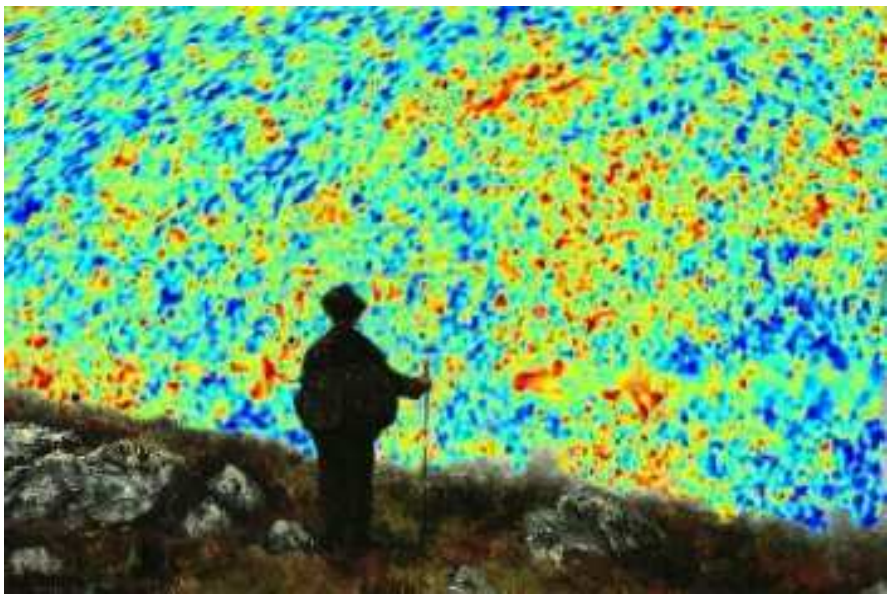
Quantum Fields

from the Hubble to the Planck Scale: An Introduction

M. Kachelrieß



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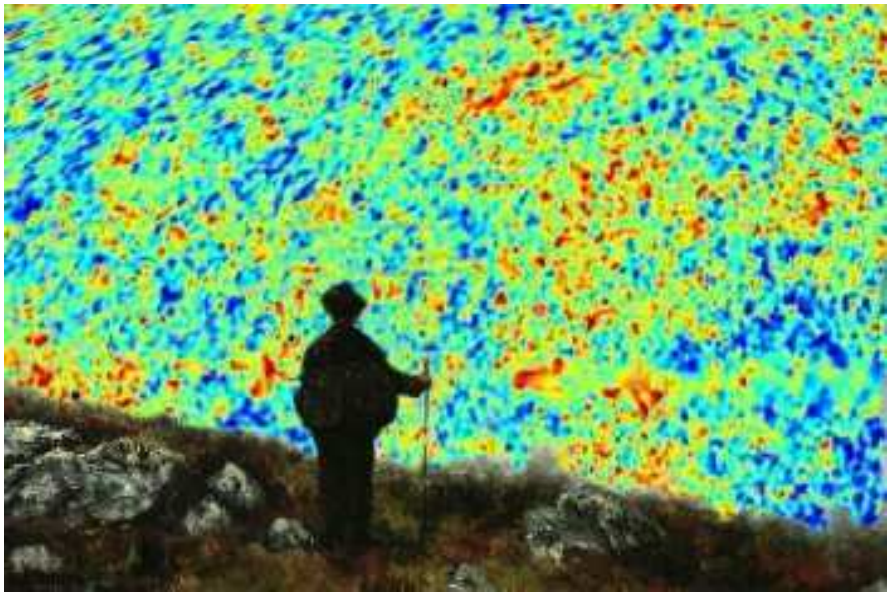

M. Kachelrieß
Institutt for fysikk
NTNU, Trondheim
Norway
email: Michael.Kachelriess@ntnu.no

If you find typos (almost sure, if you read carefully enough), conceptional errors, strange explanations, inconsistencies or have any suggestions, send me please an email!

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Contents

Notation and conventions

We use natural units with $\hbar = c = 1$, but keep mostly Newton's gravitational constant $G \neq 1$. Instead of G, we use also $\kappa = 8\pi G$, the Planck mass $M_{\rm Pl} = 1/\sqrt{G}$ or the reduced Planck mass $\widetilde{M}_{\rm Pl} = 1/\sqrt{8\pi G}$. Maxwell's equations are written in the Lorentz-Heaviside version of the cgs system. Thus there is a factor 4π in the Coulomb law, but not in Maxwell's equations. Sommerfeld's fine-structure constant is $\alpha = e^2/(4\pi) \approx 1/137$.

We choose as signature of the metric -2, thence the metric tensor in Minkowski space is $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. If not otherwise specified, Einstein's summation convention is implied.

The d'Alembert or wave operator is $\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta$, while the four-dimensional nabla operator has the components $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

A boldface italic letter denotes the components of a three-vector $\boldsymbol{V} = \{V_x, V_y, V_z\} = \{V_i, i = 1, 2, 3\}$ or the three-dimensional part of a contravariant vector with components $V^\mu = \{V^0, V^1, V^2, V^3\} = \{V^0, \boldsymbol{V}\}$; a covariant vector has in Minkowski space the components $V_\mu = (V_0, -\boldsymbol{V})$. Scalar products of four-vectors are also denoted by $p \cdot q = p_\mu q^\mu$, of three-vectors by $\boldsymbol{pq} = p^i q^i$. Vectors and tensors in index free notation are denoted by boldface roman letters, $\mathbf{V} = V^\mu \partial_\mu$ or $\mathbf{g} = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$.
Greek indices $\alpha, \beta, \ldots$ encompass the range $\alpha = \{0, 1, 2, \ldots D-1\}$, latin indices $i, j, k, \ldots$ encompass $i = \{1, 2, \ldots D-1\}$, where D denotes the dimension of the space-time. In chapter 19, latin indices $a, b, c, \ldots$ denote tensor components with respect to the vielbein field e^a_μ.

Our convention for the Fourier transformation is

$$f(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} f(k) \mathrm{e}^{-\mathrm{i}kx} \quad \text{and} \quad f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{kx}} \,.$$

Our nomenclature for disconnected, connected, and one-particle irreducible (1PI) n-point Green functions and their corresponding generating functionals is as follows:

	Green function	generating functional
disconnected	$\mathcal{G}(x_1, \ldots, x_n)$	$Z[J, \ldots]$
connected	$G(x_1, \ldots, x_n)$	$W[J, \ldots]$
1PI	$\Gamma(x_1, \ldots, x_n)$	$\Gamma[\phi, \ldots]$

We normalise Dirac spinors as $\bar{u}(p, s)u(p, s) = 2m$.

We use as covariant derivative $D_\mu = \partial_\mu + \mathrm{i}gA^a_\mu T^a$ with coupling $g > 0$, field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$ and generators T^a satisfying $[T^a, T^b] = \mathrm{i}f^{abc}T^c$ for all gauge groups. Special cases used in the SM are the groups U(1), SU(2) and SU(3) with $g = \{e, g, g_s\}$ and $T^a = \{1, \tau^a/2, \lambda^a/2\}$ in the fundamental representation. In particular, the electric charge of the positron is $q = e > 0$.

1. Classical mechanics

As a starter we review in this chapter those concepts of classical mechanics which are essential for progressing towards quantum theory. We recall first briefly the Lagrangian and Hamiltonian formulation of classical mechanics and their derivation from an action principle. We illustrate also the Green function method using as example the driven harmonic oscillator and recall the action of a relativistic point particle.

1.1. Action principle

Variational principles Fundamental laws of Nature as Newton's axioms or Maxwell equations were discovered in the form of differential equations. Starting from Leibniz and Euler, it was realised that one can re-express differential equations in the form of variational principles: In this approach, the evolution of a physical system is described by the extremum of an appropriately chosen functional. Various versions of such variational principles exist, but they have in common that the functionals used have the dimension of "energy $\times$ time", i.e. the functionals have the same dimension as Planck's constant $\hbar$. A quantity with this dimension is called action S. An advantage of using the action as main tool to describe dynamical systems is that this allows us to implement easily both space-time and internal symmetries. For instance, choosing as action a local function that transforms as a scalar under Lorentz transformations leads automatically to relativistically invariant field equations. Moreover, the action S as a scalar quantity summarises economically the information contained typically in a set of various coupled differential equations.

If the variational principle is formulated as an integral principle, then the functional S will depend on the whole path $q(t)$ described by the system between the considered initial and final time. In the formulation of quantum theory we will pursue, we will look for a direct connection from the classical action $S[q]$ of the path $[q(t)\colon q'(t')]$ to the transition amplitude $\langle q', t'|q, t\rangle$. Thus the use of the action principle will not only allow us an easy discussion of the symmetries of a physical system, but lies also at the heart of the approach to quantum theory we will follow.

1.1.1. Hamilton's principle and Lagrange's equations

A functional $F[f(x)]$ is a map from a certain space of functions $f(x)$ into the real or complex numbers. We will consider mainly functionals from the space of (at least) twice differentiable functions between fixed points a and b. More specifically, Hamilton's principle uses as functional the action S defined by

$$S[q^i] = \int_a^b \mathrm{d}t\, L(q^i, \dot{q}^i, t)\,, \tag{1.1}$$

where L is a function of the $2n$ independent functions q^i and $\dot{q}^i = \mathrm{d}q^i/\mathrm{d}t$ as well as of the parameter t. In classical mechanics, we call L the Lagrange function of the system, q^i are its

n generalised coordinates, $\dot{q}^i$ the corresponding velocities and t is the time. The extrema of this action give those paths $q(t)$ from a to b which are solutions of the equation of motions for the system described by L.

How do we find those paths that extremize the action S? First of all, we have to prescribe which variables are kept constant, which are varied and which constraints the variations have to obey. Depending on the variation principle we choose, these conditions and the functional form of the action will differ. Hamilton's principle corresponds to a smooth variation of the path,

$$q^i(t,\varepsilon) = q^i(t,0) + \varepsilon\eta^i(t)\,,$$

that keeps the endpoints fixed, $\eta^i(a) = \eta^i(b) = 0$, but is otherwise arbitrary. The scale factor ε determines the magnitude of the variation for the one-parameter family of paths $\varepsilon\eta^i(t)$. The notation $S[q^i]$ stresses that we consider the action as a functional only of the coordinates q^i: The velocities $\dot{q}^i$ are not varied independently because ε is time-independent. Since the time t is not varied in Hamilton's principle, varying the path $q^i(t,\varepsilon)$ requires only to calculate the resulting change of the Lagrangian L. Following this prescription, the action has an extremum if

$$0 = \left.\frac{\partial S[q^i(t,\varepsilon)]}{\partial\varepsilon}\right|_{\varepsilon=0} = \int_a^b \mathrm{d}t\,\left(\frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial\varepsilon} + \frac{\partial L}{\partial\dot{q}^i}\frac{\partial\dot{q}^i}{\partial\varepsilon}\right) = \int_a^b \mathrm{d}t\,\left(\frac{\partial L}{\partial q^i}\eta^i + \frac{\partial L}{\partial\dot{q}^i}\dot{\eta}^i\right)\,. \tag{1.2}$$

Here we applied—as always in the following—Einstein's convention to sum over a repeated index pair. Thus e.g. the first term in the bracket equals

$$\frac{\partial L}{\partial q^i}\,\eta^i \equiv \sum_{i=1}^{n}\frac{\partial L}{\partial q^i}\,\eta^i$$

for a system described by n generalised coordinates. We can eliminate $\dot{\eta}^i$ in favour of η^i, integrating the second term by parts, arriving at

$$\left.\frac{\partial S[q^i(t,\varepsilon)]}{\partial\varepsilon}\right|_{\varepsilon=0} = \int_a^b \mathrm{d}t\,\left[\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}^i}\right)\right]\eta^i + \left[\frac{\partial L}{\partial\dot{q}^i}\eta^i\right]_a^b\,. \tag{1.3}$$

The boundary term $[\ldots]_a^b$ vanishes, because we required that the variations η^i are zero at the endpoints a and b. Since the variations η^i are otherwise arbitrary, the sum in the first bracket has to vanish for an extremal curve. The n equations resulting from the condition $\partial S[q^i(t,\varepsilon)]/\partial\varepsilon = 0$ are called the (Euler-) Lagrange equations of the action S,

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^i} = 0\,, \tag{1.4}$$

and give the equations of motion for the system specified by L. In the future, we will use a more concise notation, calling

$$\delta q^i \equiv \lim_{\varepsilon\to 0}\frac{q^i(t,\varepsilon) - q^i(t,0)}{\varepsilon} = \left.\frac{\partial q^i(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} \tag{1.5}$$

the variation of q^i, and similarly for functions and functionals of q^i. Thus we can re-write e.g. Eq. (1.2) in a more evident form as

$$0 = \delta S[q^i] = \int_a^b \mathrm{d}t\,\delta L(q^i,\dot{q}^i,t) = \int_a^b \mathrm{d}t\,\left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial\dot{q}^i}\delta\dot{q}^i\right)\,. \tag{1.6}$$

We close this paragraph with two remarks. First, observe that the Lagrangian L is not uniquely fixed: Adding a total time-derivative, $L \to L' = L + \mathrm{d}f(q,t)/\mathrm{d}t$, does not change the resulting Lagrange equations,

$$S' = S + \int_a^b \mathrm{d}t\, \frac{\mathrm{d}f}{\mathrm{d}t} = S + f(q(b),t_b) - f(q(a),t_a)\,, \tag{1.7}$$

since the last two terms vanish varying the action with the restriction of fixed endpoints a and b. Second, we note that Hamilton's principle is often called the principle of least action. This name is somewhat misleading, since the extremum of the action can be also a maximum or a saddle point.

Lagrange function We illustrate now how one can use symmetries to constrain the possible form of a Lagrangian L. As example, we consider the case of a free non-relativistic particle with mass m subject to the Galilean principle of relativity. More precisely, we use that the homogeneity of space and time forbids that L depends on $\boldsymbol{x}$ and t, while the isotropy of space implies that L depends only on the norm of the velocity vector $\boldsymbol{v}$, but not on its direction. Thus the Lagrange function of a free particle can be only a function of v^2, $L = L(v^2)$.

Let us consider two inertial frames moving with the infinitesimal velocity $\boldsymbol{\varepsilon}$ relative to each other. (Recall that an inertial frame is defined as a coordinate system where a force-free particle moves along a straight line.) Then a Galilean transformation connects the velocities measured in the two frames as $\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{\varepsilon}$. The Galilean principle of relativity requires that the laws of motion have the same form in both frames, and thus the Lagrangians can differ only by a total time-derivative. Expanding the difference δL in ε gives with $\delta v^2 = 2\boldsymbol{v}\boldsymbol{\varepsilon}$

$$\delta L = \frac{\partial L}{\partial v^2}\,\delta v^2 = 2\boldsymbol{v}\boldsymbol{\varepsilon}\frac{\partial L}{\partial v^2}\,. \tag{1.8}$$

The difference δL has to be a total time-derivative. Since $v^i = \dot{x}^i$, the term $\partial L/\partial v^2$ has to be independent of v. Hence, the Lagrangian of a free particle has the form $L = av^2 + b$. The constant b drops out of the equation of motions, and we can set it therefore to zero. To be consistent with usual notation, we call the proportionality constant $m/2$, and the total expression kinetic energy T,

$$L = T = \frac{1}{2}mv^2\,. \tag{1.9}$$

For a system of non-interacting particles, the Lagrange function L is additive, $L = \sum_a \frac{1}{2}m_a v_a^2$. If there are interactions (assumed for simplicity to depend only on the coordinates), then we subtract a function $V(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)$ called potential energy. One confirms readily that this choice for L reproduces Newton's law of motion.

Energy The Lagrangian of a closed system depends, because of the homogeneity of time, not on time. Its total time derivative is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial q^i}\,\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\,\ddot{q}^i\,. \tag{1.10}$$

Using the equations of motion and replacing $\partial L/\partial q^i$ by $(\mathrm{d}/\mathrm{d}t)\partial L/\partial \dot{q}^i$, it follows

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dot{q}^i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial \dot{q}^i}\,\ddot{q}^i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}^i\,\frac{\partial L}{\partial \dot{q}^i}\right)\,. \tag{1.11}$$

Hence the quantity

$$E \equiv \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \tag{1.12}$$

remains constant during the evolution of a closed system. This holds also more generally, e.g. in the presence of static external fields, as long as the Lagrangian is not time-dependent.

We have still to show that E coincides indeed with the usual definition of energy. Using as Lagrange function $L = T(q, \dot{q}) - V(q)$, where the kinetic energy T is quadratic in the velocities, we have

$$\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = \dot{q}^i \frac{\partial T}{\partial \dot{q}^i} = 2T \tag{1.13}$$

and thus $E = 2T - L = T + V$.

Conservation laws In a general way, we can derive the connection between a symmetry of the Lagrangian and a corresponding conservation law as follows: Let us assume that under a change of coordinates $q^i \to q^i + \delta q^i$, the Lagrangian changes by a total time derivative,

$$L \to L + \delta L = L + \frac{\mathrm{d}\delta F}{\mathrm{d}t}\,. \tag{1.14}$$

In this case, the equation of motions are unchanged and the coordinate change $q^i \to q^i + \delta q^i$ is a symmetry of the Lagrangian. But the change $\mathrm{d}\delta F/\mathrm{d}t$ has to equal δL induced by the variation δq^i,

$$\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i - \frac{\mathrm{d}\delta F}{\mathrm{d}t} = 0\,. \tag{1.15}$$

Replacing again $\partial L/\partial q^i$ by $(\mathrm{d}/\mathrm{d}t)\partial L/\partial \dot{q}^i$ and applying the product rule gives as conserved quantity

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i - \delta F\,. \tag{1.16}$$

Thus any continuous symmetry of a Lagrangian system results in a conserved quantity. In particular, energy conservation follows for a system invariant under time-translations with $\delta q^i = \dot{q}^i \delta t$. Other conservation laws are discussed in problem 1.7.

1.1.2. Palatini's principle and Hamilton's equations

Legendre transformation and the Hamilton function In the Lagrange formalism, we describe a system by specifying its generalised coordinates and velocities using the Lagrangian, $L = L(q^i, \dot{q}^i, t)$. An alternative is to use generalised coordinates and their canonically conjugated momenta p_i defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}\,. \tag{1.17}$$

The passage from $\{q^i, \dot{q}^i\}$ to $\{q^i, p_i\}$ is a special case of a Legendre transformation[1]: Starting from the Lagrangian L we define a new function $H(q^i, p_i, t)$ called Hamiltonian or *Hamilton function* via

$$H(q^i, p_i, t) = \frac{\partial L}{\partial \dot{q}^i}\,\dot{q}^i - L(q^i, \dot{q}^i, t) = p_i \dot{q}^i - L(q^i, \dot{q}^i, t)\,. \tag{1.18}$$

[1] The concept of a Legendre transformation should be familiar from thermodynamics, where it used to change between extensive variables (e.g. entropy S) and their conjugate intensive variables (e.g. temperature T).

Here we assume that we can invert the definition (1.17) and are thus able to substitute velocities $\dot{q}^i$ by momenta p_i in the Lagrangian L.

The physical meaning of the Hamiltonian H follows immediately comparing its defining equation with the one for the energy E. Thus the numerical value of the Hamiltonian equals the energy of a dynamical system; we insist however that H is expressed as function of coordinates and their conjugated momenta. A coordinate q_i that does not appear explicitly in L is called cyclic. The Lagrange equations imply then $\partial L/\partial \dot{q}_i = \text{const.}$, so that the corresponding canonically conjugated momentum $p_i = \partial L/\partial \dot{q}^i$ is conserved.

Palatini's formalism and Hamilton's equations Previously, we considered the action S as a functional only of q^i. Then the variation of the velocities $\dot{q}^i$ is not independent and we arrive at n second order differential equations for the coordinates q^i. An alternative approach is to allow independent variations of the coordinates q^i and of the velocities $\dot{q}^i$. We trade the latter against the momenta $p_i = \partial L/\partial \dot{q}^i$ and rewrite the action as

$$S[q^i, p_i] = \int_a^b \mathrm{d}t \, \left[p_i \dot{q}^i - H(q^i, p_i, t) \right] . \tag{1.19}$$

The independent variation of coordinates q^i and momenta p_i gives

$$\delta S[q^i, p_i] = \int_a^b \mathrm{d}t \, \left[p_i \delta \dot{q}^i + \dot{q}^i \delta p_i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right] . \tag{1.20}$$

The first term can be partially integrated, and the resulting boundary terms vanishes by assumption. Collecting then the δq^i and δp_i terms and requiring that the variation is zero, we obtain

$$0 = \delta S[q^i, p_i] = \int_a^b \mathrm{d}t \, \left[- \left(\dot{p}_i + \frac{\partial H}{\partial q^i} \right) \delta q^i + \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) \delta p_i \right] . \tag{1.21}$$

As the variations δq^i and δp_i are independent, their coefficients in the round brackets have to vanish separately. Thus we obtain in this formalism directly Hamilton's equations,

$$\dot{q}^i = \frac{\partial H}{\partial p_i} , \qquad \text{and} \qquad \dot{p}_i = - \frac{\partial H}{\partial q^i} . \tag{1.22}$$

This approach can be applied also to field theories, treating e.g. the vector potential A^μ and the field-strength $F^{\mu\nu}$ in electrodynamics as independent variables.

Consider now an observable $O = O(q^i, p_i, t)$. Its time-dependence is given by

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{\partial O}{\partial q^i} \dot{q}^i + \frac{\partial O}{\partial p_i} \dot{p}_i + \frac{\partial O}{\partial t} = \frac{\partial O}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial O}{\partial p_i} \frac{\partial H}{\partial q^i} + \frac{\partial O}{\partial t} , \tag{1.23}$$

where we used Hamilton's equations. If we define the Poisson brackets $\{A, B\}$ between two observables A and B as

$$\{A, B\} = \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i} , \tag{1.24}$$

then we can rewrite Eq. (1.23) as

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \{O, H\} + \frac{\partial O}{\partial t} . \tag{1.25}$$

This equations gives us a formal correspondence between classical and quantum mechanics: The time-evolution of an operator O in the Heisenberg picture is given by the same equation as in classical mechanics, if the Poisson bracket is changed against a commutator. Since the Poisson bracket is antisymmetric, we find

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}\,. \tag{1.26}$$

Hence the Hamiltonian H is a conserved quantity, if and only if H is time-independent.

1.2. Green functions and the response method

We can test the internal properties of a physical system, if we impose an external force $J(t)$ on it and compare its measured to its calculated response. If the system is described by linear differential equations, then the superposition principle is valid: We can reconstruct the solution $x(t)$ for an arbitrary applied external force $J(t)$, if we know the response to a normalised delta-function like kick $J(t) = \delta(t - t')$. Mathematically, this corresponds to the knowledge of the Green function $G(t - t')$ for the differential equation $D(t)x(t) = J(t)$ describing the system. Even if the system is described by a non-linear differential equation, we can often use a linear approximation in case of a sufficiently small external force $J(t)$. Therefore the Green function method is extremely useful and we will apply it extensively discussing quantum field theories.

We illustrate this method with the example of the harmonic oscillator which is the prototype for a quadratic, and thus exactly solvable, action. In classical physics, causality implies that the knowledge of the external force $J(t')$ at times $t' < t$ is sufficient to determine the solution $x(t)$ at time t. We define two Green functions $\widetilde{G}$ and G_R by

$$x(t) = \int_{-\infty}^{t} \mathrm{d}t'\, \widetilde{G}(t - t')J(t') = \int_{-\infty}^{\infty} \mathrm{d}t' G_R(t - t')J(t')\,, \tag{1.27}$$

where the retarded Green function G_R satisfies $G_R(t - t') = \widetilde{G}(t - t')\vartheta(t - t')$. The definition (1.27) is motivated by the trivial relation $J(t) = \int \mathrm{d}t'\, \delta(t - t')J(t')$: An arbitrary force $J(t)$ can be seen as a superposition of delta functions $\delta(t - t')$ with weight $J(t')$. If the Green function $G_R(t - t')$ determines the response of the system to a delta function-like force, then we should obtain the solution $x(t)$ integrating $G_R(t - t')$ with the weight $J(t')$.

We convert the equation of motion $m\ddot{x} + m\omega^2 x = J$ of a forced harmonic oscillator into the form $D(t)x(t) = J(t)$ by writing

$$D(t)x(t) \equiv m\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right) x(t) = J(t)\,. \tag{1.28}$$

Inserting (1.27) into (1.28) gives

$$\int_{-\infty}^{\infty} \mathrm{d}t'\, D(t)G_R(t - t')J(t') = J(t)\,. \tag{1.29}$$

For an arbitrary external force $J(t)$, this relation can be only valid if

$$D(t)G_R(t - t') = \delta(t - t')\,. \tag{1.30}$$

Thus a Green function $G(t-t')$ is the inverse of its defining differential operator $D(t)$. As we will see, Eq. (1.30) does not specify uniquely the Green function, and thus we will omit the index "R" for the moment. Performing a Fourier transformation,

$$G(t-t') = \int \frac{\mathrm{d}\Omega}{2\pi}\, G(\Omega)\mathrm{e}^{-\mathrm{i}\Omega(t-t')} \quad \text{and} \quad \delta(t-t') = \int \frac{\mathrm{d}\Omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\Omega(t-t')}\,, \tag{1.31}$$

the differential equation (1.30) is transformed into an algebraic one,

$$\int \frac{\mathrm{d}\Omega}{2\pi}\, G(\Omega)D(t)\mathrm{e}^{-\mathrm{i}\Omega(t-t')} = \int \frac{\mathrm{d}\Omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\Omega(t-t')}\,. \tag{1.32}$$

The action of $D(t)$ on the plane-waves $\mathrm{e}^{-\mathrm{i}\Omega(t-t')}$ can be evaluated now easily. Comparing then the coefficients on both sides of this equation results in

$$G(\Omega) = \frac{1}{m}\frac{1}{\omega^2-\Omega^2}\,. \tag{1.33}$$

For the back-transformation with $\tau = t-t'$,

$$G(\tau) = \int \frac{\mathrm{d}\Omega}{2\pi m}\, \frac{\mathrm{e}^{-\mathrm{i}\Omega\tau}}{\omega^2-\Omega^2}\,, \tag{1.34}$$

we have to specify how the poles at $\Omega^2 = \omega^2$ are avoided. It is this choice by which we select the appropriate Green function. In classical physics, we implement causality ("cause always precedes its effect") selecting the retarded Green function.

We will use Cauchy's residue theorem, $\oint \mathrm{d}z\, f(z) = 2\pi\mathrm{i}\sum \mathrm{res}_{z_0} f(z)$, to calculate the integral. Its application requires to close the integration contour adding a path which gives a vanishing contribution to the integral. This is achieved, when the integrand $G(\Omega)\mathrm{e}^{-\mathrm{i}\Omega\tau}$ vanishes fast enough along the added path. Thus we have to choose for positive τ the contour $\mathcal{C}_-$ in the lower plane, $\mathrm{e}^{-\mathrm{i}\Omega\tau} = \mathrm{e}^{-|\Im(\Omega)|\tau} \to 0$ for $\Im(\Omega) \to -\infty$, while we have to close the contour in the upper plane for negative τ. If we want to obtain the retarded Green function $G_R(\tau)$ which vanishes for $\tau < 0$, we have to shift therefore the poles $\Omega_{1/2} = \pm\omega$ into the lower plane as shown in Fig. 1.1 by adding a small negative imaginary part, $\Omega_{1/2} \to \Omega_{1/2} = \pm\omega - \mathrm{i}\varepsilon$, or

$$G_R(\tau) = -\frac{1}{2\pi m}\int \mathrm{d}\Omega\, \frac{\mathrm{e}^{-\mathrm{i}\Omega\tau}}{(\Omega-\omega+\mathrm{i}\varepsilon)(\Omega+\omega+\mathrm{i}\varepsilon)}\,. \tag{1.35}$$

The residue $\mathrm{res}_{z_0} f(z)$ of a function f with a single pole at z_0 is given by

$$\mathrm{res}_{z_0}\, f(z) = \lim_{z\to z_0}(z-z_0)f(z)\,. \tag{1.36}$$

Thus we pick up at $\Omega_1 = -\omega - \mathrm{i}\varepsilon$ the contribution $2\pi\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\omega\tau}/(-2\omega)$, while we obtain $2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\omega\tau}/(2\omega)$ from $\Omega_2 = \omega - \mathrm{i}\varepsilon$. Combining both contributions, we arrive at

$$G_R(\tau) = \frac{\mathrm{i}}{2m\omega}\left[\mathrm{e}^{-\mathrm{i}\omega\tau} - \mathrm{e}^{\mathrm{i}\omega\tau}\right]\vartheta(\tau) = \frac{1}{m}\,\frac{\sin(\omega\tau)}{\omega}\,\vartheta(\tau) \tag{1.37}$$

as result for the retarded Green function of the forced harmonic oscillator.

We can now obtain a particular solution solving (1.27). For instance, choosing $J(t) = \delta(t-t')$, results in

$$x(t) = \frac{1}{m}\,\frac{\sin(\omega\tau)}{\omega}\,\vartheta(\tau)\,. \tag{1.38}$$

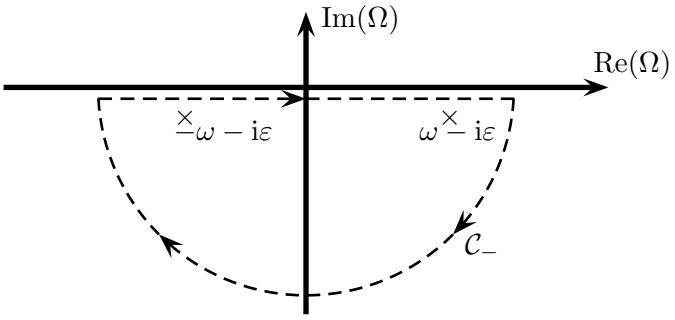


Figure 1.1.: Poles and contour in the complex Ω plane used for the integration of the retarded Green function.

Thus the oscillator was at rest for $t < 0$, got a kick at $t = 0$, and oscillates according $x(t)$ afterwards. Note the following two points: First, the fact that the kick proceeds the movement is the result of our choice of the retarded (or causal) Green function. Second, the particular solution (1.38) for a oscillator initially at rest can be generalised by adding the solution to the homogeneous equation $\ddot{x} + \omega^2 x = 0$.

1.3. Relativistic particle

In special relativity, we replace the Galilean transformations as symmetry group of space and time by Lorentz transformations. The latter are all those coordinate transformations $x^\mu \to \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu$ that keep the squared distance

$$s_{12}^2 \equiv (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 \tag{1.39}$$

between two space-time events x_1^μ and x_2^μ invariant. The distance of two infinitesimally close space-time events is called the line-element $\mathrm{d}s$ of the space-time. In Minkowski space, it is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{1.40}$$

using a Cartesian inertial frame. We can interpret the line-element $\mathrm{d}s^2$ as a scalar product, if we introduce the metric tensor $\eta_{\mu\nu}$ with elements

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.41}$$

and a scalar product of two vectors as

$$a \cdot b \equiv a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu \,. \tag{1.42}$$

In Minkowski space, we call a four-vector any four-tupel V^μ that transforms as $\tilde{V}^\mu = \Lambda^\mu{}_\nu V^\nu$. By convention, we associate three-vectors with the spatial part of vectors with upper indices, e.g. we set $x^\mu = \{t, x, y, z\}$ or $A^\mu = \{\phi, \boldsymbol{A}\}$. Lowering then the index by contraction with the metric tensor result in a minus sign of the spatial components of a four-vector, $x_\mu = \eta_{\mu\nu} x^\mu =$

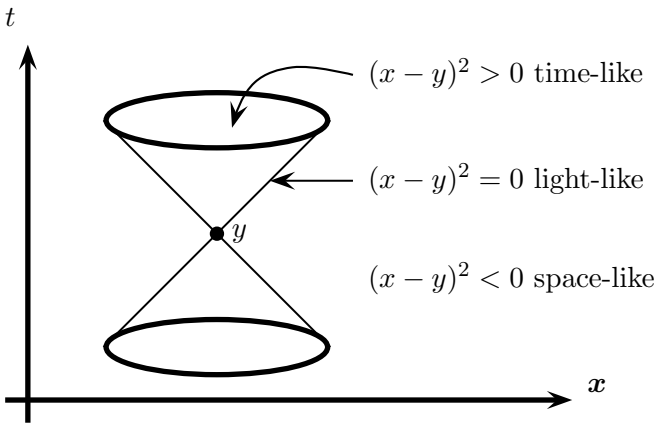


Figure 1.2.: Light-cone at the point y generated by light-like vectors. Contained in the light-cone are the time-like vectors, outside the space-like ones.

$\{t, -x, -y, -z\}$ or $A_\mu = \{\phi, -\boldsymbol{A}\}$. Summing over an index pair, typically one index occurs in an upper and one in a lower position. Note that in the denominator, an upper index counts as a lower index and vice versa, cf. e.g. with Eqs. (1.18) and (1.17). Additionally to four-vectors, we will meet tensors $T^{\mu_1\cdots\mu_n}$ of rank n which transform as $\tilde{T}^{\mu_1\cdots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n}T^{\nu_1\cdots\nu_n}$.

Since the metric $\eta_{\mu\nu}$ is indefinite, the norm of a vector a^μ can be

$$a_\mu a^\mu > 0, \quad \text{time-like}, \tag{1.43}$$
$$a_\mu a^\mu = 0, \quad \text{light-like or null-vector}, \tag{1.44}$$
$$a_\mu a^\mu < 0, \quad \text{space-like}. \tag{1.45}$$

The cone of all light-like vectors starting from a point P is called *light-cone*, cf. Fig. 1.2. The time-like region inside the light-cone consists of two parts, past and future. Only events inside the past light-cone can influence the physics at point P, while P can influence only its future light-cone. The proper-time τ is the time displayed by a clock moving with the observer. With our conventions—negative signature of the metric and $c = 1$—the proper-time elapsed between two space-time events equals the integrated line-element between them,

$$\tau_{12} = \int_1^2 \mathrm{d}s = \int_1^2 [\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu]^{1/2} = \int_1^2 \mathrm{d}t[1-v^2]^{1/2} < t_2 - t_1\,. \tag{1.46}$$

The last part of this equation, where we introduced the three-velocity $v^i = \mathrm{d}x^i/\mathrm{d}t$ of the clock, shows explicitly the relativistic effect of time dilation, as well as the connection between coordinate time t and the proper-time τ of a moving clock, $\mathrm{d}\tau = (1-v^2)^{1/2}\mathrm{d}t \equiv \mathrm{d}t/\gamma$. The line describing the position of an observer is called *world-line*. Parametrising the worldline by the parameter σ, $x = x(\sigma)$, the proper-time is given by

$$\tau = \int \mathrm{d}\sigma \left[\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right]^{1/2}. \tag{1.47}$$

Note that τ is invariant under a reparameterisation $\tilde{\sigma} = f(\sigma)$.

The only invariant differential we have at our disposal to form an action for a free point-like particle is the line-element, or equivalently the proper-time,

$$S_0 = \alpha \int_a^b \mathrm{d}s = \alpha \int_a^b \mathrm{d}\sigma \, \frac{\mathrm{d}s}{\mathrm{d}\sigma} \tag{1.48}$$

with $L = \alpha \mathrm{d}s/\mathrm{d}\sigma = \alpha \mathrm{d}\tau/\mathrm{d}\sigma$. We check now if this choice which implies the Lagrangian

$$L = \alpha \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = \alpha \left[\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \right]^{1/2} \tag{1.49}$$

for a free particle is sensible: The action has the correct non-relativistic limit,

$$S_0 = \alpha \int_a^b \mathrm{d}s = \alpha \int_a^b \mathrm{d}t \, \sqrt{1 - v^2} = \int_a^b \mathrm{d}t \, \left(-m + \frac{1}{2} m v^2 + \mathcal{O}(v^4) \right) , \tag{1.50}$$

if we set $\alpha = -m$. The mass m corresponds to a potential energy in the non-relativistic limit and has therefore a negative sign in the Lagrangian. The time t enters the relativistic Lagrangian in a Lorentz invariant way as one of the dynamical variables, $x^\mu = (t, \boldsymbol{x})$, while σ assumes now t's purpose to parametrise the trajectory, $x^\mu(\sigma)$. Since a moving clock goes slower than a clock at rest, solutions of this Lagrangian maximise the action.

Example 1.1: Relativistic dispersion relation: We extend the non-relativistic definition of the momentum, $p_i = \partial L/\partial \dot{x}^i$, to four dimensions setting $p_\alpha = -\partial L/\partial \dot{x}^\alpha$. Note the minus sign that reflects the minus in the spatial components of a covariant vector, $p_\alpha = (E, -\boldsymbol{p})$. Then

$$p_\alpha = -\frac{\partial L}{\partial \dot{x}^\alpha} = m \frac{\mathrm{d}x_\alpha/\mathrm{d}\sigma}{\mathrm{d}\tau/\mathrm{d}\sigma} = m \frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} \equiv m u_\alpha \, . \tag{1.51}$$

In the last step, we defined the four-velocity $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$. Using $\mathrm{d}t = \gamma \mathrm{d}\tau$, it follows $u_\alpha u^\alpha = 1$ and $p_\alpha p^\alpha = m^2$. The last relation expresses the relativistic dispersion relation $E^2 = m^2 + \boldsymbol{p}^2$.

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \frac{\partial L}{\partial(\mathrm{d}x^\alpha/\mathrm{d}\sigma)} = \frac{\partial L}{\partial x^\alpha} \, . \tag{1.52}$$

Consider e.g. the x^1 component, then

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \frac{\partial L}{\partial(\mathrm{d}x^1/\mathrm{d}\sigma)} = \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{1}{L} \frac{\mathrm{d}x^1}{\mathrm{d}\sigma} \right) = 0 \, . \tag{1.53}$$

Since $L = -m\mathrm{d}\tau/\mathrm{d}\sigma$, Newton's law follows for the x^1 coordinate after multiplication with $\mathrm{d}\sigma/\mathrm{d}\tau$,

$$\frac{d^2 x^1}{\mathrm{d}\tau^2} = 0 \, , \tag{1.54}$$

and similar for the other coordinates.

An equivalent, but often more convenient form for the Lagrangian of a free particle is

$$L = -m \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \, , \tag{1.55}$$

where we set $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. If there are no interactions (except gravity), we can neglect the mass m of the particle and one often sets $m \to -1$.

Next we want to add an interaction term $S_{\rm em}$ between a particle with charge q and an electromagnetic field. The simplest possible action is to integrate the potential A_μ along the world-line $x^\mu(\sigma)$ of the particle,

$$S_{\rm em} = -q \int \mathrm{d}x^\mu \, A_\mu(x) = -q \int \mathrm{d}\sigma \, \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \, A_\mu(x) \,. \tag{1.56}$$

In the last expression, we can view $q\dot{x}^\mu$ as the current j^μ induced by the particle and thus the interaction has the form $L_{\rm em} = -j^\mu A_\mu$. Any candidate for $S_{\rm em}$ should be invariant under a gauge transformation of the potential,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \,. \tag{1.57}$$

This is the case, since the induced change in the action,

$$\delta_\Lambda S_{\rm em} = -q \int_1^2 \mathrm{d}\sigma \, \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \, \frac{\partial \Lambda(x)}{\partial x^\mu} = -q \int_1^2 \mathrm{d}\Lambda = -q[\Lambda(2) - \Lambda(1)] \,, \tag{1.58}$$

depends only on the endpoints. Thus $\delta_\Lambda S_{\rm em}$ vanishes keeping the endpoints fixed. Assuming that the Lagrangian is additive,

$$L = L_0 + L_{\rm em} = -m \left[\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \right]^{1/2} - q \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \, A_\mu(x) \tag{1.59}$$

the Lagrange equations give now

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \left[\frac{m \mathrm{d}x_\alpha/\mathrm{d}\sigma}{[\eta_{\mu\nu} \mathrm{d}x^\mu/\mathrm{d}\sigma \, \mathrm{d}x^\nu/\mathrm{d}\sigma]^{1/2}} + q A_\alpha \right] = q \, \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, \frac{\partial A_\lambda(x)}{\partial x^\alpha} \,. \tag{1.60}$$

Performing then the differentiation of $A(x(\sigma))$ with respect to σ and moving it to the RHS, we find

$$m \frac{\mathrm{d}}{\mathrm{d}\sigma} \left[\frac{\mathrm{d}x_\alpha/\mathrm{d}\sigma}{\mathrm{d}\tau/\mathrm{d}\sigma} \right] = q \left(\frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, \frac{\partial A_\lambda}{\partial x^\alpha} - \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, \frac{\partial A_\alpha}{\partial x^\lambda} \right) = q \, \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, F_{\alpha\lambda} \,, \tag{1.61}$$

where we introduced the electromagnetic field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Choosing $\sigma = \tau$ we obtain the covariant version of the Lorentz equation,

$$m \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = q \, F^\alpha{}_\lambda \, u^\lambda \,. \tag{1.62}$$

You should work through problem 1.8, if this equation and the covariant formulation of the Maxwell equations are not familiar to you.

Summary

The Lagrange and Hamilton function are connected by a Legendre transformation, $L(q^i, \dot{q}^i, t) = p_i \dot{q}^i - H(q^i, p_i, t)$. Lagrange's and Hamilton's equations follow extremizing the action $S[q^i] = \int_a^b \mathrm{d}t \, L(q^i, \dot{q}^i, t)$ and $S[q^i, p_i] = \int_a^b \mathrm{d}t \, \left[p_i \dot{q}^i - H(q^i, p_i, t) \right]$, respectively, keeping the endpoints a and b in coordinate space fixed. Knowing the Green function $G(t - t')$ of a

linear system, we can find the solution $x(t)$ for an arbitrary external force $J(t)$, by integrating $G(t-t')$ with the weight $J(t)$.

Further reading

The series of Landau and Lifshitz on theoretical physics is a timeless resource for everybody studying and working in this field; its volume 1 presents a succinct treatment of classical mechanics.

Problems

1.1 Units. ♣
a.) The four fundamental constants $\hbar$ (Planck's constant), c (velocity of light), G_N (gravitational constant) and k (Boltzmann constant) can be combined to obtain the dimension of a length, time, mass, energy and temperature. Find the relations and calculate the numerical values of two of them. What is the physical meaning of these "Planck units"? b.) Find the connection between a cross section σ expressed in units of cm^2, mbarn, and GeV^{-2}.

1.2 $\mathrm{d}\delta = \delta\mathrm{d}$.
Use the definition (1.5) to show that variation and differentiation commute, i.e. that "$\mathrm{d}\delta = \delta\mathrm{d}$".

1.3 Higher derivatives.
Find the Lagrange equations for a Lagrange function containing higher derivatives, $L = L(q^i, \dot{q}^i, \ddot{q}^i, \ldots)$.

1.4 Oscillator with friction.
Consider an one-dimensional system described by the Lagrangian $L = \exp(2\alpha t)L_0$ and $L_0 = \frac{1}{2}m\dot{q}^2 - V(q)$. a.) Show that the equation of motion corresponds to an oscillator with friction term. b). Derive the energy lost per time $\mathrm{d}E/\mathrm{d}t$ of the oscillator, with $E = \frac{1}{2}m\dot{q}^2 + V(q)$. c.) Show that the result in b.) agrees with the one obtained from the Lagrange equations of the first kind, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i$, where the generalised forces Q^i perform the work $\delta A = Q_i \delta q^i$.

1.5 Classical driven oscillator.
Consider an harmonic oscillator satisfying $\ddot{q}(t) - \Omega^2 q(t) = 0$ for $0 < t < T$ and $\ddot{q}(t) + \omega^2 q(t) = 0$ otherwise, with ω and Ω as real constants. a.) Show that for $q(t) = A_1 \sin(\omega t)$ for $t < 0$ and $\Omega T \gg 1$, the solution $q(t) = A_2 \sin(\omega_0 t + \alpha)$ with $\alpha = \text{const.}$ satisfies $A_2 \approx \frac{1}{2}(1+\omega^2/\Omega^2)^{1/2}\exp(\Omega T)$. b.) If the oscillator was in the ground-state at $t < 0$, how many quanta are created?

1.6 Functional derivative. ♣
We define the derivative of a functional $F[\phi]$ by

$$\int \mathrm{d}x\, \eta(x) \frac{\delta F[\phi]}{\delta \phi(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ F[\phi + \varepsilon\eta] - F[\phi] \right\} .$$

a.) Find the functional derivative of $F[\phi] = \int \mathrm{d}x f(x)\phi(x)$ and show thereby that $\delta\phi(x)/\delta\phi(x') = \delta(x - x')$. b.) Rederive the Lagrange equations.

1.7 Conservation laws.
Discuss the symmetries of the Galilean transformations and the resulting conservation laws, following the example of time translation invariance and energy conservation.

1.8 Electrodynamics.
Compare Eq. (1.62) to the three-dimensional version of the Lorentz force and derive thereby the elements of the field-strength tensor $F_{\mu\nu}$. Find the Lorentz invariants that can be formed out of $F_{\mu\nu}$ and express them through $\boldsymbol{E}$ and $\boldsymbol{B}$. What is the meaning of the zero component of the Lorentz force?

1.9 Relativity of simultaneity. ♣
Draw a space-time diagram (in $d = 2$) for two inertial frames connected by a boost with velocity β: What are the angles between the axes t and t', x and x'? Draw lines of constant t and t' and convince yourself that the time order of two space-like events is not invariant.

1.10 Wave equation for a string.
Consider a string of length L, mass density ρ and tension κ in one spatial dimension. Denoting its deviation from its equilibrium position x_0 with

$\phi(x,t) \equiv x(t) - x_0$, write down its kinetic and potential energy (density) and the corresponding action. Derive its equation of motion. [Note: $\phi(x,t)$ depends on t and x, and the Lagrange equation for $L(\phi, \partial_t\phi, \partial_x\phi)$ will contain $\mathrm{d}/\mathrm{d}t$ and $\mathrm{d}/\mathrm{d}x$ terms.]

2. Quantum mechanics

The main purpose of this chapter is to introduce Feynman's path integral as an alternative to the standard operator approach to quantum mechanics. Most of our discussion of quantum fields will be based on this approach, and thus becoming familiar with this technique using the simpler case of quantum mechanics is of central importance. Instead of employing the path integral directly, we will use as basic tool the ground-state persistence amplitude $\langle 0,\infty|0,-\infty\rangle_J$: This quantity is the probability amplitude that a system under the influence of an external force J stays in its ground-state. Since we can apply an arbitrary force J, the amplitude $\langle 0,\infty|0,-\infty\rangle_J$ contains all information about the system. Moreover, it serves as a convenient tool to calculate Green functions which will become our main target studying quantum field theories.

2.1. Reminder of the operator approach

A classical system described by a Hamiltonian $H(q^i, p_i, t)$ can be quantised promoting q^i and p_i to operators[1] $\hat{q}^i$ and $\hat{p}_i$ which satisfy the canonical commutation relations $\left[\hat{q}^i, \hat{p}_j\right] = \mathrm{i}\delta^i_j$. The latter are the formal expression of Heisenberg's uncertainty relation. Apart from ordering ambiguities, the Hamilton operator $H(\hat{q}^i, \hat{p}_i, t)$ can be directly read from the Hamiltonian $H(q^i, p_i, t)$. The basic features of any quantum theory can be synthesised into a few principles.

General principles A physical system in a pure state is fully described by a probability amplitude

$$\psi(a,t) = \langle a|\psi(t)\rangle \in \mathbb{C}\,, \tag{2.1}$$

where $\{a\}$ is a set of quantum numbers specifying the system and the states $|\psi(t)\rangle$ form a complex Hilbert space. The probability to find the specific values a_* in a measurement is given by $p(a_*) = |\psi(a_*,t)|^2$. The possible values a_* are the eigenvalues of hermitian operators $\hat{A}$ whose eigenvectors $|a\rangle$ form an orthogonal, complete basis. In Dirac's bra-ket notation, we can express these statements by

$$\hat{A}|a\rangle = a|a\rangle\,, \qquad \langle a|a'\rangle = \delta\left(a-a'\right), \qquad \int \mathrm{d}a\,|a\rangle\langle a| = 1\,, \tag{2.2}$$

In general, operators do not commute. Their commutation relations can be obtained by the replacement $\{A,B\} \to \mathrm{i}[\hat{A},\hat{B}]$ in the definition (1.24) of the Poisson brackets.

The state of a particle moving in one dimension in a potential $V(q)$ can be described either by the eigenstates of the position operator $\hat{q}$ or of the momentum operator $\hat{p}$. Both form a complete, orthonormal basis, and they are connected by a Fourier transformation which we choose to be asymmetric. This asymmetry is reflected in the completeness relation of the states,

$$\int \mathrm{d}q\,|q\rangle\,\langle q| = 1 \quad \text{and} \quad \int \frac{\mathrm{d}p}{2\pi}\,|p\rangle\,\langle p| = 1\,. \tag{2.3}$$

[1] When there is the danger of an ambiguity, operators will be written with a "hat"; otherwise we drop it.

Time-evolution Since the states $|\psi(t)\rangle$ form a complex Hilbert space, the superposition principle is valid: If ψ_1 and ψ_2 are possible states of the system, then also

$$\psi(t) = c_1\psi_1(t) + c_2\psi_2(t)\,, \qquad c_i \in \mathbb{C}\,. \tag{2.4}$$

In quantum mechanics, a stronger version of this principle holds which states that if $\psi_1(t)$ and $\psi_2(t)$ describe the possible time-evolution of the system, then so does also the superposed state $\psi(t)$. This implies that the time-evolution is described by a linear, homogeneous differential equation. Choosing it as first-order in time, we can write the evolution equation as

$$\mathrm{i}\partial_t|\psi(t)\rangle = D|\psi(t)\rangle\,, \tag{2.5}$$

where the differential operator D on the RHS has to be still determined.

We call the operator describing the evolution of a state from $\psi(t)$ to $\psi(t')$ the time-evolution operator $U(t',t)$. This operator is unitary, $U^{-1} = U^\dagger$, in order to conserve probability and forms a group, $U(t_3,t_1) = U(t_3,t_2)U(t_2,t_1)$ with $U(t,t) = 1$. For an infinitesimal time step δt,

$$|\psi(t+\delta t)\rangle = U(t+\delta t,t)\,|\psi(t)\rangle\ , \tag{2.6}$$

we can set with $U(t,t) = 1$

$$U(t+\delta t,t) = 1 - \mathrm{i}H\delta t\,. \tag{2.7}$$

Here we introduced the generator of infinitesimal time-translations H. The analogy to classical mechanics suggests that H is the operator version of the classical Hamilton function $H(q,p)$. Inserting Eq. (2.7) into (2.6) results in

$$\frac{|\psi(t+\delta t)\rangle - |\psi(t)\rangle}{\delta t} = -\mathrm{i}H\,|\psi(t)\rangle\ . \tag{2.8}$$

Comparing then Eqs. (2.5) and (2.8) reveals that H is equal to the operator D on the RHS of Eq. (2.5). We call a time-evolution equation of this type for an arbitrary Hamiltonian H Schrödinger equation.

Next we want to determine the connection between H and U. Plugging $\psi(t) = U(t,0)\psi(0)$ in the Schrödinger equation gives

$$\left[\mathrm{i}\frac{\partial U(t,0)}{\partial t} - HU(t,0)\right]\psi(0) = 0\,. \tag{2.9}$$

Since this equation is valid for an arbitrary state $\psi(0)$, we can rewrite it as an operator equation,

$$\mathrm{i}\partial_{t'}U(t',t) = HU(t',t)\,. \tag{2.10}$$

Integrating it, we find as formal solution

$$U(t',t) = 1 - \mathrm{i}\int_t^{t'} \mathrm{d}t''H(t'')U(t'',t) \tag{2.11}$$

or, if H is time-independent,

$$U(t,t') = \exp(-\mathrm{i}H(t-t'))\,. \tag{2.12}$$

Up-to now, we have considered the Schrödinger picture where operators are constant and the time-evolution is given by the change in the state vectors $\psi(t)$. In the Heisenberg picture,

the time evolution is driven completely by the one of the operators. States and operators in the two pictures are connected by

$$O_S(t) = U(t,t_0) O_H(t) U^\dagger(t,t_0)\,, \tag{2.13}$$
$$|\psi_S(t)\rangle = U(t,t_0)\,|\psi_H(t)\rangle\ , \tag{2.14}$$

if they agree at the time t_0.

Propagator We insert the solution of U for a time-independent H into $|\psi(t')\rangle = U(t',t)|\psi(t)\rangle$ and multiply from the left with $\langle q'|$,

$$\psi(q',t') = \langle q'|\psi(t')\rangle = \langle q'|\exp[-\mathrm{i}H(t'-t)]|\psi(t)\rangle\,. \tag{2.15}$$

Then we insert $1 = \int \mathrm{d}^3q|q\rangle\langle q|$,

$$\psi(q',t') = \int \mathrm{d}^3q\,\langle q'|\exp[-\mathrm{i}H(t'-t)]|q\rangle\langle q|\psi(t)\rangle = \int \mathrm{d}^3q\,K(q',t';q,t)\psi(q,t)\,. \tag{2.16}$$

In the last step we introduced the propagator or Green function K in its coordinate representation,

$$K(q',t';q,t) = \langle q'|\exp[-\mathrm{i}H(t'-t)]|q\rangle\,. \tag{2.17}$$

The Green function K equals the probability amplitude for the propagation between two space-time points; $K(q',t';q,t)$ is therefore also called more specifically two-point Green function. We can express the propagator K by the solutions of the Schrödinger equation, $\psi_n(q,t) = \langle q|n(t)\rangle = \langle q|n\rangle\exp(-\mathrm{i}E_nt)$ as

$$\begin{aligned} K(q',t';q,t) &= \sum_{n,n'}\langle q'|n\rangle \underbrace{\langle n|\exp(-\mathrm{i}H(t'-t))|n'\rangle}_{\delta_{n,n'}\exp(-\mathrm{i}E_n(t'-t))}\langle n'|q\rangle \\ &= \sum_n \psi_n(q')\psi_n^*(q)\exp(-\mathrm{i}E_n(t'-t))\,, \end{aligned} \tag{2.18}$$

where n represents the complete set of quantum numbers specifying the energy eigenvalues of the system. Note that this result is very general and holds for any time-independent Hamiltonian[2].

Let us compute the propagator of a free particle in one dimension, described by the Hamiltonian $H = p^2/2m$. We write with $\tau = t'-t$

$$\begin{aligned} K(q',t';q,t) &= \left\langle q'\right|\mathrm{e}^{-\mathrm{i}H\tau}\left|q\right\rangle = \left\langle q'\right|\mathrm{e}^{-\mathrm{i}\tau\hat{p}^2/2m}\int\frac{\mathrm{d}p}{2\pi}\left|p\right\rangle\left\langle p\right|\left.q\right\rangle \\ &= \int\frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{-\mathrm{i}\tau p^2/2m}\left\langle q'\right|\left.p\right\rangle\left\langle p\right|\left.q\right\rangle = \int\frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{-\mathrm{i}\tau p^2/2m+\mathrm{i}(q'-q)p}\,, \end{aligned} \tag{2.19}$$

where we used $\langle q'|\,p\rangle = \exp(\mathrm{i}q'p)$ in the last step. The integral is Gaussian, if we add an infinitesimal factor $\exp(-\varepsilon p^2)$ to the integrand in order to ensure the convergence of the integral. Thus the physical value of the energy $E = p^2/(2m)$ seen as a complex variable is

[2]We will see that the relativistic propagator has the same form, including into the summation its negative energy eigenvalues and its spin.

approached from the negative imaginary plane, $E \to E - \mathrm{i}\varepsilon$. Taking afterwards the limit $\varepsilon \to 0$, we obtain

$$K(q',t';q,t) = \left(\frac{m}{2\pi \mathrm{i}\tau}\right)^{1/2} \mathrm{e}^{\mathrm{i}m(q'-q)^2/2\tau} \,. \tag{2.20}$$

Knowing the propagator, we can calculate the solution $\psi(t')$ at any time t' for a given initial state $\psi(t)$ via Eq. (2.16).

Example 2.1: Calculate the three Gaussian integrals $A = \int \mathrm{d}x \exp(-x^2/2)$, $B = \int \mathrm{d}x \exp(-ax^2/2 + bx)$, and $C = \int \mathrm{d}x \cdots \mathrm{d}x_n \exp(-x^T Ax/2 + J^T x)$ for a symmetric $n \times n$ matrix A.
a.) We square the integral and calculate then A^2 introducing polar coordinates, $r^2 = x^2 + y^2$,

$$A^2 = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \exp(-(x^2+y^2)/2) = 2\pi \int_0^{\infty} \mathrm{d}r\, r\mathrm{e}^{-r^2/2} = 2\pi \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t} = 2\pi \,,$$

where we substituted $t = r^2/2$. Thus the result for the basic Gaussian integral is $A = \sqrt{2\pi}$. All other solvable variants of Gaussian integrals can be reduced to this result.
b.) We complete the square in the exponent,

$$-\frac{a}{2}\left(x^2 - \frac{2b}{a}x\right) = -\frac{a}{2}\left(x - \frac{b}{a}\right)^2 + \frac{b^2}{2a} \,,$$

and shift then the integration variable to $x' = x - b/a$. The result is

$$B = \int_{-\infty}^{\infty} \mathrm{d}x \exp(-ax^2/2 + bx) = \mathrm{e}^{b^2/2a} \int_{-\infty}^{\infty} \mathrm{d}x' \exp(-ax'^2/2) = \sqrt{\frac{2\pi}{a}}\, \mathrm{e}^{b^2/2a} \,. \tag{2.21}$$

c.) We should complete again the square and try $X' = X - A^{-1}J$. With

$$\begin{aligned}(X - A^{-1}J)^T A (X - A^{-1}J) &= X^T AX - X^T AA^{-1}J - J^T A^{-1}AX + J^T A^{-1}AA^{-1}J \\ &= X^T AX - 2J^T X + J^T A^{-1} J \,,\end{aligned}$$

we obtain after shifting the integration vector,

$$C = \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp(-X^T AX/2 + J^T X) = \exp(J^T A^{-1} J/2) \int \mathrm{d}x'_1 \cdots \mathrm{d}x'_n \exp(-X'^T AX'/2) \,.$$

Since the matrix A is symmetric, we can diagonalise A via an orthogonal transformation $D = OAO^T$. This corresponds to a rotation of the integration variables, $Y = OX'$. The Jacobian of this transformation is one, and thus the result is

$$C = \exp(J^T A^{-1} J/2) \prod_{i=1}^{n} \int \mathrm{d}y_i \exp(-a_i y_i^2/2) = \sqrt{\frac{(2\pi)^n}{\det A}}\, \exp\left(\frac{1}{2} J^T A^{-1} J\right) \,. \tag{2.22}$$

In the last step we expressed the product of eigenvalues a_i as the determinant of the matrix A.

2.2. Path integrals in quantum mechanics

In problem 2.1 you are asked to calculate the classical action of a free particle and of a harmonic oscillator and to compare them to the corresponding propagators found in quantum mechanics. Surprisingly, you will find that in both cases the propagator can be written as $K(q',t';q,t) = N \exp(\mathrm{i}S)$ where S is the classical action along the path $[q(t) : q'(t')]$ and N a

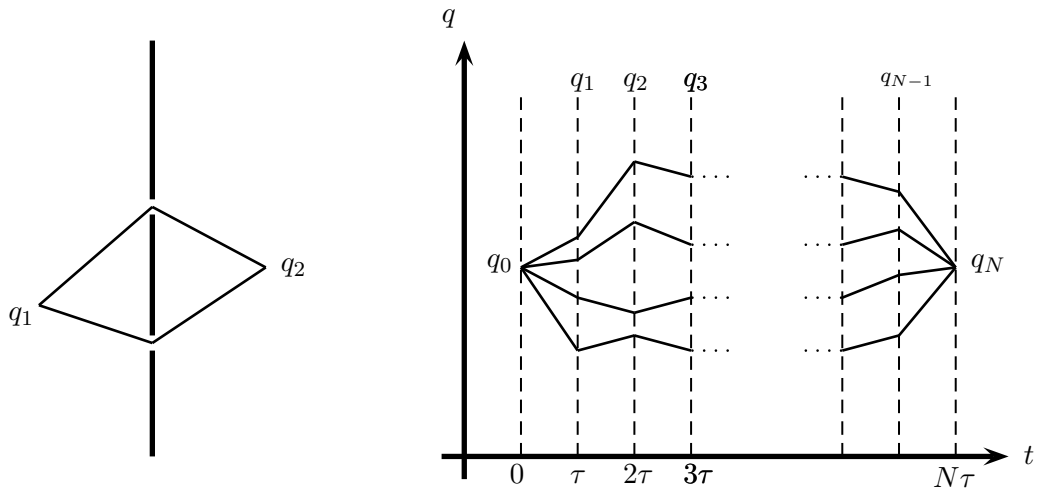


Figure 2.1.: Left: The double slit experiment. Right: The propagator $K(q_N, \tau; q_0, 0)$ expressed as a sum over all N-legged continuous paths.

normalisation constant. This suggests that we can reformulate quantum mechanics, replacing the standard operator formalism used to evaluate the propagator (2.17) "somehow" by the classical action.

To get an idea how to proceed, we look at the famous double-slit experiment: According to the superposition principle, the amplitude A for a particle to move from the source at q_1 to the detector at q_2 is the sum of the amplitudes A_i for the two possible paths,

$$A = K(q_2, t_2; q_1, t_1) = \sum_{\text{paths}} A_i \,. \tag{2.23}$$

Clearly, we could add in a Gedankenexperiment more and more screens between q_1 and q_2, increasing at the same time the number of holes. Although we replace in this way continuous space-time by a discrete lattice, the differences between these two descriptions should vanish for sufficiently small spacing τ. Moreover, for $\tau \to 0$, we can expand $U(\tau) = \exp(-\mathrm{i}H\tau) \simeq 1 - \mathrm{i}H\tau$. Applying then $H = \hat{p}^2/(2m) + V(\hat{q})$ to eigenfunctions $|q\rangle$ of $V(\hat{q})$ and $|p\rangle$ of $\hat{p}^2$, we can replace the operator H by its eigenvalues. In this way, we hope to express the propagator as a sum over paths, where the individual amplitudes A_i contain only classical quantities.

We apply now this idea to a particle moving in one dimension in a potential $V(q)$. The transition amplitude A for the evolution from the state $|q, 0\rangle$ to the state $|q', t'\rangle$ is

$$A \equiv K(q', t'; q, 0) = \left\langle q' \right| \mathrm{e}^{-\mathrm{i}Ht'} \left| q \right\rangle \,. \tag{2.24}$$

This amplitude equals the matrix-element of the propagator K for the evolution from the initial point $q(0)$ to the final point $q'(t')$.

Let us split the time evolution into two smaller steps, writing $\mathrm{e}^{-\mathrm{i}Ht'} = \mathrm{e}^{-\mathrm{i}H(t'-t_1)}\mathrm{e}^{-\mathrm{i}Ht_1}$. Inserting also $\int \mathrm{d}q_1 \, |q_1\rangle \langle q_1| = 1$, the amplitude becomes

$$A = \int \mathrm{d}q_1 \left\langle q' \right| e^{-\mathrm{i}H(t'-t_1)} \left| q_1 \right\rangle \left\langle q_1 \right| \mathrm{e}^{-\mathrm{i}Ht_1} \left| q \right\rangle = \int \mathrm{d}q_1 \, K(q', t'; q_1, t_1) K(q_1, t_1; q, 0). \tag{2.25}$$

This formula expresses simply the group property, $U(t', 0) = U(t', t_1)U(t_1, 0)$, of the time-evolution operator U evaluated in the basis of the continuous variable q. More physically, we

can view this equation as an expression of the quantum mechanical rule for combining amplitudes: If the same initial and final states can be connected by various ways, the amplitudes for each of these processes should be added. A particle propagating from q to q' must be somewhere at the intermediate time t_1. Labelling this intermediate position as q_1, we compute the amplitude for propagation via the point q_1 as the product of the two propagators in Eq. (2.25) and integrate over all possible intermediate positions q_1.

We continue to divide the time interval t' into a large number N of time intervals of duration $\tau = t'/N$. Then the propagator becomes

$$A = \langle q' | \underbrace{\mathrm{e}^{-\mathrm{i}H\tau}\mathrm{e}^{-\mathrm{i}H\tau}\cdots\mathrm{e}^{-\mathrm{i}H\tau}}_{N \text{ times}} | q \rangle \ . \tag{2.26}$$

We insert again a complete set of states $|q_i\rangle$ between each exponential, obtaining

$$\begin{aligned} A &= \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \langle q' | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q_{N-1} \rangle \langle q_{N-1} | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q_{N-2} \rangle \cdots \langle q_1 | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q \rangle \\ &\equiv \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} K_{q_N,q_{N-1}} K_{q_{N-1},q_{N-2}} \cdots K_{q_2,q_1} K_{q_1,q_0} \, , \end{aligned} \tag{2.27}$$

where we have defined $q_0 = q$ and $q_N = q'$. Note that these initial and final positions are fixed and therefore are not integrated over. Figure 2.1 illustrates that we can view the amplitude A as the integral over the partial amplitudes A_{path} of the individual N-legged continuous paths.

We ignore the problem of defining properly the limit $N \to \infty$, keeping N large but finite. We rewrite the amplitude as sum over the amplitudes for all possible paths, $A = \sum_{\text{paths}} A_{\text{path}}$, with

$$\sum_{\text{paths}} = \int dq_1 \cdots dq_{N-1}, \qquad A_{\text{path}} = K_{q_N,q_{N-1}} K_{q_{N-1},q_{N-2}} \cdots K_{q_2,q_1} K_{q_1,q_0} \, .$$

Let us look at the last expression in detail. We can expand the exponential in each propagator $K_{q_{j+1},q_j} = \langle q_{j+1} | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q_j \rangle$ for a single sub-interval, because τ is small,

$$\begin{aligned} K_{q_{j+1},q_j} &= \langle q_{j+1} | \left(1 - \mathrm{i}H\tau - \frac{1}{2} H^2 \tau^2 + \cdots \right) | q_j \rangle \\ &= \langle q_{j+1} | \, q_j \rangle - \mathrm{i}\tau \, \langle q_{j+1} | \, H \, | q_j \rangle + \mathcal{O}(\tau^2) \, . \end{aligned} \tag{2.28}$$

In the second term of (2.28), we insert a complete set of momentum eigenstates between H and $|q_j\rangle$. This gives

$$\begin{aligned} &- \mathrm{i}\tau \, \langle q_{j+1} | \left(\frac{\hat{p}^2}{2m} + V(\hat{q}) \right) \int \frac{\mathrm{d}p_j}{2\pi} \, | p_j \rangle \, \langle p_j | \, q_j \rangle \\ &= -\mathrm{i}\tau \int \frac{\mathrm{d}p_j}{2\pi} \left(\frac{p_j^2}{2m} + V(q_{j+1}) \right) \langle q_{j+1} | \, p_j \rangle \, \langle p_j | \, q_j \rangle \\ &= -\mathrm{i}\tau \int \frac{\mathrm{d}p_j}{2\pi} \left(\frac{p_j^2}{2m} + V(q_{j+1}) \right) \mathrm{e}^{\mathrm{i}p_j(q_{j+1}-q_j)} \, . \end{aligned} \tag{2.29}$$

The expression (2.29) is not symmetric in q_j and q_{j+1}. The reason for this asymmetry is that we could have inserted the factor 1 either to the right or to the left of the Hamiltonian H. In the latter case, we would have obtained p_{j+1} and $V(q_j)$ in (2.29). Since the difference

$[V(q_{j+1}) - V(q_j)]\tau \simeq V'(q_j)(q_{j+1} - q_j)\tau \simeq V'(q_j)\dot{q}_j\tau^2$ is of order τ^2, the ordering problem should not matter in the continuum limit which we will take eventually; we set therefore $V(q_{j+1}) \simeq V(q_j)$.

The first term of (2.28) gives a delta function, which we can express as

$$\langle q_{j+1}|\, q_j\rangle = \delta(q_{j+1} - q_j) = \int \frac{\mathrm{d}p_j}{2\pi}\, \mathrm{e}^{\mathrm{i}p_j(q_{j+1}-q_j)}\,. \tag{2.30}$$

Now we can combine the two terms, obtaining as propagator for the step $q_j \to q_{j+1}$

$$K_{q_{j+1},q_j} = \int \frac{\mathrm{d}p_j}{2\pi}\, \mathrm{e}^{\mathrm{i}p_j(q_{j+1}-q_j)} \left[1 - \mathrm{i}\tau \left(\frac{{p_j}^2}{2m} + V(q_j)\right) + \mathcal{O}(\tau^2)\right]\,. \tag{2.31}$$

Since we work only at $\mathcal{O}(\tau)$, we can exponentiate the factor in the square bracket,

$$1 - \mathrm{i}\tau\ H(p_j, q_j) + \mathcal{O}(\tau^2) = \mathrm{e}^{-\mathrm{i}\tau H(p_j,q_j)}\,. \tag{2.32}$$

Next we rewrite the exponent in the first factor of Eq. (2.31) using $\dot{q}_j = (q_{j+1} - q_j)/\tau$, such that we can factor out the time-interval τ. The amplitude A_{path} consists of N such factors. Combining them, we obtain

$$A_{\text{path}} = \prod_{j=0}^{N-1} \int \frac{\mathrm{d}p_j}{2\pi} \exp \mathrm{i}\tau \sum_{j=0}^{N-1} [p_j\dot{q}_j - H(p_j, q_j)]\,. \tag{2.33}$$

We recognise the argument of the exponential as the discrete approximation of the action $S[q,p]$ in the Palatini form of a path passing through the points $q_0 = q, q_1, \cdots, q_{N-1}, q_N = q'$. The propagator $K = \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} A_{\text{path}}$ becomes then

$$K = \prod_{j=1}^{N-1} \int \mathrm{d}q_j \prod_{j=0}^{N-1} \int \frac{\mathrm{d}p_j}{2\pi} \exp \mathrm{i}\tau \sum_{j=0}^{N-1} [p_j\dot{q}_j - H(p_j, q_j)]\,. \tag{2.34}$$

For $N \to \infty$, this expression approximates an integral over all functions $p(t)$, $q(t)$ consistent with the boundary conditions $q(0) = q$, $q(t') = q'$. We adopt the notation $\mathcal{D}p\mathcal{D}q$ for the *functional* or *path integral* over all functions $p(t)$ and $q(t)$,

$$K \equiv \int \mathcal{D}p(t)\mathcal{D}q(t)\mathrm{e}^{\mathrm{i}S[q,p]} = \int \mathcal{D}p(t)\mathcal{D}q(t) \exp\left(\mathrm{i}\int_0^{t'} \mathrm{d}t\, (p\dot{q} - H(p,q))\right)\,. \tag{2.35}$$

This result expresses the propagator as a path integral in phase-space. It allows us to obtain for any classical system which can be described by a Hamiltonian the corresponding quantum dynamics.

If the Hamiltonian is of the form $H = p^2/2m + V$, as we have assumed[3] in our derivation, we can carry out the quadratic momentum integrals in (2.34). We can rewrite this expression as

$$K = \prod_{j=1}^{N-1} \int \mathrm{d}q_j \exp -\mathrm{i}\tau \sum_{j=0}^{N-1} V(q_j) \prod_{j=0}^{N-1} \int \frac{\mathrm{d}p_j}{2\pi} \exp \mathrm{i}\tau \sum_{j=0}^{N-1} \left(p_j\dot{q}_j - p_j^2/2m\right)\,.$$

[3]Since we evaluated $\exp(\mathrm{i}H\tau)$ for infinitesimal τ, the result (2.35) holds also for a time-dependent potential $V(q,t)$.

The p integrals are all uncoupled Gaussians. One such integral gives

$$\int \frac{\mathrm{d}p}{2\pi}\, \mathrm{e}^{\mathrm{i}\tau(p\dot{q}-p^2/2m)} = \sqrt{\frac{m}{2\pi\mathrm{i}\tau}}\, \mathrm{e}^{\mathrm{i}\tau m\dot{q}^2/2}\,, \tag{2.36}$$

where we added again an infinitesimal factor $\exp(-\varepsilon p^2)$ to the integrand. Thence the propagator becomes

$$\begin{aligned} K &= \prod_{j=1}^{N-1} \int \mathrm{d}q_j \exp -\mathrm{i}\tau \sum_{j=0}^{N-1} V(q_j) \prod_{j=0}^{N-1} \left(\sqrt{\frac{m}{2\pi\mathrm{i}\tau}} \exp \mathrm{i}\tau \frac{m\dot{q}_j^2}{2} \right) \\ &= \left(\frac{m}{2\pi\mathrm{i}\tau}\right)^{N/2} \prod_{j=1}^{N-1} \int \mathrm{d}q_j \exp \mathrm{i}\tau \sum_{j=0}^{N-1} \left(\frac{m\dot{q}_j^2}{2} - V(q_j) \right). \end{aligned} \tag{2.37}$$

The argument of the exponential is again a discrete approximation of the action $S[q]$ of a path passing through the points $q_0 = q, q_1, \cdots, q_{N-1}, q_N = q'$, but now seen as functional of only the coordinate q. As above, we can write this in the more compact form

$$K = \langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q(t) \mathrm{e}^{\mathrm{i}S[q]} = \int \mathcal{D}q(t) \exp\left(\mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t\, L(q, \dot{q}) \right), \tag{2.38}$$

where the integration includes all paths satisfying the boundary condition $q(t_i) = q_i$ and $q(t_f) = q_f$. This is the main result of this section, and is known as the *path integral in configuration space*. It will serve us as starting point discussing quantum field theories of bosonic fields.

Knowing the path integral and thus the propagator is sufficient to solve scattering problems in quantum mechanics. In a relativistic theory, the particle number during the course of a scattering process is however not fixed, since energy can be converted into matter. In order to prepare us for such more complex problems, we will generalise in the next section the path integral to a *generating functional* for *n-point Green functions*: In this formalism, the usual propagator giving the probability amplitude that a single particle moves from $q_i(t_i)$ to $q_f(t_f)$ becomes the special case of a 2-point Green function, while Green functions with $n > 2$ describe processes involve more points. For instance, the 4-point Green function will be the essential ingredient to calculate $2 \to 2$ scattering processes in QFT. The corresponding generating functional, finally, is the quantity which n.th derivative returns the n-point Green functions.

2.3. Generating functional for Green functions

Having re-expressed the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle$ of a quantum mechanical system as a path integral, we want to generalise first this result to the matrix elements of an arbitrary potential $V(q)$ between the states $|q_i, t_i\rangle$ and $|q_f, t_f\rangle$. For all practical purposes, we can assume that we can expand $V(q)$ as a power-series in q; thus it is sufficient to consider the matrix elements $\langle q_f, t_f | q^m | q_i, t_i \rangle$. In quantum field theory, the initial and final states are generally free particles which are described mathematically as harmonic oscillators. In this case, we are able to reconstruct all excited states $|n\rangle$ from the ground-state,

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle\ .$$

Therefore it will be sufficient to study matrix elements between the ground-state $|0\rangle$. With this choice, we are able to extend the integration limit in the path integral (2.38) to $t = \pm\infty$. This will not only simplify its evaluation, but also avoid the need to choose a specific Lorentz frame. As a result, the action will have an obviously Lorentz invariant form in a relativistic theory.

Time-ordered products of operators and the path integral In a first step, we try to include the operator q^m into the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle$. We can reinterpret our result for the path integral as follows,

$$\langle q_f, t_f | \mathbf{1} | q_i, t_i \rangle = \int \mathcal{D}q(t)\; 1\; \mathrm{e}^{\mathrm{i}S[q(t)]}\,. \tag{2.39}$$

Thus we can see the LHS as matrix element of the unit operator $\mathbf{1}$, while the RHS corresponds to the path integral average of the classical function $f(q, \dot{q}) = 1$. Now we want to generalise this rather trivial statement to two operators $\hat{A}(t_a)$ and $\hat{B}(t_b)$ given in the Heisenberg picture. In evaluating

$$\int \mathcal{D}q(t)\; A(t_a) B(t_b)\; \mathrm{e}^{\mathrm{i}S[q(t)]} = \langle q_f, t_f | \; \mathbf{?} \; | q_i, t_i \rangle\,, \tag{2.40}$$

we go back to Eq. (2.27) and insert $\hat{A}(t_a)$ and $\hat{B}(t_b)$ at the correct intermediate times t_i,

$$= \begin{cases} \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \ldots \langle q_{a+1}, t_{a+1} | \, \hat{A} \, | q_a, t_a \rangle \cdots \langle q_{b+1}, t_{b+1} | \, \hat{B} \, | q_b, t_b \rangle \ldots & \text{for} \quad t_a > t_b\,, \\ \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \ldots \langle q_{b+1}, t_{b+1} | \, \hat{B} \, | q_b, t_b \rangle \cdots \langle q_{a+1}, t_{a+1} | \, \hat{A} \, | q_a, t_a \rangle \ldots & \text{for} \quad t_a < t_b\,. \end{cases} \tag{2.41}$$

Since the time along a classical path increases, the matrix-elements of the operators $\hat{A}(t_a)$ and $\hat{B}(t_b)$ are also ordered with time increasing from the left to the right. If we define the *time-ordered product* of two operators as

$$T\{\hat{A}(t_a)\hat{B}(t_b)\} = \hat{A}(t_a)\hat{B}(t_b)\vartheta(t_a - t_b) + \hat{B}(t_b)\hat{A}(t_a)\vartheta(t_b - t_a)\,, \tag{2.42}$$

then the path integral average of the classical quantities $A(t_a)$ and $B(t_b)$ corresponds to the matrix-element of the time-ordered product of these two operators,

$$\langle q_f, t_f | T\{\hat{A}(t_a)\hat{B}(t_b)\} | q_i, t_i \rangle = \int \mathcal{D}q(t)\; A(t_a) B(t_b)\; \mathrm{e}^{\mathrm{i}S[q(t)]}\,, \tag{2.43}$$

and similar for more than two operators.

External sources We want to include next in our formalism the possibility that we can change the state of our system by applying an external driving force or source term $J(t)$. In quantum mechanics, we could imagine e.g. a harmonic oscillator in the ground-state $|0\rangle$, making a transition under the influence of an external force J to the state $|n\rangle$ at the time t and back to the ground-state $|0\rangle$ at the time $t' > t$. Including such transitions, we can mimic the relativistic process of the creation and annihilation of particles as follows: We identify the vacuum (i.e. the state containing zero real particles) with the ground-state of the quantum mechanical system, and the creation and annihilation of n particle with the (de-) excitation of the n.th energy level by an external source J.

Schwinger realised that adding a linear coupling to an external source,

$$L \to L + J(t)q(t)\,, \tag{2.44}$$

will lead also to an efficient way to calculate the matrix elements of an arbitrary polynomial of operators $q(t_n)\cdots q(t_1)$: If the source $J(t)$ would be a simple number instead of a time-dependent function in the augmented path-integral

$$\langle q_f, t_f | q_i, t_i \rangle_J \equiv \int \mathcal{D}q(t) \mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t (L + Jq)} \,, \tag{2.45}$$

then we could obtain $\langle q_f, t_f | q^m | q_i, t_i \rangle_J$ simply by differentiating $\langle q_f, t_f | q_i, t_i \rangle_J$ m-times with respect to J. However, the LHS is a functional of $J(t)$ and thus we need to perform instead functional derivatives with respect to $J(t)$. By analogy with the rules for the differentiation of functions, e.g. $\partial x^l / \partial x^k = \delta^l_k$, we define[4] a functional derivative as

$$\frac{\delta}{\delta J(x)} \, 1 = 0 \quad \text{and} \qquad \frac{\delta J(x)}{\delta J(x')} = \delta(x - x') \,. \tag{2.46}$$

Thus we replace for a continuous index the Kronecker delta by a delta function, and assume that the Leibniz and the chain rule holds for sufficiently nice functions $J(x)$.

Now we are able to differentiate $\langle q_f, t_f | q_i, t_i \rangle_J$ with respect to the source J. Starting from

$$\frac{\delta}{\delta J(t_1)} \int \mathcal{D}q(t) \, \mathrm{e}^{\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}t J(t) q(t)} = \mathrm{i} \int \mathcal{D}q(t) \, q(t_1) \mathrm{e}^{\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}t J(t) q(t)} \,, \tag{2.47}$$

we obtain

$$\langle q_f, t_f | T\{\hat{q}(t_1)\cdots\hat{q}(t_n)\} | q_i, t_i \rangle = (-\mathrm{i})^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)} \langle q_f, t_f | q_i, t_i \rangle_J \bigg|_{J=0} \,. \tag{2.48}$$

Thus the source $J(t)$ is a convenient tool to obtain the functions $q(t_1)\cdots q(t_n)$ in front of $\exp(\mathrm{i}S)$. Having performed the functional derivatives, we set the source $J(t)$ to zero, coming back to the usual path integral. Physically, the expression (2.48) corresponds to the probability amplitude that a particles moves from $q_i(t_i)$ to $q_f(t_f)$, having the intermediate positions $q(t_1), \ldots, q(t_n)$.

Ground-state persistence amplitude As last step, we want to eliminate the initial and final states $|q_i, t_i\rangle_J$ and $|q_f, t_f\rangle$ in favour of the ground state or vacuum, $|0\rangle$. In this way, we convert the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle_J$ into the probability amplitude that a system which was in the ground-state $|0\rangle$ at $t_i \to -\infty$ remains in this state at $t_f \to \infty$ despite the action of the source $J(t)$. Inserting a complete set of energy eigenstates, $1 = \sum_n |n\rangle\langle n|$, into the propagator, we obtain

$$\langle q', t' | q, t \rangle = \sum_n \psi_n(q') \psi_n^*(q) \exp(-\mathrm{i} E_n (t' - t)) \,. \tag{2.49}$$

We can isolate the ground-state $n = 0$ by adding either to the energies E_n or to the time difference $\tau = (t' - t)$ a small negative imaginary part. In this case, all terms are exponentially damped in the limit $\tau \to \infty$, and the ground-state as state with the smallest energy dominates more and more the sum. Alternatively, we can add a term $+\mathrm{i}\varepsilon q^2$ to the Lagrangian.

[4] As the notation suggest, the variation of a functional defined in Eq. (1.5) is the special case of a *directional* functional derivative, cf. problem 1.6.

Remark 2.1: Wick rotation and Euclidean action:
Instead of adding the infinitesimal small term $\mathrm{i}\varepsilon q^2$ to the Lagrangian, we can do a somewhat more drastic change, rotating in the action the time axis clockwise by 90 degrees in the complex plane. Inserting $t_E = \mathrm{i}t$ into $x_\mu x^\mu$, we see that this procedure called Wick rotation corresponds to the transition from Minkowski to Euclidean space,

$$x^2 = t^2 - \boldsymbol{x}^2 = (-\mathrm{i}t_E)^2 - \boldsymbol{x}^2 = -[t_E^2 + \boldsymbol{x}^2] = -x_E^2 \,.$$

Performing the changes $t = -\mathrm{i}t_E$ and $\mathrm{d}t = -\mathrm{i}\mathrm{d}t_E$ in the action of a particle moving in an one-dimensional potential gives

$$S = -\mathrm{i}\int \mathrm{d}t_E \left(-\frac{1}{2}m\dot{q}_E^2 - V(q)\right) \equiv \mathrm{i}S_E \,. \tag{2.50}$$

Note that the Euclidean action $S_E = T + V$ is bounded from below. The phase factor in the path-integral transforms as $\mathrm{e}^{\mathrm{i}S} = \mathrm{e}^{-S_E}$, and thus contributions with large S_E are exponentially damped in the Euclidean path-integral.

