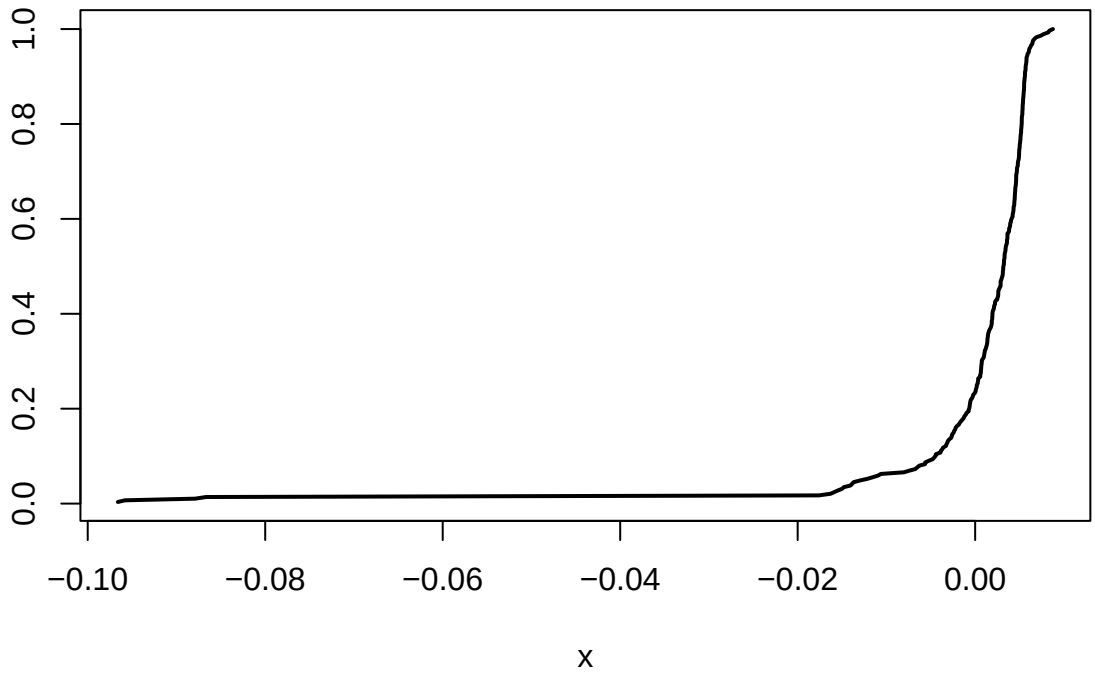
```
ndc <- read.table("Nondefcap.dat")
ndc <- ts(ndc[,2],start=c(1992,3),frequency=12,names= "NewOrders")
ndc.diff <- diff(ndc)
n <- length(ndc.diff)
gamma.hat <- acf(ndc.diff,lag=n-1,type="covariance",plot=FALSE)$acf[,,1]
ndc.acf1 <- gamma.hat[2]
```

- The lag 1 autocovariance is estimated to be -0.002489.
- We use the subsampling methodology, with $\tau_n = \sqrt{n}$.
- First consider $b = 5$.

```
b.sub <- 5
q.sub <- floor(n-b.sub+1)
sub.edf <- NULL
for(i in 1:q.sub)
{
    sub.ndc <- ndc.diff[i:(i+b.sub-1)]
    gamma.sub <- acf(sub.ndc,lag=b.sub-1,type="covariance",plot=FALSE)$acf[,,1]
    sub.edf <- c(sub.edf,sqrt(b.sub)*(gamma.sub[2]-ndc.acf1))
}
sub.edf <- sort(sub.edf)
interval <- c(sub.edf[floor(.025*q.sub)],sub.edf[floor(.975*q.sub)])
```

- The 95% confidence interval based on subsampling is [-0.0089642,0.0133861].
- We plot the subsampling edf.

```
plot(sub.edf,seq(1,q.sub)/q.sub,type="l",xlab="x",ylab="",lwd=2)
```



