

SPECIALIZATION PROJECT IN THEORETICAL PHYSICS

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# Supercurrent transport by Andreev Bound States in external field

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*Author:*

Anna Ursula Birkeland BRØYN

*Supervisor*

Prof. Jacob Linder

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**NTNU – Trondheim**  
Norwegian University of  
Science and Technology

## **Abstract**

Here you give a summary of your work and your results. This is like a management summary and should be written in a clear and easy language, without many difficult terms and without abbreviations. Everything you present here must be treated in more detail in the main report. You should not give any references to the report in the summary – just explain what you have done and what you have found out. The Summary and Conclusions should be no more than two pages.

You may assume that you have got three minutes to present to the Rector of NTNU what you have done and what you have found out as part of your thesis. (He is an intelligent person, but does not know much about your field of expertise.)

## Acknowledgement

I would like to thank the following persons for their great help during ...

If the project has been carried out in cooperation with an external partner (e.g., a company), you should acknowledge the contribution and give thanks to the involved persons.

You should also acknowledge the contributions made by your supervisor(s).

O.N.

(Your initials)

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# Abbreviations

**FTA** Fault tree analysis

**MTTF** Mean time to failure

**RAMS** Reliability, availability, maintainability, and safety

# **Chapter 1**

## **Introduction**

# Chapter 2

## Superconductivity

Two fundamental properties associated to superconductivity are 1) zero electrical resistance giving rise to *supercurrents*, for temperatures below some critical temperature  $T_c$  and 2) complete expulsion of magnetic field below  $T_c$ , known as the Meissner effect [1, 2]. The theory behind these properties was presented by Bardeen, Cooper and Schrieffer in 1957 and is known as the BCS-theory [3].

### 2.1 The Meissner effect

Meissner and Ochsenfeld discovered in 1933 [1] that applied magnetic field,  $H$ , below some critical limit  $H_c$ , would be expelled in the superconductor for temperatures below  $T_c$ , resulting in zero field inside the superconductor,  $B = \mu_0(H + M) = 0$ , so that  $M = -H$ . The superconductor is thus a perfect diamagnet with susceptibility

$$\chi = \frac{dM}{dH} = -1. \quad (2.1)$$

This is called the Meissner effect and is a consequence of induced screening supercurrents at the surface of the superconductor. No current can exist only on the surface of a material as this would imply a finite current in a layer of zero thickness requiring infinite density of free charge. Consequently, the screening current must exist at some finite distance,  $\lambda_L$ , into the superconductor and thus letting the external magnetic field penetrate to a depth  $\lambda_L$ . This penetration



depth will depend on the density of superconducting carriers (Cooper pairs) and is a result from the London equations [4] and Ampere's law.

The Meissner effect breaks down as the external field is increased to above the critical limit  $H_c$ . Depending on the material we will then get full (in type I superconductors) or partial (in type II superconductors) penetration of magnetic flux and the superconductor will go from the superconducting state into the normal or mixed state, respectively.

## 2.2 BCS theory

The BCS theory is based on the appearance of so called *Cooper pairs* which conventionally are formed by a phonon-mediated attractive interaction between two electrons overwinning the Coulomb repulsion [3]. The Cooper pairs are bosonic...

### 2.2.1 The BCS Hamiltonian

The Hamiltonian of the system will consist of two parts, describing the non-interacting and interacting electrons, respectively. A given state is defined by the momentum  $\mathbf{k}$  and spin  $\sigma$ . In the second quantization formalism the annihilation- and creation operators,  $c_{\mathbf{k},\sigma}$  and  $c_{\mathbf{k},\sigma}^\dagger$ , will destroy and create an electron in the corresponding state, respectively. The number operator  $n_{\mathbf{k},\sigma} = c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k},\sigma}$  counts the number of electrons in the state. The non-interacting part of the Hamiltonian will simply be the energy of each state,  $\epsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$ , times the number operator and summed over all states. This will thus be the first term in Hamiltonian (2.2). The interacting part of the Hamiltonian will describe a scattering process where two electrons into the states  $(\mathbf{k}, \sigma)$  and  $(\mathbf{k}', \sigma')$  are scattered to the states  $(\mathbf{k} + \mathbf{q}, \sigma)$  and  $(\mathbf{k}' - \mathbf{q}, \sigma')$ , i.e.  $(\mathbf{k}, \sigma)$  and  $(\mathbf{k} + \mathbf{q}, \sigma)$  are destroyed by the annihilation operators while  $(\mathbf{k} + \mathbf{q}, \sigma)$  and  $(\mathbf{k}' - \mathbf{q}, \sigma')$  are created by the creation operators. We must also include a matrix element  $V_{\mathbf{k},\mathbf{k}'}$  including both the attractive phonon-mediated interaction and the repulsive Coulomb interaction, between the electrons. The second term in the Hamiltonian (2.2) describe this interaction. The total Hamiltonian in-

cluding both the non-interacting and the interacting term is thus given as

$$H = \sum_{\mathbf{k}, \sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^{\dagger} c_{\mathbf{k}, \sigma} + \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}, \sigma, \sigma'} V_{\mathbf{k}, \mathbf{k}'}(\mathbf{q}, \omega) c_{\mathbf{k}+\mathbf{q}, \sigma}^{\dagger} c_{\mathbf{k}'-\mathbf{q}, \sigma'}^{\dagger} c_{\mathbf{k}, \sigma} c_{\mathbf{k}', \sigma'}. \quad (2.2)$$

We define  $\epsilon_{\mathbf{k}} \equiv \epsilon_{\mathbf{k}} - \mu$  as the energy above the Fermi surface. We have used the chemical potential,  $\mu$ , in the place of the Fermi energy,  $\epsilon_F$ , as these two quantities are essentially the same in all relevant cases. The attractive interaction will only be valid in a small energy range,  $\omega$ , above the Fermi-surface, and for electrons on opposite sides of the Fermi-surface. We may therefore let  $\mathbf{k}' = -\mathbf{k}$ . Due to the Pauli principle we will in most cases find the electrons in the Cooper pairs in opposite spin states, so we will also let  $\sigma' = -\sigma$ . By now changing the dummy indices, the Hamiltonian takes the form

$$H - \mu N = \sum_{\mathbf{k}, \sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^{\dagger} c_{\mathbf{k}, \sigma} + \sum_{\mathbf{k}, \mathbf{k}'} V_{\mathbf{k}, \mathbf{k}'} c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} c_{\mathbf{k}', \uparrow} c_{-\mathbf{k}', \downarrow}, \quad (2.3)$$

where  $N$  is the number of electrons. Henceforth we will write  $H$  in place of  $H - \mu N$ .

