Considerations on the Flow of Superfluid Helium*

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First, we show that the most important equations of the dynamics of the two types of superfluids, He II and superconductors, follow quite directly from the simple assumption that the quantum field of the particles has a mean value which may be treated as a macroscopic variable. The background of this ansatz is also discussed. Second, we apply these equations to various physical situations in He II, notably the orifice geometry and the superfluid film, and show how they, and particularly the idea of phase slippage accompanying all dissipative processes, can be applied and what kinds of macroscopic interference phenomena may be expected. The effect of synchronization in the ac interference experiment is discussed.

I. INTRODUCTION

The material of the first part of this article covers really much the same areas of basic physics which are treated by Martin and by Nozieres in their articles at the same conference at which this was presented. Nonetheless the reader will find that the emphasis is sharply different. The striking macroscopic interference phenomena, the observation of which has been stimulated by Josephson's remarkable discovery,[1] call out for a description in terms of a definite wave function with a definable phase ϕ in every part of the system, while quantum hydrodynamics as pioneered by Landau[2] has tended to emphasize the superfluid velocity v_s and its equations of motion. The identification of $v_s=(\hbar/m)\nabla\phi$ has its limitations in relating these points of view, especially in Josephson junctions; while the opposite point of view, that ϕ exists, makes theorists uncomfortable because it breaks the gauge symmetry. Nonetheless I shall choose to take the latter path, assuming either that the reader has read such a discussion as Ref. 3 which gives the reason why this broken symmetry is possible, or that he will understand that the superfluid system is to be attached at some point to a large superfluid reservoir, with respect to which the phase is to be measured.

The idea of off-diagonal long-range order was introduced by Penrose and Onsager.[4] They suggested that superfluidity be described as a state in which the density matrix

$$\rho(r,r')=\langle\psi^*(r)\psi(r')\rangle$$

could be factorized:

$$\rho(r,r')=\psi^*(r)\psi(r')+\text{small terms.} \tag{1}$$

Beliaev[5] extended this for helium to a Green's function theory in which ψ was explicitly time-dependent, and first observed that the chemical potential determined the time dependence. Sortly thereafter Gor'kov[6] observed that superconductivity theory could be cast into the same form by substituting electron pair field operators $\psi\psi$ for the He atom bose field. In superconductivity there already existed a set of phenomenological equations proposed by Ginzburg and Landau[7] which dealt with an "order parameter" η, and it was soon recognized that this order parameter was the same as the factorized ψ of the Gor'kov theory.[8] It was only much later that Gross[9] and Pitaevskii[10] proposed similar sets of equations for liquid helium.

The notion that it was possible to regard the function which appears in these treatments as essentially the mean value of the quantum particle field has long been accepted in both helium[5] and superconductivity (there the first explicit discussion of the transformation between the "ODLRO" and mean field points of view was given by Anderson[11]) but only in the case of a homogeneous system; apparently the general case has not been discussed until recently even for superconductivity.[3] The basic idea is that it is as legitimate to treat the quantum field amplitude as a macroscopic dynamical variable as it is the position of a solid body; both represent a broken symmetry which, however, cannot be conveniently repaired until one gets to the stage of quantizing and studying the quantum fluctuations of the macroscopic behavior of the system.

Here we are going to discuss less the microscopic background of this ansatz than a number of its most important consequences for He II, many of which follow without further microscopic assumptions and are therefore of fundamental interest. Only some of what we will have to say is new, in the sense that many of the

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[1] B. D. Josephson, Phys. Letters **1**, 251 (1962).
[2] L. D. Landau, J. Phys. USSR **5**, 71 (1941).
[3] P. W. Anderson, in *Lectures on the Many-Body Problem*, edited by E. R. Caianello (Academic Press Inc., New York, 1964), Vol. 2, p. 113.
[4] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956); see also O. Penrose, Phil. Mag. **42**, 1373 (1951).
[5] S. T. Beliaev, Zh. Eksperim. i Teor. Fiz. **34**, 417 (1958) [English transl.: Soviet Phys.—JETP **7**, 289 (1958)].
[6] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].
[7] V. L. Ginsburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).
[8] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [English transl.: Soviet Phys.—JETP **9**, 1364 (1958)].
[9] E. P. Gross, Nuovo Cimento **20**, 454 (1951).
[10] L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **40**, 646 (1961) [English transl.: Soviet Phys.—JETP **13**, 451 (1961)].
[11] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

appropriate equations have been written down (see the papers of Martin and Nozieres). The basic idea was previously stated in a Letter,[12] and some of the consequences have been explored in subsequent Letters by Donnelly[13] and by Zimmerman.[14] We will first discuss a particularly simple point of view on the derivation of the basic equations, following more closely than usually the corresponding ideas in superconductivity.[3,15] The emphasis will be on what can be shown to follow more or less rigorously from using this ansatz as a general semimicroscopic definition of superfluidity. Only the parameter values in the resulting equations require any other knowledge of the microscopic system. Then we will discuss the dynamical consequences for the orifice experimental geometry, as well as some more general situations such as film flow. In Appendix B we discuss briefly the interesting connection with classical ideal fluid hydrodynamics, where the basic "Josephson" theorem turns out to have a classical analogue which has not to our knowledge been clearly stated previously.

II. BASIC EQUATIONS

We take as our definition of a superfluid that it is a fluid in which the particle field operator ψ has a macroscopic mean value, in a sense which is defined shortly.

$$\langle\psi(r,t)\rangle = f(r,t)\exp[i\phi(r,t)]. \tag{2}$$

Here we allow slow (on the atomic scale) space and time variations; the essential point is that $\langle\psi\rangle$ has a mean value in the thermodynamic, quasi-equilibrium sense. One may think of the situation here as completely analogous to the definition of temperature in a nonequilibrium state; it is possible if there are equilibrating processes of shorter range in time or space than the coarse-grained scale of our averaging, which in turn is to occur over regions smaller than the macroscopic scale of physical measurements. Of course, in every system one can define a temperature and entropy if the average is allowed to extend over a long enough time, while only certain special systems will give a stable limiting value to $\langle\psi\rangle$. More explicitly, we visualize averaging ψ over some small region of space–time; if the region is small enough compared to the rates and ranges of microscopic fluctuations, we will obtain some finite value. As we increase the size ΔV of the region $\langle\psi\rangle$ will decrease to zero in a normal system very rapidly, in times of the order $h/$(mean kinetic energy) and ranges of order interparticle distances. In a superfluid system, on the other hand, we assume that at an intermediate, "coarse-grained" scale $\langle\psi\rangle$ approaches a limiting finite value, not changing until we reach a scale on which it varies because of the presence of macroscopic perturbations such as macroscopic fields and flows. (In Appendix A we discuss this definition a little more deeply.)

The whole problem of superfluid dynamics, then, reduces to the question of how to deal with $\langle\psi\rangle$ (often denoted the superfluid order parameter) as a thermodynamical variable. It is quite important that $\langle\psi\rangle$ is a *complex* order parameter; it has both an amplitude f and a phase ϕ. There is actually rather a sharp distinction between the two real thermodynamic variables f and ϕ, and it is the behavior with respect to ϕ which is responsible for specific superfluid properties. The distinction is that ϕ is coupled, like a polarization or a strain, to external forces, where f is merely an internal order parameter in the sense, for instance, of the original long-range order parameter of order–disorder systems, or of the antiferromagnetic sublattice magnetization. In principle a corresponding force might exist but in practice it does not, so f simply manifests itself as a convenient tool by which to describe the condensation process.

The point of ϕ, then, is that it is not only a thermodynamic but also a dynamical variable. The latter fact comes from its being the dynamically (not thermodynamically) canonically conjugate variable to N, the total number of particles in the system described by ϕ. (The limitations of this statement for systems of few particles[16] are irrelevant here.)