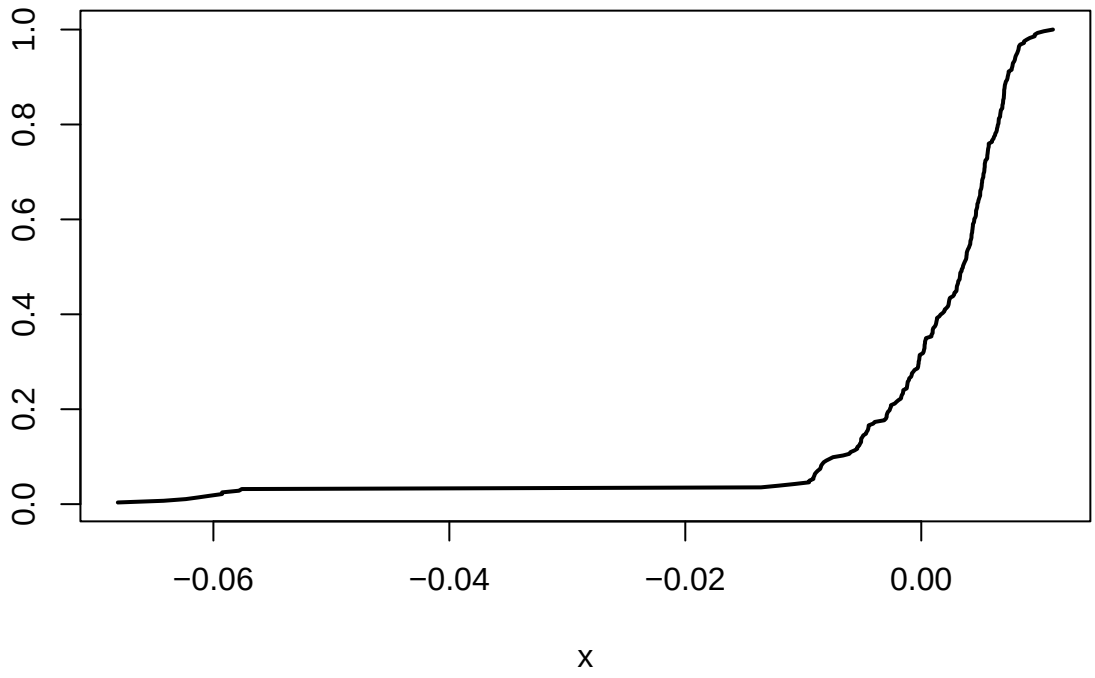
- Repeat with $b = 10$.

```
b.sub <- 10
q.sub <- floor(n-b.sub+1)
sub.edf <- NULL
for(i in 1:q.sub)
{
    sub.ndc <- ndc.diff[i:(i+b.sub-1)]
    gamma.sub <- acf(sub.ndc,lag=b.sub-1,type="covariance",plot=FALSE)$acf[,,1]
    sub.edf <- c(sub.edf,sqrt(b.sub)*(gamma.sub[2]-ndc.acf1))
```

```
}
sub.edf <- sort(sub.edf)
interval <- c(sub.edf[floor(.025*q.sub)],sub.edf[floor(.975*q.sub)])
```

- The 95% confidence interval based on subsampling is [-0.0111949,0.0567725].
- We plot the subsampling edf.

```
plot(sub.edf,seq(1,q.sub)/q.sub,type="l",xlab="x",ylab="",lwd=2)
```



Lesson 12-8: Block Bootstrap

- We now look at *block bootstrap methods*, which is similar in spirit to subsampling.
- We take sub-spans of the time series, and then patch them together to generate a synthetic time series that functions as our pseudo-sample.

Paradigm 12.8.2. Block Bootstrap for the Sample Mean

- Suppose we have a stationary time series $\{X_t\}$.
- Suppose θ is the mean, and $\widehat{\theta}$ is the sample mean. Then

$$J_n(x) = \mathbb{P}[\sqrt{n}(\widehat{\theta} - \theta) \leq x] \to J(x),$$

  where $J(x) = \Phi(x/\sigma_\infty)$.
- Let b be small relative to n, as in the subsampling method.
- Divide the sample into $Q = n - b + 1$ overlapping blocks.
- We want to construct a length n pseudo-sample from length b blocks, so we require $k = n/b$ such blocks.

- Idea: randomly draw from the Q blocks, and then paste them together!
- Denote those Q blocks of random variables as $B_1, \ldots, B_Q$.