### 2.2.2 Mean field Approximation

We will use mean field approximation to simplify the Hamiltonian and assume the fluctuations around the expectation values to be small such that we can write

$$c_{-\mathbf{k}, \downarrow} c_{\mathbf{k}, \uparrow} = \langle c_{-\mathbf{k}, \downarrow} c_{\mathbf{k}, \uparrow} \rangle + c_{-\mathbf{k}, \downarrow} c_{\mathbf{k}, \uparrow} - \langle c_{-\mathbf{k}, \downarrow} c_{\mathbf{k}, \uparrow} \rangle \equiv \langle c_{-\mathbf{k}, \downarrow} c_{\mathbf{k}, \uparrow} \rangle + \delta_{\mathbf{k}}, \quad (2.4)$$

and only keep  $\delta$  to the first order. By also defining the *gap parameter* as follows

$$\Delta_{\mathbf{k}'} = \sum_{\mathbf{k}} V_{\mathbf{k}, \mathbf{k}'} \langle c_{-\mathbf{k}, \downarrow} c_{\mathbf{k}, \uparrow} \rangle, \quad (2.5)$$

the Hamiltonian will simplify to

$$\begin{aligned}
H &= \sum_{\mathbf{k}, \sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^{\dagger} c_{\mathbf{k}, \sigma} + \sum_{\mathbf{k}} \left[ \Delta_{\mathbf{k}}^* c_{\mathbf{k}, \uparrow} c_{-\mathbf{k}, \downarrow} + \Delta_{\mathbf{k}} c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} - \Delta_{\mathbf{k}} \langle c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} \rangle \right] \\
&= - \sum_{\mathbf{k}} \Delta_{\mathbf{k}} \langle c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} \rangle + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \left[ c_{\mathbf{k}, \uparrow}^{\dagger} c_{\mathbf{k}, \uparrow} + c_{-\mathbf{k}, \downarrow}^{\dagger} c_{-\mathbf{k}, \downarrow} \right] + \sum_{\mathbf{k}} \left[ \Delta_{\mathbf{k}}^* c_{\mathbf{k}, \uparrow} c_{-\mathbf{k}, \downarrow} + \Delta_{\mathbf{k}} c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} \right] \\
&= \sum_{\mathbf{k}} \left[ \varepsilon_{\mathbf{k}} - \Delta_{\mathbf{k}} \langle c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} \rangle \right] + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \left[ c_{\mathbf{k}, \uparrow}^{\dagger} c_{\mathbf{k}, \uparrow} - c_{-\mathbf{k}, \downarrow} c_{-\mathbf{k}, \downarrow}^{\dagger} \right] + \sum_{\mathbf{k}} \left[ \Delta_{\mathbf{k}}^* c_{\mathbf{k}, \uparrow} c_{-\mathbf{k}, \downarrow} + \Delta_{\mathbf{k}} c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} \right] \quad (2.6) \\
&= E_0 + \sum_{\mathbf{k}} \begin{pmatrix} c_{\mathbf{k}, \uparrow}^{\dagger} & c_{-\mathbf{k}, \downarrow} \end{pmatrix} \begin{pmatrix} \varepsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & -\varepsilon_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}, \uparrow} \\ c_{-\mathbf{k}, \downarrow}^{\dagger} \end{pmatrix} \equiv E_0 + \sum_{\mathbf{k}} \varphi_{\mathbf{k}}'^{\dagger} H'_{\mathbf{k}} \varphi'_{\mathbf{k}}
\end{aligned}$$

where we have used the standard commutation relations for fermions (A.1c) and defined

$$E_0 \equiv \sum_{\mathbf{k}} \left[ \varepsilon_{\mathbf{k}} - \Delta_{\mathbf{k}} \langle c_{\mathbf{k}, \uparrow}^{\dagger} c_{-\mathbf{k}, \downarrow}^{\dagger} \rangle \right], \quad H'_{\mathbf{k}} = \begin{pmatrix} \varepsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & -\varepsilon_{\mathbf{k}} \end{pmatrix} \quad \text{and} \quad \varphi'_{\mathbf{k}} \equiv \begin{pmatrix} c_{\mathbf{k}, \uparrow} \\ c_{-\mathbf{k}, \downarrow}^{\dagger} \end{pmatrix}.$$

The Hamiltonian (2.6) can be diagonalized by inserting  $U_{\mathbf{k}} U_{\mathbf{k}}^{\dagger} = I$ , where  $U$  is a unitary matrix:

$$U_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}}^* \\ v_{\mathbf{k}} & u_{\mathbf{k}}^* \end{pmatrix}, \quad U_{\mathbf{k}}^{\dagger} = \begin{pmatrix} u_{\mathbf{k}}^* & v_{\mathbf{k}}^* \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \quad (2.7)$$

and  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  satisfy the relation

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1. \quad (2.8)$$

This will be satisfied if we write  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  on the form

$$u_{\mathbf{k}} = e^{i\alpha} \cos \theta_{\mathbf{k}}, \quad v_{\mathbf{k}} = e^{i\beta} \sin \theta_{\mathbf{k}}. \quad (2.9)$$

Our Hamiltonian will now be on the form

$$H = E_0 + \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^{\dagger} H_{\mathbf{k}} \varphi_{\mathbf{k}} \quad (2.10)$$

with  $H_{\mathbf{k}} = U_{\mathbf{k}}^\dagger H'_{\mathbf{k}} U_{\mathbf{k}}$  and  $\varphi_{\mathbf{k}} \equiv U_{\mathbf{k}}^\dagger \varphi'_{\mathbf{k}}$ , i.e.

$$\varphi_{\mathbf{k}} \equiv \begin{pmatrix} \gamma_{\mathbf{k},\uparrow} \\ \gamma_{-\mathbf{k},\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}}^* & v_{\mathbf{k}}^* \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k},\uparrow} \\ c_{-\mathbf{k},\downarrow}^\dagger \end{pmatrix}. \quad (2.11)$$

The new fermionic operators  $\gamma_{\mathbf{k},\uparrow}$  and  $\gamma_{-\mathbf{k},\downarrow}^\dagger$  are describing excitations of so called *quasiparticles*.

### 2.2.3 Diagonalization of the BCS Hamiltonian

We need to find what values of  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  that will satisfy the relation (2.8) and diagonalize  $H_{\mathbf{k}}$ :

$$\begin{aligned} H_{\mathbf{k}} &= U_{\mathbf{k}}^\dagger H'_{\mathbf{k}} U_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}}^* & v_{\mathbf{k}}^* \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \varepsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & -\varepsilon_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}}^* \\ v_{\mathbf{k}} & u_{\mathbf{k}}^* \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon_{\mathbf{k}}(|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2) + \Delta_{\mathbf{k}} u_{\mathbf{k}}^* v_{\mathbf{k}} + \Delta_{\mathbf{k}}^* u_{\mathbf{k}} v_{\mathbf{k}}^* & \Delta_{\mathbf{k}} u_{\mathbf{k}}^{*2} - \Delta_{\mathbf{k}}^* v_{\mathbf{k}}^{*2} - 2\varepsilon_{\mathbf{k}} u_{\mathbf{k}}^* v_{\mathbf{k}}^* \\ \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^2 - \Delta_{\mathbf{k}} v_{\mathbf{k}}^2 - 2\varepsilon_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} & -[\varepsilon_{\mathbf{k}}(|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2) + \Delta_{\mathbf{k}} u_{\mathbf{k}}^* v_{\mathbf{k}} + \Delta_{\mathbf{k}}^* u_{\mathbf{k}} v_{\mathbf{k}}^*] \end{pmatrix}. \end{aligned} \quad (2.12)$$

