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REVIEW ARTICLE

Phase-coherent transport in hybrid superconducting nanostructures

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Abstract. This article is an overview of recent experimental and theoretical work on transport in phase-coherent hybrid nanostructures, with particular emphasis on dc electrical conduction. A summary of multiple-scattering theory and the quasi-classical methods is presented and comparisons between the two are made. Several paradigms of phase-coherent transport are discussed, including zero-bias anomalies, re-entrant and long-range proximity effects, Andreev interferometers and superconductivity-induced conductance suppression.

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1. Introduction

During the past few years, phase-coherent quasi-particle transport in hybrid superconducting structures has emerged as a new field of study, bringing together the hitherto separate areas of superconductivity and mesoscopic physics (Al'tshuler *et al* 1991, Buot 1993). The seeds of this new field of research were sown for a variety of different reasons. The search for novel devices led several groups to embark on experimental programmes aimed at exploring the properties of hybrid semiconducting–superconducting structures. During the 1980s there had been great progress in understanding transport properties of normal submicron conductors and therefore towards the end of that decade, it seemed natural to ask how these were affected by the presence of superconductivity. Partly as a consequence of formidable technological problems to be overcome in growing hybrid nanostructures, the first experimental results did not emerge until the summer of 1991, when two groups (Petrashov and Antonov 1991, Kastalskii *et al* 1991) independently submitted papers on phase-coherent transport in submicron superconducting structures. The first of these reported an *increase* in resistance in metallic conductors due to the onset of superconductivity and the second the discovery of a zero-bias anomaly in the dc resistance of a superconductor–semiconductor (S–Sm) contact. The former is still the subject of theoretical investigation (Wilhelm *et al* 1997, Seviour *et al* 1997), while the latter is now well understood.

In a phase-coherent normal–superconducting (N–S) structure, the phase of quasi-particles as well as Cooper pairs is preserved and transport properties depend in detail on the quasi-particle states produced by elastic scattering from inhomogeneities and boundaries. A key physical phenomenon, which arises in the presence of superconductivity is the possibility that an electron can coherently evolve into a hole and vice versa. This phenomenon, known as Andreev scattering (Andreev 1964), occurs without phase breaking and is describable using a variety of theoretical techniques. The effect of superconductivity on transport across a N–S interface is of course an old subject. In lowest order, the classical tunnelling Hamiltonian approach ignores Andreev scattering and predicts that the dc conductance G is proportional to the density of states. Later (Shelankov 1980, Blonder *et al* (BTK) 1982, Blonder and Tinkham 1983, Shelankov 1984) it was pointed out that the contribution to the sub-gap conductance from Andreev scattering can be significant and a theory of a clean N–I–S interface was developed, which showed that for a delta-function barrier, there is indeed a marked deviation from tunnelling theory, but as the barrier strength is increased, the result of classical tunnelling theory is recovered. BTK theory applies to a one-dimensional N–I–S system or, by summing over all transverse wave vectors, to two- or three-dimensional systems with translational invariance in the plane of the barrier and yields for the current I through the contact

$$I = (2e/h)\Omega \int_{-\infty}^{\infty} dE (f(E - eV) - f(E))(1 + A(E) - B(E)) \quad (1)$$

where $A(E)$ and $B(E)$ are Andreev and normal reflection coefficients listed in table II of Blonder *et al* (1982), $f(E)$ the Fermi function and Ω a measure of the area of the junction. In the presence of disorder or other inhomogeneities, this must be replaced by the more general expressions outlined in sections 2 and 4 below.

Prior to 1991, experiments on N–I–S point contacts had been in broad agreement with BTK theory, exhibiting a conductance minimum at zero voltage $V = 0$ and a peak at $eV \approx \Delta$, where Δ is the superconducting energy gap. In the experiment of Kastalskii *et al* (1991), the dc current through a Nb–InGaAs contact is measured as function of the applied voltage. At the interface, depending on the semiconductor doping level, a Schottky barrier naturally forms, so the system behaves like a superconductor–insulator–normal (S–I–N) structure. (An exception to this is InAs, which does not form a Schottky barrier at an N–S interface.) According to BTK theory, as the barrier strength increases the sub-gap conductance should vanish. In contrast, the experiment revealed an excess sub-gap conductance at low bias, whose value was comparable with the conductance arising when the superconducting electrode is in the normal state. This zero-bias anomaly (ZBA) was later observed by Nguyen *et al* (1992) in an experiment involving InAs–AlSb quantum wells attached to superconducting Nb contacts and, by using high-transmittance Nb–Ag (or Al) contacts of varying geometry, Xiong *et al* (1993) were able to observe the evolution from BTK to ZBA behaviour. In an experiment by Bakker *et al* (1994) involving a silicon-based two-dimensional electron gas (2DEG) contacted to two superconducting electrodes, a gate voltage was also used to control the strength of the ZBA and in Magnée *et al* (1994) an extensive study of the ZBA in Nb/Si structures was performed. Since these early experiments, a great deal of effort has been aimed at observing Andreev scattering in ballistic 2DEGs, including that of van Wees *et al* (1994b), Dimoulas *et al* (1995), Marsh *et al* (1994), Takayanagi and Akazaki (1995a) and Takayanagi *et al* (1996).

Kastalskii *et al* (1991) attributed the excess conductance to a non-equilibrium proximity effect, in which superconductivity is induced in the normal electrode, giving rise to an excess pair current. Initially this phenomenon was seen as separate from Andreev reflection, but subsequent theoretical developments have shown that the distinction between the proximity effect and Andreev reflection is artificial. As will become clear later, the ZBA arises from an interplay between Andreev scattering and disorder-induced scattering in the normal electrode. Andreev scattering is sensitive to the breaking of time-reversal symmetry and as a consequence the conductance peak is destroyed by the introduction of a magnetic field.

Zero-bias anomalies constitute the first of a small number of paradigms of phase-coherent transport in hybrid N–S structures. A second paradigm is the observation of re-entrant (van Wees *et al* 1994b, Charlat *et al* 1996a, b) and long-range (Courtois *et al* 1996) behaviour signalled by the appearance of finite-bias anomalies (FBAs) in the conductance of high-quality N–S interfaces. At high temperatures $T > T^*$ and bias voltages $V > V^*$, where for a N metal of length L and diffusion coefficient D , $k_B T^* = eV^* = \sqrt{(D/L^2)}$, both the ZBA and FBA conductance peaks decay as $1/\sqrt{T}$ and $1/\sqrt{V}$. For a clean interface there also exists a conductance maximum at V^* , T^* and therefore at low temperature and voltage a re-entrance to the low-conductance state occurs. An interesting feature of this phenomenon is the long-range nature of the effect, which typically decays as a power law in L^*/L , where $L^* = \sqrt{(D/eV)}$. This behaviour is in sharp contrast with the exponential decay of the Josephson effect and has been observed in a number of experiments. In a T-shaped Ag sample with Al islands at different distances from the current–voltage probes (Petrashov *et al* 1993b), a long-range proximity effect was observed, in which the influence of the island extended over length scales greater than the thermal coherence length L^* . Similar behaviour was also observed (Petrashov *et al* 1994b) in ferromagnet–superconductor hybrids made from Ni–Sn and Ni–Pb. In an experiment involving a square Cu loop in contact with two Al electrodes Courtois *et al* (1996) clearly identified both the short- and long-range contributions to phase-coherent transport. In this interferometer experiment, they observed a phase-periodic conductance decaying as a power law in $1/T$, in parallel with a Josephson

current which decays exponentially with L/L_T .

The above re-entrance phenomenon is also observed in a third paradigm of phase-coherent transport, which arises when a normal metal is in contact with two superconductors, with order parameter phases ϕ_1 and ϕ_2 , whose difference $\phi = \phi_1 - \phi_2$ can be varied by some external means. Prior to the experimental realization of these structures, the electrical conductance of such Andreev interferometers was predicted to be an oscillatory function of ϕ . Spivak and Khmel'nitskii (1982) and Al'tshuler and Spivak (1987) identified a high-temperature ($T \gg T^*$), weak-localization contribution to the conductance of a disordered sample, whose amplitude of oscillation was less than or of order $2e^2/h$. For an individual sample, the period of oscillation was found to be 2π , but for the ensemble average a period of π was predicted. Nakano and Takayanagi (1991) and Takagi (1992) examined a clean interferometer in one dimension and again predicted a 2π -periodic conductance with an amplitude of oscillation less than or of order $2e^2/h$. Lambert (1993) examined a disordered conductor in the low-temperature limit ($T \ll T^*$) and identified a new contribution to the ensemble-averaged conductance with a periodicity of 2π . This 2π periodicity is a consequence of particle-hole symmetry, which also guarantees that at zero temperature and voltage, the conductance should possess a zero-phase extremum (Lambert 1994). Prior to experiments on such devices, the generic nature of this prediction was confirmed in numerical simulations (Hui and Lambert 1993a) encompassing the ballistic, diffusive and almost localized regimes and in a tunnelling calculation of the ensemble-averaged conductance by Hekking and Nazarov (1993).

The first experimental realizations of Andreev interferometers came almost simultaneously from three separate groups. In March of 1994, de Vegvar *et al* (1994) showed results for a structure formed from two Nb electrodes in contact with an Al wire. They found a small oscillation $10^{-3}(2e^2/h)$ with a sample-specific phase in the 2π -periodic component, suggesting that the ensemble-averaged conductance should have a periodicity of π , in agreement with Spivak and Khmel'nitskii (1982). However, in contrast with all subsequent experiments, no zero-phase extremum was observed. In April/May of that year, Pothier *et al* (1994a, b) produced an interferometer involving two tunnel junctions, which showed a 2π -periodic conductance, with a zero-phase maximum and a low-bias, low-temperature amplitude of oscillation of order $10^{-2}(2e^2/h)$, which decayed with increasing temperature. In May 1994, van Wees *et al* (1994b) (see also Dimoulas *et al* 1995) produced the first quasi-ballistic InAs 2DEG interferometer, with high-transparency N-S interfaces. This experiment showed the first re-entrant behaviour in which the amplitude of oscillation δG varied from $\delta G \approx -0.08(2e^2/h)$ at zero voltage (where a minus sign indicates a zero-phase minimum and a plus sign a zero-phase maximum) passes through zero at a bias of order 0.1 mV, reaches a maximum at a bias of order V^* and then decays to zero at higher voltages. (For a detailed study see den Hartog *et al* 1996a, b.) Unlike the Josephson current which decays exponentially with T/T^* , these conductance oscillations decayed only as a power law.

The first experiment showing an amplitude of oscillation greater than $2e^2/h$ was carried out by Petrashov *et al* (1994a, 1995). Here, silver or antimony wires in the shape of a cross, make two separate contacts with superconducting Al and the phase difference between the contacts is varied using either an external field applied to a superconducting loop or by passing a supercurrent through the Al. The amplitude was found to be $\delta G \approx 100(2e^2/h)$ for Ag and $3 \times 10^{-2}(2e^2/h)$ for Sb, and exhibited a periodicity of 2π . In this experiment, the phase difference ϕ was varied both by passing a magnetic flux through an external superconducting loop and by passing a supercurrent through a straight section of the superconductor, thereby emphasizing that the precise manner in which the order parameter

phase is controlled is not important. These experiments were crucial in demonstrating that in metallic samples, the ensemble-averaged conductance is the relevant quantity and therefore a quasi-classical description is relevant. It is perhaps worth mentioning that with hindsight, an earlier experiment reporting large-scale oscillations in a sample with two superconducting islands (Petrashov *et al* 1992, 1993a) can be regarded as a precursor to these interferometer experiments. However, the phase of the islands was not explicitly controlled, making an interpretation of the 1993 experiments more difficult.

A fourth and more recent paradigm of phase-coherent transport is the appearance of negative multi-probe conductances in structures where Andreev transmission of quasi-particles is a dominant process (Allsopp *et al* 1994). The first experiments reporting this behaviour were carried out by den Hartog *et al* (1996a, b), using a diffusive InAs 2DEG. These probe individual coefficients in the current–voltage relations and demonstrate fundamental reciprocity relations arising from time-reversal and particle–hole symmetry.

Finally, a fifth paradigm is the suppression of electrical conductance by superconductivity in metallic systems without tunnel barriers. Experimentally this phenomenon has been observed in several structures, involving both non-magnetic (i.e. silver) and magnetic (i.e. nickel) N components in contact a superconductor via clean interfaces. In the experiment of Petrashov and Antonov (1991) the conductance of a Ag wire on which several Pb islands are deposited decreases by 3% when the islands become superconducting, this corresponding to a conductance decrease of $\delta G \approx 100(2e^2/h)$. These experiments reveal two regimes: as the field decreases below H_{c2} , an initial resistance increase occurs. The resistance remains field insensitive until a lower field is reached (corresponding to the order of a flux quantum through the sample), at which point a further conductance suppression occurs. In the experiment by Petrashov *et al* (1993b) involving a T-shaped Ag sample with Al islands at different distances from the current–voltage probes, a superconductivity-induced change in the resistance by up to 30% was observed, but it was found that for different samples, the resistance could either increase or decrease. Although superconductivity-induced conductance suppression in metallic samples was predicted a number of years ago (Hui and Lambert 1993b) these experiments at first sight appeared to conflict with quasi-classical theories, which universally predict that the normal-state, zero-temperature, zero-bias conductance G_N is identical to the conductance G_{NS} in the superconducting state. This effect is addressed in Hui and Lambert (1993b), Claughton *et al* (1995), Wilhelm *et al* (1997) and Seviour *et al* (1997).

The main aim of this review is to outline the quasi-classical and multiple-scattering theories needed to describe the dc electrical conductance of phase-coherent hybrid N–S structures. For this reason we shall not discuss thermodynamic phenomena such as the Josephson effect in any detail (for a recent review see Bruder 1996), despite the fact that ground-breaking experiments using clean (Takayanagi and Akazaki 1995a, b, c) S–2DEG–S point contacts show quantization of the critical current as predicted by Furasaki *et al* (1991) and for shorter junctions by Beenakker and van Houten (1991). There are several notable theoretical papers addressing Andreev scattering in such structures (including van Wees *et al* 1991, Bagwell 1992, Furasaki *et al* 1991, Gusenheimer and Zaikin 1994, Zyuzin 1994, Hurd and Wendin 1994, 1995, Averin and Bardas 1995, 1996, Bratus *et al* 1995, Chang *et al* 1996, Koyama *et al* 1995, 1996, Levy Yeyati *et al* 1996, Martin-Rodero *et al* 1996, Wendin and Shumeiko 1996, Reidel *et al* 1996, Volkov and Takayanagi 1996b, Antonov *et al* 1997, Bardas and Averin 1997, Zaitsev and Averin 1997). Similarly, to restrict the length of this review, recent theories of thermoelectric coefficients (Bagwell and Alam 1992, Claughton and Lambert 1996) and shot noise (Datta 1995) will not be discussed, nor will we discuss work on Coulomb effects in superconducting islands (Eiles *et al* 1993, Lafarge *et al* 1993,

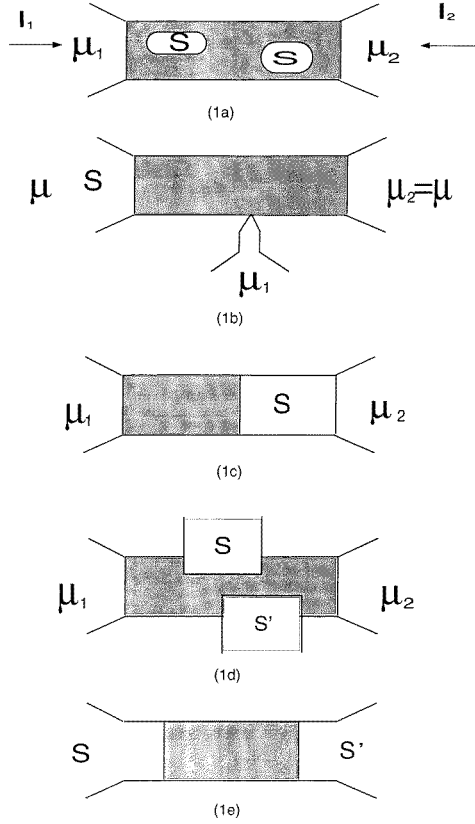


Figure 1. Various generic experimental arrangements used in measuring dc transport in superconducting hybrids. The grey areas indicate phase-coherent normal regions, whereas superconductive parts are labelled S. The widening open parts at the ends represent the reservoirs.

Tuominen *et al* 1992, 1993, Hergenrother *et al* 1994, Black *et al* 1996, Hekking *et al* 1993).

While Andreev interference effects are generic phenomena, their manifestation in a given experiment is sensitive to many parameters. For the purpose of writing this review, it is therefore convenient to adopt the simple classification of experimental arrangements sketched in figure 1. The generic structure shown in figure 1(a) represents our first class of N-S-N hybrids and has many realizations. For convenience, we distinguish these from a second class of N-S hybrids of the kind shown in figures 1(b) and 1(c), in which the superconductor effectively forms part of an external reservoir and is not simply part of the scattering region. Figure 1(d) indicates a third class of N-SS'-N structures, involving two (or more) separate superconductors S and S', with respective order parameter phases ϕ , ϕ' . As noted above, transport properties of such Andreev interferometers are periodic functions of the phase difference $\phi - \phi'$. Figure 1(e) shows a fourth class of S-N-S' structures, in which two superconducting reservoirs are connected to a normal scattering region. In this case, the structure forms a Josephson junction and in contrast with all other structures shown in figure 1, in the linear response limit, the dc conductance measured between the superconducting reservoirs is identically zero. In this case, the relevant dc quantity is the current-phase relation and the associated critical current. Clearly one could also measure the

Josephson current between the two superconductors in the N–SS'–N structure of figure 1(c) and therefore the above classification is intended to label the measurement being made, rather than the device being measured. Structures of the form shown in figure 1(e) will not be discussed.