Finally, we have only to connect the results we obtained so far. Adding a coupling to an external source $J(t)$ and a damping factor $+\mathrm{i}\varepsilon q^2$ to the Lagrangian gives us the ground-state persistence amplitude

$$Z[J] \equiv \langle 0,\infty|0,-\infty\rangle_J = \int \mathcal{D}q(t)\, \mathrm{e}^{\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t(L+Jq+\mathrm{i}\varepsilon q^2)} \tag{2.51}$$

in the presence of a classical source J. This amplitude is a functional of J which we denote by $Z[J]$. Taking derivatives w.r.t. the external sources J, and setting them afterwards to zero, we obtain

$$\left.\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0} = \mathrm{i}^n \int \mathcal{D}q(t)\, q(t_1)\cdots q(t_n)\mathrm{e}^{\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t(L+\mathrm{i}\varepsilon q^2)} \,. \tag{2.52}$$

The RHS corresponds to the path integral in Eqs. (2.43), augmented by the factor $\mathrm{i}\varepsilon q^2$. But this factor damps in the limit of large t everything except the ground state. Thus we found that $Z[J]$ is the generating functional for the vacuum expectation value of the time-ordered product of operators $\hat{q}(t_i)$,

$$(-\mathrm{i})^n \left.\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0} = \langle 0,\infty|T\{\hat{q}(t_1)\cdots\hat{q}(t_n)\}|0,-\infty\rangle = G_F(t_1,\ldots,t_n) \,. \tag{2.53}$$

In the last step, we defined also the n-point Green function $G_F(t_1,\ldots,t_n)$. The subscript F indicates that the $\mathrm{i}\varepsilon q^2$ prescription selects from the set of possible Green functions (retarded, advanced, ...) the ones suggested by Feynman. These functions will be the main building block we will use to perform calculations in quantum field theory, and the formula corresponding to Eq. (2.53) will be our master formula in field theory.

2.4. Oscillator as a one-dimensional field theory

Canonical quantisation A one-dimensional harmonic oscillator can be viewed as a free quantum field theory in one time and zero space dimensions. In order to exhibit this equivalence clearer, we rescale the usual Lagrangian

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \,, \tag{2.54}$$

where m is the mass of the oscillator and ω its frequency as

$$\phi(t) \equiv \sqrt{m}x(t)\,. \tag{2.55}$$

We call the variable $\phi(t)$ a "scalar field," and the Lagrangian now reads

$$L(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2\,. \tag{2.56}$$

After the rescaling, the kinetic term $\dot{\phi}^2$ has the dimensionless coefficient 1/2. This choice is standard in field theory and therefore such a field is called "canonically normalised."

We derive the corresponding Hamiltonian, determining first the conjugate momentum π as $\pi(t) = \partial L/\partial\dot{\phi} = \dot{\phi}(t)$. Thus the (classical) Hamiltonian follows as

$$H(\phi,\pi) = \frac{1}{2}\pi^2 + \frac{1}{2}\omega^2\phi^2\,. \tag{2.57}$$

The transition to quantum mechanics is performed by promoting ϕ and π to operators which satisfy the canonical commutation relations $[\phi,\pi] = \mathrm{i}$. The harmonic oscillator is solved most efficiently introducing creation and annihilation operators, $a^\dagger$ and a. They are defined by

$$\phi = \frac{1}{\sqrt{2\omega}}\left(a^\dagger + a\right) \quad \text{and} \quad \pi = \mathrm{i}\sqrt{\frac{\omega}{2}}\left(a^\dagger - a\right)\,, \tag{2.58}$$

and satisfy $\left[a, a^\dagger\right] = 1$. The Hamiltonian follows as

$$H = \frac{\omega}{2}\left(aa^\dagger + a^\dagger a\right) = \left(a^\dagger a + \frac{1}{2}\right)\omega\,. \tag{2.59}$$

We interpret $N \equiv a^\dagger a$ as the number operator, counting the number n of quanta with energy ω in the state $|n\rangle$.

We now work in the Heisenberg picture where operators are time dependent. The time evolution of the operator $a(t)$ can be found from the Heisenberg equation,

$$\mathrm{i}\frac{\mathrm{d}a}{\mathrm{d}t} = [a,H] = \omega a\,, \tag{2.60}$$

from which we deduce that

$$a(t) = a(0)\mathrm{e}^{-\mathrm{i}\omega t} = a_0\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{2.61}$$

As a consequence, the field operator $\phi(t)$ can be expressed in terms of the creation and annihilation operators as

$$\phi(t) = \frac{1}{\sqrt{2\omega}}\left(a_0\mathrm{e}^{-\mathrm{i}\omega t} + a_0^\dagger\mathrm{e}^{\mathrm{i}\omega t}\right)\,. \tag{2.62}$$

If we look at $\phi(t)$ as a classical variable, then a_0 and $a_0^\dagger$ have to satisfy $a_0 = a_0^\dagger \equiv a_0^*$ in order to make ϕ real: Thus they are simply the Fourier coefficients of the single eigen-mode $\sin(\omega t)$. This suggests that we can short-cut the quantisation procedure as follows: We write down the field as sum over its eigenmodes $i = 1,\ldots,k$. Then we re-interpret the Fourier coefficients as creation and annihilation operators, requiring $[a_i, a_j^\dagger] = \delta_{ij}$.

Path integral approach We solve now the same problem, the rescaled Lagrangian (2.56), in the path integral approach. Using this method, we have argued that it is convenient to include a coupling to an external force J. Let us define therefore the effective action S_{eff} as the sum of the classical action S, the coupling to the external force J and a small imaginary part $\mathrm{i}\varepsilon\phi^2$ to make the path integral well-defined,

$$S_{\text{eff}} = S + \int_{-\infty}^{\infty} \mathrm{d}t \left(J\phi + \mathrm{i}\varepsilon\phi^2\right) = \int_{-\infty}^{\infty} \mathrm{d}t \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2 + J\phi + \mathrm{i}\varepsilon\phi^2\right] . \tag{2.63}$$

The function $\mathrm{e}^{\mathrm{i}S_{\text{eff}}}$ is the integrand of the path integral. We start our work by massaging S_{eff} into a form such that the path integral can be easily performed. The first two terms in S_{eff} can be viewed as the action of a differential operator $D(t)$ on $\phi(t)$, writing

$$\frac{1}{2}\left(\dot{\phi}^2 - \omega^2\phi^2\right) = -\frac{1}{2}\phi(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right)\phi(t) = -\frac{1}{2}\phi(t)D(t)\phi(t)\,. \tag{2.64}$$

Here we performed a partial integration and dropped the boundary term: This is admissible, because the boundary term vanishes varying the action.

We can evaluate this operator going to Fourier space,

$$\phi(t) = \int \frac{\mathrm{d}E}{2\pi}\,\mathrm{e}^{-\mathrm{i}Et}\,\phi(E) \quad \text{and} \quad J(t) = \int \frac{\mathrm{d}E}{2\pi}\,\mathrm{e}^{-\mathrm{i}Et}\,J(E)\,. \tag{2.65}$$

To keep the action real, we have to write all bilinear quantities as $\phi(E)\phi(-E')$, etc. Since only the phases depend on time, the time integration gives a factor $2\pi\delta(E-E')$, expressing energy conservation,

$$S_{\text{eff}} = \frac{1}{2}\int \frac{\mathrm{d}E}{2\pi}\,\left[\phi(E)(E^2 - \omega^2 + \mathrm{i}\varepsilon)\phi(-E) + J(E)\phi(-E) + J(-E)\phi(E)\right] . \tag{2.66}$$

In the path integral, this expression corresponds to a Gaussian integral of the type (2.21), where we should "complete the square." Shifting the integration variable to

$$\tilde{\phi}(E) = \phi(E) + \frac{J(E)}{E^2 - \omega^2 + \mathrm{i}\varepsilon}\,,$$

we obtain

$$S_{\text{eff}} = \frac{1}{2}\int \frac{\mathrm{d}E}{2\pi}\,\left[\tilde{\phi}(E)(E^2 - \omega^2 + \mathrm{i}\varepsilon)\tilde{\phi}(-E) - J(E)\,\frac{1}{E^2 - \omega^2 + \mathrm{i}\varepsilon}\,J(-E)\right] . \tag{2.67}$$

Here we see that the "damping rule" for the path integral makes also the integral over the energy denominator well-defined. The physical interpretation of this way of shifting the poles—which differs from our treatment of the retarded Green function in the classical case—will be postponed to the next chapter, where we will discuss this issue in detail.

We are now in the position to evaluate the generating functional $Z[J]$. The path integral measure is invariant under a simple shift of the integration variable, $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$, and we omit the tilde from now on. Furthermore, the second term in S_{eff} does not depend on ϕ and can be factored out,

$$\begin{aligned} Z[J] = \exp\left(-\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}E}{2\pi} J(E)\,\frac{1}{E^2 - \omega^2 + \mathrm{i}\varepsilon}\,J(-E)\right) \\ \times \int \mathcal{D}\phi\,\exp\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}E}{2\pi}\,\left[\phi(E)(E^2 - \omega^2 + \mathrm{i}\varepsilon)\phi(-E)\right] . \end{aligned} \tag{2.68}$$

Setting the external force to zero, $J = 0$, the first factor becomes one and the generating functional $Z[0]$ becomes equal to the path integral in the second line. But for $J = 0$, the oscillator remains in the ground-state and thus $Z[0] = \langle 0, \infty | 0, -\infty \rangle = 1$. Therefore

$$Z[J] = \exp\left(-\frac{\mathrm{i}}{2} \int \frac{\mathrm{d}E}{2\pi} J(E) \, \frac{1}{E^2 - \omega^2 + \mathrm{i}\varepsilon} \, J(-E) \right) . \tag{2.69}$$

Inserting the Fourier transformed quantities, we arrive at

$$Z[J] = \exp\left(-\frac{\mathrm{i}}{2} \int \mathrm{d}t' \, \mathrm{d}t \; J(t') G_F(t' - t) J(t) \right) , \tag{2.70}$$

where the Feynman propagator

$$G_F(t - t') = \int \frac{\mathrm{d}E}{2\pi} \, \mathrm{e}^{-\mathrm{i}E(t-t')} \, \frac{1}{E^2 - \omega^2 + \mathrm{i}\varepsilon} \tag{2.71}$$

differs from the retarded propagator G_R defined in Eq. (1.35) by the position of its poles.

The generating functional $Z[J]$ given by (2.70) is in the form most suitable for deriving arbitrary n-point Green functions using our master formula (2.53). Note that finding $Z[J]$ required only to determine the inverse of the differential operator $D(t)$, accounting for the right boundary conditions induced by the $\mathrm{i}\varepsilon\phi^2$ term. This inverse is the Feynman propagator or two-point function $G_F(t' - t)$ which we can determine directly solving

$$-D(t) G_F(t' - t) = \delta(t' - t) \, . \tag{2.72}$$

Going to Fourier space, we find immediately

$$G_F(E) = \frac{1}{E^2 - \omega^2 + \mathrm{i}\varepsilon} \, . \tag{2.73}$$

This suggests that we can short-cut the calculation of $Z[J]$ by determining the Feynman propagator and then to use directly (2.69) or (2.70).

These results allow us also to calculate arbitrary matrix elements between oscillator states. For instance, we obtain the expectation value $\langle 0 | \, \phi^2 \, | 0 \rangle$ from

$$\langle 0 | \, T\{\phi(t')\phi(t)\} \, | 0 \rangle = (-\mathrm{i})^2 \frac{\delta^2 Z[J]}{\delta J(t') \delta J(t)} \bigg|_{J=0} = \mathrm{i} G_F(t' - t) = \frac{1}{2\omega} \, \mathrm{e}^{\mathrm{i}\omega|t-t'|} \, . \tag{2.74}$$

Here, we used in the last step the explicit expression for G_F which you should check in problem 2.5. Taking the limit $t' \searrow t$ and replacing $\phi^2 \to mx^2$, we reproduce the standard result $\langle 0 | \, x^2 \, | 0 \rangle = 1/(2m\omega)$. Matrix elements between excited states $|n\rangle = (n!)^{-1/2} (a^\dagger)^n \, |0\rangle$ are obtained by expressing the creation operator $a^\dagger$ using $\pi(t) = \dot{\phi}(t)$ as

$$a^\dagger = \sqrt{\frac{\omega}{2}} \left(1 - \frac{\mathrm{i}}{\omega} \frac{\mathrm{d}}{\mathrm{d}t} \right) \phi(t) \, . \tag{2.75}$$

2.5. The need for quantum fields

We have already argued that any relativistic quantum theory has to be a many-particle theory. Such a theory has to include infinitely many degrees of freedom—as *field theories* like electrodynamics do. Before we move on to introduce the simplest quantum field theory in the next chapter, we present a formal argument that relativity and the single particle picture are incompatible.

In classical mechanics, the principle of relativity implies that all particle trajectories of massive particles are time-like, while massless particles move along light-like trajectories. This implements causality, i.e. the requirement that no signal can be transmitted faster than light. How should we translate this principle into a quantum theory? Causality would be clearly satisfied, if the relativistic propagator $K(\boldsymbol{x}', t'; \boldsymbol{x}, t)$ vanishes for space-like distances. Another, less restrictive translation of the principle of relativity would be to ask that measurements performed at space-like separated points do not influence each other. This is achieved, if all observables $O(x)$ commute for space-like distances,

$$[\hat{O}(\boldsymbol{x}, t), \hat{O}(\boldsymbol{x}', t')] = 0 \qquad \text{for} \quad (t - t')^2 < (\boldsymbol{x} - \boldsymbol{x}')^2 \,. \tag{2.76}$$

In quantum mechanics, the Heisenberg operators $\hat{\boldsymbol{x}}(t)$ and $\hat{\boldsymbol{p}}(t)$ depend, however, only on time. Therefore we can not implement the condition (2.76) in such a framework.

The only rescue for causality in relativistic quantum mechanics is therefore the vanishing of the propagator $K(t', \boldsymbol{x}'; t, \boldsymbol{x})$ outside the light-cone. We evaluate the propagator as in the non-relativistic case,

$$K(\boldsymbol{x}', t'; \boldsymbol{x}, t) = \left\langle \boldsymbol{x}' \right| \mathrm{e}^{-\mathrm{i}H(t'-t)} \left| \boldsymbol{x} \right\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\langle \boldsymbol{x}' \right| \mathrm{e}^{-\mathrm{i}E_{\boldsymbol{p}}(t'-t)} \left| \boldsymbol{p} \right\rangle \left\langle \boldsymbol{p} \right| \boldsymbol{x} \right\rangle \tag{2.77}$$

inserting however the relativistic dispersion relation, $E_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2}$. Next we use that the momentum operator $\hat{\boldsymbol{p}}$ generates space translations, $\exp(-\mathrm{i}\hat{\boldsymbol{p}}\boldsymbol{x})|\boldsymbol{0}\rangle = |\boldsymbol{x}\rangle$, to obtain

$$K(\boldsymbol{x}', t'; \boldsymbol{x}, t) = K(x' - x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left| \langle \boldsymbol{0} | \boldsymbol{p} \rangle \right|^2 \mathrm{e}^{-\mathrm{i}p(x'-x)} \,. \tag{2.78}$$

Here we introduced also the four-vector $p^\mu = (E_p, \boldsymbol{p})$, rewriting the plan-wave thereby in a Lorentz-invariant way. In order that the complete propagator is invariant, we have to choose as integration measure $\propto \mathrm{d}^3 p / E_{\boldsymbol{p}}$, cf. problem 2.7, and we set therefore $|\langle \boldsymbol{0} | \boldsymbol{p} \rangle|^2 = 1/(2E_{\boldsymbol{p}})$. Knowing its explicit expression, it is a straight-forward exercise to show that the propagator does not vanish outside the light-cone, but goes only exponentially to zero, $K(\boldsymbol{x}', 0; \boldsymbol{x}, 0) \propto \exp(-m|\boldsymbol{x}' - \boldsymbol{x}|)$. Thus we failed to implement both versions of causality into relativistic quantum mechanics. Instead, we will develop quantum field theory with the aim to implement causality via the condition (2.76).

Before starting this endeavour, we can draw still some important conclusion from Eq. (2.78). For space-like distances, $(x - x')^2 < 0$, a Lorentz boost can change the time order of two space-time events, cf. problem 1.9 Consistency requires thus to include both time-orderings: If a particle is created at t and absorbed at $t' > t$, then it can be created necessarily also at t' and absorbed at $t > t'$. We extend therefore the propagator as

$$K(x' - x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[\vartheta(t' - t) \mathrm{e}^{-\mathrm{i}p(x'-x)} + \vartheta(t - t') \mathrm{e}^{\mathrm{i}p(x'-x)} \right] , \tag{2.79}$$

where we chose the opposite sign for the plan-wave in the second factor: In this way, the phase of the plane-waves observed in both frames agree, $-E_{\boldsymbol{p}}\tau\vartheta(\tau) < 0$ and $+E_{\boldsymbol{p}}\tau\vartheta(-\tau) < 0$, and similarly for the momenta. If we imagine that the propagating particle carries a conserved charge, then we can associate the positive frequencies to the propagation of a particle (with charge q) and the negative frequencies to the propagation of an anti-particle (with charge $-q$). Then the resulting current is frame-independent, if the antiparticle has the same mass but the opposite additive charges. This prediction of relativistic quantum field theory is experimentally confirmed with extreme accuracy: For instance, the mass and charge difference of electrons and positrons are smaller than 8×10^{-9} and 4×10^{-8}, respectively[54].

Finally, we should mention an alternative way to implement causality: Instead of defining quantum fields $\hat{\phi}(x^\mu)$ on classical space-time, we could promote time t to an operator, parametrising the world-line $\hat{x}^\mu(\tau)$ of a particle e.g. by its proper-time τ. Considering then the surface $\hat{x}^\mu(\tau, \sigma)$ generated by a set of world-lines is the starting point of string theory.

Summary

Using Feynman's path integral approach, we can express a transition amplitude as a sum over all paths weighted by a phase which is determined by the classical action, $\langle q_f, t_f | q_i, t_i\rangle = \int \mathcal{D}q(t) \exp(\mathrm{i}S[q])$. Adding a linear coupling to an external source J and a damping term to the Lagrangian, we obtain the ground-state persistence amplitude $\langle 0, \infty | 0, -\infty\rangle_J$. This quantity serves as the generating functional $Z[J]$ for n-point Green functions $G(t_1, \ldots, t_n)$ which are the time-ordered vacuum expectation values of the operators $\hat{q}(t_1), \ldots, \hat{q}(t_n)$.

Further reading

Our presentation of the path integral follows the one of[49] which contains as well as[21] useful additional material. Schweber (2005) sketches the historical development that lead to Schwinger's Green functions, including his quantum action principle.

Problems

2.1 Classical action.
Calculate the classical action $S[q]$ for a free particle and an harmonic oscillator. Compare the results with the expression for the propagator $K = \langle x', t' | x, t\rangle = N \exp(\mathrm{i}\phi)$ of the corresponding quantum mechanical system and express both ϕ and N through the action S.

2.2 Propagator as Green function.
Show that the Green function or propagator $K(x', t'; x, t) = \langle x' | \exp[-\mathrm{i}H(t' - t)] | x\rangle$ of the Schrödinger equation is the inverse of the differential operator $(\mathrm{i}\partial_t - H)$.

2.3 Commutation relations.
Show that the commutation relations for the field, $[\phi, \pi] = \mathrm{i}$, imply $[a, a^\dagger] = 1$. What happens, if we change the normalisation (2.55)?

2.4 Mode functions.
Consider the generalisation of (2.62) to $\phi(t) = u a_0 \mathrm{e}^{-\mathrm{i}\omega t} + u^* a_0^\dagger \mathrm{e}^{\mathrm{i}\omega t}$, where the functions $u(t)$ are called mode functions. a.) Show that the usual commutation relations are valid, if $\Im(u\dot{u}) = 1$. b.) Show that the standard choice $u = 1/\sqrt{2\omega}$ minimises the energy of the ground state.

2.5 Feynman propagator.
Find the explicit expression for the Feynman propagator used in (2.74) from a.) its definition as time-ordered product of fields ϕ, and b.) evaluat-

ing (2.71) using Cauchy's theorem.

2.6 Matrix elements from $Z[J]$.
Evaluate the matrix element $\langle 0|\,\phi^2\,|1\rangle$ of an harmonic oscillator from $Z[J]$.

2.7 Lorentz invariant integration measure. ♣
Show that $\mathrm{d}^3k/(2\omega_k)$ is a Lorentz invariant integration measure by a) calculating the Jacobian of a Lorentz transformation, b) showing that

$$\int \mathrm{d}^4k\,\delta(k^2-m^2)\vartheta(k^0)f(k^0,\boldsymbol{k}) = \int \frac{\mathrm{d}^3k}{2\omega_k}\,f(\omega_k,\boldsymbol{k}) \tag{2.80}$$

holds for any function f.

2.8 Propagator at large $|\boldsymbol{x}|$.
Show that the propagator (2.78) decays exponentially outside the light-cone.

2.9 Statistical mechanics.
Derive the connection between the partition function $Z = \mathrm{tr}\,\mathrm{e}^{-\beta H}$ of statistical mechanics and the path integral of quantum mechanics in Euclidean time $t_E = -\mathrm{i}t$. (Hint: compare to remark 2.1.)

2.10 Scattering at short-range potential in $d = 1$. ♣
Consider in $d = 1$ the scattering of modes with large wave-lengths λ on a short-range potential, $V(x) = 0$ for $|x| > a$ and $\lambda \gg a$. i) Show that the potential V can be approximated by $V(x) = c_0\delta(x) + c_1\delta'(x) + \mathcal{O}(Va^2/\lambda^2)$. ii) Find the transmission and reflection coefficients setting $V(x) = c_0\delta(x)$. Argue that $T \simeq \mathrm{i}p/c_0$ holds for any short-range potential in the limit $p \ll 1/a$. iii) Show that no consistent solution exists setting $c_0 = 0$.

3. Free scalar field

We extend in this chapter the path integral approach from quantum mechanics to the simplest field theory, a real scalar field $\phi(x)$. Such a field may either represent an elementary particle like the Higgs scalar, a bound-state like a scalar meson, or a scalar parameter describing a specific property of a more complex theory. Proceeding similar to our approach in quantum mechanics, we will introduce the generating functional $Z[J] = \langle 0 + |0-\rangle_J$ of n-point Green functions as our main tool to calculate the time-ordered vacuum expectation value of a product of fields $\phi(x_1) \cdots \phi(x_n)$. Calculating the vacuum energy of the scalar field, we will encounter for the first time that many calculations in quantum field theories return a formally infinite result. In order to extract sensible predictions, we have to introduce therefore the concepts of regularisation and renormalisation.

3.1. Lagrange formalism and path integrals for fields

A field is a map which associates to each space-time point x a k-tupel of values $\phi_a(x)$, $a = 1, \ldots, k$. The space of field values $\phi_a(x)$ can be characterised by its transformation properties under Poincaré transformations and internal symmetry groups. The latter are in practically all physical applications Lie groups like U(1), SU(n) or SO(n). Except for a real scalar field ϕ, these fields have several components. Thus we have to generalise Hamilton's principle to a collection of fields $\phi_a(x)$, where the index a includes all internal as well as space-time indices. Moreover, the Lagrangian for a field $\phi_a(x)$ will contain not only time but also space derivatives.

To ensure Lorentz invariance, we consider a scalar Lagrange density $\mathscr{L}(x)$ that depend as a local function on the fields and their derivatives. By analogy to $L(q, \dot{q})$, we restrict ourselves mostly to include the fields $\phi_a(x)$ and their first derivatives $\partial_\mu \phi_a(x)$. We include no explicit time-dependence, since "everything" should be explained by the fields ϕ_a and their interactions. The Lagrangian $L(\phi_a, \partial_\mu \phi_a)$ is obtained by integrating the density $\mathscr{L}$ over a given space volume V. The action S is thus the four-dimensional integral

$$S[\phi_a] = \int_{t_a}^{t_b} \mathrm{d}t \, L(\phi_a, \partial_\mu \phi_a) = \int_\Omega \mathrm{d}^4x \, \mathscr{L}(\phi_a, \partial_\mu \phi_a) \tag{3.1}$$

with $\Omega = V \times [t_a : t_b]$. A variation $\delta\phi_a(x)$ of the fields leads to a variation of the action,

$$\delta S = \int_\Omega \mathrm{d}^4x \left[\frac{\partial \mathscr{L}}{\partial \phi_a} \, \delta\phi_a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \, \delta(\partial_\mu \phi_a) \right] , \tag{3.2}$$

where we have to sum over field components ($a = 1, \ldots, k$) and the Lorentz index $\mu = 0, \ldots, 3$. The correspondence $q(t) \to \phi(x^\mu)$ implies that the scale factor ε parametrising the variations $\phi_a(x^\mu, \varepsilon)$ depends not on x^μ. We can therefore eliminate again the variation of the field gradients $\partial_\mu \phi_a$ by a partial integration using Gauß' theorem,

$$\delta S = \int_\Omega \mathrm{d}^4x \left[\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a = 0 \, . \tag{3.3}$$

The surface term vanishes, since we require that the variation is zero on the boundary $\partial\Omega$. Thus the Lagrange equations for the fields ϕ_a are

$$\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) = 0 \,. \tag{3.4}$$

If the Lagrange density $\mathscr{L}$ is changed by a four-dimensional divergence, $\delta\mathscr{L} = \partial_\mu K^\mu$, and surface terms can be dropped, the same equations of motion result. Note also that it is often more efficient to perform directly the variation $\delta\phi_a$ in the action $S[\phi_a]$ than to use the Lagrange equations.

The path integral becomes now a functional integral over the k fields ϕ_a,

$$K = \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_k \, \mathrm{e}^{\mathrm{i}S[\phi_a]} = \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_k \, \mathrm{e}^{\mathrm{i}\int_\Omega \mathrm{d}^4x \, \mathscr{L}(\phi_a, \partial_\mu \phi_a)} \,. \tag{3.5}$$

A major problem we have to address later is that the k fields ϕ_a are often not independent: For instance, in electrodynamics all potentials A^μ connected by a gauge transformation describe the same physics. This redundancy makes the path integral (3.5) ill-defined. We start therefore with the simplest case of a single, real scalar field ϕ where such problems are absent. Moreover, we restrict ourselves in this chapter to a free field without interactions.

3.2. Generating functional for a scalar field

Lagrangian The (free) Schrödinger equation $\mathrm{i}\partial_t\psi = H_0\psi$ can be obtained substituting $\omega \to \mathrm{i}\partial_t$ and $\boldsymbol{k} \to -\mathrm{i}\boldsymbol{\nabla}_x$ into the non-relativistic energy-momentum relation $\omega = \boldsymbol{k}^2/(2m)$. With the same replacements, the relativistic $\omega^2 = m^2 + \boldsymbol{k}^2$ becomes the Klein-Gordon equation

$$(\Box + m^2)\phi = 0 \qquad \text{with} \quad \Box = \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu \,. \tag{3.6}$$

$\mathrm{i}\partial_t\psi = H_0\psi$ The relativistic energy-momentum relation implies that the solutions to the free Klein-Gordon equation consist of plane-waves with positive and negative energies $\pm\sqrt{\boldsymbol{k}^2 + m^2}$. For the stability of a quantum system it is however essential that its energy eigenvalues are bounded from below. Otherwise, we could generate e.g. in a scattering process $\phi + \phi \to n\phi$ an arbitrarily high number of ϕ particles with sufficiently low energy, and no stable form of matter could exist. Interpreting the Klein-Gordon equation as a relativistic wave equation for a single particle can be therefore not fully satisfactory, since the energy of its solutions is not bounded from below.

How do we guess the correct Lagrange density $\mathscr{L}$? Plane waves can be seen as a collection of coupled harmonic oscillators at each space-time point. The correspondence $\dot{q} \to \partial_\mu\phi$ means that the "kinetic" field energy is quadratic in the field derivatives. Relativistic invariance implies that the Lagrange density is a scalar, leaving as the only two possible terms containing derivatives

$$\eta_{\mu\nu}(\partial^\mu\phi)(\partial^\nu\phi) \quad \text{and} \quad \phi\Box\phi \,.$$

Using the action principle to derive the equation of motions, we can however drop boundary terms performing partial integrations. Thus these two terms are equivalent, up to a minus sign. The Klein-Gordon equation $\Box\phi = -m^2\phi$ suggests that the mass term is also quadratic in the field ϕ. Therefore we try as Lagrange density

$$\mathscr{L} = \frac{1}{2}\eta_{\mu\nu}\,(\partial^\mu\phi)\,(\partial^\nu\phi) - \frac{1}{2}m^2\phi^2 \equiv \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}m^2\phi^2 \,. \tag{3.7}$$

From now, we will drop the parenthesis around $(\partial^\mu\phi)$ and it should be understood from the context that the derivative ∂^μ acts only on the first field ϕ. Even shorter alternative notations are $(\partial_\mu\phi)^2$ and the concise $(\partial\phi)^2$. With the short-cut $\phi_{,\alpha} \equiv \partial_\alpha\phi$ and

$$\frac{\partial}{\partial\phi_{,\alpha}}\,(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}) = \eta^{\mu\nu}\left(\delta^\alpha_\mu\phi_{,\nu} + \delta^\alpha_\nu\phi_{,\mu}\right) = \eta^{\alpha\nu}\phi_{,\nu} + \eta^{\mu\alpha}\phi_{,\mu} = 2\phi^{,\alpha}\,, \tag{3.8}$$

the Lagrange equation becomes

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\alpha\left(\frac{\partial\mathscr{L}}{\partial\phi_{,\alpha}}\right) = -m^2\phi - \partial^\alpha\partial_\alpha\phi = 0\,. \tag{3.9}$$

Thus the Lagrange density (3.7) leads indeed to the Klein-Gordon equation. We can understand the relative sign in the Lagrangian splitting the relativistic kinetic energy into the "proper" kinetic energy $(\partial_t\phi)^2/2$ and the gradient energy density $(\boldsymbol{\nabla}\phi)^2/2$,

$$\mathscr{L} = \frac{1}{2}\dot\phi^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2\,. \tag{3.10}$$

The last two terms correspond to a potential energy and carry therefore the opposite sign of the first one.

Instead of guessing, we can derive the correct Lagrangian $\mathscr{L}$ as follows: We multiply the free field equation for ϕ by a variation $\delta\phi$ that vanishes on $\partial\Omega$. Then we integrate over Ω, perform a partial integration of the kinetic term, use $\partial_\mu\delta = \delta\partial_\mu$, the Leibniz rule and ask that the variation vanishes,

$$A\int_\Omega \mathrm{d}^4x\;\delta\phi\,(\Box + m^2)\phi = A\int_\Omega \mathrm{d}^4x\;\left[-\delta(\partial_\mu\phi)\partial^\mu\phi + \delta\phi\phi m^2\right] = \tag{3.11}$$

$$= A\int_\Omega \mathrm{d}^4x\;\delta\left[-\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\phi^2m^2\right] = 0\,. \tag{3.12}$$

The term in the square brackets agrees with our guess (3.7), taking into account that the source-free field equation fixes the Lagrangian only up to the overall factor A. In analogy with a quantum mechanical oscillator, we want that the coefficients of the two terms are $\pm1/2$ and thus we set $|A| = 1$.

We can determine the correct overall sign of $\mathscr{L}$ by calculating the energy density ρ of the scalar field and requiring that it is bounded from below and stable against small perturbations. We identify the energy density ρ of the scalar field with its Hamiltonian density $\mathscr{H}$, and use the connection between the Lagrangian and the Hamiltonian known from classical mechanics. The transition from a system with a finite number of degrees of freedom to one with an infinite number of degrees of freedom proceeds as follows,

$$p_i = \frac{\partial L}{\partial\dot q^i} \qquad \Rightarrow \qquad \pi_a = \frac{\partial\mathscr{L}}{\partial\dot\phi_a}\,, \tag{3.13}$$

$$H = p_i\dot q^i - L \qquad \Rightarrow \qquad \mathscr{H} = \sum_a \pi_a\dot\phi_a - \mathscr{L}\,. \tag{3.14}$$

The canonically conjugated momentum π of a real scalar field is

$$\pi = \frac{\partial\mathscr{L}}{\partial\dot\phi} = \dot\phi\,. \tag{3.15}$$

Thus the Hamilton density is

$$\mathscr{H} = \pi\dot{\phi} - \mathscr{L} = \pi^2 - \mathscr{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \geq 0 \tag{3.16}$$

and thus obviously positive definite. Moreover, generating fluctuations $\delta\phi$ costs energy and thus the system is stable against small perturbations. Hence the transition from a single particle interpretation of the Klein-Gordon equation to a field theoretic interpretation has been sufficient to cure the problem of the negative energy solutions. Note also that we could subtract a constant ρ_0 from the Lagrangian which would drop out of the equation of motions. From Eq. (3.16) we see that such a constant corresponds to a uniform energy density in space. Such a term would act as an additional source term in the gravitational field equations, but would be otherwise unobservable.

Next we generalise the Lagrangian by subtracting a polynomial in the fields, $V(\phi)$, subject to the stability constraint discussed above. Hence the potential should be bounded from below, and we can expand it around its minimum at $\phi \equiv v$,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\phi=v} = 0\,, \qquad \left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\right|_{\phi=v} \equiv m^2 > 0\,. \tag{3.17}$$

The term $V''(v)$ acts as mass term the field ϕ. We will see soon that terms ϕ^n with $n \geq 3$ generate interactions between n particles, as expected from the analogy of a quantum field to coupled quantum mechanical oscillators. The field ϕ has the non-zero value $\phi = v$ everywhere, if the minimum v of $V(\phi)$ is not at zero, $v \neq 0$. If the value of $V(\phi)$ at the minimum v is not zero, $V(v) \neq 0$, then the non-zero potential implies a non-zero uniform energy density $\rho = -V(v)$.

Generating functional Now we move on to the quantum theory of a scalar field, which we define by the path-integral over $\exp \mathrm{i}S[\phi]$. The Green functions which encode all information about this theory can be obtained from the generating functional

$$Z[J] = \langle 0+|0-\rangle_J = \mathcal{N}\int \mathcal{D}\phi \exp \mathrm{i}\int \mathrm{d}^4x\, \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi\right)\,, \tag{3.18}$$

where we appended to the action a linear coupling between the field and an external source. To ensure the convergence of the integral, we add an infinitesimal small imaginary part to the squared mass of the particle, $m^2 \to m^2 - \mathrm{i}\varepsilon$. Next we perform an integration by part of the first term, exploiting the fact that the boundary term vanishes,

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi \exp \mathrm{i}\int \mathrm{d}^4x\, \left(-\frac{1}{2}\phi(\Box + m^2)\phi + J\phi\right)\,. \tag{3.19}$$

The first two terms, $A = -(\Box + m^2)$, are quadratic and symmetric in the field ϕ,

$$-\frac{1}{2}\int \mathrm{d}^4x\, \phi(x)(\Box_x + m^2)\phi(x) = \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\, \phi(x)A(x,x')\delta(x-x')\phi(x')\,. \tag{3.20}$$

Note that the operator A is local, $A(x) \propto A(x,x')\delta(x-x')$: Since special relativity forbids action at a distance, non-local terms like $\phi(x')A(x,x')\phi(x)$ should not appear in a relativistic Lagrangian.

If we discretise continuous space-time x^μ into a lattice, we can use Eq. (2.22) to perform the path integral,

$$Z[J] = \mathcal{N}\left(\frac{(2\pi\mathrm{i})^N}{\det[A]}\right)^{1/2} \exp\left(-\frac{1}{2}\,\mathrm{i}\,JA^{-1}J\right) \equiv \mathcal{N}Z[0]\exp(\mathrm{i}W[J])\,. \tag{3.21}$$

The pre-factor of the exponential function does not depend on J and is thus given by $\mathcal{N}Z[0] = \langle 0+|0-\rangle$. The vacuum should be stable and normalised to one in the absence of sources, $\langle 0+|0-\rangle = 1$. Therefore the proper normalisation of $Z[J]$ implies that $\mathcal{N}^{-1} = Z[0]$. Thus we can omit the normalisation factor, if we normalise the path integral measure $\mathcal{D}\phi$ such that the Gaussian integral over a free field is one.

In the last step of Eq. (3.21), we defined a new functional $W[J]$ that depends only quadratically on the source J; therefore it should be easier to handle than $Z[J]$. Going for $N \to \infty$ back to continuous space-time, the matrix multiplications become integrations,

$$Z[J] = \exp(\mathrm{i}W[J]) = \exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4x' J(x)A^{-1}(x,x')J(x')\right) \tag{3.22}$$

and

$$W[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x' J(x)A^{-1}(x,x')J(x')\,. \tag{3.23}$$

Propagator In order to evaluate the functional $W[J]$ we have to find the inverse $\Delta(x,x') \equiv A^{-1}(x,x')$ of the differential operator A, defined by

$$-(\Box + m^2)\Delta(x,x') = \delta(x-x')\,. \tag{3.24}$$

Because of translation invariance, the Green function $\Delta(x,x')$ can depend only on the difference $x-x'$. Therefore it is advantageous to perform a Fourier transformation and to go to momentum space,

$$-\int \frac{\mathrm{d}^4k}{(2\pi)^4}(\Box + m^2)\Delta(k)\mathrm{e}^{-\mathrm{i}k(x-x')} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k(x-x')}\,, \tag{3.25}$$

or

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon}\,, \tag{3.26}$$

where the pole at $k^2 = m^2$ is avoided by the $\mathrm{i}\varepsilon$. Thus the $m^2 \to m^2 - \mathrm{i}\varepsilon$ prescription introduced to ensure the convergence of the path integral tells us also how to handle the poles of the Green function. The index F specifies that the propagator Δ_F is the Green function obtained with the $m^2 - \mathrm{i}\varepsilon$ prescription proposed by Feynman. (Some authors use instead D_F for the propagator of massive bosons and Δ_F for the propagator of massless bosons.)

Note that the four momentum components k^μ are independent. Therefore $\Delta_F(k)$ describes the propagation of a *virtual* particle that has—in contrast to a real or external particle—not to be on "mass-shell:" in general

$$k_0 \neq \pm\omega_k \equiv \pm\sqrt{\boldsymbol{k}^2 + m^2}\,.$$

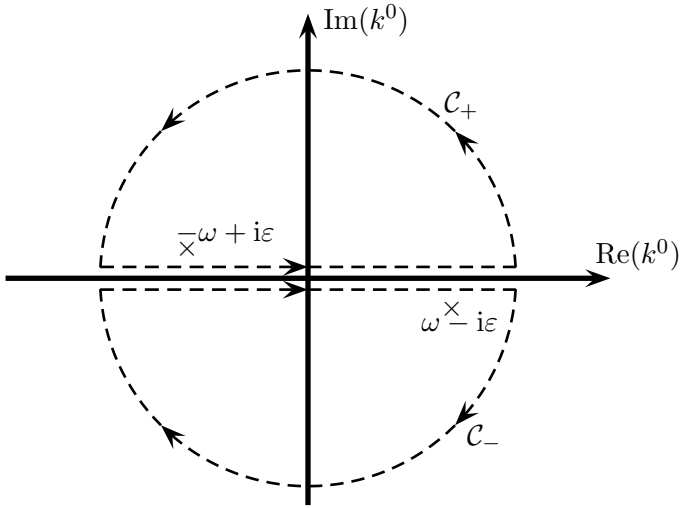


Figure 3.1.: Poles and contours in the complex k^0 plane used for the integration of the Feynman propagator.

We can evaluate the k_0 integral in the coordinate representation of $\Delta_F(x-x')$ explicitly,

$$\Delta_F(x-x') = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-\mathrm{i}k(x-x')}}{k_0^2 - \boldsymbol{k}^2 - m^2 + \mathrm{i}\varepsilon} \tag{3.27}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}k_0(t-t')}\mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}}{(k_0 - \omega_k + \mathrm{i}\varepsilon)(k_0 + \omega_k - \mathrm{i}\varepsilon)}\,, \tag{3.28}$$

using Cauchy's theorem[1]. The integrand has two simple poles at $+\omega_k - \mathrm{i}\varepsilon$ and $-\omega_k + \mathrm{i}\varepsilon$, cf. Fig. 3.1. For negative $\tau = t - t'$, we can close the integration contour $\mathcal{C}_+$ on the upper half-plane, including the pole at $-\omega_k$,

$$\int \mathrm{d}k_0 \frac{\mathrm{e}^{-\mathrm{i}k_0\tau}}{(k_0 - \omega_k + \mathrm{i}\varepsilon)(k_0 + \omega_k - \mathrm{i}\varepsilon)} = 2\pi\mathrm{i}\ \mathrm{res}_{-\omega_k} = 2\pi\mathrm{i}\ \frac{\mathrm{e}^{\mathrm{i}\omega_k\tau}}{-2\omega_k} \qquad \text{for} \quad \tau < 0\,. \tag{3.29}$$

For positive τ, we have to choose the contour $\mathcal{C}_-$ in the lower plane, picking up $2\pi\mathrm{i}\ \mathrm{e}^{-\mathrm{i}\omega_k\tau}/(2\omega_k)$ and an additional minus sign since the contour is clockwise. Combining both results, we obtain

$$\mathrm{i}\Delta_F(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \left[\mathrm{e}^{-\mathrm{i}\omega_k t}\vartheta(x^0) + \mathrm{e}^{\mathrm{i}\omega_k t}\vartheta(-x^0)\right] \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\,, \tag{3.30}$$

or after shifting the integration variable $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term,

$$\mathrm{i}\Delta_F(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \left[\mathrm{e}^{-\mathrm{i}(\omega_k t - \boldsymbol{k}\boldsymbol{x})}\vartheta(x^0) + \mathrm{e}^{\mathrm{i}(\omega_k t - \boldsymbol{k}\boldsymbol{x})}\vartheta(-x^0)\right]\,. \tag{3.31}$$

Comparing this expression to our guess (2.79) at the end of the last chapter, we see that our intuitive arguments about the structure of a Lorentz invariant propagator in a quantum theory

[1] Since ε is infinitesimal and $\omega_k > 0$, we can set $2\mathrm{i}\omega_k\varepsilon + \varepsilon^2 \to \mathrm{i}\varepsilon$.

were correct. We stress once again the salient features of the Feynman propagator: First, the propagator contains positive and negative frequencies, as expected from the existence of solutions to the Klein-Gordon equation with positive and negative energies. Second, positive frequencies propagate forward in time, while negative frequencies propagate backward. This implies the existence of anti-particles. Third, the relativistic normalisation of (on-shell) plane waves includes a factor $1/\sqrt{2\omega_k}$, or

$$\langle k|k'\rangle = 2\omega_k(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k'})\,, \tag{3.32}$$

while the non-relativistic normalisation uses $\langle \boldsymbol{k}|\boldsymbol{k'}\rangle = \delta(\boldsymbol{k}-\boldsymbol{k'})$.

Remark 3.1: Other Green functions are obtained, if we choose different prescriptions for the handling of the poles. For positive $\tau = t - t'$, we have to close the circle on the lower half-plane. Shifting both poles to the lower half-plane, $\pm\omega_k - \mathrm{i}\varepsilon$, gives thus the retarded propagator $\Delta_{\mathrm{ret}}(x)$ vanishing for all $\tau < 0$. In the opposite case, we shift both poles to the upper half-plane, $\pm\omega_k + \mathrm{i}\varepsilon$, and obtain the advanced propagator $\Delta_{\mathrm{adv}}(x)$. Both propagators are real-valued, propagating a real solution of the wave equation into another real one at a different time, as required in classical physics. Moreover, both Green functions have support inside the light-cone, the retarded in the forward and the advanced in the backward part of the light cone. This behaviour should be contrasted with the Feynman propagator Δ_F which is complex-valued and non-zero in $\mathbb{R}(1,3)$.
Another way to handle the singularities is to use Cauchy's Principal value prescription, obtaining $\Delta(x) = \frac{1}{2}[\Delta_{\mathrm{adv}}(x) + \Delta_{\mathrm{ret}}(x)]$. This choice corresponds to an action-at-distance which seems to have no relevance in physics. Finally, we can shift one pole up and the other one down. The choice $\pm(\omega_k - \mathrm{i}\varepsilon)$ used in the Feynman propagator allows us to rotate the integration contour anti-clockwise to $-\mathrm{i}\infty : +\mathrm{i}\infty$ avoiding both poles in the complex k_0 plane. Since $k_0 = \mathrm{i}\partial_t$, this transformation is consistent with the clockwise rotation in coordinate space required to obtain an Euclidean action bounded from below. Thus the Feynman prescription is the only one in which the physics in Minkowski and Euclidean space are analytically connected.

We are now in the position to evaluate the generating functional

$$W[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x' J(x)\Delta_F(x-x')J(x')\,. \tag{3.33}$$

in Fourier space. Inserting the Fourier transformations for the propagator as well as for the sources J gives

$$W[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x' \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{\mathrm{d}^4\tilde{k}}{(2\pi)^4} J(k)^*\mathrm{e}^{\mathrm{i}kx}\,\frac{\mathrm{e}^{-\mathrm{i}\tilde{k}(x-x')}}{\tilde{k}^2 - m^2 + \mathrm{i}\varepsilon}\,J(k')\mathrm{e}^{-\mathrm{i}k'x'}\,. \tag{3.34}$$

Exchanging the integration order and performing the space-time integrations leads to the conservation of the four-momenta entering and leaving the two interaction points, $(2\pi)^8\delta(k-\tilde{k})\delta(\tilde{k}-k')$: The source $J(k)^*$ produce a scalar particle with momentum k, and thus only the Fourier component k of the scalar propagator contributes. This is a very general behaviour, based solely on the translation invariance of the free particle states we are using. In the final step, we cancel two of the three momentum integrations with the two momentum delta functions and are left with

$$W[J] = -\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} J(k)^*\,\frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon}\,J(k)\,. \tag{3.35}$$

The closer the particles emitted by the source $J(k)$ are on-shell, $k^2 \to m^2$, the larger is their contribution to $W[J]$. For $k^2 = m^2$, the propagator diverges finally. Reason for this unphysical result is that our formalism assumes that the exchanged particle is stable, and a real particle can thus travel an infinite distance. If we would take the finite life-time $\tau_{1/2}$ of the exchanged particle into account, the infinitesimal $\mathrm{i}\varepsilon$ would be replaced by a finite quantity determined by $\tau_{1/2}$.

The functional $W[J]$ has the same structure as the one for the harmonic oscillator found in the last chapter. We will see that it contains, as in one-dimensional case, all information about a free scalar field, not only about its ground state.

Attractive Yukawa potential by scalar exchange From our macroscopic experience, we know the two cases of electromagnetism, where equal electric charges repel each other, and of gravity where two masses attract each other. The first physics question we want to answer with our newly developed formalism is if the scalar field falls into the category of a fundamentally attractive or repulsive interaction.

In order to address this question, we consider two static point charges as external sources, $J = J_1(x_1) + J_2(x_2)$ with $J_i = \delta(\boldsymbol{x} - \boldsymbol{x}_i)$, in $W[J]$. Multiplying out the terms in $J(x)\Delta_F(x - x')J(x')$ gives four contributions, $W_{ij} \propto J_i J_j$: The terms $W_{11}[J]$ and $W_{22}[J]$ correspond to the emission and re-absorption of the particle by the same source J_i. They are examples for self-interactions that we neglect for the moment. The interaction between two different charges is given by

$$W_{12}[J] = W_{21}[J] = -\frac{1}{2}\int \mathrm{d}^4x \mathrm{d}^4x' \int \frac{\mathrm{d}^4k}{(2\pi)^4} J_1(x)\, \frac{\mathrm{e}^{-\mathrm{i}k(x-x')}}{k^2 - m^2 + \mathrm{i}\varepsilon}\, J_2(x') \tag{3.36}$$

$$= -\frac{1}{2}\int \mathrm{d}t\mathrm{d}t' \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}k^0(t-t')}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{x}_1 - \boldsymbol{x}_2)}}{k^2 - m^2 + \mathrm{i}\varepsilon}\,. \tag{3.37}$$

Performing one of the two time integrals, e.g. the one over t', gives $2\pi\delta(k_0)$. Hence our assumption of static sources implies that the virtual particle carries zero energy and is space-like, $k^2 = -\boldsymbol{k}^2 < 0$. Eliminating then the k_0 integral with the help of the delta function, we obtain next

$$W_{12}[J] = \frac{1}{2}\int \mathrm{d}t \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}}{\boldsymbol{k}^2 + m^2} \tag{3.38}$$

with $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. The denominator is always positive, and we can therefore omit the $\mathrm{i}\varepsilon$. Before we can go on, we have to make sense out of the infinite time integral: Looking at

$$Z[J] = \langle 0|\exp(-\mathrm{i}H[J]\tau)|0\rangle = \exp(\mathrm{i}W[J])\,, \tag{3.39}$$

we see that $W[J] = -E\tau$ with$\tau = t - t'$ as the considered time interval. Hence the potential energy V of two static point charges separated by the distance $\boldsymbol{r}$ is

$$V = -(W_{12} + W_{21})/\tau = -\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}}{\boldsymbol{k}^2 + m^2} = -\frac{\mathrm{e}^{-mr}}{4\pi r} < 0\,. \tag{3.40}$$

Thus the potential energy V of two equal charges is reduced by the exchange of a scalar particle, which means that the scalar force between them is attractive. If the exchanged particle is massive, the range of the force is of order $1/m$. These two observations were the basic motivation for Yukawa to suggest in 1935 the exchange of scalar particles as model

for the nuclear force. Note also that we obtain in the limit $m \to 0$ a $1/r$ potential as in Newton's and Coulomb's law. Thus we learnt the important fact that the only two known forces of infinite range, the electromagnetic and the gravitational force, are transmitted by massless particles, the photon and the graviton, respectively. The result $V \propto 1/r$ for $m = 0$ and $n = 4$ space-time dimensions, or more generally $V \propto 1/r^{n-3}$ for $n \geq 4$, follows from simple dimensional analysis: For $m = 0$, the only remaining dimensionfull parameter after the integration over $\boldsymbol{k}$ is the distance r. From Eq. (3.40), we read off that the potential energy V has the dimension $[V] = k^{n-3}$. Thus the potential energy scales for $m = 0$ as r^{-n+3}.

Finally, note that the amplitude $W_{12} + W_{21}$ or $J(x)\Delta_F(x-x')J(x') = J(x')\Delta_F(x'-x)J(x)$ is symmetric against the exchange $1 \leftrightarrow 2$ or $x_1 \leftrightarrow x_2$, reflecting that the scalar propagator is an even function. Thus scalar particles are bosons and follow Bose-Einstein statistics.

3.3. Green functions for a free scalar field

In the last section, we obtained the scalar Feynman propagator or two-point Green function as the inverse of the Klein-Gordon operator. As next step, we want to derive n-point Green functions from the their generating functional. Moreover, we will introduce two types of Green functions, namely disconnected n-point functions $\mathcal{G}(x_1, \ldots, x_n)$ and connected n-point functions $G(x_1, \ldots, x_n)$,

Consider the expansion of the exponential in Eq. (3.21),

$$Z[J] = \exp(\mathrm{i}W[J]) = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} W^n = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \mathcal{G}_n(x_1, \cdots, x_n) J(x_1) \cdots J(x_n)\,, \tag{3.41}$$

where we assume that $Z[J]$ is normalised so that $Z[0] = 1$. The RHS serves as definition of the disconnected n-point Greens function $\mathcal{G}(x_1, \cdots, x_n)$. They can be calculated as the functional derivatives of $Z[J]$,

$$\mathcal{G}(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} Z[J] \bigg|_{J=0}\,. \tag{3.42}$$

For $n = 2$ we should rederive the Feynman propagator. Starting from

$$\begin{aligned} \frac{1}{\mathrm{i}} \frac{\delta Z[J]}{\delta J(x)} &= \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} \exp\left(-\frac{\mathrm{i}}{2} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 J(x_1) \Delta_F(x_1 - x_2) J(x_2) \right) \\ &= -\int \mathrm{d}^4x_1 \Delta_F(x - x_1) J(x_1) \exp(\mathrm{i}W[J])\,, \end{aligned} \tag{3.43}$$

we obtain

$$\begin{aligned} &\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(y)} \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} Z[J] = \mathrm{i}\Delta_F(x-y) \exp(\mathrm{i}W[J]) \\ &+ \left(\int \mathrm{d}^4x_1 \Delta_F(x - x_1) J(x_1) \right) \left(\int \mathrm{d}^4x_1 \Delta_F(y - x_1) J(x_1) \right) \exp(\mathrm{i}W[J])\,. \end{aligned} \tag{3.44}$$

Setting $J = 0$ gives the desired result for the 2-point function,

$$\mathcal{G}(x, y) = \mathrm{i}\Delta_F(x - y)\,. \tag{3.45}$$

It is straightforward to continue: Another functional derivative gives the 3-point function,

$$\begin{aligned}
&\frac{\delta}{\mathrm{i}\delta J(x_1)}\frac{\delta}{\mathrm{i}\delta J(x_2)}\frac{\delta}{\mathrm{i}\delta J(x_3)}Z[J] = \\
&-\left(\int \mathrm{d}^4x\Delta_F(x_1-x)J(x)\right)\left(\int \mathrm{d}^4x\Delta_F(x_2-x)J(x)\right)\left(\int \mathrm{d}^4x\Delta_F(x_3-x)J(x)\right)\exp(\mathrm{i}W[J]) \\
&-\mathrm{i}\Delta_F(x_2-x_3)\int \mathrm{d}^4x\Delta_F(x_1-x)J(x)\exp(\mathrm{i}W[J]) \\
&-\mathrm{i}\Delta_F(x_2-x_1)\int \mathrm{d}^4x\Delta_F(x_3-x)J(x)\exp(\mathrm{i}W[J]) \\
&-\mathrm{i}\Delta_F(x_3-x_1)\int \mathrm{d}^4x\Delta_F(x_2-x)J(x)\exp(\mathrm{i}W[J])\,. \qquad (3.46)
\end{aligned}$$

For n odd, we obtain always a source J in the pre-factor because $W[J]$ is an even polynomial in J. Hence all odd n-point functions are zero. We continue with the 4-point function: After taking another derivative and setting $J=0$, only terms linear in J of (3.46) contribute and thus

$$\begin{aligned}
\mathcal{G}(x_1,x_2,x_3,x_4) = &-[\Delta_F(x_1-x_2)\Delta_F(x_3-x_4) \\
&+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\
&+\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)]\,. \qquad (3.47)
\end{aligned}$$

We see that the 4-point function is the sum of all permutations of products of two 2-point functions. For instance, the first term $\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)$ in the 4-point function describes the independent propagation of a scalar particle from x_1 to x_2 and of another one from x_3 to x_4. Thus our approach leads indeed to a many-particle theory. Since we did not include interactions, particles are propagating independently and the n-point function factorises into products of two-point functions. Thus the functional $Z[J]$ generates disconnected Green functions. The statement that the n-point function is the sum of all permutations of the product of the $n/2$ two-point functions holds for all n and is called "Wick's theorem".

We introduce next the connected n-point functions $G(x_1,\ldots,x_n)$. Their generating functional is $W[J]$,

$$G(x_1,\ldots,x_n) = \frac{1}{\mathrm{i}^n}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\mathrm{i}W[J]\bigg|_{J=0}\,. \qquad (3.48)$$

For a free theory, W is quadratic in the sources J. Hence, all connected n-point functions $G(x_1,\ldots,x_n)$ with $n>2$ vanish and the only non-zero one is the two-point function with

$$G(x,y) = \mathrm{i}\Delta_F(x-y) = \mathcal{G}(x,y)\,. \qquad (3.49)$$

To summarise: There exists only one non-zero connected n-point function in a free theory which is determined by the Feynman propagator, $G(x,y) = \mathrm{i}\Delta_F(x-y)$. All non-zero disconnected n-point functions can be obtained by permuting the product of $n/2$ two-point functions ("Wick's theorem"). Hence any higher-order Green function can be constructed out of a single building block, the Feynman propagator.

In perturbation theory, we will recast the interacting theory – loosely speaking – in "interaction vertices times free propagators". This enables us to derive simple Feynman rules that tell us how one constructs an arbitrary Green function out of vertices and propagators.

Causality and the Feynman propagator I promised in section 2.5 that a relativistic field theory automatically implements the requirement of causality: No signal using ϕ particles as carrier should travel with a speed larger than the one of light. We saw already that the Feynman prescription leads to a relativistic consistent interpretation of the propagator, although the propagator does not vanish outside the light-cone but goes only exponentially to zero, cf. problem 2.8. One may therefore wonder, if this means that the uncertainty principle makes the light-cone "fuzzy" and thus the axiom of special relativity that no signal can be transmitted with $v > c$ is violated on scales smaller $\lesssim 1/m$.

We can address this question considering the field $\phi(x)$ as operator $\hat{\phi}(x)$ and asking then when a measurement of $\hat{\phi}(x)$ influences $\hat{\phi}(x')$. Recall first that the Feynman propagator equals the 2-point Green function which in turn corresponds to the vacuum expectation value of the time-ordered product of field operators,

$$\mathcal{G}(x_1, x_2) = \langle 0|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|0\rangle = \mathrm{i}\Delta_F(x_1 - x_2)\,. \tag{3.50}$$

The property $\Delta_F(x_1 - x_2) = \Delta_F(x_2 - x_1)$ implies that the field operators $\hat{\phi}(x_1)$ and $\hat{\phi}(x_2)$ commute,

$$\begin{aligned}\langle 0|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|0\rangle &= \langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle\vartheta(t_1 - t_2) \\ &\quad + \langle 0|\hat{\phi}(x_2)\hat{\phi}(x_1)|0\rangle\vartheta(t_2 - t_1)\,.\end{aligned} \tag{3.51}$$

Using the analogy of a free quantum field to an infinite set of oscillators, we try to express the field operator $\hat{\phi}(x)$ through annihilation and creation operators. Comparing to the expansion (2.62) of an oscillator in $d = 1$, e.g. to $\phi(t) = (2\omega)^{-1/2}(a\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger \mathrm{e}^{\mathrm{i}\omega t})$, suggests the ansatz

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\boldsymbol{k})\mathrm{e}^{+\mathrm{i}kx}\right]\,, \tag{3.52}$$

with $k^0 = \omega_k$, $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$ as annihilation and creation operators that satisfy

$$a(\boldsymbol{k})\,|0\rangle = 0\,, \qquad a^\dagger(\boldsymbol{k})\,|0\rangle = |\boldsymbol{k}\rangle \qquad \text{and} \qquad [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}')\,. \tag{3.53}$$

Hence the vacuum state $|0\rangle$ is defined by $a(\boldsymbol{k})\,|0\rangle = 0$ for all $\boldsymbol{k}$.

If this ansatz is correct, then we should be able to reproduce the known form of the Feynman propagator: Inserting our ansatz for the field into $\langle 0|\hat{\phi}(\boldsymbol{x}, t)\hat{\phi}(0)|0\rangle$ for $t > 0$, we obtain four terms containing the products aa, $aa^\dagger$, $a^\dagger a$, $a^\dagger a^\dagger$. Only $aa^\dagger$ survives, resulting into

$$\langle 0|\int \frac{\mathrm{d}^3k\,\mathrm{d}^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}} a(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}kx}a^\dagger(\boldsymbol{k}')\vartheta(t)|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\mathrm{e}^{-\mathrm{i}kx}\vartheta(t)\,. \tag{3.54}$$

In the second step, we used the commutation rule $[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}')$. Performing the same exercise for $t < 0$, we see that we reproduce also in this case the corresponding term of the Feynman propagator. Thus we conclude that our ansatz for the field and commutation rules for the annihilation and creation operators are consistent.

Note that we could create in (3.54) alternatively one-particle states, $\langle 0|a(\boldsymbol{k})a^\dagger(\boldsymbol{k}')|0\rangle = \langle \boldsymbol{k}|\boldsymbol{k}'\rangle$, and consistency requires thus that the states $|\boldsymbol{k}\rangle$ are non-relativistically normalised, $\langle \boldsymbol{k}|\boldsymbol{k}'\rangle = \delta(\boldsymbol{k} - \boldsymbol{k}')$. This should come as no surprise, since we started from the analogy to the non-relativistic oscillator. If one prefers states satisfying the relativistic normalisation, one

can rescale the creation and annihilation operators such that $[\tilde{a}(k), \tilde{a}^\dagger(k')] = (2\pi)^3 2\omega_k \delta(\boldsymbol{k}-\boldsymbol{k}')$ and the field-operator becomes

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \left[\tilde{a}(k)\mathrm{e}^{-\mathrm{i}kx} + \tilde{a}^\dagger(k)\mathrm{e}^{+\mathrm{i}kx}\right] . \tag{3.55}$$

Both normalisations lead to canonical commutation relations between the field $\hat{\phi}$ and its canonically conjugated momentum density $\hat{\pi} = \dot{\hat{\phi}}$ at equal times,

$$[\hat{\phi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{x}', t)] = \mathrm{i}\delta(\boldsymbol{x} - \boldsymbol{x}') . \tag{3.56a}$$

$$[\hat{\phi}(\boldsymbol{x}, t), \hat{\phi}(\boldsymbol{x}', t)] = [\hat{\pi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{x}', t)] = 0 . \tag{3.56b}$$

We come back to the question if the commutator of two fields vanishes for space-like separation. We evaluate first

$$\begin{aligned}[\hat{\phi}(x), \hat{\phi}(x')] &= \int \frac{\mathrm{d}^3k\,\mathrm{d}^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}} \left[a(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\boldsymbol{k})\mathrm{e}^{+\mathrm{i}kx}, a(\boldsymbol{k}')\mathrm{e}^{-\mathrm{i}k'x'} + a^\dagger(\boldsymbol{k}')\mathrm{e}^{+\mathrm{i}k'x'}\right] = \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \left(\mathrm{e}^{-\mathrm{i}k(x-x')} - \mathrm{e}^{+\mathrm{i}k(x-x')}\right) \equiv D(x - x') . \end{aligned} \tag{3.57}$$

For equal times, $t = t'$, exchanging the dummy variable $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term shows that the contribution from positive and negative energies cancel. Thus the equal-time commutator of two fields is zero, as claimed in (3.56b).

For space-like distances, $(x - x')^2 < 0$, we can find a Lorentz boost which changes the ordering of the space-time events, $x - x' \to -(x - x')$. Since the function $D(x - x')$ is the sum of two Lorentz invariant expressions, its value has to be the same in all inertial frames. But for space-like distances, we can transform $D(x)$ into $-D(x)$, and therefore $D(x)$ has to vanish, if x is outside the light-cone of x' and vice versa. Thus we have shown that also the commutator of two space-like separated fields vanishes,

$$[\hat{\phi}(x), \hat{\phi}(x')] = 0 \qquad \text{for} \quad (x - x')^2 < 0 , \tag{3.58}$$

which is the condition for causality: The trasmission of a signal corresponds not only to the propagation of a virtual particle, but includes its measurement. Thus the fact that the Feynman propagator does not vanish outside the light-cone does not contradict causality by itself.

There are two main differences between the Feynman propagator and the commutator of two fields: First, $[\hat{\phi}(x), \hat{\phi}(x')]$ is an operator, while $\mathrm{i}\Delta(x_1 - x_2)$ is a vacuum expectation value. The quantum vacuum fluctuates, and these fluctuations are correlated also on space-like distances, similar to the ERP correlations in quantum mechanics. The Feynman propagator $\mathrm{i}\Delta_F(x_1 - x_2)$ is designed to describe not only the propagation of time-like particles, but includes also the space-like propagation of virtual particles: The most "extreme" case is the *instantaneous* exchange of particles transmitting the Coulomb or Yukawa force between static sources, cf. Eq. (3.38). Second, in $[\hat{\phi}(x), \hat{\phi}(x')]$ we subtract the contribution of positive and negative frequencies, while we add them in the Feynman propagator. As a result, the contributions from a particle travelling the distance $\boldsymbol{x}$ and from an anti-particle travelling the distance $-\boldsymbol{x}$ cancel in the commutator, while they add up in the Feynman propagator. Since causality relies on the cancellation between positive and negative energy modes in $[\hat{\phi}(x), \hat{\phi}(x')]$, we conclude that a relativistic quantum theory has to incorporate anti-particles.

3.4. Vacuum energy and the Casimir effect

Vacuum energy We now aim at calculating the energy of the vacuum state of a free scalar quantum field. The energy density ρ of the field ϕ is given by the vacuum expectation value of its Hamiltonian density $\mathscr{H}$,

$$\rho = \langle 0|\mathscr{H}|0\rangle = \rho_0 + \frac{1}{2}\langle 0|\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2|0\rangle = \rho_0 + \rho_1\,. \tag{3.59}$$

Here we added the constant energy density ρ_0 to (3.16) and used that the vacuum is normalised, $\langle 0|0\rangle = 1$. For the calculation of ρ_1, we can recycle our result for the propagator of a scalar field by considering $\phi^2(x)$ as the limit of two fields at nearby points,

$$\langle 0|\phi(x')\phi(x)|0\rangle_{x'\searrow x} = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \mathrm{e}^{-\mathrm{i}k(x'-x)}\bigg|_{x'\searrow x} = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\,. \tag{3.60}$$

We perform first the differentiation in $\langle\pi^2\rangle = \langle\dot{\phi}^2\rangle$ and $\langle(\boldsymbol{\nabla}\phi)^2\rangle$ and send then $x' \searrow x$. Thus π^2 and $(\boldsymbol{\nabla}\phi)^2$ add a ω_k^2 and $\boldsymbol{k}^2$ term, respectively,

$$\rho = \langle 0|\mathscr{H}|0\rangle = \rho_0 + \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\left[\frac{1}{2}(\omega_k^2 + \boldsymbol{k}^2 + m^2)\right] = \rho_0 + \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2}\omega_k\,. \tag{3.61}$$

If we insert $\hbar$ and c into this expression, we see that ρ_0 as a classical contribution to the energy density of the vacuum is $\propto \hbar^0$, while the second term $\rho_1 \propto \hbar\omega_k/V$ as a quantum correction is linear in $\hbar$. The total energy density ρ of the vacuum state of a free scalar field has a very intuitive interpretation: Additionally to the classical energy density ρ_0, it sums up the zero-point energies of all individual modes $\boldsymbol{k}$ of a free field. Despite its simplicity, we cannot make sense out of this result: Since both the density of modes and their energy increases with $|\boldsymbol{k}|$, the integral diverges. This is the first example that momentum integrals in quantum field theories are often ill-defined and require some care and cure.

Let us now consider the case that the Hamiltonian (3.16) describes the physics correctly only up to the energy scale Λ, while the modes with $|\boldsymbol{k}| \gtrsim \Lambda$ do not contribute to ρ_1. Such a possibility exists e.g. in supersymmetric theories where the contributions of different particle types cancel each other above the scale Λ_{SUSY} where supersymmetry is broken. Integrating the contribution to the vacuum energy density by field modes up to the cutoff scale Λ, we find

$$\rho_1 = \int_0^\Lambda \frac{\mathrm{d}k\; k^2}{2\pi^2}\frac{1}{2}\omega_k \sim \Lambda^4 \tag{3.62}$$

in the limit $\Lambda \gg m$. Since only the total energy density ρ is observable, the unknown ρ_0 can be always chosen such that $\rho_0 + \rho_1$ agrees with observations, even if $|\rho_0|, |\rho_1| \gg |\rho|$. Nevertheless, the strong sensitivity of ρ_1 on the value of the cutoff scale Λ is puzzling for two reasons: First, cosmological observations determine the total vacuum energy density ρ_Λ to which all types of fields contribute as $\rho_\Lambda \sim (\mathrm{meV})^4$. On the other hand, accelerator experiments give no indications that a cancellation mechanism as supersymmetry works at energy below few TeV. Thus we expect naively at least $\rho_\Lambda \sim (\Lambda_{\mathrm{SUSY}})^4 \gtrsim (\mathrm{fewTeV})^4$, which is 60 orders of magnitude larger than observed, if no strong cancellation of the various contributions to ρ_Λ takes place. Second, the behaviour $\rho_1 \sim \Lambda^4$ implies that all scalar particles with mass $m \lesssim \Lambda$ contribute equally to ρ. This poses the question, if we have to know the physics at energy scales much larger than those we probe experimentally in order to make predictions using QFT. Such a

behaviour would be in contradiction of developing successfully chemistry, atomic or nuclear physics using only the experimental data and models of the corresponding relevant energy scale E. Something similar should happen in QFT too and, as we will see later, heavy particles with mas m "decouple" at energies $E \ll m$: Their effects are either suppressed by factors E/m, or are hidden in unobservable quantities like ρ_1.

Casimir effect Although we cannot calculate unambiguously the vacuum energy, we can determine the energy difference of different vacua. As a concrete example, we consider the suggestion by Casimir that the vacuum between two conducting plates is disturbed. As a result, the vacuum energy density between the plates becomes a function of their distance d. The difference of the vacuum energy density inside and outside the plates is finite and leads to a measurable force between them.

Let us consider two parallel, uncharged, perfectly conducting plates at distance d. Standing waves between them have the form $\sin(n\pi x/d)$ with discrete energies $\omega_n = n\pi/d$. The vacuum fluctuations of a photon have the same form as the one of massless scalar field, except that there is an additional factor two due to its two spin degrees of freedom. Thus the vacuum energy inside the box of volume dL_yL_z per single polarisation mode is given by

$$E = L_yL_z \sum_{n=1}^{\infty} \int \frac{\mathrm{d}k_y \mathrm{d}k_z}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{n\pi}{d}\right)^2 + k_y^2 + k_z^2}\,. \tag{3.63}$$

To simplify the calculations, we consider a $1+1$ dimensional system of two plates separated by the distance d. Then the energy density $\rho = E/d$ of a massless field per polarisation mode inside the plates is

$$\rho(d) = \frac{\pi}{2d^2} \sum_{n=1}^{\infty} n\,. \tag{3.64}$$

Next we introduce a cutoff function $f(a) = \exp(-an\pi/d)$ which suppresses the high-energy modes,

$$\rho(d) \to \rho(a,d) = \frac{\pi}{2d^2} \sum_{n=1}^{\infty} n\mathrm{e}^{-an\pi/d}\,. \tag{3.65}$$

This procedure is called *regularisation*: For $a > 0$, we obtain a well-defined mathematical sum which we can manipulate following the usual rules of analysis, while we recover for $a \to 0$ the original divergent sum. We have chosen as argument of the exponential $an\pi/d$, because the physically relevant quantities are the energy levels $\omega_n = n\pi/d$ of the system. Now we can evaluate the regularised sum, rewriting it as a geometrical sum,

$$\rho(a,d) = \frac{\pi}{2d^2} \sum_{n=1}^{\infty} n\mathrm{e}^{-an\pi/d} = -\frac{1}{2d}\frac{\partial}{\partial a} \sum_{n=0}^{\infty} \mathrm{e}^{-an\pi/d} \tag{3.66}$$

$$= -\frac{1}{2d}\frac{\partial}{\partial a}\frac{1}{1-\mathrm{e}^{-a\pi/d}} = \frac{\pi}{2d^2}\frac{\mathrm{e}^{-a\pi/d}}{(1-\mathrm{e}^{-a\pi/d})^2}\,. \tag{3.67}$$

Then we use $\mathrm{e}^x(1-\mathrm{e}^{-x})^2 = 4\sinh^2(x/2)$ and expand $\rho(a,d)$ for small a in a Laurent series,

$$\rho(a,d) = \frac{\pi}{8d^2}\frac{1}{\sinh^2(a\pi/2d)} = \frac{1}{2\pi a^2} - \frac{\pi}{24d^2} + \mathcal{O}(a^2d^{-4})\,. \tag{3.68}$$

Note that we isolated thereby the divergence into a term which does not depend on the distance d of the plates. Thus the divergence cancels in the difference of the vacuum energy with and without plates,

$$\rho_{\text{Cas}}(d) \equiv \lim_{a\to 0} \left[\rho(a,d) - \rho(a, d\to\infty)\right] = -\frac{\pi}{24d^2}\,. \tag{3.69}$$

This final step in order to obtain a finite result is called *renormalisation*. One can verify that the result is not only independent of the cutoff parameter a, but also of the shape of a reasonable[2] cutoff function $f(a)$. In contrast, the a dependent terms in Eq. (3.68) may depend on the form of $f(a)$.

The quantity measured in actual experiments is the force F with which the plates attract (or repel) each other. This force is given by

$$-F = \frac{\partial E}{\partial d} = \frac{\partial(d\rho_{\text{Cas}})}{\partial d} = \frac{\pi}{24d^2}\,. \tag{3.70}$$

Thus two parallel plates attract each other. The experimentally relevant case of electromagnetic waves between two parallel plates in 3+1 dimensions can be calculated analogously. The experimental confirmation of the Casimir effect has been achieved only in the 1990s, with a precision on the 1% level.

How can we understand that the Casimir force is independent on the details of the regularisation procedure? Let us compare the impact of the two plates on modes with different wave-number $k = 2\pi/\lambda$: In a typical experimental set-up, the plates are separated by a distance of the order $d \sim 1\,\text{mm}$ and thus $k_0 \equiv 2\pi/d \sim \text{meV}$. The plates eliminate all low-energy modes with $k < k_0$ between them, while the modes with $k > k_0$ attain a discrete spectrum. However, for $k_n \gg k_0$, the spacing between the modes becomes negligible and experimentally one cannot distinguish the discrete spectrum from a continuous one. In particular, we can approximate the sum over the discrete energies by an integral and the contributions of modes with $k \gg k_0$ with and without plates cancel calculating the energy difference. Since the main contribution to the Casimir energy comes from cutting off modes with $k \lesssim 2\pi/d \sim \text{meV}$, we conclude that the Casimir energy is an IR effect. Therefore the details of the UV regularisation should not influence the result and any reasonable cutoff function that makes the mathematical manipulations (3.66-3.68) well-defined should lead to the same result.

Summary

The exchange of time-like quanta with zero energy between two static sources leads to the Yukawa potential. The corresponding force mediated by a scalar field is attractive. The Feynman propagator obtained by the $m^2 - \mathrm{i}\varepsilon$ prescription is the unique Green function which can be analytically continued to an Euclidean Green function. It propagates particles (with positive frequencies) forward in time, while anti-particles (with negative frequencies) propagate backward in time. While these two contributions add up in the scalar Feynman propagator, they cancel in the commutator of field-operators at space-like distances, as required by causality. Disconnected n-point Green functions are generated by the functional $Z[J]$, while

[2] Reasonable means that $f(a)$ is normalised, $f(0) = 1$, and that all its derivatives vanish for large a, $\lim_{a\to\infty} f^{(n)}(a) = 0$.

$\mathrm{i}W[J] = \ln Z[J]$ generates connected Green functions. Wick's theorem says that a n-point function can be obtained as the permutation of products of 2-point functions. The Casimir effect shows that the zero-point energies of quantum fields have real, measurable consequences.

Further reading

The quantisation of free fields using both canonical quantisation and the path integral approach is discussed extensively in[35].

Problems

3.1 Complex scalar field.
Derive the Lagrangian and the Hamiltonian for a complex scalar field, considering $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$ and $\phi^* = (\phi_1 - \mathrm{i}\phi_2)/\sqrt{2}$ as the dynamical variables. Find the conserved current of this complex field. [Hint: Proceed similar to the case of the Schrödinger equation.]

3.2 Maxwell Lagrangian.
Derive the Lagrangian for the photon field A_μ from the source-free Maxwell equation $\partial_\mu F^{\mu\nu} = 0$ following the steps from (3.11) to (3.12) in the scalar case. What is the meaning of the unused set of Maxwell equations?

3.3 Dimension of ϕ.
a.) Determine the mass dimension of a scalar field ϕ in d space-time dimensions. b.) For which d has $\mathscr{L}_{\mathrm{int}} = \lambda\phi^3$ ($\mathscr{L}_{\mathrm{int}} = \lambda\phi^4$) a dimensionless coupling constant?

3.4 Yukawa potential.
Follow the approach of section 2.4 to evaluate the free functional $Z[J]$.

3.5 Yukawa potential.
Show that the Yukawa potential $V(r) = \mathrm{e}^{-mr}/(4\pi r)$ is the Fourier transform of $(\boldsymbol{k}^2 + m^2)^{-1}$, cf. Eq. (3.40).

3.6 Vacuum energy.
Re-derive (3.61) expressing ϕ by annihilation and creation operators. Show that rewriting all creation operators on the left of the annihilation operators results in $\rho_1 = 0$. (This prescription is called "normal ordering".)

3.7 Canonical commutation relations.
Derive (3.56a) assuming the validity of (3.53).

3.8 Feynman propagator.
Discuss the behaviour of the scalar Green function $\Delta_F(0, r)$ for large $r = |\boldsymbol{x}|$.

3.9 Green functions.
Show that the connected and the unconnected n-point Green functions are identical for $n = 2$, while they differ in general for $n \geq 3$.

3.10 ζ function regularisation.
a.) The function $f(t) = t/(\mathrm{e}^t - 1)$ is the generating function for the Bernoulli numbers B_n, i.e. $f(t) = \frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$. Calculate the first Bernoulli numbers up to B_2. b.) Connect the $\sum_{n=1}^{\infty} n\mathrm{e}^{-an}$ to $f(t)$; split the sum into a divergent and a finite part for $a \to 0$ and compare to our old result. c.) The Riemann ζ function can be defined as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$ and then analytically continued into the complex s plane. The Bernoulli numbers are connected to the Riemann ζ function with negative odd argument as $\zeta(1-2n) = -\frac{B_{2n}}{2n}$. Find the Casimir energy using the Riemann ζ function.

3.11 Casimir effect.
Repeat the calculation of the Casimir effect for the case of 3+1 dimensions.

4. Scalar field with $\lambda\phi^4$ interaction

We know from quantum mechanics that adding an anharmonic term to an oscillator forces us to use either perturbative or numerical methods. The same happens in field theory: No analytic solution for a realistic interacting theory is at present known in $n = 4$ space-time dimensions. Therefore we develop in this chapter a perturbative method to evaluate the generating functionals $Z[J]$ and $W[J]$. We continue to work with the simplest case of a single real scalar field, and choose as interaction a $\lambda\phi^4$ term. Then the coupling constant λ is dimensionless in the for us interesting case $n = 4$. If λ is small enough, we may hope that a perturbative series expansion in the coupling provides a useful approximation scheme. As motivation, we note that a scalar field with $\lambda\phi^4$ interaction can not only model a wide range of phenomena in statistical physics but describes also the Higgs field of the SM and its self-interactions.

4.1. Perturbation theory for interacting fields

General formalism The Lagrange density $\mathscr{L}$ in the functional $Z[J]$ for the scalar field considered up to now was at most quadratic in the fields and its derivatives. On one hand, this allowed us to evaluate the path integral, while on the other hand this means that the field has no interactions: Two wave packets described by the free propagator just pass each other without interaction, as the superposition principle prescribes. As next step we add therefore an interaction term $\mathscr{L}_I$ to the free Lagrangian $\mathscr{L}_0$, i.e. we set $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I$. Then the generating functional $Z[J]$ for an interacting real scalar field ϕ becomes

$$Z[J] = \int \mathcal{D}\phi \exp \mathrm{i} \int \mathrm{d}^4x \ (\mathscr{L}_0 + \mathscr{L}_I + J\phi) \ , \tag{4.1}$$

while we denote the free functionals we considered so far from now on as $Z_0[J]$ and $W_0[J]$.

Starting from the full generating functional $Z[J]$ we can define *exact* Green functions which we denote by boldface letters: For instance, the exact 2-point function or propagator is given analogous to Eq. (3.48) by

$$\boldsymbol{G}(x_1, x_2) = \frac{1}{\mathrm{i}^2} \frac{\delta^2 \, Z[J]}{\delta J(x_1) \delta J(x_2)} \bigg|_{J=0} = \int \mathcal{D}\phi \ \phi(x_1)\phi(x_2) \mathrm{e}^{\mathrm{i} \int \mathrm{d}^4x \, [\mathscr{L}_0 + \mathscr{L}_I]} \, . \tag{4.2}$$

In general, we are not able to calculate these exact Green functions, and we will apply therefore perturbation theory. We assume that the interaction term $\mathscr{L}_I$ is a polynomial $\mathcal{P}(\phi)$ of degree ≥ 3 in the field ϕ and contains an expansion parameter λ which is small in the considered kinematic regime, $\mathscr{L}_I = \lambda\mathcal{P}(\phi)$ with $\lambda \ll 1$. This suggests to expand the interaction term,

$$\exp \mathrm{i} \int \mathrm{d}^4x \mathscr{L}_I(\phi) = 1 + \mathrm{i}\lambda \int \mathrm{d}^4x \mathcal{P}(\phi(x)) + \frac{(\mathrm{i}\lambda)^2}{2!} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathcal{P}(\phi(x_1))\mathcal{P}(\phi(x_2)) + \ldots \tag{4.3}$$

Since

$$\mathrm{i}\phi(x)\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x'(\mathscr{L}_0+J\phi)} = \frac{\delta}{\delta J(x)}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x'(\mathscr{L}_0+J\phi)}\,, \tag{4.4}$$

we can perform the replacement

$$\mathscr{L}_I(\phi(x)) \to \mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\,. \tag{4.5}$$

Then the interaction $\mathscr{L}_I$ does not depend longer on ϕ and can be pulled out of the functional integral,

$$\begin{aligned}\widetilde{Z}[J] &= \mathcal{N}\exp\mathrm{i}\int \mathrm{d}^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\int \mathcal{D}\phi\exp\mathrm{i}\int \mathrm{d}^4x\ (\mathscr{L}_0 + J\phi)\\ &= \mathcal{N}\exp\mathrm{i}\int \mathrm{d}^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)Z_0[J] = \mathcal{N}\exp\mathrm{i}\int \mathrm{d}^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\mathrm{e}^{\mathrm{i}W_0[J]}\,.\end{aligned} \tag{4.6}$$

The solution of the free functionals Z_0 and W_0 was given in Eq. (3.21) as

$$Z_0[J] = Z_0[0]\exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4x' J(x)\Delta_F(x-x')J(x')\right) = Z_0[0]\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}W_0^n\,. \tag{4.7}$$

Perturbation theory consists in a double expansion of the two exponentials in $Z[J]$: One in the coupling constant λ and one in the number of external sources J. The latter is fixed by the number of external particles in a scattering process, while the former is chosen according to the desired precision of the calculation.

Choosing the interaction term Let us recall from our discussion of the free Lagrangian the physical requirements we should impose on the Lagrangian: Each term should be a Lorentz scalar which is local in the fields. The corresponding Hamiltonian has to be bounded from below, stable against small perturbations and real. These conditions assure that the vacuum in the absence of external sources is stable.

Additional restrictions follow from a surprisingly simple argument employing dimensional analysis: Using natural units, $\hbar = c = 1$, the dimension of all physical quantities can be expressed as powers of one basic unit which we choose as mass m. Then we use that the action has dimension zero, $[S] = m^0$, and thus the Lagrangian $[\mathscr{L}] = m^4$ in four space-time dimensions. We consider next the free Lagrangian: From the kinetic term, we conclude that the dimension of a scalar field is $[\phi] = m^1$. Thus simple dimensional analysis shows that the term m^2 in front of ϕ^2 has the interpretation of a mass squared. Furthermore, we can order possible self-couplings of a scalar field according to their dimension as

$$\mathscr{L}_I = g_3 M\phi^3 + g_4\phi^4 + \frac{g_5}{M}\phi^5 + \ldots\,, \tag{4.8}$$

where the coupling constants g_i are dimensionless and we introduced the mass scale M to ensure $[\mathscr{L}] = m^4$. We call ϕ^n a dimension d operator. Similar as in the case of the Fermi constant, $G_F = \sqrt{2}g^2/(8m_W^2)$, the scale M could be connected to the exchange of a heavy particle.