For the off-diagonal elements to be zero we must have  $\Delta_{\mathbf{k}}^* u_{\mathbf{k}}^2 - \Delta_{\mathbf{k}} v_{\mathbf{k}}^2 - 2\varepsilon_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} = 0$ . We write  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  on the form given in equation (2.9) and write  $\Delta_{\mathbf{k}} = |\Delta_{\mathbf{k}}| e^{i\gamma}$ . This yields

$$\begin{aligned} 0 &= \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^2 - \Delta_{\mathbf{k}} v_{\mathbf{k}}^2 - 2\varepsilon_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \\ &= |\Delta_{\mathbf{k}}| e^{i(\alpha+\beta)} \cos^2 \theta \left( e^{i(\alpha-\beta-\gamma)} - e^{-i(\alpha-\beta-\gamma)} \tan^2 \theta_{\mathbf{k}} - 2 \frac{\varepsilon_{\mathbf{k}}}{|\Delta_{\mathbf{k}}|} \tan \theta_{\mathbf{k}} \right), \end{aligned}$$

which gives

$$\alpha - \beta = \gamma \quad \text{and} \quad \tan \theta_{\mathbf{k}} = -\frac{\varepsilon_{\mathbf{k}}}{|\Delta_{\mathbf{k}}|} \pm \sqrt{\frac{\varepsilon_{\mathbf{k}}^2}{|\Delta_{\mathbf{k}}|^2} + 1}. \quad (2.13)$$

$u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  will thus be satisfied by

$$\begin{aligned} |u_{\mathbf{k}}|^2 &= \cos^2 \theta = \frac{1}{2} \left( 1 \pm \frac{\varepsilon_{\mathbf{k}}}{\sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}} \right) \\ |v_{\mathbf{k}}|^2 &= \sin^2 \theta = \frac{1}{2} \left( 1 \mp \frac{\varepsilon_{\mathbf{k}}}{\sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}} \right). \end{aligned} \quad (2.14)$$

If we let  $\varepsilon_{\mathbf{k}}^+ > 0$  and  $\varepsilon_{\mathbf{k}}^- < 0$ , we notice how we get  $|u_{\mathbf{k}}| = 1$  and  $|v_{\mathbf{k}}| = 0$  when  $\Delta_{\mathbf{k}} = 0$ , i.e. when there is no attraction between the electrons and thus in the limit of the *normal* state, according to equation (2.5). We calculate the diagonal terms of  $H_{\mathbf{k}}$  (2.12) and find

$$H_{\mathbf{k}} = \begin{pmatrix} E_{\mathbf{k}} & 0 \\ 0 & -E_{\mathbf{k}} \end{pmatrix} \quad (2.15)$$

where we have defined

$$E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}. \quad (2.16)$$

as the quasiparticle excitation energy. It is now clear why  $\Delta_{\mathbf{k}}$  is referred to as the *gap-parameter* as it gives a gap in the excitation spectrum of the quasiparticles  $\varphi_{\mathbf{k}}$ . Moreover, we get

$$k^{\pm} = k_F \sqrt{1 + \frac{\varepsilon_{\mathbf{k}}^{\pm}}{\mu}} = k_F \sqrt{1 \pm \frac{\sqrt{E_{\mathbf{k}}^2 - |\Delta_{\mathbf{k}}|^2}}{\mu}} \quad (2.17)$$

where  $\mu = \hbar^2 k_F^2 / 2m$  and  $\varepsilon_{\mathbf{k}}^{\pm} = \pm \sqrt{E_{\mathbf{k}}^2 - |\Delta_{\mathbf{k}}|^2}$  is obtained from equation (2.16). We notice how we get a fourfold degeneracy of relevant states,  $(k^+, k^-, -k^+, -k^-)$ , for each  $E_{\mathbf{k}}$ . From equation (2.14) we see that the quasiparticle excitation  $\gamma_{\mathbf{k},\uparrow}^{\dagger}$  from equation (2.11) will be electronlike, since we have  $u_{\mathbf{k}} \rightarrow 1$  and  $v_{\mathbf{k}} \rightarrow 0$  as  $\Delta \rightarrow 0$  and  $c_{\mathbf{k},\uparrow}^{\dagger}$  creates an electron while  $c_{-\mathbf{k},\downarrow}$  destroys an electron, leaving a hole. Similarly,  $\gamma_{\mathbf{k},\uparrow}$  will be holelike. Moreover, from equation (2.17) we see that  $\pm k^+$  ( $\pm k^-$ ) corresponds to energy above (below) the Fermi surface and thus  $\pm k^+$  ( $\pm k^-$ ) are electron (hole)-like excitations.

### 2.2.4 Bogoliubov-de Gennes Equations

In the description above we assumed the Hamiltonian to be position-invariant so that the wave functions could be considered as simple plane waves,  $\sim \exp(i\mathbf{k} \cdot \mathbf{r})$ . We simply replaced the Hamiltonian for a single particle system,

$$h(\mathbf{r}) = -\frac{\hbar^2 \nabla^2}{2m} - \mu(\mathbf{r}) + V(\mathbf{r}), \quad (2.18)$$

with  $\varepsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m - \mu$ . For systems where we can not do this simplification we introduce field operators:

$$\psi(\mathbf{r}, t) \equiv \sum_{\mathbf{k}} U(\mathbf{r}, t) \varphi_{\mathbf{k}}, \quad \psi^\dagger(\mathbf{r}, t) \equiv \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^\dagger U^\dagger(\mathbf{r}, t) \quad (2.19)$$

and rewrite the Hamiltonian in equation (2.6) as

$$H = E_0 + \int d^3r \psi^\dagger(\mathbf{r}, t) \begin{pmatrix} h(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -h(\mathbf{r}) \end{pmatrix} \psi(\mathbf{r}, t) \equiv E_0 + \int d^3r \psi^\dagger(\mathbf{r}, t) H(\mathbf{r}) \psi(\mathbf{r}, t). \quad (2.20)$$

Again the Hamiltonian may be diagonalized by setting  $U^\dagger(\mathbf{r}, t) H(\mathbf{r}) U(\mathbf{r}, t) = H_{\mathbf{k}}$ , or equally  $H(\mathbf{r}) U(\mathbf{r}, t) = U(\mathbf{r}, t) H_{\mathbf{k}}$ , where  $H_{\mathbf{k}}$  is on the form given in equation (2.15). By separating these equations for each eigenvalue in  $H_{\mathbf{k}}$  we get the *Bogoliubov de Gennes equations* (BdG equations) [5]:

$$\begin{pmatrix} h(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -h(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u(\mathbf{r}, t) \\ v(\mathbf{r}, t) \end{pmatrix} = E_{\mathbf{k}} \begin{pmatrix} u(\mathbf{r}, t) \\ v(\mathbf{r}, t) \end{pmatrix}, \quad (2.21)$$

$$\begin{pmatrix} -h(\mathbf{r}) & -\Delta^*(\mathbf{r}) \\ -\Delta(\mathbf{r}) & h(\mathbf{r}) \end{pmatrix} \begin{pmatrix} -v(\mathbf{r}, t) \\ u(\mathbf{r}, t) \end{pmatrix} = E_{\mathbf{k}} \begin{pmatrix} -v(\mathbf{r}, t) \\ u(\mathbf{r}, t) \end{pmatrix}.$$