1. For $1 \leq j \leq M$, draw $B_1^{*(j)}, \ldots, B_k^{*(j)}$ randomly (with replacement) from the set of available blocks $B_1, \ldots, B_Q$.
2. Concatenate for each j those draws, obtaining the jth pseudo-series $X_1^{*(j)}, \ldots, X_n^{*(j)}$.
3. Compute the statistic on each pseudo-series, denoted $\widehat{\theta}_n^{*(j)}$.
4. We estimate $J_n(x)$ with
$$\frac{1}{M} \sum_{j=1}^{M} 1_{\{\sqrt{n}(\widehat{\theta}_n^{*(j)} - \widehat{\theta}_n) \leq x\}}.$$

- Just like subsampling, we insert $\widehat{\theta}_n$ for θ.
- There is a variant called the *tapered block bootstrap*, where observations at the edge of a block B_k get down-weighted.

Exercise 12.33. Block Bootstrap for the Mauna Loa Mean

- Consider annual differences of the logged Mauna Loa CO2 time series.
- We want to estimate the mean using block bootstrap. So we apply the above method to obtain the cdf of the centered statistic.

```
mau <- read.table("mauna.dat",header=TRUE,sep="")
mau <- ts(mau,start=1958,frequency=12)
mau.gr <- 1000*diff(log(mau),lag=12)
n <- length(mau.gr)
mau.mean <- mean(mau.gr)
```

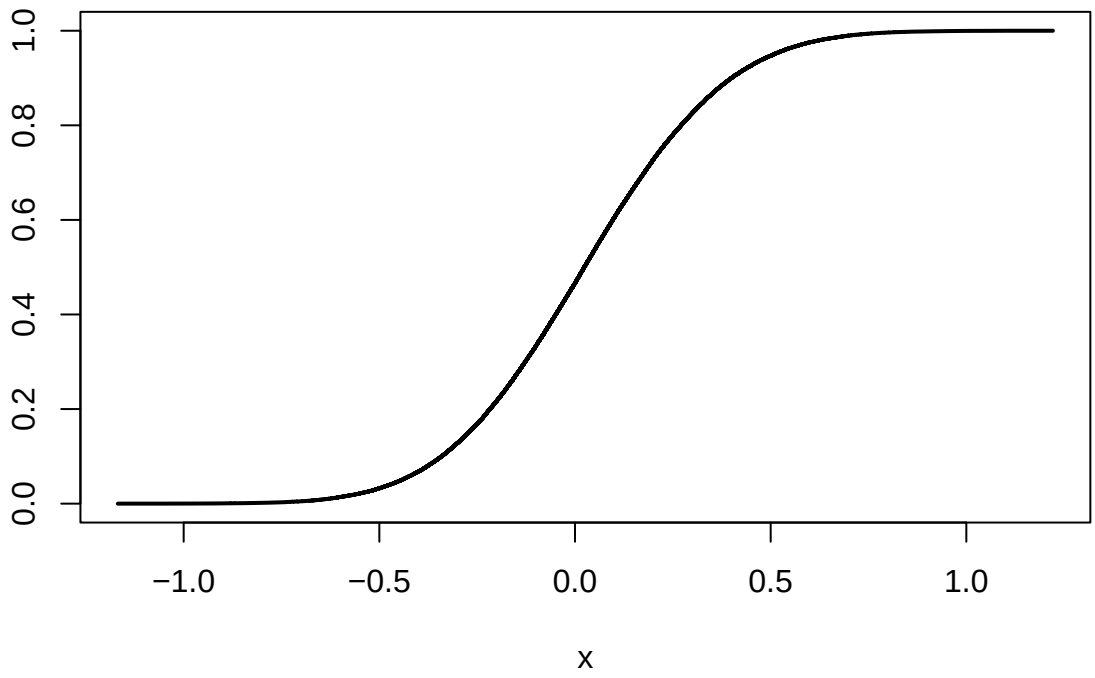
- The sample mean (after rescaling) is 3.6382049.
- First try $b = \sqrt{n}$ and $M = 10^5$.
- We can also try $b = n^{1/3}$ and $b = n^{2/3}$ by rerunning the code chunk.

```
# Make choice of b, and then run
b <- ceiling(sqrt(n))
#b <- ceiling(n^(1/3))
#b <- ceiling(n^(2/3))

k <- ceiling(n/b)
Q.big <- n - b + 1
monte.roots <- NULL
Monte <- 100000
for(i in 1:Monte)
{
    monte.sample <- NULL
    for(j in 1:k)
    {
        boot.index <- sample(Q.big,size=1)
        monte.sample <- c(monte.sample,mau.gr[seq(boot.index,boot.index+b-1)])
    }
    monte.sample <- monte.sample[1:n]
      monte.root <- mean(monte.sample) - mau.mean
    monte.roots <- c(monte.roots,monte.root)
}
interval <- c(sort(monte.roots)[floor(.025*Monte)],sort(monte.roots)[floor(.975*Monte)])
```

- The 95% confidence interval based on the bootstrap is [3.0408972,4.1694835].