Let us now estimate by dimensional analysis which energy scaling of the interaction probability we expect for the different coupling terms in $\mathscr{L}_I$. At lowest order perturbation theory

in the coupling constants, the interaction probability is $\propto |\mathscr{L}_I|^2$. Hence the interaction probability scales as $\propto (g_n/M^{n-4})^2$. Now we consider the ultra-relativistic limit, so that we can neglect the mass m of the scalar particle compared to the center-of-mass (cms) energy $\sqrt{s}$. A probability has to be dimensionless, and for $s \gg m^2$ the only remaining dimensionfull variable that can enter $\mathscr{L}_I$ is s. Thus the interaction probability has to scale as $g_n^2(s/M^2)^{n-4}$ in the limit $s \gg m^2$. Let us now distinguish the two ranges $m^2 \ll s \ll M^2$ and $s \gg M^2$. In the latter case, the interaction terms with $n > 4$ contain the large factors $(\sqrt{s}/M))^{n-4} \gg 1$ and perturbation theory becomes thus unreliable. In contrast, these terms are smaller than one below the scale M and thus suppressed relativ to the operators with dimension $d \leq 4$. In a first approach, we neglect therefore all operators with dimension $d \geq 5$. Simplifying further $\mathscr{L}_I$, we want to include only one interaction term. In this case, a ϕ^3 term would lead to an unstable vacuum. Therefore our choice for the scalar self-interaction is $\mathscr{L}_I = -\lambda\phi^4/4!$, where the factor $1/4!$ was added for later convenience. If this choice of interaction is realised in Nature for a specific particle has to be decided by experiment.

4.2. Green functions for the $\lambda\phi^4$ theory

We start now with the perturbative evaluation of Eq. (4.6) for a $\lambda\phi^4/4!$ interaction. From

$$\begin{aligned} Z[J] &= \left(1 - \frac{\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x\, \frac{\delta^4}{\mathrm{i}^4\delta J(x)^4} + \ldots\right) Z_0[J] = Z_0[J] - \frac{\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x\, \frac{\delta^4 Z_0[J]}{\delta J(x)^4} + \ldots \\ &= Z_0[J]\left(1 + \lambda z_1[J] + \lambda^2 z_2[J] + \ldots\right) \end{aligned} \tag{4.9}$$

we see that we will generate a series of the type free Green functions plus $\mathcal{O}(\lambda)$ corrections plus higher order corrections. The calculation of the first-order correction is very similar to the calculation of the free four-point function, with the difference that now the four sources sit at the same point. You should find in problem 4.1 as result

$$\begin{aligned} \left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp(\mathrm{i}W_0[J]) = \Bigg[&3(\mathrm{i}\Delta_F(0))^2 + 6\mathrm{i}\Delta_F(0)\left(\int \mathrm{d}^4y \Delta_F(x-y)J(y)\right)^2 \\ &+ \left(\int \mathrm{d}^4y \Delta_F(x-y)J(y)\right)^4\Bigg] \exp(\mathrm{i}W_0[J])\,. \end{aligned} \tag{4.10}$$

Next we introduce a graphical representation for the various terms in Eq. (4.10). Each Feynman propagator $\Delta_F(x-y)$ is represented by

$$\mathrm{i}\Delta_F(x-y) = \quad x \bullet\!\!\!\!-\!\!\!\!-\!\!\!\!\bullet\ y \tag{4.11}$$

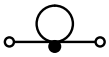
a source term $J(x)$ by

$$\mathrm{i}\int \mathrm{d}^4x\, J(x) = \quad \circ\!\!\!\!-\!\!\!\!- \tag{4.12}$$

and an interaction vertex by

$$-\,\mathrm{i}\lambda\int \mathrm{d}^4x = \bullet \tag{4.13}$$

Each source and vertex has its own coordinates and an integration over all coordinates is implied. In the case of the ϕ^4 interaction, a vertex connects four lines. Using this notation, we can express Z_1 as

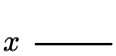
$$Z_1[J] = \frac{1}{4!}\left(3 \; \text{  } + 6 \; \text{  } + \; \text{  } \right) \exp\left(\frac{1}{2} \; \text{  } \right) . \tag{4.14}$$

A graph[1] consists of lines and dots, where the latter may be vertices or sources. We distinguish internal and external lines: A line which ends on both sides at a dot with at least two lines attached is called internal; otherwise it is an external line. The three graphs contained in $Z_1[J]$ differ by the number of loops, i.e. by the number L of closed lines. A graph with loop number $L = 0$ (as the third one in $Z_1[J]$) is called a tree graph, otherwise it is a loop graph. An inspection of the three graphs shows their loop number L is connected to the number n of lines and d of dots as $L = n - d + 1$. This formula holds in general for connected graphs, as we will see shortly. Note also that the first and the second graph contained in $Z_1[J]$ can be obtained from the third one by joining two and one lines, respectively. There are six ways to join one line, and three ways to join two lines. Thus the prefactors of the various graphs can be derived by simple symmetry arguments.

Knowing $Z_1[J]$, we can derive disconnected Green functions valid at $\mathcal{O}(\lambda)$ by performing functional derivatives,

$$\mathcal{G}^{(n)}(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} Z_0[J] \left(1 + \lambda z_1[J]\right) \Big|_{J=0} . \tag{4.15}$$

In the graphical notation, differentiating with respect to $J(x)$ amounts to replace the open dot denoting the source $\mathrm{i} \int \mathrm{d}^4 y J(y)$ by its position x,

$$\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} \; \text{  } = \; \text{  } \tag{4.16}$$

[1] This graphical notation first introduced by Stückelberg was made popular by Feynman. The graphs are therefore often called Feynman diagrams or Feynman graphs.

Vacuum diagrams We call terms in the perturbative evaluation of $Z[J]$ which contain no source vacuum diagrams. Since setting $J = 0$ eliminates all graphs containing at least one source, the vacuum diagrams correspond to loops without external lines. The corresponding Green functions are the "zero-point" Green functions $\mathcal{G}^{(0)}$.

Let us assume that the path integral measure $\mathcal{D}\phi$ is chosen such that the free vacuum is normalised, $Z_0[0] = 1$. Switching on interactions will change the free vacuum into the true vacuum of the interacting theory. As a result, the true vacuum and thus $Z[J]$ are not normalised. As example, we obtain setting $J = 0$ in our result for $Z[J]$ at lowest order perturbation theory, Eq. (4.14),

$$\mathcal{G}^{(0)} \equiv Z[0] = 1 - \frac{\mathrm{i}\lambda}{8} \int \mathrm{d}^4x (\mathrm{i}\Delta_F(0))^2 \neq 1\,. \tag{4.17}$$

Because of $\mathcal{N} = \exp\ln(\mathcal{N})$, a normalisation different from one is equivalent to adding a constant term to the Lagrangian,

$$\mathscr{L} \to \mathscr{L} + \ln(\mathcal{N})/(VT) = \mathscr{L} - \rho\,, \tag{4.18}$$

where VT is the four-dimensional integration volume in the action.

Since vacuum diagrams only change the vacuum energy density ρ but do not contribute to scattering processes, one often prefers to eliminate these diagrams multiplying $Z[J]$ with the normalisation constant

$$\mathcal{N}^{-1} = Z[0] = \int \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}S}\,. \tag{4.19}$$

Thus one uses normalised generating functional $\widetilde{Z}[J] = \mathcal{N}Z[J] = Z[J]/Z[0]$ which corresponds to a vacuum with zero energy density ρ. We now show that this normalisation eliminates all vacuum graphs. Expanding the numerator and denominator of $\widetilde{Z}[J]$ up to $\mathcal{O}(\lambda)$, we have at lowest order perturbation theory

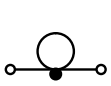
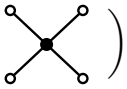
$$\widetilde{Z}[J] = \frac{Z[J]}{Z[0]} = \frac{1 + \lambda z_1[J] + \mathcal{O}(\lambda^2)}{1 + \lambda z_1[0] + \mathcal{O}(\lambda^2)}\, Z_0[J] = \{1 + \lambda(z_1[J] - z_1[0])\}\; Z_0[J] + \mathcal{O}(\lambda^2)\,. \tag{4.20}$$

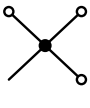
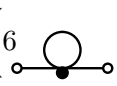
Thus dividing $Z[J]$ by the source-free functional subtracts indeed the vacuum graphs. It becomes obvious that this procedure works at any order perturbation theory, if we look at the generating functional for connected graphs, $W[J]$. As dividing $Z[J]$ by the source-free functional $Z[0]$ corresponds to

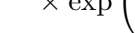
$$\mathrm{i}\widetilde{W}[J] = \ln \widetilde{Z}[J] = \ln Z[J] - \ln Z[0]\,, \tag{4.21}$$

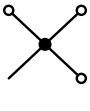
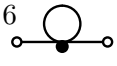
it is clear that this procedure eliminates indeed all vacuum graphs.

2-point functions We start by taking one derivative of the normalised generating functional,

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_1)}\left[1+\frac{1}{4!}\left(6\ \text{  }+\ \text{  }\right)\right]\exp\left(\frac{1}{2}\ \text{}\right)= \tag{4.22}$$

$$=\left[\frac{1}{4!}\left(6\cdot 2\ \text{}+4\ \text{  }\right)+\left(1+\frac{1}{4!}\left(6\ \text{  }+\ \text{  }\right)\right)\ \text{  }\right]$$

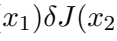
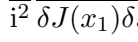
$$\times\exp\left(\frac{1}{2}\ \text{  }\right)=$$

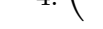
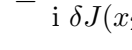
$$=\Bigg[\ \text{}+\frac{1}{4!}\Bigg(12\ \text{}+4\ \text{  }+6\ \text{  }\ \text{} \tag{4.23}$$

$$+\ \text{  }\ \text{  }\Bigg)\Bigg]\exp\left(\frac{1}{2}\ \text{  }\right).$$

Every term in this expression contains at least one source J, and the one-point function $\mathcal{G}^{(1)}(x)$ vanishes therefore. If we proceed to the two-point function $\mathcal{G}^{(2)}(x_1,x_2)$, we have to differentiate only those terms with one source,

$$\frac{1}{\mathrm{i}^2}\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\widetilde{Z}[J]= \tag{4.24}$$

$$=\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_2)}\left[\ \text{  }+\frac{1}{4!}\left(12\ \text{}+\begin{array}{c}\text{vanishing terms}\\ \text{for } J{=}0\end{array}\right)\right]\exp\left(\frac{1}{2}\ \text{  }\right) \tag{4.25}$$

$$=\left(\ \text{  }+\frac{1}{2}\ \text{  }\right)\exp\left(\frac{1}{2}\ \text{  }\right). \tag{4.26}$$

Setting then the sources J to zero, the exponential factor becomes one. Converting the graphical formula back into standard notation, we find the 2-point function $\mathcal{G}^{(2)}(x_1,x_2)$ at order $\mathcal{O}(\lambda)$ as the sum of the free 2-point function $\mathcal{G}_0^{(2)}(x_1,x_2)$ and a correction term,

$$\mathcal{G}^{(2)}(x_1,x_2)=\mathcal{G}_0^{(2)}(x_1,x_2)-\frac{\mathrm{i}\lambda}{2}\int \mathrm{d}^4x\,\mathrm{i}\Delta_F(x_1-x)\mathrm{i}\Delta_F(x-x)\mathrm{i}\Delta_F(x-x_2)\,. \tag{4.27}$$

This correction is called the self-energy $\Sigma(x_1,x_2)$ of the scalar particle. Note that the pre-factors combine to $6\times 2/4!=1/2$, so that there appears an extra factor $1/2$. Such factors are called symmetry factors. They appear because we included a factor $1/4!$ in $\mathscr{L}_I$ to compensate for the 4! permutations of four sources. If a diagram has a certain symmetry, i.e. if it can be rearranged by permutating propagators and/or vertices giving the same expression, the cancellation is only partially and a non-zero pre-factor is left over.

Example 4.1: Let us illustrate by an example how one can determine the symmetry factor of more complicated diagrams. As first step, we express the Green function that corresponds to a given Feynman diagram as the time-ordered product of fields. Consider e.g. the so-called "sunrise diagram",

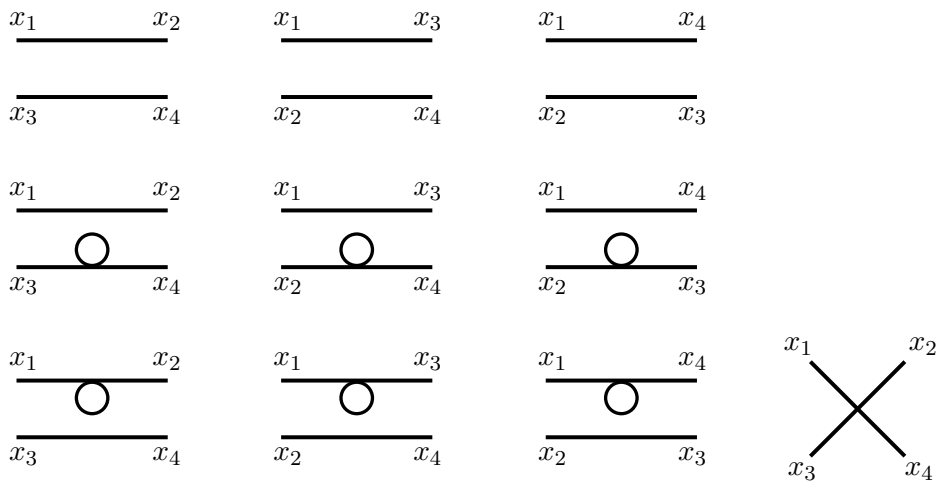
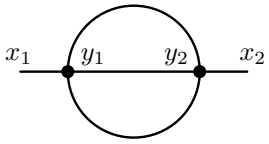


Figure 4.1.: Graphs contributing to the disconnected four-point function $\mathcal{G}(x_1, x_2, x_3, x_4)$.



which is a second order diagram, corresponding to the term

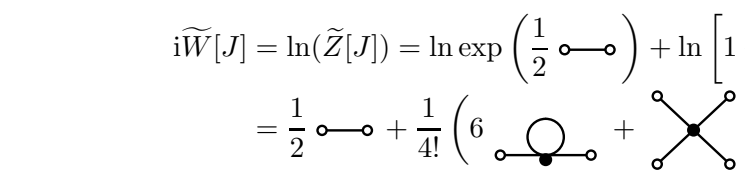
$$\frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{4!}\right)^2 \int \mathrm{d}^4y_1 \mathrm{d}^4y_2 \langle 0|T[\phi(x_1)\phi(x_2)\phi^4(y_1)\phi^4(y_2)]|0\rangle + (y_1 \leftrightarrow y_2)$$

in the perturbative expansion. The exchange graph $y_1 \leftrightarrow y_2$ is identical to the original one, canceling the factor $1/2!$ from the Taylor expansion. This cancellation takes place in general: The $1/n!$ factor from the Taylor expansion of a n.th order contains n interaction points which leads to $n!$ permutations.
Next consider how the fields ϕ are combined in $T[\cdots]$: The internal points y_1 and y_2 denote interaction points, which have four fields attached. In contrast, the external points x_1 and x_2 carry each only one field. We have to count the number of possible ways to combine the fields in the time-ordered product into the five propagators of the graph. As shorthand notation, we mark a possible combination as $\phi(x_1)\phi(y_1)$. We have four possibilities to combine $\phi(x_1)$ with $\phi(y_1)$, $\phi(x_1)\phi(y_1)$. Similarly, there are four possibilities for $\phi(x_2)\phi(y_2)$. The remaining six fields can be combined in 3! ways into pairs, as e.g. in $\phi(y_1)\phi(y_1)\phi(y_1)\phi(y_2)\phi(y_2)\phi(y_2)$. Thus the symmetry factor of this diagram is given by

$$S = \left(\frac{1}{2!} \times 2!\right)\left(\frac{1}{4!}\right)^2 (4 \times 4 \times 3!) = \frac{1}{3!}$$

4-point functions The disconnected 4-point function $\mathcal{G}(x_1, x_2, x_3, x_4)$ is shown graphically in Fig. 4.1. The first three graphs correspond to the free 4-point function from (3.47), the next six graphs are the corresponding $\mathcal{O}(\lambda)$ corrections. Finally, the last diagram corresponds to the connected 4-point function $G(x_1, x_2, x_3, x_4)$. Next we want to derive $G(x_1, x_2, x_3, x_4)$ from its generating functional $\widetilde{W}[J]$. We insert $\widetilde{Z}[J]$ into

$$\mathrm{i}\widetilde{W}[J] = \ln(\widetilde{Z}[J]) = \ln\exp\left(\frac{1}{2}\right) + \ln\left[1 + \frac{1}{4!}\left(6 \quad + \quad\right)\right] + \mathcal{O}(\lambda^2)$$
$$= \frac{1}{2} + \frac{1}{4!}\left(6 \quad + \quad\right) + \mathcal{O}(\lambda^2)\,, \tag{4.28}$$



where we expanded the logarithm, $\ln(1+x) \simeq x$. Taking four derivatives with respect to J, only the last term survives and we obtain as connected Green function

$$G(x_1, x_2, x_3, x_4) = -\mathrm{i}\lambda \int \mathrm{d}^4x\, \mathrm{i}\Delta_F(x_1 - x)\mathrm{i}\Delta_F(x_2 - x)\mathrm{i}\Delta_F(x_3 - x)\mathrm{i}\Delta_F(x_4 - x)\,. \tag{4.29}$$

Feynman rules for the $\lambda\phi^4$ theory in coordinate space We can summarise our results in few simple rules which allows us to write down Green functions directly, without the need to derive them from their generating functional. The rules refer to connected diagrams:

1. Draw all topologically different diagrams for the chosen order $\mathcal{O}(\lambda^n)$ and number of external coordinates or particles.
2. To each line connecting the points x and x' we associate a propagator $\mathrm{i}\Delta_F(x - x')$.
3. Each vertex has a factor $-\mathrm{i}\lambda$ and connects n lines for a $\lambda\phi^n$ interaction.
4. Integrate over all intermediate points.
5. Determine and add the symmetry factor.
6. The rules above give n-point Green functions. A scattering process is described by the transition amplitude $\mathrm{i}\mathcal{A}$ between a fixed initial and final state which contain real, on-shell particles. Thus the propagators of the external lines, which describe virtual particles, should be replaced by on-shell wave functions—this rule will be derived in chapter 9.2.

Feynman rules in momentum space The integration over intermediate points is trivial, because all propagators (and possible wave function on external lines) depend like $\exp(\pm\mathrm{i}kx)$ on the position of the interaction point. Hence the space integrations result in four-momentum conservation at each vertex. The Feynman rules in momentum space have thus the following changes:

2. To each line we associate a propagator $\mathrm{i}\Delta_F(k) = \mathrm{i}/(k^2 - m^2 + \mathrm{i}\varepsilon)$.
3. Fix the external momenta and impose 4-momentum conservation at each vertex.
4. Integrate over all unconstrained momenta k with $\int \mathrm{d}^4k/(2\pi)^4$. The number of independent momenta we have to integrate over equals the loop number L of the graph.

6. If the diagram should represent the transition amplitude $\mathrm{i}\mathcal{A}$ of a scattering process, propagators on the external lines should be replaced by on-shell wave functions. It is advantageous to omit the normalisation factors $N = [(2\pi)^3(2\omega_k)]^{-1/2}$ from the amplitude and to include them later into a flux (for the initial state) and phase space factor (for the final state). In this way, the amplitude $\mathrm{i}\mathcal{A}$ is Lorentz invariant and easier to manipulate. Thus for scalar particles, the Feynman rule for external particles is simply to write "1".

Finally, let us derive the loop number L of connected graphs. Each of the d dots (i.e. a source or vertex) comes with a space integration that results in a delta function expressing four-momentum conservation; $(d-1)$ of these delta functions can be used to eliminate the momentum integrations contained in each propagator, while the remaining delta function expresses the conservation of the external momenta. Thus the number of independent loop momenta in a a connected graph with n lines is $L = n - d + 1$. Expressing L via the number of vertices and sources, $d = V + S$, and the number of internal and external lines, $n = I + E$, we have $L = I + E - V - S + 1$. Since each external line comes with one source, we can express therefore the loop number also as $L = I - V + 1$.

4.3. Loop diagrams

Most of measurements in particle physics are based on scattering experiments. Therefore it might look natural to start exploring the physical consequences of the $\lambda\phi^4$ theory by calculating the scattering cross section corresponding to the 4-point Green function (4.29). According to the Feynman rules, we should replace the propagators on the external lines by "1", so that the Feynman amplitude of this tree-level process becomes simply $\mathcal{A} = -\lambda$. Thus in this case, the calculation of the cross section requires simply to multiply the number $|\mathcal{A}|^2 = \lambda^2$ with the appropriate flux and phase space factor (which we will derive in chapter 9). While the complexity of the calculation of tree-level process increases fast with the number of external particles, the resulting amplitudes are always mathematically well-behaved. In contrast, the evaluation of loop graphs may return an ill-defined result that requires regularisation and renormalisation. Aim of this section is to illustrate this procedure. We concentrate first on the technicalities involved in the evaluation of these loop diagrams, before we interpret the results. We will have time to digest these examples, before we will come back to the problem of renormalisation in Chapter 11. The basic steps in the evaluation of simple Feynman integrals are summarised in the appendix 4.A.

4.3.1. Self-energy

We consider first the only one-loop diagram contained in $Z_1[J]$, the 2-point function of a scalar particle at $\mathcal{O}(\lambda)$,

$$G^{(2)}(x_1, x_2) = \mathrm{i}\Delta_F(x_1 - x_2) - \frac{\lambda}{2}\Delta_F(0)\int \mathrm{d}^4x \Delta_F(x_1 - x)\Delta_F(x - x_2)\,. \tag{4.30}$$

The calculation of $\mathcal{G}^{(2)}(x_1, x_2)$ consists of three steps: First, we have to combine its two pieces into a single, modified propagator: As it stands, the expression seems to describes the propagation of two modes, a free one plus one consisting of the $\mathcal{O}(\lambda)$ correction, while $\mathcal{G}^{(2)}(x_1, x_2)$ should describe the propagation of a single particle with properties modified by the self-interactions. Second, we have to calculate the loop $\mathrm{i}\Delta_F(0)$ which will turn out to be infinite. Thus the final task is the question how we should interpret this result.

We start concentrating on the correction term, the self-energy $\Sigma(x_1, x_2)$ of the scalar particle, and insert the Fourier representation of the two propagators into the integral,

$$\Sigma(x_1, x_2) = -\frac{\lambda}{2}\Delta_F(0)\int \mathrm{d}^4x \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4p'}{(2\pi)^4}\,\frac{\mathrm{e}^{-\mathrm{i}p(x_1-x)}}{p^2 - m^2 + \mathrm{i}\varepsilon}\,\frac{\mathrm{e}^{-\mathrm{i}p'(x-x_2)}}{p'^2 - m^2 + \mathrm{i}\varepsilon}\,. \tag{4.31}$$

The d^4x integration results in $(2\pi)^4\delta(p-p')$, then one of the momentum integrations can be performed. Together this gives

$$\Sigma(x_1-x_2) = -\frac{\lambda}{2}\Delta_F(0)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p(x_1-x_2)}}{(p^2-m^2+\mathrm{i}\varepsilon)^2}\,. \tag{4.32}$$

Inserting also for the free Green function its Fourier representation, we arrive at

$$G^{(2)}(x_1-x_2) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p(x_1-x_2)}\left[\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}-\frac{\frac{\lambda}{2}\Delta_F(0)}{(p^2-m^2+\mathrm{i}\varepsilon)^2}\right]. \tag{4.33}$$

The Green function $G^{(2)}(p)$ in momentum space is thus given by the expression the square bracket, which we could have written down immediately using the Feynman rules in momentum space. Next we factor out one propagator,

$$G^{(2)}(p) = \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\left[1+\frac{\frac{\mathrm{i}\lambda}{2}\Delta_F(0)}{p^2-m^2+\mathrm{i}\varepsilon}\right]. \tag{4.34}$$

Assuming that perturbation theory is justified, the second term in the parenthesis should be small. Thus $[1+\lambda a] = [1-\lambda a]^{-1}+\mathcal{O}(\lambda^2)$ and

$$G^{(2)}(p) = \frac{\mathrm{i}}{p^2-m^2-\frac{\mathrm{i}\lambda}{2}\Delta_F(0)+\mathrm{i}\varepsilon}\,. \tag{4.35}$$

The residue of the free propagator $\mathrm{i}/(p^2-m^2+\mathrm{i}\varepsilon)$ defines the "bare" particle mass m at zero order in λ. Switching on interactions, we continue to define the physical (or renormalised) mass m_{phys} of the scalar particle by the residue of $G^{(2)}(p)$. Thus at order λ,

$$m^2_{\mathrm{phys}} = m^2+\delta m^2 = m^2+\frac{\mathrm{i}\lambda}{2}\Delta_F(0)\,. \tag{4.36}$$

Hence interactions shift or "renormalise" the "bare" mass m used initially in the classical Lagrangian $\mathscr{L}$. It is important to realise that such a renormalisation does not apertain to QFTs but happens in any interacting theory. A familiar example in a classical context is the Debye screening of the electric charge in a plasma.

As next step, we have to calculate (and to interpret properly)

$$\mathrm{i}\Delta_F(0) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\varepsilon}\,. \tag{4.37}$$

Since the mass correction is $\delta m^2 = \mathrm{i}\lambda\Delta_F(0)/2$, the Feynman propagator at coincident points $\Delta_F(0)$ has to be purely imaginary. Otherwise the $\lambda\phi^4$ theory would contain no stable particles.

Wick rotation The integrals appearing in loop graphs can be easier integrated, if one performs a Wick rotation from Minkowski to Euclidean space: Rotating the integration contour anti-clockwise to $-\mathrm{i}\infty : +\mathrm{i}\infty$ avoids both poles in the complex k_0 plane and is thus admissible. Introducing as new integration variable $\mathrm{i}k_4 = k_0$, it follows

$$\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{1}{k^2-m^2+\mathrm{i}\varepsilon} = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}k_0\,\frac{1}{k^2-m^2+\mathrm{i}\varepsilon} = \mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}k_4\,\frac{1}{k^2-m^2+\mathrm{i}\varepsilon}\,. \tag{4.38}$$

We next combine $\boldsymbol{k}$ and k_4 into a new four-vector $k_E = (\boldsymbol{k}, k_4)$. Since

$$k^2 = -(|\boldsymbol{k}|^2 + k_4^2) = -k_E^2 \tag{4.39}$$

we work now (apart from the overall sign) in an Euclidean space. In particular, the denominator never vanishes and we can omit the iε. Moreover, the integrand is now spherically symmetric. Thus we have

$$\mathrm{i}\Delta_F(0) = \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \, \frac{1}{k_E^2 + m^2} \, . \tag{4.40}$$

As required by our interpretation $\delta m^2 = \mathrm{i}\Delta_F(0)$, the propagator $\Delta_F(0)$ is imaginary. Because the relative sign of the momenta and the mass term indicates if we work in the Euclidean or Minkowski space, we will omit the index E in the following. Introducing furthermore spherical coordinates, we see that $\Delta_F(0)$ diverges quadratically for large k,

$$\lambda \mathrm{i}\Delta_F(0) \propto \int_0^\Lambda \mathrm{d}k \, k^3 \frac{1}{k^2 + m^2} \propto \Lambda^2 \, . \tag{4.41}$$

Dimensional regularisation Using the integral representation

$$\frac{1}{k^2 + m^2} = \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-s(k^2+m^2)} \tag{4.42}$$

and interchanging the integrals, we can reduce the momentum integral to a Gaussian integral. Manipulations like interchanging the order of integrations or a change of integration variables in divergent expressions as Eq. (4.41) are however ambiguous. Before we can proceed, we have to "regularise" therefore the integral, similar as we did introducing a cutoff function into the expression of the zero-point energy.

We will use dimensional regularisation (DR), i.e. we will calculate integrals for $d = 4 - 2\varepsilon$ dimensions where they are finite. Then we find

$$\mathrm{i}\Delta_F(0) = \int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \mathrm{e}^{-s(k^2+m^2)} = \frac{1}{(4\pi)^{d/2}} \int_0^\infty \mathrm{d}s \, s^{-d/2} \mathrm{e}^{-sm^2} \, . \tag{4.43}$$

The substitution $x = sm^2$ transforms the integral into one of the standard representations of the Gamma function (see the appendix 4.A for some useful formula),

$$\mathrm{i}\Delta_F(0) = \frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{d/2}} \, \Gamma\left(1 - \frac{d}{2}\right) \, . \tag{4.44}$$

This expression diverges for $d = 2, 4, 6, \ldots$, but is as announced finite for $d = 4 - 2\varepsilon$ and small ε. In the next step, we would like to expand the expression in a Laurent series, separating pole terms in ε and a finite remainder.

Appearance of a dimensionfull scale As the expression stands, we can not expand the prefactor of the Gamma function, because it is dimensionfull. In order to make the factor m^{d-2} dimensionless, we should supply a new mass scale. More physically, we can understand the need for an additional dimensionfull scale by the requirement that the action $S = \int \mathrm{d}^d x \mathscr{L}$ remains dimensionless if we deviate from $d = 4$ dimensions. From the kinetic term, we deduce

that the scalar field has the mass dimension $[\phi] = d/2 - 1$. The interaction term implies then that λ aquires the dimension $[\lambda] = 4-d$. In order to retain a dimensionless coupling constant, we introduce therefore a mass μ called the renormalisation scale as follows,

$$S_I = \int \mathrm{d}^4x \mathscr{L}_I = -\int \mathrm{d}^4x \frac{\lambda}{4!}\phi^4 \to -\mu^{4-d}\int \mathrm{d}^dx \frac{\lambda}{4!}\phi^4\,. \tag{4.45}$$

Adding the factor μ^{4-d} to our previous result, we obtain

$$\lambda\mu^{4-d}\mathrm{i}\Delta_F(0) = \lambda\,\frac{m^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-d/2}\Gamma(1-d/2)\,. \tag{4.46}$$

Now we expand the dimensionless last two factors in this expression around $d = 4$ using Eq. (A.41) for the Gamma function,

$$\Gamma(1-d/2) = \Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon) \tag{4.47}$$

and

$$a^\varepsilon = \mathrm{e}^{\varepsilon \ln a} = 1 + \varepsilon \ln a + \mathcal{O}(\varepsilon^2)\,. \tag{4.48}$$

Note that we require the expansion of the prefactor of the Gamma function up to $\mathcal{O}(\varepsilon^2)$ because of the pole term in (4.47). Thus the mass correction is given by

$$\lambda\mu^{4-d}\,\mathrm{i}\Delta_F(0) \propto m^2\left[-\frac{1}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon)\right]\left[1 + \varepsilon\ln\left(\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}(\varepsilon^2)\right]\,. \tag{4.49}$$

or

$$\delta m^2 = \frac{\mathrm{i}\lambda}{2}\,\mu^{4-d}\,\Delta_F(0) = \frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\left[-\frac{1}{\varepsilon} - 1 + \gamma + \ln\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\varepsilon)\right]\,. \tag{4.50}$$

This expansion has allowed us to separate the correction into a divergent term $\propto 1/\varepsilon$ and a finite remainder. The latter contains an analytic part, $-1+\gamma$, and an non-analytic piece that depends on the renormalisation scale, $\ln(m^2/\mu^2)$. This result is typical for DR: First, all divergences appear in the limit $d \to 4$ as poles of the Gamma function. Second, the renormalisation scale μ enters always via (4.48) in a logarithm. Thus the only dimensionfull parameter which can set the scale of the mass correction δm^2 is the mass of the particle in the loop. In the case of a theory with a single particle, the correction must have therefore the form $\delta m^2 \propto m^2$ using DR. You should contrast this behaviour with the one using as regularisation scheme an Euclidean cutoff Λ: Integrating up to momenta $\Lambda \gg m$, the particle mass m can be neglected, and the correction diverges as a power-law $\delta m^2 \propto \Lambda^2$.

Let us now discuss in turn the three different kind of terms present in Eq. (4.50). First, the form of the divergent terms depends on the regularisation scheme applied. However, in any scheme we can eliminate them using Eq. (4.36), requiring that the unobservable bare mass m^2 contains the same divergent terms with the opposite sign. Second, the finite analytic terms depend also on the scheme, since we can always shift a finite part from m^2 to δm^2. In order to specify precisely δm^2 we have to fix therefore a *renormalisation scheme* , i.e. a set of rules resolving these ambiguities. Finally, the finite, non-analytic terms are important predictions, which are independent of the regularisation scheme (apart from a rescaling of the arbitrary parameter μ). As we will see in chapter 9 such non-analytic terms are necessary that the S-matrix is unitary, or in other words that the theory preserves probability.

4.3.2. Vacuum energy density

We can generate out of the self-energy diagram new one-loop graphs by adding or subtracting two external lines. Subtracting two lines generates an one-loop graph without external lines[2], the "zero-point" Green function $G^{(0)}$ at order λ^0. One way to calculate this quantity is to evaluate directly $Z_0[0]$ using

$$\det A = \exp \ln \det A = \exp \operatorname{tr} \ln A \tag{4.51}$$

what gives

$$Z_0[0] = \exp\left[-\frac{1}{2}\operatorname{tr}\ln(\Box - m^2)\right] . \tag{4.52}$$

We postpone the question how such an expression can be evaluated and use instead another approach, recycling our result for the self-energy. Vacuum diagrams are generated by the functional $Z[J]$ setting $J = 0$,

$$\langle 0+|0-\rangle = Z_0[0] = \int \mathcal{D}\phi \exp \mathrm{i} \int \mathrm{d}^4x \, \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right) . \tag{4.53}$$

We saw in Eq. (3.60) that the zero-point energy is related to the propagator at coincident points. Since we suspect a connection between vacuum diagrams and the zero-point energy, we try to relate $Z[0]$ and $\Delta_F(0)$. Taking a derivative with respect to m^2 gives

$$\frac{\partial}{\partial m^2}\,\langle 0+|0-\rangle = -\frac{\mathrm{i}}{2}\int \mathrm{d}^4x \,\langle 0+|\phi(x)^2|0-\rangle = -\frac{\mathrm{i}}{2}\int \mathrm{d}^4x \,\mathrm{i}\Delta_F(0)\langle 0+|0-\rangle\,. \tag{4.54}$$

The additional factor $\langle 0+|0-\rangle = \mathcal{N}^{-1}$ on the RHS takes into account that we defined the Feynman propagator with respect to a normalised vacuum. Translation invariance implies that $\langle 0+|0-\rangle$ does not depend on x. Thus we obtain

$$\frac{\partial}{\partial m^2}\ln\langle 0+|0-\rangle = -\frac{\mathrm{i}}{2}\int \mathrm{d}^4x \,\mathrm{i}\Delta_F(0) = -\frac{\mathrm{i}}{2}\,VT\,\mathrm{i}\Delta_F(0) \tag{4.55}$$

with VT as the four-dimensional integration volume. Integrating and exponentiating the resulting formal solution, we obtain

$$\langle 0+|0-\rangle = \exp\left\{-\frac{\mathrm{i}}{2}\,VT\int \mathrm{d}m^2\,\mathrm{i}\Delta_F(0)\right\} . \tag{4.56}$$

Comparing this result to

$$\langle 0+|0-\rangle = \langle 0+|\exp\left(-\mathrm{i}HT\right)|0-\rangle = \exp\left(-\mathrm{i}\rho\,VT\right) , \tag{4.57}$$

we see that we should associate

$$\rho = \frac{1}{2}\int \mathrm{d}m^2\,\mathrm{i}\Delta_F(0) \tag{4.58}$$

with the energy density of the vacuum. On the other hand, we can connect ρ to the source-free generating functionals as

$$\rho = \frac{\mathrm{i}\ln Z[0]}{VT} = \frac{-W[0]}{VT} , \tag{4.59}$$

[2] Although $G^{(0)}$ is often represented as a closed loop, it has also no internal line; this is in agreement with our general formula $l = n - V + 1$.

Thus the contribution of quantum fluctuations to the energy density of the vacuum is given by the sum of connected vacuum graphs, in accordance with (4.21).

Next we evaluate (4.58) which gives the contribution of a free scalar field to the vacuum energy density. Using our result (4.44) for the propagator, $\mathrm{i}\Delta_F(0) = C(m^2)^{d/2-1}$, we can perform the integration over m^2,

$$\rho = C\,\frac{m^d}{d} - \rho_0 = \frac{m^d}{(4\pi)^{d/2}d}\,\Gamma\left(1-\frac{d}{2}\right) - \rho_0\,, \tag{4.60}$$

where we introduced the integration constant ρ_0.

The energy density given by Eq. (4.60) diverges for $d = 2, 4, 6\ldots$ as $1/\varepsilon$. We can make ρ finite and equal to the observed value ρ_Λ, if we choose ρ_0 as

$$\rho_0 = \mu^{d-4}\left[\frac{1}{4}\,\frac{m^4}{(4\pi)^2}\,\frac{1}{\varepsilon} - \rho_\Lambda\right]. \tag{4.61}$$

The prefactor μ^{d-4} ensures again that the action remains dimensionless also for $d \neq 4$.

Note that this implies that we should start off with $\mathscr{L} - \rho_0$ instead of $\mathscr{L}$. Even if we dismiss a non-zero vacuum energy in the classical Lagrangian, it will appear automatically by quantum corrections. More generally, every possible term that is not forbidden by a symmetry in $\mathscr{L}$ will show up calculating loop corrections. We have seen that we can absorb the vacuum energy density ρ into the normalisation of the path integral,

$$\int \mathcal{D}\phi\,\mathrm{e}^{-\int \mathrm{d}^4x\rho} = \mathcal{N}\int \mathcal{D}\phi\,.$$

Therefore, one may wonder if ρ has a real physical meaning or could be eliminated by a simple redefinition of the integration measure. The answer is no: First, we used our freedom to define the path integral measure setting $Z_0[0] = 1$. Second, ρ depends on the parameters (masses, coupling constants) of the considered theory, but the path integral measure should be independent of the details of the Lagrangian we integrate.

Remark 4.1: Equivalence to the zero-point energy:
Performing the k^0 integral in Eq. (4.37) or using (3.31)

$$\mathrm{i}\Delta_F(0) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\sqrt{m^2+\boldsymbol{k}^2}} \tag{4.62}$$

and integrating then with respect to m^2,

$$\rho = \frac{1}{2}\int \mathrm{d}m^2\,\mathrm{i}\Delta_F(0) = \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\sqrt{m^2+\boldsymbol{k}^2}\,, \tag{4.63}$$

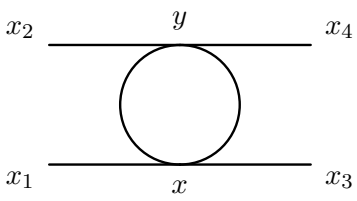
shows that the present expression for the vacuum energy agrees with the sum over zero-point energy evaluated in Eq. (3.61). However, the results for ρ differ: While Eq. (3.62) shows that $\rho \propto \Lambda^4$ using a cutoff, we have obtained $\rho \propto m^4$ in the case of dimensional regularisation. Thus in this scheme a massless particle as the photon would give a zero contribution to the cosmological constant. We will come back to this difference in chapter 26.

4.3.3. Vertex correction

For our last example we add two external lines to the self-energy diagram. This generates diagrams which describe e.g. $2 \to 2$ scattering at $\mathcal{O}(\lambda^2)$. At tree-level, the same process was given by the four-point function (4.29). The corresponding Feynman amplitude in momentum space is simply $\mathrm{i}\mathcal{A} = -\mathrm{i}\lambda$. Thus we suspect that the loop process leads to the renormalisation of the coupling constant λ.

Determining the Feynman amplitude Instead of calculating the order λ^2 term in the perturbative expansion of the generating functional $Z[J]$ we use directly the Feynman rules to obtain the Feynman amplitude for this process. According to these rules, the first steps in the calculation of the Feynman amplitude are to draw all Feynman diagrams, to find the symmetry factor and to associate then the right mathematical expressions to the graphical symbols.

In coordinate space, we have to connect four external points (say $x_1, \ldots, x_4$) with the help of two vertices (say at x and y) which combine each four lines. An example is shown here



Two additional diagrams are obtained connecting x_1 with x_2 or x_4. In order to determine the symmetry factor, we consider the expression for the four-point function corresponding to the graph shown above,

$$\frac{1}{2!}\left(-\mathrm{i}\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x \mathrm{d}^4y \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\phi^4(y)\}|0\rangle + (x \leftrightarrow y)\,, \tag{4.64}$$

and count the number of possible contractions: We can connect $\phi(x_1)$ with each one of the four $\phi(x)$, and then $\phi(x_3)$ with one of the three remaining $\phi(x)$. This gives 4×3 possibilities. Another 4×3 possibilities come by the same reasoning from the upper part of the graph. The remaining pairs $\phi^2(x)$ and $\phi^2(y)$ can be combined in two possibilities. Finally, the factor $1/2!$ from the Taylor expansion is cancelled by the exchange graph. Thus the symmetry factor is

$$S = \frac{1}{2!} 2! \left(\frac{4 \times 3}{4!}\right)^2 2 = \frac{1}{2}\,. \tag{4.65}$$

Next we associate the right mathematical expressions to the symbols of the graphs: We replace internal propagators by $\mathrm{i}\Delta(k)$, external lines by 1 and vertices by $-\mathrm{i}\lambda$. Imposing four-momentum conservation at the two vertices leaves one free loop momentum, which we call p. The momentum of the other propagator is then fixed to $p - q$, where $q^2 = s = (p_1 + p_2)^2$, $q^2 = t = (p_1 - p_3)^2$, and $q^2 = u = (p_1 - p_4)^2$ for the three graphs shown in Fig. 4.2. Thus the Feynman amplitudes in $n = 4$ are at order $\mathcal{O}(\lambda^2)$

$$\mathrm{i}\mathcal{A}_q^{(2)} = \frac{1}{2}\lambda^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{1}{[p^2 - m^2 + \mathrm{i}\varepsilon]} \frac{1}{[(p-q)^2 - m^2 + \mathrm{i}\varepsilon]}\,. \tag{4.66}$$

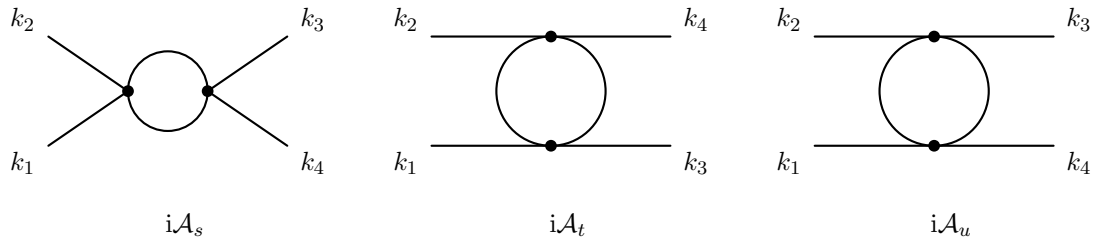


Figure 4.2.: Three graphs contributing to $\phi(k_1)\phi(k_2) \to \phi(k_3)\phi(k_4)$ at order λ^2.

The squared cms energy s and the two variables describing the momentum transfer t and u are called Mandelstam variables. For $2 \to 2$ scattering, they are connected by $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$, see problem 4.3. According to the value of q^2 one calls the diagrams the s, t and u channel.

Performing again a simple counting of the powers of loop momenta, we find that the amplitude is logarithmically divergent,

$$\mathcal{A}_q^{(2)} \propto \int \frac{\mathrm{d}^4 p}{p^4} \propto \ln(\Lambda)\,. \tag{4.67}$$

If we consider the infinite number of one-loop graphs characterised by $n = V \geq 0$, then we see that adding two external lines increases the number of propagators in the loop by one. As a result, the convergence of the loop integral improves from a quartic divergence (vacuum energy), over a quadratic divergence (self-energy energy) to a logarithmic divergence for the vertex correction. Adding two or more external lines to the vertex correction would therefore produce a finite diagram. At one-loop, the $\lambda\phi^4$ theory contains thus only three divergent Feynman graphs.

Calculating the loop integral The path to be followed in the evaluation of simple loop integrals as (4.66) can be sketched schematically as follows: Regularise the integral (and add a mass scale if you use DR). Combine then the denominators, and shift the integration variable to eliminate linear terms in the denominator by completing the square. Performing the same shift of variables in the numerator, linear terms can be dropped as they vanish after integration. Finally, Wick rotate the integrand, and reduce the integral to a known one by a suitable variable substitution. We do the last steps once in the appendix 4.A where we derive a list of useful Feynman integrals which we simply look up in the future.

We start by rewriting the integral for $d = 4 - 2\varepsilon$ dimensions as

$$\mathrm{i}\mathcal{A}_q^{(2)} = \frac{1}{2}\lambda^2(\mu^2)^{4-d} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{D}\,, \tag{4.68}$$

where we introduced also the short-cut D for the denominator in the integrand. Next we use

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2} \tag{4.69}$$

to combine the two denominators, setting $a = p^2 - m^2$ and $b = (p-q)^2 - m^2$,

$$\mathcal{D} \equiv az + b(1-z) = p^2 - m^2 - 2pq(1-z) + q^2(1-z)\,. \tag{4.70}$$

Then we eliminate the term linear in p substituting $p'^2 = [p - q(1-z)]^2$,

$$\mathcal{D} = p'^2 - m^2 + q^2 z(1-z)\,. \tag{4.71}$$

Since $\mathrm{d}^d p = \mathrm{d}^d p'$, we can drop the primes and find

$$\mathrm{i}\mathcal{A}_q^{(2)} = \frac{1}{2}\lambda^2(\mu^2)^{4-d}\int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{[p^2 - m^2 + q^2 z(1-z)]^2}\,. \tag{4.72}$$

Performing a Wick rotation requires that $q^2 z(1-z) < m^2 - \mathrm{i}\varepsilon$ for all $z \in [0:1]$, or $q^2 < 4m^2$. The integral is of the type $I(\omega, 2)$ calculated in the appendix and equals

$$I(\omega, 2) = \mathrm{i}\,\frac{1}{(4\pi)^\omega}\,\frac{\Gamma(2-\omega)}{\Gamma(2)}\,\frac{1}{[m^2 - q^2 z(1-z)]^{2-\omega}}\,. \tag{4.73}$$

Inserting the result into the Feynman amplitude gives

$$\mathcal{A}_q^{(2)} = \frac{1}{2}\lambda^2(\mu^2)^{4-d}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,[m^2 - q^2 z(1-z)]^{d/2-2} \tag{4.74}$$

$$= \frac{\lambda^2}{32\pi^2}(\mu^2)^{2-d/2}\Gamma(2-d/2)\int_0^1 \mathrm{d}z\left[\underbrace{\frac{m^2 - q^2 z(1-z)}{4\pi\mu^2}}_{f}\right]^{\frac{d}{2}-2}. \tag{4.75}$$

In the last step, we made the function f dimensionless. Now we take the limit $\varepsilon = 2 - d/2 \to 0$, expanding both the Gamma function, $\Gamma(2-d/2) = \Gamma(\varepsilon) = 1/\varepsilon - \gamma + \mathcal{O}(\varepsilon)$, and $f^{-\varepsilon}$. From

$$\frac{\lambda^2}{32\pi^2}\mu^{2\varepsilon}\left(\frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)\right)\left[1 - \varepsilon\int_0^1 \mathrm{d}z\,\ln f + \mathcal{O}(\varepsilon^2)\right] \tag{4.76}$$

we see that all diagrams give the same divergent part, while we have to replace q^2 by the value $\{s, t, u\}$ appropriate for the three diagrams,

$$\begin{aligned}\mathcal{A} &= \mathcal{A}^{(1)} + \mathcal{A}_s^{(2)} + \mathcal{A}_t^{(2)} + \mathcal{A}_u^{(2)} + \mathcal{O}(\lambda^3)\\ &= -\lambda\mu^\varepsilon + \frac{3\lambda^2\mu^{2\varepsilon}}{32\pi^2\varepsilon} - \frac{\lambda^2\mu^{2\varepsilon}}{32\pi^2}\left[3\gamma + F(s,m,\mu) + F(t,m,\mu) + F(u,m,\mu)\right],\end{aligned} \tag{4.77}$$

with

$$F(q^2, m, \mu) = \int_0^1 \mathrm{d}z\,\ln\left[\frac{m^2 - q^2 z(1-z)}{4\pi\mu^2}\right]. \tag{4.78}$$

Note that t and u are in the physical region negative and thus the condition $q^2 < 4m^2$ is always satisfied for these two diagrams, cf. with problem 4.3. By contrast, for the s channel diagram the relation $q^2 = s > 4m^2$ holds: In this case, we have to continue analytically the result (4.78) into the physical region. We will postpone this task to chapter 9 and note for the moment only that thereby the argument of the logarithm in (4.78) changes sign. Additionally, an imaginary part of the scattering amplitude is generated.

4.3.4. Basic idea of renormalisation

The regularisation of loop integrals has introduced as a new parameter the renormalisation scale μ. As we perform perturbation theory at order λ^n, we have to connect the parameters $\{m_n, \lambda_n, \rho_n\}$ of the truncated theory with the physical ones of the full theory. This process is called renormalisation and will replace the undetermined parameter μ by a physical momentum scale relevant for the considered process.

Renormalisation of the coupling Let us try to connect the amplitude $\mathrm{i}\mathcal{A}$ to a physical measurement. We assume that experimentalists measured $\phi\phi \to \phi\phi$ scattering. It is sufficient that they provide us with a single value, e.g. with the value of the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ at zero-momentum transfer close to threshold $s = 4m^2$. Then the function $F(s, m, \mu)$ is given by

$$F(4m^2, m, \mu) = C\lambda^2 \ln\left[\frac{4m^2}{\mu^2}\right] + \text{const}\,, \tag{4.79}$$

while it becomes for $s \gg m^2$

$$F(s, m, \mu) = C\lambda^2 \ln\left[\frac{s}{\mu^2}\right] + \text{const}\,. \tag{4.80}$$

Subtracting the infinite parts (and the constant term) from $\mathcal{A}$, we obtain for $s, t, u \gg m^2$,

$$\mathcal{A} = -\lambda - C\lambda^2\big[\ln(s/\mu^2) + \ln(t/\mu^2) + \ln(u/\mu^2)\big] + \mathcal{O}(\lambda^3) \equiv -\lambda - C\lambda^2 L(s/\mu^2)\,, \tag{4.81}$$

where we introduced also the sloppy notation L for the three log terms in the square bracket. This expression for $\mathcal{A}$ is finite but still arbitrary since it contains μ.

We use now the experimental measurement at the scale $s = 4m^2$ to connect via

$$\mathcal{A} = -\lambda - C\lambda^2 L(4m^2/\mu^2) \tag{4.82}$$

the measured value λ_{phys} of the coupling to our calculation,

$$-\lambda_{\text{phys}} = -\lambda - C\lambda^2 L(4m^2/\mu^2) + \mathcal{O}(\lambda^3)\,. \tag{4.83}$$

Here we indicated also that our perturbative calculation is only valid up to $\mathcal{O}(\lambda^3)$ terms. Now we solve for λ,

$$\begin{aligned} -\lambda &= -\lambda_{\text{phys}} + C\lambda^2 L(4m^2/\mu^2) + \mathcal{O}(\lambda^3) \\ &= -\lambda_{\text{phys}} + C\lambda_{\text{phys}}^2 L(4m^2/\mu^2) + \mathcal{O}(\lambda_{\text{phys}}^3)\,. \end{aligned} \tag{4.84}$$

In the second line, we could replace λ^2 by λ_{phys}^2, because their difference is of $\mathcal{O}(\lambda^3)$. Next we insert λ back into the matrix element $\mathcal{A}$ for general s and replace then again λ^2 by λ_{phys}^2,

$$\begin{aligned} \mathcal{A} &= -\lambda - C\lambda^2 L(s/\mu^2) + \mathcal{O}(\lambda^3) \\ &= -\lambda_{\text{phys}} + C\lambda_{\text{phys}}^2 L(4m^2/\mu^2) - C\lambda_{\text{phys}}^2 L(s/\mu^2) + \mathcal{O}(\lambda_{\text{phys}}^3) \\ &= -\lambda_{\text{phys}} - C\lambda_{\text{phys}}^2 L(s/4m^2) + \mathcal{O}(\lambda_{\text{phys}}^3)\,. \end{aligned} \tag{4.85}$$

Combining the log's, the scale μ has canceled and we find

$$\mathcal{A} = -\lambda_{\text{phys}} - \frac{\lambda_{\text{phys}}^2}{32\pi^2}\,\big[\ln(s/4m^2) + \ln(t/4m^2) + \ln(u/4m^2)\big] + \mathcal{O}(\lambda_{\text{phys}}^3)\,. \tag{4.86}$$

Thus the amplitude is finite and depends only on the measured value λ_{phys} of the coupling constant and the kinematical variables s and t.

Running coupling We look now from a somewhat different point of view at the problem of the apparent μ dependence of physical observables. Assume that we have subtracted the infinite parts (and the constant term) of the amplitude $\mathcal{A}$, obtaining Eq. (4.81). We now demand that the scattering amplitude $\mathcal{A}$ as a physical observable is independent of the arbitrary scale μ, $\mathrm{d}\mathcal{A}/\mathrm{d}\mu = 0$. If we did a perturbative calculation up to $\mathcal{O}(\lambda^n)$, the condition $\mathrm{d}\mathcal{A}/\mathrm{d}\mu = 0$ can hold only up to terms $\mathcal{O}(\lambda^{n+1})$. The explicit μ dependence of the amplitude, $\partial\mathcal{A}/\partial\mu \neq 0$, can be only cancelled by a corresponding change of the parameters m and λ contained in the classical Lagrangian, converting them into "running" parameters $m(\mu)$ and $\lambda(\mu)$. Then the condition[3] $\mathrm{d}\mathcal{A}/\mathrm{d}\mu = 0$ becomes

$$\left(\frac{\partial}{\partial\mu} + \frac{\partial m^2}{\partial\mu}\frac{\partial}{\partial m^2} + \frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda}\right)\mathcal{A}(s,t,m(\mu),\lambda(\mu),\mu) = 0\,. \tag{4.87}$$

The only explicit μ dependence of $\mathcal{A}$ is contained in the $F(q^2,m,\mu)$ functions, giving in the limit $\varepsilon \to 0$

$$\frac{\partial}{\partial\mu}\mathcal{A}(s,t,m(\mu),\lambda(\mu),\mu) = -3\,\frac{\lambda^2}{32\pi^2}\,\frac{\partial}{\partial\mu}F(q^2,m,\mu) = \frac{3\lambda^2}{16\pi^2\mu}\,. \tag{4.88}$$

Since the change of $m(\mu)$ and $\lambda(\mu)$ is given by loop diagrams, it includes at least an additional factor λ. Therefore the action of the derivatives ∂_{m^2} and ∂_λ on the 1-loop contribution $\mathcal{A}^{(2)}$ leads to a term of $\mathcal{O}(\lambda^3)$ which can be neglected. Thus the only remaining term will be given by ∂_μ acting on the tree-level term $\mathcal{A}^{(1)}$. Note that this holds also at higher orders and ensures that $\partial_\mu\lambda$ at $\mathcal{O}(\lambda^{n+1})$ is determined by the parameters calculated at $\mathcal{O}(\lambda^n)$. Combining the two contributions we find

$$\mu\frac{\partial\lambda}{\partial\mu} = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)\,. \tag{4.89}$$

Thus the scattering amplitude $\mathcal{A}$ is independent of the scale μ, if we transform the coupling constant λ into a scale dependent "running" coupling $\lambda(\mu)$ whose evolution is given by Eq. (4.89). Since we truncate the perturbation series at a finite order, the cancellation of the scale dependence is incomplete and a residual dependence of physical quantities on μ remains.

Separating variables in Eq. (4.89), we find

$$\lambda(\mu) = \frac{\lambda_0}{1 - 3\lambda_0/16\pi^2\ln(\mu/\mu_0)} \tag{4.90}$$

with $\lambda_0 \equiv \lambda(\mu_0)$ as initial condition. Thus the running coupling $\lambda(\mu)$ increases logarithmically for increasing μ in the $\lambda\phi^4$ theory.

Comparing (4.90) to our result for the scattering amplitude (4.86),

$$\mathcal{A} = -\lambda_0\left(1 + \frac{\lambda_0}{32\pi^2}\left[\ln(s/4m^2) + \ln(t/4m^2) + \ln(u/4m^2)\right]\right) + \mathcal{O}(\lambda_0^3)\,, \tag{4.91}$$

we see that we can rewrite the amplitude using a symmetric point $q^2 = s = t = u$ and $\mu_0^2 = 4m^2$ as

$$\mathcal{A} = -\lambda_0\left(1 + \frac{3\lambda_0}{32\pi^2}\ln(q^2/\mu_0^2)\right) = -\lambda(q^2)\,. \tag{4.92}$$

[3] Equations of the type (4.87) that describe the change of observables as function of the scale μ are called renormalisation group equations (RGE).

This shows that the q^2 dependence of the amplitude $\mathcal{A}$ in the limit $q^2 \gg m^2$ is determined completely by the scale dependence of the running coupling $\lambda(\mu)$. Therefore, we should set the renormalisation scale μ in general equal to the physical momentum scale q that characterises the considered process. We will come back to this topic in chapter 12, giving a formal definition of the running coupling.

4.A. Appendix: Evaluation of Feynman integrals

Combination of propagators The standard strategy in the evolution of loop integrals is the combination of the n propagator denominators into a single propagator-like denominator of higher power. One uses either Schwinger's proper-time representation

$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{\mathrm{i}s(p^2 - m^2 + \mathrm{i}\varepsilon)} \tag{4.93}$$

or the Feynman parameter integral

$$\begin{aligned}\frac{1}{x_1 \cdots x_n} &= \Gamma(n) \int_0^1 \mathrm{d}\alpha_1 \cdots \int_0^1 \mathrm{d}\alpha_n\, \delta(1 - \sum_i \alpha_i)\, [\alpha_1 x_1 \cdots \alpha_n x_n]^{-n} \\ &= \Gamma(n) \int_0^1 \mathrm{d}\alpha_1 \cdots \int_0^{\alpha_{n-2}} \mathrm{d}\alpha_n\ [\alpha_1(1 - x_1) + \alpha_2(x_1 - x_2) \cdots \alpha_n x_{n-1}]^{-n}\ . \end{aligned} \tag{4.94}$$

In order to derive this formula for $n = 2$, consider

$$\frac{1}{b-a} \int_a^b \frac{\mathrm{d}x}{x^2} = \frac{1}{b-a} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{ab} \tag{4.95}$$

for $a, b \in \mathbb{C}$. Setting $x = az + b(1 - z)$ and changing the integration variable, we obtain

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1 - z)]^2}\,. \tag{4.96}$$

The cases $n > 2$ can be derived by induction, rewriting e.g,. $1/(abc)$ as $1/(aB)$ with $B = bc$, and using the result for $n - 1$. Finally, we can generalise these formulae to expressions like $1/(a^n b^m)$ by taking derivatives with respect to a and b.

Evaluation of Feynman integrals We want to calculate integrals of the type

$$I_0(\omega, \alpha) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 - m^2 + \mathrm{i}\varepsilon]^\alpha} \tag{4.97}$$

defined in Minkowski space. Performing a Wick rotation to Euclidean space and introducing spherical coordinates results in

$$I_0(\omega, \alpha) = \mathrm{i}(-1)^\alpha \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 + m^2]^\alpha} = \mathrm{i}\frac{(-1)^\alpha}{(2\pi)^{2\omega}} \Omega_{2\omega} \int_0^\infty \mathrm{d}k\, \frac{k^{2\omega - 1}}{[k^2 + m^2]^\alpha}\,, \tag{4.98}$$

where we denoted the volume $\mathrm{vol}(S^{2\omega-1})$ of a unit sphere in 2ω dimensions[4] by $\Omega_{2\omega}$. You are asked in problem 4.2 to show that $\Omega_{2\omega} = 2\pi^\omega/\Gamma(\omega)$ and thus $\Omega_4 = 2\pi^2$. Substituting $k = m\sqrt{x}$ and using

[4] A $n-1$-dimensional sphere $S^{n-1}(R)$ encloses the n-dimensional volume $x_1^2 + \ldots + x_n^2 \leq R^2$, while its own $n-1$-dimensional volume is given by $x_1^2 + \ldots + x_n^2 = R^2$. The volume of a unit 1-sphere is a length, $\mathrm{vol}(S^1) = 2\pi$, of a unit 2-sphere an area, $\mathrm{vol}(S^2) = 4\pi$, and of a unit 3-sphere a volume, $\mathrm{vol}(S^3) = 2\pi^2$. If we say that the volume of a sphere is $4\pi R^3/3$, we mean in fact the volume of the 3-ball $B^3(R)$, $x_1^2 + x_2^2 + x_3^2 \leq R^2$ which is enclosed by the 2-sphere $S^2(R)$.

the integral representation (A.27) for Euler's beta function allows us to express the k integral as a product of Gamma functions,

$$\int_0^\infty \mathrm{d}k\, \frac{k^{2\omega-1}}{[k^2+m^2]^\alpha} = \frac{1}{2} m^{2\omega-2\alpha} \int_0^\infty \mathrm{d}x\, \frac{x^{\omega-1}}{[1+x]^\alpha} = \tag{4.99}$$

$$\frac{1}{2} m^{2\omega-2\alpha} B(\omega, \alpha-\omega) = \frac{1}{2} m^{2\omega-2\alpha} \frac{\Gamma(\omega)\Gamma(\alpha-\omega)}{\Gamma(\alpha)}\,. \tag{4.100}$$

Combining this result with $\Omega_{2\omega} = 2\pi^\omega/\Gamma(\omega)$ we obtain

$$I_0(\omega,\alpha) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\, \frac{1}{[k^2-m^2+\mathrm{i}\varepsilon]^\alpha} = \mathrm{i}\, \frac{(-1)^\alpha}{(4\pi)^\omega}\, \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}\, [m^2-\mathrm{i}\varepsilon]^{\omega-\alpha}\,. \tag{4.101}$$

Note that m^2 can denote any function of the external momenta and masses, since we reuqired only that it is independent of the loop momentum. We can generate additional formulae by adding first a dependence on a external momentum p^μ, shifting then the integration variable $k \to k+p$,

$$I(\omega,\alpha) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\, \frac{1}{[k^2+2pk-m^2+\mathrm{i}\varepsilon]^\alpha} = \mathrm{i}\, \frac{(-1)^\alpha}{(4\pi)^\omega}\, \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}\, [m^2+p^2-\mathrm{i}\varepsilon]^{\omega-\alpha}\,. \tag{4.102}$$

Taking then derivatives with respect to the external momentum p^μ results in

$$I_\mu(\omega,\alpha) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\, \frac{k_\mu}{[k^2+2pk-m^2+\mathrm{i}\varepsilon]^\alpha} = -p_\mu I(\omega,\alpha) \tag{4.103}$$

and

$$I_{\mu\nu}(\omega,\alpha) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\, \frac{k_\mu k_\nu}{[k^2+2pk-m^2+\mathrm{i}\varepsilon]^\alpha} = \tag{4.104}$$

$$= \mathrm{i}\, \frac{(-\pi)^\omega}{(2\pi)^{2\omega}}\, \frac{\Gamma(\alpha-\omega-1)}{\Gamma(\alpha)}\, \frac{p_\mu p_\nu(\alpha-\omega-1) - \frac{1}{2}\eta_{\mu\nu}(m^2+p^2)}{[m^2+p^2-\mathrm{i}\varepsilon]^{\alpha-\omega}}\,. \tag{4.105}$$

Contracting both sides with $k_\mu k_\nu$ and using $\eta^{\mu\nu}\eta_{\mu\nu} = 2\omega$ gives

$$I_2(\omega,\alpha) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\, \frac{k^2}{[k^2+2pk-m^2+\mathrm{i}\varepsilon]^\alpha} = \mathrm{i}\, \frac{(-\pi)^\omega}{(2\pi)^{2\omega}}\, \frac{\Gamma(\alpha-\omega-1)}{\Gamma(\alpha)}\, \frac{(\alpha-2\omega-1)p^2-\omega m^2}{[m^2+p^2-\mathrm{i}\varepsilon]^{\alpha-\omega}}\,. \tag{4.106}$$

Special cases often needed are

$$I(\omega,2) = \frac{\mathrm{i}}{(4\pi)^\omega}\, \Gamma(2-\omega)\, (m^2+p^2-\mathrm{i}\varepsilon)^{\omega-2}\,, \tag{4.107}$$

$$I_2(\omega,2) = -\frac{\mathrm{i}}{(4\pi)^\omega}\, \omega\Gamma(1-\omega)\, (m^2+p^2-\mathrm{i}\varepsilon)^{\omega-1}\,, \tag{4.108}$$

and

$$I(2,3) = -\frac{\mathrm{i}}{32\pi^2}\, \frac{1}{m^2+p^2-\mathrm{i}\varepsilon}\,. \tag{4.109}$$

Summary

Disconnected n-point Green functions are generated by the functional $Z[J]$, while $\mathrm{i}W[J] = \ln Z[J]$ generates connected Green functions. The three loop diagrams we calculated in the

$\lambda\phi^4$ theory were infinite and had to be regularised. Renormalising the three parameters contained in the classical Lagrangian of the $\lambda\phi^4$ theory, ρ_0, m^2, and λ, eliminated the divergences and converted them into "running" quantities.

Further reading

The quantisation of fields using both canonical quantisation and the path integral approach is discussed extensively in[35]. For an treatment of the $\lambda\phi^3$ theory in $n = 6$ dimensions—a theory which resembles a bit more QED than the $\lambda\phi^4$ theory we discussed—see[66].

Problems

4.1 $Z[J]$ at order λ.
Derive Eq. (4.10).

4.2 Volume of n dimensional sphere.
a.) Calculate the volume of the unit sphere S^{n-1} defined by $x_1^2 + \ldots + x_n^2 = 1$ in $\mathbb{R}^n$.
b.) Generalise the result to arbitrary (not necessarily integer) dimensions and show that it agrees with the familiar results for $n = 1, 2$ and 3.

4.3 Mandelstam variables.
Show that for $2 \to 2$ scattering $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$ is valid. Express t as function of the scattering angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ and derive the relation between $\mathrm{d}\sigma/\mathrm{d}\Omega$ and $\mathrm{d}\sigma/\mathrm{d}t$. Find the lower and upper limits of t and u.

4.4 Casimir effect.
Repeat the calculation of the Casimir effect for a scalar field in 3+1 dimensions combining ζ function and dimensional regularisation.

4.5 Renormalisation invariance of the propagator.
Derive analogously to Eq. (4.87ff) an equation $\mathrm{d}m^2(\mu)/\mathrm{d}\mu = f(m^2)$ requiring that the propagator (4.35) is independent of the scale μ.

4.6 Renormalisation with an Euclidean momentum cutoff.
Calculate the self-energy (mass correction) and vertex correction using an Euclidean momentum cutoff Λ. Derive the RGE equations $\mathrm{d}m^2(\Lambda)/\mathrm{d}\Lambda = f(m^2)$, $\mathrm{d}\lambda(\Lambda)/\mathrm{d}\Lambda = f(\lambda)$ and $\mathrm{d}\rho(\Lambda)/\mathrm{d}\Lambda = f(\rho)$ and compare them to the ones derived using DR.

4.7 Parameter integral (4.78).
Evaluate the integral (4.78).

5. Global symmetries and Noether's theorem

Emmy Noether showed 1917 that any global continuous symmetry of a classical system described by a Lagrangian leads to a locally conserved current. We can divide such symmetries into two classes: Symmetries of space-time and internal symmetries of a group of fields. Prominent examples for the latter are the global symmetries that lead to the conservation laws of electric charge or baryon number, respectively. For a quantum system, we have to study the impact of symmetries on its generating functional. If this functional remains invariant, the symmetry holds also at the quantum level. Then conserved charges exists which commute with the Hamiltonian. Analysing space-time symmetries, we restrict ourselves in this chapter to the case when we can neglect gravity. Then space-time is the familiar Minkoswki space characterised by the Poincaré symmetry group, i.e. the product of the translation and the Lorentz group, with its corresponding conservation laws.

5.1. Internal symmetries

We have used up-to now mainly space-time symmetry of Minkowski space, namely the requirement of Lorentz invariance, to deduce possible terms in the action. If we allow for more than one field, e.g. several scalar fields, the new possibility of *internal* symmetries arise. For instance, we can look at a theory of two massive scalar fields with quartic interactions,

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi_1\right)^2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{4}\lambda_1\phi_1^4 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}m_2^2\phi_2^2 - \frac{1}{4}\lambda_2\phi_2^4 - \frac{1}{2}\lambda_3\phi_1^2\phi_2^2\,. \tag{5.1}$$

In order to maintain the discrete Z_2 symmetry $\phi \to -\phi$ of the individual Lagrangians, we have omitted odd terms like $\phi_1\phi_2^3$. Then the theory contains five arbitrary parameters, two masses m_i and three coupling constants λ_i. For arbitrary values of these parameters, no new additional symmetry results. In nature, we find however often a set of particles with nearly the same mass and (partly) the same couplings. One of the first examples was suggested by Heisenberg after the discovery of the neutron, which has a mass very close to the one of the proton, $m_n \simeq m_p$: With respect to strong interactions, it is useful to view the proton and neutron as two different "isospin" states of the nucleon, similiar as an electron has two spin states. An example of an exact symmetry are particles and their anti-particles, as e.g. the charged pions $\pi^\pm$ which can be combined into one complex scalar field.

If we set in our case $m_1 = m_2$ and $\lambda_2 = \lambda_1$, the Lagrangian becomes invariant under the exchange $\phi_1 \leftrightarrow \phi_2$. Adding the further condition that $\lambda_3 = \lambda_1 = \lambda_2$, we arrive at

$$\mathscr{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] - \frac{1}{2}m^2(\phi_1^2+\phi_2^2) - \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2\,. \tag{5.2}$$

Now any orthogonal transformation $O \in \mathrm{O}(2)$ in the two-dimensional field space $\{\phi_1, \phi_2\}$ leads to the same Langrangian $\mathscr{L}$. In particular, the Lagrangian is invariant under a rotation $R(\alpha) \in \mathrm{SO}(2)$ which mixes $\{\phi_1, \phi_2\}$ as

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,. \tag{5.3}$$

The fields transform as a vector $\boldsymbol{\phi} = \{\phi_1, \phi_2\}$ and a rotation leaves the length of this vector invariant. Generalising this to n scalar fields, we can write down immediately a theory that is invariant under transformations $\phi_a \to R_{ab}\phi_b$ where R is an element of $\mathrm{O}(n)$,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\phi})^2 - \frac{1}{2}m^2\boldsymbol{\phi}^2 - \frac{\lambda}{4}(\boldsymbol{\phi}^2)^2 \,. \tag{5.4}$$

Note that the Lagrangian is only invariant under global, i.e. space-time independant rotations.

The free Lagrangian $\mathscr{L}_0$, i.e. the part quadratic in the fields, is diagonal, $\mathscr{L}_0 = \mathscr{L}_0(\phi_1) + \mathscr{L}_0(\phi_2)$. Thus the propagator $D_{ab}(x - x')$ is diagonal too, $D_{ab}(x - x') \propto \delta_{ab}$. An interaction vertex at x connects four propagators $D_{ab}(x - x_i)$. As a result of the Z_2 symmetry, an even number of ϕ_1 and ϕ_2 particles are connected at each vertex which has therefore the form $-\mathrm{i}\lambda(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$.

5.2. Noether's theorem

From our experience in classical and quantum mechanics, we expect that global continuous symmetries lead also in field theory to conservation laws for the generators of the symmetry. In order to derive such a conservation law, we consider an infinitesimal change $\delta\phi_a$ of the fields that keeps by assumption $\mathscr{L}(\phi_a, \partial_\mu\phi_a)$ invariant,

$$0 = \delta\mathscr{L} = \frac{\delta\mathscr{L}}{\delta\phi_a}\,\delta\phi_a + \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\partial_\mu\phi_a \,. \tag{5.5}$$

Now we exchange $\delta\partial_\mu = \partial_\mu\delta$ in the second term and use then the Lagrange equations, $\delta\mathscr{L}/\delta\phi_a = \partial_\mu(\delta\mathscr{L}/\delta\partial_\mu\phi_a)$, in the first one. Then we can combine the two terms using the product rule,

$$0 = \delta\mathscr{L} = \partial_\mu\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\right)\delta\phi_a + \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\partial_\mu\delta\phi_a = \partial_\mu\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\phi_a\right) . \tag{5.6}$$

Hence the invariance of $\mathscr{L}$ under the change $\delta\phi_a$ implies the existence of a conserved current, $\partial_\mu j^\mu = 0$, with

$$j^\mu = \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\phi_a \,. \tag{5.7}$$

If the transformation $\delta\phi_a$ leads to change in $\mathscr{L}$ that is a total four-divergence, $\delta\mathscr{L} = \partial_\mu K^\mu$, and boundary terms can be dropped, then the equations of motion remain invariant. The conserved current j^μ, also called Noether current, is changed to

$$j^\mu = \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a}\,\delta\phi_a - K^\mu \,. \tag{5.8}$$

In Minkowski space, we can convert this differential form of a conservation law into a global one, obtaining a globally conserved Noether charge

$$Q = \int_V \mathrm{d}^3x\, j^0 \,. \tag{5.9}$$

Often (but not always) this charge has a profound physical meaning. Finally, we note that the conserved current j^μ is not unique, since we can add a four-divergence K^μ.

Internal symmetries As an example, we can use our n scalar fields invariant under the group[1] SO(n). We need the infinitesimal generators T_i of rotations,

$$\phi'_a = R_{ab}\phi_b \approx (1 + \alpha_i T_i + \mathcal{O}(\alpha_i^2))_{ab}\phi_b \,. \tag{5.10}$$

SO(n) has an anti-symmetric Lie algebra with $n(n-1)/2$ generators. Thus a theory invariant under SO(n) has $n(n-1)/2$ conserved currents. The special case $n = 2$ has as important application e.g. the charged pions $\pi^{\pm}$. We combine the two real fields ϕ_1 and ϕ_2 into the complex field $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$, then the Lagrangian becomes

$$\mathscr{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \,. \tag{5.11}$$

Under the combined phase transformations $\phi \to \mathrm{e}^{-\mathrm{i}\vartheta}\phi$ and $\phi^\dagger \to \mathrm{e}^{\mathrm{i}\vartheta}\phi^\dagger$, the Lagrangian $\mathscr{L}$ is clearly invariant. With $\delta\phi = -\mathrm{i}\phi$, $\delta\phi^\dagger = \mathrm{i}\phi^\dagger$, the conserved current follows as

$$j^\mu = \mathrm{i}\left[\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi\right] \,. \tag{5.12}$$

The conserved charge $Q = \int \mathrm{d}^3x\, j^0$ can be also negative and thus we cannot interpret j^0 as the probability density to observe a ϕ particle. Instead, we should associate Q with a conserved additive quantum number as e.g. the electric charge.

Space-time symmetries of Minkowski space The Poincaré group as symmetry group of Minkowski space has ten generators[2]. If the Lagrangian does not depend explicitely on space-time coordinates, i.e. $\mathscr{L} = \mathscr{L}(\phi_a, \partial_\mu \phi_a)$, ten conservation laws for the fields ϕ_a follow. We consider first the behaviour of the fields ϕ_a and the Langrangian under an infinitesimal translation $x^\mu \to x^\mu + \varepsilon\xi^\mu$. As in the case of internal symmetries, we consider only global transformations and thus ε does not depend on x. From

$$\phi_a(x^\mu) \to \phi_a(x^\mu + \varepsilon\xi^\mu) \approx \phi_a(x^\mu) + \varepsilon\xi^\nu \partial_\nu \phi_a(x^\mu) \tag{5.13}$$

we find the change

$$\delta\phi_a(x) = \xi^\mu \partial_\mu \phi_a(x) = \partial_\mu[\xi^\mu \phi_a(x)] \,.$$

Since the Lagrange density $\mathscr{L}$ contains by assumption no explicit space-time dependence, it will change simply as $\mathscr{L}(x^\mu) \to \mathscr{L}(x^\mu + \varepsilon\xi^\mu)$ or

$$\delta\mathscr{L}(x) = \xi^\mu \partial_\mu \mathscr{L}(x) = \partial_\mu[\xi^\mu \mathscr{L}(x)] \,. \tag{5.14}$$

Thus $K^\mu = \xi^\mu \mathscr{L}(x)$ and inserting both in the Noether current gives

$$j^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\left[\xi^\nu \partial_\nu \phi_a\right] - \xi^\mu \mathscr{L} = \xi_\nu \left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\frac{\partial\phi_a}{\partial x_\nu} - \eta^{\mu\nu}\mathscr{L}\right] \equiv \xi_\nu T^{\mu\nu} \,, \tag{5.15}$$

where the square bracket defines the (energy-momentum) stress tensor $T^{\mu\nu}$ of the fields ϕ_a. The corresponding four conserved Noether charges are the components of the four-momentum

$$p^\nu = \int \mathrm{d}^3x\, T^{0\nu} \,. \tag{5.16}$$

[1] Although the Langrangian is invariant under the larger group O(n), we consider only the subgroup SO(n) which is continuously connected to the identity. The additional discrete transformations contained in O(n) can be used to classify solutions of the Lagrangian, but do not lead to additional conservation laws.

[2] If this sound unfamiliar, read first the appendices B.3 and B.4 before continuing

The conserved tensor defined by Eq. (5.15) is called the *canonical* stress tensor. The definition (5.15) does not guarantee that $T^{\mu\nu}$ is symmetric. A symmetric stress tensor, $T^{\mu\nu} = T^{\nu\mu}$, is however the condition for the conservation of the total angular momentum, as we will show in the next paragraph[3]. Since the Lagrange density is only determined up to a four-divergence, we can symmetrize always $T^{\mu\nu}$ adding an appropriate four-dimensional divergence.

Example 5.1: The general expression (5.15) for the canonical stress tensor becomes for a free complex scalar field

$$T^{\mu\nu} = 2\partial^\mu\phi^\dagger\partial^\nu\phi - \eta^{\mu\nu}\mathscr{L}\,. \tag{5.17}$$

Thus the canonical stress tensor of a scalar field is already symmetric. Its 00 component,

$$T^{00} = \rho = \mathscr{H} = 2|\dot\phi|^2 - \mathscr{L} = |\dot\phi|^2 + |\boldsymbol{\nabla}\phi|^2 + m^2|\phi|^2\,, \tag{5.18}$$

agrees with twice the result (3.16) for the energy-density ρ of a single real scalar field. We consider now plane-wave solutions to the Klein-Gordon equation, $\phi = N\exp(\mathrm{i}kx)$. If we insert $\partial_\mu\phi = \mathrm{i}k_\mu\phi$ into $\mathscr{L}$, we find $\mathscr{L} = 0$ and thus

$$T^{00} = 2N^2k^0k^0\,. \tag{5.19}$$

Changing from the continuum normalisation to a box of size $V = L^3$ amounts to replace $(2\pi)^3$ by L^3. Thus the normalisation constant $N^{-2} = (2\pi)^3 2\omega$ becomes for a finite volume $N^{-2} = 2\omega V$. Thence the energy-density $T^{00} = \omega/V$ agrees with the expectation for one particle with energy ω per volume V. The remaining components of $T^{\mu\nu}$ are fixed by its tensor structure,

$$T^{\mu\nu} = 2N^2k^\mu k^\nu = \frac{k^\mu k^\nu}{\omega V}\,. \tag{5.20}$$

Since the stress tensor $T^{\mu\nu}$ is symmetric, we can find a frame in which $T^{\mu\nu}$ is diagonal with $T \propto \mathrm{diag}(\omega, v_xp_x, v_yp_y, v_zp_z)/V$. The spatial part of the stress tensor agrees with the pressure tensor of an ideal fluid, cf. Eq. (15.52). Thus a scalar field can be viewed as an ideal fluid with energy density ρ and pressure P, or $T^{\mu\nu} = \mathrm{diag}(\rho, P_x, P_y, P_z)$.

Angular momentum If the tensor $T^{\mu\nu}$ is symmetric, we can construct six more conserved quantities. If we define

$$M^{\mu\nu\lambda} = x^\nu\, T^{\mu\lambda} - x^\lambda\, T^{\mu\nu}\,, \tag{5.21}$$

then $M^{\mu\nu\lambda}$ is conserved with respect to the index μ,

$$\partial_\mu M^{\mu\nu\lambda} = \delta^\nu_\mu\, T^{\mu\lambda} - \delta^\lambda_\mu T^{\mu\nu} = T^{\nu\lambda} - T^{\lambda\nu} = 0\,, \tag{5.22}$$

provided that $T^{\nu\lambda} = T^{\lambda\nu}$. In this case,

$$J^{\mu\nu} = \int \mathrm{d}^3x\, M^{0\mu\nu} = \int \mathrm{d}^3x\, \left[x^\mu T^{0\nu} - x^\nu T^{0\mu}\right]\,, \tag{5.23}$$

is a globally conserved tensor. The antisymmetry of $J^{\mu\nu}$ implies that there exist six conserved charges. The three charges

$$J^{ij} = \int \mathrm{d}^3x\, \left[x^iT^{0j} - x^jT^{0i}\right] \tag{5.24}$$

[3] Another reason to require a symmetric stress tensor $T^{\mu\nu}$ is that it serves as source term for the symmetric gravitational field.

correspond to the conservation of total angular momentum, since T^{0j} is the momentum density. The remaining three charges J^{0i} express the fact that the center-of-mass moves with constant velocity.

While $J^{\mu\nu}$ transforms as expected for a tensor under Lorentz transformations, it is not invariant under translations $x^\mu \to x^\mu + \xi^\mu$. Instead, the angular momentum changes as

$$J^{\mu\nu} \to J^{\mu\nu} + \xi^\mu p^\nu - \xi^\nu p^\mu \,. \tag{5.25}$$

Clearly, this is a consequence of the definition of the orbital angular momentum with respect to the center of rotation. We want therefore to split the total angular momentum $J^{\mu\nu}$ into the orbital angular momentum $L^{\mu\nu}$ and an intrinsic part connected to a non-zero spin of the field. The latter we require to be invariant under translations. We set

$$S_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} u^\delta \,, \tag{5.26}$$

where u^α is the four-velocity of the center-of-mass system (cms). Because of the antisymmetry in $\beta\gamma$ of $\varepsilon_{\alpha\beta\gamma\delta}$, the change in (5.25) induced by a translation drops out in S_α. In the cms, $u^\alpha = (1,0,0,0)$ and thus $S_0 = 0$ and $S_\alpha u^\alpha = 0$. The other components are $S_1 = J^{23}$, $S_2 = J^{31}$, and $S_3 = J^{12}$. Thus the vector S_α describes as desired the intrinsic angular momentum of a field. It is called the Pauli-Lubanski spin vector.

5.3. Quantum symmetries

Conserved currents We have seen that Noether's theorem guaranties on the classical level the conservation of currents generated by global continuous symmetries. In the corresponding quantum theory, we have to study the impact of this symmetry on the generating functional Z. Since we used the equation of motions to derive Noether's theorem, current conservation holds only for classically allowed paths in field space, or in other words for on-shell fields. Thus the action evaluated for off-shell fields is not invariant under global symmetry transformations. In the path integral, the fields are however only integration variables. The generating functional is therefore invariant, if we can find a field transformation $\phi_i \to \tilde{\phi}_i$ which eliminates the change of the action for off-shell fields and keeps the integration measure invariant, $\mathcal{D}\phi_i = \mathcal{D}\tilde{\phi}_i$.

Let us assume that our theory has a global symmetry under which the classical solutions transform as $\phi_a \to \phi'_a = \phi_a + \varepsilon\eta_a$. Here, ε takes the same value at all space-time points and η_a is a function of the original fields, $\eta_a = \eta_a(\phi_a(x))$. The classically forbidden solutions will be transformed into $\tilde{\phi}_a \neq \phi'_a$. We can express $\tilde{\phi}_a$ always as

$$\phi_a \to \tilde{\phi}_a = \phi_a + \varepsilon(x)\eta_a \,, \tag{5.27}$$

promoting thereby $\varepsilon(x)$ to a space-time dependent function. To be concrete, we consider again a global U(1) symmetry for a complex scalar field. Since the transformation (5.27) is *local*, kinetic terms breaks the symmetry: Direct calculation shows that the Lagrangian (5.11) changes as

$$\delta\mathscr{L} = \mathrm{i}\,[\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi]\,\partial_\mu\varepsilon = j^\mu\,\partial_\mu\varepsilon \,, \tag{5.28}$$

where j^μ is the classical Noether current. In problem 5.6, you should fill in the details of this calculation and show that the final result $\delta\mathscr{L} = j^\mu\,\partial_\mu\varepsilon$ holds more generally.