From equation (2.11) we have  $\gamma_{\mathbf{k},\uparrow}^\dagger = u(\mathbf{r}, t) c_{\mathbf{k},\uparrow}^\dagger + v(\mathbf{r}, t) c_{-\mathbf{k},\downarrow}$ . By writing  $\gamma_{\mathbf{k},\sigma}^\dagger = u_\sigma(\mathbf{r}, t) c_{\mathbf{k},\sigma}^\dagger + v_{-\sigma}(\mathbf{r}, t) c_{-\mathbf{k},-\sigma}$  and  $\gamma_{\mathbf{k},\sigma} = u_\sigma(\mathbf{r}, t) c_{-\mathbf{k},-\sigma}^\dagger + v_{-\sigma}(\mathbf{r}, t) c_{\mathbf{k},\sigma}$ , we can represent  $\gamma_{\mathbf{k},\sigma}^\dagger$  and  $\gamma_{\mathbf{k},\sigma}$  by the vectors  $\Psi_{e,\sigma}$  and  $\Psi_{h,\sigma}$ , respectively, where  $\Psi$  is a vector of the form  $(u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow)^T$ . We get:

$$\begin{aligned} \gamma_{\mathbf{k},\uparrow}^\dagger \rightarrow \Psi_{e,\uparrow} &= \begin{pmatrix} u(\mathbf{r}, t) \\ 0 \\ 0 \\ v(\mathbf{r}, t) \end{pmatrix}, & \gamma_{-\mathbf{k},\downarrow}^\dagger \rightarrow \Psi_{e,\downarrow} &= \begin{pmatrix} 0 \\ u(\mathbf{r}, t) \\ -v(\mathbf{r}, t) \\ 0 \end{pmatrix}, \\ \gamma_{\mathbf{k},\uparrow} &\rightarrow \Psi_{h,\uparrow} = \begin{pmatrix} v^*(\mathbf{r}, t) \\ 0 \\ 0 \\ u^*(\mathbf{r}, t) \end{pmatrix}, & \gamma_{-\mathbf{k},\downarrow} &\rightarrow \Psi_{h,\downarrow} = \begin{pmatrix} 0 \\ -v^*(\mathbf{r}, t) \\ u^*(\mathbf{r}, t) \\ 0 \end{pmatrix}. \end{aligned} \quad (2.22)$$

We define the  $2 \times 2$ -matrices  $\hat{H}(\mathbf{r}) \equiv \hat{\sigma}_0 h(\mathbf{r})$  and  $\hat{\Delta}(\mathbf{r}) \equiv i\hat{\sigma}_2 \Delta(\mathbf{r})$  where  $\hat{\sigma}_0$  is the identity matrix and  $\hat{\sigma}_i$  with  $(i = 1, 2, 3)$  are the Pauli matrices, see equation (A.2) in the appendix. Moreover, we define  $\vec{u}(\mathbf{r}, t) \equiv (u_\uparrow(\mathbf{r}, t) \ u_\downarrow(\mathbf{r}, t))^T$  and  $\vec{v}(\mathbf{r}, t) \equiv (v_\uparrow(\mathbf{r}, t) \ v_\downarrow(\mathbf{r}, t))^T$ . The BdG-equations (2.21) can then be written more compact:

$$\begin{pmatrix} \hat{H}(\mathbf{r}) & \hat{\Delta}(\mathbf{r}) \\ \hat{\Delta}^\dagger(\mathbf{r}) & -\hat{H}(\mathbf{r}) \end{pmatrix} \begin{pmatrix} \vec{u}(\mathbf{r}, t) \\ \vec{v}(\mathbf{r}, t) \end{pmatrix} = E_{\mathbf{k}} \begin{pmatrix} \vec{u}(\mathbf{r}, t) \\ \vec{v}(\mathbf{r}, t) \end{pmatrix}. \quad (2.23)$$

# Chapter 3

## Josephson effect

Josephson effect describes two important phenomena of supercurrents in a *Josephson junction*. A Josephson junction is a device consisting of two superconductors that is brought into contact via a *weak link*, in which the *critical current* is much lower. The critical current is the maximum supercurrent that can exist in the superconductor and is related to the density of Cooper pairs. Firstly Josephson predicted [6] that supercurrents would flow through the Josephson junction even without any applied voltage. Secondly if the junction was driven by an external current exceeding the critical current, electromagnetic waves would be radiated.

There are several ways to construct a weak link, however we will in this project only consider the *SNS-junction*, i.e. a junction consisting of two superconductors, separated by a normal metal. ‘

SKRIVE UTLEDNING men skal jeg bruke free energy, eller utledningen i sudbø?



# Chapter 4

## Andreev reflection

When an electron with momentum,  $\mathbf{q}^+$ , and spin,  $\sigma$ , in the normal metal is propagating towards the interface between the normal metal and the superconductor, it will be scattered with certain probabilities of transmission and reflection. There are two possible ways the electron could be transmitted and reflected. The electron may be transmitted into the superconductor as an electron-like quasiparticle such that the energy of the transmitted quasiparticle is on the *same* side of the Fermi surface, i.e. with momentum  $\mathbf{k}^+$  and spin  $\sigma$ , or as a hole-like quasiparticle by crossing the Fermi surface, i.e. with momentum  $-\mathbf{k}^-$  and spin  $\sigma$ . The electron may be reflected, either in the normal way, i.e. as an electron with opposite momentum,  $-\mathbf{q}^+$ , and the same spin,  $\sigma$ , or by *Andreev reflection* [7]. In Andreev reflection the incoming electron goes into the superconductor and form a Cooper pair with an electron of opposite spin, leaving a reflected hole with momentum  $\mathbf{q}^-$  and spin  $-\sigma$ . We ignore the spin degeneracy and express the wave vectors as  $\psi(\mathbf{r}) = \begin{pmatrix} u_{\mathbf{k}}(\mathbf{r}) & v_{\mathbf{k}}(\mathbf{r}) \end{pmatrix}^T$ . In the simplest case we consider plane waves, i.e. energies,  $E_{\mathbf{k}}$ , as given in equation (2.16) with corresponding wave numbers,  $\pm k^+$  and  $\pm k^-$ , and wave vectors of the form [8]

$$\psi_{\pm k^+}(\mathbf{r}) = \begin{pmatrix} u_0 e^{i\alpha} \\ v_0 e^{i\beta} \end{pmatrix} e^{\pm i\mathbf{k}^+ \cdot \mathbf{r}} \quad \text{and} \quad \psi_{\pm k^-}(\mathbf{r}) = \begin{pmatrix} v_0 e^{-i\beta} \\ u_0 e^{-i\alpha} \end{pmatrix} e^{\pm i\mathbf{k}^- \cdot \mathbf{r}}. \quad (4.1)$$

The incoming, reflected and transmitted wave vectors will in this notation take the form

$$\begin{aligned}
 \psi_i(\mathbf{r}) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\mathbf{q}^+ \mathbf{r}} \\
 \psi_r(\mathbf{r}) &= r_{ee} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\mathbf{q}^+ \mathbf{r}} + r_{eh} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\mathbf{q}^- \mathbf{r}} \\
 \psi_t(\mathbf{r}) &= t_{ee} \begin{pmatrix} u_0 e^{i\alpha} \\ v_0 e^{i\beta} \end{pmatrix} e^{i\mathbf{k}^+ \mathbf{r}} + t_{eh} \begin{pmatrix} v_0 e^{-i\beta} \\ u_0 e^{-i\alpha} \end{pmatrix} e^{-i\mathbf{k}^- \mathbf{r}},
 \end{aligned} \tag{4.2}$$

were  $r_{ee}$ ,  $r_{eh}$ ,  $t_{ee}$ , and  $t_{eh}$  represent the probabilities of normal reflection, Andreev reflection, electron-like transmission and hole-like transmission with crossing of the Fermi surface, respectively. We notice how an electron(hole) with momentum  $+\mathbf{k}$  is moving in positive(negative) direction, while momentum  $-\mathbf{k}$  gives negative(positive) direction.