- We plot the bootstrap edf.

```
plot(sort(monte.roots),seq(1,Monte)/Monte,type="l",xlab="x",ylab="",lwd=2)
```



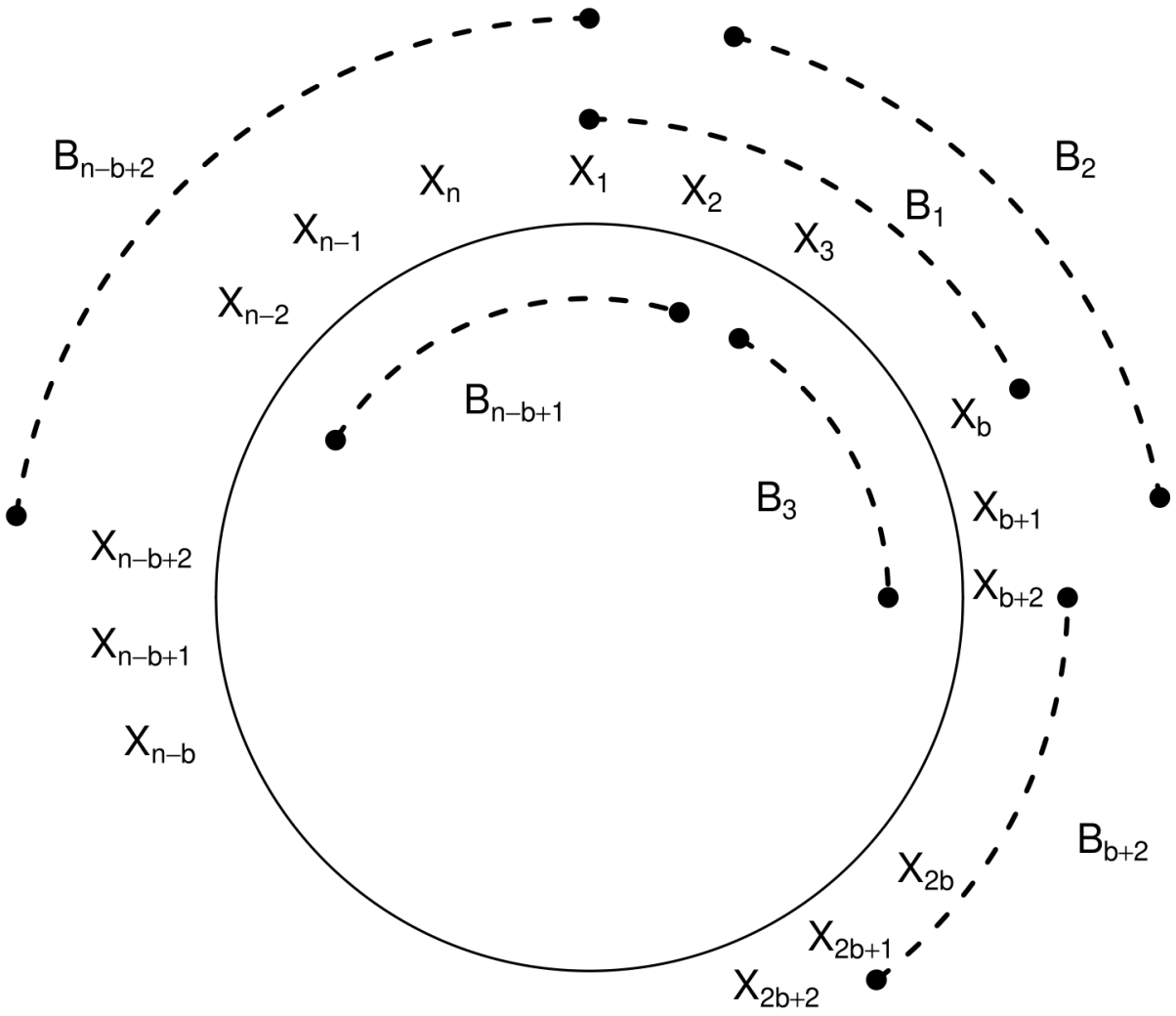


Figure 1: Wrapping a time series around the circle