For the most part, the theoretical descriptions discussed below have finessed problems of self-consistency, by computing measured quantities as a function of the superconducting order parameter pairing field $f_{\sigma\sigma'}(\mathbf{r}) = \langle \psi_{\sigma}(\mathbf{r}) \psi_{\sigma'}(\mathbf{r}) \rangle$ induced by making contact with a piece of superconductor. As an example, figure 1(a) shows a normal mesoscopic scattering region in contact with a superconducting island which plays the role of an externally controllable source of $f_{\sigma\sigma'}(\mathbf{r})$, in much the same way that the coils of a magnet are an external source of magnetic field. The coils are not of primary interest and in many cases, neither is the superconductor. It is assumed that parameters characterizing the superconductor are given and the key question is that of how the superconductivity influences transport through the scattering region. Of course once the influence of superconductivity is understood, transport properties can be used to probe the symmetry and spatial structure of the order parameter, as suggested by Cook *et al* (1996). Furthermore, in the presence of large currents which modify the order parameter, a complete self-consistent treatment is necessary (as described for example in Bruder 1990, Hara *et al* 1993, Barash *et al* 1995, Cañizares and Sols 1995, 1997, Chang *et al* 1996, Gyorffy 1995, Martin and Lambert 1995, 1996, Riedel *et al* 1996).

2. The multiple-scattering approach to dc transport in superconducting hybrids

2.1. Fundamental current–voltage relations

In this section we review the multi-channel current–voltage relations for a disordered phase-coherent scatterer connected to normal reservoirs, obtained for two normal probes by Lambert (1991) and extended to multi-probes by Lambert *et al* (1993). The coefficients (denoted A_{ij} and a_{ij} below) are directly accessible experimentally and can be combined to yield a variety of transport coefficients, the simplest of which is the electrical conductance of a normal–superconducting interface. To avoid time-dependent order parameter phases varying at the Josephson frequency, which would render a time-independent scattering approach invalid, these are derived under the condition that all superconductors share a common condensate chemical potential μ . The derivation of the fundamental current–voltage relation presented in Lambert (1991) follows closely the multi-channel scattering theory developed during the 1980s for non-superconducting mesoscopic structures (Büttiker 1986, Buot 1993). In the normal state, this approach yields, for example, the multi-channel Landauer formula (Landauer 1970) for the electrical conductance

$$G = (2e^2/h)T_0 \quad (2)$$

where T_0 is the transmission coefficient of the structure. Historically the above formula was not accepted without a great deal of debate and contradicts the corresponding expression used by practitioners of quasi-classical theories, where the alternative expression

$$G = (2e^2/h)(T_0/R_0) \quad (3)$$

is usually employed, with (in one dimension) R_0 the reflection coefficient. In fact the above two expressions refer to a two-probe measurement and to a four-probe measurement, respectively, and the crucial lesson from the debate surrounding these equations is that transport coefficients such as the electrical conductance are secondary quantities. More fundamental are the current–voltage relations describing a given mesoscopic structure.

Equations (2) and (3) are not valid in the presence of Andreev scattering, because charge transport and quasi-particle diffusion are no longer equivalent. For example when a quasi-particle Andreev reflects at an N–S interface, the energy and probability density of the excitation are reflected back into the normal conductor, whereas a charge of $2e$ is injected into the superconductor. Thus charge flows into the superconductor, even though the excitation does not and, as a consequence, a current–voltage relation should be used, which takes into account this charge–energy separation. For a scattering region connected to L normal reservoirs, labelled $i = 1, 2, \dots, L$, it is convenient to write this in the form

$$I_i = \sum_{j=1}^N A_{ij} \quad (4)$$

where I_i is the current flowing from reservoir i and the coefficients A_{ij} will be discussed in detail below. In the linear response limit, this reduces to

$$I_i = \sum_{j=1}^L a_{ij}(v_j - v). \quad (5)$$

The above expressions describe reservoirs at voltages $v_i, i = 1, 2, \dots, L$, connected to a scattering region containing one or more superconductors with a common condensate chemical μ and relate the current I_i from reservoir i to the voltage differences $v_j - v$, where $v = \mu/e$. The $L = 2$ formula describes a wide variety of experimental measurements and underpins many subsequent theoretical descriptions of disordered N–S interfaces and inhomogeneous structures. For this reason, after discussing the relationship between the coefficients A_{ij} , a_{ij} and the scattering matrix, we shall examine the two-probe formula in some detail and illustrate its application to some generic experimental measurements.

2.2. The relationship between the generalized conductance matrix and the s -matrix

The $L = 2$ analysis of Lambert (1991) is based on the observation that in the absence of inelastic scattering, dc transport is determined by the quantum mechanical scattering matrix $s(E, H)$, which yields scattering properties at energy E , of a phase-coherent structure described by a Hamiltonian H . If the structure is connected to external reservoirs by open scattering channels labelled by quantum numbers n , then this has matrix elements of the form $s_{n,n'}(E, H)$. The squared modulus of $s_{n,n'}(E, H)$ is the outgoing flux of quasi-particles along channel n , arising from a unit incident flux along channel n' . Adopting the notation of Lambert *et al* (1993), we consider channels belonging to current-carrying leads, with quasi-particles labelled by a discrete quantum number α ($\alpha = +1$ for particles, -1 for holes) and therefore write $n = (l, \alpha)$, where l labels all other quantum numbers associated with the leads. With this notation, the scattering matrix elements $s_{n,n'}(E, H) = s_{l,l'}^{\alpha,\beta}(E, H)$ satisfy the unitarity condition $s^\dagger(E, H) = s^{-1}(E, H)$, the time-reversibility condition $s^t(E, H) = s(E, H^*)$ and, if E is measured relative to the condensate chemical potential $\mu = ev$, the particle–hole symmetry relation $s_{l,l'}^{\alpha,\beta}(E, H) = \alpha\beta[s_{l,l'}^{-\alpha,-\beta}(-E, H)]^*$. (For convenience we adopt the convention of including appropriate ratios of channel group velocities in the definition of s to yield a unitary scattering matrix.)

For a scatterer connected to external reservoirs by L crystalline, normal leads, labelled $i = 1, 2, \dots, L$, it is convenient to write $l = (i, a)$, $l' = (j, b)$, where a (b) is a channel belonging to lead i (j). With this notation, the quantities entering the current–voltage relation are of the form

$$P_{i,j}^{\alpha,\beta}(E, H) = \sum_{a,b} |s_{(i,a),(j,b)}^{\alpha,\beta}(E, H)|^2 = \text{Tr}[s_{ij}^{\alpha,\beta}(E, H)\{s_{ij}^{\alpha,\beta}(E, H)\}^\dagger] \quad (6)$$

which is an expression for the coefficient for reflection ($i = j$) or transmission ($i \neq j$) of a quasi-particle of type β in lead j to a quasi-particle of type α in lead i . For $\alpha \neq \beta$, $P_{i,j}^{\alpha,\beta}(E, H)$ is an Andreev scattering coefficient, while for $\alpha = \beta$, it is a normal scattering coefficient. Since unitarity yields

$$\sum_{\beta b j} |s_{(i,a),(j,b)}^{\alpha,\beta}(E, H)|^2 = \sum_{\alpha a j} |s_{(i,a),(j,b)}^{\alpha,\beta}(E, H)|^2 = 1 \quad (7)$$

where i and j sum over all leads containing open channels of energy E , this satisfies

$$\sum_{\beta j}^L P_{ij}^{\alpha,\beta}(E, H) = N_i^\alpha(E) \quad \text{and} \quad \sum_{\alpha i}^L P_{ij}^{\alpha,\beta}(E, H) = N_j^\beta(E) \quad (8)$$

where $N_i^\alpha(E)$ is the number of open channels for α -type quasi-particles of energy E in lead i , satisfying $N_i^+(E) = N_i^-(-E)$. Similarly particle-hole symmetry yields

$$P_{ij}^{\alpha,\beta}(E, H) = P_{ij}^{-\alpha,-\beta}(-E, H) \quad (9)$$

and time-reversal symmetry yields

$$P_{ij}^{\alpha,\beta}(E, H) = P_{ji}^{\beta,\alpha}(E, H^*). \quad (10)$$

Having introduced the scattering coefficients $P_{ij}^{\alpha,\beta}(E, H)$, the coefficients A_{ij} of the fundamental formula (4) are given by

$$A_{ij} = (2e/h) \sum_{\alpha} (\alpha) \int_0^\infty dE \left\{ \delta_{ij} N_i^\alpha(E) f_i^\alpha(E) - \sum_{\beta} P_{ij}^{\alpha\beta}(E, H) f_j^\beta(E) \right\} \quad (11)$$

with $f_j^\alpha(E) = \{\exp[(E - \alpha(ev_j - \mu))/k_b T] + 1\}^{-1}$ the distribution of incoming α -type quasi-particles from lead j^\dagger .

Equation (4) yields the current-voltage characteristics of a given structure at finite voltages, provided that all scattering coefficients are computed in the presence of a self-consistently determined order parameter and self-consistent values of all of the other scattering potentials. At finite temperature, but zero voltage, where $v_i - v \rightarrow 0$, it reduces to equation (5), with a_{ij} given by

$$a_{ij} = (2e^2/h) \int_{-\infty}^\infty dE \left[-\frac{\partial f(E)}{\partial E} \right] [N_i^+(E) \delta_{ij} - P_{ij}^{++}(E, H) + P_{ij}^{--}(E, H)] \quad (12)$$

where, $f(E)$ is the Fermi function and equation (9) has been used. At finite voltages, but zero temperature, it reduces to

$$A_{ij} = (2e/h) \int_0^{(ev_i - \mu)} dE [\delta_{ij} N_i^+(E) + P_{ij}^{-+}(E, H) - P_{ij}^{++}(E, H)]. \quad (13)$$

Finally at finite voltages, the differential of equation (4) with respect to v_j (with μ and all other potentials held constant) yields

$$\partial I_i / \partial v_j = a_{ij} \quad (14)$$

where, at finite temperature,

$$a_{ij} = (2e^2/h) \sum_{\alpha} (\alpha) \int_0^\infty dE \left\{ \delta_{ij} N_i^\alpha(E) \left[-\alpha \frac{\partial f_i^\alpha(E)}{\partial E} \right] - \sum_{\beta} P_{ij}^{\alpha\beta}(E, H) \left[-\beta \frac{\partial f_j^\beta(E)}{\partial E} \right] \right\} \quad (15)$$

[†] It is perhaps worth noting that in Lambert (1991), the following notation is employed: $N_p(E) = N_1^+$, $\tilde{R}_{pp} = P_{11}^{++}(E, H)/N_p(E)$, $\tilde{R}_{hp} = P_{11}^{-+}(E, H)/N_p(E)$ etc.

and, at zero temperature,

$$a_{ij} = [2e^2/h][\delta_{ij}N_i^+(E_i) + P_{ij}^{-+}(E_j, H) - P_{ij}^{++}(E_j, H)] \quad (16)$$

where $E_i = ev_i - \mu$.

It is worth noting that replacing E by $-E$ and utilizing the particle-hole symmetry relation (9) allows equation (12) to be rewritten in the form

$$a_{ij} = (2e^2/h) \int_{-\infty}^{\infty} dE \left[-\frac{\partial f(E)}{\partial E} \right] [N_i^-(E)\delta_{ij} - P_{ij}^{--}(E, H) + P_{ij}^{+-}(E, H)] \quad (17)$$

which demonstrates that particles and holes are treated on an equal footing in equations (12) and (17). Furthermore, in view of the symmetries (9), (10), the reciprocity relation $a_{ij}(H) = a_{ji}(H^*)$ is satisfied.

2.3. Two-probe formulae in more detail

While the above notation is convenient for arbitrary L , it perhaps obscures the simplicity of the final result and therefore, in the literature, several alternative forms of notation have been employed. For the case of $L = 2$ normal probes, where the scattering matrix has the structure

$$S(E, H) = \begin{pmatrix} r(E) & t'(E) \\ t(E) & r'(E) \end{pmatrix} \quad (18)$$

it is convenient to write $s_{11}^{\alpha\beta}(E, H) = r_{\alpha\beta}(E)$, $s_{22}^{\alpha\beta}(E, H) = r'_{\alpha\beta}(E)$, $s_{21}^{\alpha\beta}(E, H) = t_{\alpha\beta}(E)$ and $s_{12}^{\alpha\beta}(E, H) = t'_{\alpha\beta}(E)$. With this notation, the submatrices r, t, r', t' have the form

$$r(E) = \begin{pmatrix} r_{++}(E) & r_{-+}(E) \\ r_{-+}(E) & r_{--}(E) \end{pmatrix} \quad (19)$$

etc. The matrix $r_{\alpha\beta}$ ($r'_{\alpha\beta}$) is a matrix of amplitudes describing the reflection of β -type quasi-particles from reservoir 1 (2) into α -type quasi-particles travelling back into reservoir 1 (2). Similarly $t_{\alpha\beta}$ ($t'_{\alpha\beta}$) is a matrix of amplitudes describing the transmission of β -type quasi-particles from reservoir 1 (2) into α -type quasi-particles of reservoir 2 (1).

Using these submatrices, the $L = 2$ current-voltage relation can be written as

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 - v \\ v_2 - v \end{pmatrix} \quad (20)$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \int_{-\infty}^{\infty} dE \left(-\frac{\partial f(E)}{\partial E} \right) \times \begin{pmatrix} N_1^+(E) - R_0(E) + R_a(E) & T'_a(E) - T'_0(E) \\ T_a(E) - T_0(E) & N_2^+(E) - R'_0(E) + R'_a(E) \end{pmatrix} \quad (21)$$

with

$$\begin{pmatrix} R_0(E) \\ T_0(E) \\ R_a(E) \\ T_a(E) \end{pmatrix} = \begin{pmatrix} P_{11}^{++}(E, H) \\ P_{21}^{++}(E, H) \\ P_{11}^{-+}(E, H) \\ P_{21}^{-+}(E, H) \end{pmatrix} = \begin{pmatrix} \text{Tr}\{r_{++}(E)r_{++}^\dagger(E)\} \\ \text{Tr}\{t_{++}(E)t_{++}^\dagger(E)\} \\ \text{Tr}\{r_{-+}(E)r_{-+}^\dagger(E)\} \\ \text{Tr}\{t_{-+}(E)t_{-+}^\dagger(E)\} \end{pmatrix} \quad (22)$$

and

$$\begin{pmatrix} R'_0(E) \\ T'_0(E) \\ R'_a(E) \\ T'_a(E) \end{pmatrix} = \begin{pmatrix} P_{22}^{++}(E, H) \\ P_{12}^{++}(E, H) \\ P_{22}^{-+}(E, H) \\ P_{12}^{-+}(E, H) \end{pmatrix} = \begin{pmatrix} \text{Tr}\{r'_{++}(E)r_{++}^\dagger(E)\} \\ \text{Tr}\{t'_{++}(E)t_{++}^\dagger(E)\} \\ \text{Tr}\{r'_{-+}(E)r_{-+}^\dagger(E)\} \\ \text{Tr}\{t'_{-+}(E)t_{-+}^\dagger(E)\} \end{pmatrix}. \quad (23)$$

Similarly the zero-temperature differential conductance (16) becomes

$$a_{ij} = [2e^2/h][\delta_{ij}N_i^+(E_i) + R_a(E_j) - R_0(E_j)]. \quad (24)$$

2.4. Applications of the two-probe conductance matrix

As noted in Lambert (1991), the two-probe current–voltage relation (20) can be used to derive generalizations of both (2) and (3). To further illustrate the versatility of the two-probe theory, we now apply it to some typical experimental measurements. For convenience in what follows, we set $2e^2/h$ equal to unity.

Example 1. The normal limit. Consider the structure of figure 1(a), where a scattering region containing superconducting inclusions is connected to normal reservoirs at potentials v_1, v_2 . In the normal limit, the condensate potential v must disappear from the fundamental current–voltage relation (4). As shown below, this occurs, because in the absence of Andreev scattering, unitarity of the scattering matrix implies that $a_{11} = -a_{12} = -a_{21} = a_{22} = T_0$. Hence equation (20) reduces to

$$I_1 = -I_2 = T_0(v_1 - v_2) \quad (25)$$

which is simply the Landauer formula (2).

Example 2. Experiments where $\mu_2 = \mu$. Figure 1(b) shows an experiment in which the superconductor and reservoir 2 are held at the same potential. In this case, equation (20) yields

$$\frac{I_1}{(v_1 - v)} = a_{11} = \int_{-\infty}^{\infty} dE - \frac{\partial f(E)}{\partial E} [N_1^+(E) - R_0(E) + R_a(E)]. \quad (26)$$

This experimental configuration is of the type used in tunnelling experiments, aimed at probing the proximity effect in the vicinity of an N–S boundary (Gueron *et al* 1996). This result describes any of the structures shown in figure 1, provided that $\mu_2 = \mu$, and is a generalization to the disordered and inhomogeneous structures of the boundary conductance formula derived by Blonder *et al* (1982).

Example 3. Experiments where $a_{12} = a_{21} = 0$. An example of such an experiment is shown in figure 1(c), where under sub-gap conditions, the presence of a long superconductor (of length greater than the superconducting coherence length) prevents the transmission of quasi-particles from reservoir 1 to reservoir 2 and vice versa. In this case, combining the unitarity condition $N_1^+(E) = R_0(E) + R_a(E)$ with equation (20) yields

$$\frac{I_1}{(v_1 - v)} = a_{11} = \int_{-\infty}^{\infty} dE \left(-\frac{\partial f(E)}{\partial E} \right) [2R_a(E)]. \quad (27)$$

In common with the current–voltage relation from which it derives, equation (27) is valid in the presence of disorder and inhomogeneities and in the presence of an arbitrary number of superconducting inclusions of arbitrary geometry.