In equation (2.16) we found that only energies above the energy gap,  $|\Delta_{\mathbf{k}}|$ , are allowed for the quasiparticles. Consequently, when  $E_{\mathbf{k}} < |\Delta_{\mathbf{k}}|$  the amplitudes  $c$  and  $d$  will be zero and only reflection (either normal or Andreev reflection) is allowed. States with such energies in SNS-junctions would thus be trapped in the normal metal and is referred to as Andreev Bound States (ABS).

# Chapter 5

## Physical System

We consider a one-dimensional SNS-junction parallel to the  $x$ -axis, with  $x = 0$  and  $x = L$  at interface between the normal metal and the leftmost (L) and rightmost (R) superconductor, respectively. We use the position-space representation  $\Psi(x) = (\vec{u}(x) \ \vec{v}(x))^T$  as used in equation (2.23). We consider s-wave superconductors such that the gap parameter,  $\Delta(x)$ , is constant in each superconductor, with equal magnitude,  $\Delta_0$ , but allow for different phases,  $\phi_L$  and  $\phi_R$ . Necessarily, the gap parameter is zero in the normal metal. The overall gap parameter is

$$\Delta(x) = \Delta_0 \left( e^{i\phi_L} \Theta(-x) + e^{i\phi_R} \Theta(x-L) \right), \quad (5.1)$$

where  $\Theta(x)$  is the Heaviside step function. The Hamiltonian of the system will be on the form given in equation (2.18). We allow for different chemical potential,  $\mu_S$  and  $\mu_N$ , and different effective mass,  $m_S$  and  $m_N$ , in the superconductors and the normal metal, respectively. Moreover, we let  $V(x)$  be a delta-potential barrier at the interfaces and allow for different strength, i.e.  $V(x) = V_L \delta(x) + V_R \delta(x-L)$ . The overall Hamiltonian is then

$$h(x) = h_S(x) (\Theta(-x) + \Theta(x-L)) + h_N(x) \Theta(x) \Theta(L-x) + V_L \delta(x) + V_R \delta(x-L) \quad (5.2)$$

where  $h_S(x) = -\frac{\hbar^2}{2m_S} \frac{d^2}{dx^2} - \mu_S$  and  $h_N(x) = -\frac{\hbar^2}{2m_N} \frac{d^2}{dx^2} - \mu_N$ . The Hamiltonian,  $h(x)$ , gap parameter,  $\Delta(x)$  and wave function,  $\Psi(x)$ , must satisfy the BdG-equations (2.23).

# Chapter 6

## Andreev Bound State energies in SNS-junction

We will in this chapter find the energies of the Andreev Bound States in an SNS-junction in cases with and without barriers and external field. blablabla

### 6.1 ABS energies without barriers or applied field

blablabla

#### 6.1.1 Andreev reflection amplitude

The probability amplitudes from chapter section 4 may be found by using the boundary conditions in the interface between the normal metal and the super conductor. If we assume no barrier in the boundary conditions yields

$$\begin{aligned}\psi_i(0) + \psi_r(0) &= \psi_t(0) \\ \psi'_i(0) + \psi'_r(0) &= \psi'_t(0).\end{aligned}\tag{6.1}$$

Solving these equations give the amplitudes

$$\begin{aligned}
r_{eh} &= \frac{2e^{-i(\alpha-\beta)}}{\frac{u_0}{v_0} \frac{k^-+q^-}{k^++k^-} \left(1 + \frac{k^+}{q^+}\right) + \frac{v_0}{u_0} \frac{k^+-q^-}{k^++k^-} \left(1 - \frac{k^-}{k^+}\right)} \\
r_{ee} &= \left( \frac{u_0}{v_0} \frac{k^++q^-}{k^++k^-} + \frac{v_0}{u_0} \frac{k^+-q^-}{k^++k^-} \right) e^{i(\alpha-\beta)} r_{eh} \\
t_{ee} &= \frac{1}{v_0} \frac{k^-+q^-}{k^++k^-} e^{-i\beta} r_{eh} \\
t_{eh} &= \frac{1}{u_0} \frac{k^+-q^-}{k^++k^-} e^{i\alpha} r_{eh}.
\end{aligned} \tag{6.2}$$

This can be simplified if we use  $k_{\pm} \approx q_{\pm} \approx k_F$  in  $r_{eh}$ :

$$r_{eh} = \frac{v_0}{u_0} e^{-i(\alpha-\beta)} \equiv e^{-i\eta} e^{-i(\alpha-\beta)}. \tag{6.3}$$

Similarly we have the amplitude of an incoming hole which is Andreev reflected as an electron with  $r_{he} = r_{eh}^*$ :

$$r_{he} = \frac{v_0}{u_0} e^{i(\alpha-\beta)} = e^{-i\eta} e^{i(\alpha-\beta)}. \tag{6.4}$$

### 6.1.2 Bohr-Sommerfeld quantization

For the Andreev Bound states we have  $E < \Delta_0$  which yields  $\eta = i \arccos(E/\Delta_0)$ . The Bohr-Sommerfeld quantization condition require the total phase obtained by the state in a whole cycle should be a multiple of  $2\pi$ . A electron starting at the left interface traveling towards the right interface would gain a phase of  $k^+ L$  before it is andreev reflected at the interface with the amplitude  $r_{eh}$  and thus is gaining a phase of  $\eta - (\alpha - \beta)$  from the reflected. The state would then continue as a hole traveling left, and thus obtaining a phase of  $-k^- L$

$$\begin{aligned}
2\pi n &= \oint d\phi = \int_{S_1}^{S_2} \pm k^{\pm} dx + \phi_{(eh)(he)}^R + \int_{S_2}^{S_1} \pm k^{\mp} dx + \phi_{(he)(eh)}^L \\
&= \pm L(k^{\pm} - k^{\mp}) - 2\eta \pm \varphi
\end{aligned} \tag{6.5}$$

where  $\phi_{(eh)(he)}^{R/L} = -\eta \mp (\alpha_{R/L} - \beta_{R/L})$  is the phase from Andreev reflection of a electron (hole) and we have defined the phase difference  $\varphi \equiv (\alpha_L - \beta_L) - (\alpha_R - \beta_R)$ . Again, we let  $k^+ \approx k^- \approx k_F$ .

### 6.1.3 ABS energy

We can then find the Andreev energy levels:

$$E = \Delta_0 \cos\left(\frac{\varphi}{2}\right). \quad (6.6)$$

## 6.2 ABS energies with barriers

We consider a one-dimensional SNS-junction parallel to the  $x$ -axis, with  $x = 0$  and  $x = L$  at interface between the normal metal and the leftmost (L) and rightmost (R) superconductor, respectively. We use the position-space representation  $\Psi(x) = (\vec{u}(x) \ \vec{v}(x))^T$  as used in equation (2.23). We consider s-wave superconductors such that the gap parameter,  $\Delta(x)$ , is constant in each superconductor, with equal magnitude,  $\Delta_0$ , but allow for different phases,  $\phi_L$  and  $\phi_R$ . Necessarily, the gap parameter is zero in the normal metal. The overall gap parameter is

$$\Delta(x) = \Delta_0 \left( e^{i\phi_L} \Theta(-x) + e^{i\phi_R} \Theta(x-L) \right), \quad (6.7)$$

where  $\Theta(x)$  is the Heaviside step function. The Hamiltonian of the system will be on the form given in equation (2.18). We allow for different chemical potential,  $\mu_S$  and  $\mu_N$ , and different effective mass,  $m_S$  and  $m_N$ , in the superconductors and the normal metal, respectively. Moreover, we let  $V(x)$  be a delta-potential barrier at the interfaces and allow for different strength, i.e.  $V(x) = V_L \delta(x) + V_R \delta(x-L)$ . The overall Hamiltonian is then

$$h(x) = h_S(x) (\Theta(-x) + \Theta(x-L)) + h_N(x) \Theta(x) \Theta(L-x) + V_L \delta(x) + V_R \delta(x-L) \quad (6.8)$$

where  $h_S(x) = -\frac{\hbar^2}{2m_S} \frac{d^2}{dx^2} - \mu_S$  and  $h_N(x) = -\frac{\hbar^2}{2m_N} \frac{d^2}{dx^2} - \mu_N$ . The Hamiltonian,  $h(x)$ , gap parameter,  $\Delta(x)$  and wave function,  $\Psi(x)$ , must satisfy the BdG-equations (2.23).