We illustrate this fact by forming wave packets in many-body wave-function space, just as in discussing the relationship of p and q it is useful to form wave packets. Let us write the wave function of one of our coarse-grained cells of volume ΔV as

$$\Psi(\Delta V) = \sum_N a_N \Psi_N(\Delta V) \tag{3}$$

where, since our cell is only a part of the superfluid, it is essential to realize that the state is a superposition of states Ψ_N with different numbers of particles N, with coefficients a_N large in some range of values $\Delta N \sim N^{\frac{1}{2}}$. An important simplifying assumption concealed in (3) is that the cell may be made big enough so that a_N and Ψ_N do not depend very much parametrically on the variables of the rest of the system: this is what is meant (see Appendix A) by a "satisfactory" local description.

We postulate that $\langle\psi\rangle$ has a limiting mean value, which must be of order $(N/\Delta V)^{\frac{1}{2}}$; we calculate this value from (3):

$$\frac{1}{\Delta V}\int_{\Delta V} d\tau \langle\psi(r)\rangle = \frac{1}{\Delta V}\sum_{N,N'} a_{N'}^* a_N \int (\Psi_{N'}, \psi(r)\Psi_N)\, d\tau$$
$$= \sum_N a_{N-1}^* a_N \int \frac{(\Psi_{N-1}, \psi(r)\Psi_N)\, d\tau}{\Delta V}. \tag{4}$$

[12] P. L. Richards and P. W. Anderson, Phys. Rev. Letters **14**, 540 (1965).
[13] R. J. Donnelley, Phys. Rev. Letters **14**, 939 (1965).
[14] W. Zimmermann, Phys. Rev. Letters **14**, 976 (1965).
[15] P. W. Anderson, N. R. Werthamer, and J. M. Luttinger, Phys. Rev. **138**, A1157 (1965).
[16] W. H. Louisell, Phys. Letters **7**, 60 (1963); L. Susskind and J. Glogower, Physics **1**, 49 (1964).

In the case of the Bose-condensed perfect gas, one may take

$$\Psi_N = (c_0^+)^N \Psi_{\text{vac}}, \tag{5}$$

where

$$c_0 = \frac{1}{(\Delta V)^{\frac{1}{2}}} \int \psi(r)\, d\tau$$

$$c_0 \Psi_N = N^{\frac{1}{2}} \Psi_{N-1}, \tag{6}$$

so that

$$\langle \psi \rangle = \sum_N a_{N-1}^* a_N (N/\Delta V)^{\frac{1}{2}}. \tag{7}$$

This is the maximum possible value of the matrix element. (Here we have taken advantage, as we will always, of our freedom to choose the phase of Ψ_N to make the matrix element real in order to keep the phase factors—which of course are *not* irrelevant—in the a_N's.) In real superfluid systems and at finite temperatures (the case of superconductivity is of course quite similar if for a single Bose ψ or c we read Fermion pair operators) the matrix element does not take on its maximum possible value—in He, for instance, as McMillan has shown,[17] the value is about 11% of the maximum when ΔV is reasonably large. Even if

$$M = (\Psi_{N-1}, \bar{\psi} \Psi_N) \tag{8}$$

is large, however, $\langle \psi \rangle$ will not be large unless the a_N preserve phase coherence. For example, we may form a wave packet using

$$a_N = (2\pi \Delta N)^{-\frac{1}{4}} \exp\left[-\tfrac{1}{2}(N-\bar{N})^2/(\Delta N)^2\right] \exp(i\phi N). \tag{9}$$

In this case if $\Delta N \gg 1$,

$$\langle \psi \rangle \cong |M| \exp(i\phi) \tag{10}$$

whereas if the phase factors are arbitrary $\langle \psi \rangle$ will be very much smaller, even if every individual Ψ_N represents a pure Bose-condensed state of the volume element ΔV. We will very shortly discuss the reason why wave packets like (9) actually occur in superfluid systems; first, however, let us dispose of a few formal prelimnaries.

Any linear combination of the set of fixed-number states Ψ_N may be written as a linear combination of our basic wave packets (3)–(9) [which we will call $\Psi(\phi)$]. In particular, the number eigenfunction Ψ_{N_0} may itself be written

$$\Psi_{N_0} \propto \int_0^{2\pi} d\phi \exp(-i\phi N_0) \Psi(\phi) \tag{11}$$

$$\left(= \sum_N a_N\, \delta(N, N_0) \Psi_N\right).$$

The operator $-i\partial/\partial\phi$ acting on $\Psi(\phi)$ has the same effect as multiplying Ψ_{N_0} by N_0:

$$\int_0^{2\pi} d\phi \exp(-i\phi N_0)\left(-i\frac{\partial}{\partial\phi}\right)\Psi(\phi)$$

$$= N_0 \int_0^{2\pi} d\phi \exp(-i\phi N_0)\Psi(\phi) = N_0 \Psi_{N_0}$$

so that we may take

$$N \leftrightarrow -i(\partial/\partial\phi) \tag{12a}$$

and correspondingly it may be shown that

$$i(\partial/\partial N) \leftrightarrow \phi \tag{12b}$$

in the limit that N may be considered a continuous variable. Thus, as we stated, N and ϕ are conjugate dynamical variables.

In general, also, N and ΔN are sufficiently large that the system's dynamics may be treated reasonably well classically and the uncertainty in ϕ is not excessive—of course (12) implies $\Delta N \Delta\phi \sim 1$.

In any case the equations of motion of N and ϕ are

$$i\hbar \dot{N} = [\mathcal{H}, N] = i(\partial\mathcal{H}/\partial\phi)$$

$$i\hbar\dot{\Phi} = [\mathcal{H}, \phi] = -i(\partial\mathcal{H}/\partial N).$$

Taking the mean values of these two equations and assuming that the wave packets are such that $\Delta N/N$ and $\Delta\phi$ are both small (quasi-classical case) is the source of the two equations which essentially characterize superfluidity: the equation for superfluid flow, corresponding to London's equation or the Ginsburg–Landau current equation in superconductivity, and the "Josephson" frequency equation, which is related to the acceleration equation for superfluid flow in both He and superconductivity. The second is somewhat simpler though less generally known: it is just

$$\hbar(d\phi/dt) = \partial E/\partial N = \mu. \tag{13}$$

Here we have used the standard definition of the chemical potential μ; if the fluid is in motion so that the particles have kinetic energy $\frac{1}{2}mv_s^2$ that is to be included in μ as well as the internal energy and any external forces. Obviously the partial derivative holds S fixed. In an isothermal, assumed incompressible bath of liquid helium, with free surface at height h in a gravitational field g,

$$\mu = m(p/\rho) + mgh + \tfrac{1}{2}mv_s^2. \tag{14}$$

In the nonisothermal case (14) should contain the thermomechanical term.

Equation (13) is of the utmost importance in understanding superfluidity. It says two things: first, that if the state of the superfluid is constant in time, because ϕ is a thermodynamic variable it will be constant and μ must be constant everywhere: there can be no potential differences in the truly steady state. Second, if there

[17] W. L. McMillan, Phys. Rev. **138**, A442 (1965).

is any potential difference $\mu_1-\mu_2$ between two elements in the superfluid $\phi_1-\phi_2$ must change in time. This can happen in two ways. The simplest is acceleration. If we take the gradient of (13), we obtain

$$(d/dt)(\hbar\nabla\phi)=\mathbf{F}, \tag{15}$$

where $\mathbf{F}$ is the total force on the particles, and making the identification (which we shall discuss shortly) of

$$\hbar\nabla\phi=p_s=mv_s, \tag{16}$$

this is the statement that the superfluid may undergo acceleration without frictional damping by whatever external forces act upon it. A potential difference may lead to continuous acceleration, then.