Example 4. Experiments where $I_1 = -I_2 = I$. Such a situation is illustrated in figure 1(a). In this case, inverting equation (20) yields

$$\begin{pmatrix} v_1 - v \\ v_2 - v \end{pmatrix} = \frac{1}{d} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} I \\ -I \end{pmatrix} \quad (28)$$

where $d = a_{11}a_{22} - a_{12}a_{21}$. Hence the two-probe conductance $G = I/(v_1 - v_2)$ takes the form

$$G = \frac{d}{a_{11} + a_{22} + a_{12} + a_{21}}. \quad (29)$$

As an example of the use of this formula, we note that in the zero-temperature limit, where all quantities are evaluated at zero energy, equation (29) can be written (Lambert 1993, Lambert *et al* 1993) as

$$G = T_0 + T_a + \frac{2(R_a R'_a - T_a T'_a)}{R_a + R'_a + T_a + T'_a}. \quad (30)$$

For a symmetric scatterer, where primed quantities equal unprimed quantities, this reduces to $G = T_0 + R_a$, whereas in the absence of transmission between the reservoirs, the resistance G^{-1} reduces to a sum of two resistances $G^{-1} = (1/2R_a) + (1/2R'_a)$. It should be noted that a combination of particle-hole symmetry and unitarity yield at $E = 0$, $T_0 + T_a = T'_0 + T'_a$ and therefore equation (30) is symmetric under an interchange of primed and unprimed coefficients.

Example 5. Experiments where $I_2 = 0$. As a final example, consider the experiment sketched in figure 1(d) where reservoir 2 acts as a voltage probe, with $I_2 = 0$. In this case equation (20) yields for the ratio of the voltages

$$\frac{v_2 - v}{v_1 - v} = -\frac{a_{21}}{a_{22}} \quad (31)$$

where, from equation (21), the coefficient a_{22} is positive. In contrast, the coefficient a_{21} is necessarily negative for a normal system, but in the presence of Andreev scattering can have arbitrary sign. Hence superconductivity can induce voltage sign reversals which are not present in the normal limit. This feature was first predicted within the context of negative four-probe conductances (Allsopp *et al* 1994) and has been confirmed in recent experiments by the Groningen group (den Hartog *et al* 1996a, b).

2.5. The Bogoliubov-de Gennes equation

The above formulae relate measurable quantities to scattering matrix elements and are fundamental, because they are independent of the method chosen to evaluate the s -matrix. To end this section we briefly introduce the Bogoliubov-de Gennes equation (de Gennes 1989), which forms a basis for computing scattering coefficients. The Bogoliubov-de Gennes equation arises during the diagonalization of the mean-field BCS Hamiltonian, which for a non-magnetic, spin-singlet superconductor takes the form

$$H_{eff} = E_0 + \int d\mathbf{r} \int d\mathbf{r}' (\psi_{\uparrow}^{\dagger}(\mathbf{r})\psi_{\downarrow}(\mathbf{r}'))H(\mathbf{r}, \mathbf{r}') \begin{pmatrix} \psi_{\uparrow}(\mathbf{r}') \\ \psi_{\downarrow}(\mathbf{r}') \end{pmatrix} \quad (32)$$

where E_0 is a constant, $\psi_{\sigma}(\mathbf{r})$ and $\psi_{\sigma}^{\dagger}(\mathbf{r})$ are field operators, destroying and creating electrons of spin σ at position \mathbf{r} , and

$$H(\mathbf{r}, \mathbf{r}') = \begin{pmatrix} \delta(\mathbf{r} - \mathbf{r}')H_0(\mathbf{r}) & \Delta(\mathbf{r}, \mathbf{r}') \\ \Delta^*(\mathbf{r}, \mathbf{r}') & -\delta(\mathbf{r} - \mathbf{r}')H_0^*(\mathbf{r}) \end{pmatrix}. \quad (33)$$

To each positive eigenvalue E_n of H , with eigenvector

$$\Psi_n(\mathbf{r}) = \begin{pmatrix} u_n(\mathbf{r}) \\ v_n(\mathbf{r}) \end{pmatrix}$$

satisfying

$$\int d\mathbf{r}' H(\mathbf{r}, \mathbf{r}') \begin{pmatrix} u_n(\mathbf{r}') \\ v_n(\mathbf{r}') \end{pmatrix} = E_n \begin{pmatrix} u_n(\mathbf{r}) \\ v_n(\mathbf{r}) \end{pmatrix} \quad (34)$$

there exists a corresponding negative eigenvalue $-E_n$ with eigenvector

$$\Psi_{-n}(\mathbf{r}) = \begin{pmatrix} -v_n^*(\mathbf{r}) \\ u_n^*(\mathbf{r}) \end{pmatrix}.$$

Consequently \hat{H}_{eff} is diagonalized by the transformation

$$\begin{pmatrix} \psi_{\uparrow}(\mathbf{r}) \\ \psi_{\downarrow}(\mathbf{r}) \end{pmatrix} = \sum_{n>0} \begin{pmatrix} u_n(\mathbf{r}) & -v_n^*(\mathbf{r}) \\ v_n(\mathbf{r}) & u_n^*(\mathbf{r}) \end{pmatrix} \begin{pmatrix} \gamma_{n\uparrow} \\ \gamma_{n\downarrow} \end{pmatrix} \quad (35)$$

where, to avoid overcounting, only one of $\Psi_n(\mathbf{r})$ or $\Psi_{-n}(\mathbf{r})$ is included in the sum over n .

It should be noted that whereas the Bogoliubov–de Gennes equation (34) refers to a model in which the variable \mathbf{r} varies continuously, in a tight-binding model, often used in numerical simulations, the corresponding Bogoliubov–de Gennes equation is

$$\begin{aligned} E\psi_i &= \epsilon_i\psi_i - \gamma \sum_{\delta} \psi_{i+\delta} + \Delta_i\phi_i \\ E\phi_i &= -\epsilon_i\phi_i + \gamma^* \sum_{\delta} \phi_{i+\delta} + \Delta_i^*\psi_i \end{aligned} \quad (36)$$

where ψ_i (ϕ_i) indicates the particle (hole) wave function on site i and $i + \delta$ labels a neighbour of i .

As discussed in Hui and Lambert (1990) and Lambert *et al* (1993), for a scattering region connected to two crystalline normal leads, the Bogoliubov–de Gennes equation may be solved by means of a transfer matrix method, which yields a transfer matrix T satisfying

$$\begin{pmatrix} O' \\ I' \end{pmatrix} = T \begin{pmatrix} I \\ O \end{pmatrix} \quad (37)$$

where O (I) refer to vectors of outgoing (incoming) plane-wave amplitudes on the left and O' (I') to corresponding amplitudes on the right, each plane wave being divided by the square root of its longitudinal group velocity to ensure unitarity of s . Whereas T connects plane-wave amplitudes in the left lead to amplitudes in the right lead, the s -matrix connects incoming amplitudes to outgoing amplitudes and satisfies

$$\begin{pmatrix} O \\ O' \end{pmatrix} = s \begin{pmatrix} I \\ I' \end{pmatrix}. \quad (38)$$

Once T is known, the s -matrix can be constructed. Indeed if s is written as

$$s = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \quad (39)$$

then T has the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} (t^\dagger)^{-1} & r'(t')^{-1} \\ -(t')^{-1}r & (t')^{-1} \end{pmatrix} \quad (40)$$

from which the following inverse relation is obtained:

$$s = \begin{pmatrix} -T_{22}^{-1}T_{21} & T_{22}^{-1} \\ (T_{11}^\dagger)^{-1} & T_{12}T_{22}^{-1} \end{pmatrix}. \quad (41)$$

An alternative method of evaluating coefficients in the above current–voltage relations is provided by the recursive Green’s function method, which uses Gaussian elimination to compute the Green’s function on sites located at the surface of the interface between external normal leads and the scatterer. Given the surface Green’s function, scattering coefficients can be obtained from generalized Fisher–Lee relations (Fisher and Lee 1981), derived by Takane and Ebisawa (1992a) and later rederived by Lambert and co-workers (Lambert 1993, Lambert *et al* 1993). This recursive technique is identical to the ‘decimation’ method employed by Lambert and Hui (1990) and is essentially an efficient implementation of Gaussian elimination.

A third method of evaluating the coefficient R_a was derived by Beenakker (1992) for the case where there is perfect Andreev reflection at the boundary of a clean superconductor. The resulting formula expresses R_a in terms of scattering properties of the normal state, as discussed in section 5 below, and facilitates the application of random-matrix theory to N–S structures (Beenakker 1998).

3. Zaitsev’s boundary conditions and multiple-scattering techniques

In the previous section, the multiple-scattering approach to dc transport was reviewed and in the following section the method of the quasi-classical Green’s function will be discussed. Whereas the multiple-scattering method is a stand-alone approach, the quasi-classical technique must be supplemented by boundary conditions describing quasi-particle scattering at the interface between different metals. These boundary conditions are provided by scattering theory and therefore provide a bridge between the two approaches. In the literature, quasi-classical boundary conditions have not been discussed using the language of modern scattering theory and as a consequence are restricted to planar surfaces with translational invariance in the transverse direction. In this section we offer a rederivation of Zaitsev’s boundary conditions (Zaitsev 1984), which not only fills this gap, but also yields a more general condition applicable to non-planar interfaces.

As noted in section 2, the starting point for a multiple-scattering description is the scattering matrix s , with matrix elements $s_{nn'}$ connecting incoming to outgoing scattering channels, and satisfying equation (38).

Clearly $s(E, H)$ is a functional of all physical potentials entering the Hamiltonian H , as well as a function of E . Since H is Hermitian, quasi-particle probability is conserved, which yields

$$s^{-1}(E, H) = s^\dagger(E, H) \quad (42)$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T(E, H)^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T(E, H). \quad (43)$$

Furthermore, time-reversal symmetry yields

$$s^*(E, H^*) = s^{-1}(E, H) \quad (44)$$

and

$$T(E, H) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T^*(E, H^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (45)$$

Hence

$$s(E, H^*) = s^t(E, H) \quad (46)$$

and

$$T^{-1'}(E, H^*) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T(E, H) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (47)$$

It should be noted that the unitary matrix s in equation (38) and the matrix T in equation (37) connect open channels to open channels only. If evanescent states which decay at large distances from the scatterer are included in the outgoing states on the left-hand side of (38) and states which grow at large distances are included on the right, then the corresponding scattering matrix \hat{s} and transfer matrix \hat{T} will not satisfy equations (42) and (43). The matrices s and T should then be constructed by eliminating closed channels from \hat{s} and \hat{T} .

Using the above definitions, the boundary conditions derived by Zaitsev are readily extended to the case of multiple-scattering channels. The problem to be solved is that of connecting the Green's function G_j in a region j to the Green's function(s) in all other regions connected to the scatterer. Although the following analysis is easily generalized to many leads attached to a scattering region, for simplicity of notation we restrict the discussion to only two leads.

For the purpose of this section, it is convenient to write the channel index n in the form $n = j, \sigma, p$, where $j = 1, 2$ labels a lead, p labels all transverse quantum numbers, $\sigma = +1$ for a right-going (or right-decaying) wave and $\sigma = -1$ for a left-going (or left-decaying) wave. If \mathbf{r}_j denotes a position in a crystalline lead j of constant cross-section, then a plane wave of unit flux incident along channel n can be written as

$$\phi_n(\mathbf{r}_j) = \chi_n(\boldsymbol{\rho}_j)(v_n(E))^{-1/2} \exp(ik_{j\sigma}^p(E)z_j) \quad (48)$$

where $\chi_n(\boldsymbol{\rho}_j)$ is the transverse mode that is associated with channel n and

$$(v_n(E))^{-1/2} \exp(ik_{j\sigma}^p(E)z_j)$$

is a plane wave of unit flux, with longitudinal wave vector $k_{j\sigma}^p(E)$. Near the boundary, the most important terms in the equations of motion are the kinetic energy and the scattering potential due to the barrier, which defines the boundary. As a consequence, the Green's function can be written in terms of scattering states and for $\mathbf{r}_i \neq \mathbf{r}_j$, with $\mathbf{r}_i, \mathbf{r}_j$ located within the leads, has the form

$$G(\mathbf{r}_i, \mathbf{r}_j) = \sum_{p, p', \sigma, \sigma'} \chi_{i, \sigma, p}(\boldsymbol{\rho}_i) \chi_{j, \sigma', p'}(\boldsymbol{\rho}_j) A_{\sigma, \sigma'}^{ij}(p, p', \Sigma) \frac{\exp i[k_{i, \sigma}^p(E)z_i + k_{j, \sigma'}^{p'}(E)z_j]}{\sqrt{v_{(i, \sigma, p)}(E)v_{(j, \sigma', p')}(E)}} \quad (49)$$

where Σ is equal to the sign of $z_i - z_j$.

For $\mathbf{r}_i \neq \mathbf{r}_j$ and a fixed value of Σ , the Green's function is simply a wave function and therefore the matrix $A_{\sigma, \sigma'}^{ij}$, with matrix elements $A_{\sigma, \sigma'}^{ij}(p, p', \Sigma)$ satisfies relations of the form of equation (37). First consider the form of the Green's function when $z_j < z_i$ and $j = 1$. When viewed as a function of z_i , for $z_j < 0$, one has

$$\hat{T}(E, H) \begin{pmatrix} A_{++}^{11} & A_{+-}^{11} \\ A_{-+}^{11} & A_{--}^{11} \end{pmatrix} = \begin{pmatrix} A_{++}^{21} & A_{+-}^{21} \\ A_{-+}^{21} & A_{--}^{21} \end{pmatrix}. \quad (50)$$

On the other hand, when viewed as a function of z_j , using the conjugate equation for the Green's function (which involves the time-reversed Hamiltonian), one obtains for $i = 2$ and $z_i > 0$

$$\begin{pmatrix} A_{++}^{22} & A_{+-}^{22} \\ A_{-+}^{22} & A_{--}^{22} \end{pmatrix} = \hat{T}(E, H^*) \begin{pmatrix} A_{++}^{21} & A_{+-}^{21} \\ A_{-+}^{21} & A_{--}^{21} \end{pmatrix}. \quad (51)$$

Hence after eliminating the off-diagonal terms, A_{++}^{21} , etc, we obtain the general boundary condition relating the Green's functions on the left of the scatterer to the Green's functions on the right:

$$\hat{T}(E, H) \begin{pmatrix} A_{++}^{11} & A_{+-}^{11} \\ A_{-+}^{11} & A_{--}^{11} \end{pmatrix} = \begin{pmatrix} A_{++}^{22} & A_{+-}^{22} \\ A_{-+}^{22} & A_{--}^{22} \end{pmatrix} \hat{T}^t(E, H^*)^{-1}. \quad (52)$$

Equation (52) is a generalization of Zaitsev's boundary condition to the case of a non-planar barrier, which in general may contain impurities and break time-reversal symmetry[†].

Since the sum on the right-hand side of equation (49) includes evanescent channels, the general boundary condition involves the matrix \hat{T} . However, at large distances from the scatterer, evanescent channels can be ignored and therefore \hat{T} can be replaced by T . In view of equation (47), if time-reversal symmetry is present, the boundary condition simplifies to

$$T(E, H) \begin{pmatrix} -A_{+-}^{11} & A_{++}^{11} \\ -A_{--}^{11} & A_{-+}^{11} \end{pmatrix} = \begin{pmatrix} -A_{+-}^{22} & A_{++}^{22} \\ -A_{--}^{22} & A_{-+}^{22} \end{pmatrix} T(E, H). \quad (53)$$

In the work of Zaitsev, where p represents the transverse momentum and all matrices are diagonal, equation (53) is satisfied for each separate p and in the notation of Zaitsev has the form

$$T(E, H) \begin{pmatrix} -\check{g}_+^1 & \check{g}_+^1 \\ -\check{g}_-^1 & \check{g}_-^1 \end{pmatrix} = \begin{pmatrix} -\check{g}_+^2 & \check{g}_+^2 \\ -\check{g}_-^2 & \check{g}_-^2 \end{pmatrix} T(E, H). \quad (54)$$

As an example, taking the trace of equation (54) yields[‡]

$$\check{g}_+^1 - \check{g}_-^1 = \check{g}_+^2 - \check{g}_-^2 \quad (55)$$

which demonstrates that the antisymmetric part of the quasi-classical Green's function is continuous across a scattering region[§]. More generally, equation (53) yields the corresponding relation

$$\text{Tr}(A_{-+}^{11} - A_{+-}^{11}) = \text{Tr}(A_{-+}^{22} - A_{+-}^{22}). \quad (56)$$

Similarly generalizations of all other boundary conditions can be expressed as traces over the matrices $A_{\sigma\sigma'}^{ij}$. Indeed multiplying both sides of (53) by each of the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & T_{22}^{-1} \end{pmatrix} \quad \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ T_{12}^{-1} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & T_{21}^{-1} \\ 0 & 0 \end{pmatrix} \quad (57)$$

and taking the trace of the resulting four equations yields

$$\text{Tr}[A_{-+}^{11} - A_{+-}^{22}] = -\text{Tr}[T_{22}^{-1} T_{21} A_{++}^{11} + T_{12} T_{22}^{-1} A_{--}^{22}] = \text{Tr}[r A_{++}^{11} - r' A_{--}^{22}] \quad (58)$$

$$\text{Tr}[A_{+-}^{11} - A_{-+}^{22}] = -\text{Tr}[T_{11}^{-1} T_{12} A_{--}^{11} + T_{21} T_{11}^{-1} A_{++}^{22}] = \text{Tr}[r^\dagger A_{--}^{11} - r'^\dagger A_{++}^{22}] \quad (59)$$

$$\text{Tr}[A_{-+}^{11} + A_{+-}^{22}] = \text{Tr}[-T_{12}^{-1} T_{11} A_{++}^{11} + T_{22} T_{12}^{-1} A_{--}^{22}] = \text{Tr}[r^{\dagger-1} A_{++}^{11} + r'^{-1} A_{--}^{22}] \quad (60)$$

$$\text{Tr}[A_{+-}^{11} + A_{-+}^{22}] = \text{Tr}[-T_{21}^{-1} T_{22} A_{--}^{11} + T_{11} T_{21}^{-1} A_{++}^{22}] = \text{Tr}[r^{-1} A_{--}^{11} + r'^{-1} A_{++}^{22}]. \quad (61)$$

Subtracting (58) from (59) yields, in view of (56),

$$\text{Tr}[r A_{++}^{11} - r^\dagger A_{--}^{11}] = \text{Tr}[r' A_{--}^{22} - r'^\dagger A_{++}^{22}] \quad (62)$$

and subtracting (60) from (61) yields

$$\text{Tr}[r^{\dagger-1} A_{++}^{11} - r^{-1} A_{--}^{11}] = \text{Tr}[r'^{\dagger-1} A_{--}^{22} - r'^{-1} A_{++}^{22}]. \quad (63)$$

[†] A similar argument with $z_j > z_i$ yields an identical result and therefore the boundary condition is independent of the choice of Σ .