Next we have to generalize the generating functional for a single, real scalar field given in Eq. (4.1) to a complex scalar field. We treat ϕ and ϕ^* as the two independent degrees of freedom, and add therefore also two independent sources J and J^*. Coupling them as $\mathscr{L}_\mathrm{s} \equiv J\phi^* + J^*\phi$ to the fields keeps the Lagrangian real. We denote the total Lagrangian as $\mathscr{L}_\mathrm{eff} = \mathscr{L}_\mathrm{cl} + \mathscr{L}_\mathrm{s}$, with $\mathscr{L}_\mathrm{cl} = \mathscr{L}_0 + \mathscr{L}_\mathrm{int}$ as the Lagrangian used to derive to the classical equation of motions. Thus the generating functional is

$$Z[J, J^*] = \int \mathcal{D}\phi\mathcal{D}\phi^* \exp \mathrm{i} \int \mathrm{d}^4x \ (\mathscr{L}_\mathrm{cl} + J\phi^* + J^*\phi) \ . \tag{5.29}$$

We want to calculate matrix elements of the operator representing the classical Noether current (5.8). Using the path integral formalism, we can derive the time-ordered vacuum expectation value of a product of fields ϕ and the current operator j^μ by adding a classical external source v_μ coupled to j^μ,

$$Z[J, J^*, v_\mu] = \int \mathcal{D}\phi\mathcal{D}\phi^* \exp \mathrm{i} \int \mathrm{d}^4x \ (\mathscr{L}_\mathrm{cl} + J\phi^* + J^*\phi + v_\mu j^\mu) \ . \tag{5.30}$$

Then we obtain e.g. the product of two fields and one current as

$$\langle 0|T\{j^\mu(x)\phi^*(x_1)\phi(x_2)\}|0\rangle = \frac{1}{\mathrm{i}^3} \frac{\delta^3}{\delta v_\mu(x)\delta J(x_1)\delta J^*(x_2)} \mathrm{i}W[J, J^*, v_\mu]\Big|_{v_\mu = J = 0} \ , \tag{5.31}$$

while the vacuum expectation value of the current is given by

$$\langle j^\mu(x)\rangle \equiv \langle 0|j^\mu(x)|0\rangle = \frac{1}{\mathrm{i}} \frac{\delta}{\delta v_\mu(x)} \mathrm{i}W[J, J^*, v_\mu]\Big|_{v_\mu = J = 0} \ . \tag{5.32}$$

Inverting this relation we find

$$\delta W[J, J^*, v_\mu] = \int \mathrm{d}^4x \, \langle j^\mu(x)\rangle \delta v_\mu(x) \, . \tag{5.33}$$

We are interested how W and Z change under a transformation of the external source v_μ. To deduce their transformation properties, it is sufficient to consider them for zero external sources J and J^*. Setting $Z[0, 0, v_\mu] \equiv Z[v_\mu]$ and choosing $\delta v_\mu(x) = -\partial_\mu \varepsilon(x)$, it is

$$\delta W[v_\mu] = W[v_\mu - \partial_\mu\varepsilon(x)] - W[v_\mu] = -\int \mathrm{d}^4x \, \langle j^\mu(x)\rangle \partial_\mu\varepsilon(x) = \int \mathrm{d}^4x \, \partial_\mu \langle j^\mu(x)\rangle \varepsilon(x) \, . \tag{5.34}$$

Thus $\delta W[v_\mu] = 0$ guarantees current conservation in the quantum theory, $\partial_\mu\langle j^\mu\rangle = 0$. The corresponding change of $Z[v_\mu]$ under the same transformation is

$$Z[v_\mu - \partial_\mu\varepsilon(x)] = \int \mathcal{D}\phi\mathcal{D}\phi^* \exp \mathrm{i} \int \mathrm{d}^4x \ \{\mathscr{L} + [v_\mu - \partial_\mu\varepsilon(x)]j^\mu\} \ . \tag{5.35}$$

We now assume that the substitution $\phi_a \to \tilde{\phi}_a = \phi_a + \varepsilon(x)\eta_a$ keeps the integration measure invariant, $\mathcal{D}\phi\mathcal{D}\phi^* = \mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\phi}^*$. Recalling then that $\delta\mathscr{L} = j^\mu \, \partial_\mu\varepsilon$, we find that the generating functional is invariant,

$$Z[v_\mu - \partial_\mu\varepsilon(x)] = \int \mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\phi}^* \exp \mathrm{i} \int \mathrm{d}^4x \ \Big(\mathscr{L}(\tilde{\phi}, \partial_\mu\tilde{\phi}) + v_\mu j^\mu\Big) = Z[v_\mu] \, . \tag{5.36}$$

In the case of the U(1) transformation, the two phases cancel in the integration measure

$$\mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\phi}^* = \prod_x \mathrm{d}\tilde{\phi}(x)\mathrm{d}\tilde{\phi}^*(x) = \prod_x \mathrm{d}\phi(x)\mathrm{d}\phi^*(x) \, . \tag{5.37}$$

As a result, the vev of the electromagnetic current is conserved, $\partial_\mu\langle j^\mu\rangle = 0$.

Anomalies The substitution $\phi_a \to \tilde{\phi}_a = \phi_a + \varepsilon(x)\eta_a$ shifts the center of the integration at each space-time point by the value $\varepsilon(x)\eta_a$. Such a linear shift seems harmless. Therefore it was taken for granted that the path integral remains invariant under this change and, consequently, that this approach predicts that all classical global symmetries hold also on the quantum level. It was only realised by Fujikawa in 1979 that the integration measure in the path integral may transform non-trivially under a symmetry: Since the path integral is divergent, we have to regularise it and this procedure may break the classical symmetry.

If the classical symmetry is broken, one speaks of an "anomaly." The three most important examples are the trace anomaly, the chiral anomaly, and the breaking of conformal invariance in string theory. We will discuss the first two cases later in some detail. The anomalous term breaking conformal invariance in string theory vanishes for a definite number of space-time dimensions, $D = 10$ or 26, what is the reason for the predictions of extra-dimensions in string theory.

Summary

Noether's theorem shows that continuous global symmetries lead classically to conservation laws. Such symmetries can be divided into space-time and internal symmetries. Minkowski space-time is invariant under global Poincaré transformations. The corresponding ten Noether charges are the four-momentum p^μ and the total angular momentum $J^{\mu\nu}$.

Since any local symmetry $O(x)$ contains as a special case global transformations O, local symmetries lead to the same conservation laws as the corresponding global symmetries. In a quantum theory, the vacuum expectation value of a Noether current is conserved, if the symmetry transformation keeps the path integral measure invariant. In the cases of the conservation of electric and colour charge, the global symmetry is a consequence of an underlying local gauge symmetry which we will study later in chapter 10 in detail. In most other cases however, as e.g. the conservation of baryon or lepton number, the global symmetry can not be generalised to a local one, and one speaks therefore of accidental symmetries. Such symmetries are not protected against quantum corrections and there is no reason to expect them to hold exactly. We will see later that baryon and lepton number are indeed broken.

Further reading

A more complete discussion of Noether's theorem can be found in Greiner (1996) and Hill (1951).

Problems

5.1 Lagrangian for N scalar fields.
The most general expression for the Lagrange density $\mathscr{L}$ of N scalar fields ϕ_i which is Lorentz invariant and at most quadratic in the fields is

$$\mathscr{L} = \frac{1}{2}A_{ij}\partial^\mu\phi_i\partial_\mu\phi_j - \frac{1}{2}B_{ij}\phi_i\phi_j - C\,.$$

Find the constraints on the coefficients and show that $\mathscr{L}$ can be recast into "canonical form"

$$\mathscr{L} = \frac{1}{2}\partial^\mu\phi_i\partial_\mu\phi_i - \frac{1}{2}b_i\phi_i\phi_i - C$$

by linear field redefinitions.

5.2 Stress tensor for point particles.
a.) Find the stress tensor for an ensemble of N non-relativistic point particles; b) on scales L such

that $\Delta N/L^3 \gg 1$, one can describe the phase space density by a smooth function $f(\boldsymbol{x}, \boldsymbol{p})$. Discuss the physical meaning of the different elements of $T^{\mu\nu}$.

5.3 Stress tensor for an ideal fluid.
a.) Find first the stress tensor of dust, i.e. of pressure-less matter.
b.) Generalise this result to an ideal fluid. (Hint: The state of an ideal fluid is completely determined by its energy density ρ and its pressure P; in the rest-frame of the fluid, the pressure is isotropic $P_{ij} = P\delta_{ij}$.)
c.) Compare $\partial_\mu T^{\mu\nu} = 0$ to the ideal fluid equations.

5.4 Stress tensor for the electromagnetic field.
Determine the canonical energy-momentum stress tensor $T^{\mu\nu}$ of the free Maxwell field. Symmetrize $T^{\mu\nu}$ and determine the energy-density $\rho = T^{00}$. Find the trace of $T^{\mu\nu}$.

5.5 Scale invariance.
Consider the effect of a scale transformation $x \to \mathrm{e}^\alpha x$ on a scalar field with a $\lambda\phi^4$ self-interaction assuming that it acts linearly on the fields, $\phi(x) \to \mathrm{e}^{D\alpha}\phi(\mathrm{e}^\alpha x)$. Here α and D are numbers. i) Write down first the infinitesimal version of the scale transformation, with the aim to show that $\mathscr{L}$ can be made invariant for a specific value of D, if $m = 0$.
ii) Find the corresponding conserved current s^μ.
iii) Show that the current s^μ can be written as $s^\mu = x^\nu \tilde{T}_{\mu\nu}$, where $\tilde{T}_{\mu\nu}$ is an "improved" energy-momentum tensor. Hint: Proceed similar as in the case of the angular momentum tensor.

5.6 $\delta\mathscr{L}$ from a local U(1) transformation.
Show that the change of the Lagrangian (5.11) under a local U(1) transformation can be written as (5.28).

6. Space-time symmetries

In the previous chapter, we discussed the symmetries of Minkowski space. In this case, we could view the Poincaré group as the group generating global symmetry transformations on Minkowski space and find the resulting conservation laws. Aim of the present chapter is to extend this discussion to the case of a Riemannian manifold, i.e. to a curved space which looks only locally Euclidean. We will show how one can find the symmetries of such manifolds and how they determine conservation laws.

Riemannian manifolds arise naturally in classical mechanics using generalised coordinates q^i, since the kinetic energy $T = a_{ik}\dot{q}^i\dot{q}^k$ defines a quadratic form a_{ik} which we can view as metric tensor on the configuration space $\{q^i\}$. However, the for us more important appearence of a (pseudo-) Riemanian manifold is in Einstein's theory of general relativity which replaces Minkowski space by a curved space-time. Most of the mathematical structures we will introduce have also a close analogue in gauge theories which we will use later on to describe the electroweak and strong interactions.

Equivalence principle As a start, we motivate why one can replace the gravitational force by the curvature of space-time discussing the equivalence principle. The idea underlying this principle emerged in the 16th century, when among others Galileo Galilei found experimentally that the acceleration g of a test mass in a gravitational field is universal. Because of this universality, the gravitating mass m_g and the inertial mass m_i are identical in classical mechanics. While $m_i = m_g$ can be achieved for one material always by a convenient choice of units, there should be in general deviations for test bodies with differing compositions. Current limits for departures from universal gravitational attraction for different materials are however very tight, $|\Delta g/g| < 10^{-12}$.

As a result, gravity has compared to the three other known fundamental interactions the unique property that it can be switched-off locally: Inside a freely falling elevator, one does not feel any gravitational effects except tidal forces. The latter arise if the gravitational field is non-uniform and tries to deform the elevator. Inside a sufficiently small freely falling system, also tidal effects plays no role. Einstein promoted the equivalence of inertial and gravitating mass to the postulate of the "strong equivalence principle:" In a small enough region around the center of a freely falling coordinate system all physics is described by the laws of special relativity.

In general relativity, the gravitational force of Newton's theory that accelerates particles in an Euclidean space is replaced by a curved space-time in which particles move force-free along geodesic lines. In particular, photons move still as in special relativity along curves satisfying $\mathrm{d}s^2 = 0$, while all effects of gravity are now encoded in the non-Euclidean geometry of space-time which is determined by the line-element $\mathrm{d}s^2$ or the metric tensor $g_{\mu\nu}$,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{6.1}$$

Switching on a gravitational field, the metric tensor $g_{\mu\nu}$ can be tranformed only locally by a coordinate change into the form $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Thus we should develop the tools

necessary to do analysis on a curved manifold $\mathscr{M}$ which geometry is described by the metric tensor $g_{\mu\nu}$.

6.1. Manifolds and tensor fields

Manifolds A manifold $\mathscr{M}$ is any set that can be continuously parametrized. The number of independent parameters needed to specify uniquely any point of $\mathscr{M}$ is its dimension n, the parameters $x = \{x^1, \ldots, x^n\}$ are called coordinates. Locally, a manifold with dimension n can be approximated by $\mathbb{R}^n$. Examples for manifolds are Lie groups, the configuration space q^i or the phase space (q^i, p_i) of classical mechanics, and space-time in general relativity.

We require the manifold to be smooth: The transitions from one set of coordinates to another one, $x^i = f(\tilde{x}^i, \ldots, \tilde{x}^n)$, should be C^∞. In general, it is impossible to cover all $\mathscr{M}$ with one coordinate system that is well-defined on all $\mathscr{M}$. An example are spherical coordinate (ϑ, ϕ) on a sphere S^2, where ϕ is ill-defined at the poles. Instead one has to cover the manifold with patches of different coordinates that partially overlap.

Vector fields A vector field $\boldsymbol{V}(x^a)$ on (a subset $\mathscr{S}$ of) $\mathcal{M}$ is a set of vectors associating to each space-time point $x^a \in \mathscr{S}$ exactly one vector. The paradigm for such a vector field is the four-velocity $\boldsymbol{u}(\tau) = \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ which is the tangent vector to the world-line $x(\tau)$ of a particle. Since the differential equation $\mathrm{d}\boldsymbol{x}/\mathrm{d}\sigma = \boldsymbol{X}(\sigma)$ has locally always a solution, we can find for any given $\boldsymbol{X}$ a curve $x(\sigma)$ which has $\boldsymbol{X}$ as tangent vector. Although the definition $\boldsymbol{u}(\tau) = \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ coincides with the one familiar from Minkowski space, there an important difference: In a general manifold, we can not imagine a vector $\boldsymbol{V}$ as an "arrow" $\overrightarrow{PP'}$ pointing from a certain point P to another point P' of the manifold. Instead, the vectors $\boldsymbol{V}$ generated by all smooth curves through P span a n-dimensional vector space at the point P called tangent space T_P. We can visualise the tangent space for the case of a two-dimensional manifold embedded in $\mathbb{R}^3$: At any point P, the tangent vectors lie in a plane $\mathbb{R}^2$ which we can associate with T_P. In general, $T_P \neq T_{P'}$ and we cannot simply move a vector $\boldsymbol{V}(x^\mu)$ to another point $\tilde{x}^\mu$. This implies in particular that we cannot add the vectors $\boldsymbol{V}(x^\mu)$ and $\boldsymbol{V}(\tilde{x}^\mu)$, if the points x^μ and $\tilde{x}^\mu$ differ. Therefore we cannot differentiate a vector field without introducing an additional mathematical structure which allows us to transport a vector from one tangent space to another.

If we want to decompose the vector $\boldsymbol{V}(x^\mu)$ into components $V^\nu(x^\mu)$, we have to introduce a basis $\boldsymbol{e}_\mu$ in the tangent space. There are two natural choices for such a basis: First, we could use Cartesian basis vectors as in a Cartesian inertial system in Minkowski space. We will follow this approach later, when we discuss gravity as a gauge theory in chapter 19. Now, we will use the more conventional approach and use as basis vectors the tangential vectors along the coordinate lines x^μ in $\mathscr{M}$,

$$\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu \,. \tag{6.2}$$

Note that the index μ with value i in $\boldsymbol{e}_\mu$ denotes the i.th basis vector $\boldsymbol{e}_\mu = (0, \ldots, 1, \ldots 0)$, with an one at the i.th position, not a component. Using this basis, a vector can be decomposed as

$$\boldsymbol{V} = V^\mu \boldsymbol{e}_\mu = V^\mu \partial_\mu \,. \tag{6.3}$$

A coordinate change

$$x^\mu = f(\tilde{x}^1, \ldots, \tilde{x}^n)\,, \tag{6.4}$$

or more briefly $x^\mu = x^\mu(\tilde{x}^\nu)$ changes the basis vectors as

$$\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\,\frac{\partial}{\partial \tilde{x}^\nu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\,\tilde{\boldsymbol{e}}_\nu\,. \tag{6.5}$$

Therefore the vector $\boldsymbol{V}$ will be invariant under general coordinate transformations,

$$\boldsymbol{V} = V^\mu \partial_\mu = \tilde{V}^\mu \tilde{\partial}_\mu = \tilde{\boldsymbol{V}}, \tag{6.6}$$

if its components transform opposite to the basis vectors $\boldsymbol{e}_\mu = \partial_\mu$, or

$$V^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}\,\tilde{V}^\nu. \tag{6.7}$$

Covectors or one-forms In quantum mechanics, we use Dirac's bracket notation to associate to each vector $|a\rangle$ a dual vector $\langle a|$ and to introduce a scalar product $\langle a|\,b\rangle$. If the vectors $|n\rangle$ form a basis, then the dual basis $\langle n|$ is defined by $\langle n|\,n'\rangle = \delta_{nn'}$. Similiarly, we define a basis $\boldsymbol{e}^\mu$ dual to the basis $\boldsymbol{e}_\mu$ in T_P by

$$\boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = \delta^\mu_\nu\,. \tag{6.8}$$

This basis can be used to form a new vector space T_P^* called the cotangent space which is dual to T_P. Its elements $\boldsymbol{\omega}$ are called covectors or one-forms,

$$\boldsymbol{\omega} = \omega_\mu \boldsymbol{e}^\mu\,. \tag{6.9}$$

Combining a vector and an one-form, we obtain a map into the real numbers,

$$\boldsymbol{\omega}(\boldsymbol{V}) = \omega_\mu V^\nu \boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = \omega_\mu V^\mu\,. \tag{6.10}$$

The last equality shows that we can calculate $\boldsymbol{\omega}(\boldsymbol{V})$ in component form without reference to the basis vectors. In order to simplify notation, we will use therefore in the future simply $\omega_\mu V^\mu$; we also write $\boldsymbol{e}^\mu \boldsymbol{e}_\nu$ instead of $\boldsymbol{e}^\mu(\boldsymbol{e}_\nu)$.

Using a coordinate basis, the duality condition (6.8) is obviously satisfied, if we choose $\boldsymbol{e}^\mu = \mathrm{d}x^\mu$. Then the one-form becomes

$$\boldsymbol{\omega} = \omega_\mu \mathrm{d}x^\mu\,. \tag{6.11}$$

Thus the familiar "infinitesimals" $\mathrm{d}x^\mu$ are actually the finite basis vectors of the cotangent space T_P^*. We require again that the transformation of the components ω_μ of a covector cancels the transformation of the basis vectors,

$$\omega_\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\,\tilde{\omega}_\nu\,. \tag{6.12}$$

This condition guarantees that the covector itself is an invariant object, since

$$\boldsymbol{\omega} = \omega_\mu \mathrm{d}x^\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\,\tilde{\omega}_\nu \frac{\partial x^\mu}{\partial \tilde{x}^\sigma}\,\mathrm{d}\tilde{x}^\sigma = \tilde{\omega}_\mu \mathrm{d}\tilde{x}^\mu = \tilde{\boldsymbol{\omega}}\,. \tag{6.13}$$

Covariant and contravariant tensors Next we generalise the concept of vectors and covectors. We call a vector $\boldsymbol{X}$ also a *contravariant tensor* of rank one, while we call a covector also a *covariant vector* or covariant tensor of rank one. A general tensor of rank (n, m) is a multilinear map

$$\boldsymbol{T} = T^{\mu,\ldots,\nu}_{\alpha,\ldots,\beta} \underbrace{\partial_\mu \otimes \ldots \otimes \partial_\nu}_{n} \otimes \underbrace{\mathrm{d}x^\alpha \otimes \ldots \otimes \mathrm{d}x^\beta}_{m} \tag{6.14}$$

which components transforms as

$$\tilde{T}^{\mu,\ldots,\nu}_{\alpha,\ldots,\beta}(\tilde{x}) = \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\rho} \cdots \frac{\partial \tilde{x}^\nu}{\partial x^\sigma}}_{n} \underbrace{\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \cdots \frac{\partial x^\delta}{\partial \tilde{x}^\beta}}_{m} T^{\rho,\ldots,\sigma}_{\gamma,\ldots,\delta}(x) \tag{6.15}$$

under a coordinate change.

Metric tensor A (pseudo-) Riemannian manifold is a differentiable manifold containing as additional structure a symmetric tensor field $g_{\mu\nu}$ which allows us to measure distances and angles. We define the scalar product of two vectors $\boldsymbol{a}(x)$ and $\boldsymbol{b}(x)$ which have the coordinates a^μ and b^μ in a certain basis $\boldsymbol{e}_\mu$ as

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a^\mu \boldsymbol{e}_\mu) \cdot (b^\nu \boldsymbol{e}_\nu) = (\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu) a^\mu b^\nu = g_{\mu\nu} a^\mu b^\nu \,. \tag{6.16}$$

Thus we can evaluate the scalar product between any two vectors, if we know the symmetric matrix $g_{\mu\nu}$ composed out of the N^2 products of the basis vectors,

$$g_{\mu\nu}(x) = \boldsymbol{e}_\mu(x) \cdot \boldsymbol{e}_\nu(x) \,, \tag{6.17}$$

at any point x of the manifold. This symmetric matrix $g_{\mu\nu}$ is called *metric tensor*. The manifold is called Riemannian, if all eigenvalues of $g_{\mu\nu}$ are positive, and thus the scalar product defined by $g_{\mu\nu}$ is positive-definite. If the scalar product is indefinite, as in the case of general relativity, one calls the manifold pseudo-Riemannian.

In the same way, we define for the dual basis $\boldsymbol{e}^\mu$ the metric $g^{\mu\nu}$ via

$$g^{\mu\nu} = \boldsymbol{e}^\mu \cdot \boldsymbol{e}^\nu \,. \tag{6.18}$$

A comparison with Eq. (6.10) shows that the metric $g^{\mu\nu}$ maps covariant vectors X_μ into contravariants vectors X^μ, while $g_{\mu\nu}$ provides a map into the opposite direction. In the same way, we can use the metric tensor to raise and lower indices of any tensor.

Next we want to determine the relation of $g^{\mu\nu}$ with $g_{\mu\nu}$. We multiply $\boldsymbol{e}^\rho$ with $\boldsymbol{e}_\mu = g_{\mu\nu}\boldsymbol{e}^\nu$, obtaining

$$\delta^\rho_\mu = \boldsymbol{e}^\rho \cdot \boldsymbol{e}_\mu = \boldsymbol{e}^\rho \cdot g_{\mu\nu}\boldsymbol{e}^\nu = g^{\rho\nu} g_{\mu\nu} \tag{6.19}$$

or

$$\delta^\rho_\mu = g_{\mu\nu} g^{\nu\rho} \,. \tag{6.20}$$

Thus the components of the covariant and the contravariant metric tensors, $g_{\mu\nu}$ and $g^{\mu\nu}$, are inverse matrices of each other. Moreover, the mixed metric tensor of rank (1,1) is given by the Kronecker delta, $g^\nu_\mu = \delta^\nu_\mu$. Note that the trace of the metric tensor is therefore not -2, but

$$\mathrm{tr}(g_{\mu\nu}) = g^{\mu\mu} g_{\mu\mu} = \delta^\mu_\mu = 4 \,, \tag{6.21}$$

because we have to contract an upper and a lower index.

6.2. Covariant derivative and the geodesic equation

Covariant derivative In an inertial system in Minkowski space, taking the partial derivative ∂_μ maps a tensor of rank (n, m) into a tensor of rank $(n+1, m)$. Additionally, this map obeys linearity and the Leibniz product rule. We will see that in general the partial derivative in a curved space does not satisfy these rules. Therefrore, we have to introduce a modified derivative which we call covariant derivative.

We start by considering the gradient $\partial_\mu\phi$ of a scalar ϕ. By definition, a scalar quantity does not depend on the coordinate system, $\phi(x) = \tilde{\phi}(\tilde{x})$. Therefore its gradient transforms as

$$\partial_\mu\phi \to \tilde{\partial}_\mu\tilde{\phi} = \frac{\partial x^\nu}{\partial\tilde{x}^\mu}\partial_\nu\phi\,. \tag{6.22}$$

Thus the gradient is a covariant vector. Simililarly, the derivative of a vector $\boldsymbol{V}$ transforms as a tensor,

$$\partial_\mu\boldsymbol{V} \to \tilde{\partial}_\mu\tilde{\boldsymbol{V}} = \frac{\partial x^\nu}{\partial\tilde{x}^\mu}\partial_\nu\boldsymbol{V}\,, \tag{6.23}$$

because $\boldsymbol{V}$ is an invariant quantity. If we consider however its components $V^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{V}$, then the moving coordinate basis in curved space-time, $\partial_\mu\boldsymbol{e}^\nu \neq 0$, leads to an additional term in the derivative,

$$\partial_\mu V^\nu = \boldsymbol{e}^\nu \cdot (\partial_\mu\boldsymbol{V}) + \boldsymbol{V} \cdot (\partial_\mu\boldsymbol{e}^\nu)\,. \tag{6.24}$$

The term $\boldsymbol{e}^\nu \cdot (\partial_\mu\boldsymbol{V})$ transforms as a tensor, since both $\boldsymbol{e}^\nu$ and $\partial_\mu\boldsymbol{V}$ are tensors. This implies that the combination of the two remaining terms has to transform as tensor too, wich we define as covariant derivative

$$\nabla_\mu V^\nu \equiv \boldsymbol{e}^\nu \cdot (\partial_\mu\boldsymbol{V}) = \partial_\mu V^\nu - \boldsymbol{V} \cdot (\partial_\mu\boldsymbol{e}^\nu)\,. \tag{6.25}$$

The first equality tells us that we can view the covariant derivative $\nabla_\mu V^\nu$ as the projection of $\partial_\mu\boldsymbol{V}$ onto the direction $\boldsymbol{e}^\nu$.

We expand now the partial derivatives of the basis vectors as a linear combination of the basis vectors,

$$\partial_\rho\boldsymbol{e}^\mu = -\Gamma^\mu{}_{\rho\sigma}\boldsymbol{e}^\sigma\,. \tag{6.26}$$

The n^3 numbers $\Gamma^\mu{}_{\rho\sigma}$ are called (affine) connection coefficients or symbols, in order to stress that they are not the components of a tensor. You are asked to derive their transformation properties in problem 6.6. Introducing this expansion into (6.25) we can rewrite the covariant derivative of a vector field as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\sigma\mu}V^\sigma\,. \tag{6.27}$$

Using $\nabla_\sigma\phi = \partial_\sigma\phi$ and requiring that the usual Leibniz rule is valid for $\phi = X_\mu X^\mu$ leads to

$$\nabla_\sigma X_\mu = \partial_\sigma X_\mu - \Gamma^\nu{}_{\mu\sigma}X_\nu \tag{6.28}$$

and to

$$\partial_\rho\boldsymbol{e}_\mu = \Gamma^\sigma{}_{\mu\rho}\boldsymbol{e}_\sigma\,. \tag{6.29}$$

For a general tensor, the covariant derivative is defined by the same reasoning as

$$\nabla_\sigma T^{\mu\ldots}_{\nu\ldots} = \partial_\sigma T^{\mu\ldots}_{\nu\ldots} + \Gamma^\mu{}_{\rho\sigma}T^{\rho\ldots}_{\nu\ldots} + \ldots - \Gamma^\rho{}_{\nu\sigma}T^{\mu\ldots}_{\rho\ldots} - \ldots \tag{6.30}$$

Note that it is the last index of the connection coefficients that is the same as the index of the covariant derivative. The *p*lus sign goes together with upper (su*p*erscripts), the minus with lower indices.

Parallel transport We say a tensor $\boldsymbol{T}$ is parallel transported along the curve $x(\sigma)$, if its components $T^{\mu\dots}_{\nu\dots}$ stay constant. In flat space, this means simply

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}T^{\mu\dots}_{\nu\dots} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\partial_\alpha T^{\mu\dots}_{\nu\dots} = 0\,. \tag{6.31}$$

In curved space, we have to replace the normal derivative by a covariant one. We define the directional covariant derivative along $x(\sigma)$ as

$$\frac{D}{\mathrm{d}\sigma} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\nabla_\alpha\,. \tag{6.32}$$

Then a tensor is parallel transported along the curve $x(\sigma)$, if

$$\frac{D}{\mathrm{d}\sigma}T^{\mu\dots}_{\nu\dots} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\nabla_\alpha T^{\mu\dots}_{\nu\dots} = 0\,. \tag{6.33}$$

Metric compatibility Relations like $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ or $g_{\mu\nu}p^\mu p^\nu = m^2$ become invariant under parallel transport only, if the metric tensor is covariantly constant,

$$\nabla_\sigma g_{\mu\nu} = \nabla_\sigma g^{\mu\nu} = 0\,. \tag{6.34}$$

A connection satisfying Eq. (6.34) is called metric compatible and leaves lengths and angles invariant under parallel transport. This requirement gurantees that we can introduce locally in the whole space-time Cartesian inertial coordinate systems where the laws of special relativity are valid. Moreover, these local inertial systems can be consistently connected by parallel transport using an affine connection satisfying the constraint (6.34).

Note that we have already built in this constraint into our definition of the covariant derivative: If the length of a vector would not be conserved under parallel transport, then we should differentiante in (6.24) also the scalar product in $V^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{V}$, leading to an additional term in Eq. (6.25).

Geodesic equation The requirement that the affine connection is metric compatible fixes the connection not uniquely, and thus the question arises which connection describes physics on a general space-time? Ultimately, the combined action for gravity and matter should select the correct connection—an approach we resume in Chapter 19. For the moment, we use a simple workaround which does not require the knowledge of the action of gravity: In flat space, we know that the solution to the equation of motions of a free particle is a straight line. Such a path is characterized by two properties: It is the shortest curve between the considered initial and final point, and it is the curve whose tangent vectors remains constant if they are parallel transported along it. Both conditions can be generalised to curved space and the curves satisfying either one of them are called geodesics.

Using the definition of a geodesics as the "straightest" line on a manifold requires as mathematical structure only the possibility to parallel transport a tensor and thus the existence of an affine connection. In contrast, the concept of an "extremal" (shortest or longest) line between two points on a manifold relies on the existence of a metric. Requiring that these two definitions agree fixes uniquely the connection to be used in the covariant derivative.

We start by defining geodesics as the "straightest" line or an autoparallel curve on a manifold—the case which is almost trivial: The tangent vector along the path $x(\tau)$ is

$u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Then the requirement (6.33) of parallel transport for u^μ becomes

$$\frac{D}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0\,. \tag{6.35}$$

Introducing $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, we obtain the geodesic equation in its standard form,

$$\ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0\,. \tag{6.36}$$

Note that a possible antisymmetric part of the connection $\Gamma^\mu{}_{\rho\sigma}$ drops out of the geodesic equation, because $\dot{x}^\rho\dot{x}^\sigma$ is symmetric: Contracting a symmetric tensor $S_{\mu\nu}$ with an antisymmetric tensor $A_{\mu\nu}$, we find

$$S_{\mu\nu}A^{\mu\nu} = -S_{\mu\nu}A^{\nu\mu} = -S_{\nu\mu}A^{\nu\mu} = -S_{\mu\nu}A^{\mu\nu}\,, \tag{6.37}$$

where we used first the antisymmetry of $A^{\mu\nu}$, then the symmetry of $S_{\mu\nu}$, and finally exchanged the dummy summation indices. Thus the contraction is zero, $S_{\mu\nu}A^{\mu\nu} = 0$.

Next we derive the defining equation for a geodesics as the extremal curve between two points on a manifold. The Lagrangian of a free particle in Minkowski space, Eq. (1.55), is generalized to a curved space-time manifold with the metric tensor $g_{\mu\nu}$ by replacing $\eta_{\mu\nu}$ with $g_{\mu\nu}$ (we set also $m = -1$),

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,. \tag{6.38}$$

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial(\dot{x}^\lambda)} - \frac{\partial L}{\partial x^\lambda} = 0\,. \tag{6.39}$$

Only the metric tensor $g_{\mu\nu}$ depends on x^μ and thus $\partial L/\partial x^\lambda = g_{\mu\nu,\lambda}\dot{x}^\mu\dot{x}^\nu$. Here we introduced also the short-hand notation $g_{\mu\nu,\lambda} = \partial_\lambda g_{\mu\nu}$ for partial derivatives. Now we use $\partial\dot{x}^\mu/\partial\dot{x}^\nu = \delta^\mu_\nu$ and apply the chain rule for $g_{\mu\nu}(x(\sigma))$, obtaining first

$$g_{\mu\nu,\lambda}\dot{x}^\mu\dot{x}^\nu = 2\,\frac{\mathrm{d}}{\mathrm{d}\sigma}\,(g_{\mu\lambda}\dot{x}^\mu) = 2(g_{\mu\lambda,\nu}\dot{x}^\mu\dot{x}^\nu + g_{\mu\lambda}\ddot{x}^\mu) \tag{6.40}$$

and then

$$g_{\mu\lambda}\ddot{x}^\mu + \frac{1}{2}(2g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda})\dot{x}^\mu\dot{x}^\nu = 0\,. \tag{6.41}$$

Next we rewrite the second term as

$$2g_{\lambda\mu,\nu}\dot{x}^\mu\dot{x}^\nu = (g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu})\dot{x}^\mu\dot{x}^\nu\,, \tag{6.42}$$

multiply everything by $g^{\kappa\mu}$ and arrive at our desired result,

$$\ddot{x}^\kappa + \frac{1}{2}g^{\kappa\lambda}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda})\dot{x}^\mu\dot{x}^\nu = \ddot{x}^\kappa + \{{}^{\kappa}_{\mu\nu}\}\dot{x}^\mu\dot{x}^\nu = 0\,. \tag{6.43}$$

Here we defined in the last step the Christoffel symbols

$$\{{}^{\mu}_{\nu\lambda}\} = \frac{1}{2}g^{\mu\kappa}(\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\kappa g_{\nu\lambda})\,. \tag{6.44}$$

They are also called Levi-Civita or Riemannian connection. A comparison with Eq. (6.36) shows that our two geodesic equations agree, if we choose as connection the Christoffel symbols. Moreover, the Christoffel symbols are symmetric in their two lower indices and, as we

will show next, compatible to the metric tensor. Following standard practise, we will denote them also with $\Gamma^{\lambda}{}_{\mu\nu}$. In the remainder of this section, we will use always as affine connection the Christoffel symbols.

We define[1]

$$\Gamma_{\mu\nu\lambda} = g_{\mu\kappa}\Gamma^{\kappa}{}_{\nu\lambda}\,. \tag{6.45}$$

Thus $\Gamma_{\mu\nu\lambda}$ is symmetric in the last two indices. Then it follows

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}(\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda})\,. \tag{6.46}$$

Adding $2\Gamma_{\mu\nu\lambda}$ and $2\Gamma_{\nu\mu\lambda}$ gives

$$\begin{aligned} 2(\Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}) &= \partial_b g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda} \\ &\quad + \partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} = 2\partial_\lambda g_{\mu\nu} \end{aligned} \tag{6.47}$$

or

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}\,. \tag{6.48}$$

Applying the general rule for covariant derivatives, Eq. (6.30), to the metric,

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^{\kappa}{}_{\mu c} g_{\kappa\nu} - \Gamma^{\kappa}{}_{\nu\lambda} g_{\mu\kappa} = \partial_\lambda g_{\mu\nu} - \Gamma_{\mu\lambda\nu} - \Gamma_{\nu\lambda\mu}\,, \tag{6.49}$$

and inserting Eq. (6.48) shows that

$$\nabla_\lambda g_{\mu\nu} = \nabla_\lambda g^{\mu\nu} = 0\,. \tag{6.50}$$

Hence ∇_λ commutes with contracting indices,

$$\nabla_\lambda(X^\mu X_\mu) = \nabla_\lambda(g_{\mu\nu}X^\mu X^\mu) = g_{\mu\nu}\nabla_\lambda(X^\mu X^\nu) \tag{6.51}$$

and conserves the norm of vectors as announced. Thus the Christoffel symbols are symmetric and compatible with the metric. These two properties specifiy uniquely the connection.

Example 6.1: Calculate the Christoffel symbols of the two-dimensional unit sphere S^2.
The line-element of the two-dimensional unit sphere S^2 is given by $\mathrm{d}s^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2$. A faster alternative to the definition (6.44) of the Christoffel coefficients is the use of the geodesic equation: From the Lagrange function $L = g_{ab}\dot{x}^a\dot{x}^b = \dot{\vartheta}^2 + \sin^2\vartheta\dot{\phi}^2$ we find

$$\frac{\partial L}{\partial \phi} = 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = \frac{\mathrm{d}}{\mathrm{d}t}(2\sin^2\vartheta\dot{\phi}) = 2\sin^2\vartheta\ddot{\phi} + 4\cos\vartheta\sin\vartheta\dot{\vartheta}\dot{\phi}$$

$$\frac{\partial L}{\partial \vartheta} = 2\cos\vartheta\sin\vartheta\dot{\phi}^2\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}} = \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{\vartheta}) = 2\ddot{\vartheta}$$

and thus the Lagrange equations are

$$\ddot{\phi} + 2\cot\vartheta\dot{\vartheta}\dot{\phi} = 0 \qquad \text{and} \qquad \ddot{\vartheta} - \cos\vartheta\sin\vartheta\dot{\phi}^2 = 0\,.$$

Comparing with the geodesic equation $\ddot{x}^\kappa + \Gamma^{\kappa}{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, we can read off the non-vanishing Christoffel symbols as $\Gamma^{\phi}{}_{\vartheta\phi} = \Gamma^{\phi}{}_{\phi\vartheta} = \cot\vartheta$ and $\Gamma^{\vartheta}{}_{\phi\phi} = -\cos\vartheta\sin\vartheta$. (Note that $2\cot\vartheta = \Gamma^{\phi}{}_{\vartheta\phi} + \Gamma^{\phi}{}_{\phi\vartheta}$.)

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[1] We showed that the metric tensor can be used to raise or to lower tensor indices, but the connection Γ is not a tensor.

We can use also the Hamiltonian formulation for a relativistic particle. From the Lagrangian $L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ we determine first the conjugated momenta $p_\mu = \partial L / \partial \dot{x}^\mu = \dot{x}_\mu$ and perform then a Legendre transformation,

$$H(x^\mu, p_\mu, \tau) = p_\mu \dot{x}^\mu - L(x^\mu, \dot{x}^\mu, \tau) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \,. \tag{6.52}$$

Since the Lagrangian of a free particle does not depend explicitely on the evolution parameter σ, there exists at least one conserved quantity. This conservation law, $H = 1/2$, expresses the fact that the tangent vector $\dot{x}^\mu$ has a constant norm. Hamilton equations give then

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = g^{\mu\nu} p_\nu \tag{6.53}$$

and

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2} \frac{\partial g^{\alpha\beta}}{\partial x^\mu} p_\alpha p_\beta \,. \tag{6.54}$$

This is a useful alternative to the standard geodesic equation: First, it makes clear that the momentum component p_μ is conserved, if the metric tensor is independent of the coordinate x^μ. Second, we can calculate $\dot{p}_\mu$ directly from the metric tensor, without knowing the Christoffel symbols. Combining the Eqs. (6.53) and (6.54) one can rederive the standard form of the geodesic equation, cf. problem 6.7.

6.3. Integration and Gauss' theorem

Having defined the covariant derivative of an arbitrary tensor field, it is natural to ask how the inverse, the integral over a tensor field, can be defined. The short answer is that this is in general impossible: Integrating a tensor field requires to sum tensors at different points in an invariant way, which is only possible for scalars. Restricting ourselves to scalar fields, we should generalise an integral like $I = \int \mathrm{d}^4x \, \phi(x)$ valid in an Cartesian intertial frame x^μ in Minkowski space to a general space-time with coordinates $\tilde{x}$. For a general coordinate transformation $x^\mu \to \tilde{x}^\mu$, we have to take into account that the Jacobi determinant $J = \det(\partial \tilde{x}^\mu / \partial x^\rho)$ of the transformation can deviate from one. We can express this Jacobian by the determinant $g \equiv \det(g_{\mu\nu})$ of the metric tensor as follows: Applying the tranformation law of the metric tensor,

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} \, g^{\rho\sigma}(x) \tag{6.55}$$

to the case where the x^ρ are inertial coordinates, we obtain with $g = \det(g_{\mu\nu}) = \det(\eta_{\mu\nu}) = -1$ that

$$\det(\tilde{g}) = J^2 \det(g) = -J^2 \tag{6.56}$$

or $J = \sqrt{|\tilde{g}|}$. Thus $I = \int \mathrm{d}^4x \sqrt{|g|} \, \phi$ is an invariant definition of an integral over a scalar field which agrees for inertial coordinates with the one known from special relativity. Now we choose as scalar ϕ the divergence of a vector field, $\phi = \nabla_\mu X^\mu$, or

$$I = \int \mathrm{d}^4x \sqrt{|g|} \, \nabla_\mu X^\mu = \int \mathrm{d}^4x \sqrt{|g|} \, \left(\partial_\mu X^\mu + \Gamma^\mu{}_{\lambda\mu} X^\lambda \right) \,. \tag{6.57}$$

Our aim is to derive a generalised Gauss' theorem. Let us recall that this theorem allows us to convert a n dimensional volume integral over the divergence of a vector field into a $n-1$ dimensional surface integral of the vector field,

$$\int_\Omega \mathrm{d}^4x\, \partial_\mu X^\mu = \int_{\partial\Omega} \mathrm{d}S_\mu X^\mu \,. \tag{6.58}$$

The only way how (6.58) may be reconciled with (6.57) is to hope that we can express the covariant divergence as $1/\sqrt{|g|}\partial_\mu(\sqrt{|g|}X^\mu)$. In order to check this possibility, we determine first the partial derivative of the metric determinant g. As preparation, we consider the variation of a general matrix M with elements $m_{ij}(x)$ under an infinitesimal change of the coordinates, $\delta x^\mu = \varepsilon x^\mu$. It is convenient to look at the change of $\ln\det M$,

$$\begin{aligned}\delta \ln\det M &\equiv \ln\det(M+\delta M) - \ln\det(M) \\ &= \ln\det[M^{-1}(M+\delta M)] = \ln\det[1+M^{-1}\delta M] = \\ &= \ln[1+\mathrm{tr}(M^{-1}\delta M)] + \mathcal{O}(\varepsilon^2) = \mathrm{tr}(M^{-1}\delta M) + \mathcal{O}(\varepsilon^2)\,. \end{aligned} \tag{6.59}$$

In the last step, we used $\ln(1+\varepsilon) = \varepsilon + \mathcal{O}(\varepsilon^2)$. Expressing now both the LHS and the RHS as $\delta M = \partial_\mu M \delta x^\mu$ and comparing then the coefficients of δx^μ gives

$$\partial_\mu \ln\det M = \mathrm{tr}(M^{-1}\partial_\mu M)\,. \tag{6.60}$$

Applied to derivatives of $\sqrt{|g|}$, we obtain

$$\frac{1}{2}g^{\mu\nu}\partial_\lambda g_{\mu\nu} = \frac{1}{2}\partial_\lambda \ln g = \frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|})\,. \tag{6.61}$$

This expression coincides with contracted Christoffel symbols,

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}g^{\mu\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) = \frac{1}{2}g^{\mu\kappa}\partial_\nu g_{\mu\kappa} = \frac{1}{2}\partial_\nu \ln g = \frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|})\,. \tag{6.62}$$

Now we can express the divergence of a vector field as

$$\nabla_\mu X^\mu = \partial_\mu X^\mu + \Gamma^\mu{}_{\lambda\mu}X^\lambda = \partial_\mu X^\mu + \frac{1}{\sqrt{|g|}}(\partial_\mu\sqrt{|g|})X^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}X^\mu)\,. \tag{6.63}$$

Gauss' theorem for the divergence of a vector field follows directly,

$$\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\nabla_\mu X^\mu = \int_\Omega \mathrm{d}^4x\,\partial_a(\sqrt{|g|}X^\mu) = \int_{\partial\Omega}\mathrm{d}S_\mu\sqrt{|g|}\,X^\mu\,. \tag{6.64}$$

This implies in particular that we can drop terms like $\nabla_\mu X^\mu$ in the action, if the vector field X^μ vanishes on the boundary. Similiarly, Gauss' theorem allows us to derive global conservation laws from $\nabla_\mu X^\mu = 0$ in the same way as in Minkowski space.

Next we consider the divergence of an anti-symmetric tensor of rank 2,

$$\nabla_\mu A^{\mu\nu} = \partial_\mu A^{\mu\nu} + \Gamma^\mu{}_{\lambda\mu}A^{\lambda\nu} + \Gamma^\nu{}_{\lambda\mu}A^{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}A^{\mu\nu})\,. \tag{6.65}$$

In the latter case, the third term $\Gamma^{\nu}{}_{\lambda\mu}A^{\mu\lambda}$ vanishes because of the antisymmetry of $A^{\mu\lambda}$ so that we could combine the first two as in the vector case. This generalises to completely anti-symmetric tensors of all orders. In contrast, we find for a symmetric tensor of rank 2,

$$\nabla_\mu S^{\mu\nu} = \partial_\mu S^{\mu\nu} + \Gamma^{\mu}{}_{\lambda\mu}S^{\lambda\nu} + \Gamma^{\nu}{}_{\lambda\mu}S^{\mu\lambda} = \frac{1}{\sqrt{|g|}}\,\partial_\mu(\sqrt{|g|}S^{\mu\nu}) + \Gamma^{\nu}{}_{\lambda\mu}S^{\mu\lambda}\,. \tag{6.66}$$

Hence the divergence of a symmetric tensor of rank two contains an additional term $(\partial_\nu g_{\mu\lambda})S^{\mu\lambda}$ which prohibits the use of Gauss' theorem.

6.4. Symmetries of a general space-time

In the case of a Riemanian space-time manifold $(\mathscr{M}, \boldsymbol{g})$, we say the space-time possess a symmetry if it looks the same as one moves from a point P along a vector field $\boldsymbol{\xi}$ to a different point $\tilde{P}$. More precisely, we mean with "looking the same" that the metric tensor transported along $\boldsymbol{\xi}$ remains the same.

Such symmetries may be obvious, if one uses coordinates adapted to these symmetries: For instance, the metric may be independent from one or several coordinates. Let us assume that the metric is e.g. independent from the time coordinate x^0. Then x^0 is a cyclic coordinate, $\partial L/\partial x^0 = 0$, of the Lagrangian $L = \mathrm{d}\tau/\mathrm{d}\sigma$ of a free test particle moving in $\mathscr{M}$. With $L = \mathrm{d}\tau/\mathrm{d}\sigma$, the resulting conserved quantity $\partial L/\partial \dot{x}^0 = \text{const.}$ can be written as

$$\frac{\partial L}{\partial \dot{x}^0} = g_{0\beta}\frac{\mathrm{d}x^\beta}{L\mathrm{d}\sigma} = g_{0\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = \boldsymbol{\xi}\cdot\boldsymbol{u} \tag{6.67}$$

with $\boldsymbol{\xi} = \boldsymbol{e}_0$ and $\boldsymbol{u}$ as the four-velocity. Hence the quantity $\boldsymbol{\xi}\cdot\boldsymbol{u} = p^0/m$ is conserved along the solutions $x^\alpha(\sigma)$ of the Lagrange equation of a free particle on $\mathscr{M}$, i.e. along geodesics: The motion of all test particles in the corresponding space-time conserve energy. The vector field $\boldsymbol{\xi}$ that points in the direction in which the metric does not change is called Killing vector field.

Since we allow arbitrary coordinate systems, space-time symmetries are however in general not evident by a simple inspection of the metric tensor. We say the metric is invariant moving along the Killing vector field ξ^μ, when the resulting change $\delta g^{\mu\nu}$ of the metric is zero. In order to use this condition, we have to be able to calculate how the tensor $g^{\mu\nu}$ changes as we transport it along a vector field ξ^μ. Clearly, it is sufficient to consider an infinitesimal distance. Then we can work in the approximation

$$\tilde{x}^\mu = x^\mu + \varepsilon\xi^\mu(x^\nu) + \mathcal{O}(\varepsilon^2)\,, \qquad \varepsilon \ll 1\,, \tag{6.68}$$

and neglect all terms quadratic in ε.

We recall first the tranformation law for a rank two tensor as the metric under an arbitrary coordinate transformation,

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}\,g^{\alpha\beta}(x)\,. \tag{6.69}$$

Applied to the transport along $\boldsymbol{\xi}$ defined in (6.68), we obtain

$$\begin{aligned}\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}\,g^{\alpha\beta}(x) &= (\delta^\mu_\alpha + \varepsilon\xi^\mu{}_{,\alpha})(\delta^\nu_\beta + \varepsilon\xi^\nu{}_{,\beta})g^{\alpha\beta}(x)\\ &= g^{\mu\nu}(x) + \varepsilon(\xi^{\mu,\nu} + \xi^{\nu,\mu}) + \mathcal{O}(\varepsilon^2)\,.\end{aligned} \tag{6.70}$$

In order to be able to compare the new $\tilde{g}^{\mu\nu}(\tilde{x})$ with $g^{\mu\nu}(x)$, we have to express $\tilde{g}^{\mu\nu}(\tilde{x})$ as function of x. A Taylor expansion gives

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \tilde{g}^{\mu\nu}(x+\varepsilon\xi) = \tilde{g}^{\mu\nu}(x) + \varepsilon\xi^{\alpha}\partial_{\alpha}\tilde{g}^{\mu\nu}(x) + \mathcal{O}(\varepsilon^2)\,. \tag{6.71}$$

Setting equal Eqs. (6.70) and (6.71), we obtain

$$g^{\mu\nu}(x) + \varepsilon(\xi^{\mu,\nu} + \xi^{\nu,\mu}) = \tilde{g}^{\mu\nu}(x) + \varepsilon\xi^{\alpha}\partial_{\alpha}\tilde{g}^{\mu\nu}(x)\,. \tag{6.72}$$

Thus the metric is kept invariant, $\tilde{g}^{\mu\nu}(x) = g^{\mu\nu}(x)$, if the condition

$$\xi^{\mu,\nu} + \xi^{\nu,\mu} - \xi^{\rho}\partial_{\rho}g^{\mu\nu} = 0 \tag{6.73}$$

or

$$g^{\mu\rho}\partial_{\rho}\xi^{\nu} + g^{\nu\rho}\partial_{\rho}\xi^{\mu} - \xi^{\rho}\partial_{\rho}g^{\mu\nu} = 0 \tag{6.74}$$

is satisfied. Expressing partial derivatives as covariant ones[2], the terms containing connection coefficients cancel and we obtain the Killing equation

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0\,. \tag{6.75}$$

Its solutions $\boldsymbol{\xi}$ are the Killing vectors of the metric.

We now check that Eq. (6.75) leads indeed to a conservation law, as required by our initial definition of a Killing vector field. We multiply the equation for geodesic motion,

$$\frac{Du^{\mu}}{\mathrm{d}\tau} = 0, \tag{6.76}$$

by the Killing vector ξ_{μ} and use Leibniz's product rule together with the definition of the absolut derivative (6.32),

$$\xi_{\mu}\frac{Du^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\xi_{\mu}u^{\mu}) - \nabla_{\mu}\xi_{\nu}u^{\mu}u^{\nu} = 0. \tag{6.77}$$

The second term vanishes for a Killing vector field ξ^{μ}, because the Killing equation implies the antisymmetry of $\nabla_{\mu}\xi_{\nu}$. Hence the quantity $\xi_{\mu}u^{\mu}$ is indeed conserved along any geodesics.

Example 6.2: Find all ten Killing vector fields of Minkowski space and specifiy the corresponding symmetries and conserved quantities.
The Killing equation $\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0$ simplifies in Minkowski space to

$$\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} = 0\,. \tag{6.78}$$

Taking one more derivative and using the symmetry of partial derivatives, we arrive at

$$\partial_{\rho}\partial_{\mu}\xi_{\nu} + \partial_{\rho}\partial_{\nu}\xi_{\mu} = 2\partial_{\mu}\partial_{\rho}\xi_{\nu} = 0\,. \tag{6.79}$$

Integrating this equation twice, we find

$$\xi^{\mu} = \omega_{\nu}{}^{\mu}x^{\nu} + a^{\mu}\,. \tag{6.80}$$

[2]Since Eq. (6.75) is tensor equation, the previous Eq. (6.74) is also invariant under arbitrary coordinate transformations, although it contains only partial derivatives. This suggests that one can introduce the derivative of an arbitrary tensor along a vector field, called Lie derivative, without the need for a connection.

The matrix $\omega^{\mu\nu}$ has to be antisymmetric in order to satisfy Eq. (6.78). Thus the Killing vector fields are determined by ten integration constants. They agree with the infinitesimal generators of Lorentz transformations, cf. appendix B.3.
The four parameters a^μ generate translations, $x^\mu \to x^\mu + a^\mu$, described by four Killing vector fields which can be chosen as the Cartesian basis vectors of Minkowski space,

$$\boldsymbol{T}_0 = \partial_t, \qquad \boldsymbol{T}_1 = \partial_x, \qquad \boldsymbol{T}_2 = \partial_y, \qquad \boldsymbol{T}_3 = \partial_z\,.$$

For a particle with momentum $p^\mu = mu^\mu$ moving along $x^\mu(\lambda)$, the existence of a Killing vector $\boldsymbol{T}^\mu$ implies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\boldsymbol{T}^\mu \cdot \boldsymbol{u}) = \frac{\mathrm{d}}{m\mathrm{d}\lambda}(\boldsymbol{T}^\mu \cdot \boldsymbol{p}) = 0\,,$$

i.e. the conservation of the four-momentum component p_μ.
Consider next the ij (=spatial) components of the Killing equation. Three additional Killing vectors are

$$\boldsymbol{J}_1 = y\partial_z - z\partial_y\,, \qquad \text{and cyclic permutations.} \tag{6.81}$$

The existence of Killing vectors $\boldsymbol{J}_i$ implies that $\boldsymbol{J}_i \cdot \boldsymbol{p}$ is conserved along a geodesics of particle. But

$$\boldsymbol{J}_1 \cdot \boldsymbol{p} = yp_z - zp_y = L_x$$

and thus the angular momentum around the origin of the coordinate system is conserved.
The other three components satisfy the 0α component of the Killing equations ($\omega_1{}^0 = \omega_0{}^1$),

$$\boldsymbol{K}_1 = t\partial_z + z\partial_t\,, \qquad \text{and cyclic permutations.} \tag{6.82}$$

The conserved quantity $tp_z - zE = \text{const.}$ now depends on time and is therefore not as popular as the previous ones. Its conservation implies that the center of mass of a system of particles moves with constant velocity $v_\alpha = p_\alpha/E$.

Global conservation laws An immediate consequence of Eq. (6.63) is a covariant form of Gauß' theorem for vector fields. In particular, we can conclude from local current conservation, $\nabla_\mu j^\mu = 0$, the existence of a globally conserved charge. If the conserved current j^μ vanishes at infinity, then we obtain also in a general space-time

$$\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\nabla_\mu j^\mu = \int_\Omega \mathrm{d}^4x\partial_\mu(\sqrt{|g|}j^\mu) = 0\,. \tag{6.83}$$

For a non-zero current, the volume integral over the charge density j^0 remains constant,

$$\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\nabla_\mu j^\mu = \int_{V(t_2)} \mathrm{d}^3x\sqrt{|g|}j^0 - \int_{V(t_1)} \mathrm{d}^3x\sqrt{|g|}j^0 = 0\,. \tag{6.84}$$

Thus the conservation of Noether charges of internal symmetries as the electric charge, baryon number, etc., is not affected by an expanding universe or other gravitational effects.

Next we consider the energy-momentum stress tensor as an example for a locally conserved symmetric tensors of rank two. Now, the second term in Eq. (6.66) prevents us to convert the local conservation law into a global one. If the space-time admits however a Killing field ξ_μ, then we can form the vector field $P^\mu = T^{\mu\nu}\xi_\nu$ with

$$\nabla_\mu P^\mu = \nabla_\mu(T^{\mu\nu}\xi_\nu) = \xi_\nu\nabla_\mu T^{\mu\nu} + T^{\mu\nu}\nabla_\mu\xi_\nu = 0\,. \tag{6.85}$$

Here, the first term vanishes since $T^{\mu\nu}$ is conserved and the second because $T^{\mu\nu}$ is symmetric, while $\nabla_\mu \xi_\nu$ is antisymmetric. Therefore the vector field $P^\mu = T^{\mu\nu}\xi_\nu$ is also conserved, $\nabla_\mu P^\mu = 0$, and we obtain thus the conservation of the component of the energy-momentum vector in direction ξ.

Global energy conservation requires thus the existence of a time-like Killing vector field. If the metric is time-dependent, as e.g. in the case of the expanding Universe, a time-like Killing vector field does not exist and the energy contained in a "comoving" volume changes with time.

Summary

In a curved space-time $\mathcal{M}$ we require a connection to compare vectors at different points. The unique connection which is symmetric and compatible with the metric are the Christoffel symbols. The symmetries of a space-time $\mathcal{M}$ are determined by its Killing vector fields ξ^μ. The momentum component parallel to ξ^μ of test particles moving in $\mathcal{M}$ is conserved. Locally conserved currents lead in general only for vector currents to globally conserved charges. In the case of locally conserved tensors (as $\nabla_\mu T^{\mu\nu} = 0$), the global conservation of the corresponding charges requires the existence of Killing vector fields.

Further reading

Volume 2 of Landau and Lifshitz introduces classical field theory including general relativity.[17] contains a clear introduction to differential geometry on a level accessible for physicists.

Problems

6.1 Derivative of $S^{\mu\nu}$.
Express the covariant derivative $\nabla_\mu S^\mu{}_\nu$ of a symmetric tensor field without Christoffel symbols.

6.2 Lie bracket.
Show that the commutator $[\boldsymbol{V}, \boldsymbol{W}]$ of two vector fields $\mathbf{V} = V^\mu \partial_\mu$ and $\mathbf{W} = W^\mu \partial_\mu$ is again a vector field.

6.3 Lie derivative.
The LHS of (6.74) defines the Lie derivative of a rank two tensor. Show in the same way that the Lie derivative of a vector field $\mathbf{V}$ is given by $\mathscr{L}_{\boldsymbol{\xi}} = [\boldsymbol{X}, \boldsymbol{\xi}]$, i.e. agrees with the Lie bracket from problem 6.2. Generalise the Lie derivative to a general tensor field, assuming linearity and the Leibniz rule.

6.4 Conformal invariance.
The condition $\tilde{g}^{\mu\nu}(\tilde{x}) = g^{\mu\nu}(\tilde{x})$ defines an isometry, i.e. a distance conserving map between two spaces, $\mathrm{d}s^2 = \mathrm{d}\tilde{s}^2$. We can relax this condition to $\tilde{g}^{\mu\nu}(\tilde{x}) = \Omega^2(\tilde{x}) g^{\mu\nu}(\tilde{x})$. Show that this transformation keeps the light-cone structure invariant. Derive the analogue of (6.75) and solve it for the case of Minkowski space.

6.5 Inertial coordinates.

The equivalence principle implies that we can find in a curved space-time locally coordinates (i.e. at a chosen point P) with $g_{\mu\nu} = \eta_{\mu\nu}$, $\partial_\rho g_{\mu\nu} = 0$ and $\Gamma^\rho{}_{\mu\nu} = 0$. Show that this requirement holds choosing as connection the Christoffel symbols.

6.6 Torsion

Derive the transformation properties of a connection. Show that he difference of two connections $\Gamma^\mu{}_{\rho\sigma}$ and $\tilde{\Gamma}^\mu{}_{\rho\sigma}$ transforms as a tensor (which is called torsion).

6.7 Geodesic equation from H.

Show that the Eqs. (6.53) and (6.54) imply the

standard form of the geodesic equation.

6.8 Alternative Lagrangian.

Use the Lagrangian (1.49) to derive the geodesic equation and the Hamiltonian. Interprete the result.

7. Spin-1 and spin-2 fields

We introduced fields transforming as tensors under coordinate general transformations. In this chapter, we will show that such fields have integer spin and obey Bose-Einstein statistics. Therefore they can exist as macroscopic fields and are thus candidates to describe the electric and the gravitational force. Since both the electric and the gravitational potential $V(r)$ follow a $1/r$ law, we expect from our discussion of the Yukawa potential that the two forces are mediated by massless particles. We will find later that no interacting theory of massless particles with spin $s > 2$ exists. Therefore it is sufficient to consider the two cases $s = 1$ and $s = 2$.

7.1. Tensor fields

The momentum modes $\propto \mathrm{e}^{-\mathrm{i}kx}$ of massive fields can be boosted to their rest-frame, where $k^\mu = (m, \mathbf{0})$. In this frame, the total angular momentum reduces to spin, and nonrelativistic quantum mechanics is valid. Thus a field with spin s has $2s + 1$ spin or polarisation states. We will see at the end of this chapter that a tensor field of rank n has spin $s = n$. This implies that fields with spin $s \geq 1$ contain unphysical degrees of freedom: For instance, a massive spin-1 field has three and a massive spin-2 field has five polarisation states. On the other hand, a vector field A^μ has four components, and a symmetric tensor field $h^{\mu\nu}$ of rank two has ten components in $d = 4$ space-time dimensions. The purpose of a relativistic wave equation is thus to impose the correct relativistic dispersion relation and to select the correct physical polarisation states in the chosen frame.

The first requirement is satisfied if each component of a free field ϕ_a satisfies the Klein-Gordon equation. Additionally, we have to impose constraints which eliminate the undesired components,

$$\left(\Box + m^2\right) \phi_a(x) = 0\,, \qquad \text{and} \quad f_i(\phi_a(x)) = 0\,. \tag{7.1}$$

The reason for this mismatch in the number of degrees of freedom is that in general a tensor of rank n is reducible, i.e. it contains components of rank $< n$. For instance, the trace h^μ_μ of a second rank tensor transforms clearly as a scalar. Therefore we should choose the constraints for massive fields with spin s such that all components with spin $< s$ are eliminated.

Example 7.1: An object which contains invariant subgroups with respect to a symmetry operation is called reducible. As example, we consider a tensor of rank two. First, we want to find the reducible subgroups of an arbitrary tensor of rank two with respect to general coordinate transformations: We can split any tensor of rank two in its symmetric and anti-symmetric part,

$$T^{\mu\nu} = \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right) + \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right) = S^{\mu\nu} + A^{\mu\nu}\,.$$

As the (anti-)symmetrisation is linear, it is invariant under the change $\tilde{T}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\sigma}\, T^{\rho\sigma}(x)$. Moreover, the trace of a tensor is an invariant. Thus we can break up $S^{\mu\nu}$ into its trace $S = S^\mu_\mu$ and its traceless part $S^\mu_\nu - S\,\delta^\mu_\nu/d$ in d dimensions.

Next we consider the reducible subgroups of a symmetric tensor $h^{\mu\nu}$ of rank two with respect to spatial rotations. (Since we consider the massive particle in its rest frame, this the relevant decomposition to find its spin states.) We split $h^{\mu\nu}$ into a scalar h^{00}, a vector h^{0i} and a reducible tensor h^{ij},

$$h^{\mu\nu} = \begin{pmatrix} h^{00} & h^{0i} \\ h^{i0} & h^{ij} \end{pmatrix} .$$

Then we decompose h^{ij} again into its trace h^{ii} and its traceless part $h^i_j - h\delta^i_j/(d-1)$. The latter has $6-1=5$ degrees of freedom in $d=4$, as required for a massive spin-2 field.

This problem is more severe for massless fields, where only two physical degrees of freedom exist[1]. In this case, even the irreducible tensors contain too many degrees of freedom. They can be consistently eliminated only, if some redundancy of the field variables exists which in turn leads to a symmetry of the fields. In this chapter, we discuss the consequences of this redundancy called gauge symmetry on the level of the wave equations and their solutions for the photon and the graviton.

Tensor structure of the propagator We can gain some insight into the general tensor structure of the Feynman propagator for fields with spin $s > 0$ using the definition of the 2-point Green function as the time-ordered vacuum expectation values of fields. In general, we can express an arbitrary solutions of a free spin $s = 0, 1$ and 2 field by its Fourier components as

$$\phi(x) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}(\omega_k t - \boldsymbol{kx})} + \text{h.c.} \right] , \tag{7.2}$$

$$A^\mu(x) = \sum_r \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a_r(\boldsymbol{k}) \varepsilon^\mu_r \mathrm{e}^{-\mathrm{i}(\omega_k t - \boldsymbol{kx})} + \text{h.c.} \right] , \tag{7.3}$$

$$h^{\mu\nu}(x) = \sum_r \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a_r(\boldsymbol{k}) \varepsilon^{\mu\nu}_r \mathrm{e}^{-\mathrm{i}(\omega_k t - \boldsymbol{kx})} + \text{h.c.} \right] , \tag{7.4}$$

where the momentum is on-shell, $k^\mu = (\omega_k, \boldsymbol{k})$ and r labels the spin or polarisation states. The constraints $f_i(\phi(x)) = 0$ are now conditions on the polarisation vector and tensor, respectively, which depend on $\boldsymbol{k}$. Proceeding as in the scalar case, (3.51), we expect that e.g. the propagator for a vector field is given by

$$\mathrm{i}D^{\mu\nu}_F(x) = \langle 0|T\{A^\mu(x)A^{\nu *}(0)\}|0\rangle = \tag{7.5}$$

$$= \sum_r \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \left[\varepsilon^\mu_r(\boldsymbol{k}) \varepsilon^{\nu *}_r(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \vartheta(x^0) + \varepsilon^\mu_r(\boldsymbol{k}) \varepsilon^{\nu *}_r(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} \vartheta(-x^0) \right] \tag{7.6}$$

$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathcal{P}^{\mu\nu}(k)\, \mathrm{e}^{-\mathrm{i}kx}}{k^2 - m^2 + \mathrm{i}\varepsilon} . \tag{7.7}$$

The expression (7.6) is in line with the interpretation of the propagator as the probability for the creation of a particle at x with any momentum k and polarisation r, its propagation to x' followed by its annihilation. In the last step, we introduced the tensor $\mathcal{P}^{\mu\nu}(k)$ which corresponds to the sum over the polarisation states $\varepsilon^\mu_r(\boldsymbol{k})\varepsilon^{\nu *}_r(\boldsymbol{k})$. We will show that the polarisation tensors are polynomials in in the momentum k, and thus Eq. (7.6) shows that $\mathcal{P}^{\mu\nu}(k)$ is even in the momentum. As a result, our discussion of causality in the scalar case applies for all tensor field, implying that these fields seen as quantum fields commute; the particles described by these fields satisfy Bose-Einstein statistics.

[1] See the appendix B.4 for a brief discussion.

7.2. Vector fields

Proca and Maxwell equations A massive vector field A^μ has four components in $d = 4$ space-time dimensions, while it has only $2s + 1 = 3$ independent spin components. Correspondingly, a four-vector A^μ transforms under a rotation as $(A^0, \boldsymbol{A})$, i.e. it contains a scalar and a three-vector. Therefore we have to add to the four Klein-Gordon equations for A^μ one constraint which eliminates A^0: The only linear, Lorentz invariant possibility is

$$\left(\Box + m^2\right) A^\mu(x) = 0 \qquad \text{and} \quad \partial_\mu A^\mu = 0\,. \tag{7.8}$$

In momentum space, this translates into $(k^2 - m^2)A^\mu(k) = 0$ and $k_\mu A^\mu(k) = 0$. In the rest frame of the particle, $k^\mu = (m, \mathbf{0})$, and the constraint becomes $A^0 = 0$. Hence a field satisfying (7.8) has only three space-like components as required for a massive $s = 1$ field. We can choose the three polarisation vectors which label the three degrees of freedom in the rest frame e.g. as the Cartesian unit vectors, $\boldsymbol{\varepsilon}_i \propto \mathbf{e}_i$.

The two equations can be combined into one equation called Proca equation,

$$(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\, A_\nu + m^2 A^\mu = 0\,. \tag{7.9}$$

To show the equivalence of this equation with (7.8), we act with ∂_μ on it,

$$(\partial^\nu\Box - \Box\partial^\nu)\, A_\nu + m^2\partial_\mu A^\mu = m^2\partial_\mu A^\mu = 0\,. \tag{7.10}$$

Hence, a solution of the Proca equation fulfils automatically the constraint $\partial_\mu A^\mu = 0$ for $m^2 > 0$. On the other hand, we can neglect the second term in (7.9) for $\partial_\nu A^\nu = 0$ and obtain the Klein-Gordon equation.

We now go over to the case of a massless spin-1 field which is described by the Maxwell equations. In classical electrodynamics, the field-strength tensor $F_{\mu\nu}$ is an observable quantity, while the potential A_μ is merely a convenient auxiliary quantity. From the definition

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.11}$$

it is clear that $F_{\mu\nu}$ is invariant under the transformations

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\Lambda(x)\,. \tag{7.12}$$

Thus $A'_\mu(x)$ is for any $\Lambda(x)$ physically equivalent to $A_\mu(x)$, leading to the same field-strength tensor and thus e.g. to the same Lorentz force on a particle. The transformations (7.12) are called gauge transformations. Note that the mass term m^2A^μ in the Proca equation breaks gauge invariance.

If we insert into the Maxwell equation the definition of the potential,

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial_\mu\partial^\nu A^\mu = j^\nu\,, \tag{7.13}$$

we see that this expression equals the $m = 0$ limit of the Proca equation. Gauge invariance allows us to choose a potential A^μ such that $\partial_\mu A^\mu = 0$. Such a choice is called fixing the gauge, and the particular case $\partial_\mu A^\mu = 0$ is denoted as Lorenz gauge. In the Lorenz gauge, the wave equation simplifies to

$$\Box A^\mu = j^\mu\,. \tag{7.14}$$

Additionally, we can add to the potential A^μ any function $\partial^\mu\chi$ satisfying $\Box\chi = 0$. We can use this freedom to set $A^0 = 0$. Inserting then a plane-wave $A^\mu \propto \varepsilon^\mu \mathrm{e}^{\mathrm{i}kx}$ into the free wave equation, $\Box A^\nu = 0$, we find that k is a null-vector and that $\varepsilon^\mu k_\mu = -\boldsymbol{\varepsilon}\cdot\boldsymbol{k} = 0$. Thus the photon propagates with the speed of light, is transversely polarised and has two polarisation states as expected for a massless particle.

Closely connected to the gauge invariance of electrodynamics is the fact that its source, the electromagnetic current, is conserved. The antisymmetry of $F^{\mu\nu}$, which is the basis for the symmetry (7.12), leads also to $\partial_\mu\partial_\nu F^{\mu\nu} = 0$. Thus the Maxwell equation $\partial_\mu F^{\mu\nu} = j^\nu$ implies the conservation of the electromagnetic current j^μ,

$$\partial_\mu\partial_\nu F^{\mu\nu} = \partial_\nu j^\nu = 0\,. \tag{7.15}$$

Propagator for massive spin-1 fields The propagator $D_{\mu\nu}$ for a massive spin-1 field is determined by

$$\left[\eta^{\mu\nu}(\Box + m^2) - \partial^\mu\partial^\nu\right] D_{\nu\lambda}(x) = \delta^\mu_\lambda\delta(x)\,. \tag{7.16}$$

Inserting the Fourier transformation of the propagator and of the delta function gives

$$\left[\left(-k^2 + m^2\right)\eta^{\mu\nu} + k^\mu k^\nu\right] D_{\nu\lambda}(k) = \delta^\mu_\lambda\,. \tag{7.17}$$

We will apply the tensor method to solve this equation: In this approach, we use first all tensors available in the problem to construct the required tensor of rank 2. In the case at hand, we have at our disposal only the momentum k_μ of the particle—which we can combine to $k_\mu k_\nu$—and the metric tensor $\eta_{\mu\nu}$. Thus the tensor structure of $D_{\mu\nu}(k)$ has to be of the form

$$D_{\mu\nu}(k) = A\eta_{\mu\nu} + Bk_\mu k_\nu \tag{7.18}$$

with two unknown scalar functions $A(k^2)$ and $B(k^2)$. Inserting this ansatz into (7.17) and multiplying out, we obtain

$$\begin{aligned}
\left[(-k^2 + m^2)\eta^{\mu\nu} + k^\mu k^\nu\right]\left[A\eta_{\nu\lambda} + Bk_\nu k_\lambda\right] &= \delta^\mu_\lambda\,,\\
-Ak^2\delta^\mu_\lambda + Am^2\delta^\mu_\lambda + Ak^\mu k_\lambda + Bm^2k^\mu k_\lambda &= \delta^\mu_\lambda\,,\\
-A(k^2 - m^2)\delta^\mu_\lambda + (A + Bm^2)k^\mu k_\lambda &= \delta^\mu_\lambda\,.
\end{aligned} \tag{7.19}$$

In the last step, we regrouped the LHS into the two tensor structures δ^μ_λ and $k^\mu k_\lambda$. A comparison of their coefficients gives then $A = -1/(k^2 - m^2)$ and

$$B = -\frac{A}{m^2} = \frac{1}{m^2(k^2 - m^2)}.$$

Thus the massive spin-1 propagator follows as

$$D_F^{\mu\nu}(k) = \frac{-\eta^{\mu\nu} + k^\mu k^\nu/m^2}{k^2 - m^2 + \mathrm{i}\varepsilon}\,. \tag{7.20}$$

Note that there is a sign ambiguity, since we could have added a minus sign to the Proca equation.

Next we check this sign and our claim that the propagator $D_F^{ab}(k)$ of spin $s > 0$ fields can be obtained as sum over their polarisation states $\varepsilon_a^{(r)}$ times the scalar propagator $\Delta(k)$. As the theory is Lorentz invariant, we can choose the frame most convenient for this comparison

which is the restframe of the massive particle. Then $k^\mu = (m, \mathbf{0})$ and the three polarisation vectors can be chosen as the Cartesian basis vectors. Comparing then

$$-\eta^{\mu\nu} + k^\mu k^\nu/m^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \sum_r \varepsilon^{\mu(r)}\varepsilon^{\nu(r)}\,, \tag{7.21}$$

shows that both methods agree and can be used to derive the Feynman propagator. In the latter approach, working from the RHS to the LHS of Eq. (7.21), we derive first the expression valid for the Feynman propagator in a specific frame. Then we have to rewrite the expression in an invariant way using the relevant tensors, here $\eta^{\mu\nu}$ and $k^\mu k^\nu$.

Propagator for massless spin-1 fields As we have seen, we can set $m = 0$ in the Proca equation and obtain the Maxwell equation. The corresponding limit of the propagator (7.20) leads however to an ill-defined result. As we know that the number of degrees of freedom differs between the massive and the massless case, this is not too surprising. If we try next the limit $m \to 0$ in Eq. (7.19), then we find

$$-Ak^2\delta^\mu_\lambda + Ak^\mu k_\lambda = \delta^\mu_\lambda. \tag{7.22}$$

This equation has for arbitrary k with $A = -1/k^2$ and $A = 0$ no solution. Moreover, the function B is undetermined. We can understand this physically, since for a massless field current conservation holds. But $\partial_\mu J^\mu(x) = 0$ implies $k_\mu J^\mu(k) = 0$ and thus the $k^\mu k^\nu$ term does not influence physical quantities: In physical measurable quantities, as e.g. $W[J]$, the propagator is always matched between conserved currents, and the longitudinal part $k^\mu k^\nu$ drops out.

We now try to construct the photon propagator from its sum over polarisation states. First we consider a linearly polarised photon with polarisation vectors $\varepsilon^{(r)}_\mu$ lying in the plane perpendicular to its momentum vector $\boldsymbol{k}$. If we perform a Lorentz boost on $\varepsilon^{(1)}_\mu$, we will find

$$\tilde{\varepsilon}^{(1)}_\mu = \Lambda^\nu{}_\mu \varepsilon^{(1)}_\nu = a_1\varepsilon^{(1)}_\mu + a_2\varepsilon^{(2)}_\mu + a_3 k_\mu\,, \tag{7.23}$$

where the coefficients a_i depend on the direction $\boldsymbol{\beta}$ of the boost. Thus, in general the polarisation vector will not be anymore perpendicular to $\boldsymbol{k}$. Similiarly, if we perform a gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \tag{7.24}$$

with

$$\Lambda(x) = \mathrm{i}\lambda\exp(-\mathrm{i}kx) + \text{h.c.}\,, \tag{7.25}$$

then

$$A'_\mu(x) = \varepsilon'_\mu\exp(-\mathrm{i}kx) + \text{h.c.} \tag{7.26}$$

with $\varepsilon'_\mu = \varepsilon_\mu + \lambda k_\mu$. Since λ is arbitrary, only the components of ε^μ transverse to $\boldsymbol{k}$ can have physical significance[2]. Using $\varepsilon'_\mu k^\mu = 0$, we can eliminate both the time-like and the longitudinal component choosing $\varepsilon'_0 = -\varepsilon'_3$. Thus we see again that a consistent theory of

[2] Choose e.g. a photon propagating in z direction, $k^\mu = (\omega, 0, 0, \omega)$, to see that the gauge transformation does not affect the transverse components.

massless spin-1 particles requires the existence of the symmetry (7.24). The gauge symmetry in turn implies that the massless spin-1 particle couples only to conserved currents.

We will now use current conservation, $k_\mu J^\mu(k) = 0$, to derive a convenient expression for the propagator of a massless vector particle. The two polarisation vectors of a photon should satisfy the normalisation $\varepsilon_\mu^{(a)*}\varepsilon^{\mu(b)} = \delta^{ab}$. For a linearly polarised photon propagating in z direction, $k^\mu = (\omega, 0, 0, \omega)$, the polarisation vectors are $\varepsilon_\mu^{(1)} = \delta_\mu^1$ and $\varepsilon_\mu^{(2)} = \delta_\mu^2$. If we perform the sum over the two polarisation states, we find

$$\sum_r \varepsilon_\mu^{(r)*}\varepsilon_\nu^{(r)} = \mathrm{diag}\{0, 1, 1, 0\}\,. \tag{7.27}$$

If we try to rewrite this expression in an invariant way using $\eta_{\mu\nu}$ and $k_\mu k_\nu/k^2$, we fail: We cannot cancel at the same time $\eta_{00} = +1$ and $\eta_{33} = -1$ by $k_\mu k_\nu/k^2$. We introduce therefore additionally the momentum vector $\tilde{k}^\mu = (\omega, 0, 0, -\omega)$ obtained by a spatial reflection from k^μ. This allows us to write the polarisation sum as an invariant tensor expression,

$$\sum_r \varepsilon_\mu^{(r)*}\varepsilon_\nu^{(r)} = -\eta_{\mu\nu} + \frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k\tilde{k}} \equiv -\pi_{\mu\nu}\,. \tag{7.28}$$

Current conservation, $k_\mu J^\mu(k) = 0$, implies that the second term in the polarisation sum does not contribute to physical observables. For the same reason, we can add an arbitrary term $\xi k_\mu k_\nu$. We use this freedom to eliminate the $\tilde{k}$ dependence and to set

$$J^{\mu*}\left(\sum_r \varepsilon_\mu^{(r)*}\varepsilon_\nu^{(r)}\right) J^\nu = J^{\mu*}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) J^\nu\,. \tag{7.29}$$

Now we can read off the photon propagator as

$$D_F^{\mu\nu}(k) = \frac{-\eta^{\mu\nu} + (1-\xi)k^\mu k^\nu/k^2}{k^2 + \mathrm{i}\varepsilon}\,. \tag{7.30}$$

A specific choice of the parameter ξ called gauge fixing parameter corresponds to the choice of a gauge in Eq. (7.13). In particular, the Feynman gauge $\xi = 1$, which leads to a form of the propagator often most convenient in calculations, correspond to the Lorenz gauge in Eq. (7.13). In this gauge, $\sum_{r=0}^3 \varepsilon_\mu^{(r)*}\varepsilon_\nu^{(r)} = \mathrm{diag}\{-1, 1, 1, 1\}$: the propagator contains nonphysical degrees of freedom, time-like and longitudinal photons, which contributions cancel however in physical observables. Similarly, for all other values ξ the propagator is explicitly Lorentz invariant but contains unphysical degrees of freedom. We will see later that it is a general feature of gauge theories as electrodynamics that we have to choose between a covariant gauge which introduces unphysical degrees of freedom and a gauge which contains only the transverse degree of freedom but selects a specific frame.

Repulsive Coulomb potential by vector exchange We consider as in the scalar case two static point charges as external sources, but use now a vector current $J^\mu = J_1^\mu(x_1) + J_2^\mu(x_2)$. Since $J^\mu = (\rho, \boldsymbol{j})$, only the zero component, $J_i^\mu = \delta_0^\mu \delta(\boldsymbol{x} - \boldsymbol{x}_i)$, contributes for a static source to $W[J]$. Moreover, we can neglect the longitudinal part $k^\mu k^\nu/m^2$ of the propagator. This is justified, since the concept of a potential energy makes only sense in the non-relativistic

limit, i.e. for $V \ll m$ or equivalently $r \gg 1/m$. Hence

$$W_{12}[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x' \int \frac{\mathrm{d}^4k}{(2\pi)^4} J_1^\mu(x)\, \frac{-\eta_{\mu\nu}\, \mathrm{e}^{-\mathrm{i}k(x-x')}}{k^2 - m^2 + \mathrm{i}\varepsilon}\, J_2^\nu(x') \tag{7.31}$$

$$= \frac{1}{2}\int \mathrm{d}t\mathrm{d}t' \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}k(x-x')}}{k^2 - m^2 + \mathrm{i}\varepsilon}\,. \tag{7.32}$$

Comparing with our earlier result for scalar exchange in Eq. (3.40), it becomes clear without further calculation that spin-1 exchange between equal charges is repulsive. In the limit $m \to 0$, we obtain the Coulomb potential with the correct sign for electromagnetic interactions.

7.3. Gravity

Wave equation From Newton's law we know that gravity is fundamentally attractive and of long-range. Thus the gravitational force has to be mediated by a massless particle which can not be a spin $s = 1$ particle. Analog to the electric field $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ we can introduce a classical gravitational field $\boldsymbol{g}$ as the gradient of the gravitational potential, $\boldsymbol{g} = -\boldsymbol{\nabla}\phi$. We obtain then $\boldsymbol{\nabla}\cdot\boldsymbol{g}(\boldsymbol{x}) = -4\pi G\rho(\boldsymbol{x})$ and as Poisson equation

$$\Delta\phi(\boldsymbol{x}) = 4\pi G\rho(\boldsymbol{x})\,, \tag{7.33}$$

where ρ is the mass density, $\rho = \mathrm{d}m/\mathrm{d}^3x$.

Special relativity gives us two hints how we should transfer this equation into a relativistic framework: First, the Laplace operator Δ on the LHS is the $c \to \infty$ limit of minus the d'Alembert operator $\Box$. Second, the RHS should be the $v/c \to 0$ limit of something incorporating not only the mass density but all types of energy densities. To proceed, consider first how the mass density ρ transforms under a Lorentz transformation: An observer moving with the speed β relative to the rest frame of the matter distribution ρ measures the energy density $\rho' = \gamma\mathrm{d}m/(\gamma^{-1}\mathrm{d}V) = \gamma^2\rho$, with $\gamma = 1/\sqrt{1-\beta^2}$. This is the transformation law of the 00 component of a tensor of rank 2, alas the energy-momentum stress tensor $T^{\mu\nu}$.