### 6.2.1 Boundary conditions

Charge conservation yields the boundary condition

$$\begin{aligned}\Psi_L(0) &= \Psi_N(0) \equiv \Psi(0), \\ \Psi_R(L) &= \Psi_N(L) \equiv \Psi(L).\end{aligned}\tag{6.9}$$

Since we have a delta potential at the interfaces we will not have conservation of the derivative of the wave vectors and integrate the BdG-equations (2.23) in order to find the boundary conditions for the derivatives:

$$\begin{aligned}0 &= \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} E_k \Psi(x) dx = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \begin{pmatrix} \hat{H}(x) & \hat{\Delta}(x) \\ \hat{\Delta}^\dagger(x) & -\hat{H}(x) \end{pmatrix} \Psi(x) dx \\ &= \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{0^-} \begin{pmatrix} h_S(x) \hat{\sigma}_0 & i\Delta(x) \hat{\sigma}_2 \\ -i\Delta^*(x) \hat{\sigma}_2 & -h_S(x) \hat{\sigma}_0 \end{pmatrix} \Psi_L(x) dx + \begin{pmatrix} V_L \hat{\sigma}_0 & 0 \\ 0 & -V_L \hat{\sigma}_0 \end{pmatrix} \Psi(0) \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_{0^+}^{\epsilon} \begin{pmatrix} h_N(x) \hat{\sigma}_0 & 0 \\ 0 & -h_S(x) \hat{\sigma}_0 \end{pmatrix} \Psi_N(x) dx \\ &= \begin{pmatrix} \hat{\sigma}_0 & 0 \\ 0 & -\hat{\sigma}_0 \end{pmatrix} \left( V_L \Psi(0) - \frac{\hbar^2}{2} \lim_{\epsilon \rightarrow 0} \left( \frac{1}{m_S} \int_{-\epsilon}^{0^-} \frac{d^2}{dx^2} \Psi_L(x) dx + \frac{1}{m_N} \int_{0^+}^{\epsilon} \frac{d^2}{dx^2} \Psi_N(x) dx \right) \right) \\ &= \begin{pmatrix} \hat{\sigma}_0 & 0 \\ 0 & -\hat{\sigma}_0 \end{pmatrix} \left( V_L \Psi(0) + \frac{\hbar^2}{2m_S} \Psi'_L(0) - \frac{\hbar^2}{2m_N} \Psi'_N(0) \right),\end{aligned}$$

and the boundary condition at the  $x = 0$  interface for the derivatives is thus

$$\frac{\hbar^2}{2m_N} \Psi'_N(0) - \frac{\hbar^2}{2m_S} \Psi'_L(0) = V_L \Psi(0).\tag{6.10}$$

Similarly, we get

$$\frac{\hbar^2}{2m_S} \Psi'_R(L) - \frac{\hbar^2}{2m_N} \Psi'_N(L) = V_R \Psi(L)\tag{6.11}$$

as boundary condition at the  $x = L$  interface.

### 6.2.2 Superconduction region

The solution of the BdG-equations (2.23) that satisfy the boundary conditions will be on the form

$$\Psi_k^\pm(x) = \begin{pmatrix} \vec{u}_k \\ \vec{v}_k \end{pmatrix} e^{ik^\pm x} \quad (6.12)$$

and

$$h(x)\Psi_k^\pm(x) = \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \Psi_k^\pm(x) \equiv \varepsilon_k^\pm \Psi_k^\pm(x), \quad (6.13)$$

where  $h(x)$  is either  $h_S$  or  $h_N$  depending on the region, as defined under equation (6.8). We find the eigenvalues,  $E_k$  of an equation of the form  $A\Psi_k(x) = E_k\Psi_k(x)$ , for a matrix  $A$  by setting the determinant of the matrix  $(A - E_k I)$  to zero. For the eigenvalue problem in the BdG-equations (2.23) we must calculate

$$\begin{aligned} 0 &= \det \begin{pmatrix} (\varepsilon_k^\pm - E_k) \hat{\sigma}_0 & i\Delta \hat{\sigma}_2 \\ -i\Delta^* \hat{\sigma}_2 & (-\varepsilon_k^\pm - E_k) \hat{\sigma}_0 \end{pmatrix} \\ &= \det \left( (\varepsilon_k^\pm - E_k) \hat{\sigma}_0 \right) \det \left( -(\varepsilon_k^\pm + E_k) \hat{\sigma}_0 + i\Delta^* \hat{\sigma}_2 \frac{1}{\varepsilon_k^\pm - E_k} i\Delta \hat{\sigma}_2 \right) \\ &= (\varepsilon_k^\pm - E_k)^2 \left( \varepsilon_k^\pm + E_k + \frac{\Delta_0^2}{\varepsilon_k^\pm - E_k} \right)^2, \end{aligned} \quad (6.14)$$

where we have used the relation

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B). \quad (6.15)$$

As  $E_k = \varepsilon_{k,S}^\pm$  would give zero in the denominator when  $\Delta_0$  is non-zero, the only solution to the above equation (6.14) is

$$E_k^2 = \varepsilon_k^{\pm 2} + \Delta_0^2 \quad (6.16)$$

which agrees with the energies obtained in equation (2.16) and  $k^\pm$  and  $\varepsilon_k^\pm$  will be as in equation (2.17). We will here only consider positive energies,  $E_k = \sqrt{\varepsilon_k^{\pm 2} + \Delta_0^2}$  and find the corresponding eigenvectors. For  $\varepsilon_k^+ = +\sqrt{E_k^2 - \Delta_0^2}$  we get the wave-functions describing electronlike quasipar-



titles:

$$\Psi_{e,\uparrow}^{\pm}(x) = \begin{pmatrix} u_0 e^{i\alpha} \\ 0 \\ 0 \\ \nu_0 e^{i\beta} \end{pmatrix} e^{\pm i k^+ x} \quad \text{and} \quad \Psi_{e,\downarrow}^{\pm}(x) = \begin{pmatrix} 0 \\ u_0 e^{i\alpha} \\ -\nu_0 e^{i\beta} \\ 0 \end{pmatrix} e^{\pm i k^+ x}. \quad (6.17)$$