[‡] More precisely, one has to multiply equation (54) by T^{-1} and exploit the cyclic property of the trace.

[§] As will become clear in the next section, this corresponds to the conservation of the current.

Adding (58) to (59) and (60) to (61) yields

$$\text{Tr}[A_{+-}^{11} + A_{-+}^{11}] - \text{Tr}[A_{+-}^{22} + A_{-+}^{22}] = \text{Tr}[rA_{++}^{11} + r^\dagger A_{--}^{11}] - \text{Tr}[r'A_{--}^{22} + r'^\dagger A_{++}^{22}] \quad (64)$$

and

$$\begin{aligned} & \text{Tr}[A_{+-}^{11} + A_{-+}^{11}] + \text{Tr}[A_{+-}^{22} + A_{-+}^{22}] \\ &= \text{Tr}[r^{\dagger-1} A_{++}^{11} + r^{-1} A_{--}^{11}] + \text{Tr}[r'^{-1} A_{++}^{22} + r'^{\dagger-1} A_{--}^{22}]. \end{aligned} \quad (65)$$

For the translationally invariant case considered by Zaitsev, writing $t = t' = |t| \exp i\theta$, $r = |r| \exp i\phi$, $r' = -|r| \exp i(2\theta - \phi)$, $A_{\sigma\sigma}^{ii} = \check{G}_\sigma^i$ and $A_{\sigma-\sigma}^{ii} = \check{g}_\sigma^i$, one obtains from either of equations (62) and (63)

$$\check{G}_a^1 = \check{G}_a^2 \quad (66)$$

from (64) and (65)

$$\check{g}_s^1 - \check{g}_s^2 = |r|(\check{G}_s^1 + \check{G}_s^2) \quad (67)$$

$$\check{g}_s^1 + \check{g}_s^2 = \frac{1}{|r|}(\check{G}_s^1 - \check{G}_s^2) \quad (68)$$

and from (55)

$$\check{g}_a^1 = \check{g}_a^2 \quad (69)$$

where

$$\check{G}_{s,a}^1 = [\check{G}_+^1 \exp i\phi \pm \check{G}_-^1 \exp -i\phi]/2 \quad (70)$$

$$\check{G}_{s,a}^2 = [\check{G}_+^2 \exp i(\phi - 2\theta) \pm \check{G}_-^2 \exp i(2\theta - \phi)]/2 \quad (71)$$

and

$$\check{g}_{s,a}^i = [\check{g}_+^i \pm \check{g}_-^i]/2. \quad (72)$$

Equations (66) and (69), which are a limiting case of the more general relation (53), are identical to the boundary conditions of Zaitsev and will be discussed in the next section.

4. The quasi-classical Green's function approach

4.1. Quasi-classical equations

The quasi-classical approach to the theory of superconductivity, initiated by Eilenberger (1968) and Larkin and Ovchinnikov (1969) and developed by several authors (Usadel 1970, Eliashberg 1971, Larkin and Ovchinnikov 1973, 1975) has been largely used to analyse transport phenomena in dirty hybrid systems. This approach can be used to study thermodynamic and kinetic properties of superconductors, whose dimensions significantly exceed the Fermi wavelength $\lambda_F = 2\pi/k_F$. For the purpose of reviewing recent theoretical work, we briefly summarize the quasi-classical Green's function approach, following mainly Larkin and Ovchinnikov (1986). There exist in the literature several excellent reviews on the subject including that of Rammer and Smith (1986) and we refer the reader to these for a more detailed exposition. The derivation of the equation for quasi-classical Green's function starts from the Dyson equation for the matrix Green's function \check{G} in the non-equilibrium Keldysh formalism (Keldysh 1965):

$$(\check{G}_0^{-1} - \check{\Sigma})\check{G} = \check{1} \quad (73)$$

where

$$\check{G} = \begin{pmatrix} \hat{G}^R & \hat{G} \\ 0 & \hat{G}^A \end{pmatrix}. \quad (74)$$

Following standard notation, we indicate with a ‘hat’ matrices in Nambu space and define retarded, advanced and Keldysh Green’s functions

$$\hat{G}^R(1, 2) = \theta(t_1 - t_2)(\hat{G}^{-+}(1, 2) - \hat{G}^{+-}(1, 2)) \quad (75)$$

$$\hat{G}^A(1, 2) = -\theta(t_2 - t_1)(\hat{G}^{-+}(1, 2) - \hat{G}^{+-}(1, 2)) \quad (76)$$

$$\hat{G}(1, 2) = \hat{G}^{-+}(1, 2) + \hat{G}^{+-}(1, 2) \quad (77)$$

where

$$i\hat{G}^{-+} = \begin{pmatrix} \langle \psi_\uparrow(1)\psi_\uparrow^\dagger(2) \rangle & \langle \psi_\uparrow(1)\psi_\downarrow(2) \rangle \\ -\langle \psi_\downarrow^\dagger(1)\psi_\uparrow^\dagger(2) \rangle & -\langle \psi_\downarrow^\dagger(1)\psi_\downarrow(2) \rangle \end{pmatrix} \quad (78)$$

$$i\hat{G}^{+-} = -\begin{pmatrix} \langle \psi_\uparrow^\dagger(2)\psi_\uparrow(1) \rangle & \langle \psi_\downarrow(2)\psi_\uparrow(1) \rangle \\ -\langle \psi_\uparrow^\dagger(2)\psi_\downarrow^\dagger(1) \rangle & -\langle \psi_\downarrow(2)\psi_\downarrow^\dagger(1) \rangle \end{pmatrix}. \quad (79)$$

In these expressions, $1 \equiv (\mathbf{r}_1, t_1)$ and $2 \equiv (\mathbf{r}_2, t_2)$, which allow equation (73) to be written in the less compact form

$$\int d2 (\check{G}_0^{-1}(1, 2) - \check{\Sigma}(1, 2))\check{G}(2, 3) = \check{\delta}(1 - 3).$$

The equation conjugate to equation (73) is

$$\check{G}(\check{G}_0^{-1} - \check{\Sigma}) = \check{1} \quad (80)$$

and taking the difference between equations (73) and (80) yields

$$[\check{G}_0^{-1} - \check{\Sigma}, \check{G}] = 0. \quad (81)$$

This equation is simplified by going to the centre-of-mass and relative coordinates (\mathbf{R}, T) and (\mathbf{r}, t) defined as

$$\mathbf{r}_{1,2} = \mathbf{R} \pm \mathbf{r}/2 \quad t_{1,2} = T \pm t/2$$

and Fourier transforming with respect to \mathbf{r} and t , and introducing the quasi-classical Green’s function defined by

$$\check{g}(\mathbf{R}, T; \hat{\mathbf{p}}, \epsilon) = \frac{i}{\pi} \int_{-\infty}^{\infty} d\xi \check{G}(\mathbf{R}, T; \mathbf{p}, \epsilon) \quad (82)$$

where $\xi = p^2/2m - \mu$ is the energy measured with respect to the Fermi level and $\hat{\mathbf{p}}$ is the unit vector in the direction of the momentum \mathbf{p} . On the assumption that the self-energy depends weakly on the energy ξ , one sets $\xi = 0$ in $\check{\Sigma}$, which yields

$$\partial_T \{\check{\tau}_z, \check{g}\} + v_F \hat{\mathbf{p}} \cdot \partial_{\mathbf{R}} \check{g} - i\epsilon [\check{\tau}_z, \check{g}] + i[\check{\Sigma}, \check{g}] = 0 \quad (83)$$

where the curly brackets denote the anticommutator. This is the equation for the quasi-classical Green’s function first derived by Eilenberger (1968). $\check{\tau}_z$ is a block-diagonal matrix with diagonal block entries $\hat{\sigma}_z$. In deriving equation (80), the non-homogeneous term present in the Dyson equation (73) was eliminated and, as a consequence, equation (83) determines \check{g} only up to a constant. As discussed in Eilenberger (1968) and Larkin and Ovchinnikov (1973, 1975), a useful normalization condition[†] for \check{g} is

$$\check{g}\check{g} = 1. \quad (84)$$

[†] A more recent discussion about the meaning of the normalization condition can be found in Shelankov (1985). See also Zaitsev (1984).

In terms of the quasi-classical Green's function, the physical current is given by

$$\mathbf{j}(\mathbf{R}, T) = -\frac{1}{4}N_0v_F \int_{-\infty}^{\infty} d\epsilon \langle \mu \text{Tr}(\sigma_z \check{g}(\epsilon, \hat{\mathbf{p}}, \mathbf{R}, T)) \rangle \quad (85)$$

where $N_0 = m^2v_F/2\pi^2$ is the free single-particle density of states per spin, σ_z the usual Pauli matrix and v_F the Fermi velocity. Here $\langle \cdots \rangle$ indicates the average over the angle θ formed by $\hat{\mathbf{p}}$ and the direction of \mathbf{R} , i.e., $(1/2) \int_{-1}^1 d\cos(\theta) (\cdots)$. It should be noted that, because of the angular average, only the antisymmetric part of \check{g} enters the expression for the current.

In the dirty limit, in the case of an isotropic scattering potential with elastic scattering time τ , the effect of non-magnetic impurities can be described by the self-energy

$$\check{\Sigma} = -\frac{i}{2\tau} \langle \check{g} \rangle \quad (86)$$

and equation (83) can be considerably simplified (Usadel 1970). In this case one expands \check{g} in spherical harmonics keeping only the s- and p-wave terms:

$$\check{g}(\cos(\theta)) = \check{g}_0 + \cos(\theta)\check{g}_1 \quad (87)$$

with \check{g}_0 and \check{g}_1 not depending on $\cos(\theta)$ and $\cos(\theta)\check{g}_1 \ll \check{g}_0$. On inserting equation (87) in equation (83), \check{g}_1 is expressed in terms of \check{g}_0 :

$$\check{g}_1 = -l\check{g}_0 \partial_R \check{g}_0 \quad (88)$$

and for \check{g}_0 one obtains a diffusion-like equation:

$$D \partial_R \check{g}_0 \partial_R \check{g}_0 + i\epsilon [\check{\tau}_z, \check{g}_0] - \partial_T \{\check{\tau}_z, \check{g}_0\} = 0 \quad (89)$$

with $D = v_F l \tau / 3$ the diffusion coefficient and $l = v_F \tau$ the mean free path.

Equations (83) and (89) are valid for bulk systems and, in the presence of boundaries, must be supplemented by boundary conditions which connect the Green's functions evaluated in different regions. These boundary conditions have turned out to be crucial to the recent development of transport theory in hybrid structures. For this reason, we present a brief discussion of these in the following subsection.

4.2. Boundary conditions for quasi-classical Green's functions

In this subsection, we discuss the boundary conditions for the quasi-classical Green's function (Zaitsev 1984), given by equations (66) to (69) of section 3. The antisymmetric functions $\check{g}_a = \check{g}_a^1 = \check{g}_a^2$ and $\check{G}_a = \check{G}_a^1 = \check{G}_a^2$ are continuous across the boundary, while the symmetric ones \check{g}_s^i and \check{G}_s^i experience a jump determined by the transparency of the barrier. The size of the jump vanishes for perfectly transmitting interfaces.

The function \check{g} is the quasi-classical Green's function satisfying equation (83), whereas the Green's function \check{G} arises from reflected waves at the boundary. It is therefore necessary to derive an equation of motion for \check{G} to be used together with equation (83). We will not show this derivation here, because in actual calculations, one only uses the Green's function \check{g} . Zaitsev has shown that the final result is of the form

$$\check{g}^i(k_i) \check{G}^i(k_i) = (-1)^i \text{sgn}(k_i) \check{G}^i(k_i) \quad (90)$$

which must be used in conjunction with the normalization condition

$$\check{g}^i \check{g}^i = \check{1}. \quad (91)$$

The boundary conditions for the antisymmetric components are already decoupled with respect to the \check{g}_a and \check{G}_a . To decouple the symmetric components, we express equations (90) and (91) in terms of the symmetric and antisymmetric parts to yield

$$\begin{aligned}\check{g}_s^i \check{G}_s^i + \check{g}_a \check{G}_a &= (-1)^i \check{G}_a \\ \check{g}_s^i \check{G}_a + \check{g}_a \check{G}_s^i &= (-1)^i \check{G}_s^i\end{aligned}\quad (92)$$

and

$$\check{g}_s^i \check{g}_s^i + \check{g}_a \check{g}_a = \check{1} \quad \check{g}_s^i \check{g}_a + \check{g}_a \check{g}_s^i = 0. \quad (93)$$

By manipulating equation (92) one obtains

$$\check{g}_s^1 \check{G}_s^1 + \check{g}_s^2 \check{G}_s^2 = \check{g}_a (\check{g}_s^1 \check{G}_s^1 - \check{g}_s^2 \check{G}_s^2) \quad (94)$$

which is the extra condition to be used together with equations (67) and (68). By means of equations (67) and (68) one expresses \check{G}_s^2 and \check{G}_s^1 in terms of \check{g}_s^1 and \check{g}_s^2 and substitutes in equation (94), so the final boundary condition reads

$$\check{g}_a [R(1 - \check{g}_a \check{g}_a) + (T/4)(\check{g}_s^1 - \check{g}_s^2)^2] = (T/4)(\check{g}_s^2 \check{g}_s^1 - \check{g}_s^1 \check{g}_s^2) \quad (95)$$

where $R = |r|^2$, $T = 1 - R$ are the reflection and transmission coefficients of the barrier.

As a simple example, we consider the normal case, when the Green's function is a two-by-two matrix

$$\check{g} = \begin{pmatrix} 1 & g \\ 0 & -1 \end{pmatrix} \quad (96)$$

with $g = 2f$, f being the usual distribution function entering the Boltzmann kinetic equation. On observing that

$$[\check{g}_{1s}, \check{g}_{2s}] = 4 \begin{pmatrix} 0 & f_{1s} - f_{2s} \\ 0 & 0 \end{pmatrix} \quad (\check{g}_1 - \check{g}_2)^2 = \check{0} \quad \check{g}_a^2 = \check{0} \quad (97)$$

the boundary condition (95) assumes the form

$$f_a = (T/2R)(f_{1s} - f_{2s}). \quad (98)$$

It is worth noticing that this result could have been obtained directly from simple counting arguments of the form

$$f_1(k) = T f_2(k) + R f_1(-k) \quad f_2(-k) = T f_1(-k) + R f_2(k). \quad (99)$$

In view of equation (85) for the current, equation (98) yields for the conductance of a tunnel junction in the normal case

$$G_T = e^2 N_0 v_F \langle \mu T / R \rangle \quad (100)$$

where $\mu = \theta$. In the general case, equation (95) can be simplified in the low-transparency limit. If $T \ll 1$, then $\check{g}_a \approx T$ and the boundary condition equation (95) reduces to

$$\check{g}_a = (T/4R)[\check{g}_{1s}, \check{g}_{2s}]. \quad (101)$$

In the dirty limit, $\check{g}_a = -\mu l \check{g}_s \partial_R \check{g}_s$ and equations (88) and (89) yield the boundary condition (Kuprianov and Lukichev 1988)

$$l \check{g}_{1s} \partial_R \check{g}_{1s} = l \check{g}_{2s} \partial_R \check{g}_{2s} \quad (102)$$

which after multiplying equation (101) by μ and taking the angular average yields

$$l \check{g}_{1s} \partial_R \check{g}_{1s} = \frac{3}{4} \langle \mu T / R \rangle [\check{g}_{2s}, \check{g}_{1s}]. \quad (103)$$

By defining a conserved ‘super’-current \check{I} (cf. equation (85)) one finally arrives at the following boundary condition:

$$\check{I} = \frac{\sigma}{e} \check{g}_{2s} \partial_R \check{g}_{2s} = \frac{G_T}{2e} [\check{g}_{2s}, \check{g}_{1s}], \quad (104)$$

where $\sigma = 2e^2 N_0 v_F l / 3$ is the Drude electrical conductivity. Equation (104) is the desired boundary condition to be used together with the diffusion equation (89) in the presence of boundaries. Equation (104) is strictly valid in the case of small barrier transparency. Such restriction has been recently relaxed by Lambert *et al* (1997). In the next subsection we show how to compute the current–voltage relation of the two fundamental elements of a hybrid system, namely a tunnel junction and a diffusive region.