Thus the field equation for a purely scalar theory of gravity would be

$$\Box\phi = -4\pi G T^\mu_\mu. \tag{7.34}$$

Such a theory predicts no coupling between photons and gravitation, since $T^\mu_\mu = 0$ for the electromagnetic field, and is therefore in contradiction to the observed gravitational lensing of light. A purely vector theory for gravity fails too, since it predicts not attraction but repulsion of two masses. Hence we are forced to consider a symmetric spin-2 field $\bar{h}_{\mu\nu}$ as mediator of the gravitational force; its source is the energy-momentum stress tensor

$$\Box\bar{h}^{\mu\nu} = -2\kappa T^{\mu\nu}. \tag{7.35}$$

The normalisation constant $\kappa \propto G_N$ has to be determined such that in the non-relativistic limit the Poisson equation (7.33) holds.

Let us consider as a warm-up first the massive case: A symmetric, massive spin-2 field has ten independent components, but only $2s + 1 = 5$ physical spin degrees of freedom. Thus we have to impose five constraints additional to the source-free equation

$$(\Box + m^2)\bar{h}^{\mu\nu} = 0\,. \tag{7.36}$$

Proceeding as in the $s = 1$ case, (7.8), we use as constraint $\partial_\mu \bar{h}^{\mu\nu} = 0$ which provides now four conditions. We can use them to set $\bar{h}^{0\mu} = 0$. We obtain the missing fifth constraint subtracting the trace $\bar{h}^{ii}$ which transforms as a scalar from $\bar{h}^{ij}$.

We move now to the massless case considering a plane wave $h_{\mu\nu} = \varepsilon_{\mu\nu} \exp(-\mathrm{i}kx)$. In analogy to the photon case, we expect that also the graviton has only two, transverse degrees of freedom. If we choose the plane wave propagating in the z direction, $\boldsymbol{k} = k\boldsymbol{e}_z$, then we expect that the polarisation tensor can be expressed as

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{7.37}$$

Here we used that the polarisation tensor has to be symmetric and traceless. The choice (7.37) is called the *transverse traceless* (TT) gauge.

Metric perturbations as a tensor field In the case of the photon, we could reduce the degrees of freedom from four to two, because of the redundancy implied by the gauge symmetry of electromagnetism. Moreover, the gauge symmetry lead to the conservation of the electromagnetic current. The two obvious questions to address next are which symmetry and which conservation law are connected to gravitation.

The second part of the questions is the simpler one, since we know already that in flat space $\partial_\mu T^{\mu\nu} = 0$ holds. Thus for gravity energy-momentum conservation will play the role of current conservation, implying that a gravitational wave is transverse, $k_\mu T^{\mu\nu}(k) = 0$. In order to answer the first part of the question, we have to consider the properties of $h^{\mu\nu}$.

The equivalence principle implies that all test-particles move along the same world-line, if they are released at the same initial point and move only under the influence of the gravitational force. This universality motivated Einstein to describe the effect of gravity by the curvature of space-time. We associate therefore the symmetric tensor field[3] $\bar{h}_{\mu\nu}$ with small perturbations around the Minkowski metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} , \qquad \varepsilon \ll 1 . \tag{7.38}$$

We choose a Cartesian coordinate system x^μ and ask ourselves which transformations are compatible with the splitting (7.38) of the metric. If we consider global (i.e. space-time independent) Lorentz transformations $\Lambda^\nu{}_\mu$, then $x'^\nu = \Lambda^\nu{}_\mu x^\mu$. The metric tensor transforms as

$$g'_{\alpha\beta} = \Lambda^\rho{}_\alpha \Lambda^\sigma{}_\beta g_{\rho\sigma} = \Lambda^\rho{}_\alpha \Lambda^\sigma{}_\beta (\eta_{\rho\sigma} + h_{\rho\sigma}) = \eta_{\alpha\beta} + \Lambda^\rho{}_\alpha \Lambda^\sigma{}_\beta h_{\rho\sigma} = \eta'_{\alpha\beta} + \Lambda^\rho{}_\alpha \Lambda^\sigma{}_\beta h_{\rho\sigma} . \tag{7.39}$$

Since $h'_{\alpha\beta} = \Lambda^\rho{}_\alpha \Lambda^\sigma{}_\beta h_{\rho\sigma}$, we see that global Lorentz transformations respect the splitting (7.38). Thus $h_{\mu\nu}$ transforms as a rank-2 tensor under global Lorentz transformations. We can view therefore the perturbation $h_{\mu\nu}$ as a symmetric rank-2 tensor field defined on Minkowski space that satisfies the wave equation (7.35), similar as the photon field is a rank-1 tensor field fulfilling Maxwell's equations.

[3] (We drop the bar, anticipating that $\bar{h}^{\mu\nu}$ may be a function of $h^{\mu\nu}$: we will derive this connection, $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\alpha_\alpha$, in section 19.3.

The splitting (7.38) is however clearly not invariant under general coordinate transformations, as they allow e.g. the finite rescaling $g_{\mu\nu} \to \Omega g_{\mu\nu}$. We restrict therefore ourselves to infinitesimal coordinate transformations,

$$\bar{x}^\mu = x^\mu + \varepsilon \xi^\mu(x^\nu) \tag{7.40}$$

with $\varepsilon \ll 1$. Then the Killing equation (6.74) simplifies to

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \,, \tag{7.41}$$

because the term $\xi^\rho \partial_\rho h_{\mu\nu}$ is quadratic in the small quantities $\varepsilon h_{\mu\nu}$ and $\varepsilon \xi_\mu$ and can be neglected. Recall that the $\xi^\rho \partial_\rho h_{\mu\nu}$ term appeared, because we compared the metric tensor at different points. In its absence, it is more fruitful to view Eq. (7.41) not as a coordinate but as a gauge transformation analogous to (7.12): In this interpretation, we stay in Minkowski space and the fields $h'_{\mu\nu}$ and $h_{\mu\nu}$ describe the same physics, since the (linearized) Einstein equations do not fix uniquely $h_{\mu\nu}$ for a given source. In momentum space, (7.41) becomes a condition on the polarisation tensor,

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + \xi_\mu k_\nu + \xi_\nu k_\mu. \tag{7.42}$$

We can use this freedom to eliminate four components of $\varepsilon_{\mu\nu}$. After that, we can still add four functions χ_μ with $\Box \chi_\mu = 0$, eliminating four additional components. This justifies the use of the TT gauge.

Graviton propagator We follow the same approach as in the derivation of the photon propagator. For a graviton propagating in z direction, $k^\mu = (\omega, 0, 0, \omega)$, we choose as the two polarisation states $\varepsilon^{(1)}_{\mu\nu}$ setting $\varepsilon_{11} = 1/\sqrt{2}$ and $\varepsilon_{12} = 0$ and $\varepsilon^{(2)}_{\mu\nu}$ setting $\varepsilon_{11} = 0$ and $\varepsilon_{12} = 1/\sqrt{2}$, respectively. They satisfy the normalisation $\varepsilon^{(a)}_{\mu\nu} \varepsilon^{\mu\nu(b)} = \delta^{ab}$.

Now we should perform the sum over the two polarisation states, $\sum_r \varepsilon^{(r)}_{\mu\nu} \varepsilon^{(r)}_{\rho\sigma}$, and express the result as a linear combination of $\eta_{\mu\nu}$ and $k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu$. A straightforward way to do this is to combine first the ten independent quantities of the symmetric tensors in ten-dimensional vectors, $\varepsilon_{\mu\nu} \to E_a$, $\eta_{\mu\nu} \to N_a$, and $k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu \to K_a$, to calculate the tensor products of these vectors and to compare then the resulting $10 \otimes 10$ matrices. An alternative, shorter way is to use the requirement that the propagator is transverse in all indices, $k^\mu \sum_r \varepsilon^{(r)}_{\mu\nu} \varepsilon^{(r)}_{\rho\sigma} = \ldots = k^\sigma \sum_r \varepsilon^{(r)}_{\mu\nu} \varepsilon^{(r)}_{\rho\sigma} = 0$, because of energy-momentum conservation, $\partial_\mu T^{\mu\nu}(x) = 0$. This implies that the graviton propagator should be composed of the projection operators $\pi_{\mu\nu}$ used for the photon (cf. Eq. (7.28)) as follows

$$\sum_r \varepsilon^{(r)}_{\mu\nu} \varepsilon^{(r)}_{\rho\sigma} = A\, \pi_{\mu\nu} \pi_{\rho\sigma} + B \left[\pi_{\mu\rho} \pi_{\nu\sigma} + \pi_{\mu\sigma} \pi_{\nu\rho} \right] . \tag{7.43}$$

The last two terms have a common coefficient, since the LHS is invariant under exchanges of $\mu \leftrightarrow \nu$ or $\rho \leftrightarrow \sigma$. We fix A and B by evaluating this expression for two sets of indices. Since the only non-zero elements of $\pi_{\mu\nu}$ are $\pi_{11} = \pi_{22} = -1$, we obtain choosing e.g. $\{1212\}$ as indices

$$\sum_r \varepsilon^{(r)}_{12} \varepsilon^{(r)}_{12} = \frac{1}{2} = B \pi_{11} \pi_{22}$$

and thus $B = 1/2$. Similiarly, it follows $A = -1/2$ choosing e.g. as indices $\{1111\}$. Thus we found—with surprising ease—the polarisation sum required for the graviton propagator,

$$\sum_r \varepsilon^{(r)}_{\mu\nu}\varepsilon^{(r)}_{\rho\sigma} = -\frac{1}{2}\pi_{\mu\nu}\pi_{\rho\sigma} + \frac{1}{2}\pi_{\mu\rho}\pi_{\nu\sigma} + \frac{1}{2}\pi_{\mu\sigma}\pi_{\nu\rho}\,. \tag{7.44}$$

We continue to proceed in the same way as for the photon. Energy-momentum conservation, $k_\mu T^{\mu\nu}(k) = 0$, implies that the $k_\mu\tilde{k}_\nu + \tilde{k}_\mu k_\nu$ term in $\lambda_{\mu\nu}$ does not contribute to physical observables. We drop therefore again all terms proportional to the graviton momentum k_μ,

$$T^{\mu\nu *}\left(\sum_r \varepsilon^{(r)*}_{\mu\nu}\varepsilon^{(r)}_{\rho\sigma}\right)T^{\rho\sigma} = T^{\mu\nu *}\left(-\frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} + \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma} + \frac{1}{2}\eta_{\mu\sigma}\eta_{\nu\rho}\right)T^{\rho\sigma}. \tag{7.45}$$

Thus the graviton propagator in the Feynman gauge is given by

$$D_F^{\mu\nu;\rho\sigma}(k) = \frac{\frac{1}{2}(-\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})}{k^2 + \mathrm{i}\varepsilon}\,. \tag{7.46}$$

Other gauges are obtained by the replacement $\eta^{\mu\nu} \to \eta^{\mu\nu} - (1-\xi)k^\mu k^\nu/k^2$. In the case of gravity, the Feynman gauge $\xi = 1$ is most often called harmonic gauge, but also the names Hilbert, Loren(t)z, de Donder and confusingly many others are in use.

Attractive potential by spin-2 exchange We consider now the potential energy created by two point masses as external sources interacting via a tensor current $T^{\mu\nu} = T_1^{\mu\nu}(x_1) + T_2^{\mu\nu}(x_2)$. Specialising to static sources, only the zero-zero component, $T_i^{\mu\nu} = \delta_0^\mu\delta_0^\nu\delta(\boldsymbol{x} - \boldsymbol{x}_i)$, contributes to $W[J]$. Hence

$$W_{12}[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x' \int \frac{\mathrm{d}^4k}{(2\pi)^4} T_1^{00}(x)\, D_{F\,00;00}(k)\, \mathrm{e}^{-\mathrm{i}k(x-x')}\, T_2^{00}(x')\,. \tag{7.47}$$

Looking at the numerator of the graviton propagator, we find $-1+1+1 = 1 > 0$. Thus spin-2 exchange is attractive, as required for the force mediating gravity. Comparing Eq. (7.47) to Newton's gravitational potential, we see that the graviton couples with the strength $(8\pi G)^{1/2} \equiv \kappa^{1/2}$ to the stress tensor $T^{\mu\nu}$ of matter, $\mathscr{L}_I = \lambda h_{\mu\nu}T^{\mu\nu}$.

Helicity We determine now how a metric perturbation $h_{\mu\nu}$ transforms under a rotation with the angle α. We choose the wave propagating in z direction, $\boldsymbol{k} = k\boldsymbol{e}_z$, the TT gauge, and the rotation in the xy plane. Then the general Lorentz transformation $\Lambda_\mu^{\ \nu}$ becomes

$$\Lambda_\mu^{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{7.48}$$

Since $\boldsymbol{k} = k\boldsymbol{e}_z$ and thus $\Lambda_\mu^{\ \nu}k_\nu = k_\mu$, the rotation affects only the polarisation tensor. We rewrite $\varepsilon'_{\mu\nu} = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}\varepsilon_{\rho\sigma}$ in matrix notation, $\varepsilon' = \Lambda\varepsilon\Lambda^T$. In TT gauge, it is sufficient to perform the calculation for the xy sub-matrices. The result after introducing circular polarisation states $\varepsilon_\pm = \varepsilon_{11} \pm \mathrm{i}\varepsilon_{12}$ is

$$\varepsilon'^{\mu\nu}_\pm = \exp(\mp 2\mathrm{i}\alpha)\varepsilon^{\mu\nu}_\pm\,. \tag{7.49}$$

The same calculation for a circularly polarised photon gives $\varepsilon'^{\mu}_{\pm} = \exp(\mp \mathrm{i}\alpha)\varepsilon^{\mu}_{\pm}$.

Any plane wave ψ which is transformed into $\psi' = \mathrm{e}^{-\mathrm{i}h\alpha}\psi$ by a rotation of an angle α around its propagation axis is said to have helicity h. Thus if we say that a photon has spin 1 and a graviton has spin 2, we mean more precisely that electromagnetic and gravitational plan waves have helicity one and two, respectively. Doing the same calculation in an arbitrary gauge, one finds that the remaining, unphysical degrees of freedom transform as helicity one and zero (problem 19.5). In general, a tensor field of rank (n, m) has helicity $h = n + m$. Thus we can rephrase the statement that tensor fields follow Bose-Einstein statistics as fields with integer helicity (or spin) are bosons.

7.4. Source of gravity

The dynamical energy-momentum tensor If we compare the wave equation for a photon and a graviton, then there is an important difference: The former is in the classical limit exact. The photon carries no charge and does not contribute to its source term. As a result, the wave equation is linear. In contrast, a gravitational wave carries energy and momentum acts thus as its own source. The LHS of (7.35) should be therefore the limit of a more complicated equation, which we write symbolically as $G_{\mu\nu} = -\kappa T_{\mu\nu}$. The LHS of these field equations should be given as the variation of an appropriate action of gravity, called the Einstein-Hilbert action S_{EH}, with respect to the metric tensor $g_{\mu\nu}$. Even without knowing the action S_{EH}, we can derive an important conclusion. If the total action is the sum of S_{EH} and the action S_{m} including all relevant matter fields,

$$S = \frac{1}{2\kappa} S_{\mathrm{EH}} + S_{\mathrm{m}} \,,$$

then the variation of the matter action S_{m} should give the energy-momentum tensor as the source of the gravitational field,

$$\frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g_{\mu\nu}} = -T^{\mu\nu} \,. \tag{7.50}$$

Here we included a factor $\sqrt{|g|}$ because $T^{\mu\nu}$ is a density, while the factor 2 is required to obtain agreement with the usual definition of $T^{\mu\nu}$. Since the presence of gravity implies a curved space-time, the replacements $\{\partial_\mu, \eta_{\mu\nu}, \mathrm{d}^4x\} \to \{\nabla_\mu, g_{\mu\nu}, d^4x\sqrt{|g|}\}$ have to performed in S_{m} before the variation is performed. The tensor $T^{\mu\nu}$ defined by this equation is called *dynamical* energy-momentum stress tensor. In order to show that this rather bold definition makes sense, we have to prove that the tensor is locally conserved, $\nabla_\mu T^{\mu\nu} = 0$, and we have to convince ourselves that this definition reproduces the standard results we know already.

Conservation of the stress tensor We start by proving that the dynamical energy-momentum tensor defined by Eq. (7.50) is locally conserved. We consider the change of the matter action under a variation of the metric[4],

$$\delta S_{\mathrm{m}} = -\frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{|g|}\, T^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{|g|}\, T_{\mu\nu}\delta g^{\mu\nu} \,. \tag{7.51}$$

[4]We should view $g_{\mu\nu}$ (and not $g^{\mu\nu}$) as "the" gravitational field: In the Lagrangian of a point particle or the line-element, the coordinates x^μ are contracted with $g_{\mu\nu}$. For our sign choices, the Einstein equations are $G_{\mu\nu} = -\kappa T_{\mu\nu}$, i.e. the source of the gravitational field is $-T_{\mu\nu}$. Having understood this point, we use simply the second relation in (7.51) in the future.

We allow infinitesimal but otherwise arbitrary coordinate transformations,

$$\bar{x}^\mu = x^\mu + \xi^\mu(x)\,. \tag{7.52}$$

For the resulting change in the metric $\delta g_{\mu\nu}$ we can use Eqs. (6.70) and (6.71),

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\,. \tag{7.53}$$

We use that $T^{\mu\nu}$ is symmetric and that general covariance guarantees that $\delta S_m = 0$ for a coordinate transformation,

$$\delta S_\mathrm{m} = -\int_\Omega \mathrm{d}^4x \sqrt{|g|}\, T^{\mu\nu}\nabla_\mu \xi_\nu = 0\,. \tag{7.54}$$

Next we apply the product rule,

$$\delta S_\mathrm{m} = -\int_\Omega \mathrm{d}^4x \sqrt{|g|}\, (\nabla_\mu T^{\mu\nu})\xi_\nu + \int_\Omega \mathrm{d}^4x \sqrt{|g|}\, \nabla_\mu (T^{\mu\nu}\xi_\nu) = 0\,. \tag{7.55}$$

The second term is a four-divergence and thus a boundary term that we can neglect. The remaining first term vanishes for arbitrary ξ_μ only, if the energy-momentum tensor is conserved,

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{7.56}$$

Hence the local conservation of energy-momentum is a consequence of the general covariance of the gravitational field equations, in the same way as current conservation follows from gauge invariance in electromagnetism. You should convince yourself that the dynamical energy-momentum stress tensor evaluated for the examples of the Klein-Gordon and the Maxwell field agrees with the symmetrized canonical stress tensor, cf. problem 7.9.

7.A. Appendix: Large extra dimensions and massive gravity

Large extra dimensions As mentioned in chapter 5, quantum corrections break the conformal invariance of string theory except we live in a world with $d = 10$ or 26 space-time dimensions. There are two obvious answers to this result: First, one may conclude that string theory is disproven by nature or, second, one may adjust reality. Consistency of the second approach with experimental data could be achieved, if the $d-4$ dimensions are compactified with a sufficiently small radius R, such that they are not visible in experiments sensible to wave-lengths $\lambda \gg R$.

Let us check what happens to a scalar particle with mass m, if we add a fifth, compact dimension y. The Klein-Gordon equation for a scalar field $\phi(x^\mu, y)$ becomes

$$(\Box_5 + m^2)\phi(x^\mu, y) = 0 \tag{7.57}$$

with the five-dimensional d'Alembert operator $\Box_5 = \Box - \partial_y^2$. The equation can be separated, $\phi(x^\mu, y) = \phi(x^\mu) f(y)$, and since the fifth dimension is compact, the spectrum of f is discrete. Assuming periodic boundary conditions, $f(x) = f(x+R)$, gives

$$\phi(x^\mu, y) = \phi(x^\mu)\cos(n\pi y/R)\,. \tag{7.58}$$

The energy eigenvalues of these solutions are $\omega^2_{\boldsymbol{k},n} = \boldsymbol{k}^2 + m^2 + (n\pi/R)^2$. From a four-dimensional point of view, the term $(n\pi/R)^2$ appears as a mass term, $m_n^2 = m^2 + (n\pi/R)^2$. Since we usually consider states with different masses as different particles, we see the five-dimensional particle as a tower of

particles with mass m_n but otherwise identical quantum numbers. Such theories are called Kaluza-Klein theories, and the tower of particles Kaluza-Klein particles. If $R \ll \lambda$, where λ is the length-scale experimentally probed, only the $n = 0$ particle is visible and physics appears to be four-dimensional.

Since string theory includes gravity, one often assumes that the radius R of the extra-dimensions is determined by the Planck length, $R = 1/M_{\mathrm{Pl}} = (8\pi G_N)^{1/2} \sim 10^{-34}\,\mathrm{cm}$. In this case it is difficult to imagine any observational consequences of the additional dimensions. More interesting is the possibility that some of the extra dimensions are large,

$$R_{1,\dots,\delta} \gg R_{\delta+1,\dots,6} = 1/M_{\mathrm{Pl}}\,.$$

Since the $1/r^2$ behaviour of the gravitational force is not tested below $d_* \sim$mm scales, one can imagine that large extra dimensions exists that are only visible to gravity: Relating the $d = 4$ and $d > 4$ Newton's law $F \sim \frac{m_1 m_2}{r^{2+\delta}}$ at the intermediate scale $r = R$, we can derive the "true" value of the Planck scale in this model: Matching of Newton's law in 4 and $4 + \delta$ dimensions at $r = R$ gives

$$F(r = R) = G_N \frac{m_1 m_2}{R^2} = \frac{1}{M_D^{2+\delta}} \frac{m_1 m_2}{R^{2+\delta}}\,. \tag{7.59}$$

This equation relates the size R of the large extra dimensions to the true fundamental scale M_D of gravity in this model,

$$G_N^{-1} = 8\pi M_{\mathrm{Pl}}^2 = R^\delta M_D^{\delta+2}\,, \tag{7.60}$$

while Newton's constant G_N becomes just an auxiliary quantity useful to describe physics at $r \gtrsim R$. (You may compare this to the case of weak interactions where Fermi's constant $G_F \propto g^2/m_W^2$ is determined by the weak coupling constant g and the mass m_W of the W-boson).

Next we ask, if $M_D \sim$ TeV is possible, i.e. if one may test such theories at accelerators as LHC? Inserting the measured value of G_N and $M_D = 1\,\mathrm{TeV}$ in Eq. (7.60) we find the required value for the size R of the large extra dimension as

δ	1	2	6
R/cm	10^{13}	0.1	10^{-12}

Thus the case $\delta = 1$ is excluded by the agreement of the dynamics of the solar system with 4-dimensional Newtonian physics. The cases $\delta \geq 2$ are possible, because Newton's law is experimentally tested only for scales $r \gtrsim 1\,\mathrm{mm}$.

Massive gravity Theories with extra dimensions contain often from our 4-dimensional point of view a Kaluza-Klein tower of massive gravitons. Such modified theories of gravity have found large interest since one may hope to find an alternative explanation for the accelerated expansion of the Universe.

A striking difference between the spin-2 and the spin-0 and 1 cases is that the limit $m \to 0$ of the massive spin-2 propagator and thus of the potential energy V_{12} is not smooth: In problem 7.5, you are asked to derive the massive spin-2 propagator. As result, you should find

$$D_F^{\mu\nu;\rho\sigma}(k) = \frac{1}{2} \frac{-\frac{2}{3} G^{\mu\nu} G^{\rho\sigma} + G^{\mu\rho} G^{\nu\sigma} + G^{\mu\sigma} G^{\nu\rho}}{k^2 - m^2 + \mathrm{i}\varepsilon}\,, \tag{7.61}$$

where

$$G^{\mu\nu}(k) = -\eta^{\mu\nu} + k^\mu k^\nu / m^2 \tag{7.62}$$

is the polarisation tensor for a massive spin-1 particle. Thus the nominator in the massive spin-2 propagator is as in the massless case a linear combination of the tensor products of two spin-1 polarisation tensors. However, the coefficients of the first term differ and thus the $m \to 0$ limit of the massive propagator does not agree with the massless case. In particular, the difference cannot be compensated by a rescaling of the coupling constant $\tilde{G}_N$, because it is not an overall factor: Imagine for instance that we determine the value of $\tilde{G}_N$ by calculating the potential energy of two non-relativistic sources like the Sun and the Earth. This requires in massive gravity—for an arbitrarily small graviton

mass—a gravitational coupling constant $\tilde{G}_N$ a factor 3/4 smaller than in the massless case. Having fixed $\tilde{G}_N$, we can predict the deflection of light by the Sun. Since the first term in the propagator couples the traces T^μ_μ of two sources, it does not contribute to the deflection of light. As a result of the reduced coupling strength, the deflection angle of light by the Sun decreases by the same factor and any non-zero graviton mass would be in conflict with observations.

When this result was first derived in 1970, its authors explained this discontinuity by the different number of degrees of freedom in the two theories: Even if the Compton wave-length of a massive graviton is larger than the observable size of the Universe, and thus the Yukawa factor $\exp(-mr)$ indistinguishable from one, the additional spin states of a massive graviton may change physics. Two years later, Vainshtein realised that perturbation theory may break down in massive gravity and thus a calculation using one-graviton exchange is not reliable. More precisely, a theory of massive gravity contains an additional length scale $R_V = (GM/m^4)^{1/5}$ and for distances $r \ll R_V$ the theory has to be solved exactly.

Summary

Tensor fields satisfy second-order differential equations; their propagators are quadratic in p and thus even functions of x. As a result, tensor fields describe bosons, i.e. their field operators are commuting operators. Massless fields have only two, transverse degrees of freedom. A Lorentz invariant description for such fields is only possible, if the remaining number of non-physical degrees of freedom is redundant. This redundancy implies that fields connected by a gauge transformation are equivalent and describe the same physical system. In the case of photons, the gauge symmetry implies that they couple to a conserved current, in the case of gravitons that they couple to the conserved energy-momentum conservation tensor.

Further reading

[50] discusses in more detail (massive) gravity as a spin-2 field in Minkowski space. The history of the gauge principle is reviewed in [40].

Problems

7.1 Irreducible tensor components.
Show that the splitting into symmetric and anti-symmetric tensor components is in invariant under general coordinate transformations.

7.2 Polarisation of spin-1 particle.
a.) Determine the polarisation vectors $\varepsilon^{(r)}_\mu(k)$ of a massive spin-1 particle for arbitrary k^μ. b.) Find the sum $\sum_{r=0}^{3} \varepsilon^{(r)*}_\mu \varepsilon^{(r)}_\nu$ of the polarisation states of a massles spin-1 particle using the tensor method and imposing the two constraints $k_\mu \varepsilon^\mu = 0$ and $n_\mu \varepsilon^\mu = 0$, where n^μ is an arbitrary vector satisfying $n_\mu k^\mu \neq 0$.

7.3 Feynman propagator as sum over solutions.
Follow the steps from (3.27) to (3.31) in the scalar case for a massive spin-1 propagator.

7.4 Gravitational coupling.
Determine the coupling λ in $\mathscr{L}_I = \lambda h^{\mu\nu} T_{\mu\nu}$ comparing (7.47) to Newton's gravitational potential.

7.5 Massive spin-2 propagator.
Derive the propagator (7.61) of a massive spin-2 particle: Use the tensor method to write down the most general combination of tensors built from $\eta_{\mu\nu}$ and k_μ (or from $G_{\mu\nu}$ and k_μ), compatible with the symmetries and constraints on the polarisation tensor.

7.6 Helicity.

Show that h in $\psi' = \mathrm{e}^{-\mathrm{i}h\alpha}\psi$ for a plane wave is the eigenvalue of the helicity operator $\boldsymbol{J}\cdot\boldsymbol{p}/|\boldsymbol{p}|$.

7.7 Scalar QED ♣.

a.) Introduce in the Lagrangian (5.11) of a complex scalar field the interaction with photons via the minimal substitution, $\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}eA_\mu$, and read off the interaction vertices.
b. Calculate the matrix element for "scalar Compton scattering", $\phi\gamma \to \phi\gamma$ and show that it is gauge invariant. [Try to simplify the matrix element choosing a "good" frame and polarisation vectors.]

7.8 The $\phi\phi h_{\mu\nu}$ vertex.
Find from the stress tensor of on-shell scalar particle the matrix elements $\langle p|' T_{\mu\nu} |p\rangle$ and determine thereby the Feynman rule for the $\phi\phi h_{\mu\nu}$ vertex.

7.9 Dynamical stress tensor.
Derive the dynamical energy-momentum stress tensor $T^{\mu\nu}$ of a.) a real scalar field ϕ and b.) the photon field A^μ.

8. Fermions and the Dirac equation

Up to now we have discussed fields which transform as tensors under Lorentz transformations. Such particles have integer spin or helicity. Since we can boost massive particles into their rest-frame, we can use our knowledge of non-relativistic quantum mechanics to anticipate that additional representations of the rotation group SO(3) and thus also of the Lorentz group SO(1, 3) exist which correspond to particles with half-integer spin. Such particles are described by anti-commuting variables what is the fundamental reason for the Pauli principle and thus the stability of matter.

8.1. Spinor representation of the Lorentz group

In order to introduce spinors we have to find the corresponding representation of the Lorentz group. As always it is simpler to work at "linear order," which is in this case the Lie algebra. The Lie algebra of the Poincaré group[1] contains ten generators, the three generators $\boldsymbol{J}$ of rotations, the three generators $\boldsymbol{K}$ of Lorentz boosts and the four generators $\boldsymbol{T}$ of translations. The Killing vector fields $\boldsymbol{V}$ of Minkowski space generate these symmetries, and therefore the generators are given by the Killing vector fields. Thus we can use Eqs. (6.81) and (6.82) to calculate their commutation relations as (problem 8.1)

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k \,, \tag{8.1a}$$

$$[J_i, K_j] = \mathrm{i}\varepsilon_{ijk} K_k \,, \tag{8.1b}$$

$$[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk} J_k \,. \tag{8.1c}$$

Here we followed physicist's convention and identified $\mathrm{i}\boldsymbol{V}$ as the generators, so that they are hermitian. Moreover, we restrict our attention to the Lorentz group which is sufficient to derive the concept of a Weyl spinor. Note that the algebra of the boost generators $\boldsymbol{K}$ is not closed. Thus in contrast to rotations, boosts do not form a subgroup of the Lorentz group.

The commutation relations above are satisfied by $\pm\mathrm{i}\boldsymbol{\sigma}/2$, suggesting that we can rewrite the Lorentz group as a product of two SU(2) factors. We try to decouple the two sets of generators $\boldsymbol{J}$ and $\boldsymbol{K}$ by introducing two non-hermitian ladder operators

$$\boldsymbol{J}^{\pm} = \frac{1}{2}(\boldsymbol{J} \pm \mathrm{i}\boldsymbol{K}) \,. \tag{8.2}$$

Their commutations relations are

$$[J_i^+, J_j^+] = \mathrm{i}\varepsilon_{ijk} J_k^+ \,, \tag{8.3a}$$

$$[J_i^-, J_j^-] = \mathrm{i}\varepsilon_{ijk} J_k^- \,, \tag{8.3b}$$

$$[J_i^+, J_j^-] = 0 \qquad i, j = 1, 2, 3 \,. \tag{8.3c}$$

[1] See the Appendices B.3 and B.4 for a brief review of the Poincaré group.

Thus $\boldsymbol{J}^-$ and $\boldsymbol{J}^+$ commute with each other and generate each a SU(2) group. The Lorentz group is[2] therefore $\sim \mathrm{SU}(2) \otimes \mathrm{SU}(2)$, and states transforming in a well-defined way are labeled by a pair of angular momenta, (j^-, j^+), corresponding to the eigenvalues of J_z^- and J_z^+, respectively. From our knowledge of the angular momentum algebra in nonrelativistic quantum mechanics, we conclude that the dimension of the representation (j^-, j^+) is $(2j^- + 1)(2j^+ + 1)$. Because of $\boldsymbol{J} = \boldsymbol{J}^- + \boldsymbol{J}^+$, the representation (j^-, j^+) contains all possible spins j in integer steps from $|j^- - j^+|$ to $j^- + j^+$.

The representation $(0, 0)$ has dimension one, transforms trivially, $\boldsymbol{J} = \boldsymbol{K} = 0$, and corresponds therefore to the scalar representation. The two smallest non-trivial representations are $\boldsymbol{J}^+ = 0$, i.e. $(j^-, 0)$ with $\boldsymbol{J}^{(1/2)} = -\mathrm{i}\boldsymbol{K}^{(1/2)}$, and $\boldsymbol{J}^- = 0$, i.e. $(0, j^+)$ with $\boldsymbol{J}^{(1/2)} = \mathrm{i}\boldsymbol{K}^{(1/2)}$. Both representations have spin 1/2 and dimension two. We define therefore two types of two-component spinors,

$$\phi_L: \quad (1/2, 0), \quad \boldsymbol{J}^{(1/2)} = \boldsymbol{\sigma}/2, \quad \boldsymbol{K}^{(1/2)} = +\mathrm{i}\boldsymbol{\sigma}/2\,, \tag{8.4a}$$

$$\phi_R: \quad (0, 1/2), \quad \boldsymbol{J}^{(1/2)} = \boldsymbol{\sigma}/2, \quad \boldsymbol{K}^{(1/2)} = -\mathrm{i}\boldsymbol{\sigma}/2\,, \tag{8.4b}$$

which we call left-chiral and right-chiral Weyl spinors. These Weyl spinors form the fundamental representation of the Lorentz group: All higher spin states can be obtained as tensor products involving them. Their transformation properties under an (active) finite Lorentz transformation with parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\eta}$ follow by exponentiating their generators as $\exp(-\mathrm{i}\boldsymbol{J}\boldsymbol{\alpha} + \mathrm{i}\boldsymbol{K}\boldsymbol{\eta})$ (compare B.3 for our choice of signs),

$$\phi_L \to \phi_L' = \exp\left[-\frac{\mathrm{i}\boldsymbol{\sigma}\boldsymbol{\alpha}}{2} - \frac{\boldsymbol{\sigma}\boldsymbol{\eta}}{2}\right]\phi_L \equiv S_L\phi_L\,, \tag{8.5a}$$

$$\phi_R \to \phi_R' = \exp\left[-\frac{\mathrm{i}\boldsymbol{\sigma}\boldsymbol{\alpha}}{2} + \frac{\boldsymbol{\sigma}\boldsymbol{\eta}}{2}\right]\phi_R \equiv S_R\phi_R\,. \tag{8.5b}$$

While the transformation matrices S_L and S_R agree for rotations, the terms describing Lorentz boosts have opposite signs. Note also that only rotations are described by a unitary transformation, while Lorentz boosts lead to a non-unitary transformation of the Weyl spinors.

We ask now if we can convert a left- into a right-chiral spinor and vice versa. Thus we should find a spinor $\tilde{\phi}_L$ constructed out of ϕ_L which transforms as $S_R\tilde{\phi}_L$. Changing S_L into S_R requires reversing the relative sign between the rotation and the boost term, which we achieve by complex conjugating ϕ_L,

$$\phi_L^{*\prime} = \left[1 + \frac{\mathrm{i}\boldsymbol{\sigma}^*\boldsymbol{\alpha}}{2} - \frac{\boldsymbol{\sigma}^*\boldsymbol{\eta}}{2} + \ldots\right]\phi_L^*\,. \tag{8.6}$$

Because of $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$, $\sigma_3^* = \sigma_3$, and $\sigma_1\sigma_2 = -\sigma_2\sigma_1$, $\sigma_2\sigma_3 = -\sigma_3\sigma_2$, we obtain the desired transformation property multiplying $\phi_L^{*\prime}$ with σ_2,

$$\sigma_2\phi_L^{*\prime} = \sigma_2\left[1 + \frac{\mathrm{i}(\sigma_1, -\sigma_2, \sigma_3)\boldsymbol{\alpha}}{2} - \frac{(\sigma_1, -\sigma_2, \sigma_3)\boldsymbol{\eta}}{2} + \ldots\right]\phi_L^* \tag{8.7}$$

$$= \left[1 - \frac{\mathrm{i}\boldsymbol{\sigma}\boldsymbol{\alpha}}{2} + \frac{\boldsymbol{\sigma}\boldsymbol{\eta}}{2} + \ldots\right]\sigma_2\phi_L^* = S_R\sigma_2\phi_L^*\,. \tag{8.8}$$

[2] More precisely, they have the same Lie algebra and are thus locally isomorphic but differ globally.

Thus ϕ_L and ϕ_R are not connected by a unitary transformation, $\phi_R \neq U\phi_L$, and therefore ϕ_L and ϕ_R describe different physics. Obviously, we can add to $\sigma_2\phi_L^*$ an arbitrary phase $\mathrm{e}^{\mathrm{i}\delta}$ without changing the transformation properties. We define

$$\phi_R^c \equiv -\mathrm{i}\sigma_2\phi_L^* \quad \text{and} \quad \phi_L^c \equiv \mathrm{i}\sigma_2\phi_R^*\,, \,, \tag{8.9}$$

which ensures $(\phi_L^c)^c = \phi_L$ and $(\phi_R^c)^c = \phi_R$. When we discuss later the coupling of a fermion to an external field, we will see that ϕ_L^c is the charge conjugated spinor of ϕ_L.

For the construction of a Lagrangian we need for the mass term scalars and for the kinetic energy vectors built out of the Weyl fields; they should be real to provide a real Lagrangian. In contrast to the real Lorentz transformation $\Lambda^\mu{}_\nu$ acting on tensor fields, the matrices $S_{L/R}$ are however complex and thus the Weyl fields are complex too. This suggests together with the fact that a measurement device should be the same after a rotation by 2π that observables are bilinear quantities in the fermion fields, such that they transform tensorial and their eigenvalues are real.

Out of the two Weyl spinors, we can form four different products $\phi_{L/R}^\dagger\phi_{L/R}$ leading to the combinations $S_L^\dagger S_L$, $S_R^\dagger S_R$, $S_L^\dagger S_R$, and $S_R^\dagger S_L$. The rotation $\mathrm{i}\boldsymbol{\sigma\alpha}/2$ cancels in all four products, since it enters with the same sign in S_L and S_R, and the Pauli matrices are hermitian, $\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}$. By contrast, the cancelation of the boost $\boldsymbol{\sigma\eta}/2$ requires a combination of a left- and right-chiral field,

$$\phi_L'^\dagger\phi_R' = \phi_L^\dagger\left[1 + \mathrm{i}\frac{\boldsymbol{\sigma\alpha}}{2} - \frac{\boldsymbol{\sigma\eta}}{2} + \ldots\right]\left[1 - \mathrm{i}\frac{\boldsymbol{\sigma\alpha}}{2} + \frac{\boldsymbol{\sigma\eta}}{2} + \ldots\right]\phi_R = \phi_L^\dagger\phi_R\,, \tag{8.10}$$

and similiarly for $\phi_R^\dagger\phi_L$. Thus $\phi_L^\dagger\phi_R$ and $\phi_R^\dagger\phi_L$ transform as Lorentz scalars, but not $\phi_L^\dagger\phi_L$ and $\phi_R^\dagger\phi_R$. So what are the transformation properties of the latter two products? Performing an infinitesimal boost along the z axis, we find

$$\phi_R'^\dagger\phi_R' = \phi_R^\dagger\left[1 + \frac{\sigma_3\eta}{2} + \ldots\right]\left[1 + \frac{\sigma_3\eta}{2} + \ldots\right]\phi_R = \phi_R^\dagger\phi_R + \eta\phi_R^\dagger\sigma_3\phi_R\,. \tag{8.11}$$

This looks like an infinitesimal Lorentz transformation of the time-like component $j^0 = \phi_R^\dagger\phi_R$ of a four-vector j^μ. If this interpretation is correct, we should be able to associate the spatial part $\boldsymbol{j}$ with $\phi_R^\dagger\boldsymbol{\sigma}\phi_R$. Checking thus how $\boldsymbol{j}$ transforms, we find using $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ijk}\sigma^k$ that j^1 and j^2 are invariant, while j^3 transforms as

$$\phi_R'^\dagger\sigma_3\phi_R' = \phi_R^\dagger\left[1 + \frac{\sigma_3\eta}{2} + \ldots\right]\sigma_3\left[1 + \frac{\sigma_3\eta}{2} + \ldots\right]\phi_R = \eta\phi_R^\dagger\phi_R + \phi_R^\dagger\sigma_3\phi_R\,. \tag{8.12}$$

Thus $\phi_R^\dagger\sigma^\mu\phi_R$ with $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$ transforms as a four-vector, $j^0 \to j^0 + \eta j^3$ and $j^3 \to \eta j^0 + j^3$. Performing the same calculation for the left-chiral fields reproduces the same result except for an opposite sign of η. The wrong sign can be remedied setting now $\bar\sigma^\mu \equiv (1, -\boldsymbol{\sigma})$, so that a four-vector bilinear in ϕ_L is given by $\phi_L^\dagger\bar\sigma^\mu\phi_L$.

The transformations S_L and S_R that belong to the restricted Lorentz group do not mix the left- and right-chiral Weyl spinors. Consider however the effect of a parity transformation, $P\boldsymbol{x} = -\boldsymbol{x}$, on the generators $\boldsymbol{K}$ and $\boldsymbol{J}$. The velocity changes sign, $\boldsymbol{v} \to -\boldsymbol{v}$, i.e. transforms as a polar vector, while the angular momentum $\boldsymbol{J}$ as axial vector remains invariant. Thus parity interchanges $(1/2, 0)$ and $(0, 1/2)$ and hence ϕ_L and ϕ_R, as one would expect from a left- and right-chiral object. If parity is a symmetry of the theory examined, one can therefore not

consider separately the two spinors ϕ_L and ϕ_R. Instead, it proves useful to combine them into a four-spinor called Dirac (or bi-spinor)

$$\psi = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} . \tag{8.13}$$

Another reason to consider Dirac spinors is that the scalar terms $\phi_L^\dagger \phi_R$ and $\phi_R^\dagger \phi_L$ that qualify as mass terms combine a left- and a right-chiral field: Thus the description of a particle with such a mass term requires the use of both left- and right-chiral Weyl spinors. Next we will derive a field equation for this type of spinor and discuss its properties.

8.2. Dirac equation

From Weyl spinors to the Dirac equation We can obtain the spinor $\phi_{L/R}(p)$ describing a particle with momentum p by boosting the one describing a particle at rest, $\phi_{L/R}(0)$,

$$\phi_R(p) = \exp\left[\frac{\boldsymbol{\sigma\eta}}{2}\right]\phi_R(0) = \exp\left[\frac{\eta\boldsymbol{\sigma n}}{2}\right]\phi_R(0) = \left[\cosh(\eta/2) + \boldsymbol{\sigma n}\sinh(\eta/2)\right]\phi_R(0) . \tag{8.14}$$

If we replace the boost parameter η by the Lorentz factor[3] $\gamma = \cosh\eta$ and use the identities $\cosh(\eta/2) = \sqrt{(\cosh\eta + 1)/2}$ and $\sinh(\eta/2) = \sqrt{(\cosh\eta - 1)/2}$, we can express the spinor as

$$\phi_R(p) = \left[\left(\frac{\gamma+1}{2}\right)^{1/2} + \boldsymbol{\sigma\hat{p}}\left(\frac{\gamma-1}{2}\right)^{1/2}\right]\phi_R(0) . \tag{8.15}$$

Here $\boldsymbol{\hat{p}} = \boldsymbol{p}/|\boldsymbol{p}|$ is the unit vector in direction of $\boldsymbol{p}$. Inserting $\gamma = E/m$ and combining the two terms in the angular bracket, we arrive at

$$\phi_R(p) = \frac{E + m + \boldsymbol{\sigma p}}{\sqrt{2m(E+m)}}\,\phi_R(0) . \tag{8.16}$$

Similiarly, we find

$$\phi_L(p) = \frac{E + m - \boldsymbol{\sigma p}}{\sqrt{2m(E+m)}}\,\phi_L(0) . \tag{8.17}$$

Thus ϕ_L and ϕ_R differ only by the sign of the operator $\boldsymbol{\sigma p}$ which measures the projection of the spin $\boldsymbol{\sigma}$ on the momentum $\boldsymbol{p}$ of the particle. For a particle at rest, this difference disappears and we set therefore $\phi_L(0) = \phi_R(0)$. This allows us to eliminate the zero momentum spinors, giving

$$\phi_L^R(p) = \frac{E \pm \boldsymbol{\sigma p}}{m}\,\phi_R^L(p) . \tag{8.18}$$

In matrix form, these two equations correspond to

$$\begin{pmatrix} -m & E - \boldsymbol{\sigma p} \\ E + \boldsymbol{\sigma p} & -m \end{pmatrix}\begin{pmatrix} \phi_L(p) \\ \phi_R(p) \end{pmatrix} = \begin{pmatrix} -m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & -m \end{pmatrix}\begin{pmatrix} \phi_L(p) \\ \phi_R(p) \end{pmatrix} = 0 . \tag{8.19}$$

We introduce the 4×4 matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{8.20}$$

[3] Recall the relations $E = m\cosh\eta$ and $p = m\sinh\eta$ connecting E, p, and the rapidity η.

Then we arrive with $\gamma^\mu p_\mu = \gamma^0 E - \boldsymbol{\gamma p}$ at the compact expression

$$(\gamma^\mu p_\mu - m)\,\psi(p) = 0\,. \tag{8.21}$$

Identifying $p_\mu = \mathrm{i}\partial_\mu$ we obtain the Dirac equation. The representation used for the Dirac spinor and the gamma matrices is called chiral or Weyl representation. Other representations can be obtained performing a unitary transformation, $U\tilde{\gamma}^\mu U^\dagger = \gamma^\mu$ and $U\tilde{\psi} = \psi$.

We can apply the tensor method to derive a definition of the gamma matrices and their properties which is independent of the considered representation. The only invariant (and parity conserving) tensor at our disposal is the metric tensor $\eta^{\mu\nu}$ and thus the gamma matrices have to satisfy $\{\gamma^\mu, \gamma^\nu\} = A\eta^{\mu\nu}$. Considering $\{\gamma^0, \gamma^0\}$ shows that $A = 2$, or

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,. \tag{8.22}$$

These anti-commutation relations define a Clifford algebra, implying

$$\left(\gamma^0\right)^2 = 1\,, \qquad \left(\gamma^i\right)^2 = -1 \qquad \text{and} \qquad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \tag{8.23}$$

for $\mu \neq \nu$. The last condition shows that the Clifford algebra cannot be satisfied by normal numbers.

The definition (8.22) implies that we can apply in the usual way the metric tensor to raise or to lower the indices of the gamma matrices, $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$. Thus we can write $\gamma^\nu\partial_\nu = \gamma_\nu\partial^\nu$. Since the contraction of the gamma matrices γ^μ with a four-vector A^μ will appear frequently, we introduce the so-called Feynman slash,

$$\not{A} \equiv A_\mu\gamma^\mu\,, \tag{8.24}$$

as useful shortcut. This notation also stresses that the gamma matrices γ^μ allow us to map a four-vector A^μ onto an element $\not{A}$ of the Clifford algebra which then can be applied on a spinor ψ. Although we suppress the spinor indices, you should keep in mind that the matrices $(\gamma_{ab})^\mu$ carry both tensor and spinor indices.

Dirac's way towards the Dirac equation The Klein-Gordon equation was historically the first wave equation derived in relativistic quantum mechanics. Applied to the hydrogen atom, it failed to reproduce the correct energy spectrum. Dirac tried to derive as an alternative an equation linear in the derivatives ∂_μ. Since Lorentz invariance requires that ∂_μ has to be contracted with another four-vector, a first order equation has the form

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0\,. \tag{8.25}$$

Main task for Dirac was to uncover the nature of the quantities γ^μ in this equation. They cannot be normal numbers, since then they would form a four-vector, specify one direction in space-time and thus break Lorentz invariance. Multiplying the Dirac equation with $-(\mathrm{i}\gamma^\mu\partial_\mu + m)$ and comparing the result to the Klein-Gordon equation, we find

$$-(\mathrm{i}\gamma^\mu\partial_\mu + m)(\mathrm{i}\gamma^\nu\partial_\nu - m)\psi = (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = (\Box + m^2)\psi = 0\,. \tag{8.26}$$

Using the symmetry of partial derivatives, we can rewrite

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu\,. \tag{8.27}$$

Remembering next the definition of the d'Alembert operator, $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, we re-derive that the γ^μ form a Clifford algebra.

Lagrange density For a complex scalar field, we could rewrite after a partial integration the Lagrange density as $\mathscr{L} = -\phi^\dagger(\Box + m^2)\phi$. This expression corresponds to the Klein-Gordon operator $-(\Box + m^2)$ sandwiched between the quadratic form $\phi^\dagger\phi$, as the correspondance of the propagator and a two-point Green function requires. This suggests to try for the Dirac field as Lagrangian

$$\mathscr{L} = \psi^\dagger A(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = \bar\psi(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi\,, \tag{8.28}$$

where we have used as quadratic from $\psi^\dagger A\psi$ with a matrix A yet to be determined. In the second step, we defined the *adjoint* spinor $\bar\psi \equiv \psi^\dagger A$. Varying then the action $S[\psi, \bar\psi]$, we obtain

$$\delta S = \int \mathrm{d}^4 \left\{\delta\bar\psi(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi - \bar\psi(\mathrm{i}\gamma^\mu\overleftarrow{\partial_\mu} + m)\delta\psi\right\}\,. \tag{8.29}$$

Here we made a partial integration of the $\partial_\mu\delta\psi$ term, and thus the derivative $\overleftarrow{\partial_\mu}$ acts to the left. Since we treat ψ and $\bar\psi$ as two independent variables, we obtain from $\delta S = 0$ two equations of motion,

$$\bar\psi(\mathrm{i}\gamma^\mu\overleftarrow{\partial_\mu} + m) = 0 \quad \text{and} \quad (\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0\,. \tag{8.30}$$

Next we determine the unknown matrix A: Taking the hermitian of the RHS of (8.30) results in

$$\psi^\dagger(-\mathrm{i}\gamma^{\mu\dagger}\overleftarrow{\partial_\mu} - m) = 0\,. \tag{8.31}$$

This agrees with the LHS of (8.30), if A satisfies

$$A^{-1}\gamma^{\mu\dagger}A = \gamma^\mu\,. \tag{8.32}$$

One can readily check that the γ^μ matrices in the Weyl representation fulfil this relation, if we set $A = \gamma^0$. With $(\gamma^0)^2 = 1$ and $(\gamma^0)^{-1} = (\gamma^0)^\dagger = \gamma^0$, we can express this condition as

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 = \begin{cases} (\gamma^0)^2\gamma^0 = \gamma^0\,, \\ -\gamma^i(\gamma^0)^2 = -\gamma^i\,. \end{cases} \tag{8.33}$$

Thus the action principle implies that γ^0 is hermitian, while the γ^i are anti-hermitian matrices.

Using the gamma matrices and the Dirac spinor in the chiral representation it is straightforward to express the Dirac Lagrangian (8.28) by Weyl fields,

$$\mathscr{L} = \mathrm{i}\phi_R^\dagger\sigma^\mu\partial_\mu\phi_R + \mathrm{i}\phi_L^\dagger\bar\sigma^\mu\partial_\mu\phi_L - m(\phi_L^\dagger\phi_R + \phi_R^\dagger\phi_L)\,. \tag{8.34}$$

This implies that the Dirac Lagrangian and the Dirac equation are invariant under Lorentz transformations, because we have already checked that all ingredients of (8.34) are invariant. Note also that out of the two possible combinations of the two Lorentz scalars we found, only the one invariant under parity, $P\phi_L = \phi_R$, entered the mass term. Moreover, $P(\sigma^\mu\partial_\mu) = \bar\sigma^\mu\partial_\mu$, and thus the combination of the kinetic energies of ϕ_L and ϕ_R is also invariant under parity.

Hamiltonian form The Dirac equation can be transformed into Hamiltonian form by multiplying with γ^0,

$$\mathrm{i}\partial_t\psi = H_D\psi = (-\mathrm{i}\gamma^0\gamma^i\partial_i + \gamma^0 m)\psi\,. \tag{8.35}$$

Looking back at the (anti-) hermiticity properties (8.33) of the γ^μ matrices, we see that they correspond to the one required to make the Dirac Hamiltonian hermitian. By tradition, one re-writes H_D often with $\beta = \gamma^0$ and $\alpha^i = \gamma^0\gamma^i$ as

$$\mathrm{i}\partial_t\psi = H_D\psi = (\boldsymbol{\alpha}\boldsymbol{p} + \beta m)\psi\,. \tag{8.36}$$

Considering the semi-classical limit, one sees that the matrix $\boldsymbol{\alpha}$ has the meaning of a velocity operator, see problem 8.6.

Clifford algebra and bilinear quantities We now determine the minimal matrix representation for the Clifford algebra defined by Eq. (8.22). Thus we should find the number of independent products that we can form out of the four gamma matrices. Five obvious elements are the unit matrix $1 = (\gamma^0)^2$ and the four gamma matrices γ^μ themselves. Because of $(\gamma^\mu)^2 = \pm 1$, the remaining products should consist of γ^μ matrices with different indices. Thus the only product of four γ^μ matrices that we have to consider is $\gamma^0\gamma^1\gamma^2\gamma^3$. This combination will appear very often and deserves therefore a special name. Including the imaginary unit to make it hermitian, we define

$$\gamma^5 \equiv \gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{8.37}$$

Because the four gamma matrices in γ^5 anti-commute, we can rewrite its definition introducing the completely anti-symmetric tensor $\varepsilon_{\alpha\beta\gamma\delta}$ in four dimensions as

$$\gamma^5 = \frac{\mathrm{i}}{24}\,\varepsilon_{\alpha\beta\gamma\delta}\,\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\,. \tag{8.38}$$

This suggests that bilinear quantities containing one γ^5 matrix transform as pseudo-tensors, i.e. change sign under a parity transformation $\boldsymbol{x} \to -\boldsymbol{x}$. Two important properties of the γ^5 matrix are $(\gamma^5)^2 = 1$ and $\{\gamma^\mu, \gamma^5\} = 0$.

Next we consider products of three γ^μ matrices. For instance,

$$\gamma^1\gamma^2\gamma^3 = \underbrace{\gamma^0\gamma^0}_{1}\gamma^1\gamma^2\gamma^3 = -\mathrm{i}\gamma^0\gamma^5\,. \tag{8.39}$$

Hence these products are equivalent to $\gamma^\mu\gamma^5$, giving us four more basis elements.

Finally, we are left with products of two γ^μ matrices. We start from the defining equation (8.22),

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu}\,, \tag{8.40}$$

divide by two, add $\gamma^\mu\gamma^\nu/2$ and introduce the commutator of two gamma matrices,

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\gamma^\nu\gamma^\mu + \eta^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \eta^{\mu\nu}\,. \tag{8.41}$$

Adding again for later convenience an imaginary unit, we define the anti-symmetric tensor $\sigma^{\mu\nu}$ as

$$\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]\,. \tag{8.42}$$

The six matrices $\sigma^{\mu\nu}$ are the remaining basis elements of the four-dimensional Clifford algebra. All together, the basis has dimension 16,

$$\Gamma = \{1, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}\,, \tag{8.43}$$

as the 4×4 matrices. Hence an arbitrary 4×4 matrix can be decomposed into a linear combination of the 16 basis elements of the Clifford algebra. Moreover, the smallest matrix representation of the Clifford algebra are 4×4 matrices. Some useful properties of gamma matrices are collected in the Appendix A.2.

Knowing the dimension of the γ matrices, we can count the number of degrees of freedom represented by a Dirac spinor ψ. As the γ matrices and the Lorentz transformation acting on spinors are complex, the field ψ is complex too and has thus four complex degrees of freedom. We know already that the Dirac equation describes spin 1/2 particles, which come with $2s + 1 = 2$ spin degrees of freedom for a particle plus 2 for its anti-particle. Thus in this case the number of physical states matches the four components of the fields ψ. Note also the difference to the case of a complex scalar or vector field: There we introduced two complex fields $\phi^{\pm} = (\phi_1 \pm \mathrm{i}\phi_2)/\sqrt{2}$, which are connected by $(\phi^{\pm})^* = \phi^{\mp}$. The real fields ϕ_1 and ϕ_2 are not mixed by Lorentz transformations and thus we count them as two real degrees of freedom.

We come now to the construction of bilinear quantities out of the Dirac spinors. Since the Lagrangian is a scalar, we know already that $\bar{\psi}\psi$ transforms as a scalar while $j^\mu = \psi^\dagger \gamma^\mu \psi$ is vector. In general, bilinear quantities are constructed as

$$\psi^\dagger \gamma^0 \Gamma \psi \equiv \bar{\psi} \Gamma \psi \,, \tag{8.44}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ is the adjoint spinor and Γ is any of the 16 basis elements given in Eq. (8.43). In this way, the complex conjugated of a bilinear becomes

$$(\bar{\psi}\Gamma\psi')^* = (\bar{\psi}\Gamma\psi')^\dagger = \psi'^\dagger \Gamma^\dagger \gamma^0 \psi = \psi'^\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 \psi \equiv \bar{\psi}' \bar{\Gamma} \psi \tag{8.45}$$

with

$$\bar{\Gamma} \equiv \gamma^0 \Gamma^\dagger \gamma^0 \,. \tag{8.46}$$

For $\psi = \psi'$, these bilinears are real as desired. The analogue $\psi^\dagger\psi$ to the probability density $\psi^*\psi$ of the Schrödinger equation is thus the zero-component of a four-current, $\psi^\dagger\psi = \psi^\dagger\gamma^0\gamma^0\psi = \bar{\psi}\gamma^0\psi = j^0$, as one should expect in a relativistic theory.

Finally, we note that γ^0 and γ^5 are involutory matrices, i.e. they satisfy the relation $A^2 = 1$. Because of $(1 \pm A)^2 = 2(1 \pm A)$, we can construct the projection operators $P_\pm = (1 \pm A)/2$, satisfying

$$P_\pm^2 = P_\pm, \qquad P_\pm P_\mp = 0, \quad \text{and} \quad P_+ + P_- = 1 \,.$$

Thus we should be able to classify the four independent solutions of the Dirac equation with the help of $(1 \pm \gamma^0)/2$ and $(1 \pm \gamma^5)/2$ (or their suitable covariant generalisations).

Lorentz transformations Our derivation of the Weyl spinors as the fundamental representation of the Lorentz group provided automatically their transformation properties under a finite Lorentz transformation. Using the Weyl representation, the tranformation law for a Dirac spinor follows as

$$\psi(x) \to \psi(x') = S(\Lambda)\psi(x) = \begin{pmatrix} \phi_L(x') \\ \phi_R(x') \end{pmatrix} = \begin{pmatrix} S_L & 0 \\ 0 & S_R \end{pmatrix} \begin{pmatrix} \phi_L(x) \\ \phi_R(x) \end{pmatrix} . \tag{8.47}$$

We want to express the transformation matrix $S(\Lambda)$ by gamma matrices, such that it is representation independent and manifestly Lorentz invariant. We set

$$S(\Lambda) = \exp\left(-\mathrm{i}\omega_{\mu\nu} J^{\mu\nu}/2\right) , \tag{8.48}$$

where the antisymmetric matrix $\omega_{\mu\nu}$ parametrises the Lorentz transformation and the six generators $(J_{ab})_{\mu\nu}$ have to be determined. Since the generators are the relativistic generalisation of $\boldsymbol{\sigma}$, we suspect that they are connected to $\sigma_{\mu\nu}$. Using Eq. (8.20), we obtain as explicit expression for the $\sigma^{\mu\nu}$ matrices in the Weyl representation

$$\sigma^{0i} = \mathrm{i} \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad \text{and} \quad \sigma^{ij} = \varepsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{8.49}$$

We split $J^{\mu\nu}$ into boosts and rotations,

$$\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} = \omega_{i0}J^{i0} + \omega_{12}J^{12} + \omega_{13}J^{13} + \omega_{23}J^{23}\,. \tag{8.50}$$

Identifying $\eta_i = \omega_{i0}$ and $\alpha_i = (1/2)\varepsilon_{ijk}\omega_{jk}$, we obtain $J^{\mu\nu} = \sigma^{\mu\nu}/2$. In contrast to (8.47), the expression $S(\Lambda) = \exp\left(-\mathrm{i}\sigma_{\mu\nu}\omega^{\mu\nu}/4\right)$ is valid for any representation of the gamma matrices.

Solutions We search for plane wave solutions $u\mathrm{e}^{-\mathrm{i}px}$ and $v\mathrm{e}^{+\mathrm{i}px}$ of the Dirac equation with $m > 0$ and $E = p^0 = |\boldsymbol{p}| > 0$. The algebra is simplified, if we construct the solutions first in the rest frame of the particle. Then $\not{p} = m\gamma^0$, and thus the use of the Dirac representation,

$$\gamma^0 = 1 \otimes \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \gamma^i = \sigma^i \otimes \mathrm{i}\tau_2 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{8.51}$$

where γ^0 is diagonal is most convenient. Here σ_i and τ_i are the Pauli matrices, $\otimes$ denotes the tensor product, 0 and 1 are 2×2 matrices. In the Dirac representation, the γ^5 matrix is off-diagonal,

$$\gamma^5 = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.52}$$

The Dirac equation becomes

$$(\not{p} - m)u = m(\gamma^0 - 1)u = 0 \tag{8.53a}$$

$$(\not{p} + m)v = m(\gamma^0 + 1)v = 0\,. \tag{8.53b}$$

The RHS shows that $(1 \pm \gamma^0)/2$ project a general spinor at rest on the subspaces of solutions with positive or negative energy. Inserting the explicit form of γ^0 into (8.53), the four solutions in the rest frame of the particle follow as

$$u(m,+) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u(m,-) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v(m,-) = \mathcal{N}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v(m,+) = \mathcal{N}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8.54}$$

The additional $\pm$ label should be the quantum number of a suitable operator labelling the two spin states of a Dirac particle. Note the opposite order of the spin label in the v spinor compared to u. We will see later that this choice is required by the structure of the relativistic spin operator s^μ. As an intuitiv argument, we add that this labelling corresponds to our interpretation of antiparticles as particles moving backwards in time: The spinor v describes two states with negative energy, negative 3-momemtum $\boldsymbol{p}$ and negative spin s relative to u.

The solutions are orthogonal,

$$\bar{u}(p,s)u(p,s') = \mathcal{N}^2\delta_{s,s'} \quad \text{and} \quad \bar{v}(p,s)v(p,s') = -\mathcal{N}^2\delta_{s,s'}\,, \tag{8.55}$$

but not normalised to one. Note also the minus sign introduced in $\bar{v}v$ by the corresponding minus in the (3,4) corner of γ^0. Since we know that $\psi^\dagger\psi$ is the zero component of a four-vector, the normalisation of the corresponding spinor products is

$$u^\dagger(p,s)u(p,s') = \mathcal{N}^2\frac{E_p}{m}\delta_{s,s'} \quad \text{and} \quad v^\dagger(p,s)v(p,s') = -\mathcal{N}^2\frac{E_p}{m}\delta_{s,s'}\,. \tag{8.56}$$

Summing over spins, we obtain in the rest frame

$$\sum_s u_a(m,s)\bar{u}_b(m,s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{ab}\mathcal{N}^2 = \frac{1}{2}(\gamma^0+1)_{ab}\mathcal{N}^2\,, \tag{8.57}$$

$$\sum_s v_a(m,s)\bar{v}_b(m,s) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}_{ab}\mathcal{N}^2 = \frac{1}{2}(\gamma^0-1)_{ab}\mathcal{N}^2\,. \tag{8.58}$$

We saw that $\gamma^0 \pm 1$ corresponds in an arbitrary frame to $(\not{p} \pm m)/m$. Thus in general these relations become

$$\Lambda_+ \equiv \sum_s u_a(p,s)\bar{u}_b(p,s) = \mathcal{N}^2\left(\frac{\not{p}+m}{2m}\right)_{ab}\,, \tag{8.59}$$

$$\Lambda_- \equiv -\sum_s v_a(p,s)\bar{v}_b(p,s) = \mathcal{N}^2\left(\frac{-\not{p}+m}{2m}\right)_{ab}\,, \tag{8.60}$$

where we defined $\Lambda_\pm$ as the projection operator on states with positive and negative energy, respectively.

The two most common normalisation conventions for the Dirac spinors are $\mathcal{N} = \sqrt{2m}$ and $\mathcal{N} = 1$. We will use the former, $\mathcal{N} = \sqrt{2m}$, which has three advantages: First, the expressions for $\Lambda_\pm$ which appear frequently become more compact. Second, spurious singularities in the limit $m \to 0$ disappear. Finally, the normalisation of fermion states and thus also the phase space volume becomes identical to the one of bosons.

The solutions of the Dirac equation for an arbitrary frame can be simplest obtained remembering $(\not{p}-m)(\not{p}+m) = p^2 - m^2 = 0$, i.e.

$$u(p,\pm) = \frac{\not{p}+m}{\sqrt{m+E}}\,u(0,\pm) \quad \text{and} \quad v(p,\pm) = \frac{-\not{p}+m}{\sqrt{m+E}}\,v(0,\pm)\,. \tag{8.61}$$

Here, the normalisation was fixed using (8.55).

Spin We have seen that the Dirac equation describes a particle with helicity one-half. Thus the $\pm$ degeneracy of the u and v spinors should correspond to the different helicity or spin states of a Dirac particle. We introduce the spin operator

$$\mathbf{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\,, \tag{8.62}$$

as an obvious generalisation of the non-relativistic spin matrices. This operator has the eigenvalues $\Sigma_z u(m,\pm) = \pm u(m,\pm)$ and $\Sigma_z v(m,\pm) = \pm v(m,\pm)$ and can therefore be used to

classify the spin states of a Dirac particle in the rest frame, where $[H_D, \Sigma_z] \propto [\gamma^0, \Sigma_z] = 0$. Note however that $[H_D, \Sigma_z] \neq 0$ for $p^2 \neq m^2$, and thus the eigenvalue of Σ_z is not conserved for a moving particle. This comes not as a surprise, because the total angular momentum $\boldsymbol{L} + \boldsymbol{s}$ and not only the spin $\boldsymbol{s}$ should be conserved.

We are looking now for the relativistic generalisation of the three-dimensional spin operator $\boldsymbol{\Sigma}$. It should be a product of gamma matrices which contains in the rest frame of the particle $\boldsymbol{\sigma}$ in the diagonal. We note first that $\gamma^5\boldsymbol{\gamma} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{smallmatrix}\right)$ has the required structure. Then we define the spin vector s^μ with the properties $s^2 = -1$, $s^\mu = (0, \boldsymbol{s})|_{p=m}$ and thus $s \cdot p = 0$. Since

$$\gamma^5 \not{s}|_{p=m} = -\gamma^5 \boldsymbol{s}\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\sigma}\boldsymbol{s} & 0 \\ 0 & -\boldsymbol{\sigma}\boldsymbol{s} \end{pmatrix}, \tag{8.63}$$

we see that $\gamma^5 \not{s}$ measures in the rest frame the projection of the spin along the chosen axis $\boldsymbol{s}$. Moreover, $\gamma^5 \not{s}$ commutes with the Dirac Hamiltonian, $[\gamma^5 \not{s}, \not{p}] = 0$, and has, because of $(\gamma^5 \not{s})^2 = 1$, as eigenvalues ± 1. If we apply $\gamma^5 \not{s}$ on the spinor $v(p, s)$—what is easiest done in the rest frame—

$$\gamma^5 \not{s}\, v(m, s) = \begin{pmatrix} \boldsymbol{\sigma}\boldsymbol{s} & 0 \\ 0 & -\boldsymbol{\sigma}\boldsymbol{s} \end{pmatrix} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ -\boldsymbol{\sigma}\boldsymbol{s}\chi \end{pmatrix} = s_j \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \tag{8.64}$$

we see that χ_1 has the eigenvalue $s_1 = -1$, while χ_2 has the eigenvalue $s_2 = +1$. This explains the "wrong" order of the two spin states of $v(p, s)$ in (8.54). Finally, we can define a projection operator on a definite spin state by

$$\Lambda_s = \frac{1}{2}\left(1 + \gamma^5 \not{s}\right). \tag{8.65}$$

Thus we can obtain from an arbitrary Dirac spinor ψ a state with definite sign of the energy and spin by applying the two projection operators $\Lambda_\pm$ and Λ_s.

Helicity An important special case of the spin operator $\gamma^5 \not{s}$ is the helicity operator $h \equiv \boldsymbol{s}\boldsymbol{p}/|\boldsymbol{p}|$ which measures the projection of the spin $\boldsymbol{s} = \boldsymbol{\Sigma}/2$ on the momentum $\boldsymbol{p}$ of a particle,

$$\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{p}}{2|\boldsymbol{p}|}\, \psi = h\psi\,. \tag{8.66}$$

The helicity operator and the Dirac Hamiltonian commute, $[H_D, \boldsymbol{\Sigma}\boldsymbol{p}] = 0$, because there is no orbital angular momentum in the direction of $\boldsymbol{p}$. Therefore common eigenfunctions of H_D and h called helicity states can be constructed, cf. problem 8.9. Positive helicity particles are called right-handed, negative helicity particles left-handed. For a massive particle, helicity is a frame-dependent quantity: If we choose e.g. a frame with $\boldsymbol{\beta} \| \boldsymbol{p}$ and $\beta > p$, then the particles moves in the opposite direction and h changes sign. Since we cannot "overtake" a massless particle, helicity becomes in this case a Lorentz invariant quantity.

Axial and vector U(1) symmetries Out of the 16 billinear forms, two transform as vectors under proper Lorentz transformations, $j^\mu = \bar{\psi}\gamma^\mu\psi$ and $j_5^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$. We now want to check if these two currents are conserved. Inspection of the Lagrange density shows immediately that global U(1) transformations,

$$\psi(x) \to \psi'(x) = \mathrm{e}^{\mathrm{i}\phi}\psi(x) \qquad \text{and} \qquad \bar{\psi}(x) \to \bar{\psi}'(x) = \mathrm{e}^{-\mathrm{i}\phi}\bar{\psi}(x)\,, \tag{8.67}$$

keep the Lagrangian invariant, $\delta\mathscr{L} = 0$. Noether's theorem leads then to the conserved current $j^\mu = \bar\psi\gamma^\mu\psi$. In the second case, the underlying symmetry is using $\{\gamma^5, \gamma^\mu\} = 0$,

$$\psi'(x) \to \mathrm{e}^{\mathrm{i}\phi\gamma^5}\psi(x) \quad \text{and} \quad \bar\psi(x) \to \bar\psi'(x) = (\mathrm{e}^{\mathrm{i}\phi\gamma^5}\psi(x))^\dagger\gamma^0 = \bar\psi(x)\mathrm{e}^{\mathrm{i}\phi\gamma^5}\,. \tag{8.68}$$

The resulting (infinitesimal) change is

$$\delta\mathscr{L} = 2mi\bar\psi\gamma^5\psi\,. \tag{8.69}$$

Thus the axial or chiral symmetry $\mathrm{U_A}(1)$ is broken by the mass term, leading to the non-conservation of the current j_5^μ for a massive fermion.

Chirality To understand this better we re-express the Dirac Langrangian using eigenfunctions of γ^5. We can split any solution ψ of the Dirac equation into

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \equiv P_L\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi \equiv P_R\psi\,. \tag{8.70}$$

Since $\gamma^5\psi_L = -\psi_L$ and $\gamma^5\psi_R = \psi_R$, $\psi_{L,R}$ are eigenfunctions of γ^5 with eigenvalue ±1. Expressing the mass term through these fields as

$$\bar\psi\psi = \bar\psi\left(P_L^2 + P_R^2\right)\psi = \psi^\dagger\left(P_R\gamma^0P_L + P_L\gamma^0P_R\right)\psi = \bar\psi_R\psi_L + \bar\psi_L\psi_R \tag{8.71}$$

and similiarly for the kinetic term,

$$\bar\psi\partial\!\!\!/\psi = \bar\psi\left(P_L^2 + P_R^2\right)\partial\!\!\!/\psi = \psi^\dagger\left(P_R\gamma^0\gamma^\mu P_R + P_L\gamma^0\gamma^\mu P_L\right)\partial_\mu\psi = \bar\psi_L\partial\!\!\!/\psi_L + \bar\psi_R\partial\!\!\!/\psi_R\,, \tag{8.72}$$

the Dirac Lagrange density becomes

$$\mathscr{L} = \mathrm{i}\bar\psi_L\partial\!\!\!/\psi_L + \mathrm{i}\bar\psi_R\partial\!\!\!/\psi_R - m(\bar\psi_L\psi_R + \bar\psi_R\psi_L)\,. \tag{8.73}$$

Comparing this expression to (8.34), we see that we can identify the Dirac fields $\psi_{L/R}$ in the chiral representation with the Weyl fields $\phi_{L/R}$ as follows,

$$\psi_L = \begin{pmatrix} \phi_L \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} 0 \\ \phi_R \end{pmatrix}\,. \tag{8.74}$$

Thus the projection operators (8.70) allows us to define the left- and right-chiral components of a Dirac field in an arbitrary representation. If the mass or interaction terms treat ψ_L and ψ_R not symmetrically, one calls them a chiral fermion.

The two kinetic terms which are invariant under chiral transformations connect left- to left-chiral and right- to right-chiral fields, while the mass term mixes left- and right-chiral fields. Such a mass term is called Dirac mass. The distinction between left- and right-chiral fields is Lorentz invariant: In terms of Weyl spinors, we saw that the Lorentz transformations S_L and S_R do not mix ϕ_L and ϕ_R—which qualified them to form the irreducible representation of the Lorentz group. In terms of Dirac spinors, the relation $[\gamma^5, \sigma^{\mu\nu}] = 0$ guarantees that left and right chiral fields transform separately under a Lorentz transformation, $\psi'_{L/R} = S(\Lambda)\psi_{L/R}$. However, the mass term of a massive Dirac particle will mix left- and right-chiral fields as they evolve in time.

Helicity and chirality eigenstates can be seen as complimentary states. The former one is a conserved, frame-dependent quantum number, while the latter is frame-independent, but not conserved. Thus helicity states are e.g. useful to describe scattering processes where the detector measures spin in a definite frame. If on the other hand the interactions of a fermion are spin-dependent, then one should choose chiral fields, since the Lagrangian should be Lorentz invariant.

Charge conjugation From $\mathscr{L} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$ and $[\mathscr{L}] = m^4$ in four dimensions, we see that the dimension of a fermion field in four dimension is $[\psi] = m^{3/2}$. Thus we can order possible couplings of a fermion to spin-1 particles according to their dimension as

$$\mathscr{L}_I = c_1 A_\mu \bar{\psi}\gamma^\mu\psi + c_2 A_\mu \bar{\psi}\gamma^5\gamma^\mu\psi + \frac{c_3}{M} F_{\mu\nu}\bar{\psi}\sigma^{\mu\nu}\psi + \dots\,, \tag{8.75}$$

where the coupling constants c_i are dimensionless and we introduced the mass scale M. The only coupling to the photon fielf one with a dimensionless coupling $e > 0$ that also respects parity is $\mathscr{L}_I = -ej^\mu A_\mu = -e\bar{\psi}\gamma^\mu\psi A_\mu$. Solving the Lagrange equations for $\mathscr{L}_0 + \mathscr{L}_I$ gives the Dirac equation including a coupling to the electromagnetic field as

$$[\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu) - m]\psi(x) = 0\,. \tag{8.76}$$

This corresponds to the "minimal coupling" prescription known from quantum mechanics.

Having defined the coupling to an external electromagnetic field, we can ask ourselves how the Dirac equation for a charged conjugated field ψ_c should look like. In the case of a scalar particle, complex conjugation transformed a positively charged particle into a negative one and vice versa. We try the same for the Dirac equation,

$$[-\mathrm{i}\gamma^{\mu *}(\partial_\mu - \mathrm{i}eA_\mu) - m]\psi^*(x) = 0\,. \tag{8.77}$$

The matrix $\gamma^{\mu *}$ satisfies also the Clifford algebra. Hence we should find the unitary transformation $U^{-1}\gamma^\mu U = -\gamma^{\mu *}$ or setting $U \equiv C\gamma^0$

$$(C\gamma^0)^{-1}\gamma^\mu C\gamma^0 = -\gamma^{\mu *}\,. \tag{8.78}$$

If it exists, then the charge-conjugated field $\psi^c \equiv C\gamma^0\psi^*$ satisfies the Dirac equation with $q = -e$,

$$[\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu) - m]C\gamma^0\psi^*(x) = 0\,. \tag{8.79}$$

Explicit calculation shows that we may choose $C = \mathrm{i}\gamma^2\gamma^0$, see problem 8.11.

In the chiral representation, $\psi^c = C\gamma^0\psi^* = \mathrm{i}\gamma^2\psi^*$ becomes

$$\psi^c = \begin{pmatrix} 0 & \mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \phi_L^* \\ \phi_R^* \end{pmatrix} = \begin{pmatrix} \mathrm{i}\sigma^2\phi_R^* \\ -\mathrm{i}\sigma^2\phi_L^* \end{pmatrix}, \tag{8.80}$$

which is in agreement with $\phi_L^c = \mathrm{i}\sigma^2\phi_R^*$ and $\phi_R^c = -\mathrm{i}\sigma^2\phi_L^*$ found earlier.

Example 8.1: Since the γ^2 matrix has the same form in the Dirac and the chiral representation, we find applying C on the spinors $u(p,\pm)$ and $v(p,\pm)$ immediately that

$$u^c(p,s) = C\gamma^0 u^*(p,s) = v(p,s) \quad \text{and} \quad v^c(p,s) = C\gamma^0 v^*(p,s) = u(p,s)\,.$$

Inserted into Eq. (8.88) this implies that $S_F^T(x) = CS_F(-x)C^{-1}$.

Feynman propagator The Green functions of the Dirac equation are defined by

$$(\mathrm{i}\partial\!\!\!/ - m)S(x,x') = \delta(x-x')\,. \tag{8.81}$$

Translation invariance implies $S(x,x') = S(x-x')$ and thus it is again convenient to perform a Fourier transformation. Then the Fourier components $S(p)$ have to obey

$$(p\!\!\!/ - m)S(p) = 1\,. \tag{8.82}$$

After multiplication with $p\!\!\!/ + m$ and use of $a\!\!\!/^2 = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}a_\mu a_\nu = a^2$, we can solve for the propagator in momentum space,

$$\mathrm{i}S_F(p) = \mathrm{i}\frac{p\!\!\!/ + m}{p^2 - m^2 + \mathrm{i}\varepsilon} = \frac{\mathrm{i}}{p\!\!\!/ - m + \mathrm{i}\varepsilon}\,, \tag{8.83}$$

where the last step is only meant as a symbolical short-cut. Here, we chose again with the $-\mathrm{i}\varepsilon$ prescription the causal or Stückelberg-Feynman propagator for the electron and, more generally, for spin 1/2 particles. Note also the connection to the scalar propagator Δ_F,

$$\mathrm{i}S_F(x) = -(\mathrm{i}\partial\!\!\!/ - m)\mathrm{i}\Delta_F(x)\,. \tag{8.84}$$

Example 8.2: Express the Feynman propagator as sum over the solutions $u(p,s)$ and $v(p,s)$:
We follow the steps from (3.27) to (3.31) in the scalar case, finding now

$$S_F(x) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}p_0}{2\pi}\frac{(p\!\!\!/ + m)\mathrm{e}^{-\mathrm{i}p_0 t}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}}}{(p_0 - E_p + \mathrm{i}\varepsilon)(p_0 + E_p - \mathrm{i}\varepsilon)} \tag{8.85}$$

$$= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[-\mathrm{i}\,\frac{p\!\!\!/ + m}{2E_p}\mathrm{e}^{-\mathrm{i}E_p t}\vartheta(x^0) + \mathrm{i}\,\frac{-E_p\gamma^0 - \boldsymbol{p}\boldsymbol{\gamma} + m}{-2E_p}\mathrm{e}^{\mathrm{i}E_p t}\vartheta(-x^0)\right]\mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}}\,. \tag{8.86}$$

Now we change as in the bosonic case the integration variable as $\boldsymbol{p}\to-\boldsymbol{p}$ in the second term,

$$\mathrm{i}S_F(x) = \int\frac{\mathrm{d}^3p}{2E_p(2\pi)^3}\,\left[(p\!\!\!/ + m)\mathrm{e}^{-\mathrm{i}(E_p t - \boldsymbol{p}\boldsymbol{x})}\vartheta(t) + (-p\!\!\!/ + m)\mathrm{e}^{\mathrm{i}(E_p t - \boldsymbol{p}\boldsymbol{x})}\vartheta(-t)\right]\,. \tag{8.87}$$

Using finally (8.59), we arrive at

$$\mathrm{i}S_F(x) = \int\frac{\mathrm{d}^3p}{2E_p(2\pi)^3}\sum_s\left[u(p,s)\bar{u}(p,s)\mathrm{e}^{-\mathrm{i}(E_p t - \boldsymbol{p}\boldsymbol{x})}\vartheta(x^0) - v(p,s)\bar{v}(p,s)\mathrm{e}^{\mathrm{i}(E_p t - \boldsymbol{p}\boldsymbol{x})}\vartheta(-x^0)\right]\,. \tag{8.88}$$

Thus the phase space volume of fermionic states is the same as the one of bosons for the normalisation of the Dirac spinors chosen by us. The minus sign between the positive energy solution propagating forward in time and the negative energy solution propagating backward in time is a direct consequence of our $-\mathrm{i}\varepsilon$ prescription. It implies that fermionic fields anti-commute,

$$\mathrm{i}S_{F,ab}(x) = \langle 0|T\{\psi_a(x)\bar{\psi}_b(0)\}|0\rangle = \langle 0|\psi_a(x)\bar{\psi}_b(0)|0\rangle\vartheta(t) - \langle 0|\bar{\psi}_b(0)\psi_a(x)|0\rangle\vartheta(-t)\,, \tag{8.89}$$

(we have added for clarity the spinor indices) and explains thereby the Pauli exclusion principle and thus the stability of matter.
Let us have a look back to understand why the sign appears. To simplify the discusion, we neglect the inessential mass term. In the positive frequency term $(p^0\gamma^0 - \boldsymbol{p}\boldsymbol{\gamma})\mathrm{e}^{\mathrm{i}px}$, we pick up relativ to the bosonic case an additional minus rom the variable change, $\boldsymbol{p}\to-\boldsymbol{p}$, resulting in $p\!\!\!/\to-p\!\!\!/$ and

$$S_F(x) \propto p\!\!\!/\,\mathrm{e}^{-\mathrm{i}px}\vartheta(t) - p\!\!\!/\,\mathrm{e}^{\mathrm{i}px}\vartheta(-t)\,.$$

Thus the relative minus sign has its origin in the fact that the fermion propagator $S_F(p)$ is odd in the momentum, while a bosonic propagator is even. In turn, the fermion propagator is linear in the momentum, because the fermion wave equation is a first-order equation.

8.3. Quantizing Dirac fermions

Spin-statistic connection We have noted that fermionic fields should anti-commute examing the Feynman propagator in expample 8.2. In a relativistic quantum field theory the spin and the statistics of a field is coupled:

- The wave equations of bosons are second-order differential equations. Therefore bosonic fields have mass dimension 1 and their propagators $\Delta(p)$ are even in the momentum p. As a result, bosonic fields commute and satisfy Bose-Einstein statistics.
- In contrast, fermions satisfy first-order differential equations and have mass dimension 3/2. Therefore the fermion propagator $S_F(p)$ is odd in p. This implies that fermions are decribed by anti-commuting classical spinors or operators, and satisfy Fermion-Dirac statistics.

This leads to a practical and a principal question: First, the practical one: How do we implement that classical functions which enter the path integral do anti-commute? And second, does the anticommutation of fermionic variables lead to a consistent picture? In particular is the Hamiltonian of such a theory bounded from below?

