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Conductance Formula for Mesoscopic Systems with a Superconducting Segment

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Simple conductance formula for small systems with a single superconducting segment is derived through the linear response theory. It is an extension of the Landauer formula. The expression consists of the usual transmission and reflection coefficients as well as the off-diagonal elements which are given by the processes where an electron is reflected to a hole due to the Andreev reflection.

[mesoscopic system, NS system, Landauer formula, Andreev reflection]

§1. Introduction

In theories of the quantum transport in mesoscopic systems, the Landauer formulae[1,2] are widely used by many authors for an expression of a conductance in terms of transmission and reflection coefficients, because of its simplicity. For example, they are employed in simulations of the conductance fluctuations[3] and phenomenological treatments[4] or simulations[5] of the Aharonov-Bohm oscillations.

Recently, it was shown theoretically that the NS mesoscopic systems which consist of normal segment (N) and a superconducting segment (S) may reveal unusual transport features owing to the Andreev reflection.[6] However there need much detailed studies to be done, for example numerical studies for typical geometries. At this stage, if we have a simple conductance formula which is applicable for systems now in consideration, it is quite helpful in various respects. For example, it reduces the cost of numerical studies compared with ordinary use of the Kubo formula and it makes phenomenological arguments possible.

Several years ago, Blonder *et al.*[7] studied model NS systems as shown in Fig. 1(d) and obtained the nonlinear *I-V* relation of junctions in terms of the reflection coefficients. Their arguments, however, are restricted in a one-dimensional ballistic system with a potential barrier at the NS interface. The extensions of their arguments to impure three dimensional cases and to more complex structures are not trivial. In this paper, we present a conductance formula which is applicable to arbitrary shaped three dimensional systems with disorder, which contain one superconducting segment. We confine our arguments in weak nonequilibrium state. We will see that our expression is identical to that of the model systems of ref. 7 in this limit.

To start with, we express the conductance coefficients in terms of the Green functions through linear response theory.[8-10] Then we relate those with the transmission and reflection coefficients with the help of formal scattering theory.[8-10] As to the basic formalism we follow Baranger and Stone[10] and extend their arguments so as to be applicable to the systems with S by incorporating the effects of the Andreev reflections.

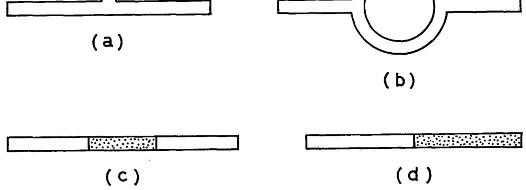

Fig. 1. Sample geometries. Dotted regions represent superconducting segments.

§2. Formulations

We consider arbitrary shaped samples with disorder as shown in Fig. 1. We take model systems for those samples, which is constructed by attaching infinite perfect-conducting leads to the both ends of each sample, for convenience of the microscopic calculations. Two leads are taken to be on the z-axis with the right hand side being positive direction. We assume that the electrical potential in S and that in the perfect leads far from the disordered region are constant even an electric field is applied in the disordered region. So we denote the voltage in S as V_s and the voltage at the left (right) hand side lead in asymptotic region as $V_1(V_2)$.

We suppose that the supercurrent does not penetrate in normal region. Under such assumption the current density in N is expressed in terms of the nonlocal conductivity tensor as

$$j(r)=\int d^3r' \underline{\sigma}(r, r')\cdot E(r'), \tag{1}$$

where the integration extends over all normal regions. Let c_m be the cross section of the perfect lead in asymptotic region at $z=z_m$, for $m=1$ and 2 where z_1 is the coordinate variable defined in the left hand side and z_2 in the right hand side. The total current passing through the sample is given in two ways by integrating $j(r)$ over a cross section at $z=z_1$ or z_2. Replacing $E(r)$ with the gradient of a electric potential and integrating by parts,[10] one obtains the total current,