4.3. Quasi-classical theory at work

(i) A tunnel junction. Consider the current through a tunnel junction as given by the commutator on the right-hand side of equation (104). The physical current is obtained from the Keldysh component of equation (104):

$$[\check{g}_2, \check{g}_1]_K = \hat{g}_2^R \hat{g}_1 + \hat{g}_2 \hat{g}_1^A - \hat{g}_1^R \hat{g}_2 - \hat{g}_1 \hat{g}_2^A$$

where we have dropped the subscript ‘s’. The normalization condition, $\check{g}\check{g} = \check{I}$, allows us to choose

$$\hat{g}_{1,2} = \hat{g}_{1,2}^R \hat{f}_{1,2} - \hat{f}_{1,2} \hat{g}_{1,2}^A$$

where the matrix \hat{f} can be taken to be diagonal (Larkin and Ovchinnikov 1975):

$$\hat{f}_{1,2} = f_{1,2}^0 \hat{\sigma}_0 + f_{1,2}^z \hat{\sigma}_z.$$

As a result, multiplying by $\hat{\sigma}_z$ and taking the trace yields (cf. equation (85))

$$j = \frac{G_T}{16e} \int_{-\infty}^{\infty} d\epsilon \text{Tr}(\hat{\sigma}_z(f_1^0 \hat{I}_a + f_2^0 \hat{I}_b + f_1^z \hat{I}_c + f_2^z \hat{I}_d)) \quad (105)$$

where

$$\begin{aligned} \hat{I}_a &= [\hat{g}_2^R(\hat{g}_1^R - \hat{g}_1^A) - (\hat{g}_1^R - \hat{g}_1^A)\hat{g}_2^A] \\ \hat{I}_b &= -[\hat{g}_1^R(\hat{g}_2^R - \hat{g}_2^A) - (\hat{g}_2^R - \hat{g}_2^A)\hat{g}_1^A] \\ \hat{I}_c &= [\hat{g}_2^R(\hat{g}_1^R \hat{\sigma}_z - \hat{\sigma}_z \hat{g}_1^A) - (\hat{g}_1^R \hat{\sigma}_z - \hat{\sigma}_z \hat{g}_1^A)\hat{g}_2^A] \\ \hat{I}_d &= -[\hat{g}_1^R(\hat{g}_2^R \hat{\sigma}_z - \hat{\sigma}_z \hat{g}_2^A) - (\hat{g}_2^R \hat{\sigma}_z - \hat{\sigma}_z \hat{g}_2^A)\hat{g}_1^A]. \end{aligned}$$

Due to the normalization condition $\hat{g}^{R(A)} \hat{g}^{R(A)} = \hat{1}$, we have

$$\hat{g}^{R(A)} = \mathbf{g}^{R(A)} \cdot \boldsymbol{\sigma} \equiv \sum_{i=1}^3 g_i^{R(A)} \sigma_i$$

where $\mathbf{g}^{R(A)} = (iF^{R(A)} \sin(\phi), iF^{R(A)} \cos(\phi), G^{R(A)})$ and ϕ is the phase of the superconducting order parameter. Hence

$$j = \frac{G_T}{8e} \int_{-\infty}^{\infty} d\epsilon (I_J + I_{PI}) \quad (106)$$

where

$$I_J = i \sin(\phi_1 - \phi_2) [f_2^0 (F_2^R - F_2^A) (F_1^R + F_1^A) + f_1^0 (F_2^R + F_2^A) (F_1^R - F_1^A)]$$

and

$$I_{PI} = [(G_1^R - G_1^A)(G_2^R - G_2^A) + \cos(\phi_1 - \phi_2)(F_2^R + F_2^A)(F_1^R + F_1^A)](f_1^z - f_2^z).$$

In equation (106), I_J is the Josephson current, while I_{PI} is sometimes referred to as the quasi-particle and interference current. To clarify equation (106), consider the case where the two regions on the left and right of the barrier are at equilibrium and $\phi_1 = \phi_2$, and hence the Josephson current vanishes. If the distribution functions $f_{1,2}^z$ have their equilibrium form,

$$f_{1,2}^z = \frac{1}{2}(\tanh((\epsilon + eV_{1,2})/2T) - \tanh((\epsilon - eV_{1,2})/2T))$$

then the current through the junction becomes

$$j = \frac{G_T}{2e} \int_{-\infty}^{\infty} d\epsilon (f_1^z - f_2^z) M_{12} \quad (107)$$

where

$$M_{12} = \frac{1}{4}((G_1^R - G_1^A)(G_2^R - G_2^A) + (F_2^R + F_2^A)(F_1^R + F_1^A)).$$

At $T = 0$, this reduces to

$$I = G_T M_{12}|_{\epsilon=0}(V_1 - V_2). \quad (108)$$

Equation (108) shows that conductance of the tunnel junction is renormalized by a term M_{12} which depends on the amount of superconducting pairing on the two sides of the junction. In the limit of normal systems, $M_{1,2} = 1$, and one recovers the conductance of the normal state (cf. equation (100)).

(ii) A diffusive region. We derive now the equivalent of equation (107) for a diffusive region in the absence of inelastic processes. In this case the equation for the ‘hat’ components of the Green’s function (89) reads

$$\partial_R \hat{g}^{R(A)} \partial_R \hat{g}^{R(A)} = i\epsilon [\hat{\sigma}_z, \hat{g}^{R(A)}] \quad \partial_R (\hat{g}^R \partial_R \hat{g} + \hat{g} \partial_R \hat{g}^A) = 0.$$

Using $\hat{g} = \hat{g}^R \hat{f} - \hat{f} \hat{g}^A$ and the equation for $\hat{g}^{R(A)}$, the equation for the Keldysh component becomes

$$\partial_R [\partial_R \hat{f} - \hat{g}^R (\partial_R \hat{f}) \hat{g}^A] - (\hat{g}^R \partial_R \hat{g}^R)(\partial_R \hat{f}) - (\partial_R \hat{f})(\hat{g}^A \partial_R \hat{g}^A) = 0.$$

The last equation yields \hat{f} , once \hat{g}^R and \hat{g}^A have been determined. Choosing

$$\hat{f} = f^0 \hat{\sigma}_0 + f^z \hat{\sigma}_z \quad \text{and} \quad \hat{g}^R = G^R \hat{\sigma}_z + iF^R \hat{\sigma}_y$$

yields, after multiplying by $\hat{\sigma}_z$ and taking the trace,

$$\partial_R [(1 - G^R G^A - F^R F^A) \partial_R f^z] = 0. \quad (109)$$

If we consider a diffusive region of length L , the distribution function depends only on the longitudinal coordinate x , and $f^z(x)$ has the form

$$f^z(x) = m(x) \frac{f^z(L) - f^z(0)}{m(L)} + f^z(0) \quad (110)$$

where

$$m(x) = \int_0^x dx' \frac{1}{1 - G^R(x') G^A(x') - F^R(x') F^A(x')}.$$

Using the formula for the physical current (85) one obtains

$$I = \frac{\sigma}{8e} \int_{-\infty}^{\infty} d\epsilon (f^z(L) - f^z(0)) \frac{1}{m(L)}. \quad (111)$$

To use equation (111) one has to solve the equation for \hat{g}^R . Since $\hat{g}^R \hat{g}^R = 1$, one can write $G^R = \cosh(u)$ and $F^R = \sinh(u)$, which yields

$$D \partial_x^2 u + 2i\epsilon \sinh(u) = 0. \quad (112)$$

Equation (112) is a non-linear equation, for which approximate analytical (Zaitsev 1990, Volkov *et al* 1993, Volkov 1994b, Zaitsev 1994) and numerical (Zhou *et al* 1995, Yip 1995, Nazarov and Stoof 1996, Golubov *et al* 1997) solutions have been obtained. For the purposes of the present discussion, we note that, at zero energy, the solution may be obtained easily and takes the form

$$u(x) = \frac{u(L) - u(0)}{L}x + u(0).$$

Since

$$1 - G^R G^A - F^R F^A = 2 \cosh^2(\text{Re}(u))$$

the current (111) becomes

$$I = \frac{\sigma}{L} \frac{\tanh(\text{Re}(u(L))) - \tanh(\text{Re}(u(0)))}{\text{Re}(u(L)) - \text{Re}(u(0))} V \quad (113)$$

where we have used the fact that, for $T \rightarrow 0$,

$$\frac{1}{4e} \int_{-\infty}^{\infty} d\epsilon (f^z(L) - f^z(0)) \rightarrow V.$$

The values $u(L)$ and $u(0)$ are determined by attaching the diffusive region to electrodes which are assumed to be in equilibrium. For a normal system $u \equiv 0$, while for a superconducting one $\text{Re}(u) = 0$ and $\text{Im}(u) = -1$. As a result the resistance of a diffusive region does not depend on the nature of the electrodes.

Nazarov recognized (Nazarov 1994) that the simplicity of the results of equations (108) and (113) can be generalized to an arbitrary circuit with normal and superconducting electrodes. In fact, the evaluation of the conductance can be reduced to the use of a set of simple rules, which we now state. A circuit is made of dispersive elements (diffusive regions and tunnel junctions) connected to one another and to the electrodes by nodes. To each circuit element is associated a ‘scalar’ current, j , which is just the physical current, and a ‘vector’ current I . To each node is associated a ‘spectral’ unit vector s . The vector current flowing through a dispersive element depends on the spectral vectors at the two ends. It is useful to imagine the spectral vector as the spherical coordinate in the northern hemisphere. The rules are as follows.

(I) The conductance in the presence of superconducting electrodes is the same as in normal circuits, with renormalized tunnel junction conductances. The renormalization factor is given by the scalar product of the spectral vectors at the two sides of the junction.

(II) The spectral vector of a normal electrode is at the north pole (\hat{z}), while that of a superconducting electrode lies on the equator, with the longitude given by the phase of the order parameter.

(III) The vector current is perpendicular to both the spectral vectors at the ends of a given element. If ϕ is the angle between the spectral vectors at the two ends of a given element, the magnitude of the vector current is $I = G_D \phi$ and $I = G_T \sin(\phi)$ for diffusive and tunnel junction elements, respectively.

(IV) The vector current is conserved at each node (the generalized Kirchhoff rule).

These rules will be applied to a N–I–S interface in the following section. In addition to the above, one might also consider a ballistic point contact (for a review see van Wees and Takayanagi 1997).

5. The N–S interface and zero-bias anomalies

In this section, we show how the multiple-scattering approach and quasi-classical theory can be used to describe the simplest example of phase-coherent structure, namely a N–S interface.

5.1. BTK theory and the Andreev approximation

The simplest problem of a one-dimensional N–I–S system was examined by Blonder *et al* (1982). This solution is also valid in higher dimensions, provided that there is translational invariance in the direction perpendicular to the electronic motion. To see this, imagine a two-dimensional system with a finite width in the y -direction and infinite in the x -direction. The motion along y is quantized, and for fixed k_y the problem becomes one-dimensional and the Bogoliubov–de Gennes equations reduce to

$$\begin{aligned} E\psi(x) &= \left(-\frac{\hbar^2}{2m}\partial_x^2 - \mu\right)\psi(x) + \Delta(x)\phi(x) \\ E\phi(x) &= \left(\frac{\hbar^2}{2m}\partial_x^2 + \mu\right)\phi(x) + \Delta(x)\psi(x) \end{aligned} \quad (114)$$

where $\bar{\mu} = \mu - \hbar^2 k_y^2 / 2m$ is the effective chemical potential in a given channel. For a pairing potential with a step-like spatial variation of the form

$$\Delta(x) = \Delta_0 \Theta(x) \quad (115)$$

the Bogoliubov–de Gennes equations can be solved by matching wave functions. For $x < 0$ the solution (ψ, ϕ) for an incident electron from the left is

$$\begin{aligned} \psi^L(x) &= e^{ikx} + r_0 e^{-ikx} \\ \phi^L(x) &= r_a e^{iqx} \end{aligned} \quad (116)$$

and for $x > 0$

$$\begin{aligned} \psi^R(x) &= t_0 u e^{i\bar{k}x} + t_a \bar{u} e^{-i\bar{q}x} \\ \phi^R(x) &= t_0 v e^{i\bar{k}x} + t_a \bar{v} e^{-i\bar{q}x}. \end{aligned} \quad (117)$$

The coherence factors u and v correspond to a particle-like excitation of energy E and momentum \bar{k} , while \bar{u} and \bar{v} correspond to a hole-like excitation at the same energy and with momentum \bar{q} , where $E = (\hbar^2/2m)k^2 - \bar{\mu} = -(\hbar^2/2m)q^2 + \bar{\mu}$ and $E^2 = [(\hbar^2/2m)\bar{k}^2 - \bar{\mu}]^2 + \Delta^2 = [(\hbar^2/2m)\bar{q}^2 + \bar{\mu}]^2 + \Delta^2$. Also

$$\begin{aligned} u^2(v^2) &= \frac{1}{2}(1 \pm \sqrt{E^2 - \Delta^2}/E) \\ \bar{v}^2(\bar{u}^2) &= \frac{1}{2}(1 \pm \sqrt{E^2 - \Delta^2}/E) \end{aligned} \quad (118)$$

with

$$\text{sgn}(u/v) = \text{sgn}(\Delta/(E - \epsilon_{\bar{k}})) \quad \text{sgn}(\bar{u}/\bar{v}) = \text{sgn}(\Delta/(E - \epsilon_{\bar{q}}))$$

where $\epsilon_{\bar{k}} = (\hbar^2/2m)\bar{k}^2 - \bar{\mu}$ and $\epsilon_{\bar{q}} = (\hbar^2/2m)\bar{q}^2 + \bar{\mu}$ for $E > \Delta$, while for $E < \Delta$ we have

$$\begin{aligned} u/v &= \Delta/(E - i\sqrt{\Delta^2 - E^2}) \\ \bar{u}/\bar{v} &= \Delta/(E + i\sqrt{\Delta^2 - E^2}). \end{aligned} \quad (119)$$

To solve for the scattering coefficients r_0 , r_a , t_0 and t_a one needs matching conditions at the interface $x = 0$. These are the usual continuity condition for (ψ, ϕ) and their derivatives and read

$$\begin{aligned} 1 + r_0 &= t_0 u + t_a \bar{u} \\ r_a &= t_0 v + t_a \bar{v} \\ 1 - r_0 &= \frac{\bar{k}}{k} t_0 u - \frac{\bar{q}}{k} t_a \bar{u} + (1 + r_0) U_0 / i k \\ r_a &= \frac{\bar{k}}{k} t_0 v - \frac{\bar{q}}{k} t_a \bar{v} + r_a U_0 / i k \end{aligned} \quad (120)$$

where the term in U_0 in the last two equations of (120) allows for a delta-like potential $U(x) = U_0 \delta(x)$ in order to reproduce the BTK model of an N–I–S interface. The system (120) completely determines the four scattering coefficients. The detailed solution is of course straightforward, but the resulting expressions look rather cumbersome, so it is convenient to resort to Andreev's approximation of setting $k = q = \bar{k} = \bar{q}$, which amounts to neglecting terms of the order Δ / E_F . Writing $z = 2mU_0 / 2k\hbar^2$ yields

$$\begin{aligned} t_0 &= \frac{\bar{v}(1 - iz)}{u\bar{v}(1 + z^2) - v\bar{u}z^2} \\ t_a &= \frac{ivz}{u\bar{v}(1 + z^2) - v\bar{u}z^2} \\ r_a &= \frac{v\bar{v}}{u\bar{v}(1 + z^2) - v\bar{u}z^2} \\ r_0 &= \frac{(v\bar{u} - u\bar{v})(z^2 + iz)}{u\bar{v}(1 + z^2) - v\bar{u}z^2}. \end{aligned} \quad (121)$$

For $E < \Delta$, where $T_0 = T_a = 0$, this yields

$$R_a = \frac{\Delta^2}{E^2 + (1 + 2z^2)^2(\Delta^2 - E^2)} \quad (122)$$

and, in view of unitarity, $R_0 = 1 - R_a$. For $E > \Delta$ one obtains

$$\begin{aligned} R_a &= \frac{\Delta^2}{(E + (1 + 2z^2)^2 \sqrt{E^2 - \Delta^2})^2} \\ R_0 &= \frac{4z^2(1 + z^2)(E^2 - \Delta^2)}{(E + (1 + 2z^2)\sqrt{E^2 - \Delta^2})^2} \\ T_0 &= \frac{2E(1 + z^2)(E + \sqrt{E^2 - \Delta^2})}{(E + (1 + 2z^2)\sqrt{E^2 - \Delta^2})^2} \\ T_a &= \frac{2Ez^2(E - \sqrt{E^2 - \Delta^2})}{(E + (1 + 2z^2)(\sqrt{E^2 - \Delta^2}))^2}. \end{aligned} \quad (123)$$

The above treatment of a planar interface has been extended by a number of authors, including Riedel and Bagwell (1993). In one dimension and at zero temperature, the boundary conductance of equation (26) reduces to the BTK result

$$G = (2e^2/h)(1 - R_0 + R_a) \quad (124)$$

which for $E < \Delta$ becomes $G = (2e^2/h)2R_a$, while for $E \gg \Delta$, $R_a \rightarrow 0$ and one recovers the Landauer expression (2), $G = (2e^2/h)(1 - R_0) = (2e^2/h)T_0$. It is interesting to

compare the sub-gap conductance

$$G_{NS} = (2e^2/h) \frac{2}{(1 + 2z^2)^2} \quad (125)$$

with that in the normal state:

$$G_N = (2e^2/h) T_N = (2e^2/h) \frac{1}{1 + z^2}. \quad (126)$$

This allows one to express G_{NS} in terms of the dimensionless normal-state conductance T_N :

$$G_{NS} = (2e^2/h) \frac{2T_N^2}{(2 - T_N)^2} \quad (127)$$

which is a consequence of the fact that the superconducting electrode in this approximation only enters as a boundary condition.

For a disordered N region in contact with a superconductor the result (127) can be extended to higher dimensions by introducing the eigenvalues T_n of the transmission matrix $T = t_{pp} t_{pp}^\dagger$ of the N region and summing equation (127) over all n (Beenakker 1992), to yield equation (132) below. For $z \neq 0$, equation (124) predicts that the low-energy conductance is suppressed by a factor $(1 + z^2)^{-2}$, as compared to the $z = 0$ case. Already for values of $z \sim 3$ this implies an almost vanishing sub-gap conductance. At the gap energy, the expression for R_a achieves a peak value of unity within the Andreev approximation, with R_a decaying to zero for energies above the gap. This picture clearly resembles that of the tunnelling approach, where the conductance is controlled by the superconducting density of states, but is at variance with the experiment of Kastalskii *et al* (1991). It was soon recognized that the low-energy peak observed by Kastalskii *et al* was due to Andreev scattering (van Wees *et al* 1992), but it was not possible to explain it using the theory discussed above, because the interplay between scattering due to disorder in the normal region and Andreev scattering at the N-S interface is crucial. The first insight into the ZBA was based on a detailed description using quasi-classical Green's function methods (Zaitsev 1990, Volkov and Klapwijk 1992, Volkov *et al* 1993). Hekking and Nazarov (1993, 1994) obtained similar results using a tunnelling approach and Beenakker and co-workers (Beenakker *et al* 1994, Marmorkos *et al* 1993) confirmed these results using multiple-scattering methods.