Similarly, for  $\varepsilon_k^- = -\sqrt{E_k^2 - \Delta_0^2}$  we get the wave-functions describing holelike quasiparticles:

$$\Psi_{h,\uparrow}^{\pm}(x) = \begin{pmatrix} \nu_0 e^{-i\beta} \\ 0 \\ 0 \\ u_0 e^{-i\alpha} \end{pmatrix} e^{\pm i k^- x} \quad \text{and} \quad \Psi_{h,\downarrow}^{\pm}(x) = \begin{pmatrix} 0 \\ -\nu_0 e^{-i\beta} \\ u_0 e^{-i\alpha} \\ 0 \end{pmatrix} e^{\pm i k^- x}. \quad (6.18)$$

where we have used the relation from equation (2.9) and (2.14). We see that this agrees with the quasiparticle-representation in equation (2.22). We have here, as in equation (2.14) defined  $u_0$  and  $\nu_0$  as

$$u_0 = \sqrt{\frac{1}{2} \left( 1 + \frac{(E_k^2 - \Delta_0^2)^{1/2}}{E_k} \right)}, \quad (6.19)$$

$$\nu_0 = \sqrt{\frac{1}{2} \left( 1 - \frac{(E_k^2 - \Delta_0^2)^{1/2}}{E_k} \right)}$$

We introduce  $\eta$  defined as

$$\eta = \begin{cases} \operatorname{arccosh} \frac{E_k}{\Delta_0}, & E_k > \Delta_0 \\ i \arccos \frac{E_k}{\Delta_0}, & E_k < \Delta_0 \end{cases} \quad (6.20)$$

such that  $u_0$  and  $\nu_0$  can be written as

$$u_0 = \frac{e^{\eta/2}}{\sqrt{2 \cosh \eta}} \quad (6.21)$$

$$\nu_0 = \frac{e^{-\eta/2}}{\sqrt{2 \cosh \eta}}.$$

### 6.2.3 Normal region

In the normal region the gap parameter,  $\Delta(x)$ , is zero, and so  $u_0 = 1$  and  $v_0 = 0$  and the eigenvalues are  $E_k = \varepsilon_k^\pm$  with corresponding eigenvectors

$$\begin{aligned} \Psi_{e,\uparrow}^\pm(x) &= \begin{pmatrix} e^{i\alpha_N} \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{\pm ik^+x}, & \Psi_{e,\downarrow}^\pm(x) &= \begin{pmatrix} 0 \\ e^{i\alpha_N} \\ 0 \\ 0 \end{pmatrix} e^{\pm ik^+x}, \\ \Psi_{h,\uparrow}^\pm(x) &= \begin{pmatrix} 0 \\ 0 \\ e^{-i\alpha_N} \\ 0 \end{pmatrix} e^{\pm ik^-x}, & \Psi_{h,\downarrow}^\pm(x) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i\alpha_N} \end{pmatrix} e^{\pm ik^-x}. \end{aligned} \tag{6.22}$$

We allow for different phases in each region and in the superconducting region we will only consider states propagating *away* from the intersections. We can write down the total wavefunction for each region; the left superconductor (L), the right superconductor (R) and the normal metal

(N):

$$\begin{aligned}
\Psi_L(x) &= a_1 \begin{pmatrix} u_0 e^{i\alpha_L} \\ 0 \\ 0 \\ \nu_0 e^{i\beta_L} \end{pmatrix} e^{-ik^+x} + a_2 \begin{pmatrix} 0 \\ u_0 e^{i\alpha_L} \\ -\nu_0 e^{i\beta_L} \\ 0 \end{pmatrix} e^{-ik^+x} + a_3 \begin{pmatrix} \nu_0 e^{-i\beta_L} \\ 0 \\ 0 \\ u_0 e^{-i\alpha_L} \end{pmatrix} e^{ik^-x} + a_4 \begin{pmatrix} 0 \\ -\nu_0 e^{-i\beta_L} \\ u_0 e^{-i\alpha_L} \\ 0 \end{pmatrix} e^{ik^-x} \\
\Psi_R(x) &= b_1 \begin{pmatrix} u_0 e^{i\alpha_R} \\ 0 \\ 0 \\ \nu_0 e^{i\beta_R} \end{pmatrix} e^{ik^+x} + b_2 \begin{pmatrix} 0 \\ u_0 e^{i\alpha_R} \\ -\nu_0 e^{i\beta_R} \\ 0 \end{pmatrix} e^{ik^+x} + b_3 \begin{pmatrix} \nu_0 e^{-i\beta_R} \\ 0 \\ 0 \\ u_0 e^{-i\alpha_R} \end{pmatrix} e^{-ik^-x} + b_4 \begin{pmatrix} 0 \\ -\nu_0 e^{-i\beta_R} \\ u_0 e^{-i\alpha_R} \\ 0 \end{pmatrix} e^{-ik^-x} \\
\Psi_N(x) &= c_1 \begin{pmatrix} e^{i\alpha_N} \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{ik^+x} + c_2 \begin{pmatrix} e^{i\alpha_N} \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-ik^+x} + c_3 \begin{pmatrix} 0 \\ e^{i\alpha_N} \\ 0 \\ 0 \end{pmatrix} e^{ik^+x} + c_4 \begin{pmatrix} 0 \\ e^{i\alpha_N} \\ 0 \\ 0 \end{pmatrix} e^{-ik^+x} \\
&\quad + c_5 \begin{pmatrix} 0 \\ 0 \\ e^{-i\alpha_N} \\ 0 \end{pmatrix} e^{ik^-x} + c_6 \begin{pmatrix} 0 \\ 0 \\ e^{-i\alpha_N} \\ 0 \end{pmatrix} e^{-ik^-x} + c_7 \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i\alpha_N} \end{pmatrix} e^{ik^-x} + c_8 \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i\alpha_N} \end{pmatrix} e^{-ik^-x}
\end{aligned} \tag{6.23}$$

#### 6.2.4 ABS energy

We use the boundary conditions and insert the equations in a homogeneous matrix equation of the form  $M(a_1 \dots a_4 \ b_1 \dots b_4 \ c_1 \dots c_8)^T = 0$  where  $M$  is a  $16 \times 16$ -matrix, see appendix !!!APPENDIX!!!. The determinant of  $M$  is found to be

$$\det(M) = \left( \frac{4k_F^2}{\cosh \eta} \right)^4 \left[ \sin^2 \frac{\varphi}{2} + (1 + \zeta) \sinh^2 \eta \right]^2 \tag{6.24}$$

which must be zero in order for the equations to have non-trivial solutions, and thus the energy is

$$E = \Delta_0 \cosh \eta = \Delta_0 \sqrt{\frac{\cos^2 \frac{\varphi}{2} + \zeta}{\zeta + 1}}. \tag{6.25}$$

In these equations we have let  $\varphi = (\alpha_L - \beta_L) - (\alpha_R - \beta_R)$  be the phase difference between the left and right semiconductor. We have also introduced  $\zeta$  which measures the effect of the barriers and is given as

$$\zeta = Z^2 + z^2 \sin(k_F L) \left[ Z \cos(k_F L) + \left( \frac{z^2}{4} - 1 \right) \sin(k_F L) \right] \quad (6.26)$$

where  $Z$  and  $z$  are defined as

$$Z = \frac{Z_L + Z_R}{2} \quad \text{and} \quad z = \sqrt{Z_L Z_R} \quad (6.27)$$

and  $Z_L$  and  $Z_R$  are the barrier strengths of respectively the left and right barrier:

$$Z_L = \frac{k_F V_L}{\mu_L} \quad \text{and} \quad Z_R = \frac{k_F V_R}{\mu_R}. \quad (6.28)$$

In the limit with no barrier, i.e.  $zeta = 0$  we see that equation (6.25) yields (6.6).