$$I=g_{11}V_1-g_{12}V_2 \tag{2a}$$

$$=g_{21}V_1-g_{22}V_2, \tag{2b}$$

with the conductance coefficients,

$$g_{mn}=\int_{c_m} d\rho \int_{c_n} d\rho' \sigma_{zz}(r, r'), \tag{3}$$

where σ_{zz} is the z, z-component of the conductivity tensor and the integrations in terms of ρ and ρ' are performed over the cross section c_m and c_n, respectively. Here $V_1>V_2$ is supposed implicitly, and we omit the term depends on V_s by choosing the voltage distribution so as $V_s=0$. Note that V_1 and V_2 are not independent of each other. In use of eqs. (2), the two probe conductance Γ is found to be

$$\Gamma=\frac{g_{11}g_{22}-g_{12}g_{21}}{g_{11}+g_{22}-g_{12}-g_{21}}. \tag{4}$$

In the special cases such as Fig. 1(d), the voltage V_2 is also zero because of direct attaching of the right hand side lead to S so that Γ is equal to g_{11} itself (eq. (2b) make no sence in this case).

In order to derive the conductance coefficients, it is necessary to get the microscopic expression of the conductivity tensor. We assume that mutual interactions in N only affect the phase coherence length, so that we do not consider that explicitly in this paper. At this stage we employ the Bogoliubov-de Gennes picture,[11] whose Hamiltonian is

$$H=\begin{bmatrix} -\dfrac{1}{2m}\left[\nabla-\dfrac{ie}{c}A(r)\right]^2+V(r)-\varepsilon_F & \Delta(r) \\ \Delta^*(r) & \dfrac{1}{2m}\left[\nabla+\dfrac{ie}{c}A(r)\right]^2-V(r)+\varepsilon_F \end{bmatrix}, \tag{5}$$

where $V(r)$ is the impurity potential and $\Delta(r)$ is the pair potential, which has non zero values only in S and has to be determined self-consistently. For simplicity we suppose that the applied magnetic field is perpendicular to the system and limited in a finite region which contains the sample. Under this assumption one can choose the gauge so as the z-component of $A(r)$ is zero in perfect leads far from the sample. It is in use later. We take unit of $\hbar=1$ until the final expression is written down.

Let $\{\psi_\alpha(r)\}$ be a two component, complete and orthonormal set of eigenfunctions of H, with eigenvalues $\{\varepsilon_\alpha\}$, where α denots both the continuous and discrete energy variable of a state. In this model the standard Kubo formula[12] yields the conductivity tensor,

$$\underline{\sigma}(r,r')=\frac{e^2\pi}{8m^2}\int \mathrm{d}\alpha\int \mathrm{d}\beta\left[f'(\varepsilon_\alpha)\delta(\varepsilon_\beta-\varepsilon_\alpha)+\frac{\mathrm{i}}{\pi}\cdot\frac{f(\varepsilon_\beta)-f(\varepsilon_\alpha)}{\varepsilon_\beta-\varepsilon_\alpha}\mathrm{P}\left[\frac{1}{\varepsilon_\beta-\varepsilon_\alpha}\right]\right]$$
$$\times\psi_\beta^\dagger(r)\overleftrightarrow{\mathrm{D}}(r)\psi_\alpha(r)\cdot\psi_\alpha^\dagger(r')\overleftrightarrow{\mathrm{D}}(r')\psi_\beta(r'), \tag{6}$$

where the integral signs α and β mean both the integration over the continuous variable and the summation over the discrete variable and operator $\overleftrightarrow{\mathrm{D}}(r)$ is defined as

$$\overleftrightarrow{\mathrm{D}}(r)=\begin{bmatrix}\left[\overrightarrow{\nabla}-\frac{\mathrm{i}e}{c}A(r)\right]-\left[\overleftarrow{\nabla}+\frac{\mathrm{i}e}{c}A(r)\right] & 0\\ 0 & \left[\overrightarrow{\nabla}+\frac{\mathrm{i}e}{c}A(r)\right]-\left[\overleftarrow{\nabla}-\frac{\mathrm{i}e}{c}A(r)\right]\end{bmatrix}. \tag{7}$$