Slightly more subtle and much more physically important is the concept of phase slippage by quantized vortex motion. As we have emphasized, ϕ is the phase of the thermodynamic variable $\langle\psi\rangle$, which is of course necessarily single-valued. ϕ, however, being a phase, need not be single-valued in a multiply connected system such as a toroid; it need merely return to its original value $\pm 2n\pi$ on traversing a path around a nonsuperfluid obstacle:

$$\oint\nabla\phi\cdot d\mathbf{s}=2n\pi. \tag{17}$$

In terms of the superfluid momentum p_s this expresses the idea of the quantization of angular momentum in units of $\hbar$ or of vorticity $\boldsymbol{\omega}=\frac{1}{2}\nabla\times v_s$ through a closed curve in units of $h/2m$. As we shall see, however, the quantity v_s is not necessarily a measurable particle velocity and so (17) is somewhat more fundamental than the usual concept of vorticity quantization.

A bucket of superfluid may be made multiply connected not only by the presence of actual solid obstacles but by the introduction of one-dimensional regions of nonsuperfluidity within the liquid itself: lines where $\langle\psi(r)\rangle=0$. These are "vortex cores" and may of course move according to the usual laws of hydrodynamics along with the surrounding fluid. Around such a line there can be a circulation of one or, less usually, an integral number of quanta, according to (17).

Equation (17) shows that the integral of $\nabla\phi$ along a path on one side of a vortex core differs from that on the other by 2π (see Fig. 1). Thus when a vortex core moves across the line between points 1 and 2, this may cause a time rate of change of $\phi_1-\phi_2$. Mathematically, we may write

$$\langle\mu_1-\mu_2\rangle_{\text{Av}}=T^{-1}\int_0^T dt\hbar\frac{d(\phi_1-\phi_2)}{dt}=\frac{\hbar}{T}\int_0^T dt\frac{d}{dt}\int_1^2\nabla\phi\cdot dl, \tag{C}$$

where $\langle\ \rangle_{\text{Av}}$ denotes time average, where we consider the limit T very long, and C is any path from 1 to 2. If the liquid is assumed to be in a reasonably steady

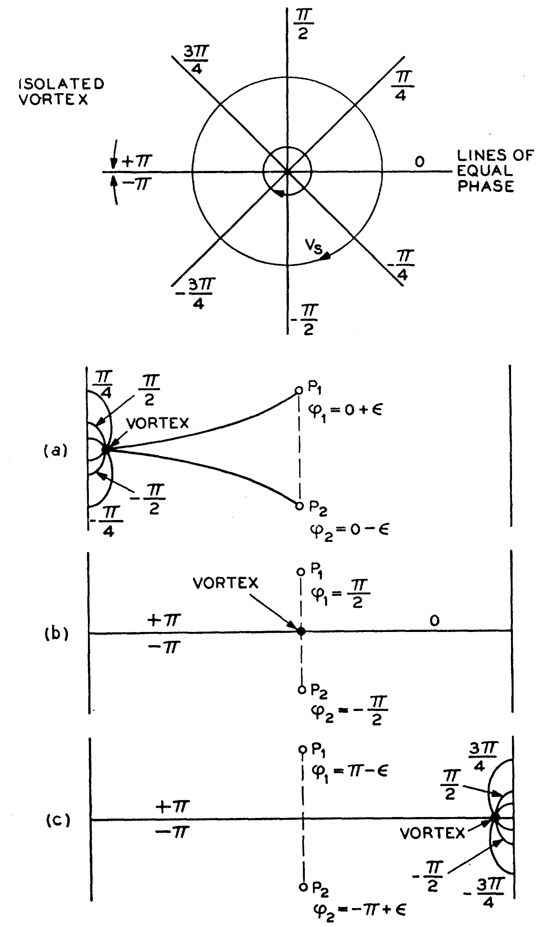

FIG. 1. Illustration of phase changes at two points P_1 and P_2 in a channel as a vortex moves between them. From a to b to c the vortex moves across from left to right; as it moves from one wall to the other the relative phase changes by 2π.

state, so that at 0 and T the positions of the various vortices do not differ very much, the integral is just equal to 2π times the number of vortices which have crossed C in this time interval. Thus we have

$$\langle\mu_1-\mu_2\rangle_{\text{Av}}=h\langle dn/dt\rangle_{\text{Av}}, \tag{18}$$

where $\langle dn/dt\rangle_{\text{Av}}$ is the average rate of motion of vortices across a path from 1 to 2. This is the "phase slippage" concept which is used to explain the various "ac Josephson"-like experiments.[18,19] Of course, this is not incompatible with the acceleration equation (15), so in a sense this phenomenon too is merely a manifestation of the fact that potential differences occur only in an accelerated superfluid; but it is a point of view which had not previous to Ref. 12 found its way into the He literature, nor until recently, with the discovery

[18] S. Shapiro, Phys. Rev. Letters **11**, 80 (1963).
[19] P. W. Anderson and A. H. Dayem, Phys. Rev. Letters **13**, 195 (1964).

of the Josephson effect and of flux creep and flow, into that of superconductivity.

Let us now take up the current equation, the theory of which is somewhat more complex. There are two aspects to this. In the first place there is the quasi-rigorous Eq. (16) the meaning of which becomes a bit vague with more careful consideration. If we simply suppose that the state of a volume element ΔV moving with velocity v_s is obtained from that of a stationary element by a pure Galilean transformation, multiplying ψ by $\exp(imv_s x/\hbar)$, this equation is trivially valid. That is the usual derivation of it, in one form or another. I know of no acceptable proof of (16), however, in the sense of showing v_s to be a real particle velocity, in physically important situations such as counterflow of normal and superconducting fluids, or where v_s is varying reasonably rapidly, as near a vortex core. Since (16) is the statement which leads to vorticity quantization, this means that that concept, often claimed to be exact, is apparently not so.

The statement has been made in the literature[13,14] that the results—specifically the "ac Josephson effect"—which follow from (13) or the phase slippage concept can equally well be "derived" from vorticity quantization plus perfect fluid hydrodynamics for the superfluid. Neither of these latter ideas, however, has at the moment a very quantitative experimental background, while theoretically as we have just seen the phase has a much more secure theoretical meaning than v_s; it seems to us that (13) is a much more rigorous and complete theoretical statement than the hydrodynamic equations, which are derived via (15) and (16) from it. Normal fluid counterflow and dissipation do not affect it. In a perfect classical fluid, for instance, the vortices cannot move across stream lines, so clearly He II is not such a fluid.

Perhaps a more rigorous general reason for using (16) at least as a *definition* for v_s (other than that it allows us to use ϕ as a velocity potential for the superfluid flow) is (15), which shows that $m\,dv_s/dt$ does then give the rate of exchange of momentum per superfluid particle with external forces, an excellent operational definition of v_s.

The superflow, however, is best determined not from the expression for v_s but from the other of the two Hamilton equations for the conjugate variables N and ϕ:

$$\hbar(dN/dt) = \partial E/\partial\phi. \tag{19}$$

Gauge invariance assures us that in fact the energy of an isolated bit of superfluid is independent of ϕ, so that in the absence of a coupling to its neighbors $dN/dt=0$. Consider, on the other hand, a pair of neighboring volume elements ΔV_1 and ΔV_2. Our definition of superfluid implies that $\langle\psi\rangle$ has a tendency to constancy, so that the mean phases ϕ_1 and ϕ_2 of the two neighboring elements will be coupled by some energy which is a minimum when $\phi_1-\phi_2=0$:

$$E = U(\phi_1-\phi_2).$$

Assuming the effects of other neighboring elements can be treated independently (suitable arguments can be found for this step) we see that the flow across the boundary between ΔV_1 and ΔV_2 is given by

$$J_{\text{tot}} = \frac{dN_1}{dt} = -\frac{dN_2}{dt} = \frac{1}{\hbar}\,\frac{\partial U(\phi_1-\phi_2)}{\partial(\phi_1-\phi_2)}. \tag{20}$$

This is the expression used in the theory of the Josephson current[1,3] across a barrier between two macroscopic pieces of superconductor; at this point we are carrying the same reasoning over to the continuous interior of a superfluid or superconductor. As in that case, we may pause now to point out that it is this coupling energy which enforces the phase coherence of each individual volume element of the superfluid. The kinetic energy matrix element which transfers particles across the boundary can cause transitions like

$$\{\Psi_1^N \to \Psi_1^{N-1};\ \Psi_2^{N'} \to \Psi_2^{N'+1}\}, \tag{21}$$

with a matrix element we may call M_{12}. (Here $\Psi_{1,2}$ is the many-body wave function of $\Delta V_{1,2}$.) If the wave packets (3) have coefficients a_N which, like (9), have coherent phase relationships, all transitions $N'\to N'+1$ can occur coherently with all transitions $N\to N-1$. Mathematically, the energy due to transitions like (21), inserting wave packets like (3), is

$$(\Psi_1\Psi_2,\ (\text{K.E.})\Psi_1\Psi_2) = \Big(\sum_N a_N^* a_{N-1}\Big)\Big(\sum_{N'} a_{N'}^* a_{N'+1}\Big) M_{12}. \tag{22}$$

As we see, this energy is orders of magnitude larger for coherent wave packets like (9), for which ϕ is determinate, than in the incoherent case.