### 6.3 ABS energies with applied field

We will now apply a magnetic field,  $\vec{B}$ , to the junction. This imply modifying the Hamiltonian to

$$h_N(x) = \frac{1}{2m_N} \left( \frac{\hbar}{i} \nabla - q \vec{A}(x) \right)^2 + q\varphi - \mu_N \quad (6.29)$$

with  $\vec{A}(x)$  as the vector potential and  $\varphi$  as the scalar potential.

#### 6.3.1 Gauge transformation

As we must have Gauge invariance we may do the transformation

$$\begin{aligned} \vec{A}' &= \vec{A} + \nabla f \\ \varphi' &= \varphi - \frac{\partial f}{\partial t} \end{aligned} \quad (6.30)$$

where  $f$  is any function of position and time. Doing such transformation imply a transformation in the wavefunction  $\Psi$  as well. Considering the time-dependent Schrödinger equation yields

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} &= \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - q\vec{A} \right)^2 + q\varphi - \mu_N \right] \Psi \\ &= \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - q\vec{A}' + q\nabla f \right)^2 + q\varphi' + q\frac{\partial f}{\partial t} - \mu_N \right] \Psi \end{aligned} \quad (6.31)$$

which yields

$$i\hbar \frac{\partial \Psi}{\partial t} - q\frac{\partial f}{\partial t} \Psi = \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \frac{d}{dx} - q\vec{A}' + q\nabla f \right)^2 + q\varphi' - \mu_N \right] \Psi \quad (6.32)$$

or

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \left( \Psi e^{iqf/\hbar} \right) &= e^{iqf/\hbar} \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - q\vec{A}' + q\nabla f \right)^2 + q\varphi' \right] \Psi \\ &= \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \frac{d}{dx} - q\vec{A}' \right)^2 + q\varphi' \right] \left( \Psi e^{iqf/\hbar} \right) \end{aligned} \quad (6.33)$$

where we have used that

$$e^{iqf/\hbar} \left( \frac{\hbar}{i} \nabla + q\nabla f \right) \Psi = \frac{\hbar}{i} \nabla \left( e^{iqf/\hbar} \Psi \right). \quad (6.34)$$

The Schrödinger equation in the transformed system is now on the same form as the original system:

$$i\hbar \frac{\partial \Psi'}{\partial t} = \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - q\vec{A}' \right)^2 + q\varphi' \right] \Psi' \quad (6.35)$$

with  $\Psi' = e^{iqf/\hbar} \Psi$ .

### 6.3.2 Bohr-Sommerfeld quantization condition

By choosing  $f$  such that  $\nabla f = -\vec{A}$  and  $\partial f / \partial t = \varphi$  we get  $\vec{A}' = 0$  and  $\varphi' = 0$ , such that the Schrödinger equation takes the form of a system without applied field:

$$i\hbar \frac{\partial \Psi'}{\partial t} = -\frac{\hbar^2 \nabla^2}{2m} \Psi'. \quad (6.36)$$

Consequently, the applied field imply a change of the wave function

$$\Psi \rightarrow e^{iqf/\hbar}\Psi \quad (6.37)$$

with  $\nabla f = -\vec{A}$  and  $\partial f/\partial t = \varphi$ . We will find the energy of the Andreev states in the same manner as in section ???. With applied field the phase of the wave function will be  $k^\pm x + qf/\hbar$ , where  $q = \pm e$  for an electron (hole) with  $e$  as the elementary charge. A infinitesimal change,  $dx$ , in  $x$  would then imply a phase change

$$k^\pm dx + \frac{q}{\hbar} (f(x+dx) - f(x)) = k^\pm dx + \frac{q}{\hbar} \frac{\partial f}{\partial x} dx = k^\pm dx - \frac{q}{\hbar} \vec{A}(x) \cdot d\vec{x} \quad (6.38)$$

equation (??) will be modified in the following way

$$\begin{aligned} 2\pi n &= \pm L(k^\pm - k^\mp) - 2\eta \pm \varphi \pm \frac{e}{\hbar} \int_{S_1}^{S_2} \vec{A} \cdot d\vec{x} \mp \frac{e}{\hbar} \int_{S_2}^{S_1} \vec{A} \cdot d\vec{x} \\ &= \pm L(k^\pm - k^\mp) - 2\eta \pm \varphi \pm \frac{2e}{\hbar} \int_{S_1}^{S_2} \vec{A} \cdot d\vec{x}. \end{aligned} \quad (6.39)$$

### 6.3.3 ABS energy

The energy is then

$$E = \Delta_0 \cos\left(\frac{\varphi}{2} + \frac{2e}{\hbar} \int_{S_1}^{S_2} \vec{A} \cdot d\vec{x}\right). \quad (6.40)$$

## 6.4 ABS energies with barriers and applied field

# Chapter 7

## Andreev Bound State Current in SNS-junction

### 7.1 ABS current without barriers or applied field

We find the current by using equation (??) and the free energy (??). In the case with no barriers we get

$$I = -\frac{e\Delta_0}{\hbar} \sum_{\sigma} \sin \frac{\varphi}{2}. \quad (7.1)$$

### 7.2 ABS current with barriers

We find the current by using equation (??) and the free energy (??). This yields

$$I = \frac{2e}{\hbar} \frac{dF}{d\varphi} = -\frac{2e}{\hbar} \sum_{\sigma} \tanh \left( \frac{E}{2k_B T} \right) \frac{dE}{d\varphi} = -\frac{e\Delta_0}{2\hbar} \sum_{\sigma} \frac{\sin \varphi}{\sqrt{(\zeta + 1)(\cos^2 \frac{\varphi}{2} + \zeta)}}. \quad (7.2)$$

### 7.3 ABS current with applied field

As we did in equation (??) we find the Josephson current, but now with external field and without barriers:

$$\begin{aligned}
 I &= -\frac{2e}{\hbar} \sum_{\sigma} \tanh\left(\frac{E}{2k_B T}\right) \frac{dE}{d\varphi} \\
 &= \frac{e\Delta_0}{\hbar} \sum_{\sigma} \tanh\left(\frac{\Delta_0 \cos\left(\frac{\varphi}{2} + \frac{2e}{\hbar} \int_{S_1}^{S_2} \vec{A} \cdot d\vec{x}\right)}{2k_B T}\right) \sin\left(\frac{\varphi}{2} + \frac{2e}{\hbar} \int_{S_1}^{S_2} \vec{A} \cdot d\vec{x}\right)
 \end{aligned} \tag{7.3}$$

### 7.4 ABS current with applied field and barriers



## **Chapter 8**

### **Conclusion and outlook**

# Appendix A

## Additional Information

### A.1 Commutation relations

$$\left[ c_{\mathbf{k},\sigma}^\dagger, c_{\mathbf{k}',\sigma'} \right]_+ = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\sigma,\sigma'} \quad (\text{A.1a})$$

$$\left[ c_{\mathbf{k},\sigma}^\dagger, c_{\mathbf{k}',\sigma'}^\dagger \right]_+ = 0 \quad (\text{A.1b})$$

$$\left[ c_{\mathbf{k},\sigma}, c_{\mathbf{k}',\sigma'} \right]_+ = 0 \quad (\text{A.1c})$$

### A.2 The pauli matrices

$$\sigma \quad (\text{A.2})$$

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