We will start to address the latter question calculating the energy density $\rho = \mathscr{H}$ of the Dirac field assuming that the classical spinors u and v *do* commute. We determine first the canonically conjugated momenta as

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\psi}} = \mathrm{i}\bar{\psi}\gamma^0 = \mathrm{i}\psi^\dagger \tag{8.90}$$

and $\bar{\pi} = 0$. Thus the Hamilton density is

$$\mathscr{H} = \pi\dot{\psi} - \mathscr{L} = \mathrm{i}\psi^\dagger\partial_t\psi - \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = \mathrm{i}\psi^\dagger\partial_t\psi\,, \tag{8.91}$$

where we used the Dirac equation in the last step. To make this expression more explicit, we express now ψ by plane wave solutions,

$$\psi(x) = \sum_s \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b_s(p)u_s(p)e^{-\mathrm{i}px} + d_s^\dagger(p)v_s(p)e^{+\mathrm{i}px}\right]\,, \tag{8.92a}$$

$$\psi^\dagger(x) = \sum_s \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b_s^\dagger(p)u_s^\dagger(p)e^{+\mathrm{i}px} + d_s(p)v_s^\dagger(p)e^{-\mathrm{i}px}\right]\,. \tag{8.92b}$$

Inserting these expressions into (8.91) gives schematically $(b^\dagger + d)(b - d^\dagger)$, where the relative minus sign comes from ∂_t acting on ψ. Since the spinors u and v are orthonormal, cf. (8.56), only the diagonal terms survive, $(b^\dagger + d)(b - d^\dagger) \to b^\dagger b - dd^\dagger$. Hence the energy of a Dirac field is given by

$$H = \int \mathrm{d}^3x\, \mathscr{H} = \sum_s \int \mathrm{d}^3p\, E_p \left[b_s^\dagger(p)b_s(p) - d_s(p)d_s^\dagger(p)\right]\,. \tag{8.93}$$

If d and $d^\dagger$ would be normal Fourier coefficients of an expansion into plan waves, the second term would be negative and the energy density of a fermion field could be made arbitrarily negative.

This conclusion is avoided if the fermion fields anticommute: In canonical quantisation, one promotes the Fourier coefficients to operators. Requiring then anti-commutation relations between the creation and annihilation operators for particles and anti-particles,

$$\{b_s(\boldsymbol{p}), b^\dagger_{s'}(\boldsymbol{p}')\} = \delta_{s,s'}\delta(\boldsymbol{p}-\boldsymbol{p}') \quad \text{and} \quad \{d_s(\boldsymbol{p}), d^\dagger_{s'}(\boldsymbol{p}')\} = \delta_{s,s'}\delta(\boldsymbol{p}-\boldsymbol{p}')\,, \tag{8.94}$$

compensates the sign in the second term.

If we restore units, then we have to add a factor $\hbar$ on the RHS of the anti-commutation relations (8.94). Since the RHS vanishes in the classical limit $\hbar \to 0$, classical spinors should anti-commute. Thus we should perform the path integral of fermionic fields over anti-commuting numbers which are called Graßmann numbers. Either way, we find the reassuring result that in a relativistic quantum field theory the spin determines the statistics in such a way the Hamiltonian is bounded from below.

Graßmann variables We now proceed to the question how we can implement the analogue of the anticommutation relations for operators into the path integral formalism. We define a Graßmann algebra $\mathcal{G}$ requiring that for $a, b \in \mathcal{G}$ the anticommutation rules

$$\{a, a\} = \{a, b\} = \{b, b\} = 0 \tag{8.95}$$

and thus $a^2 = b^2 = 0$ are valid. Then any smooth function f of a and b can be expanded into a power-series as

$$\begin{aligned} f(a,b) &= f_0 + f_1 a + \tilde{f}_1 b + f_2 ab \\ &= f_0 + f_1 a + \tilde{f}_1 b - f_2 ba\,. \end{aligned} \tag{8.96}$$

Defining the derivative as acting to the right, $\partial \equiv \overrightarrow{\partial}$, we find

$$\frac{\partial f}{\partial a} = f_1 + f_2 b\,, \qquad \frac{\partial f}{\partial b} = \tilde{f}_1 - f_2 a\,, \tag{8.97}$$

and

$$\frac{\partial^2 f}{\partial a \partial b} = -\frac{\partial^2 f}{\partial b \partial a} = -f_2\,. \tag{8.98}$$

As integration rules for Graßmann variables, we require linearity and that the infinitesimals $\mathrm{d}a$, $\mathrm{d}b$ are also Graßmann variables,

$$\{a, \mathrm{d}a\} = \{b, \mathrm{d}b\} = \{a, \mathrm{d}b\} = \{\mathrm{d}a, b\} = \{\mathrm{d}a, \mathrm{d}b\} = 0\,. \tag{8.99}$$

Multiple integrals are iterated,

$$\int \mathrm{d}a\mathrm{d}b f(a,b) = \int \mathrm{d}a \left(\int \mathrm{d}b f(a,b) \right)\,. \tag{8.100}$$

We have to determine the value of $\int \mathrm{d}a$ and $\int \mathrm{d}a a$. For the first, we write

$$\left(\int \mathrm{d}a\right)^2 = \left(\int \mathrm{d}a\right)\left(\int \mathrm{d}b\right) = \int \mathrm{d}a\mathrm{d}b = -\int \mathrm{d}b\mathrm{d}a = -\left(\int \mathrm{d}a\right)^2 \tag{8.101}$$

and thus $\int \mathrm{d}a = 0$. We are left with $\int \mathrm{d}a a$: Since there is no intrinsic scale—states are empty or occupied—we are free to set

$$\int \mathrm{d}a a = 1\,. \tag{8.102}$$

This implies also that there is no difference between definite and indefinite integrals for Graßmann variables. Moreover, differentiation and integration are equivalent for Graßmann variables.

Assume now that η_1 and η_2 are real Graßmann variables and $A \in \mathbb{R}$. Then

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \; \mathrm{e}^{\eta_2 A \eta_1} = \int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \; (1 + \eta_2 A \eta_1) = \int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \; \eta_2 A \eta_1 = \int \mathrm{d}\eta_1 \; A\eta_1 = A\,. \tag{8.103}$$

Next we consider a two-dimensional integral with an anti-symmetric matrix A and $\eta = (\eta_1, \eta_2)$. Then

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \tag{8.104}$$

and $\eta^T A \eta = 2a\eta_1\eta_2$. Using an arbitrary matrix would lead to the same result, since its symmetric part cancels. Expanding again the exponential gives

$$\int \mathrm{d}^2\eta \; \exp\left(\frac{1}{2}\eta^T A \eta\right) = a = (\det(A))^{1/2}. \tag{8.105}$$

An arbitrary antisymmetric matrix can be transformed into block diagonal form, where the diagonal is composed of matrices of the type (8.104). Thus the last formula holds for arbitrary n.

Finally, we introduce complex Grassmann variables $\eta = (\eta_1, \ldots, \eta_n)$ and their complex conjugates $\eta^* = (\eta_1^*, \ldots, \eta_n^*)$. For any complex matrix A,

$$\int \mathrm{d}^n\eta \mathrm{d}^n\eta^* \; \exp\left(\eta^\dagger A \eta\right) = \prod_{i=1}^{n} a_i = \det(A)\,. \tag{8.106}$$

We can compare this to the result over commuting complex variables, $z_i = (x_i + \mathrm{i}y_i)/\sqrt{2}$ and $\bar{z}_i = (x_i - \mathrm{i}y_i)/\sqrt{2}$, with $\mathrm{d}x\mathrm{d}y = \mathrm{d}z\mathrm{d}z^*$ and

$$\int \mathrm{d}^n z \mathrm{d}^n z^* \; \exp\left(-z^\dagger A z\right) = \frac{(2\pi)^n}{\det(A)}\,. \tag{8.107}$$

Thus for Graßmann variables the determinant appearing in the evaluation of a Gaussian integral is in the numerator, while it is in the denominator for real or complex valued functions.

Path integral for fermions In the bosonic case, the action $S[\phi, \pi]$ is quadratic in the canonically conjugated momenta π. The path integral over the momenta can thus be performed and we started directly with the path integral in configuration space. For a fermion, $\pi = \mathrm{i}\psi^\dagger$, and thus the path integral in phase space is

$$Z[0] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \; \mathrm{e}^{\mathrm{i}S[\psi,\bar{\psi}]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \; \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \, \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi}\,, \tag{8.108}$$

where we changed to $\bar\psi$ as integration variable. For its evaluation, we use (8.106) in the limit $n \to \infty$. Since the action is quadratic in the fields, we can perform the path integral formally,

$$Z[0] = \det(\mathrm{i}\partial\!\!\!/ - m) = \exp \operatorname{tr} \ln(\mathrm{i}\partial\!\!\!/ - m)\,. \tag{8.109}$$

Using the cyclic property of the trace, we write

$$\begin{aligned}\operatorname{tr}\ln(\mathrm{i}\partial\!\!\!/ - m) &= \operatorname{tr}\ln\gamma^5(\mathrm{i}\partial\!\!\!/ - m)\gamma^5 = \operatorname{tr}\ln(-\mathrm{i}\partial\!\!\!/ - m) = \\ &= \frac{1}{2}[\operatorname{tr}\ln(\mathrm{i}\partial\!\!\!/ - m) + \operatorname{tr}\ln(-\mathrm{i}\partial\!\!\!/ - m)] = \frac{1}{2}[\operatorname{tr}\ln(\Box + m^2)]\,. \end{aligned}\tag{8.110}$$

Thus $Z[0] = \exp[+\operatorname{tr}\ln(\Box + m^2)/2]$: We have found the remarkable result that the zero-point energy of fermions has the opposite sign compared to the one of bosons. We arrive at the same conclusion, using anti-commutation relations $\{d_s(\boldsymbol{p}), d^\dagger_{s'}(\boldsymbol{p}')\} = \delta_{s,s'}\delta(\boldsymbol{p} - \boldsymbol{p}')$ in the Hamiltonian (8.93),

$$H = \sum_s \int \mathrm{d}^3p\, E_p \left[b^\dagger b + d^\dagger d - \delta^{(3)}(0)\right]\,. \tag{8.111}$$

With $\delta^{(3)}(0) = \int \mathrm{d}^3x/(2\pi)^3$ we see that the last term corresponds to the negative zero-point energies of a fermion.

Note that this opens the possibility that the zero-point energies of (groups of) bosons and fermions cancel exactly, provided that i) the degrees of freedom of fermions and bosons agree. (For instance, the trace in Eq. (8.110) includes the trace over the 4×4 matrix in spinor space, leading to a factor four larger results than for a single scalar.) ii) Their masses are the same, $m_f = m_b$. iii) Their interactions match, so that also higher-order corrections are identical for fermions and bosons. The corresponding symmetry that guarantees automatically that the conditions i)-iii) are satisfied is called "supersymmetry." In an unbroken supersymmetric theory, the cosmological constant would be zero. Clearly, condition ii) is most problematic, since e.g. no bosonic partner of the electron has been found (yet). Hence supersymmetry must be a broken symmetry, but as long as the mass spitting $m_f^2 - m_b^2$ between fermions and bosons is not too large, it might be still "useful."

Feynman rules Next we add Graßmannian sources η and $\bar\eta$ to the action, $S[\psi,\bar\psi] + \bar\eta\psi + \bar\psi\eta$. Then we have to complete the square,

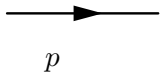
$$\bar\psi A\psi + \bar\eta\psi + \bar\psi\eta = (\bar\psi + \bar\eta A^{-1})A(\psi + A^{-1}\eta) - \bar\eta A^{-1}\eta\,, \tag{8.112}$$

obtaining

$$\begin{aligned} Z[\eta,\bar\eta] &= \int \mathcal{D}\psi\mathcal{D}\bar\psi\ \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \bar\psi A\psi + \bar\eta\psi + \bar\psi\eta} = Z[0]\,\mathrm{e}^{-\mathrm{i}\bar\eta A^{-1}\eta} \\ &= Z[0]\,\exp\left(-\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4x'\ \bar\eta(x) S_F(x-x')\eta(x')\right)\,. \end{aligned}\tag{8.113}$$

Here, $A^{-1}(x,x') = S_F(x-x') = -S_F^T(x'-x)$ which corresponds to the fact that the matrix A is antisymmetric.

The propagator of a Dirac fermion is a line with an arrow representing the flow of the conserved charge which distinguishes particles and anti-particles. Thus a fermion line cannot split, and the arrow cannot change direction.

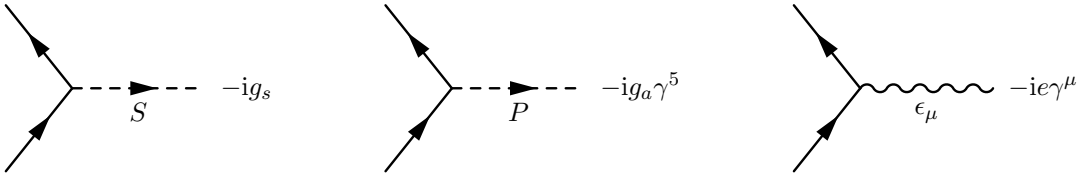


$$= \mathrm{i}S_F(p) = \frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\varepsilon}$$

We look at possible interaction terms of a Dirac fermion with scalars and photons, restricting ourselves to dimensionless coupling constants,

$$\mathscr{L}_I = -g_s S\bar{\psi}\psi - g_a P\bar{\psi}\gamma^5\psi - e\bar{\psi}\gamma^\mu\psi A_\mu\,. \tag{8.114}$$

Both interaction terms of the fermion with the scalars respect parity, if the field S is a true scalar and the field P a pseudo-scalar. Analogous to the $-\mathrm{i}\lambda$ coupling in the case of a scalar self-interactions, we read off from the Lagrangian the following interaction vertices in momentum space,



Fermion loops A closed fermion loop with n propagators corresponds to

$$\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2)...\psi^\dagger(x_n)\psi(x_n)$$

In order to combine $\bar{\psi}(x_1)$ and $\psi(x_n)$ into $T\{\psi(x_n)\bar{\psi}(x_1)\}$, we have to anticommute $\bar{\psi}(x_1)$ with the $2n-1$ fields $\psi(x_1)\cdots\psi(x_n)$, generating a minus sign. Thus we have to add to our set of Feynman rule that each fermion loop generates a minus sign. Another way to understand the minus sign of fermion loops is to look at the generating functional for connected graphs setting the sources to zero, $\mathrm{i}W[0] = \ln Z[0] = \ln\det A$. The generated graphs are single-closed loops with n Feynman propagators. The change from $1/\det A$ in Z for bosonic fields to $\det A$ in Z for fermionic fields implies an additional minus sign for closed fermion loops.

Similarly, diagrams contributing to the same process which differ only by the exchange of two identical external fermion lines carry a relative minus sign. This applies also the exchange of a particle and anti-particle in the initial and final state (cf. also the discussion of crossing symmetry in section 9.3.1).

Furry's theorem What is the relation between diagrams containing fermion loops with opposite orientation in QED? A fermion loop with n external photons attached corresponds to a trace over n fermion propagators separated by gamma matrices,

$$G_1 = \mathrm{tr}[\gamma_{\mu_1} S_F(y_1,y_n)\gamma_{\mu_n} S_F(y_n,y_{n-1})\cdots\gamma_{\mu_2} S_F(y_2,y_1)]\,. \tag{8.115}$$

If we insert $CC^{-1} = 1$ between all factors in the trace, use $C\gamma^\mu C^{-1} = -\gamma^{\mu T}$ and $CS_F(-x)C^{-1} = S_F^T(x)$, then we find

$$\begin{aligned} G_1 &= (-1)^n \mathrm{tr}[\gamma^T_{\mu_1} S_F^T(y_n,y_1)\gamma^T_{\mu_n} S_F^T(y_{n-1},y_n)\cdots\gamma^T_{\mu_2} S_F^T(y_1,y_2)] &(8.116)\\ &= (-1)^n \mathrm{tr}[\gamma_{\mu_1} S_F(y_1,y_2)\cdots\gamma_{\mu_n} S_F(y_n,y_1)] = (-1)^n G_2\,. &(8.117)\end{aligned}$$

Here we used $B^T A^T = (AB)^T$ in the last step. Except for the factor $(-1)^n$, the last expression corresponds to the loop G_2 with opposite orientation. Hence for an odd number of propagators, the two contributions cancel, while they are equal for an even number of propagators.

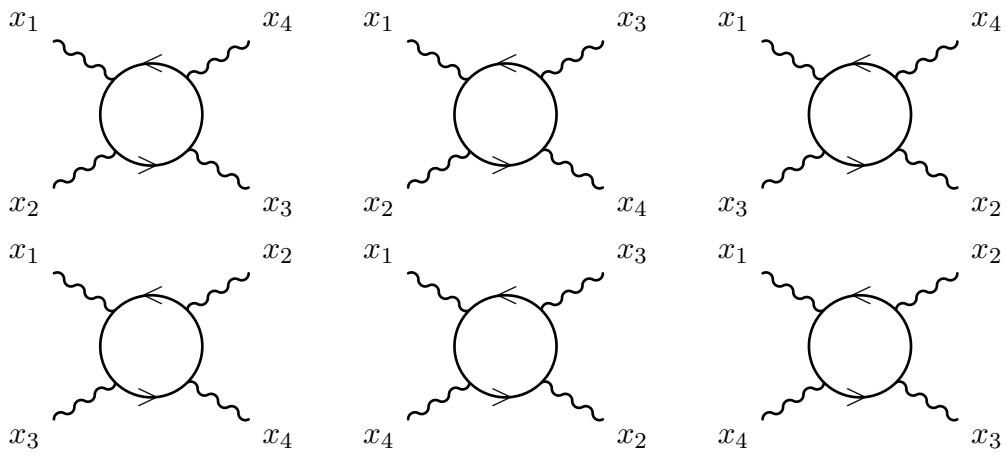
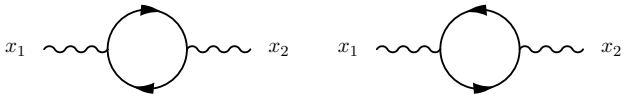


Figure 8.1.: The six topologically distinct diagrams contributing to the photon four-point function.

Symmetry factors in QED The last issue we want to address in this chapter is the question if the interactions in (8.114) lead to symmetry factors. We recall that drawing Feynman diagrams we should include only diagrams which are topologically distinct after the integration over internal coordinates. For instance, the two diagrams



are not topologically distinct, because a rotation around the x_1–x_2 axis interchanges them. Therefore, the corresponding two-point function $G(x_1, x_2)$ has to be invariant under a change of the orientation of the fermion loop – as it is guarantied by the Furry theorem. The two-point function $G(x_1, x_2)$ consists of two identical diagrams obtained by exchanging the integration variables y_1 and y_2. Thus the factor 2! compensates the 1/2! from the Taylor expansion of $\exp(\mathrm{i}\mathscr{L}_{\mathrm{int}})$, if we draw only one diagram, and its symmetry factor is one.

Next we consider the four-point function $G(x_1, x_2, x_3, x_4)$ which describes photon-photon scattering. The four-point function $G(x_1, x_2, x_3, x_4)$ contains $4! \times 3!$ diagrams, obtained by permutating y_1, y_2, y_3, y_4 and x_2, x_3, x_4. After integration over the free y_i variables, the factor 4! compensates the 1/4! from the Taylor expansion of $\exp(\mathrm{i}\mathscr{L}_{\mathrm{int}})$. In configuration space, the $3! = 6$ topologically distinct diagrams shown in Fig. 8.1 remain which carry no additional symmetry factor. Thus the resulting rule for QED is very simple: We do not need symmetry factors, if we draw all diagrams which are toplogically distinct after the integration over internal coordinates. Fermion loops with an odd number of fermions are zero and can be omitted. Independent of the type of interaction, any fermion loop leads to an additional minus sign.

8.4. Weyl and Majorana fermions

Up to now we have discussed the Dirac equation, having in mind a massive particle carrying a conserved U(1) charge that allows us to distinguish particles and anti-particles. We call such particles Dirac fermions. In the SM, all particles except neutrinos carry a non-zero electric charge, are massive and are therefore Dirac fermions. In this section, we consider the case where one of these two conditions is not fullfilled.

Weyl fermions, $q \neq 0$ and $m = 0$: The Dirac equation (8.19) in the chiral representation decouples for $m = 0$ into two equations called Weyl equations,

$$(E + \boldsymbol{\sigma p})\phi_L(p) = 0 \quad \text{and} \quad (E - \boldsymbol{\sigma p})\phi_R(p) = 0\,. \tag{8.118}$$

A fermion described by the Weyl equations is called a Weyl fermion. The correct dispersion relation, $E = |\boldsymbol{p}|$, requires that ϕ_L is an eigenstate of the helicity operator $h = \boldsymbol{\sigma p}/(2|\boldsymbol{p}|)$ with eigenvalue $h = -1/2$, while ϕ_R has the eigenvalue $h = +1/2$. Recall also that helicity is frame independent for a massless particle; in this case positive helicity agrees with right chirality[4]. Until the 1990's, the experimental data on neutrino masses were consistent with zero and neutrinos were incorporated into the SM as Weyl fermions. Since only left-chiral particles and right-chiral antiparticles participate in weak interactions, one set $\nu = \left(\begin{smallmatrix}\phi_L\\0\end{smallmatrix}\right)$, while antineutrinos were described by the CP transformed state. The lepton number L_α of the three flavours $\alpha = \{e, \mu, \tau\}$ of leptons played the role of the conserved U(1) charge that distinguishes neutrinos and antineutrinos. As result, the difference in the number of leptons and antileptons of each individual flavour was conserved. Neutrino oscillations that occur if neutrinos are massive conserve the total lepton number $L = \sum_\alpha L_\alpha$ but interchange the individual neutrino flavours L_α. Thus the observation of neutrino oscillations showed that neutrinos are not Weyl fermions.

Majorana fermions, $m > 0$ and $q = 0$: The Dirac field ψ_D has to be complex, because it transforms under the complex representation $S(\Lambda)$ of the Lorentz group. In the case of a neutral fermion, where we cannot distinguish particles and antiparticles, we should have only half of the degrees of freedom of a charged Dirac field. By analogy with the scalar case, we expect that we can halve the number of degrees of the complex Dirac field by imposing a reality condition, $\psi_M = \psi_M^*$. But this condition can be Lorentz invariant only in a special representation of the gamma matrices where $\sigma_{\mu\nu}^* = -\sigma_{\mu\nu}$ and thus $S(\Lambda)$ is real. This condition defines the Majorana representation of the γ matrices in which all γ^μ and thus $\sigma_{\mu\nu}$ are imaginary, and the charge conjugation matrix C is the unity matrix, $C = 1$. Since the spinors are real in this representation, no phase invariance $\psi(x) \to \psi'(x) = \exp(\mathrm{i}\phi)\psi(x)$ as in (8.67) can be implemented for a Majorana fermion[5] and thus they cannot carry any conserved U(1) charge.

We can halve the number of degrees of freedom of a Dirac fermion in a representation independent way by using a self-conjugated field $\psi^c = \psi$. A fermion described by a self-conjugated field ψ_M is called a Majorana fermion and the corresponding spinor a Majorana

[4] Most authors call $\psi_{L/R}$ and $\phi_{L/R}$ not left and right-chiral but left and right-handed, although this identification holds only for massless particles.

[5] Note that this argument does not forbid that a Majorana fermion carries conserved charges which transform under a real representation of a symmetry group: An example are gluinos, the supersymmetric partners of the gluons, which are Majorana fermions and transform under the adjoint representation of SU(3).

spinor. The field operator of a Majorana field contains only one type of annihilation and creation operator,

$$\psi_M(x) = \sum_s \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} \left[a_s(p) u_s(p) e^{-\mathrm{i}px} + a_s^\dagger(p) v_s(p) e^{+\mathrm{i}px} \right] . \tag{8.119}$$

Using then

$$\psi_M^c(x) = C\gamma^0 \psi_M^*(x) = \sum_s \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} \left[a_s^\dagger(p) C\gamma^0 u_s^*(p) e^{+\mathrm{i}px} + a_s(p) C\gamma^0 v_s^*(p) e^{-\mathrm{i}px} \right] , \tag{8.120}$$

we confirm immediately the Majorana property $\psi_M^c(x) = \psi_M(x)$. Expressed by Weyl spinors, a Majorana spinor becomes

$$\psi^c = \psi = \begin{pmatrix} \phi_L \\ -\mathrm{i}\sigma^2 \phi_L^* \end{pmatrix} = \begin{pmatrix} \mathrm{i}\sigma^2 \phi_R^* \\ \phi_R \end{pmatrix} . \tag{8.121}$$

A Majorana four-spinor is composed of only one Weyl spinor, which shows again that it describes two degrees of freedom. Thus a Majorana fermion ($m > 0$, $q = 0$) has two degrees of freedom, consisting either of a left-chiral two-spinor $\phi_L(\sigma)$ with positive and negative helicity or a right-chiral two-spinor $\phi_R(\sigma)$ with both helicities.

We can replace any Dirac field ψ_D by a pair of self-conjugated fields,

$$\psi_{M,1} = \frac{1}{\sqrt{2}} \left(\psi_D + \psi_D^c \right) , \tag{8.122a}$$

$$\psi_{M,2} = \frac{1}{\sqrt{2}} \left(\psi_D - \psi_D^c \right) . \tag{8.122b}$$

and vice versa inverting these relations.

Dirac versus Majorana mass terms Charge conjugated Dirac spinors were defined by

$$\psi^c = C\gamma^0 \psi^* = C\bar{\psi}^t, \qquad \bar{\psi}^c = \psi^t C .$$

We define also

$$\psi_L^c \equiv (\psi_L)^c = \frac{1}{2}(1+\gamma^5)\psi^c = (\psi^c)_R , \tag{8.123}$$

which is consistent with our previous definition for Weyl spinors. As we saw, a Dirac mass term connects the left- and right-chiral components of the same field and $\psi = \psi_L + \psi_R$ is a mass eigenstate. We now use the observation that $(\psi_L)^c = (\psi^c)_R$ allows us to obtain new mass terms[6] called Majorana mass terms,

$$-\mathscr{L}_L = m_L(\bar{\psi}_L^c \psi_L + \bar{\psi}_L \psi_L^c) \tag{8.124}$$

$$-\mathscr{L}_R = m_R(\bar{\psi}_R^c \psi_R + \bar{\psi}_R \psi_R^c) \tag{8.125}$$

which connect the left- and right-chiral components of charge-conjugated fields. The corresponding mass eigenstates are the self-conjugated fields

$$\chi = \psi_L + \psi_L^c = \chi^c \quad \text{and} \quad \omega = \psi_R + \psi_R^c = \omega^c \tag{8.126}$$

[6] Note that the terms $\bar{\psi}_L^c \psi_L = \psi_L^t C \psi_L$, etc., vanish because of $C^t = -C$, if one does not already assumes on the classical level that fields are anticommuting Graßmann variables.

with $\mathscr{L}_L = -m_L\bar{\chi}\chi$ and $\mathscr{L}_R = -m_R\bar{\omega}\omega$. In the general case, both Dirac and Majorana mass terms may be present,

$$\begin{aligned} -\mathscr{L}_{DM} &= m_D\bar{\psi}_L\psi_R + m_L\bar{\psi}^c_L\psi_L + m_R\bar{\psi}^c_R\psi_R + \text{h.c.} = \\ &= \frac{1}{2}m_D(\bar{\chi}\omega + \bar{\omega}\chi) + m_L\bar{\chi}\chi + m_R\bar{\omega}\omega\,, \end{aligned}$$

or in matrix form

$$-\mathscr{L}_{DM} = (\bar{\chi}, \bar{\omega}) \begin{pmatrix} m_L & m_D/2 \\ m_D/2 & m_R \end{pmatrix} \begin{pmatrix} \chi \\ \omega \end{pmatrix}.$$

Physical states have a definite mass and thus we have to diagonalise the mass matrix. Its eigenvalues are

$$m_{1,2} = \frac{1}{2}\left\{(m_L + m_R) \pm \sqrt{(m_L - m_R)^2 + m_D^2}\right\} \tag{8.127}$$

and its eigenvectors

$$\begin{aligned} \eta_1 &= \cos\vartheta\chi - \sin\vartheta\omega \\ \eta_2 &= \sin\vartheta\chi + \cos\vartheta\omega \end{aligned}$$

with $\tan 2\vartheta = m_D/(m_L - m_R)$.

The *seesaw model* tries to explain why neutrinos have much smaller masses than all other particles in the standard model. Let us assume that there exist both left- and right-chiral neutrinos and that they obtain Dirac masses as the other fermions, say of order $m_D \sim 100\,\text{GeV}$. The right-chiral ν_R does not participate in any SM interaction and suffers the same fate as a scalar particle: Its mass will be driven by quantum corrections to a value close to the cutoff scale used, and so we expect $m_R \gg m_D$. Expanding then

$$m_{1,2} \approx \frac{1}{2}\left\{m_R \pm m_R\sqrt{1 + m_D^2/m_R^2}\right\}, \tag{8.128}$$

the two eigenvalues are $m_1 \approx m_D^2/(2m_R)$ and $m_2 \approx m_R$. For $m_R \sim 10^{14}\,\text{GeV}$, the light neutrino mass is in the eV or sub-eV range as required by experimental data.

Summary

The fundamental representation of the proper Lorentz group for massive particles is given by left and right-chiral Weyl spinors. These two-spinors are mixed by parity and thus one combines them into a Dirac four-spinor for parity conserving theories like electromagnetic and strong interactions.

Fermions satisfy first-order differential equations and have mass dimension 3/2. Therefore the fermion propagator $S_F(p)$ is linear in p and thus $S_F(x)$ is an antisymmetric function in x. As a result, fermions satisfy Fermion-Dirac statistics and are decribed either by Grassmann variables or by anti-commuting operators.

A Weyl fermion has $m = 0$ and $q \neq 0$ and satisfies the Weyl equation; its solution has two degrees of freedom, a left-chiral field with negative helicity and a right-chiral field with positive helicity. A Majorana fermion is a self-conjugated field with $m \neq 0$ and $q = 0$ which

has therefore also only two degrees of freedom. It is decribed either by a left-chiral or a right-chiral 2-spinor with both helicities. In the Majorana representation, this spinor can be chosen to be real.

Further reading

The symmetries of the Dirac equation as well as of other relativistic wave equations are extensively discussed in W. Greiner *Relativistic Quantum Mechanics*. Two-component Weyl and Majorana spinor are discussed in more detail in Srednicki who introduces also the notation of dotted and undotted spinor indices.

Problems

8.1 Lie algebra of the Poincaré group.
Derive the commutation relation (8.1) plus the missing ones involving translation using that the (hermitian) generator T^a are connected to the Killing fields V^a by $\mathrm{i}T^a = V^a$. Rewrite the expressions in an Lorentz invariant form.

8.2 Little group
Find the Lorentz transformations $\Lambda_\mu^{\ \nu}$ which keep the momentum of a particle invariant, $\Lambda_\mu^{\ \nu} p^\mu = p^\nu$, for a a) massive particle and b) massless particle, considering an infinitesimal transformation $\omega_{\mu\nu}$. By how many parameters is $\omega_{\mu\nu}$ determined in the two cases?

8.3 Derivation of the Dirac equation.
Fill in the details in the derivation from (8.14) to (8.21).

8.4 g_s-factor of the electron.
Square the Dirac equation with an external field A^μ and show that the spin of the electron leads to the additional interaction term $\sigma_{\mu\nu}F^{\mu\nu}$ compared to the Klein-Gordon equation. Show that the Dirac equation predicts for the interaction of non-relativistic electron with a static magnetic field $H_{\rm int} = (\boldsymbol{L} + 2\boldsymbol{s})e\boldsymbol{B}$, i.e. a g_s equal to two for the electron.

8.5 Pauli equation.
Linearise the Schrödinger equation following the logic in Sec. 5.1 and show that the Pauli equation with $g_s = 2$ follows.

8.6 Zitterbewegung.
Use Ehrenfest theorem to show that $\boldsymbol{\alpha}$ can be interpreted as velocity operator in the semi-classical limit. Project out the negative energy solutions and show that the solution $\langle \boldsymbol{x}(t) \rangle$ contains an oscillating term.

8.7 Spin vector.
i) Show that the Pauli-Lubanski spin vector (5.26), setting $J^{\mu\nu} = \sigma^{\mu\nu}$, becomes in the rest-frame

$$W_i = -m\Sigma_i \,. \tag{8.129}$$

ii) Show that $W_\mu s^\mu$ can be used as relativistic spin operator.

8.8 Projection operators ♣.
Show that the inverse of the matrix $A = \sum_i a_i P_i$, where the P_i are projection operators (i.e. $\sum_i P_i = 1$ and $P_i P_j = \delta_{ij} P_j$), is given by $A^{-1} = \sum_i a_i^{-1} P_i$. Use this to derive the propagator of massive spin-1 particle.

8.9 Helicity.
a.) Show that the helicity operator $h = \boldsymbol{s} \cdot \boldsymbol{p}/|\boldsymbol{p}|$ is the special case of the spin operator $\not{s}\gamma^5$. b.) Derive the common eigenfunctions of the Dirac Hamiltonian and h. c.) Consider these helicity states in the limit $m/E \to 0$.

8.10 Lorentz transformation and gamma's.
Show that $S(\Lambda) = \exp\left(-\mathrm{i}\sigma_{\mu\nu}\omega^{\mu\nu}/4\right)$ is valid in any representation of the gamma matrices. Derive the behavior of the gamma matrices under Lorentz transformation, $S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \Lambda_\nu^{\ \mu}\gamma'^\nu$.

8.11 Properties of C.
a.) Prove the following properties of the charge conjugation matrix C, $C^T = -C$ and $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. b.) Show that $\mathrm{i}\gamma^2\gamma^0$ has the required properties.

8.12 Dirac Lagrangian.

Show that the Lagrangian (8.28) is not real. Find a suitable generalisation satisfying $\mathscr{L}^* = \mathscr{L}$.

8.13 CPT theorem.
Construct all possible Lorentz scalars combining the bilinear quantities $\bar{\psi}\Gamma\psi$ with tensors. Find their properties under CPT transformations and show that all Lorentz scalars are CPT even.

8.14 Majorana flip properties ♣.
Derive the properties of Majorana bilinears $\bar{\xi}\Gamma\eta = \pm\bar{\eta}\Gamma\xi$ under an exchange $\bar{\xi} \leftrightarrow \eta$.

8.15 Fierz identities ♣.
Derive the following identities, $\bar{\sigma}^\mu\sigma_\mu = 4$, $(\bar{\sigma}^\mu)_{ab}(\bar{\sigma}_\mu)_{cd} = 2(\mathrm{i}\sigma^2)_{ac}(\mathrm{i}\sigma^2)_{bd}$, and $(\mathrm{i}\sigma^2)_{ab}(\sigma^\mu)_{bc} = (\bar{\sigma}^{\mu T})_{ab}(\mathrm{i}\sigma^2)_{bc}$.

8.16 Fermionic vacuum energy.
Calculate the contribution of a Dirac fermion to the vacuum enrgy density, following the steps (4.53) to (4.60) for a scalar.

8.17 Current conservation.
Show that the change of the Dirac Lagrangian under a local U(1) transformation can be written as (5.28), leading to current conservation.

9. Scattering processes

Most information about the properties of fundamental interactions and particles is obtained from scattering experiments. In a scattering process, the initial and final state contains widely separated particles which can be approximated as free, real particles which are on mass-shell. By contrast, n-point Green functions describe the propagation of virtual particles. In order to make contact with experiments, we have to find therefore the link between Green functions and experimental results from scattering experiments. The latter can be predicted knowing the scattering matrix S which is an unitary operator mapping an initial state at $t = -\infty$ on a final state at $t = +\infty$. We introduce first the scattering matrix S and show then that the unitarity of the S-matrix restricts the analytic structure of Feynman amplitudes; in particular it implies the optical theorem. Then we derive the connection between n-point Green functions and scattering amplitudes, before we perform some explicit calculations of few tree-level processes. Finally, we consider the special case when in a scattering event additional soft particles are emitted.

The relation between Feynman amplitudes and cross sections or decay widths is essentially the same as in non-relativistic quantum mechanics; it is reviewed in appendix 9.A.

9.1. Unitarity of the S-matrix and its consequences

A scattering process is fully described in the Schrödinger picture by the knowledge how initial states $|i,t\rangle$ at $t \to -\infty$ are transformed into final states $|f,t\rangle$ at $t \to \infty$. This knowledge is encoded in the S-matrix elements

$$|f, t = \infty\rangle = S_{fi} \, |i, t = -\infty\rangle \ . \tag{9.1}$$

An intuitive, but mathematically delicate definition of the scattering operator S is the $t \to \infty$ limit of the time-evolution operator $U(t,-t)$

$$S = \lim_{t\to\infty} U(t,-t) \, . \tag{9.2}$$

Thus the scattering operator S evolves an eigenstate $|n,t\rangle$ of the Hamiltonian from $t = -\infty$ to $t = +\infty$,

$$S \, |n, -\infty\rangle = |n, \infty\rangle \ . \tag{9.3}$$

The unitarity of the scattering operator, $S^\dagger S = S S^\dagger = 1$, expresses the fact that we (should) use a complete set of states for the initial and final states in a scattering process,

$$1 = \sum_n |n, +\infty\rangle \, \langle n, +\infty| = \sum_n S \, |n, -\infty\rangle \, \langle n, -\infty| \, S^\dagger = S S^\dagger \, . \tag{9.4}$$

Optical theorem We split the scattering operator S into a diagonal part and the transition operator T, $S = 1 + \mathrm{i}T$, and thus

$$1 = (1 + \mathrm{i}T)(1 - \mathrm{i}T^\dagger) = 1 + \mathrm{i}(T - T^\dagger) + TT^\dagger \tag{9.5}$$

or

$$\mathrm{i}TT^\dagger = T - T^\dagger \,. \tag{9.6}$$

Note that in perturbation theory the LHS is $\mathcal{O}(g^{2n})$, while the RHS is $\mathcal{O}(g^n)$. Hence this equation implies a non-linear relation between the transition operator evaluated at different orders. At lowest order perturbation theory, the LHS vanishes and T is real, $T = T^\dagger$.

We consider now matrix elements between the initial and final state,

$$\langle f|\, T - T^\dagger \,|i\rangle = T_{fi} - T^*_{if} = \mathrm{i}\,\langle f|\, TT^\dagger \,|i\rangle = \mathrm{i}\sum_n T_{fn}T^*_{in} \,. \tag{9.7}$$

If we set $|i\rangle = |f\rangle$, we obtain a connection between the forward scattering amplitude T_{ii} and the total cross section σ_{tot} called the *optical theorem*,

$$2\Im T_{ii} = \sum_n |T_{in}|^2 \,. \tag{9.8}$$

Thus the optical theorem relates the attenuation of a beam of particles in the state i, $\mathrm{d}N_i \propto -|\Im T_{ii}|^2 N_i$, to the probability that they scatter into all possible states n. Its RHS is given by the total cross section σ_{tot} up to a factor depending on the flux of initial particles and possible symmetry factors. For the case of two particles in the initial state, comparison with (9.167) and (9.171) from the appendix shows that

$$\Im\mathcal{A}_{ii} = 2p_{\mathrm{cms}}\sqrt{s}\,\sigma_{\mathrm{tot}} \,. \tag{9.9}$$

Note also that the forward scattering amplitude T_{ii} means *scattering* without change in any conserved quantum number, since we extracted already the identity part, $T_{ii} = (S_{ii} - 1)/\mathrm{i}$.

Imaginary part of the amplitude Consider the Feynman amplitude $\mathcal{A}$ as a complex function of the squared center-of-mass (c.m.) energy s. The threshold energy $\sqrt{s_0}$ in the c.m. system equals the minimal energy for which the reaction is kinematically allowed. The optical theorem implies that $\mathcal{A}$ is real for $s < s_0$ and $s \in \mathbb{R}$. Thence $s = s^*$ and $\mathcal{A}(s) = [\mathcal{A}(s)]^*$ and therefore

$$\mathcal{A}(s) = [\mathcal{A}(s^*)]^* \qquad \text{for} \quad s < s_0 \,. \tag{9.10}$$

If $\mathcal{A}(s)$ is an analytic function, then also $[\mathcal{A}(s^*)]^*$ is analytic and we can continue this relation into the complex s plane. In particular, along the real axis we have for $s > s_0$

$$\Re\mathcal{A}(s + \mathrm{i}\varepsilon) = \Re\mathcal{A}(s - \mathrm{i}\varepsilon) \quad \text{and} \quad \Im\mathcal{A}(s + \mathrm{i}\varepsilon) = -\Im\mathcal{A}(s - \mathrm{i}\varepsilon) \,. \tag{9.11}$$

Thus starting from s_0, the amplitude $\mathcal{A}$ has a discontinuity along the real s axis. Since the amplitude $\mathcal{A}$ should be single-valued, it has to contain a branch cut along the real s axis starting at s_0. Feynman's $m^2 - \mathrm{i}\varepsilon$ prescription tells us then which side of the cut we should pick out as the "physical" one.

The second relation in (9.11) allows us to obtain the imaginary part of a Feynman amplitude from its discontinuity,

$$\mathrm{disc}(\mathcal{A}) \equiv \mathcal{A}(s+\mathrm{i}\varepsilon) - \mathcal{A}(s-\mathrm{i}\varepsilon) = 2\mathrm{i}\Im\mathcal{A}(s+\mathrm{i}\varepsilon)\,. \tag{9.12}$$

The prototype of a function having a discontinuity and a branch cut (along $\mathbb{R}^-$) is the logarithm,

$$\mathrm{Ln}(z) = \mathrm{Ln}(r\mathrm{e}^{\mathrm{i}\vartheta}) = \ln(r\mathrm{e}^{\mathrm{i}\vartheta}) + 2k\pi\mathrm{i} = \ln(r) + (\vartheta + 2k\pi)\mathrm{i} \tag{9.13}$$

with $\Im\ln(x+\mathrm{i}\varepsilon) = \pi$. How does an imaginary part in a Feynman diagram arise? Applying the relation

$$\frac{1}{x \pm \mathrm{i}\varepsilon} = P\left(\frac{1}{x}\right) \mp \mathrm{i}\pi\delta(x) \tag{9.14}$$

to the propagator of a virtual particle, we see that virtual particles which propagate on-shell lead to poles and to imaginary terms in the amplitude.

Example 9.1: Verify the optical theorem for $\phi\phi \to \phi\phi$ scattering in the $\lambda\phi^4$ theory at $\mathcal{O}(\lambda^2)$:
The logarithmic terms in the scattering amplitude (4.77) for $\phi\phi \to \phi\phi$ scattering at one-loop have the form

$$F(q^2, m) = \int_0^1 \mathrm{d}z\ \ln\left[m^2 - q^2 z(1-z)\right] \tag{9.15}$$

with $q^2 = \{s, t, u\}$. In the physical region, the relation $q^2 > 4m^2$ holds only for the s chanel diagram. The argument of the logarithm becomes negative for

$$z_{1/2} = \frac{1}{2}\left[1 \pm \sqrt{1 - 4m^2/s^2}\right] = \frac{1}{2} \pm \frac{1}{2}\beta \tag{9.16}$$

with $\beta = \sqrt{1-4m^2/s^2}$ as the velocity of the ϕ particles in the center of mass system. Using now $\mathrm{Im}[\ln(-q^2 - \mathrm{i}\varepsilon)] = -\pi$, the imaginary part follows as

$$\Im(A) = \pi\frac{\lambda^2}{32\pi^2}\int_{\frac{1}{2}-\frac{1}{2}\beta}^{\frac{1}{2}+\frac{1}{2}\beta} \mathrm{d}z = \frac{\lambda^2}{32\pi}\beta\,. \tag{9.17}$$

The optical theorem implies thus that the total cross section $\sigma_{\mathrm{tot}}(\phi\phi \to \mathrm{all})$ at $\mathcal{O}(\lambda^2)$ equals

$$\sigma_{\mathrm{tot}} = \frac{\Im\mathcal{A}_{ii}}{2p_{\mathrm{cms}}\sqrt{s}} = \frac{\lambda^2}{32\pi s}\,, \tag{9.18}$$

where we used $2p_{\mathrm{cms}} = \sqrt{s}\beta$. On the other hand, the Feynman amplitude at tree level is simply $\mathcal{A} = -\lambda$ and thus the elastic cross section for $\phi\phi \to \phi\phi$ scattering follows as $\sigma_{\mathrm{el}} = \lambda^2/(32\pi s)$. At $\mathcal{O}(\lambda^2)$, the only reaction contributing to the total cross section is elastic scattering, and thus the two cross sections agree. Note also the treatement of the symmetry factors: In the loop diagram, the symmetry factor $S = 1/2!$ is already included, while the corresponding factor for the two identical particles in the final state is added only integrating the cross section.

9.2. LSZ reduction formula

We defined the generating functional $Z[J] = \langle 0, \infty | 0, -\infty\rangle_J$ as the vacuum-vacuum transition amplitude in the presence of a classical source J. Thus the generating functional contains the boundary condition $\phi(x) \to 0$ for $t \to \pm\infty$. We have two options to find a bridge between

S-matrix elements and the formalism we have derived up to now: One possibility is to find the connection between the Green functions derived from $Z[J]$ and S-matrix elements. Another one is to define first a new functional $Z'[J]$ with the correct boundary conditions, and to establish then the connection between $Z[J]$ and $Z'[J]$. We choose the first way, restricting ourselves for simplicity to the case of a real scalar field.

Let us start with the case of a $2 \to 2$ scattering process. We can generate a free two-particle[1] state composed of plane-waves by applying two creation operators on the vacuum,

$$|\boldsymbol{k}_1, \boldsymbol{k}_2\rangle = a^\dagger(\boldsymbol{k}_1) a^\dagger(\boldsymbol{k}_2) \, |0\rangle \ . \tag{9.19}$$

We obtain localized wave packets defining new creation operators

$$a_i^\dagger = \int \mathrm{d}^3k \ f_i(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) \, , \tag{9.20}$$

where $f_i(\boldsymbol{k})$ is e.g. a Gaussian centered around $\boldsymbol{k}_i$,

$$f_i(\boldsymbol{k}) \propto \exp[-(\boldsymbol{k} - \boldsymbol{k}_i)^2/(2\sigma^2)] \, . \tag{9.21}$$

We assume that the initial state of the scattering process at $t = -\infty$ can be described by freely propagating wave packets,

$$|i\rangle = \lim_{t \to -\infty} a_1^\dagger(t) a_2^\dagger(t) \, |0\rangle = |\boldsymbol{k}_1, \boldsymbol{k}_2; -\infty\rangle \ , \tag{9.22}$$

and similarly the final state as

$$|f\rangle = \lim_{t \to \infty} a_{1'}^\dagger(t) a_{2'}^\dagger(t) \, |0\rangle = |\boldsymbol{k}_{1'}, \boldsymbol{k}_{2'}; +\infty\rangle \ . \tag{9.23}$$

Here we changed to the Heisenberg picture, since our Green functions are time-dependent.

Our task is to connect this transition amplitude $\langle f \, | i \rangle$ to the corresponding four-point Green function. The latter is the time-ordered vacuum expectation value of field operators. The first property, time-ordering, is automatically satisfied for the transition amplitude $\langle f \, | i \rangle$, since we can write

$$\langle f \, | i \rangle = \lim_{t \to \infty} \langle 0 | \, a_{1'}(t) a_{2'}(t) a_1^\dagger(-t) a_2^\dagger(-t) \, |0\rangle \tag{9.24a}$$

$$= \lim_{t \to \infty} \langle 0 | \, \mathrm{T}\{a_{1'}(t) a_{2'}(t) a_1^\dagger(-t) a_2^\dagger(-t)\} \, |0\rangle \ . \tag{9.24b}$$

Thus we only have to re-express the creation and annihilation operators as (projected) field operators. We define a scalar product for solutions of the Klein-Gordon equation as follows,

$$(\phi, \chi) = \mathrm{i} \int \mathrm{d}^3x \ \phi^*(x) \overleftrightarrow{\partial_0} \chi(x) \equiv \mathrm{i} \int \mathrm{d}^3x \left[\phi^*(x) \frac{\partial \chi(x)}{\partial t} - \frac{\partial \phi^*(x)}{\partial t} \chi(x) \right] . \tag{9.25}$$

Comparing this definition to Eq. (5.12), we see that the scalar product is the zero component of the conserved current j^μ. Thus the value of the scalar product (ϕ, χ) is time-independent and corresponds to the number of particles minus the number of anti-particles.

[1] To reduce clutter, we assume $\boldsymbol{k}_1 \neq \boldsymbol{k}_2$.

For plane-wave components with definite momentum,

$$\phi_{\boldsymbol{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}}\, \mathrm{e}^{-\mathrm{i}kx} = N_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}kx}\,, \tag{9.26}$$

the scalar product is given by

$$(\phi_{\boldsymbol{k}}, \phi_{\boldsymbol{k}'}) = \mathrm{i}N_{\boldsymbol{k}}N_{\boldsymbol{k}'} \int \mathrm{d}^3x \left[\mathrm{e}^{\mathrm{i}k'x}(-\mathrm{i}\omega_{\boldsymbol{k}})\mathrm{e}^{-\mathrm{i}kx} - \mathrm{i}\omega_{\boldsymbol{k}'}\mathrm{e}^{\mathrm{i}k'x}\mathrm{e}^{-\mathrm{i}kx}\right] \tag{9.27}$$

$$= N_{\boldsymbol{k}}^2(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}')2\omega_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}})t} = \delta(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{9.28}$$

Similarly it follows $(\phi^*_{\boldsymbol{k}}, \phi^*_{\boldsymbol{k}'}) = -\delta(\boldsymbol{k}-\boldsymbol{k}')$, while the two terms in the scalar product cancel otherwise,

$$(\phi_{\boldsymbol{k}}, \phi^*_{\boldsymbol{k}'}) = (\phi^*_{\boldsymbol{k}}, \phi_{\boldsymbol{k}'}) = 0.$$

Thus we can invert the free field operator

$$\phi(x) = \int \mathrm{d}^3k \left[a(\boldsymbol{k})\phi_{\boldsymbol{k}}(x) + a^\dagger(\boldsymbol{k})\phi^*_{\boldsymbol{k}}(x)\right] = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\boldsymbol{k})\mathrm{e}^{+\mathrm{i}kx}\right] \tag{9.29}$$

to obtain

$$a^\dagger(\boldsymbol{k}) = -(\phi^*_{\boldsymbol{k}}, \phi) = -\mathrm{i}N_{\boldsymbol{k}} \int \mathrm{d}^3x\; \mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_t}\, \phi(x)\,. \tag{9.30}$$

Next we want to rewrite this expression in a way that shows explicitely its Lorentz invariance. Using the identity

$$a^\dagger(\boldsymbol{k}, \infty) - a^\dagger(\boldsymbol{k}, -\infty) = \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\partial}{\partial t}\, a^\dagger(\boldsymbol{k}, t) \tag{9.31}$$

we insert first (9.30) assuming a wave-package localized around $\boldsymbol{k}_1$ and perform then the time differentiation,

$$a^\dagger(\boldsymbol{k}_1, \infty) - a^\dagger(\boldsymbol{k}_1, -\infty) = -\mathrm{i} \int \mathrm{d}^3k f_1(\boldsymbol{k}) \int \mathrm{d}^4x\; \partial_t \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_t}\, \phi(x)\right) \tag{9.32}$$

$$= -\mathrm{i} \int \mathrm{d}^3k f_1(\boldsymbol{k}) \int \mathrm{d}^4x\; \left(\mathrm{e}^{-\mathrm{i}kx}\partial_t^2\phi(x) - \phi(x)\partial_t^2\mathrm{e}^{-\mathrm{i}kx}\right)\,,$$

where the two terms linear in ∂_t canceled. Then we use that the field is on-shell, $k^2 = m^2$, for the replacement

$$\partial_t^2\mathrm{e}^{-\mathrm{i}kx} = (\boldsymbol{\nabla}^2 - m^2)\mathrm{e}^{-\mathrm{i}kx}.$$

Since the field is localized in space, we can perform two partial integrations moving thereby $\boldsymbol{\nabla}^2$ to the left, obtaining

$$a^\dagger(\boldsymbol{k}_1, \infty) - a^\dagger(\boldsymbol{k}_1, -\infty) = -\mathrm{i} \int \mathrm{d}^3k f_1(\boldsymbol{k}) \int \mathrm{d}^4x\; \mathrm{e}^{-\mathrm{i}kx} \left(\Box + m^2\right) \phi(x)\,. \tag{9.33}$$

In a free theory, $\phi(x)$ satisfies the Klein-Gordon equation and the RHS would vanish. In an interacting theory with e.g. $\mathscr{L}_I = -\lambda\phi^4/4!$, the RHS is however proportional to $\left(\Box + m^2\right)\phi(x) = \lambda\phi^3/3! \neq 0$.

Having performed the partial integrations, we can forget the wave-packets, $\sigma \to 0$, and write simply

$$a^\dagger(\boldsymbol{k}, -\infty) = a^\dagger(\boldsymbol{k}, \infty) + \mathrm{i}N_{\boldsymbol{k}} \int \mathrm{d}^4x \; \mathrm{e}^{-\mathrm{i}kx} \left(\Box + m^2\right) \phi(x) \,. \tag{9.34}$$

Taking the hermitian conjugate, we obtain for the annihilation operator

$$a(\boldsymbol{k}, \infty) = a(\boldsymbol{k}, -\infty) + \mathrm{i}N_{\boldsymbol{k}} \int \mathrm{d}^4x \; \mathrm{e}^{\mathrm{i}kx} \left(\Box + m^2\right) \phi(x) \,. \tag{9.35}$$

When we insert these expressions into $\langle f \,|i\rangle$, we obtain a four-point function combining the second terms from the RHS of (9.34) and (9.35). Including the terms $a^\dagger(\boldsymbol{k}, \infty)$ and $a(\boldsymbol{k}, -\infty)$ generates particles propagating from $t = -\infty$ to $t = +\infty$ with momenta unchanged, i.e. to terms corresponding to disconnected graphs. Hence we do not need to consider these contributions, restricting ourselves to connected Green functions[2]. For n particles in the initial and m particles in the final state, we obtain

$$\begin{aligned} \langle \boldsymbol{k}'_1, \ldots, \boldsymbol{k}'_m; +\infty \,|\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n; -\infty\rangle &= \mathrm{i}^{n+m} \prod_{i=1}^{n} \int \mathrm{d}^4x_i \; N_{\boldsymbol{k}_i} \, \mathrm{e}^{-\mathrm{i}k_i x_i} \left(\Box_{x_i} + m^2\right) \\ \times \prod_{j=1}^{m} \int \mathrm{d}^4y_j \; N_{\boldsymbol{k}_i} \, \mathrm{e}^{\mathrm{i}k'_j y_j} \left(\Box_{y_j} + m^2\right) \langle 0| \, \mathrm{T}\{\phi(x_1) \cdots \phi(y_m) \,|0\rangle \;. \end{aligned} \tag{9.36}$$

This is the reduction formula of Lehmann, Symanzik and Zimmermann (LSZ): For each external particle we obtained the corresponding plan wave component and a Klein-Gordon operator. Since the latter is the inverse of the free 2-point functions, we can rephrase the content of the LZS formula simply as follows: Replace the 2-point functions of external lines by appropriate wave functions, e.g. $\phi_{\boldsymbol{k}}(x)$ for scalar particles in the initial state and $\phi^*_{\boldsymbol{k}}(x)$ for scalar particles in the final state.

Since we started from field operators in the Heisenberg picture, the matrix element is in the Heisenberg picture, too. In the Schrödinger picture, it is

$$\langle \boldsymbol{k}'_1, \ldots, \boldsymbol{k}'_m | \mathrm{i}T \,|\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle \equiv \mathrm{i}T_{fi} \,,$$

where we used also $S = 1 + \mathrm{i}T$ and the fact that we neglected disconnected parts. Finally, we define the Fourier transformed n-point function as

$$G(x_1, \ldots, x_n) = \int \prod_{i=1}^{n} \frac{\mathrm{d}^4k_i}{(2\pi)^4} \; \exp\left(-\mathrm{i}\sum_i k_i x_i\right) G(k_1, \ldots, k_n) \,. \tag{9.37}$$

Then we obtain the LSZ reduction formula in momentum space,

$$\begin{aligned} \mathrm{i}T_{fi} &= \mathrm{i}^{n+m} N_{\boldsymbol{k}_1} \cdots N_{\boldsymbol{k}_n} N_{\boldsymbol{k}'_1} \cdots N_{\boldsymbol{k}'_m} (k_1^2 - m^2) \cdots (k_n^2 - m^2)(k_1'^2 - m^2) \cdots (k_m'^2 - m^2) \\ &\times G(k_1, \cdots, k_n, -k'_1, \ldots, -k'_m) \,. \end{aligned} \tag{9.38}$$

The Green function $G(k_1, \cdots, k_n, -k'_1, \ldots, -k'_m)$ is multiplied by zeros, since the external particles satisfy $k^2 = m^2$. Thus T_{fi} vanishes, except when poles $1/(k^2 - m^2)$ of

[2] You may wonder that we eliminate in this way also the forward scattering amplitudes T_{ii}: Analyticity of scattering amplitudes guaranties that the amplitude calculated for, e.g., scattering angle $\vartheta > 0$ contains also correctly the case $\vartheta = 0$.

$G(k_1, \cdots, k_n, -k'_1, \ldots, -k'_m)$ cancel these zeros. In the case of external scalar particles, only their normalization factors are left. As they are not essential for the calculation of the transition amplitudes, one include these normalization factors into the phase space of final state particle and in the flux factor of initial particles. This explains our Feynman rule to replace the scalar propagator by one for amplitudes in momentum space.

The derivation of the LSZ formula for particles with spin $s > 0$ proceeds in the same way. Their wave-functions contain additionally polarization vectors $\varepsilon^\mu(k)$, tensors $\varepsilon^{\mu\nu}(k)$, or spinors $u(p)$ and $\bar{u}(p)$ and their charge conjugated states. In the case of a photon (graviton), we have to add $\varepsilon^\mu(k)$ ($\varepsilon^{\mu\nu}(k)$) in the initial state and the complex conjugated $\varepsilon^{*\mu}(k)$ ($\varepsilon^{*\mu\nu}(k)$) in the final state. In the case of Dirac fermions, we have to assign four different spinors to the four possible combinations of particle and anti-particles in the initial and final state. Having chosen $u(p)$ as particle state with an arrow along the direction of time, the simple rule that a fermion line corresponds to a complex number $\bar{\psi} \cdots \psi$ fixes the designation of the other spinors as shown in Fig. 9.1: Connecting the upper fermion lines, $\bar{u}(p') \cdots u(p)$, corresponds to the scattering $e^-(p) + X \to e^-(p') + X'$, while $\bar{v}(q) \cdots u(p)$ describes the annihilation process $e^+(q) + e^-(p) \to X + X'$. Connecting the lower fermion lines, $\bar{v}(q) \cdots v(q)$, corresponds to the scattering $e^+(q) + X \to e^+(q') + X'$, while $\bar{u}(p') \cdots v(q)$ describes the pair creation process $X + X' \to e^+(q') + e^-(p')$. Recall that the Feynman amplitude $\mathcal{A}$ is defined omitting the normalisation factor $N_p = [2\omega_p(2\pi)^3]^{-1/2}$ from all wave-functions—the splitting of S-matrix elements into Feynman amplitudes and phase space is discussed in appendix 9.A.

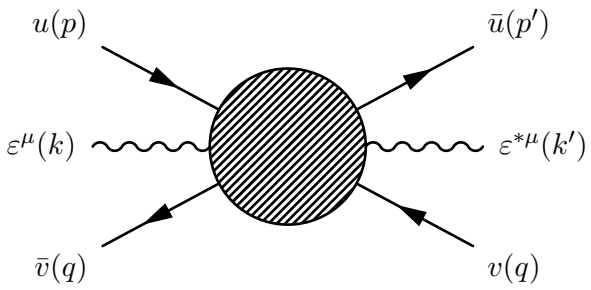


Figure 9.1.: Feynman rules for external particles in momentum space; initial state on the left, final state on the right.

Wave function renormalization Up to now we have pretended that we can describe the fields in the initial and final state as free particles. Although e.g. Yukawa interactions between two, by assumption, widely separated particles at $t = \pm\infty$ are negligibly small, self-interactions persist. These interactions lead to a renormalization of the external wave-functions.

We can rephrase the problem as follows: If the creation operator $a^\dagger(\boldsymbol{k}, -\infty)$ corresponds to the one of a free theory, then it can only connect one-particle states with the vacuum. In contrast, the interacting field can also connect many-particle states to the vacuum and

therefore its overlap with single-particle states is reduced,

$$a^\dagger(\boldsymbol{k},-\infty)\,|0\rangle = \sqrt{Z}\,|\boldsymbol{k}\rangle + \sqrt{1-Z}\,\{|\boldsymbol{k}',\boldsymbol{k}'',\boldsymbol{k}'''\rangle + \ldots\} \tag{9.39}$$

$$= \sqrt{Z}a_0^\dagger(\boldsymbol{k}_1)\,|0\rangle + \sqrt{1-Z}\,\{|\boldsymbol{k}',\boldsymbol{k}'',\boldsymbol{k}'''\rangle + \ldots\}\;. \tag{9.40}$$

Therefore the free and the interacting fields are connected by

$$\phi(x) \to \sqrt{Z}\phi_0(x) \tag{9.41}$$

for $t \to \pm\infty$, where we call the factor Z the wave-function (or the field-strength) renormalisation constant. We will show in section 11.4.2 that this factor can be extracted from the self-energy diagrams of the corresponding field. More precisely, including a factor $\sqrt{Z_k}$ for each external line of type k takes into account self-energy corrections in the external lines. Therefore it is enough to calculate the self-energy and to extract $\sqrt{Z_k}$ once; after that we can omit self-energy corrections in the external lines adding simply factors $\sqrt{Z_k}$. Finally, note that we can set $Z = 1$ in tree-level processes, since in perturbation theory $Z = 1+\mathcal{O}(g)$ holds.

9.3. Specific processes

We consider now in detail a few specific processes. First, we derive the Klein-Nishina formula for the Compton scattering cross section using the standard "trace method." Then we calculate polarized $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \gamma\gamma$ scattering applying helicity methods.

9.3.1. Trace method and Compton scattering

Matrix element The Feynman amplitude $\mathcal{A}$ of Compton scattering $e^-(p)+\gamma(k) \to e^-(p')+\gamma(k')$ at $\mathcal{O}(e^2)$ consists of two diagrams, ,

$$\mathrm{i}\mathcal{A} = -\mathrm{i}e^2\bar{u}(p')\left[\not{\varepsilon}^{*\prime}\,\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\,\not{\varepsilon} + \not{\varepsilon}\,\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\,\not{\varepsilon}^{*\prime}\right]u(p)\,. \tag{9.42}$$

Since the denominator is a non-zero light-like vector, we have omitted the $\mathrm{i}\varepsilon$. Note that the two amplitudes can be transformed into each other replacing $\varepsilon \leftrightarrow \varepsilon^{*\prime}$ and $k \leftrightarrow -k'$. This symmetry called crossing symmetry relates processes where a particle is replaced by an anti-particle with negative momentum on the other side of the reaction.

We evaluate the process in the rest-frame of the initial electron. Then $p = (m,\boldsymbol{0})$ and choosing $\varepsilon^\mu = (0,\boldsymbol{\varepsilon})$ as well as $\varepsilon'^\mu = (0,\boldsymbol{\varepsilon}')$, it follows

$$p\cdot\varepsilon = p\cdot\varepsilon' = 0\,. \tag{9.43}$$

Moreover, the photons are transversely polarized,

$$k\cdot\varepsilon = k'\cdot\varepsilon' = 0\,, \tag{9.44}$$

and we choose real polarization vectors. We anti-commute $\not{p}$ in the numerator to the right, $\not{p}\not{\varepsilon}' = 2p\cdot\varepsilon' - \not{\varepsilon}'\not{p} = -\not{\varepsilon}'\not{p}$ and use the Dirac equation, $\not{p}u(p) = mu(p)$. Then we simplify also the denominator using $p^2 = m^2$ and obtain

$$\mathcal{A} = -e^2\bar{u}(p')\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k} + \frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]u(p)\,. \tag{9.45}$$

Typically the electron target is not polarized, and the spin of the final electron is not measured. Thus we sum the squared matrix element over the final and average over the initial electron spin,

$$|\overline{\mathcal{A}}|^2 = \frac{1}{2}\sum_{s,s'}|\mathcal{A}|^2 = \frac{e^4}{2}\sum_{s,s'}\left|\bar{u}(p')\left(\frac{\not\varepsilon'\not k\not\varepsilon}{2p\cdot k}+\frac{\not\varepsilon\not k'\not\varepsilon'}{2p\cdot k'}\right)u(p)\right|^2 . \tag{9.46}$$

Calculating $|\mathcal{A}|^2 = \mathcal{A}\mathcal{A}^*$ requires the knowledge of $\mathcal{A}^* = \mathcal{A}^\dagger$. Recall that we defined $\bar\psi = \psi^\dagger\gamma^0$ such that a general amplitude $\mathcal{A}$ composed of spinors,

$$\mathcal{A} = \bar\psi(p')\Gamma\psi(p) = \psi^\dagger(p')\gamma^0\Gamma\psi(p) \tag{9.47}$$

with Γ denoting a product of the basis elements given in Eq. (8.43), becomes

$$\mathcal{A}^* = \bar\psi(p)\gamma^0\Gamma^\dagger\gamma^0\psi(p') \equiv \bar\psi(p)\overline{\Gamma}\psi(p') . \tag{9.48}$$

Important special cases worth to memorize are $\overline{\gamma^\mu} = \gamma^\mu$, $\overline{\gamma^5} = -\gamma^5$, and $\overline{\not a\not b\cdots\not z} = \not z\cdots\not b\not a$. We now write out the spinor indices,

$$|\overline{\mathcal{A}}|^2 = \frac{e^4}{2}\sum_{s,s'}\bar u_a(p')\left[\frac{\not\varepsilon'\not k\not\varepsilon}{2p\cdot k}+\frac{\not\varepsilon\not k'\not\varepsilon'}{2p\cdot k'}\right]_{ab}u_b(p)\bar u_c(p)\left[\frac{\not\varepsilon\not k\not\varepsilon'}{2p\cdot k}+\frac{\not\varepsilon'\not k'\not\varepsilon}{2p\cdot k'}\right]_{cd}u_d(p') . \tag{9.49}$$

Using the property (8.59) of the Dirac spinors, $\sum_s u_a(p,s)\bar u_b(p,s') = (\not p + m)_{ab}$, we obtain

$$|\overline{\mathcal{A}}|^2 = \frac{e^4}{2}\left[\not p' + m\right]_{da}\left[\frac{\not\varepsilon'\not k\not\varepsilon}{2p\cdot k}+\frac{\not\varepsilon\not k'\not\varepsilon'}{2p\cdot k'}\right]_{ab}\left[\not p + m\right]_{bc}\left[\frac{\not\varepsilon\not k\not\varepsilon'}{2p\cdot k}+\frac{\not\varepsilon'\not k'\not\varepsilon}{2p\cdot k'}\right]_{cd} . \tag{9.50}$$

Since $\not p + m$ combines one spinor in $\mathcal{A}$ and one in $\mathcal{A}^*$, the result is a trace over gamma matrices,

$$|\overline{\mathcal{A}}|^2 = \frac{e^4}{8}\mathrm{tr}\left\{(\not p' + m)\left[\frac{\not\varepsilon'\not k\not\varepsilon}{p\cdot k}+\frac{\not\varepsilon\not k'\not\varepsilon'}{p\cdot k'}\right](\not p + m)\left[\frac{\not\varepsilon\not k\not\varepsilon'}{p\cdot k}+\frac{\not\varepsilon'\not k'\not\varepsilon}{p\cdot k'}\right]\right\} . \tag{9.51}$$

Working out some more examples of this type (e.g. in problem 9.2), you should convince yourself that each fermion line in $\mathcal{A}$ is converted into a trace in $|\overline{\mathcal{A}}|^2$. Useful identities for the evaluation of such traces are given in the appendix.

We simplify this trace by anti-commuting identical variables, such that they become neighbours. Then we can use $\not a\not a = a^2$ and reduce thereby the number of gamma matrices in each step by two. Multiplying out the terms in the trace, we obtain three contributions that we denote by

$$\mathrm{tr}\{\quad\} = \frac{S_1}{(p\cdot k)^2}+\frac{S_2}{(p\cdot k')^2}+\frac{2S_3}{(p\cdot k)(p\cdot k')} . \tag{9.52}$$

We consider only the first term S_1 in detail. Starting from

$$S_1 = \mathrm{tr}\left\{\left(\not p' + m\right)\not\varepsilon'\not\varepsilon\not k\left(\not p + m\right)\not k\not\varepsilon\not\varepsilon'\right\} \tag{9.53}$$

$$= \mathrm{tr}\{\not p'\not\varepsilon'\not\varepsilon\underbrace{\not k\not p}_{2kp-\not p\not k}\not k\not\varepsilon\not\varepsilon'\} + m^2\mathrm{tr}\{\not\varepsilon'\not\varepsilon\underbrace{\not k\not k}_{k^2=0}\not\varepsilon\varepsilon'\} \tag{9.54}$$

$$= 2\,(k\cdot p)\,\mathrm{tr}\,\{\not p'\not\varepsilon'\not\varepsilon\not k\not\varepsilon\not\varepsilon'\} - \mathrm{tr}\,\{\not p'\not\varepsilon'\not\varepsilon\not p\not k\not k\not\varepsilon\not\varepsilon'\} \tag{9.55}$$

we arrived at an expression with only six gamma matrices. We continue the work,

$$S_1 = 2\,(k\cdot p)\,\mathrm{tr}\,\left\{\not p'\not\varepsilon'\not\varepsilon\not k\not\varepsilon\not\varepsilon'\right\} = -2\,(k\cdot p)\,\mathrm{tr}\,\left\{\not p'\not\varepsilon'\not k\underbrace{\not\varepsilon\not\varepsilon}_{\varepsilon^2=-1}\not\varepsilon'\right\} \tag{9.56}$$

$$= 2\,(k\cdot p)\,\mathrm{tr}\,\left\{p'\not\varepsilon'\underbrace{\not k\not\varepsilon'}_{2k\cdot\varepsilon'-\not\varepsilon'\not k}\right\} = 2\,(k\cdot p)\left[2\left(k\cdot\varepsilon'\right)\mathrm{tr}\,\left(\not p'\not\varepsilon'\right) - \mathrm{tr}\,\left\{\not p'\not\varepsilon'\not\varepsilon'\not k\right\}\right] \tag{9.57}$$

$$= 8\,(k\cdot p)\left[2\left(k\cdot\varepsilon'\right)\left(p'\cdot\varepsilon'\right) + \left(p'\cdot k\right)\right]\,. \tag{9.58}$$

We want to eliminate as next step the two scalar products that include p'. Because of four-momentum conservation $p'+k'=p+k$, it is

$$\left(p'-k\right)^2 = \left(p-k'\right)^2 \tag{9.59}$$

and thus

$$p'\cdot k = p\cdot k'\,. \tag{9.60}$$

Multiplying the four-momentum conservation equation by ε', it follows moreover

$$p+k=p'+k' \quad\Rightarrow\quad \underbrace{\varepsilon'\cdot p}_{0} + \varepsilon'\cdot k = \varepsilon'\cdot p' + \underbrace{\varepsilon'\cdot k'}_{0}\,. \tag{9.61}$$

Thus our final result for S_1 is

$$S_1 = 8\,(k\cdot p)\left[2\left(k\cdot\varepsilon'\right)^2 + k'\cdot p\right]\,. \tag{9.62}$$

S_2 can be obtained observing the crossing symmetry of the amplitude by the replacements $\varepsilon\leftrightarrow\varepsilon'$ and $k\leftrightarrow -k'$. The cross term S_3 has to be calculated and we give here only the final result for the combination of the three terms, where some terms cancel

$$\left|\overline{\mathcal{A}}\right|^2 = e^4\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+4(\varepsilon\cdot\varepsilon')^2-2\right]\,. \tag{9.63}$$

Cross section To obtain the cross section, we have to calculate the flux factor and to perform the integration over the phase space of the final state,

$$\mathrm{d}\sigma = \frac{1}{4I}\,(2\pi)^4\,\delta^{(4)}(P_i-P_f)|\mathcal{A}_{fi}|^2\prod_{f=1}^{n}\frac{\mathrm{d}^3p_f}{2E_f(2\pi)^3} = \frac{1}{4I}\,|\mathcal{A}_{fi}|^2\,\mathrm{d}\Phi^{(n)}\,, \tag{9.64}$$

with the final state phase space $\mathrm{d}\Phi^{(n)}$. The flux factor I in the rest system of the electron is simply

$$I \equiv v_{\mathrm{rel}}\,p_1\cdot p_2 = m\omega\,. \tag{9.65}$$

Using Eq. (2.80),

$$\frac{\mathrm{d}^3p'}{2E'} = \mathrm{d}^4p'\delta^{(4)}(p'^2-m^2)\vartheta(p'_0)\,, \tag{9.66}$$

the phase space integration becomes

$$\mathrm{d}\Phi^{(2)} = \frac{1}{(2\pi)^2}\int\mathrm{d}\Omega_{k'}\frac{|\boldsymbol{k'}|^2\mathrm{d}k'}{2|\boldsymbol{k'}|}\int\frac{\mathrm{d}^3p'}{2E'}\,\delta^{(4)}\left(p'+k'-p-k\right) = \tag{9.67}$$

$$= \frac{1}{8\pi^2}\int\mathrm{d}\Omega_{k'}|\boldsymbol{k'}|\mathrm{d}k'\,\delta\left(\left(p+k-k'\right)^2-m^2\right)\,. \tag{9.68}$$

The argument of the delta function is

$$\left(p+k-k'\right)^2 - m^2 = m^2 + 2p\cdot k - 2p\cdot k' - 2k\cdot k' - m^2 \tag{9.69}$$

$$= 2m|\boldsymbol{k}| - 2m|\boldsymbol{k'}| - 2|\boldsymbol{k}||\boldsymbol{k'}|\left(1-\cos\vartheta\right) \tag{9.70}$$

$$= 2m\left(\omega-\omega'\right) - 2\omega\omega'\left(1-\cos\vartheta\right) \equiv f(\omega')\,. \tag{9.71}$$

In order to evaluate the delta function we have to determine the derivative $f'(\omega')$,

$$f'(\omega') = -2m - 2\omega\left(1-\cos\vartheta\right)\,, \tag{9.72}$$

and the zeros of $f(\omega')$,

$$0 = 2m\left(\omega-\omega'\right) - 2\omega\omega'\left(1-\cos\vartheta\right)\,. \tag{9.73}$$

Solving for ω' gives $\omega'\left[\omega\left(1-\cos\vartheta\right)+m\right] = m\omega$ and

$$\omega' = \frac{\omega}{1+\frac{\omega}{m}\left(1-\cos\vartheta\right)}\,. \tag{9.74}$$

This is the famous relation for the frequency shift of a photon found first experimentally in the scattering of X-rays on electrons by Compton 1921. The observed energy change of photons was crucial in accepting the quantum nature ("particle-wave duality") of photons. Combining everything, we obtain

$$\mathrm{d}\Phi^{(2)} = \frac{1}{8\pi^2}\int \mathrm{d}\Omega_{k'}|\omega'|\mathrm{d}\omega'\,\delta\left(2m\left(\omega-\omega'\right) - 2\omega\omega'(1-\cos\vartheta)\right) \tag{9.75}$$

$$= \frac{1}{8\pi^2}\int \mathrm{d}\Omega_{k'}\,\frac{\omega'}{2m\left[1+\frac{\omega}{m}\left(1-\cos\vartheta\right)\right]} = \frac{1}{16\pi^2\,m\omega}\int \mathrm{d}\Omega_{k'}\,\omega'^2 \tag{9.76}$$

and thus as differential Klein-Nishina cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4m\omega}\,\frac{\omega'^2}{16\pi^2\,m\omega}\,\left|\overline{\mathcal{A}}\right|^2 = \frac{\alpha^2}{4m^2}\frac{\omega'^2}{\omega^2}\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+4(\varepsilon\cdot\varepsilon')^2-2\right] \tag{9.77}$$

with $\alpha \equiv e^2/(4\pi)$. For scatterings in the forward direction, $\vartheta \to 0$ and thus $\omega' \to \omega$, the scattered photon retains (in the lab frame) its energy even in the ultra-relativistic limit $\omega \gg m$. The same holds in the classical limit, $\omega \ll m$, but now for all directions. Thus we obtain as classical limit of the Klein-Nishina formula the polarized Thomson cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \frac{\alpha^2}{m^2}\left(\varepsilon\cdot\varepsilon'\right)^2 = r_0^2\left(\varepsilon\cdot\varepsilon'\right)^2\,, \tag{9.78}$$

with $r_0^2 = \alpha/m$ as the classical electron radius.

Averaging and summing over the photon polarization vectors is simplest, if we choose the angle between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon'}$ as ϑ. Then

$$\sum_{r,r'}\left(\varepsilon\cdot\varepsilon'\right)^2 = 1+\cos^2\vartheta\,. \tag{9.79}$$

The integration over the scattering angle ϑ can be done analytically. We use $x = \cos\vartheta$ and set $\tilde{\omega} \equiv \omega/m$,

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1}\mathrm{d}x\left[\frac{1}{[1+\tilde{\omega}(1-x)]^3}+\frac{1}{1+\tilde{\omega}(1-x)}-\frac{1-x^2}{[1+\tilde{\omega}(1-x)]^2}\right] \tag{9.80}$$

$$= \frac{\pi\alpha^2}{2m^2}\left\{\frac{1+\tilde{\omega}}{\tilde{\omega}^3}\left[\frac{2\tilde{\omega}(1+\tilde{\omega})}{(1+2\tilde{\omega}}-\ln(1+2\tilde{\omega})\right]+\frac{\ln(1+2\tilde{\omega})}{2\tilde{\omega}}-\frac{1+3\tilde{\omega}}{(1+2\tilde{\omega})^2}\right\}\,. \tag{9.81}$$

Since in the electron rest frame $s = (p+k)^2 = m^2 + 2m\omega = m^2(1+2\tilde{\omega})$, we can use $\tilde{\omega} = (s/m^2 - 1)/2$ to express σ in an explicit Lorentz invariant form.

Approximations for the non-relativistic and the ultra-relativistic limit are

$$\sigma = \sigma_{\text{Th}} \times \begin{cases} 1 - 2\tilde{\omega} + \mathcal{O}(\tilde{\omega}^2) & \text{for } \tilde{\omega} \ll 1\,, \\ \frac{3}{8\tilde{\omega}} \left(\ln(2\tilde{\omega}) + \frac{1}{2}\right) + \mathcal{O}(\tilde{\omega}^{-2}) & \text{for } \tilde{\omega} \gg 1\,, \end{cases} \tag{9.82}$$

where the Thomson cross section is given by $\sigma_{\text{Th}} = 8\pi\alpha^2/(3m^2)$. These approximations are shown together with the exact result in the left panel of Fig. 9.2. In the ultra-relativistic limit $s \gg m^2$, the total cross section for Compton scattering decreases as $\sigma \propto 1/s$. On the other hand, the differential cross section in the forward direction is constant. As a result, the relative importance of the forward region $\vartheta \sim 0$ increases for increasing s: While $\mathrm{d}\sigma/\mathrm{d}x$ is symmetric around $x = 0$ in the classical limit $\omega \to 0$, it becomes more and more asymmetric with a a shrinking peak around the forward region at $\vartheta \sim 0$, cf. the right panel of Fig. 9.2.

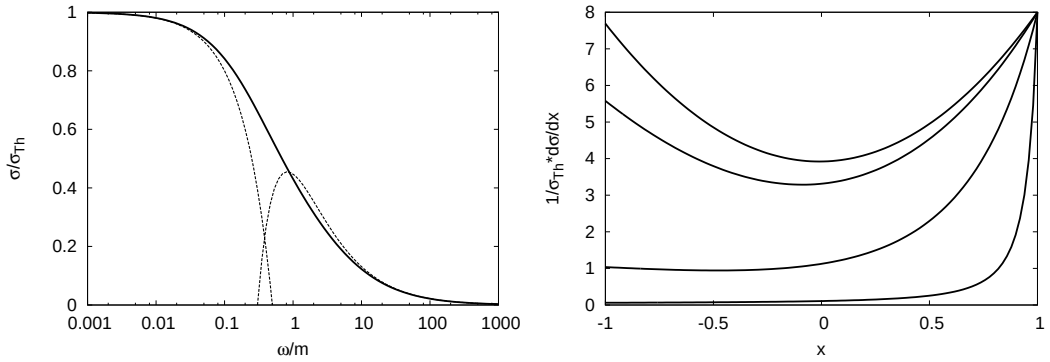


Figure 9.2.: Left: The total cross section $\sigma/\sigma_{\text{Th}}$ as function of $\tilde{\omega}$ together with the classical and ultra-relativistic limits given in Eq. (9.82). Right: The normalised differential cross section $\sigma_{\text{Th}}^{-1}\,\mathrm{d}\sigma/\mathrm{d}x$ as function of $x = \cos\vartheta$ for $\tilde{\omega} = 0.01$, 0.1, 1, and 10 (from top to down).

Crossing symmetry We noticed that the two amplitudes in Compton scattering can be transformed into each other replacing $\varepsilon \leftrightarrow \varepsilon^{*\prime}$ and $k \leftrightarrow -k'$. This is an example of a general symmetry of relativistic quantum field theories called crossing symmetry. Using the Feynman rules for in and out particles, it follows that matrix elements where an in-going particle is replaced by an out-going anti-particle or vice versa are related by the following substitutions,

- exchange the momentum $k \leftrightarrow -k'$;
- exchange particle and anti-particle wave functions; thus in momentum space, $1 \leftrightarrow 1$ for spinless particles, $\varepsilon \leftrightarrow \varepsilon^{*\prime}$ for spin-1 and $u \leftrightarrow v$ for fermions.
- multiply by -1 for each exchanged fermion pair.

The additional minus for fermions is required, because the spin sums of fermions and antifermions are related by

$$\sum_s u(p,s)\bar{u}(p,s) = (\not{p} + m) = -\left(\not{p}' - m\right) = -\sum_s v(p',s)\bar{v}(p',s)\,. \tag{9.83}$$

Note that this symmetry allows us to obtain the matrix elements of different processes: For instance, we can relate the processes $e^-e^+ \to \mu^-\mu^+$ with $e^-\mu^- \to e^-\mu^-$ and $\mu^-\mu^+ \to e^-e^+$.

In a more formal approach the crossing symmetry is derived not by relying on perturbation theory and the Feynman rules, but using the analytical properties of S-matrix elements in a relativistic quantum field theory. The LSZ reduction formula distinguishes in and out particles only by the sign of the momenta used in the Fourier transformation. If one can analytically continue the residue of a pole in an S-matrix element from p^0 to $-p^0$, then one converts the S-matrix for a particle with $\phi(p)$ into the one for an anti-particle with $\phi^*(-p)$. Remarkably, the basic properties of a relativistic quantum field theory, locality and causality, are sufficient to proof that this analytical continuation is possible.

Finally, note that the factor -1 for each exchanged external fermion pair implies a relative minus sign for diagrams connected by crossing which contribute to the same process. Thus there is a relative minus sign between e.g. the t and the u channel diagrams for $e^-e^- \to e^-e^-$ scattering.

9.3.2. Helicity method and polarized QED processes

The number of terms that have to be calculated grows using the trace method as $\sim n^2/2$ with the number n of diagrams. For large n, it should be therefore favorable to calculate the amplitude $\mathcal{A}(s_1, \ldots, s_f)$ for fixed polarizations of the external particles: The amplitude is a complex number and can be trivially squared. An efficient way to calculate polarized amplitudes uses helicity wave functions, an approach used also in most modern computer programs for the calculation of scattering processes.