In the interior of the superfluid, it is more convenient to go over to a continuum representation of the coupling energy which maintains phase coherence. We write U as a functional of the gradient of ϕ (as well as f, of course)

$$U = \int U[(f),\ \nabla\phi]\,d\tau$$

$$\simeq \int E(f,S)(\nabla\phi)^2\,d\tau \tag{23}$$

and then by considering a cell of wall area A and thickness d we find

$$\frac{J}{\text{unit area}} = \frac{dN_1}{dt}\Big/A = \frac{\delta U}{\hbar\,\delta\nabla\phi} = \hbar^{-1}E\nabla\phi. \tag{24}$$

If we use the pseudo-identity

$$v_s = \hbar\nabla\phi/m,$$

we may write

$$J_s = n_s(f,\, S)v_s$$
$$= [\hbar n_s(T)/m]\nabla\phi, \qquad (25a)$$

so that we can identify the parameter E of (24):

$$E = \delta^2 U/\delta(\nabla\phi)^2 = \hbar^2 n_s(T)/m. \qquad (25b)$$

Thus the current equation contains a completely arbitrary parameter n_s. In pure, homogeneous systems at absolute zero Galilean invariance can be used to show that $n_s = n$, the total number of particles; but in impure systems at $T=0$, or any system at $T\neq 0$, no such identity holds, though in general n_s is of the same order of magnitude as n.

The supercurrent (24) exists even if the phase ϕ is completely stationary in time, which as we have shown implies the absence of accelerating forces. In fact, in the presence of accelerating forces and time-dependent ϕ we must expect additional quasi-particle currents, in general; the system will exhibit a two-fluid hydrodynamics, the complexities of which need not concern us here. So far as I know their proper treatment requires more knowledge of the actual physical system than we are assuming here.

One very important point about (24) taken together with (15) is that it makes it at least highly probable that there is both a necessary and a sufficient connection between the existence of supercurrents and our definition of superfluidity (1) (which is essentially equivalent to what Yang[20] has named ODLRO). Namely, $\langle\psi\rangle$ and therefore ϕ will not exist if the energy is not such as to maintain spatial coherence of ϕ, so $\delta^2 U/\delta(\nabla\phi)^2$ must exist and be positive in a superfluid in this sense, meaning necessarily supercurrents by (24). (15) shows that they flow with zero forces and are therefore supercurrents. Hypothetical phases with $\langle\psi\rangle$ but no supercurrents (Cohen[21]) seem to ignore this half of the argument.

Conversely, the only dynamically conjugate variable to N is the phase as we have defined it, so that the existence of a dN/dt in a stationary state implies a $\delta U/\delta\phi$, which implies that ϕ is a meaningful variable. Various hypothetical superconducting phases (e.g., Frohlich[22]) do not satisfy *this* half of the argument.

This concludes our general discussion of the basic equations of superfluidity. We restate the conclusions: the phase equation (13) and the corresponding equation of phase slippage are exact in the "integrated" sense that they give the phase difference between two distant points in undisturbed regions of superfluid. The existence of the order parameter alone guarantees the existence of quantized vortices, and according to (24) these are indeed vortices in that they contain a superfluid circulation. However, the quantization of vorticity in any true sense is dependent on the imprecise assumption (15) that the phase is the velocity potential with fixed coefficient. This need not be true unless we treat (15) merely as a definition of v_s, for instance near a vortex core, just as the quantization of flux in superconductivity is not necessarily precise. Operationally, for example, the measurement of h/e by the ac Josephson effect, or of h/mg by the helium counterpart, is more precise in principle than by flux or vorticity quantization. In practice, of course, present-day methods are not capable yet of distinguishing these niceties, but it is of value to have a clear idea of the theoretical assumptions behind the various equations, since it is foreseeable that the most precise measurements of many important physical quantities will involve quantum coherence.

III. SOME DYNAMICAL CONSEQUENCES

The macroscopic quantum interference effects promised by the existence of $\langle\psi\psi\rangle$ in superconductivity have been relatively easy to observe for a number of reasons: the light mass of the electron, permitting weak superfluid connections to be made easily by use of the tunneling phenomenon, the coupling to the electromagnetic field which leads to flux quantization, and most particularly the fact that that coupling provides a second parameter, the penetration depth λ, which in screening out the current and magnetic field from the interior of the superconductor creates the Meissner effect, that is ensures that every superconductor exhibits a finite critical magnetic field H_{c_1} below which it contains no vortices and thus essentially exhibits constant ϕ. In He, $\lambda\rightarrow\infty$ so that the corresponding $\omega_{c_1}=0$; no rotation of a sufficiently large He sample is too small for vortices to be energetically favorable. Indeed, experiments show that few samples are ever free of vorticity. Even worse, the coherence length—the length given by the ratio of $\delta U/\delta\,|\,\psi\,|^2$ to $\delta U/\delta\,|\,\nabla\psi\,|^2$, which determines how rapidly the order parameter can vary and thus how large a vortex core is—is of order a few Å, so that no channel through which He can flow is too small to contain a vortex. All this means that the useful idealization in superconductivity of the "ideal Josephson junction," a weak link between two reservoirs having constant phase, is probably not relevant to the helium case. Any barrier in which the flow occurred only by quantum-mechanical tunneling would have to be of subatomic dimensions, especially in thickness, to permit any measurable current to flow, and could not be supported mechanically. Any attempt to replace the ideal junction by a thin channel, on the other hand, runs into exactly the same difficulties that are encountered with long thin bridges in the case of superconductivity,[23,19] namely, that the device acts like a

[20] C. N. Yang, Rev. Mod. Phys. **34,** 694 (1962).
[21] M. H. Cohen, Phys. Rev. Letters **12,** 664 (1964).
[22] H. Frohlich and C. Terreaux, Proc. Phys. Soc. (London) **86,** 233 (1965).

[23] R. D. Parks and J. M. Mochel, Rev. Mod. Phys. **36,** 284 (1964).

sequence of weak links in series and one is never sure exactly where the phase rigidity is breaking down. The closest approximation we could imagine to a single definable "weak superfluid junction" was the orifice geometry, which is analogous to the "short" thin film bridge of Anderson and Dayem[19] in superconductivity.

Let us analyze this system in some detail, as we did the Josephson junction previously.[3] First, a note as to the driving term to be inserted to represent any externally applied conditions. In the case of normal systems it is natural to place the ends of a specimen in contact with reservoirs at different chemical potential levels μ_1, μ_2, giving an "applied" potential gradient $\nabla\mu$, and to describe any situation in terms of solutions of the resulting applied field problem; we calculate J as a function of the pressure gradient or field, even though actually we may be driving the system with a constant current generator. The microscopic theory is done by inserting μN terms into the Hamiltonian, as is well-understood in the calculation of resistance, for instance, or thermoelectric power.