To illustrate how these different methods explain the ZBA, the following subsection briefly summarizes quasi-classical predictions for the zero-energy conductance and in subsection 5.3 the multiple-scattering approach is discussed.

5.2. The quasi-classical approach: circuit theory

To apply the 'circuit rules' of quasi-classical theory (Nazarov 1994) to a N-I-S system, we consider two resistances in series; the first is associated with the normal disordered region (N), while the second is associated with the tunnel junction (I). We assume that the normal region progressively widens in a macroscopic electrode and that the superconductor plays the role of the other electrode. In this circuit there are three spectral vectors. The vector associated with the normal electrode is $s_N = z$ and that associated with the superconducting electrode is $s_S = x$, both of which are fixed (cf. rule II of circuit theory). The third spectral vector is located at the node connecting the diffusive element and the tunnel junction and lies in the x - z plane: $s = \cos(\theta)x + \sin(\theta)z$. According to the circuit rules, one computes

the conductance of this simple circuit in the usual way, but the resistance of the tunnel junction is renormalized (cf. rule I of circuit theory):

$$G_{NS} = \frac{1}{1/G_D + 1/(G_T \sin(\theta))} \quad (128)$$

where G_D and G_T are the conductances of the normal diffusive region and of the tunnel junction in the normal state, respectively. The angle θ is determined from the conservation of the vector current (cf. rules III–IV of circuit theory) and satisfies the equation

$$G_D \theta = \cos(\theta) G_T. \quad (129)$$

This equation has two simple limits: (i) $G_D/G_T \gg 1$, $\theta \approx G_T/G_D$, $G_{NS} \approx G_T^2/G_D$; (ii) $G_D/G_T \ll 1$, $\theta \approx \pi/2$, $G_{NS} \approx G_D + G_T$. This shows that, on varying the parameter G_T/G_D , the conductance switches from a quadratic to a linear dependence on the tunnel junction conductance. A two-particle process behaves effectively as a one-particle process ($G_{NS} \approx G_T$). The smallness of the probability for a two-particle transmission is compensated by the relative increase of disorder-induced scattering on the normal side. A detailed comparison between equation (128) and a numerical s -matrix investigation can be found in Claughton *et al* (1996).

5.3. The scattering approach: random-matrix theory

To illustrate the multiple-scattering approach to the ZBA, we now show how equation (128) may be derived using random-matrix theory (Beenakker *et al* 1994); for a more detailed introduction to random-matrix theory applied to quantum transport see also Stone *et al* (1991) and the recent review by Beenakker (1998)). The starting point is the following scaling equation for the eigenvalue density $\rho(\lambda, s) = \langle \sum_{n=1}^N \delta(\lambda - \lambda_n) \rangle$ (Mello and Pichard 1989):

$$\frac{\partial}{\partial s} \rho(\lambda, s) = -\frac{2}{N} \frac{\partial}{\partial \lambda} \lambda (1 + \lambda) \rho(\lambda, s) \frac{\partial}{\partial \lambda} \int_0^\infty d\lambda' \rho(\lambda', s) \ln |\lambda - \lambda'| \quad (130)$$

where λ_n is related to the transmission eigenvalue T_n of the n th transverse mode by

$$\lambda_n = (1 - T_n)/T_n \quad (131)$$

and the average $\langle \dots \rangle$ is over the appropriate ensemble. Here $s = L/l$, where L is the size of the system and l the mean free path. N is the number of open channels in the normal region. The above equation yields the ensemble-averaged eigenvalue density for a diffusive system and it is physically equivalent to the quasi-classical theory outlined in the previous section. Once $\rho(\lambda, s)$ is known, the conductance of the N–I–S system is given by (cf. equation (127))

$$G_{NS} = \frac{2e^2}{h} \sum_{n=1}^N \frac{T_n^2}{(2 - T_n)^2} = \frac{2e^2}{h} 2 \int_0^\infty d\lambda \rho(\lambda, s) \frac{1}{(1 + 2\lambda)^2}. \quad (132)$$

With respect to the scaling variable s , the presence of a barrier at the N–S interface acts as an ‘initial condition’ instead of a ‘boundary condition’. In fact, at the interface, where $s = 0$, the eigenvalue density in a mode-independent approximation is given by

$$\rho(\lambda, 0) = N \delta(\lambda - (1 - T)/T) \quad (133)$$

where T is the transmission coefficient of the barrier. To solve the evolution equation (130) with the initial condition (133) one introduces the auxiliary function $F(z, s)$

$$F(z, s) = \int_0^\infty d\lambda' \frac{\rho(\lambda', s)}{z - \lambda'} \quad (134)$$

which is analytic in the z -complex plane with a cut along the positive real axis. In terms of F , the evolution equation (130) becomes

$$N \frac{\partial}{\partial s} F + \frac{\partial}{\partial z} z(1+z) F^2 = 0. \quad (135)$$

Writing $z = \sinh^2(\xi)$ and

$$U(\xi, s) = \frac{\sinh(2\xi)}{2N} F(z(\xi), s) \quad (136)$$

yields

$$\frac{\partial}{\partial s} U + U \frac{\partial}{\partial \xi} U = 0 \quad (137)$$

which is Euler's equation of an ideal two-dimensional fluid. The real and imaginary parts of U are the cartesian components of the velocity field $U = U_x + iU_y$. The solution can then be obtained directly in terms of the initial condition $U_0(\xi) \equiv U(\xi, s)$, as

$$U(\xi, s) = U_0(\xi - sU(\xi, s)). \quad (138)$$

The initial condition (133) for the eigenvalue density translates U into

$$U_0(\xi) = \frac{\sinh(2\xi)}{2(\cosh^2(\xi) - T^{-1})} \quad (139)$$

which in the limit of small transparency barrier, i.e., $T \ll 1$, is approximated by (compare equation (101))

$$U_0(\xi) = -\frac{T}{2} \sinh(2\xi). \quad (140)$$

To evaluate the conductance, we note that in terms of the function U , the equation (132) becomes

$$G_{NS} = \frac{2e^2}{h} N \lim_{\xi \rightarrow -i\pi/4} \frac{\partial U}{\partial \xi}. \quad (141)$$

By using (138) together with (140) and writing out the real and imaginary part of U at $\xi = -i\pi/4$, we obtain the pair of equations

$$\begin{aligned} U_x &= -\frac{T}{2} \sinh(2sU_x) \sin(2sU_y) \\ U_y &= \frac{T}{2} \cosh(2sU_x) \cos(2sU_y). \end{aligned} \quad (142)$$

The first equation requires $U_x = 0$, and by defining $\theta = 2sU_y$, we rewrite the second equation as

$$\theta = sT \cos(\theta). \quad (143)$$

By recalling that $G_D = (2e^2/h)(l/L)$ and $G_T = (2e^2/h)T$ are the conductances of the disordered normal region and of the tunnel barrier, we recover the equation for the conservation of the vector current (129) obtained within circuit theory. The expression for the conductance (141) may now be obtained by noticing that

$$\frac{\partial U}{\partial \xi} = \frac{1}{s - 1/[T \cosh(2\xi - 2sU)]} \quad (144)$$

and setting $\xi = -i\pi/4$. Insertion of equation (144) into equation (141) leads to an expression for the conductance identical to that given by circuit theory (cf. equation (128)).

The above ideas have been further developed in a number of treatments of Andreev scattering in chaotic and resonant structures (including Altland and Zirnbauer 1996, Berkovits 1995, Beenakker *et al* 1995, Brouwer and Beenakker 1995a, b, Bruun *et al* 1995, Claughton *et al* 1995, Argaman and Zee 1996, Frahm *et al* 1996, Melsen *et al* 1996, Slevin *et al* 1996).

6. Re-entrant and long-range proximity effects

6.1. Re-entrant behaviour of the conductance

As well as the ZBA discussed in the previous section, a number of recent experiments have revealed a non-monotonic behaviour of the voltage and temperature dependence of the conductance. In this section, we use the quasi-classical approach to highlight the origin of this phenomenon. For simplicity, we consider the case of a normal diffusive wire located in the region $0 < x < L$ between a normal and a superconducting reservoir. As explained in subsection 4.3 the equation for the Green's function F^R , which is conveniently written as

$$D \partial_x^2 u(x) + 2i\epsilon \sinh(u(x)) = 0 \quad (145)$$

must be solved with the boundary conditions $u(0) = 0$ and $u(L) = i\pi/2$. The voltage-dependent conductance is then obtained from

$$\frac{G(\epsilon)}{G_0} = \left\{ \frac{1}{L} \int_0^L dx \frac{1}{\cosh^2(\text{Re } u(x, \epsilon))} \right\}^{-1} \quad (146)$$

where G_0 is the conductance in the normal state. Using the identity $\cosh^{-2} w = 1 - \tanh^2 w$, equation (146) becomes

$$\frac{G(\epsilon)}{G_0} = \left\{ 1 - \frac{1}{L} \int_0^L dx \tanh^2(\text{Re } u(x, \epsilon)) \right\}^{-1}. \quad (147)$$

When $\text{Re } u(x, \epsilon)$ is a small quantity, one can expand the denominator of equation (147) to yield

$$\frac{\delta G(\epsilon)}{G_0} = \frac{1}{L} \int_0^L dx \tanh^2(\text{Re } u(x, \epsilon)) \quad (148)$$

which allows us to obtain approximate solutions in both the low- and high-energy limits. In the limit $\epsilon \ll \epsilon_T$ where $\epsilon_T = D/L^2$, one can treat the term linear in the energy as a perturbation and write $u(x, \epsilon) = u_0(x) + \delta u(x, \epsilon)$. When inserted into equation (145), this yields

$$u(x, \epsilon) = -i\frac{\pi}{2} \frac{x}{L} + \frac{8}{\pi^2} \frac{\epsilon}{\epsilon_T} \left(\sin\left(\frac{\pi x}{2L}\right) - \frac{x}{L} \right). \quad (149)$$

In the opposite limit of large energies, $\epsilon \gg \epsilon_T$, one notices that among the solutions of the original non-linear equation, there exists a subset satisfying the following equation:

$$\partial_x u = 2k \sinh\left(\frac{u}{2}\right). \quad (150)$$

Solutions of equation (150) are also solutions of the original equation (145)[†]. The solution of equation (150) reads

$$\tanh\left(\frac{u(x, \epsilon)}{4}\right) = \tanh\left(\frac{u(L, \epsilon)}{4}\right)e^{k(x-L)} \quad (151)$$

where $k = \sqrt{-2i\epsilon/D}$. At the normal reservoir, the value of the solution cannot be imposed, though in the limit of large energies, the r.h.s. of (151) is exponentially small and the solution satisfies the condition that the pairing function must vanish. We also note that in both the low- and high-energy limits, the real part of $u(x, \epsilon)$ is a small quantity and therefore the assumption made at the beginning is justified. With these two solutions we may finally write

$$\frac{\delta G(\epsilon)}{G_0} = A\left(\frac{\epsilon}{\epsilon_T}\right)^2 \quad \epsilon \ll \epsilon_T \quad (152)$$

$$\frac{\delta G(\epsilon)}{G_0} = B\sqrt{\frac{\epsilon_T}{\epsilon}} \quad \epsilon \gg \epsilon_T \quad (153)$$

with A and B numerical constants.

The above analysis illustrates the key ingredients required to obtain a re-entrant effect, which was noted prior to the experiments by Artemenko *et al* (1979). The effect originates from the presence of a distribution function which is spatially not in equilibrium and consequently will be most relevant in situations where the voltage drop is distributed along the wire, rather than dropping predominantly across the tunnel junction.

The fact that the conductance variation appears to be a small quantity both in the low-energy and in the high-energy limit, controlled respectively by the small parameters $(\epsilon/\epsilon_T)^2$ and $\sqrt{\epsilon_T/\epsilon}$, is not accidental. It has been noticed that the conductance correction resembles the fluctuation correction in the theory of paraconductivity, the so-called Maki–Thompson term (Volkov 1994a). In the high-energy limit, the proximity effect induces Cooper pairs in the normal metal, which act like a superconducting fluctuation above the critical temperature. At low energy, the induced pair wave function is not small, so at first sight the correction should be sizable. However the ‘longitudinal’ component of the induced pair wave function gives a contribution which is equal to the normal-state conductance and is not temperature dependent, while the ‘transverse’ component, which is small at low energies, gives the additional temperature-dependent contribution. On general grounds, one expects that the low-energy ϵ^2 -dependence should be universal, while the high-energy behaviour may differ from the inverse-square-root law, depending on the quality of the interface and on the effective dimensionality of the mesoscopic wire.

6.2. Long-range proximity effects

We now discuss briefly the origin of the long-range proximity effects, observed in the experiments of Petrashov *et al* (1993b, 1995), Dimoulas *et al* (1995) and Courtois *et al* (1996), where the conductance of a mesoscopic loop between two superconducting electrodes was measured. In such structures, the total current is the sum of two terms. The first is a Josephson current between the two superconducting electrodes and decays exponentially as L/L_T in agreement with known theory of superconducting weak links

[†] Although the converse is not generally true, because there may be solutions of the original equation, which are not solutions of the above equation. This in general will depend on the boundary conditions, as is evident from the fact that the original equation is second order while the equation derived above is first order.

(Likharev 1979). The second was shown to be due to a phase-periodic conductance with the amplitude decaying as the inverse of the temperature. A detailed theory of the effect has been provided by Volkov *et al* (1996) and Volkov and Takayanagi (1996a) using both quasi-classical and multiple-scattering methods. The different behaviours of the two contributions to the current occur because the Josephson current has a thermodynamic origin and is obtained by integrating over a range of energies of the order of $k_B T$, while the second contribution giving rise to the re-entrant behaviour is kinetic in nature. In the structure considered in the experiments, a non-vanishing value of the pair wave function $F(x, \epsilon)$ is induced in the normal region. Away from the superconductor the pair wave function decays exponentially as

$$F(x, \epsilon) \approx \exp(-kx) \quad (154)$$

where $k = \sqrt{(\epsilon/D)}$. The contribution to the Josephson critical current comes from an integral over the energy involving the products of anomalous Green's functions $F^R F^R$ and $F^A F^A$. Since these products contain Green's functions with poles on the same side of the real axis, one can deform the contour of integration and transform the integral over the energy to a sum over Matsubara frequencies $\epsilon_n = \pi T(2n + 1)$. As a result, the main contribution comes from energies of the order of the temperature. At a distance L , the Josephson current decays exponentially as L/L_T where $L_T = \sqrt{(D/T)}$. On introducing the Thouless energy $\epsilon_T = D/L^2$, the condition for a vanishing Josephson current becomes $\epsilon_T \ll T$. In addition to the above, the proximity effect gives rise to an additional contribution to the normal-state conductance, which involves products of the type $F^R F^A$. As a consequence, the integral over the energy cannot be transformed to a sum over Matsubara frequencies. (This expresses the fact that the main contribution to the integral comes from energies of the order of the Thouless energy.) As a consequence, if the condition $\epsilon_T \ll T$ is satisfied, then the pair wave function spreads over the entire sample.

7. Andreev interferometers

When a quasi-particle Andreev reflects from a normal–superconducting interface, the phase of the outgoing excitation is shifted by the phase of the superconducting order parameter. Consequently if a phase-coherent normal conductor is in contact with two superconductors with order parameter phases ϕ_1 and ϕ_2 , transport properties will be oscillatory functions of the phase difference $\phi = \phi_1 - \phi_2$. Following a number of theoretical proposals (Spivak and Khmel'nitskii 1982, Al'tshuler and Spivak 1987, Nakano and Takayanagi 1991, Takagi 1992, Lambert 1993, Hui and Lambert 1993a, Hekking and Nazarov 1993), several experimental realizations of Andreev interferometers have been reported (de Vegvar *et al* 1994, Pothier *et al* 1994a, b, van Wees *et al* 1994b, Dimoulas *et al* 1995, Petrashov *et al* 1995, Courtois *et al* 1996). In addition to interferometers formed when a normal metal makes contact with two superconductors, the conductance of a single extended N–S structure is predicted to be an oscillatory function of the phase gradient across the interface (Cook *et al* 1995), although no experiments on such phase gradiometers have been reported to date.

These papers address a number of issues including the fundamental periodicity of the conductance oscillation, the nature of the zero-phase extremum, the magnitude of the oscillation amplitude and the role of disorder, geometry and dimensionality. In the main, experiments have probed either diffusive or ballistic structures, which in turn can be divided into metallic samples with a conductance G much greater than e^2/h and semiconducting structures with G less than or of order e^2/h . For the former, the ensemble-averaged conductance $\langle G \rangle$ is the relevant quantity and mesoscopic fluctuations are unimportant. These

have confirmed the 2π -periodicity for $\langle G \rangle$ predicted by Lambert (1993) and Hui and Lambert (1993a) and by all subsequent theories (Hekking and Nazarov 1993, 1994, Zaitsev 1994, Takane 1994, Allsopp *et al* 1996, Kadigrobov *et al* 1996, Nazarov and Stoof 1996, Volkov and Zaitsev 1996, Zagoskin *et al* 1996, Leadbeater and Lambert 1997). For the latter, mesoscopic fluctuations can dominate and the non-averaged conductance G is of interest, which depends in detail on the geometry and impurity realization of a particular sample. These confirm the 2π -periodicity for the non-averaged conductance predicted initially for diffusive systems by Al'tshuler and Spivak (1987) and for one-dimensional clean systems by Nakano and Takayanagi (1991) and Takagi (1992).