In the limit of no superconducting order (i.e. $\Delta(r)\equiv 0$), the expression of the nonlocal conductivity tensor, eq. (6), reduces to that of ref. 10. The substitution of eq. (6) to eq. (3) yields the conductance coefficients,

$$\mathrm{g}_{\mathrm{mn}}=\frac{e^2\pi}{8m^2}\int \mathrm{d}\alpha\int \mathrm{d}\beta\left[f'(\varepsilon_\alpha)\delta(\varepsilon_\beta-\varepsilon_\alpha)+\frac{\mathrm{i}}{\pi}\cdot\frac{f(\varepsilon_\beta)-f(\varepsilon_\alpha)}{\varepsilon_\beta-\varepsilon_\alpha}\mathrm{P}\left[\frac{1}{\varepsilon_\beta-\varepsilon_\alpha}\right]\right]$$
$$\times\int_{C_\mathrm{m}}\mathrm{d}\rho\int_{C_\mathrm{n}}\mathrm{d}\rho'\,\psi_\beta^\dagger(r)\overleftrightarrow{\mathrm{D}}_z(r)\psi_\alpha(r)\cdot\psi_\alpha^\dagger(r')\overleftrightarrow{\mathrm{D}}_z(r')\psi_\beta(r'), \tag{8}$$

where $\overleftrightarrow{\mathrm{D}}_z(r)$ is the z-component of $\overleftrightarrow{\mathrm{D}}(r)$. In asymptotic region, z-component of the vector potential is zero so that the operator $\overleftrightarrow{\mathrm{D}}_z(r)$ is simplified to

$$\overleftrightarrow{\mathrm{D}}_z=\begin{bmatrix}\frac{\overrightarrow{\partial}}{\partial z}-\frac{\overleftarrow{\partial}}{\partial z} & 0\\ 0 & \frac{\overrightarrow{\partial}}{\partial z}-\frac{\overleftarrow{\partial}}{\partial z}\end{bmatrix}. \tag{9}$$

Let us introduce the retarded and advanced Green functions,

$$\boldsymbol{G}^\pm(r,r')=\int \mathrm{d}\alpha\,\psi_\alpha(r)\psi_\alpha^\dagger(r')/(-\varepsilon_\alpha\pm\mathrm{i}\delta). \tag{10}$$

This is the 2×2 matrix form function and its elements are generally given by

$$\boldsymbol{G}^\pm(r,r')=\begin{bmatrix}G^\pm(r,r') & F^\pm(r,r')\\ F^\pm(r,r')^* & -G^\pm(r,r')^*\end{bmatrix}. \tag{11}$$

Noting the following identity,

$$\psi_\beta^\dagger(r)\overleftrightarrow{\mathrm{D}}_z\psi_\alpha(r)\cdot\psi_\alpha^\dagger(r')\overleftrightarrow{\mathrm{D}}_z'\psi_\beta(r')=-\mathrm{tr}\,[\psi_\alpha(r)\psi_\alpha^\dagger(r')\overleftrightarrow{\mathrm{D}}_z\overleftrightarrow{\mathrm{D}}_z'\psi_\beta(r')\psi_\beta^\dagger(r)], \tag{12}$$

one can express g_{mn} of eq. (8) in terms of these Green functions. Carrying out several manipulations which are quite similar to those given in ref. 10 in detail, one finds the resultant expressions at zero temperature. The diagonal element (i.e. m=n) is given by