It is precisely the nature of superfluids that they cannot assume a stationary state under a field or pressure gradient, but will, as explained by Anderson, Werthamer, and Luttinger,[15] have to be described by phases with a time dependence obeying (13). This condition follows when we insert the appropriate μN terms, as there described; but that does not lead to a way to discuss the equally interesting case of an imposed current. One obvious technique is to impose a fixed phase difference by postulating an infinitely tight coupling to reservoirs consisting of large superfluids of fixed phase, but that is often quite unphysical. We have used without discussion[3] the technique of inserting a term

$$\Delta E = J_{12}(\phi_1 - \phi_2)$$

in the energy to describe the effect of a fixed supercurrent J_{12} between regions 1 and 2. In superconductivity a rather rough physical derivation of this can be given in terms of the electromagnetic interaction between J and the magnetic flux of a vortex line. It is analogous to, but more general than, the technique of inserting a $p\cdot v_s$ term sometimes used to derive the hydrodynamic equations.

In the case of He we may return very simply to Eq. (19). We showed that the particle accumulation rate dN_1/dt in a volume element ΔV_1 is given by:

$$dN_1/dt = \hbar^{-1}(\partial U/\partial\phi_1).$$

If no net accumulation is to occur, and if the current leaving ΔV_1 to neighboring elements is given in terms of the coupling energy between them by

$$J_{12} = dN_1/dt)_{\text{into superfluid}} = \hbar^{-1}(\partial U_{\text{coupling}}/\partial\phi_1)$$

there must be a compensating term

$$dN_1/dt)_{\text{current generator}} = -\hbar J_{12}\phi_1.$$

Similarly, to make the corresponding current leave element (2) we must have $dN_2/dt)_{\text{c.g.}} = \hbar J_{12}\phi_2$ so we may represent a constant current generator by

$$\mathcal{H}_{\text{gen}} = \hbar J_{12}(\phi_2 - \phi_1). \tag{26}$$

Using this let us discuss the orifice problem in the presence of a constant driving current. Current acceleration can be important only on the time-scale of the U-tube oscillations, which is usually longer than the time necessary to create a vortex or otherwise change the phase but can be included if desired. Also, it will greatly simplify one's thinking without falsifying any important physical features to assume $T\rightarrow 0$; i.e., only incompressible, superfluid flows.

Under these conditions $\hbar/m\nabla\phi = v_s$ and

$$\nabla\cdot v_s = 0$$

so

$$\nabla^2\phi = 0.$$

Thus we can solve for the flow in the absence of vortices by a simple potential calculation. Let the radius of the orifice be a. The equipotentials and streamlines are along coordinate surfaces in a set of oblate spheriodal coordinates, defined in terms of cylindrical coordinates r, θ, z through the axis of the orifice by

$$\begin{gathered}(r^2/a^2\cosh^2 u) + (z^2/a^2\sinh^2 u) = 1;\\ (r^2/a^2\sin^2 v) - (z^2/a^2\cos^2 v) = 1;\\ \theta = \theta;\\ -\infty \leq u \leq \infty, \qquad 0 \leq v \leq \pi/2, \qquad 0 \leq \theta < 2\pi;\end{gathered} \tag{27}$$

or

$$\begin{gathered} r = a\cosh u\sin v\\ z = a\sinh u\cos v.\end{gathered} \tag{28}$$

The potential (i.e., phase) is

$$\phi(r, z) = C\int\frac{du}{\cosh u} = 2C\tan^{-1}e^u, \tag{29}$$

so that the total phase difference between the two large reservoirs separated by the orifice is

$$\phi_1 - \phi_2 = \pi C = \phi(+\infty) - \phi(-\infty). \tag{30}$$

The velocity is in the "u" direction and is

$$v_s = \frac{\hbar}{ma}\frac{C}{\cosh u(\sinh^2 u + \cos^2 v)^{\frac{1}{2}}}. \tag{31}$$

Some special values of the velocity field are:

Along the axis:

$$r = 0, \qquad \sin v = 0, \qquad z = a\sinh u$$

$$v_s = \frac{\hbar C}{ma}(\cosh u)^{-2} = \frac{\hbar C}{m}\frac{a}{a^2 + z^2}. \tag{32a}$$

In the orifice:

$$z=0, \qquad r=a\sin v, \qquad u=0$$

$$v_s=(\hbar/m)[C/(a^2-r^2)^{\frac{1}{2}}]. \tag{32b}$$

On the plane:

$$r=\pm a\cosh u, \qquad \cos v=0$$

$$v_s=(\hbar C/mr)[a/(r^2-a^2)^{\frac{1}{2}}]. \tag{32c}$$

The total particle flow is

$$J=\frac{\rho}{m}\int v_s\cdot dS=\frac{2\pi\hbar Ca\rho}{m^2}=\frac{2\hbar a\rho}{m^2}(\phi_1-\phi_2), \tag{33}$$

[using (30)] and the total kinetic energy is

$$K=\tfrac{1}{2}\rho\int v_s^2\,d\tau$$

$$=\tfrac{1}{2}\rho\frac{\hbar^2}{m^2}\int(\nabla\phi)^2\,d\tau$$

$$=\tfrac{1}{2}\rho\frac{\hbar^2}{m^2}\left(\int dS_{+\infty}\cdot-\int dS_{-\infty}\cdot\right)\phi\nabla\phi$$

$$K=\tfrac{1}{2}J\hbar(\phi_1-\phi_2)=(\rho a/m)[\hbar^2(\phi_1-\phi_2)^2/m]. \tag{34}$$

Thus this energy is a quadratic function of the phase difference $(\phi_1-\phi_2)$. However, it is essential to realize that this is only one branch of a multiple-valued function, because by gauge invariance K must be a periodic function of $\phi_1-\phi_2$ with period 2π. That is, if we were to pin down the phases in the two reservoirs by coupling to reservoirs of fixed relative phase $(\phi_1-\phi_2)_0$ we could satisfy the boundary conditions by any flow

$$J=(2\hbar a\rho/m^2)[(\phi_1-\phi_2)_0\pm 2n\pi]$$

and the corresponding kinetic energy

$$K=(\rho a/m)(\hbar^2/m)[(\phi_1-\phi_2)_0\pm 2n\pi]^2. \tag{34}$$

Equation (34) must also be used, then, in the presence of a current generator (27); thus the total energy is

$$E=(\rho a\hbar^2/m^2)[(\phi_1-\phi_2)\pm 2n\pi]^2-\hbar J_{12}(\phi_1-\phi_2). \tag{35}$$

This energy considered as a function of $\phi_1-\phi_2$ has an infinity of points of metastable equilibrium where

$$J=J_{12}$$

$$(\phi_1-\phi_2)=2n\pi+(m^2J_{12}/2\hbar a\rho).$$

$$E=-J_{12}[nh+(m^2J_{12}/4a\rho)]. \tag{36}$$

The situation is diagrammed in Fig. 2. The first drawing assumes a small current generator, $J_{12}<ha\rho/m^2$, which is the current value when $\phi_1-\phi_2=\pi$, the value at which the parabolas cross. For this current, although none of the energy minima are truly stable each is at least the lowest energy state for a given fixed phase difference (this has not been proved but seems obvious).