Massless fermions We restrict our short introduction into helicity methods to massless particles. In the case of fermions, we know that then the use of Weyl spinors in the chiral representation is most convenient,

$$u_L(p) = \begin{pmatrix} \phi_L(p) \\ 0 \end{pmatrix} \quad \text{and} \quad u_R(p) = \begin{pmatrix} 0 \\ \phi_R(p) \end{pmatrix} . \tag{9.84}$$

We do not need to consider $v_{L,R}(p)$, since they correspond to particle spinors of opposite helicity, $u_{R,L}(p)$. Moreover, two out of the fours possible scalar products involving $u_{L,R}$ are zero for massless fermions,

$$\bar{u}_L(p)u_L(q) = \bar{u}_R(p)u_R(q) = 0 . \tag{9.85}$$

This motivates us to introduce a bracket notation for the helicity spinors as follows

$$\bar{u}_L(p) = \langle p \ , \quad \bar{u}_R(p) = [p \ , \quad u_L(p) = p] \ , \quad u_R(p) = p\rangle . \tag{9.86}$$

Then the only non-zero Lorentz-invariant spinor products are given by a pair of brackets of the same type,

$$\bar{u}_L(p)u_R(q) = \langle pq\rangle \qquad \text{and} \qquad \bar{u}_R(p)u_L(q) = [pq] . \tag{9.87}$$

We call the quantities on the RHSs *angle* and *square brackets*. Next we consider the tensor product of the spinors,

$$p\rangle[p = u_R(p)\bar{u}_R(p) = P_R\not{p} \ , \quad \text{and} \quad p][p = u_L(p)\bar{u}_L(p) = P_L\not{p} . \tag{9.88}$$

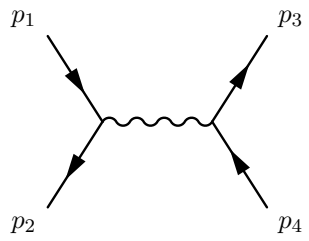


Figure 9.3.: Feynman diagrams for the process $e^-e^+ \to \mu^+\mu^-$.

These identities connect the massless spinors $p\rangle$ and $[p$ to the light-like four-vector p^μ.

We are now in position to derive some basic properties of the brackets. First, we can connect the two types of spinor products as

$$\langle pq\rangle = \bar{u}_L(p)u_R(q) = [\bar{u}_R(q)u_L(p)]^* = [qp]^* \,. \tag{9.89}$$

Multiplying then $p\rangle[p$ and $q]\langle q$ and taking the trace gives

$$\langle pq\rangle[qp] = \mathrm{tr}\{P_R \not{q} P_L \not{p}\} = \mathrm{tr}\{P_R \not{q} \not{p}\} = 2p\cdot q \,, \tag{9.90}$$

so that

$$|\langle pq\rangle|^2 = |[qp]|^2 = 2p\cdot q \ . \tag{9.91}$$

Next, we express the spinor products through Weyl spinors and use $u_R(p) = \mathrm{i}\sigma^2 u_L^*(p)$,

$$\langle pq\rangle = \phi_L^\dagger(p)\phi_R(q) = \phi^*_{La}(p)(\mathrm{i}\sigma^2)_{ab}\phi^*_{Lb}(q) \,. \tag{9.92}$$

Then the antisymmetry of $(\mathrm{i}\sigma^2)_{ab} = \varepsilon_{ab}$ implies

$$\langle pq\rangle = -\langle qp\rangle \quad \text{and} \quad [pq] = -[qp] \,. \tag{9.93}$$

Thus the brackets are square roots of the corresponding Lorentz vector products which are antisymmetric in their two arguments. Finally, we note that the Fierz identity applied to the sigma matrices (cf. with problem 8.15),

$$(\bar{\sigma}^\mu)_{ab}(\bar{\sigma}_\mu)_{cd} = 2(\mathrm{i}\sigma^2)_{ac}(\mathrm{i}\sigma^2)_{bd} \,, \tag{9.94}$$

allows the simplification of contracted spinor expressions,

$$\langle p\gamma^\mu q]\langle k\gamma_\mu \ell] = 2\langle pk\rangle[\ell q] \,, \qquad \langle p\gamma^\mu q][k\gamma_\mu \ell\rangle = 2\langle p\ell\rangle[kq] \,. \tag{9.95}$$

$e^-e^+ \to \mu^-\mu^+$ scattering It is now time to apply our new "bracket" formalism. We consider the tree-level amplitude for, e.g., $e_L^-(1)e_R^+(2) \to \mu_L^-(3)\mu_R^+(4)$ in QED, given by the single diagram shown in Fig. 9.3. considering all momenta as outgoing. Then the amplitude is

$$\mathrm{i}\mathcal{A} = (-\mathrm{i}e)^2 \frac{-\mathrm{i}}{q^2}\ \bar{u}_L(3)\gamma^\mu u_L(4)\,\bar{u}_L(2)\gamma_\mu u_L(1) \tag{9.96}$$

$$= \frac{\mathrm{i}e^2}{q^2}\langle 3\gamma^\mu 4]\langle 2\gamma_\mu 1] = \frac{2\mathrm{i}e^2}{q^2}\langle 32\rangle[14] \,, \tag{9.97}$$

where we have employed the Fierz identity (9.94) in the last step. Since $\langle 32\rangle$ and $[14]$ are both square roots of

$$u = (k_2 + k_3)^2 = (k_1 + k_4)^2 \,, \tag{9.98}$$

we can replace them by the Mandelstam invariant u. We consider the process in the cm frame of the e^+e^- pair. With $u = -2E^2(1 + \cos\vartheta)$, and $q^2 = s = 4E^2$, the amplitude becomes

$$|\mathrm{i}\mathcal{A}|^2 = e^4(1 + \cos\vartheta)^2 \,. \tag{9.99}$$

You should rederive this result using the more familiar trace formalism and compare the amount of algebra required in the two approaches (problem 9.6).

Massless gauge bosons In the next step, we incorporate massless gauge bosons as the photon into this framework. We claim that the polarization vectors of a massless vector boson in the final-state can be represented as

$$\epsilon_R^{*\mu}(k) = \frac{1}{\sqrt{2}} \frac{\langle r\gamma^\mu k]}{\langle rk\rangle} \,, \qquad \epsilon_L^{*\mu}(k) = -\frac{1}{\sqrt{2}} \frac{[r\gamma^\mu k\rangle}{[rk]} \,. \tag{9.100}$$

Here, k is the momentum of the vector boson, and r is a fixed lightlike 4-vector, called the reference vector, which is assumed to be not collinear with k.

Now we show that this definition makes sense: First, we note that the vectors satisfy $[\varepsilon_R^*(k)]^* = \varepsilon_L^*(k)$. One can also check that the polarization vectors are correctly normalized. Moreover, the Dirac equation, $\not{k}\, k] = 0$, guaranties that the polarization vectors (9.100) are transverse,

$$k_\mu \varepsilon_{R,L}^{*\mu}(k) = 0 \,. \tag{9.101}$$

Finally, we have to show that a change from one reference vector r to another lightlike vector s corresponds to a gauge transformation and thus does not affect physics. The change of a polarization vector under a change of reference vector $r \to s$ is

$$\varepsilon_R^{*\mu}(k;r) - \varepsilon_R^{*\mu}(k;s) = \frac{1}{\sqrt{2}} \left(\frac{\langle r\gamma^\mu k]}{\langle rk\rangle} - \frac{\langle s\gamma^\mu k]}{\langle sk\rangle} \right) \tag{9.102}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\langle rk\rangle\langle sk\rangle} \big\{ -\langle r\gamma^\mu k]\langle ks\rangle + \langle s\gamma^\mu k]\langle kr\rangle \big\} \,. \tag{9.103}$$

Now we use first the tensor products (9.88), and then the antisymmetry of the brackets,

$$\varepsilon_R^{*\mu}(k;r) - \varepsilon_R^{*\mu}(k;s) = \frac{1}{\sqrt{2}} \frac{1}{\langle rk\rangle\langle sk\rangle} \big\{ -\langle r\gamma^\mu \not{k} s\rangle + \langle s\gamma^\mu \not{k} r\rangle \big\} \tag{9.104}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\langle rk\rangle\langle sk\rangle} \big\{ \langle s(\not{k}\gamma^\mu + \gamma^\mu \not{k}) r\rangle \big\} \tag{9.105}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\langle rk\rangle\langle sk\rangle} \langle sr\rangle \; 2k^\mu \,. \tag{9.106}$$

In the last line, we have applied the Clifford algebra of Dirac matrices. Thus the difference of the polarization vectors induced by a change of the reference vector is a function proportional to the photon momentum,

$$\varepsilon_R^{*\mu}(k;r) - \varepsilon_R^{*\mu}(k;s) = f(r,s)k^\mu \,. \tag{9.107}$$

Contracted into an on-shell amplitude, $\mathcal{A} = \varepsilon^\mu \mathcal{A}_\mu$, current conservation implies that this expression vanishes. Thus we can use the most convenient reference vector s which can be chosen differently in any gauge-invariant subset of Feynman diagrams.

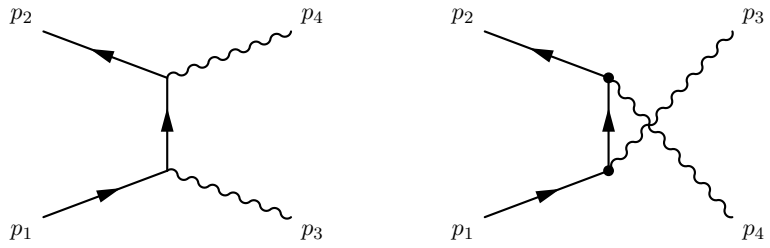


Figure 9.4.: Feynman diagrams for the process $e^-e^+ \to \gamma\gamma$.

$e^-e^+ \to \gamma\gamma$ scattering As second example, we consider now a scattering process with photons as external particles, $e^-e^+ \to \gamma\gamma$, as illustration for the use of the polarization vectors. We label the momenta as in Fig. 9.4, taking all momenta as outgoing.

Then the amplitude for this process is

$$\mathrm{i}\mathcal{A} = (-\mathrm{i}e)^2 \langle 2 \left(\not{\varepsilon}(4) \frac{\mathrm{i}(\not{2} + \not{4})}{s_{24}} \not{\varepsilon}(3) + \not{\varepsilon}(3) \frac{\mathrm{i}(\not{2} + \not{3})}{s_{23}} \not{\varepsilon}(4) \right) 1] \,, \tag{9.108}$$

where we use the shorthand $(\not{2} + \not{4})$ for $(\not{k}_2 + \not{k}_4)$ and define $s_{ij} = (i+j)^2$.

There are four possible choices for the photon polarizations. Exchanging the momenta 3 and 4 relates the cases $\gamma_R\gamma_L$ and $\gamma_L\gamma_R$, while parity connects $\gamma_R\gamma_R$ and $\gamma_L\gamma_L$. We start showing that the latter two amplitudes are zero in the massless limit we consider. Considering $\gamma_R\gamma_R$, we choose as reference vector $r = 2$ for both polarization vectors,

$$\varepsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{[2\gamma^\mu 3\rangle}{[23]} \,, \qquad \varepsilon^\mu(4) = \frac{1}{\sqrt{2}} \frac{[2\gamma^\mu 4\rangle}{[24]} \,. \tag{9.109}$$

Inserting the polarisation vectors into Eq. (9.108), we obtain using the Fierz identity (9.94)

$$\langle 2\gamma^\mu \varepsilon_\mu(4) \propto \langle 2\gamma^\mu [2\gamma_\mu 4\rangle = 2\langle 22\rangle [4 = 0 \,. \tag{9.110}$$

In the last step, we used the antisymmetry of the brackets, $\langle 22\rangle = 0$. A similar cancellation occours with $\varepsilon(3)$ and hence the entire matrix element vanishes. Party implies then that the amplitude $\mathcal{A}(\gamma_L\gamma_L)$vanishes too. Alternatively, we can show the same cancellation using $r = 1$ in both polarization vectors.

Next we compute the amplitude for the case $\gamma_R\gamma_L$, choosing

$$\varepsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{[2\gamma^\mu 3\rangle}{\langle 23\rangle} \quad \text{and} \quad \varepsilon^\mu(4) = -\frac{1}{\sqrt{2}} \frac{[1\gamma^\mu 4\rangle}{[14]} \,. \tag{9.111}$$

Then the second diagram in Fig. 9.4 vanishes because of (9.110). Using the Fierz identity, the first diagram results in

$$\mathrm{i}\mathcal{A} = \frac{-\mathrm{i}e^2}{s_{24}} \frac{2 \cdot 2}{(-2)\langle 23\rangle [14]} \langle 24\rangle [1(\not{2} + \not{4})2\rangle [31] \,. \tag{9.112}$$

Now we use the Dirac equation, $\not{2}|2\rangle = 0$, and replace the vector $\not{4}_L$ by an angle bracket,

$$\mathrm{i}\mathcal{A} = \frac{2\mathrm{i}e^2}{s_{13}\langle 23\rangle[14]}\langle 24\rangle[14]\langle 42\rangle[31] \tag{9.113}$$

$$= \frac{2\mathrm{i}e^2}{\langle 13\rangle[31]\langle 23\rangle}\langle 24\rangle\langle 42\rangle[31] = 2\mathrm{i}e^2\frac{\langle 24\rangle^2}{\langle 23\rangle^2}\,. \tag{9.114}$$

Here we used also four-momentum conservation to convert s_{24} into such brackets that we end up with an expression containing only Mandelstam variables, $s_{23} = u$, $s_{13} = s_{24} = t$,

$$|\mathrm{i}\mathcal{A}|^2 = 4e^4\frac{t}{u} = 4e^4\frac{1-\cos\vartheta}{1+\cos\vartheta}\,. \tag{9.115}$$

This short introduction into helicity methods convinced you hopefully of their efficiency. The advantage of this method over the traditional trace method increases with the number of diagrams involved, since the step $\mathcal{A} \to |\mathcal{A}|^2$ is trivial in this approach. Massive fermions can be treated using the helicity states (A.16), and efficient extensions to massive gauge bosons exist.

9.4. Soft photons

The addition of an additional vertex introduces typically a factor $\alpha/\pi \sim 0.2\%$ into a QED cross section. Thus one may hope that perturbation theory in QED converges, at least initially, reasonably fast. An exception to this rule is the emission of an additional soft or collinear photon from an external line shown in Fig. 9.5. The denominator of the additional propagator goes for $k \to 0$ to

$$\frac{1}{(p+k)^2 - m^2} \to \frac{1}{2p\cdot k} \sim \frac{1}{2E\omega(1-\cos\vartheta)}\,, \tag{9.116}$$

where we assumed $|\boldsymbol{p}| \gg m$ in the last step. Hence the denominator can blow up in two different limits: Firstly, in case of emission of soft photons, $\omega \to 0$. Secondly, in case of collinear emission of photons, $\vartheta \to 0$, if the mass of the emitting particle can be neglected. We have seen in the example of Compton scattering that both soft and colinear emission correspond to the classical limit.

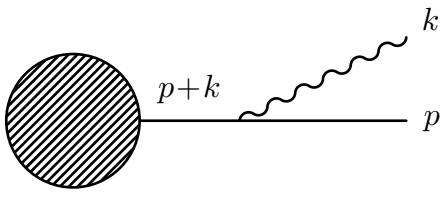


Figure 9.5.: Emission of an additional soft or collinear photon from an external line in the final state.

Universality and factorization The fact that a photon sees in the soft limit $k \to 0$ a classical current should lead to considerable simplifications: In particular, interference effects should disappear and the amplitude $\mathcal{A}_{n+1}$ for the emission of an additional soft photon should factories into an universal factor $\varepsilon^\mu S_\mu$ and the amplitude $\mathcal{A}_n$ for the original process.

Let us start considering the emission of a soft photon by a spinless particle. If a scalar in the final state with momentum p and charge q emits a photon with momentum $k \to 0$, then[3]

$$\mathcal{A}_{n+1} = q\,\frac{\varepsilon_\mu(2p^\mu + k^\mu)}{(p-k)^2 - m^2 + \mathrm{i}\varepsilon}\,\mathcal{A}_n \to -q\,\frac{\varepsilon\cdot p}{p\cdot k - \mathrm{i}\varepsilon}\,\mathcal{A}_n\,. \tag{9.117}$$

For the emission of a soft photon from an initial state particle, the corresponding factor is $+q\,\varepsilon\cdot p/(p\cdot k + \mathrm{i}\varepsilon)$. In the case of an internal line, in general no factor $(p\cdot k)^{-1}$ appears for $k \to 0$, since the virtual particle is off-shell.

For a spin-1/2 particle in the initial state, the emission of a soft photon adds the factor

$$q\,\frac{\varepsilon_\mu \bar{u}(p,s)\gamma^\mu(\not{p} + \not{k} - m)}{(p-k)^2 - m^2 + \mathrm{i}\varepsilon} \to -q\,\frac{\varepsilon_\mu \bar{u}(p,s)\gamma^\mu(\not{p} - m)}{-2p\cdot k + \mathrm{i}\varepsilon} \tag{9.118}$$

to the amplitude $\mathcal{A}_n$. Now we replace $\not{p} + m$ by the spin sum $\sum_s u(p,s)\bar{u}(p,s)$, and use

$$\bar{u}(p,s)\gamma^\mu u(p,s') = 2E_p\,\frac{p^\mu}{p^0}\delta_{s,s'} = 2p^\mu\delta_{s,s'}\,. \tag{9.119}$$

This relation can be checked by direct calculation, or by noting that the current $j^\mu = \bar{u}(p,s)\gamma^\mu u(p,s)$ should become $j^\mu = (\rho, \rho\boldsymbol{v})$ in the classical limit $k \to 0$. Thus we obtain the same universal factor describing the emission of a soft photon,

$$S^\mu = -q\,\frac{p^\mu}{p\cdot k - \mathrm{i}\varepsilon}\,, \tag{9.120}$$

as in the case of a scalar. Moreover, we confirmed that the amplitude indeed factorizes, $\mathcal{A}_{n+1} = \varepsilon_\mu S^\mu \mathcal{A}_n \equiv \varepsilon_\mu A^\mu_{n+1}$. If we allow for the emission of m soft photons from external particles with charge q_i, then

$$\mathcal{A}^{\mu_1\cdots\mu_m}_{n+m} \to \sum_{i=1}^{m}\frac{s_i q_i\, p^{\mu_m}}{p\cdot k + \mathrm{i}s_i\varepsilon}\,\mathcal{A}_n\,, \tag{9.121}$$

where the signs are $s_i = -1$ for an initial and $s_i = +1$ for a final state particle.

We have seen that the polarization vector $\varepsilon_\mu(k)$ of a photon does not transform as a four-vector, cf. Eq. (7.23), but acquires a term proportional to k^μ. As we exploited already at various places, amplitudes containing polarization vectors $\varepsilon_\mu(k)$ of external photons have to vanish therefore when contracted with k^μ. Thus Eq. (9.121) implies in the limit $k \to 0$

$$k_{\mu_j}\mathcal{A}^{\mu_1\cdots\mu_m}_{n+m} \to \sum_{i=1}^{m} s_i q_i\,\mathcal{A}_n = 0\,. \tag{9.122}$$

The prefactor of $\mathcal{A}_n$ is the total charge in the final state minus the total charge in the initial state. In order to obtain a Lorentz invariant matrix element for the soft emission of massless spin-1 particles, we have therefore to require that such particles couple to a conserved charge.

$$\sum_i q_i = \sum_f q_f\,.$$

[3] We use the Feynman rule for a $\phi\phi A^\mu$ vertex derived in problem 7.7.

Thus Lorentz invariance is sufficient to guaranty the conservation of the electromagnetic current in the low-energy limit. While this argument does not rely on gauge invariance, it tells us nothing about the behavior of "hard" photons.

Spin $s > 1$ We can apply the same line of arguments to the emission of massless particles with spin $s > 1$. In the case of gravitons, $s = 2$, one finds that the universal factor becomes

$$S^{\mu\nu} = -f \, \frac{p^\mu p^\nu}{p \cdot k - \mathrm{i}s\varepsilon} \,, \tag{9.123}$$

where f denotes the coupling to the graviton. Requiring again that an amplitude containing the polarization tensor $\varepsilon_{\mu\nu}(k)$ of external gravitons vanishes when contracted with k^μ gives the constraint

$$k_{\mu_j} \mathcal{A}_{n+m}^{\mu_1 \cdots \mu_m} \to \sum_{i=1}^{m} s_i f_i p_i^\nu \, \mathcal{A}_n = 0 \,. \tag{9.124}$$

Now the sum $\sum_i f_i p_i^\nu$ is conserved. For $f_i \neq f_j$, a linear combination of the individual four-momenta other than the total four-momentum would be conserved in the scattering process—a condition which is not possible to satisfy in a non-trivial scattering process. Thus we have to conclude that any massless $s = 2$ particle has a *universal* coupling to all types of particle. Recalling from (5.20) the form of the stress tensor, $T^{\mu\nu} = 2N^2 p^\mu p^\nu$, we see moreover that a massless spin-2 particles couples with universal strength to the stress tensor, $S^{\mu\nu} \propto T^{\mu\nu}$. This result can be viewed as the basis of the weak equivalence principle. Going further to $s = 3$, the universal factor becomes $S^{\mu\nu\lambda} \propto p^\mu p^\nu p^\lambda$, requiring that sums quadratic in the momenta, $\sum_i \tilde{f}_i p_i^\nu p_i^\mu$, are conserved. This is not possible for any scattering angles except $\vartheta = 0$ and $180°$, and thus no consistent theory of interacting massless particles with spin $s \geq 2$ is possible.

Bremsstrahlung We discuss now as a concrete example the case of bremsstrahlung, i.e. the emission of a real photon in the scattering of a charged particle in the Coulomb field $A^0 = -Ze/(4\pi|\boldsymbol{x}|)$ of a static charge. The S-matrix element of this process is

$$\mathrm{i}S_{fi} = 2\pi\delta(E' + \omega - E') \, \frac{-Ze^3}{|\boldsymbol{q}|^2} \bar{u}(p') \left[\not{\varepsilon} \, \frac{\not{p}' + \not{k} + m}{2p' \cdot k} \, \gamma^0 + \gamma^0 \, \frac{\not{p} - \not{k} + m}{-2p \cdot k} \not{\varepsilon} \right] u(p) \,, \tag{9.125}$$

where $1/|\boldsymbol{q}|^2$ is the Fourier transform of A^0. Note that the external field breaks translation invariance and the momentum is not conserved. We commute now $\not{p}$ and $\not{p}'$,

$$\mathrm{i}S_{fi} \propto e^2 \bar{u}(p') \left[\frac{2\varepsilon \cdot p' - (\not{p}' - m)\not{\varepsilon} + \not{k}\not{\varepsilon}}{2p' \cdot k} \, \gamma^0 + \gamma^0 \, \frac{2\varepsilon \cdot p - \not{\varepsilon}(\not{p} - m) + \not{k}\not{\varepsilon}}{-2p \cdot k} \right] u(p) \,, \tag{9.126}$$

such that we can use in the next step the Dirac equation. Neglecting additionally in the soft limit the $\not{k}$ term in the numerator, we find

$$\mathrm{i}S_{fi} \propto e^2 \bar{u}(p') \gamma^0 u(p) \, \left[\frac{\varepsilon \cdot p'}{p' \cdot k} - \frac{\varepsilon \cdot p}{p \cdot k} \right] . \tag{9.127}$$

As we have shown in the previous paragraph in general, the amplitude factorizes into the amplitude describing the "hard" process and the universal correction term. The latter consists

of the two terms expected for the emission of a soft photon from an initial line with momentum p and a final line with momentum p'. The probability $\mathcal{P}$ for the emission of an additional soft photon is given integrating the square bracket over the phase space,

$$\mathrm{d}P_{n+1} = \frac{\mathrm{d}\sigma_{n+1}}{\mathrm{d}\sigma_n} = \left[\frac{\varepsilon\cdot p'}{p'\cdot k} - \frac{\varepsilon\cdot p}{p\cdot k}\right]^2 \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \propto \frac{\mathrm{d}\omega_k}{\omega_k}\,. \tag{9.128}$$

This probability diverges for $\omega \to 0$ and therefore the process is called infrared (IR) divergent.

The resolution to this IR problem lies in the fact that soft photons with energy below the energy resolution E_{th} of the used detector are not detectable. Therefore the emission of n real soft photons with $E < E_{\mathrm{th}}$ is indistinguishable from the scattering process including virtual photons and thus these cross sections should be added. The IR divergences in the real and virtual corrections cancel, leading to a finite result for the combined cross section. We will discuss a detailed example for how this cancellation works in chapter 18.3.

9.A. Appendix: Decay widths and cross sections

We establish first the connection between the normalised transition matrix element $\mathcal{M}$ and the Feynman amplitude $\mathcal{A}$, where the normalisation factors of external particles are omitted. Then we derive decay widths and cross sections describing $1 \to n$ and $2 \to n$ processes.

Normalisation We have split the scattering operator S into a diagonal part and the transition operator T, $S = 1 + \mathrm{i}T$. Taking matrix elements, we obtain

$$S_{fi} = \delta_{fi} + (2\pi)^4\,\delta^{(4)}(P_i - P_f)\mathrm{i}\mathcal{M}_{fi} \tag{9.129}$$

where we set also $T_{fi} = (2\pi)^4\,\delta^{(4)}(P_i - P_f)\mathcal{M}_{fi}$.

The Feynman amplitude $\mathcal{A}$ neglects all normalisation factors of external particles, while the matrix element T_{fi} defined by (9.38) and thus $\mathcal{M}_{fi}$ contains a factor $N_{\boldsymbol{k}}$ for each external particle. Thus the transition between the matrix element $\mathcal{M}_{fi}$ and the Feynman amplitude $\mathcal{A}$ for a process with n particles in the initial and m in the final states is given by

$$\mathcal{M}_{fi} = \prod_{i=1}^{n}(2E_iV)^{-1/2}\prod_{f=1}^{m}(2E_fV)^{-1/2}\mathcal{A}_{fi}\,. \tag{9.130}$$

Here we changed also to a finite normalisation volume, $2E_{\boldsymbol{p}}(2\pi)^3 \to 2E_{\boldsymbol{p}}V$ what makes defining decay widths and cross sections easier.

9.A.1. Decay widths

We consider the decay of a particle into n particles in the final state. Squaring the scattering amplitude S_{fi} for $i \neq f$ using $(2\pi)^4\,\delta^{(4)}(0) = VT$ gives as differential transition probability

$$\mathrm{d}W_{fi} = (2\pi)^4\,\delta^{(4)}(P_i - P_f)VT|\mathcal{M}_{fi}|^2\prod_{f=1}^{n}\frac{V\mathrm{d}^3p_f}{(2\pi)^3}\,. \tag{9.131}$$

The decay rate $\mathrm{d}\Gamma$ is the transition probability per time,

$$\mathrm{d}\Gamma_{fi} = \lim_{T\to\infty}\frac{\mathrm{d}W_{fi}}{T} = (2\pi)^4\,\delta^{(4)}(P_i - P_f)V|\mathcal{M}_{fi}|^2\prod_{f=1}^{n}\frac{V\mathrm{d}^3p_f}{(2\pi)^3}\,. \tag{9.132}$$

Going over to the Feynman amplitude $\mathcal{A}$ eliminates the volume factors V,

$$\mathrm{d}\Gamma_{fi} = (2\pi)^4 \delta^{(4)}(P_i - P_f)\, \frac{1}{2E_i}\, |\mathcal{A}_{fi}|^2 \prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2E_f (2\pi)^3}\,. \tag{9.133}$$

Moreover, the phase space integrals in the final state are now Lorentz invariant, $\mathrm{d}^3 p_f/(2E_f)$. Introducing the n-particle phase space volume

$$\mathrm{d}\Phi^{(n)} = (2\pi)^4\, \delta^{(4)}(P_i - P_f)\, \prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2E_f (2\pi)^3}\,, \tag{9.134}$$

the decay rate becomes

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2E_i}\, |\mathcal{A}_{fi}|^2\, \mathrm{d}\Phi^{(n)}\,. \tag{9.135}$$

Since both $|\mathcal{A}_{fi}|^2$ and the phase space $\mathrm{d}\Phi^{(n)}$ are Lorentz invariant, the decay rate $\Gamma \propto 1/E_i = 1/(\gamma_i m_i)$ shows explicitly the time dilation effect for a moving particle.

Two-particle decays We evaluate the two particle phase space $\mathrm{d}\Phi^{(2)}$ in the rest frame of the decaying particle,

$$\mathrm{d}\Phi^{(2)} = (2\pi)^4\, \delta(M - E_1 - E_2)\, \delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\, \frac{\mathrm{d}^3 p_1}{2E_1 (2\pi)^3}\, \frac{\mathrm{d}^3 p_2}{2E_2 (2\pi)^3} \tag{9.136}$$

We perform the integration over $\mathrm{d}^3 p_1$ using the momentum delta function. In the resulting expression,

$$\mathrm{d}\Phi^{(2)} = \frac{1}{(2\pi)^2}\, \frac{1}{4E_1 E_2}\, \delta(M - E_1 - E_2)\, \mathrm{d}^3 p_2\,, \tag{9.137}$$

E_1 is now a function of p_2, $E_1^2 = p_2^2 + m_1^2$. Introducing spherical coordinates, $d^3 p_2 = \mathrm{d}\Omega p_2^2 \mathrm{d}p_2$,

$$\mathrm{d}\Phi^{(2)} = \frac{1}{(2\pi)^2} \mathrm{d}\Omega \int_0^\infty \delta(M - E_1 - E_2)\, \frac{p_2^2 \mathrm{d}p_2}{4E_1 E_2}\,, \tag{9.138}$$

and evaluating the delta function with $M - E_1 - E_2 = M - x$ and $\mathrm{d}p_2/\mathrm{d}x = p_2 x/(E_1 E_2)$ gives

$$\mathrm{d}\Phi^{(2)} = \frac{|p'_{\mathrm{cms}}|}{4\pi^2} \mathrm{d}\Omega\,, \tag{9.139}$$

where

$$p_{\mathrm{cms}}^2 = \frac{\lambda(s, m_1^2, m_2^2)}{4s} = \frac{1}{4M^2} \left[M^2 - (m_1 + m_2)^2\right] \left[M^2 - (m_1 - m_2)^2\right] \tag{9.140}$$

is the cms momentum of the two final state particles. The Kibble function $\lambda(x, y, z)$ satisfies

$$\lambda(x, y, z) = \left[(x^2 + y^2 + z^2) - 2xy - 2yz - 2xz\right]^{1/2} = \left[x^2 - (\sqrt{y} + \sqrt{z})^2\right]^{1/2} \left[x^2 - (\sqrt{y} - \sqrt{z})^2\right]^{1/2}\,. \tag{9.141}$$

Three-particle decays The three particle phase space $\mathrm{d}\Phi^{(3)}$ is

$$\mathrm{d}\Phi^{(3)} = (2\pi)^4\, \delta(M - E_1 - E_2 - E_3)\, \delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)\, \frac{\mathrm{d}^3 p_1}{2E_1 (2\pi)^3}\, \frac{\mathrm{d}^3 p_2}{2E_2 (2\pi)^3}\, \frac{\mathrm{d}^3 p_3}{2E_3 (2\pi)^3}\,. \tag{9.142}$$

We can use again the momentum delta function to perform the integration over $\mathrm{d}^3 p_3$,

$$\mathrm{d}\Phi^{(3)} = \frac{1}{(2\pi)^5} \delta(M - E_1 - E_2 - E_3)\, \frac{\mathrm{d}^3 p_1 \mathrm{d}^3 p_2}{8E_1 E_2 E_3}\,, \tag{9.143}$$

To proceed we have to know the dependence of the matrix element on the integration variables. If there is no preferred direction (either for scalar particles or after averaging over spins), we obtain

$$\begin{aligned} \mathrm{d}\Phi^{(3)} &= \frac{1}{8(2\pi)^5} \frac{4\pi p_1^2 \mathrm{d}p_1 \, 2\pi \mathrm{d}\cos\vartheta \mathrm{d}p_2}{E_1 E_2 E_3} \, \delta(M - E_1 - E_2 - E_3) \\ &= \frac{1}{32\pi^3} \frac{p_1 \mathrm{d}p_1 \, (p_1 p_2 \mathrm{d}\cos\vartheta)(p_2 \mathrm{d}p_2)}{E_1 E_2 E_3} \, \delta(M - E_1 - E_2 - E_3) \,. \end{aligned} \tag{9.144}$$

We rewrite next the momentum integrals as energy integrals. The energy-momentum relation $E_i^2 = m_i^2 + p_i^2$ gives $E_i \mathrm{d}E_i = p_i \mathrm{d}p_i$ for $i = 1, 2$. Furthermore,

$$E_3^2 = (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 + m_3^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\vartheta + m_3^2 \tag{9.145}$$

and thus $E_3 \mathrm{d}E_3 = p_1 p_2 \mathrm{d}\cos\vartheta$ for fixed $\boldsymbol{p}_1, \boldsymbol{p}_2$. Performing the angular integral, we obtain

$$\mathrm{d}\Phi^{(3)} = \frac{1}{32\pi^3} \, \mathrm{d}E_1 \mathrm{d}E_2 \mathrm{d}E_3 \delta(M - E_1 - E_2 - E_3) \,, \tag{9.146}$$

and finally

$$\mathrm{d}\Phi^{(3)} = \frac{1}{32\pi^3} \, \mathrm{d}E_1 \mathrm{d}E_2 \,. \tag{9.147}$$

The last step is only valid, if the argument of the delta function is non-zero. Thus the remaining task is to determine the boundary of the integration domain. Introducing the invariant mass of the pair (i, j)

$$m_{23}^2 = (p - p_1)^2 = (p_2 + p_3)^2 = M^2 - 2ME_1 \tag{9.148}$$

$$m_{13}^2 = (p - p_2)^2 = (p_1 + p_3)^2 = M^2 - 2ME_2 \tag{9.149}$$

$$m_{12}^2 = (p - p_3)^2 = (p_1 + p_2)^2 = M^2 - 2ME_3 \,, \tag{9.150}$$

where the last column is valid in the rest frame of the decaying particle with mass M. With $E_1 + E_2 + E_3 = M$ one finds $m_{23}^2 + m_{13}^2 + m_{12}^2 = M^2 + m_2^2 + m_3^2$. Therefore only two out of the three variables are independent. Let us choose m_{23}^2 and m_{13}^2 as integration variables, with m_{23}^2 as the outer one. Then

$$(m_2 + m_3)^2 \leq m_{23}^2 \leq (M - m_1)^2 \,. \tag{9.151}$$

For a given value of m_{23}^2, we have now to determine the allowed range of m_{13}^2. Inserting energy and momentum conservation into $E_3^2 = p_3^2 + m_3^2$, we obtain

$$(M - E_1 - E_2)^2 = m_3^2 + \boldsymbol{p}_1^2 + \boldsymbol{p}_2^2 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 \,. \tag{9.152}$$

The extrema correspond to

$$\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = \pm p_1 p_2 = \pm\sqrt{(E_1^2 - m_1^2)(E_2^2 - m_2^2)} \,. \tag{9.153}$$

9.A.2. Cross sections

We consider now the interaction of two particles in the rest system of either particle 1 or 2. For simplicity, we consider two uniform beams. They may produce n final state particles. The total number of such scatterings is

$$\mathrm{d}N \propto v_{\mathrm{M\o l}} n_1 n_2 \mathrm{d}V \mathrm{d}t \tag{9.154}$$

where n_i is the density of particle type $i = 1, 2$. The Møller velocity $v_{\mathrm{M\o l}}$ is a quantity which coincides in the rest frame of particle 1 or 2 with the norm of $|\boldsymbol{v}_2|$ and $|\boldsymbol{v}_1|$, respectively. Therefore it is often denoted simply as their relative velocity $\boldsymbol{v}_{\mathrm{rel}}$. The proportionality constant in (9.154) has the dimension of an area and is called cross section σ. We define in the rest system of either particle 1 or 2

$$\mathrm{d}N = \sigma v_{\mathrm{M\o l}} \, n_1 n_2 \, \mathrm{d}V \mathrm{d}t \,, \tag{9.155}$$

while we set in an arbitrary frame

$$\mathrm{d}N = An_1n_2\mathrm{d}V\mathrm{d}t\,. \tag{9.156}$$

We determine now A. Since both $\mathrm{d}N$ and $\mathrm{d}V\mathrm{d}t = \mathrm{d}^4x$ are Lorentz invariant, the expression An_1n_2 has to be Lorentz invariant too. The densities transform as

$$n_i = n_{i,0}\gamma = n_{i,0}\frac{E_i}{m_i}\,, \tag{9.157}$$

and thus the expression

$$A\,\frac{E_1E_2}{p_1\cdot p_2} \tag{9.158}$$

is also Lorentz invariant. In the rest system of particle 1, it becomes

$$A\,\frac{E_1E_2}{E_1E_2 - \boldsymbol{p}_1\boldsymbol{p}_2} = A = \sigma v_{\mathrm{M\o l}}\,. \tag{9.159}$$

Thus we found that A in an arbitrary frame is given by

$$A = \sigma v_{\mathrm{M\o l}}\,\frac{p_1\cdot p_2}{E_1E_2}\,. \tag{9.160}$$

A more handy expression for A is obtained as follows: In the rest frame 1, we have

$$p_1\cdot p_2 = m_1E_2 = m_1\,\frac{m_2}{\sqrt{1-v_{\mathrm{M\o l}}^2}}\,. \tag{9.161}$$

Thus the Møller velocity is given in general by

$$v_{\mathrm{M\o l}} = \sqrt{1-\frac{m_1^2m_2^2}{(p_1\cdot p_2)^2}}\,. \tag{9.162}$$

Since this expression is Lorentz invariant, we see that the notion of the Møller velocity as relative velocity is misleading.

Next we define the flux factor

$$I \equiv v_{\mathrm{M\o l}}\,p_1\cdot p_2 = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}\,. \tag{9.163}$$

Inserting (9.160) together with the definition of I into (9.156), we obtain

$$\mathrm{d}N = \sigma\,\frac{I}{E_1E_2V}\,(n_1V)(n_2\mathrm{d}V)\mathrm{d}t\,. \tag{9.164}$$

Here, we re-grouped the terms to make clear that after integration the total number N of scattering events is proportional to the number $N_1 = n_1V$ and $N_2 = \int n_2\mathrm{d}V$ of particles of type 1 and 2, respectively. The number N of scattering events per time and per particles 1 and 2 is however simply the transition probability per time,

$$\frac{\mathrm{d}W_{fi}}{T} = \frac{\mathrm{d}N}{N_1N_2T} = \mathrm{d}\sigma\,\frac{I}{E_1E_2V}\,. \tag{9.165}$$

Inserting the expression (9.132) for $\mathrm{d}W_{fi}$, we find

$$\mathrm{d}\sigma = \frac{E_1E_2V^2}{I}\,(2\pi)^4\,\delta^{(4)}(P_i-P_f)|\mathcal{M}_{fi}|^2\prod_{f=1}^{n}\frac{V\mathrm{d}^3p_f}{(2\pi)^3}\,. \tag{9.166}$$

Changing from the normalised matrix element $\mathcal{M}$ to the Feynman amplitude $\mathcal{A}$ introduces a factor $(2E_1V)^{-1}(2E_2V)^{-1}$ for the initial state and $\prod_f (2E_fV)^{-1}$ for the final state. Thus the arbitrary normalisation volume V cancels and we obtain

$$\mathrm{d}\sigma = \frac{1}{4I}\,(2\pi)^4\,\delta^{(4)}(P_i - P_f)|\mathcal{A}_{fi}|^2 \prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2E_f(2\pi)^3} = \frac{1}{4I}\,|\mathcal{A}_{fi}|^2\,\mathrm{d}\Phi^{(n)} \tag{9.167}$$

with the final state phase space $\mathrm{d}\Phi^{(n)}$. The three pieces composing the differential cross section, the flux factor I, the Feynman amplitude $\mathcal{A}$, and the final state phase space $\mathrm{d}\Phi^{(n)}$, are each Lorentz invariant. Finally, we note that a symmetry factor $S = 1/n!$ has to be added to the total decay width or cross section, if there are n identical particles in the final state.

2–2 scattering For a $1+2 \to 3+4$ scattering process, it is useful to introduce Mandelstam variables s, t, and u as

$$s = (p_1 + p_2)^2 = (p_2 + p_4)^2\,, \tag{9.168}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2\,, \tag{9.169}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2\,. \tag{9.170}$$

Since $s + t + u = \sum_{i=1}^{4} m_i^2$, the scattering amplitude $\mathcal{A}$ depends only on two variables, e.g $\mathcal{A}(s,t)$. In the cms, the flux factor becomes

$$I^2 = (p_1 \cdot p_2)^2 - m_1^2 m_2^2 = p_{\mathrm{cms}}^2 (E_1 + E_2)^2 = p_{\mathrm{cms}}\sqrt{s}\,. \tag{9.171}$$

Adding the expression for the 2-particle phase space gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\,\frac{p'_{\mathrm{cms}}}{p_{\mathrm{cms}}}\,|\mathcal{A}_{fi}|^2\,. \tag{9.172}$$

Here the cm momentum of the initial state is $p_{\mathrm{cms}}^2 = \lambda(s, m_1^2, m_2^2)/(4s)$, while $p'^2_{\mathrm{cms}} = \lambda(s, m_3^2, m_4^2)/(4s)$ is the one of the final state. Using as variable the momentum transfer t, the differential cross section becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s p_{\mathrm{cms}}^2}\,|\mathcal{A}_{fi}|^2\,, \tag{9.173}$$

where the allowed range of t has to be determined from Eq. (9.169) and $-1 \le \cos\vartheta \le 1$.

The optical theorem connects the imaginary part of the forward amplitude $\Im T_{ii}$ with the total cross section as

$$\sigma_{\mathrm{tot}} = \frac{\Im T_{ii}}{\lambda^{1/2}(s, m_1^2, m_2^2)}\,. \tag{9.174}$$

Summary

The LSZ reduction formula shows that S-matrix elements are obtained from connected Green functions by a replacement of the propagators on external lines with the corresponding wavefunctions times the wavefunction renormalisation constant $\sqrt{Z}$. Cross sections and decay rates are calculated from squared Lorentz-invariant Feynman amplitudes $\mathcal{A}$, the Lorentz-invariant final state phase space $\mathrm{d}\Phi^{(n)}$ and a flux factor I. Squared Feynman amplitudes can be obtained using "Casimir's trick". If the number of diagrams increases, it is more convinient to calculate directly the amplitude using helicity methods. The amplitude for the emission of additional soft particles factorises in the amplitude of the hard process and an

universel factor. Lorentz invariance requires then that a massless spin-1 particle couples in the low-energy limit to a conserved charge, while a massless spin-2 particle has to couples with universal strength to the energy-momentum stress tensor.

Further reading

The optical theorem and its connection to cut diagrams is discussed in more detail in[68]. The LSZ formula for particles with spin $s > 0$ is derived e.g. in[35], while[7] derive S-matrix elements defining a new functional $Z'[J]$ with the correct boundary conditions. For additional information about the helicity formalisms see Haber[38] and Peskin[56] from which our examples are taken.[5] provides a tutorial for several software tools useful for the calculation of scattering processes. The discussion of soft photon emission follows closely the one in Weinberg, for an introduction see[73].

Problems

9.1 Optical theorem.
Consider two light scalar fields ϕ_1 and ϕ_2 with mass m coupled to one heavy scalar Φ with mass $M > 2m$, $\mathscr{L} = \mathscr{L}_0 + g\phi_1\phi_2\Phi$ where $\mathscr{L}_0$ is the free Lagrangian. a.) Calculate the width Γ of the decay $\Phi \to \phi_1\phi_2$. b.) Draw the Feynman diagram(s) and write down the Feynman amplitude $\mathrm{i}\mathcal{A}$ for the scattering process $\phi_1(p_1)\phi_2(p_2) \to \phi_1(p_1')\phi_2(p_2')$. What is your interpretation of the behaviour of the amplitude for $s = (p_1 + p_2)^2 \to M^2$? c.) Consider the one-loop correction $\mathrm{i}\Sigma$ to the mass of Φ. Write down $\mathrm{i}\Sigma$ first for an arbitrary momentum p of the external particle Φ, then for its rest frame, $p = (M, 0)$. Find the poles of the integrand and use the theorem of residues to perform the q^0 part of the loop integral. Finally, use the identity (9.14) to find the imaginary part of the amplitude. You should find $M\Gamma(\Phi \to \phi_1\phi_2) = \mathrm{Im}\Sigma$, a special case of the optical theorem.

9.2 Fermion lines.
Calculate in the trace formalism $|\overline{\mathcal{A}}|^2$ for the processes $e^+e^- \to e^+e^-$ and $e^+e^- \to 2\gamma$.

9.3 Gauge invariance of Compton amplitude.
Show that the Compton amplitude $\mathcal{A} = \varepsilon_\mu(k)\varepsilon'_\nu(k')\mathcal{A}^{\mu\nu}$ satisfies $\varepsilon_\mu(k)\mathcal{A}^{\mu\nu} = \varepsilon'_\nu(k')\mathcal{A}^{\mu\nu} = 0$.

9.4 Identities for gamma matrices.
Evaluate $\gamma^\mu \not{a} \gamma_\mu$, $\gamma^\mu \not{a} \not{b} \gamma_\mu$, $\mathrm{tr}[\not{a}\not{b}]$, and $\mathrm{tr}[\not{a}\not{b}\not{c}\not{d}]$.

9.5 Polarised Thomson cross section.
Derive from (9.78) the cross section for linearly polarised photons.

9.6 Polarised cross sections in the trace formalism.
Calculate the squared matrix element of the process $e_L^-(1)e_R^+(2) \to \mu_L^-(3)\mu_R^+(4)$ using the trace formalism. (Hint: Use the spin projection operators (8.65) to obtain the required polarisations.)

9.7 Muon decay ♥.
Derive the differential decay rate of the process $\mu^-(p_1) \to e^-(p_4)\nu_\mu(p_3)\bar{\nu}_e(p_2)$, via the exchange of a W-boson described by the vertex $-\frac{\mathrm{i}g}{\sqrt{2}}\bar{f}\gamma_\mu(1 - \gamma^5)f\,W^\mu$. You can neglect lepton masses.

9.8 Emission of two soft photons.
Consider the emission of two soft photons from the same external line and show that the matrix element $\mathcal{A}_{n+2}$ contains a product of the same factor as in the emission of single photons from two separate lines.

9.9 Light deflection in gravity ♥.
Consider the scattering of a scalar particle and a photon via graviton exchange. Calculate the scattering amplitude and the cross section in the static limit $p^\mu = (M_\odot, \mathbf{0})$ for small-angle scattering ($k^2 = \vartheta = 0$ in the numerator) and show that it agrees with Einstein's prediction for light deflection by the Sun.

10. Gauge theories

We discuss in this chapter field theories in which the Lagrangian is invariant under a continuous group of *local* transformations in internal field space. The symmetry group of these transformations is called the gauge group and the vector fields associated to the generators of the group the gauge fields. We introduce as a first step unbroken gauge theories, i.e. theories with massless gauge bosons, and defer the more complex case of broken gauge symmetries to the chapters 13 and 14. The Standard Model (SM) of particle physics contains with quantum electrodynamics (QED) and quantum chromodynamics (QCD) two examples for unbroken gauge theories. While QED is an abelian gauge theory based on the gauge group U(1), QCD which describes the strong interactions is an non-abelian gauge theory with group SU(3). Non-abelian gauge theories were first studied by Yang and Mills and are therefore also often called Yang-Mills theories. The structure of Yang-Mills theories has many similiarities with gravity. We use this property to introduce the curvature of a space-time as the analogue of the field-strength in the Yang-Mills case.

10.1. Electrodynamics as abelian gauge theory

In classical electrodynamics, the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is an observable quantity, while the potential A_μ is merely a convenient auxiliary quantity. From its definition as an anti-symmetric tensor, it is clear that $F_{\mu\nu}$ is invariant under local gauge transformations

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x) \tag{10.1}$$

of the potentials. Thus $A'_\mu(x)$ is for any smooth $\Lambda(x)$ physically equivalent to $A_\mu(x)$, leading to the same field-strength tensor and thus e.g. to the same Lorentz force on a particle.

Consider now e.g. a free Dirac field $\psi(x)$ with electric charge q. We saw already that this field is invariant under *global* phase transformations $\exp[\mathrm{i}q\Lambda] \in \mathrm{U}(1)$, implying a conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$ via Noether's theorem. Can we promote this global U(1) symmetry to a local one,

$$\psi(x) \to \psi'(x) = U(x)\psi(x) = \exp[\mathrm{i}q\Lambda(x)]\psi(x)\,, \tag{10.2}$$

by making the phase U space-time dependent as in (10.1)? The partial derivatives in the Dirac Lagrangian will lead to an additional term $\propto \partial_\mu U(x)$, destroying the invariance of the free Lagrangian. However, if we add a field $A_\mu(x)$ which transforms as defined in (10.1) and couples to the Noether current j^μ of the complex field as $\mathscr{L}_I = -q\, j^\mu A_\mu$, the two gauge-dependent terms will cancel. Thus local U(1) gauge invariance of the Dirac field requires the existence of a massless gauge boson and fixes its interaction with matter: The coupling of matter to photons is obtained by replacing the normal derivative by the covariant derivative,

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}qA_\mu\,, \tag{10.3}$$

which transforms as the matter fields,

$$D_\mu\psi(x) \to D'_\mu\psi'(x) = \{\partial_\mu + \mathrm{i}q[A_\mu(x) - \partial_\mu\Lambda(x)]\}\exp[\mathrm{i}q\Lambda(x)]\psi(x) = \tag{10.4}$$

$$= \exp[\mathrm{i}q\Lambda(x)]\{\partial_\mu + \mathrm{i}qA_\mu(x)]\}\psi(x) = U(x)D_\mu\psi(x)\,. \tag{10.5}$$

We can rewrite the gauge transformation of A_μ as

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\Lambda(x) = A_\mu(x) - \frac{\mathrm{i}}{q}U(x)\partial_\mu U^\dagger(x)\,, \tag{10.6}$$

expressing the change $\delta A_\mu(x)$ through the group elements $U(x)$. Finally, we note that we can connect the field-strength tensor to the commutator of covariant derivatives,

$$[D_\mu, D_\nu]\psi = \mathrm{i}q([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi = \mathrm{i}q(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = \mathrm{i}qF_{\mu\nu}\psi\,. \tag{10.7}$$

To summarise: The invariance of complex (scalar or Dirac) fields under *global* phase transformations $\exp[\mathrm{i}q\Lambda] \in \mathrm{U}(1)$ implies a conserved current, promoting it to a *local* U(1) symmetry requires the existence of a massless U(1) gauge boson coupled via gauge-invariant derivatives to other fields.

10.2. Non-abelian gauge theories

10.2.1. Gauge invariant interactions

We want to generalise now electrodynamics, using as symmetry group instead of the abelian group U(1) larger groups like SO(n) or SU(n). A group like SU(n) will describe the interactions of $n^2 - 1$ gauge bosons with matter, using as a single parameter the gauge coupling g. The gauge transformations will moreover mix fermions living in the same representation of the group, requiring that these fermions have the same interactions and the same mass if the symmetry is unbroken. In this way, non-abelian gauge theories lead to a partial *unification* of matter fields and interactions. Note the difference to an abelian symmetry: The emission of a photon does not change any quantum number (apart from the momentum) and thus does not "mix" different particles. Therefore there is also no connection between the electric charge of different particles.

The two non-abelian groups used in the SM are SU(2) for weak and SU(3) for strong interactions. A matrix representation for the fundamental representation of these two groups are the Pauli matrices, $T^a = \sigma^a/2$, and the Gell-Mann matrices, $T^a = \lambda^a/2$, respectively. Under the fundamental representation the fermions transform as doublets for SU(2), as triplets for SU(3), etc. Since the number of generators is $m = n^2 - 1$ for SU(n), the groups SU(2) contains three gauge bosons, while SU(3) contains eight bosons carrying strong interactions. The most important difference of these non-abelian groups compared to U(1) is that the generators $T^a \equiv T^a_{ij}$ of such groups do not commute with each other. As a result, we may expect that both the expression for the field-strength tensor, Eq. (7.11), and the transformation law for the gauge field, Eq. (7.12), becomes more complicated. In contrast, we postulate that the interaction law $j_\mu A^\mu$ remains valid, with the sole difference that now $A_\mu = A^a_\mu T^a$. Thus A_μ is a Lorentz vector with values in the Lie algebra of the gauge group.

We derive now the transformation laws and structure of the gauge sector, requiring that the transformation of the fermions and their interaction with the gauge field are locally invariant.

A local gauge transformation

$$U(x) = \exp[\mathrm{i}g \sum_{a=1}^{m} \vartheta^a(x) T^a] \equiv \exp[\mathrm{i}g\vartheta(x)] \tag{10.8}$$

changes a vector of fermion fields $\boldsymbol{\psi}$ with components $\{\psi_1, \ldots, \psi_n\}$ as[1]

$$\psi(x) \to \psi'(x) = U(x)\psi(x)\,. \tag{10.9}$$

Already global gauge invariance of the fermion mass term requires $m_1 = m_2 = \ldots = m_n$ and for simplicity we set $m_i = 0$. We can implement local gauge invariance, if derivatives transform in the same way as ψ. Hence we define a new covariant derivative D_μ requiring

$$D_\mu\psi(x) \to [D_\mu\psi(x)]' = U(x)[D_\mu\psi(x)]\,. \tag{10.10}$$

The gauge field should compensate the difference between the normal and the covariant derivative,

$$D_\mu\psi(x) = [\partial_\mu + \mathrm{i}gA_\mu(x)]\psi(x)\,. \tag{10.11}$$

In the non-abelian case, the gauge field A_μ is a matrix that is connected to its component fields by

$$A_\mu = A_\mu^a T^a\,. \tag{10.12}$$

We now determine the transformation properties of D_μ and A_μ demanding that (10.9) and (10.10) hold. Combining both requirements gives

$$D_\mu\psi(x) \to [D_\mu\psi]' = UD_\mu\psi = UD_\mu U^{-1}U\psi = UD_\mu U^{-1}\psi'\,, \tag{10.13}$$

and thus the covariant derivative transforms as $D'_\mu = UD_\mu U^{-1}$. Using its definition (10.11), we find

$$[D_\mu\psi]' = [\partial_\mu + \mathrm{i}gA'_\mu]U\psi = UD_\mu\psi = U[\partial_\mu + \mathrm{i}gA_\mu]\psi\,. \tag{10.14}$$

We compare now the second and the fourth term, after having performed the differentiation $\partial_\mu(U\psi)$. The result

$$[(\partial_\mu U) + \mathrm{i}gA'_\mu U]\psi = \mathrm{i}gUA_\mu\psi \tag{10.15}$$

should be valid for arbitrary ψ and hence we arrive after multiplying from the right with U^{-1} at

$$A_\mu \to A'_\mu = UA_\mu U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1} = UA_\mu U^{-1} - \frac{\mathrm{i}}{g}U\partial_\mu U^{-1}\,. \tag{10.16}$$

Here we used also $\partial_\mu(UU^{-1}) = 0$. In most cases, the gauge transformation U is an unitary transformation and one sets $U^{-1} = U^\dagger$. A term changing as $U(x)D_\mu(x)U^\dagger(x)$ is called to transform homogenously, while the potential A_μ is said to transforms inhomogenously.

Example 10.1: We can determine the transformation properties of A_μ also by demanding that (10.11) defines the interaction term in a gauge invariant way. Replacing $\partial_\mu \to D_\mu$ in the free Lagrange density of fermions and inserting then $U^{-1}U = 1$ gives

$$\begin{aligned}\mathscr{L}_f + \mathscr{L}_I &= \mathrm{i}\bar\psi\gamma^\mu D_\mu\psi = \mathrm{i}\bar\psi\gamma^\mu\partial_\mu\psi - g\bar\psi\gamma^\mu A_\mu\psi = \\ &= \mathrm{i}\bar\psi U^{-1}U\gamma^\mu\partial_\mu U^{-1}U\psi - g\bar\psi U^{-1}U\gamma^\mu A_\mu U^{-1}U\psi\,.\end{aligned} \tag{10.17}$$

[1] We suppress in the following most indices; writing them out gives e.g. $\psi'_i(x) = U_{ij}(x)\psi_j(x)$ with $U_{ij}(x) = \exp[\mathrm{i}g\sum_{a=1}^m \vartheta^a(x)T^a_{ij}]$.

Using then $\psi' = U\psi$, we obtain

$$\begin{aligned}\mathscr{L}_f + \mathscr{L}_I &= \mathrm{i}\bar{\psi}'\gamma^\mu U\partial_\mu U^{-1}\psi' - g\bar{\psi}'\gamma^\mu U A_\mu U^{-1}\psi' \\ &= \mathrm{i}\bar{\psi}'\gamma^\mu\partial_\mu\psi' - g\bar{\psi}'\gamma^\mu\left\{UA_\mu U^{-1} - \frac{\mathrm{i}}{g}U(\partial_\mu U^{-1})\right\}\psi' \,. \end{aligned} \tag{10.18}$$

The Lagrange density $\mathscr{L}_f + \mathscr{L}_I$ is thus invariant, if the gauge field transforms as given in Eqs. (10.16).

Specialising to infinitesimal transformations,

$$U(x) = \exp(\mathrm{i}g\vartheta^a(x)T^a) = 1 + \mathrm{i}g\vartheta(x) + \mathcal{O}(\vartheta^2)\,, \tag{10.19}$$

it follows

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \mathrm{i}g[A_\mu(x), \vartheta(x)] - \partial_\mu\vartheta(x)\,. \tag{10.20}$$

In the abelian U(1) case, the commutator term is not present and the transformation law reduces to the known $A_\mu \to A_\mu - \partial_\mu\vartheta$. For a (semi-simple) Lie group one defines

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c \tag{10.21}$$

with structure constants f^{abc} that can be chosen to be completely antisymmetric. Thus

$$\begin{aligned} A^a_\mu(x) \to A^{a\prime}_\mu(x) &= A^a_\mu(x) + gf^{abc}A^b_\mu(x)\vartheta^c(x) - \partial_\mu\vartheta^a(x) \\ &= A^a_\mu(x) - [\delta^{ac}\partial_\mu - gf^{abc}A^b_\mu(x)]\vartheta^c(x) \\ &\equiv A^a_\mu(x) - D^{ac}_\mu\vartheta^c(x)\,, \end{aligned} \tag{10.22}$$

where the last line defines how the covariant derivative acts on the gauge fields: Comparing this expression to the general definition $D_\mu = \partial_\mu + \mathrm{i}gA^a_\mu T^a$, we see that the gauge fields live in the adjoint representation of the gauge group[2], cf. problem 10.4. The infinitesimal change of the gauge fields A^a_μ is given by the covariant derivative acting on the parameters ϑ^a of the gauge transformation.

Finally, we have to derive the field strength tensor $\boldsymbol{F}$ and the Lagrange density $\mathscr{L}_{\mathrm{YM}}$ of the gauge field. The quantity $\boldsymbol{F}^2$ requires now additionally a summation over the group index a,

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = -\frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}\,, \tag{10.23}$$

where we assumed in the second step that the standard normalisation $\mathrm{tr}\,T^aT^b = \delta^{ab}/2$ for the group generators T^a holds. The last equation shows that it is sufficient for the gauge invariance of the action that the field-strength tensor transforms homogeneously,

$$\boldsymbol{F}(x) \to \boldsymbol{F}'(x) = U(x)\boldsymbol{F}(x)U^\dagger(x)\,. \tag{10.24}$$

There are several ways to derive the relation between $\boldsymbol{F}$ and $\boldsymbol{A}$. The field-strength tensor should be antisymmetric. Thus we should construct it out of the commutator of gauge invariant quantities that in turn should contain $\boldsymbol{A}$. An obvious try is $\mathrm{i}gF_{\mu\nu} = [D_\mu, D_\nu]$ that

[2]The n complex fermion and $n^2 - 1$ real gauge fields of SU(n) live in different representations of the group, as already the mismatch of their number indicates, see also Appendix B. Note also that the gauge transformations of the gauge fields have to be real, in contrast to the ones of the fermion fields.

worked in the abelian case. Now, additionally the non-zero commutator of the gauge fields contributes,

$$F_{\mu\nu} = F^a_{\mu\nu} T^a = \frac{1}{\mathrm{i}g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu] \,. \tag{10.25}$$

In components, this equation reads explicitly

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \,. \tag{10.26}$$

Remark 10.1: Antisymmetric tensors of rank n can also be seen as differential forms. We know already functions as forms of order $n = 0$ and co-vectors as forms of order $n = 1$. Since differentials of functions are forms of order $n = 1$,

$$\mathrm{d}f = \frac{\partial f}{\partial x^i} \mathrm{d}x^i \,, \tag{10.27}$$

the $\mathrm{d}x^i$ form a basis, and one can write in general $\mathbf{A} = A_i \, \mathrm{d}x^i$. For $n > 1$, the basis has to be antisymmetrized. Hence, a two-form as the field-strength tensor is given by

$$\mathbf{F} = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{10.28}$$

with $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$. Looking at $\mathrm{d}f$ suggestions to define the differentiation of a form ω with coefficients w and degree n as an operation that increases its degree by one to $n + 1$,

$$\mathrm{d}\omega = \frac{1}{n!} (\partial_\beta w_{\alpha_1 \ldots, \alpha_n}) \mathrm{d}x^\beta \wedge \mathrm{d}x^{\alpha_1} \wedge \ldots \wedge \mathrm{d}x^{\alpha_n} \,. \tag{10.29}$$

Thus we have $\mathbf{F} = \mathrm{d}\mathbf{A}$. Moreover, it follows $\mathrm{d}^2\omega = 0$ for all forms. Hence we can write an abelian gauge transformation as $\mathbf{F}' = \mathrm{d}(\mathbf{A} + \mathrm{d}\chi) = \mathbf{F}$.

10.2.2. Gauge fields as connection

There is a close analogy between the covariant derivative ∇_μ introduced for a space-time containing a gravitational field and the gauge invariant derivative D_μ required for a space-time containing a gauge field. In the former case, the moving coordinate basis in curved space-time, $\partial_\mu \boldsymbol{e}^\nu \neq 0$, introduces an additional term in the derivative of vector components $V^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{V}$. Analogously, a non-zero gauge field A^μ leads to a rotation of the basis vectors $\boldsymbol{e}_i$ in group space which in turn produces an additional term $\boldsymbol{\psi} \cdot (\partial_\mu \boldsymbol{e}_i)$ performing the derivative of a $\psi_i = \boldsymbol{\psi} \cdot \boldsymbol{e}_i$.

Let us rewrite our formulas such that the analogy between the covariant gauge derivative D_μ and the covariant space-time derivative ∇_μ becomes obvious. The vector $\boldsymbol{\psi}$ of fermion fields with components $\{\psi_1, \ldots, \psi_n\}$ transforming under a representation of a gauge group can be written as

$$\boldsymbol{\psi}(x) = \psi_i(x) \boldsymbol{e}_i(x) \,. \tag{10.30}$$

We can pick out the component ψ_j by multiplying with the corresponding basis vector $\boldsymbol{e}_j$,

$$\psi_j = \boldsymbol{\psi} \cdot \boldsymbol{e}_j(x) \,. \tag{10.31}$$

If the coordinate basis in group space depends on x^μ, then the partial derivative of ψ_i acquires a second term,

$$\partial_\mu \psi_i = (\partial_\mu \boldsymbol{\psi}) \cdot \boldsymbol{e}_i + \boldsymbol{\psi} \cdot (\partial_\mu \boldsymbol{e}_i) \,. \tag{10.32}$$

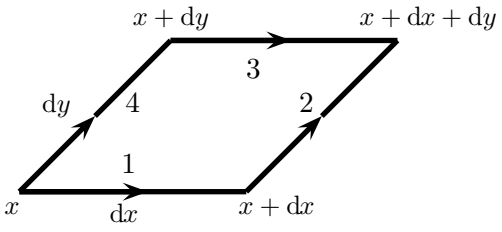


Figure 10.1.: Parallelogram used to calculate the rotation of a test field ψ_i moved along a closed loop in the presence of a non-zero gauge field A^μ.

We can argue as in section 6.2 that $(\partial_\mu \boldsymbol{\psi}) \cdot \boldsymbol{e}_i$ is an invariant quantity, defining therefore as gauge invariant derivative

$$D_\mu \psi_i = (\partial_\mu \boldsymbol{\psi}) \cdot \boldsymbol{e}_i = \partial_\mu \psi_i - \boldsymbol{\psi} \cdot (\partial_\mu \boldsymbol{e}_i)\,. \tag{10.33}$$

The change $\partial_\mu \boldsymbol{e}_i$ of the basis vector in group space should be proportional to gA_μ. Setting

$$\partial_\mu \boldsymbol{e}_i = -\mathrm{i}g(A_\mu)_{ij}\boldsymbol{e}_j \tag{10.34}$$

we are back to our old notation.

Gauge loops The correspondance between the derivatives ∇_μ and D_μ suggests that we can use the gauge field A_μ to transport fields along a curve $x^\mu(\sigma)$. In empty space, we can use the partial derivative $\partial_\mu \psi(x)$ to compare fields at different points,

$$\partial_\mu \psi(x) \propto \psi(x + \mathrm{d}x^\mu) - \psi(x)\,. \tag{10.35}$$

If there is an external gauge field present, the field ψ is additionally rotated in group space moving it from x to $x + \mathrm{d}x$,

$$\begin{aligned} \tilde{\psi}(x + \mathrm{d}x) &= \psi(x + \mathrm{d}x) + \mathrm{i}gA_\mu(x)\psi(x)\mathrm{d}x^\mu & (10.36)\\ &= \psi(x) + \partial_\mu\psi(x)\mathrm{d}x^\mu + \mathrm{i}gA_\mu(x)\psi(x)\mathrm{d}x^\mu\,. & (10.37)\end{aligned}$$

Then the total change is

$$\tilde{\psi}(x + \mathrm{d}x) - \psi(x) = [\partial_\mu + \mathrm{i}gA_\mu(x)]\psi(x)\mathrm{d}x^\mu = D_\mu\psi(x)\mathrm{d}x^\mu\,. \tag{10.38}$$

Thus we can view[3]

$$P_{\mathrm{d}x}(x) = 1 - \mathrm{i}gA_\mu(x)\mathrm{d}x^\mu \tag{10.39}$$

as an operator which allows us to transport a gauge-dependent field the infinitesimal distance from x to $x + \mathrm{d}x$.

We ask now what happens to a field $\psi_i(x)$, if we transport it along an infinitesimal parallelogram, as shown in Fig. 10.1. Calculating the path 2, we find

$$\begin{aligned} P_{\mathrm{d}y}(x + \mathrm{d}x) &= 1 - \mathrm{i}gA_\nu(x + \mathrm{d}x)\mathrm{d}y^\nu \\ &= 1 - \mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}g\partial_\mu A_\nu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu\,, & (10.40)\end{aligned}$$

[3]Note the sign change compared to the covariant derivative: there we pull-back the field from $x + \mathrm{d}x$ to x.

where we Taylor expanded $A_\nu(x+\mathrm{d}x)$. Combining the paths 1 and 2, we arrive at

$$\begin{aligned} P_{\mathrm{d}y}(x+\mathrm{d}x)P_{\mathrm{d}x}(x) &= [1-\mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}g\partial_\mu A_\nu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu][1-\mathrm{i}gA_\mu(x)\mathrm{d}x^\mu] \\ &= 1-\mathrm{i}gA_\mu(x)\mathrm{d}x^\mu - \mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}g\partial_\mu A_\nu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu \\ &\quad - g^2A_\nu(x)A_\mu(x)\mathrm{d}y^\nu\mathrm{d}x^\mu + \mathcal{O}(\mathrm{d}x^3)\,. \end{aligned} \tag{10.41}$$

Instead of performing the calculation for a round trip $1\to2\to3\to4$, we evaluate next $4\to3$ which we then subtract from $1\to2$. In this way, we can re-use our result for $1\to2$ after exchanging labels, $A_\mu\mathrm{d}x^\mu \leftrightarrow A_\nu\mathrm{d}y^\nu$, obtaining

$$\begin{aligned} P_{\mathrm{d}x}(x+\mathrm{d}y)P_{\mathrm{d}y}(x) &= 1-\mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}gA_\mu(x)\mathrm{d}x^\mu - \mathrm{i}g\partial_\nu A_\mu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu \\ &\quad - g^2A_\mu(x)A_\nu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu + \mathcal{O}(\mathrm{d}x^3)\,. \end{aligned} \tag{10.42}$$

The first three terms on the RHS's of (10.41) and (10.42) cancel in the result $P(\square)$ for the round-trip, leaving us with

$$\begin{aligned} &P(\square) \equiv P_{\mathrm{d}y}(x+dx)P_{\mathrm{d}x}(x) - P_{\mathrm{d}x}(x+dy)P_{\mathrm{d}y}(x) = \\ &-\mathrm{i}g\left\{\partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu]\right\}\mathrm{d}x^\mu\mathrm{d}y^\nu\,. \end{aligned} \tag{10.43}$$

Maxwell's equations inform us that the line integral of the vector potential equals the enclosed flux: The area of the parallelogram corresponds to $\mathrm{d}x^\mu\mathrm{d}y^\nu$, and the prefactor has to be therefore the field-strength tensor. If the enclosed flux is non-zero, then $P(\square)\psi_i \neq \psi_i$ and thus the field is rotated.

10.2.3. Curvature of space-time

Curvature and the Riemann tensor We continue to work out the analogy between Yang-Mills theories and gravity. Both the gauge field A_μ and the connection $\Gamma^\mu{}_{\kappa\rho}$ transform inhomogenously. Therefore we can not use them to judge if a gauge or gravitational field is present. In the gauge case, we introduced therefore the field-strength $F_{\mu\nu}$: It transforms homogenously and thus the statement $F_{\mu\nu}(x)=0$ holds in any gauge. This suggests to transform (10.25) into a definition for a tensor measuring the curvature of space-time,

$$(\nabla_\alpha\nabla_b - \nabla_b\nabla_\alpha)T^{\mu\cdots}_{\nu\cdots} = [\nabla_\alpha,\nabla_\beta]T^{\mu\cdots}_{\nu\cdots} \neq 0\,, \tag{10.44}$$

Thus the curvature of space-time should be proportional to the area of a loop and the amount a tensor is rotated.

For the special case of a vector V^α we obtain with

$$\nabla_\rho V^\alpha = \partial_\rho V^\alpha + \Gamma^\alpha{}_{\beta\rho}V^\beta \tag{10.45}$$

first

$$\nabla_\sigma\nabla_\rho V^\alpha = \partial_\sigma(\partial_\rho V^\alpha + \Gamma^\alpha{}_{\beta\rho}V^\beta) + \Gamma^\alpha{}_{\kappa\sigma}(\partial_\rho V^\kappa + \Gamma^\kappa{}_{\beta\rho}V^\beta) - \Gamma^\kappa{}_{\rho\sigma}(\partial_\kappa V^\alpha + \Gamma^\alpha{}_{\beta\kappa}V^\beta)\,. \tag{10.46}$$

The second part of the commutator follows from the simple relabelling $\sigma\leftrightarrow\rho$ as

$$\nabla_\rho\nabla_\sigma V^\alpha = \partial_\rho(\partial_d V^\alpha + \Gamma^a{}_{\beta\sigma}V^\beta) + \Gamma^\alpha{}_{\kappa\rho}(\partial_\sigma V^\kappa + \Gamma^\kappa{}_{b\sigma}V^\beta) - \Gamma^\kappa{}_{\sigma\rho}(\partial_\kappa V^\alpha + \Gamma^\alpha{}_{\beta\kappa}V^\beta)\,. \tag{10.47}$$

Now we subtract the two equations using that $\partial_\rho\partial_\sigma = \partial_\sigma\partial_\rho$ and $\Gamma^\alpha{}_{\beta\rho} = \Gamma^\alpha{}_{\rho\beta}$,

$$[\nabla_\rho, \nabla_\sigma]V^\alpha = \left[\partial_\rho\Gamma^\alpha{}_{\beta\sigma} - \partial_\sigma\Gamma^\alpha{}_{\beta\rho} + \Gamma^\alpha{}_{\kappa\rho}\Gamma^\kappa{}_{\beta\sigma} - \Gamma^\alpha{}_{\kappa\sigma}\Gamma^\kappa{}_{\beta\rho}\right] V^\beta \equiv R^\alpha{}_{\beta\rho\sigma}V^\beta \,. \tag{10.48}$$

The tensor $R^\alpha{}_{\beta\rho\sigma}$ is called *Riemann* or *curvature tensor*. In problem 19.4, you are asked to show that the tensor $R_{\alpha\beta\rho\sigma} = g_{\alpha\gamma}R^\gamma{}_{\beta\rho\sigma}$ is antisymmetric in the indices $\rho \leftrightarrow \sigma$, antisymmetric in $\alpha \leftrightarrow \beta$ and symmetric against an exchange of the index pairs $(\alpha\beta) \leftrightarrow (\rho\sigma)$. Therefore, we can construct out of the Riemann tensor only one non-zero tensor of rank two, contracting α either with the third or fourth index, $R^\rho{}_{\alpha\rho\beta} = -R^\rho{}_{\alpha\beta\rho}$. We define[4] the *Ricci tensor* for a pseudo-Riemannian metric by $R_{\alpha\beta} = R^\rho{}_{\alpha\beta\rho}$, while we set $R_{ab} = R^c{}_{abc}$ for a Riemannian metric (e.g. for the spatial part of the metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}l^2$). Then the Ricci tensor is given by

$$R_{\alpha\beta} = R^\rho{}_{\alpha\beta\rho} = -R^\rho{}_{\alpha\rho\beta} = \partial_\beta\Gamma^\rho{}_{\alpha\rho} - \partial_\rho\Gamma^\rho{}_{\alpha\beta} + \Gamma^\sigma{}_{\beta\rho}\Gamma^\rho{}_{\alpha\sigma} - \Gamma^\rho{}_{\alpha\beta}\Gamma^\sigma{}_{\rho\sigma} \,. \tag{10.49}$$

A further contraction gives the curvature scalar,

$$R = R_{\alpha\beta}g^{\alpha\beta} \,. \tag{10.50}$$

Example 10.2: Sphere S^2. Calculate the Ricci tensor $R_{\alpha\beta}$ and the scalar curvature R of the two-dimensional unit sphere S^2.
We have already determined the non-vanishing Christoffel symbols of the sphere S^2 as $\Gamma^\phi{}_{\vartheta\phi} = \Gamma^\phi{}_{\phi\vartheta} = \cot\vartheta$ and $\Gamma^\vartheta{}_{\phi\phi} = -\cos\vartheta\sin\vartheta$. We will show later that the Ricci tensor of a maximally symmetric space as a sphere satisfies $R_{ab} = Kg_{ab}$. Since the metric is diagonal, the non-diagonal elements of the Ricci tensor are zero too, $R_{\phi\vartheta} = R_{\vartheta\phi} = 0$. We calculate with

$$R_{ab} = R^c{}_{acb} = \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac} + \Gamma^c{}_{ab}\Gamma^d{}_{cd} - \Gamma^d{}_{bc}\Gamma^c{}_{ad}$$

the $\vartheta\vartheta$ component, obtaining

$$\begin{aligned} R_{\vartheta\vartheta} &= 0 - \partial_\vartheta(\Gamma^\phi{}_{\vartheta\phi} + \Gamma^\vartheta{}_{\vartheta\vartheta}) + 0 - \Gamma^d{}_{\vartheta c}\Gamma^c{}_{\vartheta d} = 0 + \partial_\vartheta\cot\vartheta - \Gamma^\phi{}_{\vartheta\phi}\Gamma^\phi{}_{\vartheta\phi} \\ &= 0 - \partial_\vartheta\cot\vartheta - \cot^2\vartheta = 1 \,. \end{aligned}$$

From $R_{ab} = Kg_{ab}$, we find $R_{\vartheta\vartheta} = Kg_{\vartheta\vartheta}$ and thus $K = 1$. Hence $R_{\phi\phi} = g_{\phi\phi} = \sin^2\vartheta$.
The scalar curvature is (diagonal metric with $g^{\phi\phi} = 1/\sin^2\vartheta$ and $g^{\vartheta\vartheta} = 1$)

$$R = g^{ab}R_{ab} = g^{\phi\phi}R_{\phi\phi} + g^{\vartheta\vartheta}R_{\vartheta\vartheta} = \frac{1}{\sin^2\vartheta}\,\sin^2\vartheta + 1 \times 1 = 2 \,.$$

Note that our definition of the Ricci tensor guaranties that the curvature of a sphere is also positive, if we consider it as subspace of a four-dimensional space-time.

We can push the analogy further by remembering that the field-strength defined in (10.25) is a matrix. Writing out the implicit matrix indices of $F_{\mu\nu}$ in Eq. (10.25) gives

$$(F_{\mu\nu})_{ij} = \partial_\mu(A_\nu)_{ij} - \partial_\nu(A_\mu)_{ij} + \mathrm{i}g\left\{(A_\mu)_{ik}(A_\nu)_{kj} - (A_\nu)_{ik}(A_\mu)_{kj}\right\} \,. \tag{10.51}$$

Comparing this expression to

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha{}_{\beta\nu} - \partial_\nu\Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu} \tag{10.52}$$

we see that the first two indices of the Riemann tensor, α and β, correspond to the group indices ij in the field-strength tensor. This is in line with the relation of the potential $(A_\mu)_{ij}$ and the connection $\Gamma^\alpha{}_{\beta\nu}$ implied by (10.34).

[4] For a comparison of various sign conventions see (A.4d) in Appendix A.

10.3. Quantization of gauge theories

10.3.1. Abelian case

We discussed already in Sec. 7.2 that we can derive the photon propagator only fixing a gauge. Now we reconsider this problem, and ask how we should modify the Lagrange density in order to be able to derive the photon propagator. The Lagrange density that leads to the Maxwell equation is

$$\begin{aligned}\mathscr{L}_{\rm em} &= -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\,(\partial_\mu A^\nu \partial^\mu A_\nu - \partial^\nu A_\mu \partial^\mu A_\nu) \\ &= \frac{1}{2}\,(A^\nu \partial_\mu \partial^\mu A_\nu - A_\mu \partial^\mu \partial^\nu A_\nu) = \frac{1}{2} A_\nu \left[\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\right] A_\mu = \frac{1}{2}\,A^\nu D^{-1}_{\mu\nu} A^\mu\,,\end{aligned}$$

where we made a partial integration dropping as usual the surface term. Deriving the photon propagator requires to invert the term in the square bracket. Performing a Fourier transformation, we see that we should find the inverse of the operator

$$D^{-1}_{\mu\nu}(k) = k^2 P^{\mu\nu}_T(k) = k^2\left(\eta^{\mu\nu} - k^\mu k^\nu/k^2\right)\,. \tag{10.53}$$

We have already seen that this operator projects any four-vector on the three-dimensional subspace orthogonal to k. This ensures that the physical degrees of the photon contain no longitudinal component. More formally, we see that $P^{\mu\nu}_T(k)$ is a projection operator,

$$P^{\mu\nu}_T P_{T\nu}{}^{\lambda} = P^{\mu\lambda}_T\,, \tag{10.54}$$

and has thus as only eigenvalues 0 and 1. Since $P^{\mu\nu}_T(k)$ is not the unit operator, it has at least one zero eigenvalue and is thus not invertible. Its trace is

$$P^\mu_{T\mu} = \eta_{\mu\nu}P^{\mu\nu}_T = \delta^\mu_\mu - 1 = 3\,, \tag{10.55}$$

and thus three eigenvalues are one and one eigenvalue is zero. The latter eigenvalue corresponds to $k_\mu P^{\mu\nu}_T = 0$, as required for a projection operator on the three-dimensional subspace orthogonal to k. The orthogonal part $\delta^\mu_\nu - P_{T\nu}{}^{\mu}$ is given by the longitudinal projection operator $P^{\mu\nu}_L = k^\mu k^\nu/k^2$.

We can invert $D^{-1}_{\mu\nu}$, if we choose a gauge such that the subspace parallel to k is included. The simplest choice is the Lorenz gauge. Imposing this gauge on the level of the Lagrangian means adding the term

$$\mathscr{L} \to \mathscr{L}_{\rm eff} = \mathscr{L} + \mathscr{L}_{\rm gf} = \mathscr{L} - \frac{1}{2}(\partial^\mu A_\mu)^2\,. \tag{10.56}$$

More generally, we can add the term

$$\mathscr{L}_{\rm gf} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2 \tag{10.57}$$

that depends on the arbitrary parameter ξ. This group of gauges is employed in the proof of the renormalizability of gauge theories and is therefore called R_ξ gauge. The combined effective Lagrange density is thus

$$\mathscr{L}_{\rm eff} = \mathscr{L}_{\rm em} + \mathscr{L}_{\rm gf} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 = \frac{1}{2}\,A_\nu\left[\eta^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\mu\,. \tag{10.58}$$

Inverting the term in the square brackets in Fourier space, we obtain

$$P^{\mu\nu} = -k^2\eta^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu \,. \tag{10.59}$$

Now we split this expression into its transverse and longitudinal parts,

$$\begin{aligned} P^{\mu\nu} =& -k^2\left(P_T^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right) + (1-\xi^{-1})k^\mu k^\nu \\ =& -k^2 P_T^{\mu\nu} - \xi^{-1}k^2 P_L^{\mu\nu} \,. \end{aligned} \tag{10.60}$$

Since $P_T^{\mu\nu}$ and $P_L^{\mu\nu}$ project on orthogonal subspaces, we obtain the inverse $P^{-1}_{\mu\nu}$ simply by inverting their pre-factors, cf. problem 8.8. Thus the photon propagator in R_ξ gauge is given by

$$\mathrm{i}D_F^{\mu\nu}(k^2) = \frac{-\mathrm{i}P_T^{\mu\nu}}{k^2+\mathrm{i}\varepsilon} + \frac{-\mathrm{i}\xi P_L^{\mu\nu}}{k^2+\mathrm{i}\varepsilon} = \frac{-\mathrm{i}}{k^2+\mathrm{i}\varepsilon}\Big[\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\Big] \,. \tag{10.61}$$

Special cases are the Feynman gauge $\xi = 1$, the Landau gauge $\xi = 0$, while $\xi \to \infty$ corresponds to the unitary gauge. Electromagnetic gauge invariance implies current conservation, $\partial_\mu J^\mu(x) = 0$ or $k_\mu J^\mu(k) = 0$. Thus the arbitrary, ξ dependent part of the photon propagator vanishes in physical quantities, because it is matched between conserved currents. For finite ξ we see that the propagator is proportional to k^{-2} and we expect that less problems arise in loop calculations compared to the unitary gauge.

10.3.2. Non-abelian case

In the non-abelian case, the gauge transformation (10.22) adds not only a term $\partial_\mu\vartheta$ but mixes also the fields via the term $f^{abc}A^b_\mu\vartheta^c$. Therefore, we cannot expect that the unphysical degrees of freedom decouple and the quantisation of non-abelian theories becomes more challenging.