$$\mathrm{g}_{\mathrm{mm}}=-\frac{e^2}{32\pi m^2}\int_{C_\mathrm{m}}\mathrm{d}\rho\int_{C_\mathrm{m}}\mathrm{d}\rho'\,\mathrm{tr}\,[[\boldsymbol{G}^+(r,r')-\boldsymbol{G}^-(r,r')]\overleftrightarrow{\mathrm{D}}_z\overleftrightarrow{\mathrm{D}}_z'[\boldsymbol{G}^+(r',r)-\boldsymbol{G}^-(r',r)]], \tag{13}$$

and the off-diagonal element (i.e. m$\neq$n) by

$$\mathrm{g}_{\mathrm{mn}}=\frac{e^2}{16\pi m^2}\int_{C_\mathrm{m}}\mathrm{d}\rho\int_{C_\mathrm{n}}\mathrm{d}\rho'\,\mathrm{tr}\,[\boldsymbol{G}^+(r,r')\overleftrightarrow{\mathrm{D}}_z\overleftrightarrow{\mathrm{D}}_z'\boldsymbol{G}^-(r',r)],\quad (\mathrm{m}\neq\mathrm{n}) \tag{14}$$

where the trace is over Nambu space.

In the asymptotic region of each lead, one can introduce a set of complete and orthonormal functions $\{\theta_a(\rho)\}$ for the transversal direction, where a represents a channel index. Here we assume the common functions for both leads for simplicity. When r and r' are in the asymptotic regions, the matrix elements of the Green functions between channel a and b are given by

$$\boldsymbol{G}_{ab}^{\pm}(z_{\mathrm{m}}, z_{\mathrm{n}})=\int_{c_{\mathrm{m}}}\mathrm{d}\rho\int_{c_{\mathrm{n}}}\mathrm{d}\rho'\,\boldsymbol{G}^{\pm}(r, r')\begin{bmatrix}\theta_a^*(\rho)\theta_b(\rho') & 0\\ 0 & \theta_a(\rho)\theta_b^*(\rho')\end{bmatrix}. \tag{15}$$

Making use of these elements, we rewrite eqs. (13) and (14) as

$$g_{\mathrm{mm}}=-\frac{e^2}{32\pi m^2}\mathrm{tr}\sum_{a,b}[[\boldsymbol{G}_{ba}^{+}(z, z')-\boldsymbol{G}_{ba}^{-}(z, z')]\overleftrightarrow{\mathrm{D}}_z\overleftrightarrow{\mathrm{D}}_z[\boldsymbol{G}_{ab}^{+}(z', z)-\boldsymbol{G}_{ab}^{-}(z', z)]]|_{z=z_{\mathrm{m}}, z'=z_{\mathrm{m}}}, \tag{16}$$

$$g_{\mathrm{mn}}=\frac{e^2}{16\pi m^2}\mathrm{tr}\sum_{a,b}[\boldsymbol{G}_{ba}^{+}(z, z')\overleftrightarrow{\mathrm{D}}_z\overleftrightarrow{\mathrm{D}}_z'\boldsymbol{G}_{ab}^{-}(z', z)]|_{z=z_{\mathrm{m}}, z'=z_{\mathrm{n}}}. \quad (\mathrm{m}\neq\mathrm{n}) \tag{17}$$

It remains to make connections between the asymptotic form of the Green functions and the transmission or reflection coefficients. Let us define a set of plane wave functions of the electron and the hole in channel a as

$$\chi_{ae}^{\pm}(r)=\begin{bmatrix}\dfrac{1}{\sqrt{v_a}}\mathrm{e}^{\pm\mathrm{i}k_a z}\cdot\theta_a(\rho)\\ 0\end{bmatrix}, \tag{18}$$

$$\chi_{ah}^{\pm}(r)=\begin{bmatrix}0\\ \dfrac{1}{\sqrt{v_a}}\mathrm{e}^{\pm\mathrm{i}k_a z}\cdot\theta_a(\rho)\end{bmatrix}, \tag{19}$$

where the wave vector k_a is determined so as the energy of these states equal to ε_F at the asymptotic region and v_a is the velocity defined as $v_a=k_a/m$. When χ_{ae}^{+} is injected to the sample from the left hand side lead, in analogy with the scattering problem, the asymptotic form of the stationary scattering state ψ_{ae}^{+} is given by