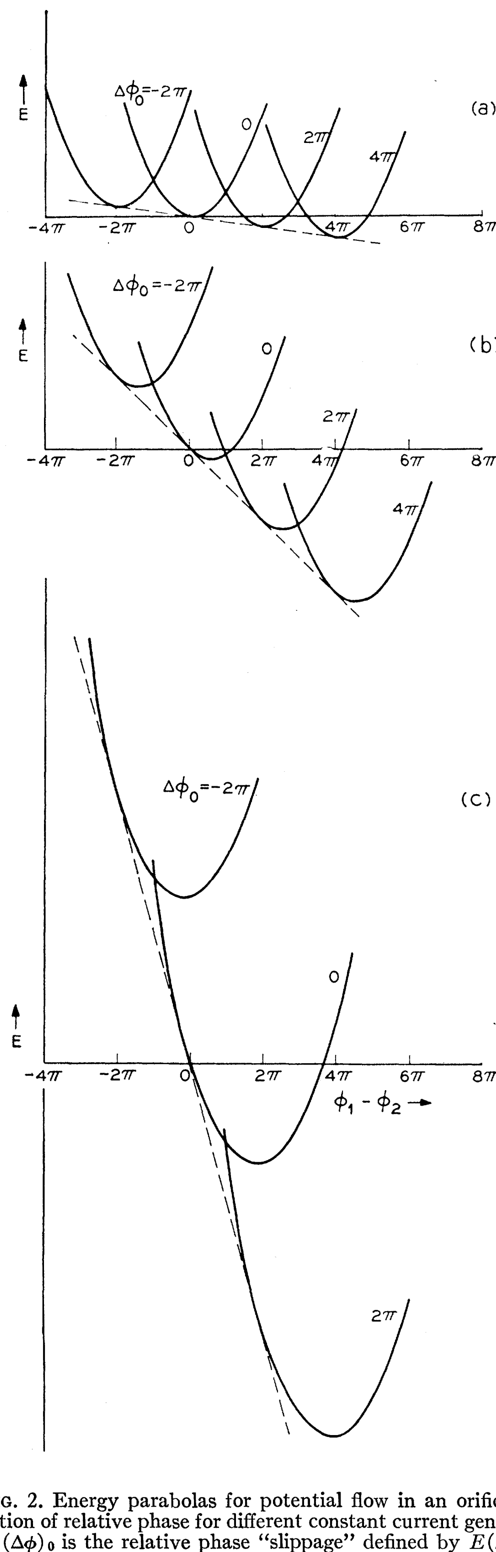

FIG. 2. Energy parabolas for potential flow in an orifice as a function of relative phase for different constant current generators J_{12}. $(\Delta\phi)_0$ is the relative phase "slippage" defined by $E(\Delta\phi_0)=J_{12}\Delta\phi_0$. (a) Low current; (b) $J_{12}\simeq v_c^0$; (c) J_{12} approaching observed critical current.

The system can still absorb energy from the current generator by "running downhill" in phase, but only if some fluctuation or external perturbation lifts it over the energy barriers between parabolas. In principle, if $J_{12} > ha\rho/m^2$ the third drawing is correct, and there is no true energy minimum; this corresponds to a "zeroth-order" critical velocity (we take the mean value over the area of the orifice)

$$(v_s)^0 = 2\hbar/ma \qquad (37)$$

which is $\frac{1}{3}$ cm/sec for $a=10\ \mu$. Actually, the crossing points between parabolas do not represent possible transitions, because the two parabolas represent entirely different "sheets" of the energy connected only by passage of a vortex across the orifice, and we must consider the activation energy problem for creation of a vortex, as has been discussed by Vinen.[24] At the very least a length of vortex line of the order $2a$ must be created, which has energy of order

$$(E_{\text{vortex}})_{\min} = (\pi\rho\hbar^2/m^2)\ \ln\ (a/\xi) \times 2a,$$

where ξ is the coherence length, ~ 1 Å. This must be compared to the energy gained when a vortex line is halfway across the orifice, which is of order

$$\pi\hbar J_{12} = (\pi^2\rho a^2\hbar/m)v_s.$$

The result is another "critical" velocity

$$v_s^{(1)} = (2/\pi)(\hbar/ma)\ \ln\ (a/\xi)$$
$$\cong 7.5\hbar/ma. \qquad (37')$$

This is also far smaller than observed critical velocities, indicating as discussed by Vinen that the great difficulty in forming vortices in most situations is probably nucleating them at the walls.

Yet another "critical" velocity may be estimated if we suppose that the mechanism for phase slippage is the most plausible one in a simple orifice geometry, that of blowing vortex rings out on the downstream side of the orifice, of approximately the size of the orifice itself. The energy of a vortex ring of radius a is

$$E_{\text{ring}} = (2\pi^2\rho a\hbar^2/m^2)\ \ln\ (a/\xi)$$

which can be produced from an energetic point of view only if the energy available from the current source in each cycle, $2\pi\hbar J_{12}$, is equal to E_{ring}. From this we get

$$v_s^{(2)} = (\hbar/ma)\ \ln\ (a/\xi) \cong 11.5(\hbar/ma). \qquad (37'')$$

[Note that the momentum conservation equation, as opposed to these energy considerations, simply gives us the frequency condition (13) as expected from (15)].

The essential physical point here is that all of these "critical" velocities are much less than real observed superflow velocities, indicating that in all cases the generation and motion of vortices is controlled by large random fluctuations, presumably either in the generation near the walls or in the motion of vortices already present. In general, the working point of an orifice is found to be not near the lowest intersection of two energy curves, where

$$v_s = 2\hbar/ma \quad \text{and} \quad (\phi_2 - \phi_1)_0 = \pi,$$

but at a phase-difference of the order of 10π, where the energy available is much greater than that necessary to form a vortex and thus we may expect rather irregular and unstable behavior. When vortices are created under such conditions they are accelerated rather strongly and give up considerable energy to the normal excitations.

It is because of this large value of the phase difference that the orifice geometry—and, correspondingly, to a lesser extent superconducting thin film bridges—are more difficult to demonstrate spacial interference effects with than the Josephson tunnel junction, for which the system moves adiabatically from one energy minimum to the next, 2π away, whenever that is energetically possible. Incidentally, it is clear that since v_c is not very dependent on channel length, the total phase difference for a long channel at v_c is even greater than for a short one, and as a result even more randomness in the creation of vortices and even less sensitivity to the precise value of phase difference is to be expected for long channels.

It is probably for these reasons that of all the macroscopic quantum interference effects, only the driven ac experiment has as yet succeeded in He. This experiment depends on the principle of synchronization, in which a strong external ac signal is introduced which can override the internal fluctuations.

First let us consider a free-running orifice connecting two reservoirs with a height difference Z. This height difference will decrease at a rate

$$\frac{dZ}{dt} \equiv \frac{J\rho}{mA} = v_s\left(\frac{\pi a^2}{A}\right), \qquad (38)$$

where A is the area of the surface in the smaller reservoir. J also determines the rate of generation of vortices; if J is greater than the observed J_c, vortices will be generated very rapidly, and conversely; but actually of course there is a functional dependence of the rate on the current:

$$dn/dt = f(v_s) \qquad (39)$$

which will be rather steep, as shown in Fig. 3(a), but at least finite at very low v_s. Finally, we have the Josephson relation (13):

$$mgZ = -h(dn/dt) = -hf[(A/\pi a^2)(dZ/dt)]. \qquad (40)$$

Equation (40) neglects the possibility of phase change by acceleration, i.e., it really contains a term in dJ/dt or d^2Z/dt^2, which will have no effect in the mean but does lead to the U-tube oscillations. If f were a step

[24] W. F. Vinen, *Proceedings of the International School of Physics "Fermi" (Varenna) 1961*, edited by G. Careri (Academic Press Inc., New York, 1963), p. 336.

function, dZ/dt would be fixed and the height difference would decay linearly, but presumably f is somewhat "soft," and as Z decreases the decay rate will do so also, but perhaps less slowly. Figure 3(b) shows a hypothetical decay curve which fits qualitatively with the vortex generation rate shown in 3(a).

Now suppose that as the height difference drops, we are causing an ac flow to be superimposed on the dc. During half the cycle we will be increasing the tendency to form vortices, during the other half decreasing it. When the height difference is such that $dn/dt=h\nu$, one vortex per cycle will be formed—presumably in the positive half-cycle, with quite high probability. When formed it uses up some considerable fraction of the available energy so another cannot be formed immediately; thus there will be a strong tendency for exactly one vortex per cycle to be formed, since the second half-cycle is not available. Because of (40) this will mean a tendency to fixed Z, i.e., a plateau in dn/dt [Fig. 3(a)] and thus in Z. Another way of putting it is that when the vortex formation is in a definite phase relationship with the ac signal, power can be transferred from the ac generator to the system as a whole, enough power to appreciably change the flow rate. If the alternating current is larger than the dc—as in our experiment[12]—clearly it is quite possible to stop the dc flow entirely, because we can control vortex formation wholly with the ac.