We consider first as a toy model for the generating functional of a Yang-Mills theory the two-dimensional integral

$$Z \propto \int \mathrm{d}x\mathrm{d}y\; \mathrm{e}^{\mathrm{i}S(x)} \,. \tag{10.62}$$

Since the integration extends from $-\infty$ to ∞, the y integration does not merely change the normalisation of Z but makes the integral ill-defined. We can eliminate the dangerous y integration by introducing a delta function,

$$Z \propto \int \mathrm{d}x\mathrm{d}y\; \delta(y)\mathrm{e}^{\mathrm{i}S(x)} \,. \tag{10.63}$$

Since the value of y in the delta function plays no role, we can replace $\delta(y)$ by $\delta(y-f(x))$ with an arbitrary function $f(x)$. If $y = f(x)$ is the solution of $g(x,y) = 0$, we obtain with

$$\delta(g(x,y)) = \frac{\delta(y-f(x))}{|\partial g/\partial y|} \tag{10.64}$$

assuming that $\partial g/\partial y > 0$

$$Z \propto \int \mathrm{d}x\mathrm{d}y\; \frac{\partial g}{\partial y}\delta(g)\mathrm{e}^{\mathrm{i}S(x)} \,. \tag{10.65}$$

Generalising this to n dimensions, we need n delta functions and have to include the Jacobian,

$$Z \propto \int \mathrm{d}^n x \mathrm{d}^n y \ \det\left(\frac{\partial g_i}{\partial y_j}\right) \prod_i \delta(g_i) \mathrm{e}^{\mathrm{i}S(x)} \,. \tag{10.66}$$

We now translate this toy example to the Yang-Mills case. The functions g are the gauge fixing conditions that we choose as

$$g^a(x) = \partial^\mu A^a_\mu(x) - \omega^a(x) \,, \tag{10.67}$$

where the $\omega^a(x)$ are arbitrary functions. The discrete index i corresponds to $\{x, a\}$, explaining why the gauge freedom results in an infinity: Although the integration measure of a compact gauge group is finite, the summation over $\mathbb{R}^4$ gives an infinite answer. Finally, we see that from the transformation law $A^a_\mu(x) \to A^{a\prime}_\mu(x) = A^a_\mu(x) - D^{ac}_\mu \vartheta^c(x)$ that the parameters ϑ^a correspond to the redundant coordinates y_i.

The generating functional for a Yang-Mills theory is thus with $\mathcal{D}A \equiv \prod_{\mu=0}^{3} \prod_{a=1}^{m} \mathcal{D}A^a_\mu$ as short-cut given by

$$Z[0] \propto \int \mathcal{D}A \ \det\left(\frac{\delta g^a}{\delta \vartheta^b}\right) \prod_{x,a} \delta(g^a) \mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}} \,, \tag{10.68}$$

where we set for the moment the sources to zero. Our task is to evaluate first $\delta g^a / \delta \vartheta^b$ and then to transform the determinant into the Lagrangian of new, auxiliary fields such that we can use the language of Feynman diagrams to perform perturbative calculations in the usual way. Inserting into the gauge fixing condition (10.67) an infinitesimal gauge transformation, we obtain

$$g^a(x) \to g^a(x) - \partial^\mu D^{ab}_\mu \vartheta^b(x) \,. \tag{10.69}$$

Thus the required functional derivative is

$$\frac{\delta g^a(x)}{\delta \vartheta^b(y)} = -\partial^\mu D^{ab}_\mu \delta(x-y) \,. \tag{10.70}$$

We can eliminate the determinant remembering $\int \mathrm{d}\eta \mathrm{d}\bar{\eta} \ \mathrm{e}^{\bar{\eta} A \eta} = \det A$ from Eq. (8.106), expressing the Jacobian as a path integral over Graßmann variables c^a and $\bar{c}^a$,

$$\det\left[\frac{\delta g^a(x)}{\delta \vartheta^b(y)}\right] \propto \int \mathcal{D}c \mathcal{D}\bar{c} \ \mathrm{e}^{\mathrm{i}S_{\mathrm{FP}}} \,. \tag{10.71}$$

The corresponding Lagrangian is

$$\mathscr{L}_{\mathrm{FP}} = -\bar{c}^a \partial^\mu D^{ab}_\mu c^b = (\partial^\mu \bar{c}^a)(D^{ab}_\mu c^b) \tag{10.72}$$

$$= \partial^\mu \bar{c}^a \partial_\mu c^a - g f^{abc} A^c_\mu \partial^\mu \bar{c}^a c^b \,, \tag{10.73}$$

where we made a partial integration and inserted the definition of the covariant derivatives acting on the gauge field, Eq. (10.22).

As a result, we have recast the determinant as the kinetic energy of complex scalar fields c^a that interact with the gauge fields. Since we had to use for the scalar fields Graßmann variables c^a, their statistics is fermionic. Clearly, such fields should be seen as a purely mathematical construct to be used only as virtual fields inside loop graphs. They are therefore called Faddeev-Popov ghosts. In an abelian theory as U(1), the interaction term in Eq. (10.73) is

absent and ghost fields decouple. Since they change then only the normalisation of the path integral, they can be omitted all together in QED.

Next we have to eliminate the $\delta(g^a(x))$. They contain the arbitrary functions $\omega^a(x)$, but the path integral does not dependent on them. Thus we have the freedom to multiply with an arbitrary function $f(\omega^a)$, thereby changing only the normalisation. Our aim is to generate after integrating over the delta functions a term $\exp(\mathrm{i}S_{\mathrm{gf}})$, as in the case of electrodynamics. Choosing

$$Z \to \exp\left(-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\, \omega^a(x)\omega^a(x)\right) Z\,, \tag{10.74}$$

integrating $\prod_{x,a}\delta(g^a)\exp\left(-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\,\omega^a(x)\omega^a(x)\right)$ with the help of $\delta(g^a)$ and (10.67), we obtain as gauge-fixing term the desired

$$\mathscr{L}_{\mathrm{gf}} = -\frac{1}{2\xi}\partial^\mu A^a_\mu \partial^\nu A^a_\nu\,. \tag{10.75}$$

The complete Lagrange density $\mathscr{L}_{\mathrm{eff}}$ of a non-abelian gauge theory consists thus of four parts,

$$\mathscr{L}_{\mathrm{eff}} = \mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{gf}} + \mathscr{L}_{\mathrm{FP}} + \mathscr{L}_{\mathrm{s}}\,, \tag{10.76}$$

where the last one couples sources linearly to the fields,

$$\mathscr{L}_{\mathrm{s}} = J^\mu A_\mu + \bar\eta c + \bar c \eta\,. \tag{10.77}$$

We break both $\mathscr{L}_{\mathrm{YM}}$ and $\mathscr{L}_{\mathrm{FP}}$ into a piece of $\mathcal{O}(g^0)$ defining the free propagator, and pieces of $\mathcal{O}(g)$ corresponding to a three gluon and a two ghost-gluon vertex, respectively, and a four gluon vertex of $\mathcal{O}(g^2)$. After a partial integration of the free part, we obtain

$$\begin{aligned}\mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{FP}} = {}& \frac{1}{2}A^a_\nu\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)A^a_\mu - g f^{abc}A^a_\mu A^b_\nu \partial^\mu A^{c\nu} - \frac{1}{4}g^2 f^{abe}f^{cde}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu} \\ & - \bar c^a \Box c^a - g f^{abc}A^c_\mu \partial^\mu \bar c^a c^b\,.\end{aligned} \tag{10.78}$$

The Feynman rules can now be read off after Fourier transforming into momentum space, cf. problem 10.6. Combining the resulting expression with $\mathscr{L}_{\mathrm{gf}}$, we see that the gluon propagator is diagonal in the group indices and otherwise identical to the photon propagator in R_ξ gauge.

The ghost propagator is the one of a massless scalar particle,

$$\Delta_{ab}(k) = \frac{\delta_{ab}}{k^2 + \mathrm{i}\varepsilon}\,. \tag{10.79}$$

Being a fermion, a closed ghost loop introduces however a minus sign.

Non-covariant gauges The introduction of ghost fields can be avoided, if one uses non-covariant gauges which depend on an arbitrary vector n^μ. An example used often in QED is the Coulomb or radiation gauge,

$$\partial_\mu A^\mu - (n_\mu\partial^\mu)(n_\mu A^\mu) = 0 \tag{10.80}$$

with $n_\mu = (1,0,0,0)$. In QCD, one employs often the axial gauge,

$$n_\mu A^\mu_a = 0, \quad a = 1,\ldots,8 \tag{10.81}$$

with $n^2 = 0$. Then the Fadeev-Popov determinant does not depend on A^μ_a, and can be absorbed in the normalisation of the path integral. While non-covariant gauges bypass the introduction of unphysical particles in loop graphs, the resulting propagators are unhandy. Moreover, they contain spurious singularities which require care. Therefore in practical all applications the use of the R_ξ gauge is advantageous.

Let us finally comment on the case of external gluons. In the case of photons, we can sum their polarisation states using $\sum_{r=0}^{3} \varepsilon_\mu^{(r)*} \varepsilon_\nu^{(r)} = -\eta_{\mu\nu}$, since the nonphysical degrees of freedom cancel in physical observables. In the non-abelian case, we can use this "trick" only in the case of a single external gluon. For two or more external gluons, we have to employ the polarisation sum derived in problem 7.2, since the no-abelian vertices mix physical and non-physical degrees of freedom.

10.A. Appendix: Feynman rules for an unbroken gauge theory

The Feynman rules for a non-broken Yang-Mills theory as QCD are given; for the abelian case of QED set the structure constants $f_{abc} = 0$, $T = 1$ and replace $g_s \to eq_f$, where q_f is the electric charge of the fermion in units of the elementary charge $e > 0$.

Propagators

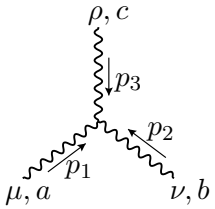
μ, a ~~~ A ~~~ ν, b

$$-\mathrm{i}\delta_{ab}\left[\frac{\eta_{\mu\nu}}{k^2+\mathrm{i}\epsilon} - (1-\xi)\frac{k_\mu k_\nu}{(k^2)^2}\right] \tag{10.82}$$

a ..... ω ..... b

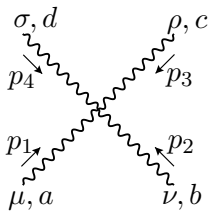
$$\delta_{ab}\,\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon} \tag{10.83}$$

Triple Gauge Interactions



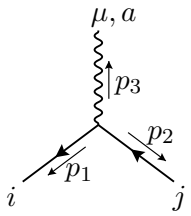
$$\begin{aligned} -g_s f^{abc}[\quad & \eta^{\mu\nu}(p_1-p_2)^\rho + \eta^{\nu\rho}(p_2-p_3)^\mu \\ & +\eta^{\rho\mu}(p_3-p_1)^\nu] \\ & p_1+p_2+p_3 = 0 \end{aligned} \tag{10.84}$$

Quartic Gauge Interactions



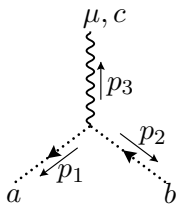
$$
\begin{aligned}
-\mathrm{i}g_s^2\Big[\quad & f_{eab}f_{ecd}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\
& + f_{eac}f_{edb}(\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}) \\
& + f_{ead}f_{ebc}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})\Big] \\
& p_1 + p_2 + p_3 + p_4 = 0
\end{aligned}
\tag{10.85}
$$

Fermion Gauge Interactions



$$
-\,\mathrm{i}\,g_s\gamma^\mu T^a_{ij} \tag{10.86}
$$

Ghost Interactions



$$
\begin{aligned}
& -g_s\ f^{abc}p_1^\mu \\
p_1 \quad & + p_2 + p_3 = 0
\end{aligned}
\tag{10.87}
$$

Summary

Requiring local symmetry under a gauge group as SU(n) or SO(n) specifies the self-interactions of massless gauge boson as well as their couplings to fermions and scalars. The presence of self-interactions implies that a pure Yang-Mills theory is non-linear. The gauge invariant derivative D_μ is the analogon to the covariant derivative ∇_μ of gravity, while the field-strength corresponds to the Riemann tensor: Both measure the rotation of a vector which is parallel-transported along a closed loop. The quantisation of Yang-Mills theories in the covariant R_ξ gauge leads to ghost particles: These fermionic scalars compensate the unphysical degrees of freedom still contained in the gauge fields A_μ using a covariant gauge fixing condition as $\partial_\mu A^\mu = 0$.

Note also the interplay between local and global symmetries: A global symmetry transformation U maps a physical state onto a different physical state with the same properties, implying via Noether's theorem a conserved current. A local symmetry transformation $U(x)$ maps a physical state on itself, implying a redundancy in our description of the system. Since local symmetries contain global transformations as a subgroup, they imply always also the

conservation of global charges via Noether's theorem.

Further reading

The Feynman rules in the appendix are taken from[61]. This article contains all Feynman rules for the SM in a convention independent notation which allows an easy comparison of references with differing conventions. Current conservation in non-abelian theories is discussed e.g. in Pederazzi. The extension of the helicity formalism to QCD, where it leads to both phenomenological useful and theoretically interesting results, is discussed in Peskin[56] and in Schwartz.

Problems

10.1 Non-abelian Maxwell equations.
Derive the non-abelian analogue of the Maxwell equations. What are the conserved Noether currents, do gauge invariant currents exist? Derive the constraint on the allowd gauge transformations such that the conserved charges transform covariantly.

10.2 Stress tensor.
Show that the stress tensor for a single quark in the background of a classical gluon field A^a_μ is given by $T^{\mu\nu} = \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{D}^\nu \psi$.

10.3 Palatini approach.
Consider the Yang-Mills action as a functional of the potential and the field-strength,

$$S_{\mathrm{YM}}[A_\mu, F_{\mu\nu}] = \qquad (10.88)$$
$$-\frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\right) .$$

Derive the non-abelian Maxwell equations in the Palatini approach, i.e. by varying A_μ and $F_{\mu\nu}$ independently.

10.4 Adjoint representation.
a.) Derive the Jacobi identity $[[A,B],C] = [A,[B,C]] - [B,[A,C]]$ and show that the structure constants $\mathrm{i}f^{abc}$ of a Lie algebra satisfy the Lie algebra by themselves.
b.) Insert $(T^a_{\mathrm{A}})_{bc} = -\mathrm{i}f^{abc}$ for the adjoint representation in the general definition (10.11) for the covariant derivative and show that the result agrees with (10.22).

10.5 Hyperbolic plane H^2.
The line-element of the Hyperbolic plane H^2 is given by $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$ with $y \geq 0$.
a.) Write out the geodesic equations and deduce the Christoffel symbols $\Gamma^a{}_{bc}$.
b.) Calculate the Riemann (or curvature) tensor $R^a{}_{bcd}$ and the scalar curvature R.

10.6 Three and four gauge boson vertex.
Derive the tensor structure $\mathbf{V}^{rst}(k_1^\rho, k_2^\sigma, k_3^\tau)$ of the three-gluon vertex by Fourier-transforming the part of $\mathscr{L}_I$ containing three gluon fields to momentum space,

$$F = \int \mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3\,(2\pi)^4\delta(p_1+p_2+p_2)$$
$$\times\ f(A^a_\mu(p_1)A^{b\mu}(p_2)A^{c\nu}(p_3))$$

and then eliminating the fields by functional derivatives with respect to them

$$V^{rst}(k_1^\rho, k_2^\sigma, k_3^\tau) = \frac{\delta^3 F}{\delta A^r_\rho(k_1)\,\delta A^s_\sigma(k_2)\,\delta A^t_\tau(k_3)}$$

Similiarly, derive the tensor structure $\mathbf{V}^{rst}(k_1^\rho, k_2^\sigma, k_3^\tau, , k_4^\lambda)$ of the four-gluon vertex. Use alternatively symmetry arguments, if possible.

11. Renormalisation I: Perturbation theory

We encountered three examples of divergent loop integrals discussing the $\lambda\phi^4$ theory. In these cases, it was possible to subtract the infinities in such a way that we obtained finite observables which depend only on the experimentally measured values of m, λ and ρ. Aim of this and the following chapter is to obtain a better understanding of this renormalisation procedure. We will see that the $\lambda\phi^4$ theory as well as the electroweak and strong interactions of the SM are examples for renormalisable theories: For such theories, the renormalisation of the finite number of parameters contained in the classical Lagrangian is sufficient to make all observables finite in any order perturbation theory.

11.1. Overview

Why renormalisation at all? We are using perturbation theory with the free, non-interacting Lagrangian as starting point to evaluate non-linear quantum field theories. Interactions change however the parameters of the free theory, as we know already both from classical electrodynamics and quantum mechanics. In the former case, Lorentz studied 1904 the connection between the measured electron mass $m_{\rm phy}$, its mechanical or inertial mass m_0 and its electromagnetic self-energy $m_{\rm el}$ in a toy model. He described the electron as a spherically symmetric uniform charge distribution with radius r_e, obtaining

$$m_{\rm phy} = m_0 + m_{\rm el} = m_0 + \frac{4e^2}{5r_e}\,. \tag{11.1}$$

Special relativity forces us to describe the electron as a point particle: Taking thus the limit $r_e \to 0$, classical electrodynamics implies an infinite "renormalisation" of the "bare" electron mass m_0 by its electromagnetic self-energy $m_{\rm el}$.

Another familiar example for renormalisation appears in quantum mechanics. Perturbation theory is possible, if the Hamilton operator H can be split into a solvable part $H^{(0)}$ and an interaction λV,

$$H = H^{(0)} + \lambda V\,, \tag{11.2}$$

and the parameter λ is small. Using then as starting point the normalised solutions $|n^{(0)}\rangle$ of $H^{(0)}$,

$$H^{(0)}|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle\,, \tag{11.3}$$

we can find the eigenstates $|n\rangle$ of the complete Hamiltonian H as a power-series in λ,

$$|n\rangle = |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \ldots \tag{11.4}$$

Since we started with normalised states, $\langle n^{(0)}|n^{(0)}\rangle = 1$, the new states $|n\rangle$ are not longer correctly normalised. Thus going from free (or "bare") to interacting states requires to renormalise the states,

$${}_R\langle n|\,n\rangle_R = 1 \quad \Rightarrow \quad |n\rangle_R \equiv Z^{1/2}\,|n\rangle\ . \tag{11.5}$$

A very similar problem we encountered introducing the LSZ formalism. In the parlance of field theory, we continue often to call this procedure wave-function renormalisation, although Z renormalises field operators.

Why regularisation at all? The familiar process of renormalisation becomes more obscure by the fact that the renormalisation constants are infinite in most quantum field theories. Mathematical manipulations as shifting the integration variable in a divergent loop integral are only well-defined, if we convert first these integrals into convergent ones. Thus we have to regularise as first step, i.e. employing a method which makes our expressions finite, so that our mathematical manipulations are well-defined and we can perform the renormalisation. You should keep in mind that the two operations, regularisation and renormalisation, are logically independent: Renormalisation of the parameters in the free theory is necessary because they are changed by interactions. This change may be finite, as the change of the photon mass in a plasma, and no regularisation is necessary.

The second question to ask is why the renormalisation constants are infinite, or in other words why do we have regularise at all? There are (at least) two possible answers to this question: Either we use a bad theory as starting point, i.e. the full quantum theory defined non-perturbatively by its generating functional $Z[J]$ is ill-defined. Or we employ a bad expansion scheme evaluating $Z[J]$ in perturbation theory.

Example 11.1: An example for a bad expansion is the following toy model for the $\lambda\phi^4$ interaction,

$$Z(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}x^2 - \lambda x^4} \stackrel{?}{=} \int_{-\infty}^{\infty} \mathrm{d}x \left[1 - \lambda x^4 + \frac{(\lambda x^4)^2}{2!} - \ldots\right] \mathrm{e}^{-\frac{1}{2}x^2} . \tag{11.6}$$

The LHS is well-defined for $\lambda > 0$. Doing perturbation theory and summing up the first N terms of the expansion on the RHS results in an alternating series,

$$Z_N(\lambda) = \sum_{n=0}^{N} \frac{(-\lambda)^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}x\, x^{4n} \mathrm{e}^{-x^2/2} = \sum_{n=0}^{N} a_n \lambda^n$$

with

$$a_n = \frac{(-1)^n}{n!}\, 2^{2n+1/2}\, \Gamma(2n + 1/2) .$$

The coefficients a_n of this series grow like a factorial and thus the convergence radius of the expansion is zero. Plotting $Z_N(\lambda)/Z(0)$ for the first few N as function of λ, problem 11.1, you see firstly that adding more terms makes the expansion worse beyond at certain value $\lambda_{\max}(N)$, and secondly that $\lambda_{\max}(N) \to 0$ for $N \to \infty$.

It should be not too surprising that the expansion (11.6) has a zero convergence radius: Moving from $\lambda > 0$ to $\lambda < 0$ changes fundamentally physics, since the vacuum is unstable for arbitrarily small negative λ. An interesting consequence of the failure of perturbation theory is that the complete theory may contain additional non-perturbative physics. Next we look at an example where we start from a bad theory.

Example 11.2: We discussed in problem 2.10 the scattering on a short-range potential in $d = 1$, and found that no consistent solution exists for an odd potential. We rephrase this problem now in

a language close to the one used in QFT. The perturbative expansion of the S-matrix in quantum mechanics is given by

$$\langle p_f|\, S\,|p_i\rangle = 2\pi\delta(E_i - E_f)\left[\langle p_f|\, V\,|p_i\rangle + \int \mathrm{d}p\, \langle p_f|\, V\,|p\rangle\, \frac{\mathrm{i}}{E - p^2/2 + \mathrm{i}\varepsilon}\, \langle p|\, V\,|p_i\rangle + \cdots\right]$$

for $p_i \neq p_f$. Recoiling on the infinitely heavy static source, the (virtual) particle in the intermediate states can have any momentum p while its energy is conserved. With $V_0(x) = c_0\delta(x)$, it follows $\langle p_f|\, V_0\,|p\rangle = c_0/(2\pi)$ and then the momentum integral in the 2.nd order correction becomes

$$\Big(\frac{c_0}{2\pi}\Big)^2 \int \mathrm{d}p \frac{\mathrm{i}}{E - p^2/2 + \mathrm{i}\varepsilon}\,.$$

Thus this momentum integral, and similiarly those at higher orders, are well-defined. Next we set $c_0 = 0$. Using then $\delta' f = \delta f'$, we obtain $\langle p_f|\, V_1\,|p\rangle = \mathrm{i}c_1(p_f - p)/(2\pi)$ and thus the momentum integral

$$\Big(\frac{c_1}{2\pi}\Big)^2 \int \mathrm{d}p \frac{\mathrm{i}(p - p_i)(p - p_f)}{E - p^2/2 + \mathrm{i}\varepsilon}$$

is linearly divergent. The divergence means that the scattering probability is sensitive to arbitrarily high momentum modes. We can understand this behaviour looking at the wave-function $\psi(x)$: Because the potential is odd, also $\psi(x)$ is odd and thence has to change rapidly within $|x| < a$. As result, its Fourier transform $\psi(k)$ necessarily contains also high-frequency modes.
In this simple toy-model, the natural way to solve the UV divergence problem is to replace the mathematical idealisation of a delta-function like potential by the true, smooth potential. If we either do not know the "true" potential or if we insist that a delta-function like potential captures all the physics contained in a scattering process at a short-range potential, then we have to regularize the potential, replacing $V(x) = c_1\delta(x)$ by

$$V(x) = c_1 \frac{\delta(x + a) - \delta(x - a)}{2a}\,.$$

In this way, we eliminate high-frequency modes with $p \gg 1/a$. Repeating the computation of the transmission amplitude, we find $T \simeq \mathrm{i}ap/c_i^2$. Hence $c_R \equiv c_1^2/a$ plays the role of an effective coupling constant in the regularised theory. Physical obervables like the transmission amplitude depend only on the single parameter c_R, if we rescale $c_1(a) \propto a^{-1/2}$. Thus this simple example from quantum mechanics exhibits the key features of a UV divergent QFT: We regularise the theory, cutting off UV modes. Requiring the independence of physical obervables from the cutoff scale, we obtain running parameters.

It is very likely that our favorite $\lambda\phi^4$ interaction suffers from both diseases: First, the expansion in λ is not convergent but results in an asymptotic series. Second, the full theory contains only the trivial $\lambda = 0$ case as consistent solution. Even if the interacting theory may be mathematically inconsistent, it can however be used as an effective model describing physics up to a finite energy scale.

Regularisation methods We have already seen that the regularisation of divergent loop integrals can be done in various ways. In general, one reparametrises the integral in terms of a parameter Λ (or ε) called regulator such that the integral becomes finite for a finite value of the regulator, while the limit $\Lambda \to \infty$ (or $\varepsilon \to 0$) returns to the original integral.

- We can avoid UV divergences evaluating loop-integrals introducing an (Euclidean) momentum cutoff Λ. Somewhat more sophisticated, we could introduce instead of a hard

cutoff a smooth function which suppresses large momenta. Using Schwinger's proper-time representation (4.93) we can cut-off large momenta setting

$$\frac{1}{p^2+m^2} \to \frac{\mathrm{e}^{-(p^2+m^2)/\Lambda^2}}{p^2+m^2} = \int_{\Lambda^{-2}}^{\infty} \mathrm{d}s\, \mathrm{e}^{-s(p^2+m^2)}\,. \tag{11.7}$$

Although conceptual easy, both regularisation schemes violate generically all symmetries of our theory. This is not a principal flaw, since we should be able to recover these symmetries in the limit $\Lambda \to \infty$. However, this "recovery process" may be non-trivial to perform. Moreover, intermediate calculations become much more transparent if we can use the symmetries of the theory, and therefore these schemes are in practise not useful except for the simplest cases.

- Pauli-Villars regularisation is a scheme where one adds heavy particles having the same quantum numbers and couplings as the originals ones. Thus the propagator of a massless scalar particle is changed to

$$\frac{1}{k^2+\mathrm{i}\varepsilon} \to \frac{1}{k^2+\mathrm{i}\varepsilon} + \sum_i \frac{a_i}{k^2-M_i^2+\mathrm{i}\varepsilon}\,.$$

  For $k^2 \ll M_i^2$, physics is unchanged, while for $k^2 \gg M_i^2$ and $a_i < 0$ the combined propagator scales as M_i^2/k^4 and the convergence of loop integrals improves. Since the heavy particles enter with the wrong sign, they are unphysical ghosts and serve only as a mathematical tool to regularise loop diagrams. Pauli-Villars regularisation respects the gauge invariance of QED, if the heavyy particles are coupled gauge invariantly to the photon.

- Lattice regularisation replaces the continuous space-time by a discrete lattice. The finite lattice spacing a introduces a momentum cutoff, eliminating all UV divergences. Moreover the (Euclidean) path-integral becomes well-defined and can be calculated numerically without the need to do perturbation theory. Thus this approach is particularly useful in the strong-coupling regime of QCD where it has been used to calculate static quantities as e.g. the hadron mass spectrum. Note that lattice regularisation for finite a respects gauge symmetries, but spoils the translation and Lorentz symmetry of the underlying QFT. Nevertheless, one recovers in the limit $a \to 0$ a relativistic QFT. A longstanding problem of lattice theory was how to implement correctly chiral fermions. This question was solved around the year 2000 and thus the SM can be now in a mathematically consistent, non-perturbative way defined as a lattice theory.

- Dimensional Regularisation (DR) is the method we applied in the calculations of the one-loop diagrams of the $\lambda\phi^4$ theory. While DR has the important virtue of preserving Lorentz and gauge invariance, it is one of the least intuitive regularization methods. We will show later that an integral without mass scale is zero in DR, e.g. $\int \mathrm{d}^d k k^{-2} = 0$. This example shows that the integration measure we implement using physical requirements with DR is not positive—as a mathematician would require. In Problem 11.2, we examine how DR modifies the range of momentum values contributing to Feynman integrals.

  Using DR with fermions, we have to extend the Clifford algebra to d dimensions. A natural choice is $\mathrm{tr}(\gamma^\mu\gamma^\nu) = d\eta^{\mu\nu}$ and $\mathrm{tr}(1) = 4$. Problematic is however the treatment of $\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ relying heavily on $d = 4$.

- Various other regularisation methods as e.g. zeta function regularisation or point splitting methods exists.

Even fixing a regularisation method as e.g. DR, we can choose various renormalisation *schemes*. Four popular choices are

- on-shell renormalisation. In this scheme, we choose the subtraction such that the on-shell masses and couplings coincide with the corresponding values measured in processes with zero momentum transfer q. For instance, we define the renormalised electric charge via the Thomson limit of the Compton scattering amplitude. While this choice is very intuitive, it is not practical for QCD: We will see soon that in this theory scattering amplitudes calculated in perturbation theory become ill-defined in the limit $q^2 \to 0$.
- The momentum subtraction (MoM) scheme is a generalisation of the on-shell scheme which can be applied also to QCD. Here we subtract from the Green functions counter terms such that the corrections are zero for a fixed space-like four-momentum $p^2 = -\mu^2$. In this way, divergences in the limit $q^2 \to 0$ are avoided.
- In the minimal subtraction (MS) scheme, we subtract only the divergent $1/\varepsilon$ poles.
- In the modified minimal subtraction $\overline{\text{MS}}$ (read em-es-bar) scheme, we subtract also the $\ln(4\pi) - \gamma$ term appearing frequently. This scheme gives more compact expressions than the others and is most often used in theoretical calculations.

The main advantage of the MS and $\overline{\text{MS}}$ schemes is that they are mass independent, i.e. that the subtraction terms do not depend on the particle masses. This independence simplifies the derivation of "running couplings" (cf. with the calculation of $\lambda(\mu)$ in section 4.3.4). As a drawback of the MS and $\overline{\text{MS}}$ schemes, quantities like the electon mass calculated in these schemes, m_e^{MS} or $m_e^{\overline{\text{MS}}}$, have to be translated into the physical mass m_e.

At a fixed order perturbation theory, predictions and reliability of different schemes vary for given external parameters: A simple example is the change from the MS to the $\overline{\text{MS}}$ scheme which are connected by $\tilde{\mu}^2 = 4\pi\mu^2 \mathrm{e}^{-\gamma}$. Thus this transition is equivalent to a change of the renormalisation scale, altering thereby the size of the $\ln(\mu^2)$ term and thus the strength of the running coupling. More drastic changes result moving from a mass independent to a mass dependent scheme, or comparing DR with other schemes. As a result, running couplings which are small enough to allow perturbation theory in one scheme may be prohibitive large in other schemes.

11.2. Anomalous magnetic moment of the electron

After this overview, let us move on to the calculation of the magnetic moment of the electron which is shifted by loop corrections from the tree-level value $g = 2$ you derived in problem 8.4. Apart from being the first successful loop calculation in the history of QFT, this process illustrates also several generic properties of loop graphs in renormalisable theories like QED.

Vertex function The tree-level interaction $\mathscr{L}_{\text{int}} = e\bar{\psi}\gamma^\mu\psi A_\mu$ between an electron and a photon corresponds in momentum space to $e\bar{u}(p')\gamma^\mu u(p)\varepsilon_\mu(q)$. Since loop integrals depend generally on the external momenta, the tree-level vertex γ^μ is modified by loop graphs as

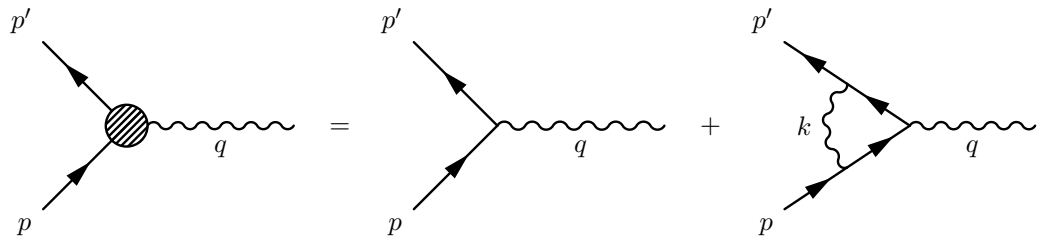


Figure 11.1.: The general vertex for the interaction of a fermion with a photon and its perturbative expansion up to $\mathcal{O}(e^3)$ within QED.

the one shown in Fig. 11.1 and becomes a function of the momenta, $\Lambda^\mu(p, p', q)$. We want to write down the most general form of the vertex function Λ^μ for the coupling between an external electromagnetic field and an on-shell Dirac fermion, consistent with the symmetries of the problem. It is as usually convinient to apply the tensor method, i.e. to express Λ^μ as a sum of linearly independent rank-1 tensors multiplied by scalar functions.

Translation invariance implies $q = p' - p$ and thus Λ^μ is only a function of two momenta which we choose as p and p'. Since $p^2 = p'^2 = m^2$, the only non-trivial scalar variable in the problem is $p \cdot p'$. We choose to use the equivalent quantity $q^2 = (p' - p)^2$ as the variable on which the arbitrary scalar functions in our ansatz for Λ^μ depend. Next we have to form all possible vectors out of the momenta p_μ and p'_μ and the 16 basis elements (8.43) of the Clifford algebra. Restricting ourselves to QED, we have to impose additionally parity conservation what forbids the use of γ^5. Hence the most general ansatz compatible with Poincaré invariance and parity is

$$\Lambda^\mu(p, p') = A(q^2)\gamma^\mu + B(q^2)p^\mu + C(q^2)p'^\mu + D(q^2)\sigma^{\mu\nu}p_\nu + E(q^2)\sigma^{\mu\nu}p'_\nu \,. \tag{11.8}$$

Current conservation requires $q_\mu \Lambda^\mu(p, p') = 0$ and leads to $C = B$ and $E = -D$. Hence

$$\Lambda^\mu(p, p') = A(q^2)\gamma^\mu + B(q^2)(p^\mu + p'^\mu) + D(q^2)\sigma^{\mu\nu}q_\nu \,. \tag{11.9}$$

Hermeticity finally implies that A, B are real and D is purely imaginary.

Gordon decomposition We derive now an identity that allows us to eliminate one of the three terms in Eq. (11.9), if we sandwich Λ^μ between two spinors which are on-shell. We evaluate

$$F^\mu = \bar{u}(p') \left[\not{p}' \gamma^\mu + \gamma^\mu \not{p} \right] u(p) \tag{11.10}$$

first using the Dirac equation for the two on-shell spinors, finding

$$F^\mu = 2m\bar{u}(p')\gamma^\mu u(p) \,. \tag{11.11}$$

Secondly, we can use $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}$, obtaining

$$F^\mu = \bar{u}(p') \left[(p' + p)^\mu + \mathrm{i}\sigma^{\mu\nu}(p' - p)_\nu \right] u(p) \,. \tag{11.12}$$

Equating (11.11) and (11.12) gives the Gordon identity: It allows us to separate the Dirac current into a part proportional to $(p+p')^\mu$, i.e. with the same structure as a scalar current, and a part vanishing for $q = p' - p \to 0$ which couples to the spin of the fermion,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{\mathrm{i}\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p)\,. \tag{11.13}$$

Using the results from problem 8.4, we can identify in the non-relativistic limit the second term as contribution to the magnetic moment of the fermion.

The Gordon identity shows that the three terms in Eq. (11.9) are not independent. Depending on the context, we can eliminate therefore the most annoying term in the vertex function. We follow conventions and introduce the (real) form-factors $F_1(q^2)$ and $F_2(q^2)$ by

$$\Lambda^\mu(p,p') = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m} = \tag{11.14}$$

$$= F_1(q^2)\frac{(p'+p)^\mu}{2m} + [F_1(q^2)+F_2(q^2)]\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\,. \tag{11.15}$$

The form-factor F_1 is the coefficient of the electric charge, $eF_1(q^2)\gamma^\mu$, and should thus go to one for small momentum transfer, $F_1(0) = 1$. Therefore the magnetic moment of an electron is shifted by $1 + F_2(0)$ from the tree-level value $g = 2$. The deviation $a \equiv (g-2)/2$ is called anomalous magnetic moment, the two form-factors are often called electric and magnetic form-factors.

Note the usefulness of the procedure to express the vertex function using only general symmetry requirements but not a specific theory for the interaction: Equation (11.14) allows experimentalists to present their measurements using only two scalar functions which in turn can be easily compared to predictions of specific theories.

Anomalous magnetic moment After having discussed the general structure of the electromagnetic vertex function, we turn now to its calculation in perturbation theory for the case of QED. The Feynman diagrams contributing to the matrix element at $\mathcal{O}(e^3)$ with wave-functions as external lines are shown in Fig. 11.1, where we omit self-energy corrections in the external lines: As we will see soon, the later do not contribute to the anomalous magnetic moment. We separate the matrix element into the tree-level part and the one-loop correction, $-\mathrm{i}e\bar{u}(p')\left[\gamma^\mu + \Gamma^\mu\right]u(p)$. Using the Feynman gauge for the photon propagator, we obtain

$$\Gamma^\mu(p,p') = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-\mathrm{i}}{k^2+\mathrm{i}\varepsilon}(-\mathrm{i}e\gamma^\nu)\frac{\mathrm{i}}{\not{p}'+\not{k}-m+\mathrm{i}\varepsilon}\gamma^\mu\frac{\mathrm{i}}{\not{p}+\not{k}-m+\mathrm{i}\varepsilon}(-\mathrm{i}e\gamma_\nu)\,. \tag{11.16}$$

This integral is logarithmically divergent for large k,

$$\int^\Lambda \mathrm{d}k\,\frac{k^3}{k^2k^2} \propto \ln\Lambda\,. \tag{11.17}$$

Before we perform the explicit calculation, we want to understand if this divergence is connected to a specific kinematical configuration of the momenta. We split therefore the vertex correction into an on-shell and an off-shell part,

$$\Gamma^\mu(p,p') = \Gamma^\mu(p,p) + [\Gamma^\mu(p,p') - \Gamma^\mu(p,p)] \equiv \Gamma^\mu(p,p) + \Gamma^\mu_{\text{off}}(p,p')\,. \tag{11.18}$$

Next we rewrite[1] the first fermion propagator for small $p' - p$ as

$$\frac{1}{\not{p}' + \not{k} - m} = \frac{1}{\not{p} + \not{k} - m + (\not{p}' - \not{p})} = \tag{11.19}$$

$$= \frac{1}{\not{p} + \not{k} - m} - \frac{1}{\not{p} + \not{k} - m}(\not{p}' - \not{p})\frac{1}{\not{p} + \not{k} - m} + \dots \tag{11.20}$$

The first term of this expansion leads to the logarithmic divergence of the loop integral for large k. In contrast, the remainder of the expansion that vanishes for $p' - p = q \to 0$ contains additional powers of $1/k$ and is thus convergent. Hence the UV divergence is contained solely in the on-shell part of the vertex correction, while the function $\Gamma^\mu_{\text{off}}(p, p') = \Gamma^\mu(p, p') - \Gamma^\mu(p, p)$ is well-behaved. Moreover, we learn from Eq. (11.14) that the divergence is confined to $F_1(0)$, while $F_2(0)$ is finite. This is good news: The divergence is only connected to a quantity already present in the classical Lagrangian, the electric charge. Thus we can predict the function $\Gamma^\mu(p, p')$ for all values $p' \neq p$, after we have renormalised the electric charge in the limit of zero momentum transfer.

We now calculate the vertex function (11.16) explicitly. We set

$$\Gamma^\mu(q) = -\mathrm{i}e^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathcal{N}^\mu(k)}{[(p'+k)^2 - m^2]\,[(p+k)^2 - m^2]\;k^2} \tag{11.21}$$

with

$$\mathcal{N}^\mu = \gamma^\nu(\not{p}' + \not{k} + m)\gamma^\mu(\not{p} + \not{k} + m)\gamma_\nu\,. \tag{11.22}$$

Then we combine the propagators introducing as Feynman parameter integrals

$$\frac{1}{xyz} = 2\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \frac{1}{[z + \alpha(x - z) + \beta(y - z)]^3} = 2\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \frac{1}{D}\,. \tag{11.23}$$

Setting $z = k^2$, we obtain

$$D = \{k^2 + \alpha[(p'+k)^2 - m^2 - k^2] + \beta[(p+k)^2 - m^2 - k^2]\}^3\,. \tag{11.24}$$

The complete calculation of the vertex function (11.16) for arbitrary off-shell momenta is already quite cumbersome. In order to shorten the calculation, we restrict ourselves therefore to the part contributing to the magnetic form factor $F_2(0)$. Because of

$$\Lambda^\mu(p, p') = \left[F_1(q^2) + F_2(q^2)\right]\gamma^\mu - F_2(q^2)\frac{(p'+p)^\mu}{2m} \tag{11.25}$$

we can simplify the calculation of $\mathcal{N}^\mu(k)$, throwing away all terms proportional to γ^μ which do not contribute to the magnetic moment. This justifies also why we can neglect diagrams with self-energy corrections in the external lines. Moreover, we can consider the limit that the electrons are on-shell and the momentum transfer to the photon vanishes.

Using the on-shell condition, $p^2 = p'^2 = m^2$, the two square brackets in D simplify to $2p' \cdot k$ and $2p \cdot k$, respectively,

$$D = \left\{k^2 + 2k \cdot (\alpha p' + \beta p)\right\}^3\,. \tag{11.26}$$

[1] The identity $(A + B)^{-1} = A^{-1} - A^{-1}\,B\,A^{-1} + \dots$ can be checked by multiplying with $A + B$.

Next we eliminate the term linear in k completing the square,

$$D = \Big\{(\underbrace{k + \alpha p' + \beta p}_{\ell})^2 - (\alpha p' + \beta p)^2\Big\}^3 = \big\{[\ell^2 - (\alpha^2 m^2 + \beta^2 m^2 + 2\alpha\beta p' \cdot p)\big\}^3 \,. \tag{11.27}$$

Since the momentum transfer to the photon vanishes, $q^2 = 2m^2 - 2p' \cdot p \to 0$, we can replace $p' \cdot p \to m^2$ and obtain as final result for the denominator

$$D = \big\{\ell^2 - (\alpha + \beta)^2 m^2\big\}^3 \,. \tag{11.28}$$

Now we move on to the evaluation of the numerator $\mathcal{N}^\mu(k)$. Performing the change of our integration variable from $k = \ell - (\alpha p' + \beta p)$ to ℓ, the numerator becomes

$$\mathcal{N}^\mu(\ell) = \gamma^\nu(\not{P}' + \not{\ell} + m)\gamma^\mu(\not{P} + \not{\ell} + m)\gamma_\nu \tag{11.29}$$

with $\not{P}' \equiv (1-\alpha)\not{p}' - \beta\not{p}$ and $\not{P} \equiv (1-\beta)\not{p} - \alpha\not{p}'$.

Multiplying out the two brackets and ordering the result according to powers of m, we observe first that the term $\propto m^2$ leads to $\propto \gamma^\mu$ and thus does not contribute to $F_2(0)$. Next we split further the term linear in m according to powers of ℓ: The term linear in ℓ vanishes after integration, while the term $m\not{\ell}^0$ results in

$$m(\gamma^\nu \not{P}'\gamma^\mu\gamma_\nu + \gamma^\nu\gamma^\mu\not{P}\gamma_\nu) = 4m(P'^\mu + P^\mu) = 4m[(1-2\alpha)p'^\mu + (1-2\beta)p^\mu] \,. \tag{11.30}$$

Using the symmetry in the integration variables α and β, we can rewrite this expression as

$$\to 4m[(1-\alpha-\beta)(p'^\mu + p^\mu)] \,. \tag{11.31}$$

We split the m^0 term in the same way according to the powers of $\not{\ell}$. The $m^0\not{\ell}^2$ term gives a γ^μ term, the $m^0\not{\ell}$ vanishes after integration, and the $m^0\not{\ell}^0$ gives after some work

$$\gamma^\nu \not{P}'\gamma^\mu\not{P}\gamma_\nu \to 2m[\alpha(1-\alpha) + \beta(1-\beta)](p' + p)^\mu \,. \tag{11.32}$$

Finally, the m^0 term contributes to the anomalous magnetic moment

$$\to -2m(p' + p)^\mu[2(1-\alpha)(1-\beta)] \,. \tag{11.33}$$

Combining all terms, we find

$$\begin{aligned}\mathcal{N}^\mu &= 4m(1-\alpha-\beta)(p'+p)^\mu + 2m[\alpha(1-\alpha) + \beta(1-\beta)](p'+p)^\mu \\ &\quad - 4m(1-\alpha)(1-\beta)(p'+p)^\mu = \\ &= 2m[(1-\alpha-\beta)(\alpha+\beta)](p'+p)^\mu \,.\end{aligned} \tag{11.34}$$

Thus

$$\Gamma_2^\mu(0) = -2\mathrm{i}e^2 \int \mathrm{d}\alpha\mathrm{d}\beta \int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{\mathcal{N}^\mu}{[\ell^2 - (\alpha+\beta)^2 m^2]^3} \,, \tag{11.35}$$

where the subscript 2 indicates that we account only for the contribution to the anomalous magnetic moment. We expressed also the loop integral in 2ω dimensions, such that we can apply the general formula derived in the appendix 4.A. Using Eq. (4.109) for $I(\omega, a)$ with $\omega = 2$ and $a = 3$,

$$I(2,3) = -\frac{\mathrm{i}}{32\pi^2} \frac{1}{(\alpha+\beta)^2 m^2 + \mathrm{i}\varepsilon} \,, \tag{11.36}$$

we obtain as expected a finite result. As last step, we perform the integrals over the Feynman parameters α and β,

$$\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \, \frac{1-\alpha-\beta}{\alpha+\beta} = \frac{1}{2}\,, \tag{11.37}$$

and find thus

$$\Gamma_2^\mu(0) = -\frac{e^2}{8\pi^2}\,\frac{1}{2m}\,(p'+p)^\mu\,. \tag{11.38}$$

Recalling Eq. (11.25), we can identify the factor $e^2/(8\pi^2)$ with the magnetic form factor $F_2(0)$. We have thus reproduced the result of the first successful calculation of a loop correction in a QFT, performed by Schwinger, and independently by Feynman and Tomonaga, in 1948, $F_2(0) = \alpha/(2\pi)$. Together with Bethe's previous estimate of the Lamb shift in the hydrogen energy spectrum, this stimulated the view that a consistent renormalisation of QED is possible.

The currently most precise experimental value for the electron anomalous magnetic moment $a_e \equiv F_2(0)$ is

$$a_e^{\mathrm{exp}} = 0.001\,159\,652\,180\,73 \pm 2.4 \times 10^{-10}]\,. \tag{11.39}$$

The calculation of the universal (i.e. common to all charged leptons) QED contribution has been completed up to fourth order. There exists also an estimate of the dominant fifth order contribution,

$$\begin{aligned} a_\ell^{\mathrm{uni}} &= 0.5\,\left(\frac{\alpha}{\pi}\right) - 0.328\,478\,965\,579\,193\,78\ldots\,\left(\frac{\alpha}{\pi}\right)^2 \\ &\quad + 1.181\,241\,456\,587\ldots\,\left(\frac{\alpha}{\pi}\right)^3 - 1.9144(35)\,\left(\frac{\alpha}{\pi}\right)^4 + 0.0(4.6)\,\left(\frac{\alpha}{\pi}\right)^5 \\ &= 0.001\,159\,652\,176\,30(43)(10)(31)\cdots \end{aligned} \tag{11.40}$$

The three errors given in round brackets are the error from the uncertainty in α, the numerical uncertainty of the α^4 coefficient and the error estimated for the missing higher order terms[41]. Comparing the measured value and the prediction using QED, we find an extremely good agreement. First of all, this is strong support that the methods of perturbative QFT we developed so far can be successfully applied to weakly coupled theories as QED. Secondly, it means that additional contributions to the anomalous magnetic moment of the electron have to be tiny.

Electroweak and other corrections The lowest order electroweak corrections to the anomalous magnetic moment contain in the loop virtual gauge bosons ($W^\pm$, Z) or a Higgs boson h and are shown in Fig. 11.2. We will consider the electroweak theory describing these diagrams only later; for the present discussion it is sufficient to know that the weak coupling constant is $g \sim 0.6$ and that the scalar and weak gauge bosons are much heavier than leptons, $M \gg m$.

The second diagram corresponds schematically to the expression

$$\sim g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2-M^2}\,\frac{A(m^2,k)}{[(p-k)-m^2]^2}\,. \tag{11.41}$$

As in QED, this integral has to be finite and we expect that it is dominated by momenta up to the mass M of the gauge bosons, $k \lesssim M$. Therefore its value should be proportional to g^2m^2/M^2 (times a possible logarithm $\ln(M^2/m^2)$) and electroweak corrections to the

anomalous magnetic moment of the electron are suppressed by a factor $(m/M)^2 \sim 10^{-10}$ compared to the QED contribution. The property that the contribution of virtual heavy particles to loop processes is suppressed in the limit $|q^2| \ll M^2$ is called "decoupling". Note the difference to the case of the mass of a scalar particle or the cosmological constant: In these examples, the loop corrections are infinite and we cannot predict these quantities. In contrast, the anomalous magnetic moment is finite but, as we include loop momenta up to infinity, depends in principal on all particles coupling to the electron, even if they are arbitrarily heavy. Only if these heavy particles "decouple," we can calculate a_e without knowing e.g. the physics at the Planck scale. Thus the decoupling property is a necessary ingredient of any reasonable theory of physics, otherwise no predictions would be possible before knowing the "theory of everything".

Clearly, the contribution of heavy particles (either electroweak gauge and Higgs bosons or other not yet discovered particles) is more visible in the anomalous magnetic moment of the muon than of the electron. Moreover, a relativistic muon lives long enough that a measurement of its magnetic moment is feasible. This is one example how radiative corrections (here evaluated at $q^2 = 0$) are sensitive to physics at higher scales M: If an observable can be measured and calculated with high enough precision, one can be sensitive to suppressed corrections of order $g^2 m^2/M^2$. Other examples are rare processes like $\mu \to e+\gamma$ or $B_s \to \mu^+\mu^-$ which are suppressed by a specific property of the SM which one does not expect to hold in general. The achieved precision in measuring and calculating such processes is high enough to probe generically scales of $M \sim 100\,\text{TeV}$, i.e. much higher than the mass scales that can be probed directly at current accelerators as LHC.

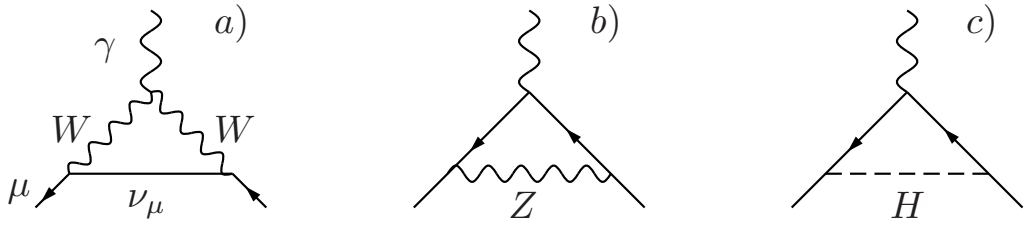


Figure 11.2.: Lowest order electroweak corrections to the anomalous magnetic moment of fermions.

Finite versus divergent parts of loop corrections We found that the vertex correction could be split into two parts

$$\Lambda^\mu(p,p') = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\,, \tag{11.42}$$

where the form factor $F_2(q^2)$ is finite for all q^2, while the form factor $F_1(q^2)$ diverges for $q^2 \to 0$. The important observation is that $F_2(q^2)$ corresponds to a Lorentz structure that is not present in the original Lagrangian of QED. This suggests that we can require from a "nice" theory that

- all UV divergences are connected to structures contained in the original Lagrangian, all new structures are finite. The basic divergent structures are also called "primitive" divergent graphs.

- If there are no anomalies, then loop corrections respect the original (classical) symmetries. Thus, e.g., the photon propagator should be at all orders transverse, respecting gauge invariance. We will see that as consequence the high-energy behaviour of the theory improves.

In such a case, we are able to hide all UV divergences in a renormalisation of the original parameters of the Lagrange density.

11.3. Power counting and renormalisability

We try to make the requirements on a "nice" theory a bit more precise. Let us consider the set of $\lambda\phi^n$ theories in $d = 4$ space-time dimensions and check which graphs are divergent. We define the superficial degree D of divergence of a Feynman graph as the difference between the number of loop momenta in the numerator and denominator of a Feynman graph. We can restrict our analysis to those diagrams called 1P irreducible (1PI) which cannot be disconnected by cutting an internal line: All 1P reducible diagrams can be decomposed into 1PI diagrams which do not contain common loop integrals and can be therefore analysed separately. Moreover, we are only interested in the loop integration and define therefore the *1PI Green functions*[2] as graphs where the propagators on the external lines were stripped off. In $d = 4$ space-time dimensions, the degree D of divergence of a 1PI Feynman graph is thus

$$D = 4L - 2I\,, \tag{11.43}$$

where L is the number of independent loop momenta and I the number of internal lines. The former contributes a factor d^4p, while the latter corresponds to a scalar propagator with $1/(p^2 - m^2) \sim 1/p^2$ for $p \to \infty$.

Momentum conservation at each vertex leads for an 1PI-diagram to

$$L = I - (V - 1)\,, \tag{11.44}$$

where V is the number of vertices and the -1 takes into account the delta function leading to overall momentum conservation: The latter constrains only the external not the loop momenta. Thus

$$D = 2I - 4V + 4\,. \tag{11.45}$$

Each vertex connects n lines and any internal line reduces the number of external lines by two. Therefore the number E of external lines is given by

$$E = nV - 2I\,. \tag{11.46}$$

As result, we can express the superficial degree D by the order of perturbation theory (V), the number of external lines E and the degree n of the interaction polynomial ϕ^n,

$$D = (n - 4)V + 4 - E\,. \tag{11.47}$$

From this expression, we see that

[2] Similiar to their relatives, the (dis-) connected Green functions, also the 1PI Green functions can be derived from a generating functional which we will introduce in the next chapter.

- for $n = 3$, the coefficient of V is negative. Therefore only a finite number of terms in the perturbative expansion are infinite. Such a theory is called super-renormalisable, the corresponding terms in the Hamiltonian are also called relevant.
- For $n = 4$, we find $D = 4 - E$. Thus the degree of divergence is independent of the order of perturbation theory being only determined by the number of external lines. Such theories contain an infinite number of divergent graphs, but they all correspond to a finite number of divergent structures—the so-called primitive divergent graphs. These interactions are also called marginal and are candidates for a renormalisable theory.
- Finally, for $n > 4$ the degree of divergence increases with the order of perturbation theory. As result, there exists an infinite number of divergent structures, and increasing the order of perturbation theory requires more and more input parameter to be determined experimentally. Such a theory is called non-renormalisable, the interaction irrelevant.

In particular, the $\lambda\phi^4$ theory as an example for a renormalisable theory has only three divergent structures: i) the case $E = 0$ and $D = 4$ corresponding a contribution to the cosmological constant, ii) the case $E = 2$ and $D = 2$ corresponding to the self-energy, and iii) the case $E = 4$ and $D = 0$, i.e. logarithmically divergence, to the four-point function. As we saw in chapter 3, the three primitive divergent diagrams of the $\lambda\phi^4$ theory correspond to the following physical effects: Vacuum bubbles renormalise the cosmological constant. The effect of self-energy insertions is twofold: Inserted in external lines it renormalises the field, while self-energy corrections in internal propagators lead to a renormalisation of its mass. The vertex correction finally renormalises the coupling strength λ.

Let us move to the case of QED. Repeating the discussion, we obtain the analogue to Eq. (11.47), but accounting now for the different dimension of fermion and bose fields,

$$D = 4 - B - \frac{3}{2}F\,, \tag{11.48}$$

where B and F count the number of external bosonic and fermionic lines, respectively. There are six different superficially divergent primitive graphs in QED: The photon and the fermionic contribution to the cosmological constant ($D = 4$), the vacuum polarisation ($D = 2$), the fermion self-energy ($D = 1$), the vertex correction ($D = 0$) and light-by-light scattering ($D = 0$). Recall that Furry's theorem implies that loops with a an odd number of fermion propagator vanish in QED. Therefore we have not included in our list of primitive divergent graphs of QED the tadpole ($B = 1$ and $D = 3$) and the "photon splitting" graph ($B = 3$ and $D = 1$).

In a theory with symmetries such as a gauge theory, the true degree of divergence can be smaller than the superficial one. For instance, light-by-light scattering corresponds to a term $\mathscr{L} \sim A^4$ that violates gauge invariance. Thus either the gauge symmetry is violated by quantum corrections or such a term is finite.

Because of the correspondence of the dimension of a field and the power of its propagator, we can connect the superficial degree of divergence of a graph to the dimension of the coupling constants at its vertices. The superficial degree $D(G)$ of divergence of a graph is connected to the one of its vertices D_v by

$$D(G) - 4 = \sum_v (D_v - 4) \tag{11.49}$$

which in turn depends as

$$D_v = \delta_v + \frac{3}{2} f_v + b_v = 4 - [g_v] \tag{11.50}$$

on the dimension of the coupling constant g at the vertex v. Here, f_v and b_v are the number of fermion and boson fields at the vertex, while δ_v counts the number of derivatives.

Thus the dimension of the coupling constant plays a crucial role deciding if a certain theory is "nice" in the naive sense defined above. Clearly $D = 0$ or $[g] = 0$ is the border-line case:

- If at least one coupling constant has a negative mass dimension, $[g] < 0$ and $D_v > 4$, the theory is non-normalisable. Examples are the Fermi theory of weak interactions, $[G_F] = -2$, and gravitation, $[G_N] = -2$.
- If all coupling constants have positive mass dimension, $[g] > 0$ and $D_v < 4$, the theory is super-normalisable. An example is the $\lambda\phi^3$ theory in $D = 4$ with $[\lambda] = 1$.
- The remaining cases, with all $[g_i] = 0$, are candidates for renormalisable theories. Examples are Yukawa interactions, $\lambda\phi^4$, Yang-Mills theories that are unbroken (QED and QCD) or broken by the Higgs mechanism (electroweak interactions).

Theories with massive bosons We have assumed that bosonic propagators behave as $\propto 1/k^2$ for large (Euclidean) momenta k. This is true both for massive and massless scalars, while it holds only for massless particles with spin $s \geq 1$: As we have seen, the massless spin-1 and spin-2 propagators in the R_ξ gauge decrease like $\propto 1/k^2$ for large k. In contrast, the massive spin-1 propagator behaves as $D_F^{\mu\nu}(k) \propto \text{const}$. Thus the divergences in loop diagrams are more severe for massive vector particles than for massless ones. For a massive bosonic field of spin s, the polarisation tensors contains s tensor products of $k_\mu k_\nu$ and therefore its propagator scales as $D_{\mu_1,\cdots,\mu_s;\nu_1,\cdots,\nu_s}(k) \propto k^{2s-2}$. This implies that the divergences of loop diagrams aggravate for higher spin fields. In particular, inserting additional massive propagators into a loop graph does not improve its convergence and thus a theory with massive $s > 0$ particles contains an infinite number of divergent diagrams at each loop order. Including an explicit mass term for gauge bosons leads therefore to a non-renormalisable theory. A solution to this problem is the introduction of gauge boson masses via the Higgs mechanism, which we will introduce in chapter 13.3. Combined with our finding that interacting theories of massless bosons are only possible for $s \leq 2$, we can conclude that elementary particles should have spin $s \leq 2$.

11.4. Renormalisation of the $\lambda\phi^4$ theory

We have argued that a theory with dimensionless coupling constant is renormalisable, i.e. that a multiplicative shift of the parameters contained in the classical Lagrangian is sufficient to obtain finite Green functions. The simplest theory of this type in $n = 4$ is the $\lambda\phi^4$ theory for which we will discuss now the renormalisation procedure at one loop level. As starter, we examine the general structure of the UV divergences.

11.4.1. Structure of the divergences

We learnt that the degree of divergence decreases increasing the number of external lines, since the number of propagators increases. The same effect has taking derivatives w.r.t. external

momenta p,

$$\frac{\partial}{\partial \not{p}} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{\not{k}+\not{p}-m} = -\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(\not{k}+\not{p}-m)^2}\,.$$

This means that

1. we can Taylor expand loop integrals, confining the divergences in the lowest order terms. Choosing e.g. $p=0$ as expansion point in the fermion self-energy,

$$\Sigma(p) = A_0 + A_1\not{p} + A_2 p^2 + \ldots \qquad \text{with} \qquad A_n = \frac{1}{n!}\frac{\partial^n}{\partial \not{p}^n}\Sigma(p)\,,$$

   we know that A_0 is (superficially) linear divergent. Thus A_1 can be maximally logarithmically divergent, while all the other A_n are finite.
2. We could choose a different expansion point, leading to different renormalisation conditions (within the same regularisation scheme).
3. The divergences can be subtracted by local operators, i.e. by polynomials of the fields and their derivatives. These terms called counter-terms have for a renormalisable theory the same structure as the terms present in the classical Lagrangian. For instance, the linear divergent term A_0 can be associated to a counter-term $\delta A_0 \bar{\psi}\psi$, while $A_1\not{p}$ corresponds to the counter-term $\delta A_1 \bar{\psi}\not{\partial}\psi$.

It is easy to show that the counter-terms are local operators at the one-loop level, where diagrams contain only one integration variable. Any loop integral $I(p)$ with superfical degree of divergence $n-1$ becomes finite after taking n derivatives w.r.t. an external momentum p. Using a cutoff Λ as regulator, this implies that in

$$\frac{\partial^n}{\partial p^n} I(p) = f(p) + \mathcal{O}(p/\Lambda) \tag{11.51}$$

the function $f(p)$ is finite and independent of Λ, while the remainder vanishes in the limit $\Lambda \to \infty$. Integrating this expression n times, we obtain

$$I(p) = F(p) + P_n(p) + \mathcal{O}(p/\Lambda)\,, \tag{11.52}$$

where $F(p)$ is also finite and independent of Λ. The function $P_n(p)$ is a n.th order polynomial containing the integration constants. Since $F(p)$ is finite, $P_n(p)$ comprises all divergencies. They are therefore the coefficients of polynomials in the external momentum p and can be subtracted by local operators, as we claimed. This argument shows also that all non-trivial analytical structures like cuts have to be contained in $F(p)$. Moreover, choosing a different regularisation scheme or point leads to the same $f(p)$ in (11.51), and thus all the scheme dependence is contained in the polynomial $P_n(p)$. As a result, the differences caused by different schemes reside only in local terms which are absorbed in the renormalisation of the parameters.

Going to higher loop orders, non-local terms as e.g. $\ln(p^2/\mu^2)$ can be generated by subdivergences. These are divergences connected to integration regions where one or more loop momenta are finite, while the remaining ones are send to infinity. Such terms are cancelled by counter-terms determined at lower order. A sketch why this should be true goes as follows:

Green functions become singular for coinciding points, i.e. when the convergence factor e^{-kx} in the Euclidean Green function becomes one. In the simplest cases as $\langle 0|\phi(x')\phi(x)|0\rangle_{x'\to x}$, the infinities are

eliminated by normal ordering, i.e. by rewriting all creation operators on the left of the annihilation operators, cf. problem 3.6. More complicated are overlapping divergences where two or more divergent loops share a propagator. Wilson suggested to expand the product of two fields as the sum of local operators O_i times coefficient functions $C_i(x-y)$ as

$$\phi(x)\phi(y) = \sum_i C_i(x-y)O_i(x)\,,$$

where the whole spatial dependence is carried by the coefficients and the local operators O_i are of the type $O_i(x) = \phi(x)\partial_\mu \cdots \partial^\mu\phi(x)$. For a massless scalar field, dimensional analysis dictates that $C_i(x) \propto x^{-2+d_i}$, if the local operator O_i has dimension d_i. Note that only the unity operator has a singular coefficient function $1/x^2$ corresponding to the massless scalar propagator. Similarly we can expand products of operators,

$$O_n(x)O_m(y) = \sum_i C^i_{nm}(x-y)O_i(x)\,,$$

where now $C^i_{nm}(x) \propto x^{-d_n-d_m+d_i}$. Thus we can use this operator product expansion (or briefly "OPE") to rewrite the overlapping divergences in terms of (singular) coefficient functions and *local* operators. Moreover, the sub-divergence occurring at order k, when $p < k$ points coincide, are eliminated by the counter-terms found at order p.
Elaborating this argument in detail, one can conclude that non-local terms due to overlapping divergencies are cancelled by the counter terms found at lower order. We will see how this works in praxis in the next section, when we calculate the vacuum energy at two-loop.

11.4.2. The $\lambda\phi^4$ theory at $\mathcal{O}(\lambda)$

There are two equivalent ways to perform perturbative renormalisation. In the one called often "conventional" perturbation theory we use the "bare" (unrenormalised) parameters in the Lagrangian,

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\,\phi_0^4\,. \tag{11.53}$$

Then we introduce a renormalised field $\phi_R = Z_\phi^{-1/2}\phi_0$ and choose the parameters Z_ϕ, m_0 and λ_0 as function of the regularisation parameter (ε, $\Lambda, \ldots$) such that the field ϕ_R has finite Green functions. In the following, we discuss the renormalisation procedure at the one-loop level for the Green functions of the $\lambda\phi^4$ theory in this scheme. Since any 1P reducible diagram can be decomposed into 1PI diagrams which do not contain common loop integrals, we can restrict our analysis again to 1PI Green functions.

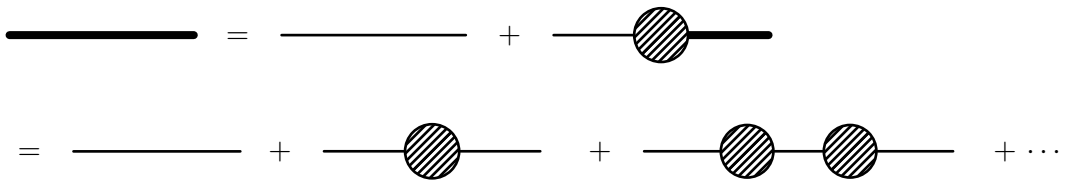
Mass and wave-function renormalisation We defined the exact or full propagator $\mathrm{i}\boldsymbol{\Delta}_F(x_1,x_2)$ in Eq. (4.2) as the path integral average of the two fields $\phi(x_1)\phi(x_2)$. Now we want to find a definition which is useful for calculations in perturbation theory: We claim that

$$[\mathrm{i}\boldsymbol{\Delta}_F(p)]^{-1} = [\mathrm{i}\Delta_F(p)]^{-1} - \boldsymbol{\Sigma}(p) = p^2 - m^2 - \boldsymbol{\Sigma}(p) - \mathrm{i}\varepsilon \tag{11.54}$$

is the exact propagator, where the exact self-energy $\boldsymbol{\Sigma}(p)$ represents the 1PI corrections to the scalar mass, $m^2_{\text{phys}} = m_0^2 + \boldsymbol{\Sigma}$. Multiplying this definition from the right with $\mathrm{i}\boldsymbol{\Delta}_F(p)$ and from the left with $\mathrm{i}\Delta_F(p)$, the so-called Dyson equation follows,

$$\mathrm{i}\boldsymbol{\Delta}_F(p) = \mathrm{i}\Delta_F(p) + \mathrm{i}\Delta_F(p)\boldsymbol{\Sigma}(p)\mathrm{i}\boldsymbol{\Delta}_F(p) = \mathrm{i}\Delta_F(p)\left[1 + \boldsymbol{\Sigma}(p)\mathrm{i}\boldsymbol{\Delta}_F(p)\right]\,. \tag{11.55}$$

Graphically, we can express this equation as



where the second line follows by iteration. Hence, $\mathrm{i}\boldsymbol{\Delta}_F(p)$ sums up the amplitudes to propagate at momentum p with zero, one,... self-energy Σ insertions, and corresponds therefore to the full propagator. At $\mathcal{O}(\lambda)$, we see that this relation holds comparing it to Eq. (4.35).

Next we have to show that the self-energy $\Sigma(p^2)$ is finite after renormalisation. The 1-loop expression is

$$-\mathrm{i}\Sigma(p^2) = \frac{-\mathrm{i}\lambda_0}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}}{k^2-m_0^2+\mathrm{i}\varepsilon}\,, \tag{11.56}$$

i.e. quadratically divergent. As a particularity of the ϕ^4 theory, the p^2 dependence of the self-energy Σ shows up only at the 2-loop level. We perform a Taylor expansion of $\Sigma(p^2)$ around the arbitrary point μ,

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2-\mu^2)\Sigma'(\mu^2) + \tilde{\Sigma}(p^2)\,, \tag{11.57}$$

where $\Sigma(\mu^2)\propto\Lambda^2$, $\Sigma'(\mu^2)\propto\ln\Lambda$ and $\tilde{\Sigma}(p^2)$ is the finite remainder. A term linear in Λ is absent, since we cannot construct a Lorentz scalar out of p^μ. Note also $\tilde{\Sigma}(\mu^2)=0$.

Now we insert (11.57) into (11.54),

$$\frac{\mathrm{i}}{p^2-m_0^2-\Sigma(p^2)+\mathrm{i}\varepsilon} = \frac{\mathrm{i}}{\underbrace{p^2-m_0^2-\Sigma(\mu^2)}_{p^2-\mu^2}-(p^2-\mu^2)\Sigma'(\mu^2)-\tilde{\Sigma}(p^2)+\mathrm{i}\varepsilon}\,, \tag{11.58}$$

where we see that we can identify μ with the renormalised mass given by the pole of the propagator.

We aim at rewriting the remaining effect for $p^2\to\mu^2=m^2$ of the self-energy insertion, $\Sigma'(m^2)$, as a multiplicative rescaling. In this way, we could remove the divergence from the propagator by a rescaling of the field. At leading order in λ, we can write

$$\tilde{\Sigma}(p^2) = \left[1-\Sigma'(m^2)\right]\tilde{\Sigma}(p^2) + \mathcal{O}(\lambda_0^2) \tag{11.59}$$

and thus

$$\mathrm{i}\boldsymbol{\Delta}_F(p) = \frac{1}{1-\Sigma'(m^2)}\,\frac{\mathrm{i}}{p^2-m^2-\tilde{\Sigma}(p^2)+\mathrm{i}\varepsilon} = \frac{\mathrm{i}Z_\phi}{p^2-m^2-\tilde{\Sigma}(p^2)+\mathrm{i}\varepsilon} \tag{11.60}$$

with the wave-function renormalisation constant

$$Z_\phi = \frac{1}{1-\Sigma'(m^2)} = 1+\Sigma'(m^2)\,. \tag{11.61}$$

Close to the pole, the propagator is the one of a free particle with mass m,

$$\mathrm{i}\boldsymbol{\Delta}_F(p) = \frac{\mathrm{i}Z_\phi}{p^2-m^2+\mathrm{i}\varepsilon} + \mathcal{O}(p^2-m^2)\,. \tag{11.62}$$

Thus the renormalisation constant Z_ϕ equals the wave-function renormalisation constant Z we had to introduce into the LSZ formalism to obtain correctly normalised states.

We define the renormalised field $\phi = Z_\phi^{-1/2}\phi_0$ such that the renormalised propagator

$$\mathrm{i}\Delta_R(p) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \langle 0|\, T\{\phi(x)\phi(0)\}\, |0\rangle = Z_\phi^{-1}\, \mathrm{i}\Delta(p) = \frac{\mathrm{i}}{p^2 - \mu^2 - \tilde{\Sigma}(p^2) + \mathrm{i}\varepsilon} \tag{11.63}$$

is finite. Similarly, we define renormalised n-point functions by

$$G_R^{(n)}(x_1, \ldots, x_n) = \langle 0|\, T\{\phi(x_1)\cdots\phi_n(x_n)\}\, |0\rangle = Z_\phi^{-n/2}\, G_0^{(n)}(x_1, \ldots, x_n)\,. \tag{11.64}$$

Since the 1PI n-point Green functions miss n field renormalisation constants compared to connected n-point Green functions, the connection between renormalised and bare 1PI n-point functions is given by

$$\Gamma_R^{(n)}(x_1, \ldots, x_n) = Z_\phi^{n/2}\, \Gamma_0^{(n)}(x_1, \ldots, x_n)\,. \tag{11.65}$$

Coupling constant renormalisation We can choose an arbitrary point inside the kinematical region, $s + t + u = 4\mu^2$ and $s \geq 4\mu^2$, to connect the coupling to a physical measurement at this point. For our convenience and less writing work, we choose instead the symmetric point

$$s_0 = t_0 = u_0 = \frac{4\mu^2}{3}\,.$$

The bare four-point 1PI Green function is (see section 4.3.3)

$$\Gamma_0^{(4)}(s,t,u) = -\mathrm{i}\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)\,, \tag{11.66}$$

the renormalised four-point function at (s_0, t_0, u_0) is

$$\Gamma_R^{(4)}(s_0,t_0,u_0) = -\mathrm{i}\lambda\,. \tag{11.67}$$

Next we expand the bare 4-point function around s_0, t_0, u_0,

$$\Gamma_0^{(4)}(s,t,u) = -\mathrm{i}\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \tag{11.68}$$

where the $\tilde{\Gamma}(x)$ are finite and zero at x_0. Now we define a vertex (or coupling constant) renormalisation constant by

$$-\mathrm{i}Z_\lambda^{-1}\lambda_0 = -\mathrm{i}\lambda_0 + 3\Gamma(s_0) \tag{11.69}$$

Inserting this definition in (11.68) we obtain

$$\Gamma_0^{(4)}(s,t,u) = -\mathrm{i}Z_\lambda^{-1}\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \tag{11.70}$$

what simplifies at the renormalisation point to

$$\Gamma_0^{(4)}(s_0,t_0,u_0) = -\mathrm{i}Z_\lambda^{-1}\lambda_0\,. \tag{11.71}$$

We use now the connection between renormalised and bare Green functions,

$$\Gamma_R^{(4)}(s,t,u) = Z_\phi^2\Gamma_0^{(4)}(s,t,u)\,, \tag{11.72}$$

and thus

$$-\mathrm{i}\lambda = Z_\phi^2 Z_\lambda^{-1}(-\mathrm{i}\lambda_0)\,. \tag{11.73}$$

The relation between the renormalised and bare coupling in the $\lambda\phi^4$ theory is thus

$$\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0\,. \tag{11.74}$$

Now we have to show that $\Gamma_R^{(4)}(s,t,u)$ is finite. Inserting (11.70) into (11.72), we find

$$\begin{aligned}\Gamma_R^{(4)}(s,t,u) &= -\mathrm{i}Z_\phi^2 Z_\lambda^{-1}\lambda_0 + Z_\phi^2[\tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u)] \\ &= -\mathrm{i}\lambda + Z_\phi^2[\tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u)]\end{aligned} \tag{11.75}$$

Since $Z_\phi = 1+\mathcal{O}(\lambda^2)$ and $\tilde\Gamma = \mathcal{O}(\lambda^2)$, this is equivalent to

$$\Gamma_R^{(4)}(s,t,u) = -\mathrm{i}\lambda + [\tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u)] + \mathcal{O}(\lambda^3) \tag{11.76}$$

consisting only of finite expressions. This completes the proof that at one-loop order all Green functions in the $\lambda\phi^4$ theory are finite, renormalising the field ϕ, its mass and coupling constant as

$$\phi = Z_\phi^{-1/2}\phi_0 \tag{11.77a}$$

$$\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0 \equiv \bar Z_\lambda \lambda_0\,, \tag{11.77b}$$

$$m^2 = m_0^2 + \Sigma(m^2) = m_0^2 + \delta m^2\,. \tag{11.77c}$$

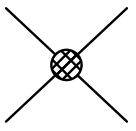
Renormalised perturbation theory In this approach, we rescale first the bare parameters in the classical Langragian, $\phi_0 = Z_\phi^{1/2}\phi_R$, $m_0 = Z_m m_R$, and $\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0 = \bar Z_\lambda\lambda_0$. The renormalisation constants Z_i vanish at tree-level and allow for a perturbative expansion. Setting $Z_i = 1+\delta_i$, we can re-express the Lagrangian through the renormalised quantities as

$$\begin{aligned}\mathscr{L} = &\frac{1}{2}Z_\phi(\partial_\mu\phi)^2 - \frac{1}{2}Z_\phi Z_m m^2\phi^2 - \frac{\lambda}{4!}\,\bar Z_\lambda\phi^4 \\ &+ \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}(\delta_\phi+\delta_m)m^2\phi^2 - \frac{\lambda}{4!}\,\delta_{\bar\lambda}\phi^4\,.\end{aligned} \tag{11.78}$$

The terms in the second line contain the divergent renormalisation constants, and this part is called the counter term Lagrangian $\mathscr{L}_{\mathrm{ct}}$. An advantage of renormalised perturbation theory is that now the expansion parameter is the renormalised coupling λ. Treating $\mathscr{L}_{\mathrm{ct}}$ as a perturbation, $Z_i = 1+\sum_{n=1}^\infty \delta_i^{(n)}$, we obtain in momentum space as additional Feynman vertices

$$= \mathrm{i}[\delta_\phi p^2 - (\delta_\phi+\delta_m)m^2] \tag{11.79}$$

and



$$= -\mathrm{i}\delta_{\bar\lambda}\lambda\,. \tag{11.80}$$

Applying renormalised perturbation theory consists of the following steps:

1. Starting from (11.78) with $n = 0$, i.e. $\delta_i^{(0)} = 0$, one derives propagator and vertices.
2. One calculates 1-loop 1PI diagrams and finds the divergent parts which determine the counter-terms $\delta_i^{(1)}$ at order $\mathcal{O}(\lambda)$. Then all other one-loop diagrams can be calculated.
3. Moving to two-loops, one generates 2-loop 1PI diagrams using the Lagrangian with the one-loop counter-terms $\delta_i^{(1)}$. They are used to extract the counter-terms $\delta_i^{(2)}$ at order $\mathcal{O}(\lambda^2)$.
4. The procedure is iterated moving to higher orders.

We illustrate the use of renormalised perturbation theory with the calculation of the remaining loop diagram at $\mathcal{O}(\lambda)$, the vacuum energy density. This example for a two-loop diagram shows also how sub-divergences are cancelled by counter-terms found at lower loop order. Including the vacuum energy density, the Lagrangian (11.78) becomes $\mathscr{L} \to \mathscr{L} + Z_\rho \rho + \delta_\rho \rho$.

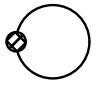
Example 11.3: Vacuum energy density at two-loop:
According to step 2., we should determine first the counter terms in $\mathscr{L}_{\rm ct}^{(1)}$ from the already calculated 1-loop 1PI diagrams. We collect first from chapter 4 the relevant results,

$$\rho^{(1)} = -\frac{m^4}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \ln(\mu^2/m^2)\right], \qquad \delta_\rho^{(1)} = \frac{m^4}{(4\pi)^2}\frac{1}{\varepsilon}, \quad \text{and} \quad \delta_m^{(1)} = \frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\frac{1}{\varepsilon}, \tag{11.81}$$

where we use the MS scheme and re-scaled $\mu^2 \to 4\pi\mu^2 \exp(-\gamma)$. Inserting the one-loop self-energy into the two-loop expression $\rho_a^{(2)} = \lambda/8\,\Delta_F^2(0)$ results in

$$\rho_a^{(2)} = \frac{\lambda}{8}\Delta_F^2(0) = \frac{\lambda}{8}\frac{m^4}{(4\pi)^4}\left[\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\left(\frac{\mu^2}{m^2}\right) + \ln^2\left(\frac{\mu^2}{m^2}\right)\right]. \tag{11.82}$$

Here a mixed term, combining a pole term $1/\varepsilon$ and a logarithm with argument μ^2/m^2, has appeared. In general, the logarithm will depend both on the masses of the loop particles and the external momenta p, $\ln[f(\mu^2/m^2, \mu^2/p^2)]$. Such terms cannot be subtracted by local polynomials in the momenta p as counter terms. In a renormalisable theory, they have to be therefore cancelled by counter-terms determined at lower loop order.
In our concrete case, we have to add only the Feynman diagram generated by the counter-term $-\frac{1}{2}\delta_m^{(1)} m^2\phi^2$, since $\delta_{\bar\lambda}$ contributes only from the two-loop level on. This interaction generates at $\mathcal{O}(\lambda)$ the following contribution to the vacuum energy density

$\rho_b^{(2)} =$  $= \frac{1}{2}\delta_m^{(1)} m^2 \Delta_F(0) = -\frac{\lambda}{8}\frac{m^4}{(4\pi)^4}\frac{2}{\varepsilon}\left[\frac{1}{\varepsilon} + \ln(\mu^2/m^2)\right]$. (11.83)

Combining the two contributions, the mixed terms disappear as expected and the remaining $1/\varepsilon^2$ pole can be subtracted by the counter term

$$\delta_\rho^{(2)} = \frac{\lambda}{8}\frac{m^4}{(4\pi)^4}\frac{1}{\varepsilon^2}. \tag{11.84}$$

Thus the two-loop contribution to the vacuum energy density is

$$\rho^{(2)} = \frac{m^4}{(4\pi)^4}\ln^2(\mu^2/m^2) = \left(\frac{m^{(1)}(\mu)}{4\pi}\right)^4. \tag{11.85}$$

Summary

Using a power counting argument for the asymptotic behaviour of the free Green functions, we singled out theories with dimensionless coupling constants: Such theories with marginal interactions are renormalisable, i.e. are theories with a finite number of primitive divergent diagrams. In this case, the multiplicative renormalisation of the finite number of parameters contained in the classical (effective) Lagrangian is sufficent to obtain finite Green functions at any order perturbation theory.

Further reading

The renormalisation of the $\lambda\phi^4$ theory is dicussed in more detail e.g. by Pokorski, non-renormalisable theories are treated by by Schwartz.[41] reviews the status of electroweak precision calculations like a_μ.

Problems

11.1 Asymptotic expansion.
Plot $Z_N(\lambda)/Z(0)$ from example 11.1 for the first few N as function of λ.

11.2 Dimensional regularisation.
Regularisation methods modify the short-distance behaviour. Discuss how DR modifies the typical integral $I_0(\omega, \alpha)$ (cf. 4.97) performing the integral over the -2ε extra dimensions.

11.3 Dimensional regularisation and γ^5.
Show that the properties $\mathrm{tr}(\gamma^\mu\gamma^\nu) = d\eta^{\mu\nu}$, $\mathrm{tr}(1) = 4$ and $\{\gamma^\mu, \gamma^5\} = 0$ lead to an inconsistency in $d \neq 4$ dimensions. (Hint: Consider first $d\,\mathrm{tr}[\gamma^5] = -d\,\mathrm{tr}[\gamma^5]$, then $d\,\mathrm{tr}[\gamma^5\gamma_\alpha\gamma_\beta] = (4-d)\mathrm{tr}[\gamma^5\gamma_\alpha\gamma_\beta]$ and finally $(4-d)\mathrm{tr}\big[\gamma^5\gamma_\alpha\gamma_\beta\gamma_\rho\gamma_\sigma\big] = 0$.]

11.4 g_s-factor of gauge bosons.
Derive the tree-level value of th g_s-factor for Yang-Mills gauge bosons from the non-abelian Maxwell equations.

11.5 Effective vertex for $\mu \to e\gamma$.
Derive the effective vertex Γ^μ for the tranition $\mu \to e\gamma$ where all three particles are on-shell and the process violates parity. Use current conservation and that the photon is on-shell to show that $\Gamma^\mu = \mathrm{i}q^\nu\sigma_{\mu\nu}([A(q^2) + B(q^2)\gamma^5]$, where q^ν is four-momentum of the photon.

11.6 Primitive divergent diagrams of scalar QED.
Find the basic primitive diagrams of scalar QED,

$$\mathscr{L} = \frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

and their superficial degree of divergence.

11.7 Comparison of cutoff and DR.
Recalculate the three basic primitive diagrams of a scalar $\lambda\phi^4$ theory using as regularisation a cutoff Λ. Find the correspondence between the coefficients of poles in DR and divergent terms in Λ.

11.8 β function of the $\lambda\phi^4$ theory.
The β function determines the logarithmic change of the coupling constants. a) Consider mass independent schemes in DR and show that the β function can be written as $\beta(\lambda) = -\varepsilon\lambda - \frac{\mu}{\tilde{Z}}\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}\mu}\lambda$ with $\tilde{Z}^{-1} = Z_\lambda^{-1}Z_\phi^2$. b.) Show that $\tilde{Z}_\lambda^{-1} = 1 - 3\lambda/(16\pi^2\varepsilon)$ in one loop approximation and find the β-function. c.) Up to which order is the β function scheme independent? d.) Solve the differential equation for $\lambda(\mu)$.