$$\begin{aligned}\psi_{ae}^{+}(r)&\underset{z\to+\infty}{\sim}\sum_b \mathrm{t}_{be,ae}^{\mathrm{L}}\chi_{be}^{+}(r)+\sum_b \mathrm{t}_{bh,ae}^{\mathrm{L}}\chi_{bh}^{-}(r),\\ &\underset{z\to-\infty}{\sim}\chi_{ae}^{+}(r)+\sum_b \mathrm{r}_{be,ae}^{\mathrm{L}}\chi_{be}^{-}(r)+\sum_b \mathrm{r}_{bh,ae}^{\mathrm{L}}\chi_{bh}^{+}(r),\end{aligned} \tag{20}$$

where, for example, $\mathrm{t}_{be,ae}^{\mathrm{L}}$ is the transmission amplitude for the electron incoming in channel a from the left to the electron outgoing to the right in channel b while $\mathrm{r}_{bh,ae}^{\mathrm{L}}$ is the reflection amplitude for the electron incoming from the left to the hole outgoing to the left. In the case of neither disorder nor superconducting order (i.e. $\Delta\equiv0$ and $V\equiv0$), the stationary scattering state ϕ_{ae}^{+} can be also defined and its asymptotic form is

$$\begin{aligned}\phi_{ae}^{+}(r)&\underset{z\to+\infty}{\sim}\sum_b \tilde{\mathrm{t}}_{be,ae}^{\mathrm{L}}\chi_{be}^{+}(r),\\ &\underset{z\to-\infty}{\sim}\chi_{ae}^{+}(r)+\sum_b \tilde{\mathrm{r}}_{be,ae}^{\mathrm{L}}\chi_{be}^{-}(r).\end{aligned} \tag{21}$$

According to the formal scattering theory, these two stationary states are related through the Green function as

$$\psi_{ae}^{+}(r)=\phi_{ae}^{+}(r)+\int\mathrm{d}^3r'\,\boldsymbol{G}^{+}(r, r')U(r')\phi_{ae}^{+}(r'), \tag{22}$$

where $U(r)$ is given by

$$U(r)=\begin{bmatrix}V(r) & \Delta(r)\\ \Delta^*(r) & -V(r)\end{bmatrix}. \tag{23}$$

Eliminating $U(r)$ in use of the equation of motion of the Green function, and after some manipulations as was done in refs. 9 and 10, the stationary scattering state is written as

$$\psi_{ae}^{+}(r)=\frac{1}{2m}\int_{c_1} \mathrm{d}\rho'\, \boldsymbol{G}^{+}(r,r')\overleftrightarrow{\mathrm{D}}_z'\chi_{ae}^{+}(r'). \tag{24}$$

Comparing this with its asymptotic form, eq. (20), one can express the transmission and the reflection coefficients using the Green function. For example,

$$\mathrm{t}_{be,ae}^{\mathrm{L}}=-\frac{\mathrm{i}}{4m^2}\int_{c_2}\mathrm{d}\rho\int_{c_1}\mathrm{d}\rho'\,\chi_{be}^{+\dagger}(r)\overleftrightarrow{\mathrm{D}}_z\boldsymbol{G}^{+}(r,r')\overleftrightarrow{\mathrm{D}}_z'\chi_{ae}^{+}(r'), \tag{25}$$

$$\mathrm{t}_{bh,ae}^{\mathrm{L}}=\frac{\mathrm{i}}{4m^2}\int_{c_2}\mathrm{d}\rho\int_{c_1}\mathrm{d}\rho'\,\chi_{bh}^{-\dagger}(r)\overleftrightarrow{\mathrm{D}}_z\boldsymbol{G}^{+}(r,r')\overleftrightarrow{\mathrm{D}}_z'\chi_{ae}^{+}(r'), \tag{26}$$