I have found a very simple mechanical analogy useful for understanding the ac Josephson effect. (See Fig. 4.) The relative phase of the two reservoirs I think of as the angular position coordinate ϕ of a set of locomotive wheels, and the velocity of the locomotive is the height difference Z; the ratio of position to phase is then the radius of the wheels corresponding to h/mg. The equation (40) for the rate of generation of vortices as a function of the flow (acceleration) may be inverted to give an effective nonlinear frictional force on the "locomotive"

$$\frac{A}{\pi a^2}\frac{dZ}{dt}=-f^{-1}\frac{mgZ}{h}=-f^{-1}\left(\frac{dn}{dt}\right). \tag{41}$$

If f is a step function, this gives a constant frictional

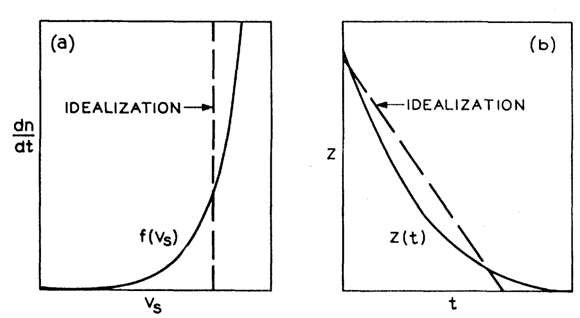

FIG. 3. (a) Rate of vortex formation as a function of v_s. "idealization" is the sharp critical current assumption; reality is probably more like $f(v_s)$ shown. (b) Decay of a helium head through an orifice or channel for critical current idealization and real situation.

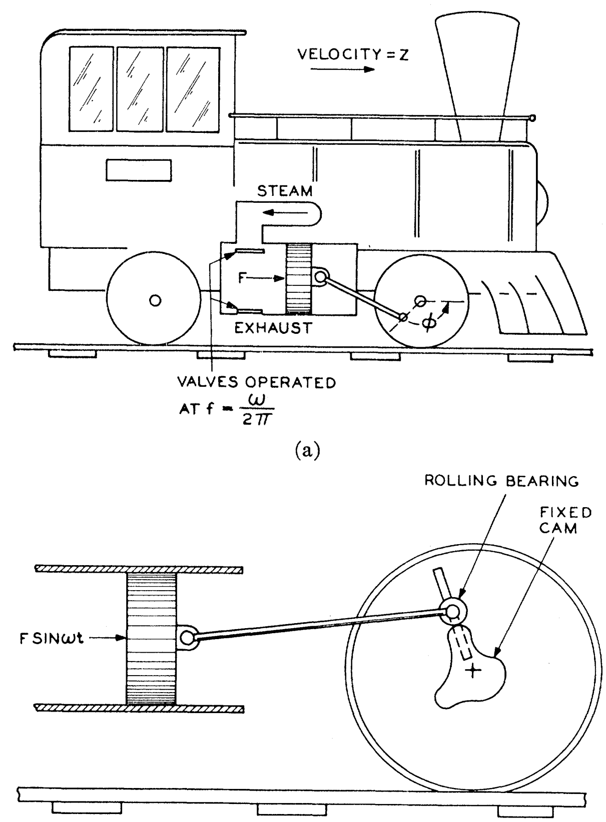

FIG. 4. (a) Illustration of "locomotive" mechanism of ac Josephson-type effect. (b) "Locomotive" system which could be driven at submultiple velocities.

force—i.e., the height (velocity) decreases linearly to zero. If f is as in Fig. 3, we get a "braking" action which leads to a decay between linear and exponential. It is an oversimplification to think of this as a constant force in time. Think of the locomotive as having square wheels and rusty bearings, so that the losses occur in some definite but not simple way during each cycle.

Now let us introduce the driven alternating current. This gives us an energy proportional to the phase coordinate, and alternating at frequency ω; it may be schematized by attaching a piston through a simple linkage to our locomotive wheels, and applying a force on the piston by (for instance) admitting steam intermittently at a rate $\omega/2\pi$:

$$F=F_0\cos(\omega t+\phi_0). \tag{42}$$

(See Fig. 4.)

As we very well know, such an alternating force is capable of keeping the locomotive going at a steady velocity Z if it is large enough, and if

$$\omega=2\pi(dn/dt),$$

i.e., the force is applied once per revolution of the wheels. Also, as is less well-known but obvious, this mechanism can act as a brake or an accelerator at this

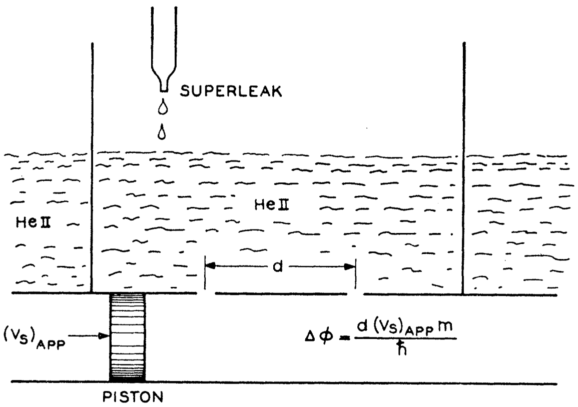

Fig. 5. Suggested space interference experiment with HeII.

velocity. The system will attempt to synchronize itself, even where the power available is not adequate to hold it in synchronization.

Another possibility is to run the wheels at n times the frequency of the valves which admit the steam. Clearly if the velocity of the locomotive is not perfectly uniform, or the valves do not regulate the steam harmonically, the two can run in synchronism and drive the locomotive. This kind of harmonic ($V = n\hbar\omega/2e$) is often observed in the true ac Josephson effect.

In helium and in the thin film bridges, the phase slippage takes place by means of vortices. This means that the motion to which the driving ac is coupled is highly anharmonic (square wheels). Another valid schematization of this is to make the piston linkage highly anharmonic—let it roll on a queer-shaped cam, for instance [see Fig. 4(b)]. Then not only can the wheels be driven faster than the frequency of the force, but also they can be driven at subharmonics, since, for example, the vortex may be formed only every nth cycle of the driving current. Thus we expect and observe both harmonics and subharmonics, in the Anderson–Dayem and Richards–Anderson experiments.[12,19]

Both of these experiments, for final quantitative description, must wait for quantitative theoretical treatments of the generation and motion of vortices. But the basic principle of phase synchronization of the external signal with the relative phase of the order parameter in the reservoirs is independent of detailed mechanism.

As for spacial interference experiments, the most promising seems to be the orifice analog of the Mercereau "current" experiment.[25] Here one drives a supercurrent through two orifices in parallel, and at the same time causes a superflow past the orifices on one side (see Fig. 5).

The second superflow enforces a phase difference $\Delta\phi$ at the two orifices on one side so that both orifices cannot simultaneously be at metastable minima of their energy curves (see Fig. 2), unless that phase difference is $2n\pi$. Another way of putting it is that the circulation through the two orifices must be quantized, leading to an additional circulating current which may aid breakdown (vortex creation) at one or both orifices. This effect would be periodic in the phase. Unfortunately, it is very sensitive to fluctuations and instabilities.

A phenomenon in which the idea of phase slippage must play an important role is the superfluid creep of films. It has been suggested that vortices form at the critical velocity with axes parallel to the film and perpendicular to the flow.[13] That is almost certainly correct—it gives dimensionally the correct critical velocity, which again is of order $\sim 10\hbar/md$. However, I would speculate somewhat differently on certain details.