In this section we briefly review the various theoretical approaches and comment on their applicability to the experiments.

7.1. Quasi-classical theory: the dirty limit

We start by using quasi-classical theory to compute the ensemble-averaged conductance when the mesoscopic normal conductor is in the diffusive transport regime. As noted in previous sections, the influence of superconductivity on the transport properties of a mesoscopic conductor may change in a qualitative way, depending on the quality of the normal-superconducting interfaces. Consider first the zero-temperature conductance of an interferometer comprising a tunnel junction (labelled as 1) connected by diffusive 1D wires to a fork. Each of the two arms of the fork is a diffusive wire, connected via tunnel junctions (labelled as 2) to infinitely long superconductors. The conductance of the diffusive wires is assumed to be much greater than that of the tunnel junctions. In this structure the superconductors play the role of electrodes and the main voltage drop occurs at the tunnel barriers. The conductance can be analysed using the circuit theory outlined in section 4 (Nazarov 1994, Zaitsev 1994), which yields

$$G = \frac{4G_1^2 G_2^2 \cos^2(\phi/2)}{\{G_1^2 + 4G_2^2 \cos^2(\phi/2)\}^{3/2}} \quad (155)$$

where G_1 is the conductance of the tunnel junction 1, G_2 is the conductance of the tunnel junction 2 and ϕ is the phase difference between the two superconductors. In the limit $G_1 \gg G_2$, this simplifies to

$$G = 4 \frac{G_2^2}{G_1} \cos^2(\phi/2) \quad (156)$$

whereas, if $G_2 \gg G_1$,

$$G = \frac{1}{2} \frac{G_1^2}{G_2} \frac{1}{|\cos^2(\phi/2)|}. \quad (157)$$

One can see that for $G_1 = G_2$ there is a zero-phase minimum, as there is for $G_1 \ll G_2$. However, when $G_1 \gg G_2$, there arises a zero-phase maximum. In all cases the conductance vanishes when the phase difference between the superconductors is π . Furthermore, the ratio G_1/G_2 controls the amplitude of the conductance oscillations which are greater when $G_1 \approx G_2$.

The results of quasi-classical theory can of course be compared with the results of numerical multiple-scattering theory. If the parameters used in the numerical simulations are chosen in such a way as to satisfy the assumptions underlying the quasi-classical treatment, then a detailed analysis shows that this is indeed the case (Claughton *et al* 1996), although certain features, such as the vanishing of the conductance at $\phi = \pi$, appear to be an artifact of the one-dimensional nature of the analysis leading to equation (155). It is perhaps worth

emphasizing that the exact numerical techniques used in Claughton *et al* (1996) are not confined to the diffusive regime and an interesting result of these simulations is that when the resistance of a given structure is dominated by tunnel barriers, the diffusive nature of the wires is not relevant and similar results are obtained for both ballistic and diffusive wires. The geometry is clearly important, but when dissipation occurs along the wire, the almost one-dimensional approximation used in many quasi-classical calculations works well.

Another structure illustrating the role of finite temperature and voltage is suggested by the experiment by Petrashov *et al* (1995), who measured the conductance of a mesoscopic wire, which makes contact with superconductors at points between the normal electrodes. At zero temperature, this structure is predicted to have only a small amplitude of oscillation arising from mesoscopic fluctuations, in contrast with the large-amplitude oscillations observed experimentally. In the experiment however, a finite voltage drop is distributed along the wire and the temperature is non-zero. Consequently the phase-periodic conductance is dominated by an effect similar to the re-entrant behaviour discussed in the previous section. The analysis of this structure has been carried out in detail by Nazarov and Stoof (1996), Volkov *et al* (1996) and Stoof and Nazarov (1996), using both quasi-classical and numerical multiple-scattering methods.

7.2. Description of the ballistic limit: a simple two-channel model

Having discussed diffusive interferometers, we now examine the clean limit, which was first described in one dimension by Nakano and Takayanagi (1991) and Takagi (1992). An analytic theory of the ballistic limit in higher dimensions has also been developed (Allsopp *et al* 1996), based on a multiple-scattering description of a clean N–S interface. In the absence of disorder, when there is no phase difference between the two superconductors, translational invariance in the direction parallel to the interface (and transverse to the current flow) allows one to reduce the two-dimensional system to a sum over many independent one-dimensional channels. When a phase difference between the superconductors is imposed, inter-channel coupling is introduced, but as shown below, this coupling typically involves only pairs of channels in such a way that an accurate description is obtained by summing over independent *pairs* of coupled channels. This considerably simplifies evaluation of the boundary conductance

$$G_B(\phi) = 2R_a = 2\text{Tr } r_a r_a^\dagger = 2 \sum_{i,j=1}^N (R_a)_{ij} \quad (158)$$

where $(R_a)_{ij} = |(r_a)_{ij}|^2$ is the Andreev reflection probability from channel j to channel i . Indeed the Andreev reflection coefficient is of the form $R_a = R_{\text{diag}} + R_{\text{off-diag}}$ where $R_{\text{diag}} = \sum_{i=1}^N (R_a)_{ii}$ and $R_{\text{off-diag}}$ is the remaining contribution from inter-channel scattering, $R_{\text{off-diag}} = \sum_{i \neq j=1}^N (R_a)_{ij}$. If channels only couple in pairs, both the off-diagonal scattering and the diagonal scattering will scale as the number of channels. Consider now a normal barrier in the N region of a N–S interface. Particles (holes) impinging on the normal scatterer are described by a scattering matrix s_{pp} (s_{hh}), and those arriving at the N–S interface by a reflection matrix ρ , where

$$s_{pp} = \begin{pmatrix} r_{pp} & t'_{pp} \\ t_{pp} & r'_{pp} \end{pmatrix} \quad \rho = \begin{pmatrix} \rho_{pp} & \rho_{ph} \\ \rho_{hp} & \rho_{hh} \end{pmatrix}.$$

The elements of s and ρ are themselves matrices describing scattering between open channels of the external leads. For an ideal interface, where Andreev's approximation is valid, ρ_{pp} and ρ_{hh} can be neglected and, as a consequence, ρ_{hp} and ρ_{ph} are unitary and

one obtains a generalization of a formula due to Beenakker (1992): $r_a = t'_{hh}\rho_{hp}M_{pp}^{-1}t_{pp}$, with $M_{pp} = 1 - r'_{pp}\rho_{ph}r'_{hh}\rho_{hp}$. In contrast with the analysis of Beenakker (1992), where ρ_{hp} is proportional to the unit matrix, the interference effect of interest here is contained in the fact that ρ_{hp} induces off-diagonal scattering. Substituting r_a into equation (158) and taking advantage of particle-hole symmetry at $E = 0$ yields

$$G = 2 \text{Tr}(T Q^{-1} T (Q^\dagger)^{-1}) \quad (159)$$

where $Q = \rho'_{ph} + (r')_{pp}\rho_{ph}(r')_{pp}^\dagger$, with $T = t_{pp}t_{pp}^\dagger$ the transmission matrix of the normal scattering region. This multiple-scattering formula for the boundary conductance is valid in the presence of an arbitrary number of channels and in any dimension.

Equation (159) is very general and makes no assumption about the nature of the matrices ρ_{ph} and s_{pp} . In a two-channel model, ρ_{ph} is chosen to be an arbitrary two-dimensional unitary matrix and, since in the absence of disorder, t_{pp} and r_{pp} are diagonal, inter-channel coupling arises from ρ_{ph} only. Substituting these matrices into equation (159) yields an expression for r_a involving a single phase θ , whose value is a linear combination of phase shifts due to normal reflection at the barrier, Andreev reflection at the N-S interface and the phase accumulated by an excitation travelling from the barrier to the interface. After averaging over the rapidly varying phase θ , the diagonal and off-diagonal terms are determined and found to vary periodically with ϕ with period 2π as predicted in one dimension by Nakano and Takayanagi (1991). A key new feature of equation (159) is that in the absence of the normal barrier, unitarity ensures that the total Andreev reflection obtained by summing these two contribution is independent of ϕ and therefore the amplitude of conductance oscillation vanishes. Thus to obtain a maximum amplitude of oscillation, the tunnel barrier must be tuned to a non-zero value.

An interesting mechanism capable of producing large conductance oscillations which scale with the number of channels has been proposed by Kadigrobov *et al* (1995), based on the simultaneous resonance of Andreev levels corresponding to different channels. To illustrate this, consider for example the clean S-N-S structure analysed by Bardeen and Johnson (1972) and Kulik (1970). For a long junction, the Andreev levels have energies

$$E_n = \frac{\hbar v_{F,z}}{2L} [-\Delta\phi \pm (2n+1)\pi] \quad (160)$$

where $v_{F,z}$ is the component of the Fermi velocity perpendicular to the N-S interfaces and $\Delta\phi$ the phase difference between the superconductors. In general these energies do not coincide, but when $\Delta\phi = (2m+1)\pi$ all levels pass through the Fermi energy $E = 0$, yielding large-scale oscillations in transport properties. In the calculation of Kadigrobov *et al* (1995), normal scattering arises from the beam splitters which form an integral part of the structure considered and therefore no additional tunnel barriers are needed to break the unitarity condition of Allsopp *et al* (1996).

8. Conductance suppression by superconductivity

When a superconducting island is added to a normal host, one naïvely expects that the electrical conductance of the composite material will increase. In contrast, detailed calculations based on the multiple-scattering formula (Hui and Lambert 1993b, Claughton *et al* 1995) predicted that the change δG in the two-probe conductance of a mesoscopic sample, due to the onset of superconductivity can have arbitrary sign. Indeed for a sample with a high enough conductance, a simple theorem (Hui and Lambert 1993b) states that δG is guaranteed to be negative and furthermore the magnitude of δG is predicted to scale with

G . This ‘anomalous proximity effect’ occurs in the absence of tunnel barriers and at first sight appears to conflict with the quasi-classical description presented in section 4, which predicts that at zero temperature, the conductance of equation (27), which takes the form $G = (4e^2/h)R_a$, is unchanged by the onset of superconductivity. In this section we briefly discuss the origin of this effect and how it can be reconciled with quasi-classical theory.

For convenience we separately discuss the clean, diffusive and almost Anderson localized limits.

8.1. The clean limit

The Landauer formula (2) predicts that the conductance of a normal ballistic wire is $G = (2e^2/h)N$, where N is the number of open scattering channels at the Fermi energy E_F . Consequently, in the absence of spin splitting, if E_F is varied, G exhibits a series of steps of height $2e^2/h$ corresponding to the opening or closing of scattering channels. The effect of inducing a uniform superconducting order parameter of magnitude Δ_0 in such a wire is shown in figure 9 of Claughton and Lambert (1995). In the vicinity of normal-state steps, the conductance is predicted to be extremely sensitive to the onset of superconductivity. With increasing Δ_0 , the steps are destroyed and G decreases. This behaviour is attributed to the breakdown of Andreev’s approximation, which is most pronounced for scattering channels whose wave vector k_x normal to the N–S interface tends to zero. Such channels occur precisely at a normal-state conductance step.

For a clean N–S–N structure, provided that the superconductor is much longer than the superconducting coherence length $\xi = k_F^{-1}E_F/\Delta_0$, there is negligible quasi-particle transmission through the S region and therefore the total resistance reduces to the sum of two boundary resistances. Since the boundaries are identical, this yields, for the total conductance defined in equation (30), $G = R_a$ and since the system consists of decoupled channels, R_a can be obtained by solving the Bogoliubov–de Gennes equation at a one-dimensional N–S interface. By insisting that scattered wave functions and their first derivatives be continuous at the boundary, one finds (Claughton and Lambert 1995)

$$R_a = \sum_{n=1}^N R_n \quad (161)$$

where

$$R_n = \frac{2}{1 + [1 + (\Delta_0/\mu_n)^2]^{1/2}} \quad (162)$$

and $\mu_n = \hbar^2(k_n^x)^2/2m$ is the longitudinal kinetic energy of a quasi-particle incident along channel n . Clearly those channels corresponding to low-angle quasi-particles with $\mu_n < \Delta_0$ possess a small Andreev reflection probability R_n and since there is no transmission, the corresponding normal reflection probability $(1 - R_n)$ approaches unity. Finally one obtains for the conductance change δG due to the switching on of a uniform order parameter in such a clean system

$$\delta G = \sum_{n=1}^N [R_n - 1] = \sum_{n=1}^N \frac{1 - [1 + (\Delta_0/\mu_n)^2]^{1/2}}{1 + [1 + (\Delta_0/\mu_n)^2]^{1/2}}. \quad (163)$$

It is perhaps worth noting that although the suppression of Andreev reflection for low-angle quasi-particles is quite general, the destruction or otherwise of conductance steps in a quantum point contact depends on the geometry of the contact. If quasi-particles are adiabatically accelerated before encountering the N–S interface, then Andreev reflection

need not be suppressed and conductance steps can survive. The criterion for the survival of the n th step is clearly

$$\Delta_0/\mu_n \ll 1. \quad (164)$$

8.2. The localized limit and resonant transport

The results of the previous subsection reveal that δG is sensitive to fine-scale structure in the normal-state conductance G_N . For clean systems this structure takes the form of well-known conductance steps. For strongly disordered systems, it is known that G_N can exhibit sharp resonances and therefore the behaviour of G with increasing Δ_0 is sensitive to such features. This behaviour is illustrated in figure 10 of Claughton *et al* (1995), which shows that negative changes in the conductance with increasing Δ_0 are associated with resonances in the normal-state transport.

In view of tunnelling theory, one might regard the suppression of conductance in an almost insulating N–S structure as unsurprising, in which case the increase in the conductance due to the onset of superconductivity would be regarded as anomalous. A detailed description of such resonant transport in N–S and N–S–N structures and in resonant interferometers is provided by Claughton *et al* (1995), which follows an earlier zero-voltage theory (Beenakker 1992) and a one-dimensional description of a delta-like potential well (Khulus *et al* 1994).

8.3. The diffusive limit

The above results show that the conductance of a clean or very dirty conductor can either increase or decrease when superconductivity is induced, depending on the microscopic impurity configuration and on the geometry. This qualitative behaviour has been observed in experiments by Petrashov and Antonov (1991) and Petrashov *et al* (1993b, 1996), which also exhibit the weak-magnetic-field behaviour shown in figure 12 of Claughton *et al* (1995). Nevertheless a quantitative comparison between these calculations and experiments is not possible, because of the differing geometries and multi-probe measurements.

A first attempt at a quantitative comparison has been made recently by Wilhelm *et al* (1997) who noted that although the one-dimensional approximation used in sections 5 and 6 to study transport properties of N–S structures successfully explains a range of experiments, the experiment by Petrashov and Antonov (1991) requires a multi-probe description. The key observation is that the kinetic properties of a two-dimensional metallic film, in contact with a superconductor, differ substantially from those of a quasi-one-dimensional wire, because of an inhomogeneous distribution of the currents in the sample. Although a detailed explanation of the effect can be obtained by solving the equations for the quasi-classical Green's function in a two-dimensional structure, it is possible to gain considerable insight by adopting an effective-circuit model, which captures the main features. For example, consider four wires arranged in a square. The top two corners of the square (labelled A and B) are the voltage probes, while the lower two corners (labelled C and D) are the current probes. Let the leftmost vertical wire between corners A and C be placed in contact with a superconductor and labelled 1, the rightmost vertical wire between corners B and D be labelled 2 and the upper and lower wires be labelled 3 and 4 respectively. Then the four-probe conductance reads

$$G_{four} = G_3 G_4 (G_1^{-1} + G_2^{-1} + G_3^{-1} + G_4^{-1}) \quad (165)$$

where G_i is the conductance of wire i . The analysis of sections 4, 5 and 6 shows that the conductance of a wire in contact with a superconductor is changed in several ways.

(1) The local conductivity is *increased* by the presence of a superconductor, but this increase decays over a distance from the superconductor of the order of the coherence length of the normal metal $D/\max\{T, E\}$.

(2) The conductance has a non-monotonic behaviour as a function of the energy and temperature.

To a first approximation one may consider that when $T \gg D/L^2$, only the wire attached to the superconductor will be appreciably affected by the presence of the superconductor. In such a situation the conductances G_2 , G_3 and G_4 are the same as in the normal state, while G_1 is *increased*. In this case, according to equation (165), the four-probe conductance is *decreased* by superconductivity. On the other hand, at $T \ll D/L^2$, all conductances increase and so does the four-probe conductance. As Wilhelm *et al* (1997) have shown, this leads to a much richer structure in the temperature and voltage dependence of a two-dimensional film compared with that of a one-dimensional wire.

However, a key prediction of Wilhelm *et al* (1997) is that at zero bias and zero temperature this effect should vanish and the conductance should be unchanged by the onset of superconductivity. In contrast, Seviour *et al* (1997) identify an additional large-scale effect, which is not contained in standard quasi-classical theory and which will survive at zero temperature and zero bias.

9. Conclusions

We conclude this review by indicating a few lines of future research. The results of the first experiments in hybrid structures have been successfully explained in terms of almost one-dimensional models. This has been possible because the physical phenomena responsible for the transport properties of a given structure mainly occurred at the N–S interface, so the actual geometrical arrangement of current and voltage probes was not important. This is clearly the case in the presence of low-transmittance interfaces, which tend to dominate the overall transport properties. The improvement of the quality of N–S interfaces between separate parts of the hybrid structure and the contact leads makes it necessary to take into account the detailed geometrical arrangement of a given measurement set-up (Allsopp *et al* 1994). In this respect, the last twelve months have witnessed a considerable experimental effort at paying particular attention to the geometry of a given measurement. As we have seen in the previous section, taking into account the actual disposition of the current and voltage probes has been crucial for describing the effect of the conductance suppression, even though the structure itself has a one-dimensional geometry (Wilhelm *et al* 1997). The recent experiment by den Hartog *et al* (1996a, b) has shown beautifully the emergence of new physical effects when the geometry of the experimental set-up is considered carefully.