$$\mathrm{r}_{be,ae}^{\mathrm{L}}=\frac{\mathrm{i}}{4m^2}\int_{c_1}\mathrm{d}\rho\int_{c_1'}\mathrm{d}\rho'\,\chi_{be}^{-\dagger}(r)\overleftrightarrow{\mathrm{D}}_z\boldsymbol{G}^{+}(r,r')\overleftrightarrow{\mathrm{D}}_z'\chi_{ae}^{+}(r'), \tag{27}$$

$$\mathrm{r}_{\mathrm{bh},ae}^{\mathrm{L}}=-\frac{\mathrm{i}}{4m^2}\int_{c_1}\mathrm{d}\rho\int_{c_1'}\mathrm{d}\rho'\,\chi_{bh}^{+\dagger}(r)\overleftrightarrow{\mathrm{D}}_z\boldsymbol{G}^{+}(r,r')\overleftrightarrow{\mathrm{D}}_z'\chi_{ae}^{+}(r'). \tag{28}$$

With these equations and the asymptotic form of the Green functions, one can relate the coefficients with the Green functions, for example,

$$\mathrm{r}_{be,ae}^{\mathrm{L}}=\mathrm{i}\sqrt{v_a v_b}\cdot \mathrm{e}^{\mathrm{i}k_b z_1+\mathrm{i}k_a z_1'}[G_{ba}^{+}(z_1,z_1')+\mathrm{i}v_a^{-1}\delta_{ab}\,\mathrm{e}^{\mathrm{i}k_a|z_1-z_1'|}], \tag{29}$$

$$\mathrm{r}_{bh,ae}^{\mathrm{L}}=\mathrm{i}\sqrt{v_a v_b}\cdot \mathrm{e}^{-\mathrm{i}k_b z_1+\mathrm{i}k_a z_1'}F_{ba}^{+}(z_1,z_1')^{*}. \tag{30}$$

By solving above relations conversely, the asymptotic form of the Green function, $\boldsymbol{G}_{ba}^{+}(z_1, z_1')$, is expressed in terms of the reflection coefficients. In the same way, we can get other asymptotic forms of the Green functions that are necessary for further calculations.

Now we turn to eqs. (16) and (17), and substitute the above obtained expressions. After some simple calculations we reach the final results.

$$\mathrm{g}_{11}=\frac{e^2}{h}\sum_{a,b}[\delta_{ab}-|\mathrm{r}_{be,ae}^{\mathrm{L}}|^2+|\mathrm{r}_{bh,ae}^{\mathrm{L}}|^2], \tag{31}$$

$$\mathrm{g}_{12}=\frac{e^2}{h}\sum_{a,b}[|\mathrm{t}_{be,ae}^{\mathrm{L}}|^2-|\mathrm{t}_{bh,ae}^{\mathrm{L}}|^2]. \tag{32}$$

Similarly, g_{21} and g_{22} are expressed with t^{R} and r^{R} that denote the transmission and reflection coefficient from the right hand side.

§3. Discussions

We obtain unusual terms in eqs. (31) and (32). The third term of g_{11} indicates that the conductance is increased by the process that an incoming electron is reflected to a hole and goes back to the incoming side. It is in accordance with ref. 7 and intuitively acceptable. As to the second term of g_{12}, it originates from the physical process, where an electron in the channel a is transmited to a hole in the channel b. One can easily see that this reduces the net current, therefore, it is reasonable that this anomalous term decreses the conductance coefficient. In both cases, these unusual terms are caused by the Andreev reflections at NS boundaries and disapper when the superconducting order is absent.

So far, we treat two probe systems only. There is no difficulty to extend the arguments to apply to multiprobe structures.[9,10] However, an extension to cases of multi-superconducting segments seems not so easy. In such cases the voltages at each superconducting segments are different from each other. Thus we need to include those terms that depend on the voltages of superconductors in eqs. (2).

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