A second important simplifying assumption in examining the experiments in hybrid structures consisted in neglecting the effect of electron–electron Coulomb interaction. This is a reasonable assumption in many metallic and semiconducting wires used in the experiments. In these cases, a mean-field treatment of electron–electron interaction is sufficient and can be easily incorporated in the effective electronic parameters. In the last decade there has been however continuous progress in downsizing fabrication of electronic systems, so it is now possible to measure transport through almost zero-dimensional structures, or quantum dots. In these systems, the effect of Coulomb interaction is no longer negligible, and various new effects arise. Besides quantum dots, quantum wires also have been a subject of noticeable

attention. It is well known that in one-dimensional systems electron–electron interaction drastically modifies the low-energy properties and leads to the concept of the Luttinger liquid. Quantum wires are believed to be an experimental realization of the Luttinger liquid state. As a consequence, in the last few years a considerable theoretical literature has accumulated in which the intent is to study phase-coherent transport in interacting systems. It is almost a logical consequence to foresee in the forthcoming years the emergence of a new field of research aimed at combining the fields of mesoscopic superconductivity and strongly interacting systems. Indeed, already quite a few papers have started to appear (Fisher 1994, Fazio *et al* 1995, Maslov *et al* 1996, Fazio and Raimondi 1997, Takane and Koyama 1997) and the time is ripe for new exciting developments.

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References

- Allsopp N K, Hui V C, Lambert C J and Robinson S K 1994 *J. Phys.: Condens. Matter* **6** 10 475
 Allsopp N K, Sanchez-Cañizares J, Raimondi R and Lambert C J 1996 *J. Phys.: Condens. Matter* **8** L377
 Altland A and Zirnbauer M R 1996 *Phys. Rev. Lett.* **76** 7406
 Al'tshuler B L, Lee P A and Webb R A (ed) 1991 *Mesoscopic Phenomena in Solids* (Amsterdam: North-Holland)
 Al'tshuler B L and Spivak B Z 1987 *Sov. Phys.-JETP* **65** 343
 Andreev A F 1964 *Sov. Phys.-JETP* **19** 1228
 Antonov V N, Volkov A F and Takayanagi H 1997 *Phys. Rev. B* **55** 3836
 Argaman N and Zee A 1996 *Phys. Rev. B* **54** 7406
 Artemenko S N, Volkov A F and Zaitsev A V 1979 *Solid State Commun.* **30** 771
 Averin D V and Bardas A 1995 *Phys. Rev. Lett.* **75** 1831
 ——— 1996 *Phys. Rev. B* **53** R1705
 Bagwell P F 1992 *Phys. Rev.* **46** 12 573
 Bagwell P F and Alam M A 1992 *Workshop on Physics and Computation, PhysComp '92* (Los Alamos, CA: IEEE Computer Society Press)
 Bakker S J, van der Drift E and Klapwijk T M 1994 *Phys. Rev. B* **49** 13 275
 Barash Y S, Galaktionov A V and Zaikin A D 1995 *Phys. Rev. B* **52** 665
 Bardas A and Averin D V 1997 *Preprint* cond-mat/9706087
 Bardeen J and Johnson J L 1972 *Phys. Rev. B* **5** 72
 Beenakker C W J 1992 *Phys. Rev. B* **46** 12 841
 ——— 1998 *Rev. Mod. Phys.* at press
 Beenakker C W J, Melsen J A and Brouwer P W 1995 *Phys. Rev. B* **51** 13 883
 Beenakker C W J, Rejaei B and Melsen J A 1994 *Phys. Rev. Lett.* **72** 2470
 Beenakker C W J and Van Houten H 1991 *Phys. Rev. Lett.* **66** 3056
 Berkovits R 1995 *J. Phys.: Condens. Matter* **7** 4105
 Black C T, Ralph D C and Tinkham M 1996 *Phys. Rev. Lett.* **76** 688
 Blonder G E and Tinkham M 1983 *Phys. Rev. B* **27** 112
 Blonder G E, Tinkham M and Klapwijk T M 1982 *Phys. Rev. B* **25** 4515
 Bratus E N, Shumeiko V S and Wendin G 1995 *Phys. Rev. Lett.* **74** 2110
 Brouwer P W and Beenakker C W J 1995a *Phys. Rev. B* **51** 7739
 ——— 1995b *Phys. Rev. B* **52** R3868
 Bruder C 1990 *Phys. Rev. B* **41** 4017
 ——— 1996 *Supercond. Rev.* **1** 261
 Bruun J, Evangelou S N and Lambert C J 1995 *J. Phys.: Condens. Matter* **7** 4033
 Buot F 1993 *Phys. Rep.* **234** 73
 Büttiker M 1986 *Phys. Rev. Lett.* **57** 1761
 Cañizares J S and Sols F 1995 *J. Phys.: Condens. Matter* **7** L317
 ——— 1997 *Phys. Rev. B* **55** 531

- Chang L, Chaudhuri S and Bagwell P F 1996 *Phys. Rev. B* **54** 9399
- Charlat P, Courtois H, Gandit Ph, Mailly D, Volkov A F and Pannetier B 1996a *Phys. Rev. Lett.* **77** 4950
- 1996b *Czech. J. Phys.* **46** 3107
- Claughton N R and Lambert C J 1995 *Phys. Rev. B* **51** 11 635
- 1996 *Phys. Rev. B* **53** 1
- Claughton N R, Leadbeater M and Lambert C J 1995 *J. Phys.: Condens. Matter* **7** 8757
- Claughton N R, Raimondi R and Lambert C J 1996 *Phys. Rev. B* **53** 9310
- Cook P M A, Hui V C and Lambert C J 1995 *Europhys. Lett.* **30** 355
- Cook P M A, Raimondi R and Lambert C J 1996 *Phys. Rev. B* **54** 9491
- Courtois H, Gandit Ph, Mailly D and Pannetier B 1996 *Phys. Rev. Lett.* **76** 130
- Datta S 1995 *Preprint*
- de Gennes P G 1989 *Superconductivity of Metals and Alloys* (New York: Addison-Wesley)
- de Vegvar P G N, Fulton T A, Mallison W H and Miller R E 1994 *Phys. Rev. Lett.* **73** 1416
- den Hartog S G, Kapteyn C M A, van Wees B J, Klapwijk T M and Borghs G 1996a *Phys. Rev. Lett.* **77** 4954
- 1996b *Phys. Rev. Lett.* **76** 4592
- Dimoulas A, Heida J P, van Wees B J and Klapwijk T M 1995 *Phys. Rev. Lett.* **74** 602
- Eilenberger G 1968 *Z. Phys.* **214** 195
- Eiles T M, Martinis J M and Devoret M H 1993 *Phys. Rev. Lett.* **70** 1862
- Eliashberg G M 1971 *Sov. Phys.-JETP* **34** 668
- Fazio R, Hekking F W J and Odintsov A A 1995 *Phys. Rev. Lett.* **74** 1843
- Fazio R and Raimondi R 1997 unpublished
- Fisher D S and Lee P A 1981 *Phys. Rev. B* **23** 6851
- Fisher M P A 1994 *Phys. Rev. B* **49** 14 550
- Frahm K M, Brouwer P W, Melsen J A and Beenakker C W J 1996 *Phys. Rev. Lett.* **76** 2981
- Furasaki A, Takayanagi H and Tsukada M 1991 *Phys. Rev. Lett.* **67** 132
- Golubov A A and Kupriyanov M Y 1995 *Pis. Zh. Eksp. Teor. Fiz.* **61** 830
- Golubov A A, Wilhelm F K and Zaikin A D 1997 *Phys. Rev. B* **55** 1123
- Gueron S, Pothier H, Birge O N, Esteve D and Devoret M H 1996 *Preprint*
- Gusenheimer U and Zaikin A D 1994 *Phys. Rev. B* **50** 6317
- Gyorffy B 1995 *Preprint*
- Hara J, Ashida M and Nagai K 1993 *Phys. Rev. B* **47** 11 263
- Hekking F W J, Glazman L I, Matveev K A and Shekhter R 1993 *Phys. Rev. Lett.* **70** 4138
- Hekking F W J and Nazarov Y V 1993 *Phys. Rev. Lett.* **71** 1625
- 1994 *Phys. Rev. B* **71** 6847
- Hergenrother J M, Tuominen M T and Tinkham M 1994 *Phys. Rev. Lett.* **72** 1742
- Hui V C and Lambert C J 1990 *J. Phys.: Condens. Matter* **2** 7303
- 1993a *Europhys. Lett.* **23** 203
- 1993b *J. Phys.: Condens. Matter* **5** L651
- Hurd M and Wendin G 1994 *Phys. Rev. B* **49** 15 258
- 1995 *Phys. Rev. B* **51** 3754
- Kadigrobov A, Shekter R I and Jonson M 1996 *Physica B*
- Kadigrobov A, Zagorkin A, Shekter R I and Jonson M 1995 *Phys. Rev. B* **52** R8662
- Kastalskii A, Kleinsasser A W, Greene L H, Milliken F P and Harbison J P 1991 *Phys. Rev. Lett.* **67** 3026
- Keldysh L V 1965 *Sov. Phys.-JETP* **20** 1018
- Khlus V A, Dyomin A V and Zazunov A L 1994 *Physica C* **214** 413
- Kleinsasser A W, Miller R E, Maltison W H and Arnold G B 1994 *Phys. Rev. Lett.* **72** 1738
- Koyama Y, Takane Y and Ebisawa H 1995 *J. Phys. Soc. Japan* **64** 1466
- 1996 *J. Phys. Soc. Japan* **65** 683
- Kulik I O 1970 *Sov. Phys.-JETP* **30** 944
- Kupriyanov M Y and Lukichev V F 1988 *Sov. Phys.-JETP* **64** 139
- Lafarge P, Joyez P, Esteve D, Urbina C and Devoret M H 1993 *Phys. Rev. Lett.* **70** 994
- Lambert C J 1991 *J. Phys.: Condens. Matter* **3** 6579
- 1993 *J. Phys.: Condens. Matter* **5** 707
- 1994 *Physica B* **203** 201
- Lambert C J and Hui V C 1990 *Physica B* **165** 1107
- Lambert C J, Hui V C and Robinson S J 1993 *J. Phys.: Condens. Matter* **5** 4187
- Lambert C J, Raimondi R, Sweeney V and Volkov A F 1997 *Phys. Rev. B* **55** 6015
- Landauer R 1970 *Phil. Mag.* **21** 863

- Larkin A I and Ovchinnikov Y N 1969 *Sov. Phys.-JETP* **28** 1200
 —1977a *Sov. Phys.-JETP* **41** 960
 —1977b *Sov. Phys.-JETP* **46** 155
 —1986 *Non-equilibrium Superconductivity* ed D N Langeberg and A I Larkin (Amsterdam: North-Holland)
 —1997 *Phys. Rev. B*
- Levy Yeyati A, Martin-Rodero A and Cuevas J C 1996 *J. Phys.: Condens. Matter* **8** 449
- Likharev K K 1979 *Rev. Mod. Phys.* **51** 101
- Magnée P H C *et al* 1994 *Phys. Rev. B* **50** 4594
- Marmorkos I K, Beenakker C W J and Jalabert R A 1993 *Phys. Rev.* **48** 2811
- Marsh A M, Williams D A and Ahmed H 1994 *Phys. Rev. B* **50** 8118
- Martin A and Lambert C J 1995 *Phys. Rev. B* **24** 17999
 —1996 *J. Phys.: Condens. Matter* **8** L731
- Martin-Rodero A, Levy Yeyati A and Garcia-Vidal F J 1996 *Phys. Rev. B* **53** R8891
- Maslov D L, Stone M, Goldbart P M and Loss D 1996 *Phys. Rev. B* **53** 1548
- Mello P A and Pichard J-L 1989 *Phys. Rev.* **40** 5276
- Melsen J A, Brouwer P W, Frahm K M and Beenakker C W J 1996 *Europhys. Lett.* **35** 7
- Nakano H and Takayanagi H 1991 *Solid State Commun.* **80** 997
- Nazarov Y V 1994 *Phys. Rev. Lett.* **73** 1420
- Nazarov Y V and Stoof T H 1996 *Phys. Rev. Lett.* **76** 823
- Nguyen C, Kroemer H and Hu E L 1992 *Phys. Rev. Lett.* **69** 2847
- Petrashov V T and Antonov V N 1991 *JETP Lett.* **54** 241
- Petrashov V T, Antonov V N, Delsing P and Claeson T 1993a *Phys. Rev. Lett.* **70** 347
 —1994a *JETP Lett.* **60** 606
 —1995 *Phys. Rev. Lett.* **74** 5268
- Petrashov V T, Antonov V N, Maksimov S V and Shaikhaidarov R Sh 1993b *JETP Lett.* **58** 49
 —1994b *JETP Lett.* **59** 551
- Petrashov V T, Antonov V N and Persson M 1992 *Phys. Scr. T* **42** 136
- Petrashov V T, Shaikhaidarov R Sh and Sosnin I A 1996 *JETP Lett.* **64** 839
- Pothier H, Guéron S, Esteve D and Devoret M H 1994a *Phys. Rev. Lett.* **73** 2488
 —1994b *Physica B* **203** 226
- Rammer J and Smith H 1986 *Rev. Mod. Phys.* **58** 323
- Riedel R A and Bagwell P 1993 *Phys. Rev. B* **48** 15198
- Riedel R A, Chang L and Bagwell P 1996 *Phys. Rev. B* **54** 16082
- Seviour R, Leadbeater M and Lambert C J 1998 *Preprint*
- Shelankov A L 1980 *JETP Lett.* **32** 111
 —1984 *Sov. Phys.-Solid State* **26** 981
 —1985 *J. Low Temp. Phys.* **60** 29
- Slevin K, Pichard J-L and Mello P A 1996 *J. Physique I* **6** 529
- Spivak B Z and Khmel'nitskii D E 1982 *JETP Lett.* **35** 413
- Spivak B and Zhou F 1995 *Phys. Rev. Lett.* **74** 2800
- Stone A D, Mello P A, Muttalib K A and Pichard J L 1991 *Mesoscoic Phenomena in Solids* ed B L Al'tshuler, P A Lee and R A Webb (Amsterdam: North-Holland)
- Stoof T H and Nazarov Y V 1996 *Phys. Rev. B* **53** 14496
- Takagaki Y and Takayanagi H 1996 *Phys. Rev. B* **53** 14530
- Takagi S 1992 *Solid State Commun.* **81** 579
- Takane Y 1994 *J. Phys. Soc. Japan* **63** 2668
- Takane Y and Ebisawa H 1991 *J. Phys. Soc. Japan* **60** 3130
 —1992a *J. Phys. Soc. Japan* **61** 1685
 —1992b *J. Phys. Soc. Japan* **61** 2858
- Takane Y and Koyama Y 1997 *J. Phys. Soc. Japan* **66** 119
- Takayanagi H and Akazaki T 1995a *Phys. Rev. B* **52** R8633
 —1995b *Japan. J. Appl. Phys.* **34** 4552
 —1995c *Japan. J. Appl. Phys.* **34** 6977
- Takayanagi H, Akazaki T and Nitta J 1995 *Phys. Rev. Lett.* **75** 3533
- Takayanagi H, Toyoda E and Akazaki T 1996 *Czech. J. Phys.* **46** 2507
- Tuominen M T, Hergenrother J M, Tighe T S and Tinkham M 1992 *Phys. Rev. Lett.* **69** 1997
 —1993 *Phys. Rev. B* **47** 11599
- Usadel K D 1970 *Phys. Rev. Lett.* **25** 507

- van Wees B J, de Vries P, Magnée P H C and Klapwijk T M 1992 *Phys. Rev. Lett.* **69** 510
- van Wees B J, Dimoulas A, Heida J P, Klapwijk T M, van der Graaf W and Borghs G 1994a *Physica B* **203** 285
- van Wees B J, Lenssen K-M H and Harmans C J P M 1991 *Phys. Rev. B* **44** 470
- van Wees B J and Takayanagi H 1997 *Mesoscopic Electron Transport* ed L L Sohn, L P Kouwenhoven and G Schön (Dordrecht: Kluwer Academic) p 469
- van Wees B J *et al* 1994b *Physica B* **203** 285
- Volkov A F 1992 *Phys. Lett.* **174A** 144
- 1994a *Physica B* **203** 267
- 1994b *Phys. Lett.* **187A** 404
- Volkov A F, Allsopp N and Lambert C J 1996 *J. Phys.: Condens. Matter* **8** L45
- Volkov A F and Klapwijk T M 1992 *Phys. Lett.* **168A** 217
- Volkov A F, Magnée P H C, van Wees B J and Klapwijk T M 1995 *Physica C* **242** 261
- Volkov A F and Takayanagi H 1996a *Phys. Rev. Lett.* **76** 4026
- 1996b *Phys. Rev. B* **53** 15 162
- Volkov A F and Zaitsev A V 1996 *Phys. Rev. B* **53** 9267
- Volkov A F, Zaitsev A V and Klapwijk T M 1993 *Physica C* **210** 21
- Wendin G and Shumeiko V S 1996 *Phys. Rev. B* **53** R6006
- Wilhelm F K, Zaikin A D and Courtois H 1997 *Preprint cond-mat/9705225*
- Xiong P, Xiao P G and Laibowitz R B 1993 *Phys. Rev. Lett.* **71** 1907
- Yip S 1995 *Phys. Rev. B* **52** 15 504
- Zagoskin A, Kadigrobov A, Shekhter R I and Jonson M 1996 *Appl. Phys. Rep.* **94** 39
- Zaitsev A V 1984 *Sov. Phys.-JETP* **59** 863
- 1990 *JETP Lett.* **51** 41
- 1994 *Phys. Lett.* **194A** 315
- 1997 *Preprint cond-mat/9708190*
- Zaitsev A V and Averin D V 1997
- Zhou F, Spivak B and Zyuzin A 1995 *Phys. Rev. B* **52** 4467
- Zyuzin A Y 1994 *Phys. Rev. B* **50** 323