First, the motion of the vortices. Examination of typical estimates indicates that the frictional forces on He vortices allow them to move with a velocity component parallel to the Magnus force of about 1% of the flow velocity. Thus vortices of the type postulated above will be generated at the solid surface and move out of the film into the free surface only a few thousand Å downstream (or vice versa, but this seems less likely). This will be the predominant dissipative mechanism if it occurs. It is hard to believe that vortex flow into the bulk fluids at either end can play an important role in a direct fashion.

There is, however, a somewhat more subtle question to be considered. If we are to take seriously the usual vortex creation critical velocity expression, h/md, it is not obvious that the smaller dimension of the film is really the value of d which must be considered. Why do not vortices form perpendicular to the film and move across it from one edge to the other? While at velocities $\sim h/mW$, where W is the width of the film ~ 1 cm (velocities $\sim 10^{-3}$ cm/sec) one would expect the formation of such vortices to be difficult dynamically, at velocities of 10^2–10^3 times that, still small compared with critical film velocities, there appears to be no such process at work.

It is suggested here that the predominant mechanism in film flow is the *pinning* of such vortices by surface flaws and thin spots in the film. Thus the film is a "hard superfluid": its flow is maintained by a pinning effect rather than by an absence of suitable vortices.

Vortices of the parallel type can also become pinned at either end and retard the motion of other vortices by their mutual repulsion. This is probably a mechanism which increases the critical velocity for rough substrates. Finally, the interaction of the pinned perpendicular vortices and the moving parallel vortices can lead to quite complicated effects: such things as the vortices reattaching themselves after a crossing in such a way that the pinned end attaches itself to the parallel vortex can occur, and become a mechanism for pinning of parallel vortices which may increase the pinning, and thus the critical velocity, as a function of the vorticity flowing into the film from the bulk liquids. Another mechanism which may play a role is motion of the free end of a pinned perpendicular vortex under

[25] R. C. Jaklevic, J. Lambe, J. E. Mercereau, and A. H. Silver, Phys. Rev. **140**, A1628 (1965).

the Magnus force until it becomes a pinned parallel vortex. This could be a copious source of vorticity.

In conclusion, then, the fundamental point made here is that in helium II, as in superconductivity, the Josephson equation and the associated concept of phase slippage are the most fundamental and exact consequences of our present theoretical understanding of superfluidity. Where phase slippage in superconductivity can occur in the absence of identifiable flux quanta, in helium with present technology it will always involve vortex lines because their core size is only a few Å, and no tunneling medium is available. Thus the crucial problem in helium flow is to find the vortex lines and study how they move across the flow path into the walls or disappear into the bulk. The complicated dynamics of vorticity is beyond the scope of this paper; we have merely speculated, with little quantitative study, in order to present concrete examples of the central ideas.

ACKNOWLEDGMENTS

I have benefitted throughout from the close collaboration of P. L. Richards. Discussions with P. C. Hohenberg, J. M. Luttinger, and W. L. McMillan were of value. A suggestion of D. J. Scalapino was reworked into the form of the space-interference experiment using orifices suggested in Sec. III. Questions asked by F. Reif and P. A. Wolff stimulated the first part of the work.

APPENDIX A. ODLRO VS MACROSCOPIC PARTICLE FIELDS

As explained in the introduction, Penrose's initial definition of the order parameter[4] in terms of a large eigenvalue of the density matrix was later extended by Beliaev,[5] by Gor'kov[6] (in terms of Green's functions), and a generalization conveniently named "off-diagonal long-range order" (ODLRO) by Yang.[20] That is, one writes

$$\langle 0_N \mid \psi^*(x)\psi(x') \mid 0_N \rangle = \sum_n \lambda_n f_n^*(x) f_n(x') \quad \text{(A1)}$$

and "ODLRO" is present when $\lambda_1 \sim N$, giving a contribution to the sum comparable to the sum of all others. One may then define a "ground state" $\langle 0_{N-1} \mid$ of the system with a different number of particles so that f_1 becomes a matrix element

$$[\lambda_1 f_1(x)]^{\frac{1}{2}} = \langle 0_{N-1} \mid \psi(x) \mid 0_N \rangle. \quad \text{(A2)}$$

In this way the necessity for dealing with states which are coherent mixtures of states with different numbers N of particles in the system is avoided, apparently, and for this reason most of the above authors prefer this scheme.

We argue that this approach is physically unnecessary, though valid, and occasionally inconvenient. For one, this definition does not permit convenient subdivision of a system. The over-all phase of f_1 is quite arbitrary—as it correctly should be for an isolated system with no particle exchanges permitted. On the other hand, once the phase at any space–time point is fixed the phase of the rest of the system is. Thus one cannot use the same description for any subdivision of the system; the λ and f for half of a bucket of He II simply do not describe it adequately. On the other hand, if one abandons the attempt to hold on to the broken gauge symmetry and ascribes a fixed, measurable phase to every superfluid system, recognizing that in principle the relative phase of any two may always be measured by a Josephson-type experiment, one immediately has a usable local description.

This is a satisfactory expedient unless the full generality of (A1) is meaningful—i.e., unless it is conceivable that more than one eigenvalue λ_1 is "large" and more than one intermediate state $\mid 0_{N-1} \rangle$ is involved. That is, we may ask whether the system may ever in any sense be a superposition of several distinct types of ODLRO. An attempt at such a theory was made by Gor'kov and Galitskii[26] for the d-state BCS theory, and proven invalid by various groups.[27] The question enters in many other cases—even, for example, in discussing flux quantization, one must be assured that one type of ODLRO only is present.

The most generally applicable argument here is that made by the author[28] in the Gor'kov–Galitskii case. It is that the two intermediate states $\mid 0_{N-1,1} \rangle$ and $\mid 0_{N-1,2} \rangle$ are demonstrably in entirely distinct Hilbert spaces in the limit $N \to \infty$, in the sense that $\sim N$ different particle states must be changed a finite amount to get from one to the other. Thus the $\mid 0_N \rangle$ state must be simply a superposition of a $\mid 0_{N,1} \rangle$ state communicating with $\mid 0_{N-1,1} \rangle$ and a $\mid 0_{N,2} \rangle$ state, and no interference effects whatever can connect the two types of states. In particular, every measurable quantity—energy, current, etc.—is the simple linear superposition of the two values. Then such a state is no more meaningful than Schrödinger's famous superposition of the quantum states of a dead cat and a live cat: a possible mathematical description of a physical absurdity.

APPENDIX B. A "NEW" COROLLARY IN CLASSICAL HYDRODYNAMICS?

Euler's equation of motion in a classical ideal fluid is

$$(\partial v/\partial t) + \nabla[(v^2/2) + \mu] = v \times \nabla \times v. \quad \text{(B1)}$$

μ is an appropriately defined chemical potential per unit mass. We now consider a general flow and draw a path C entirely inside the fluid—otherwise general—between two points P_1 and P_2 in the fluid. Points P_1 and P_2 are to be thought of eventually as being in reasonably quiet regions where the flow is steady over a long time T.

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[26] L. P. Gor'kov and V. M. Galitskii, Zh. Eksperim. i Teor. Fiz. **40**, 1124 (1961) [English transl.: Soviet Phys.—JETP **13**, 792 (1961)].
[27] D. Hone, Phys. Rev. Letters **8**, 370 (1963); R. Balian, L. H. Nosanow, and N. R. Werthamer, *ibid.* **8**, 372 (1962).
[28] P. W. Anderson, Bull. Am. Phys. Soc. **7**, 465 (1962).

Let us now perform two integrations on (B1): first, along C from P_1 to P_2

$$\int_{P_1(C)}^{P_2} \frac{\partial v}{\partial t}\, d\mathbf{l} + (\tfrac{1}{2}v^2+\mu)_{P_1}{}^{P_2} = \int_C (v \times \nabla \times v)\, d\mathbf{l}. \quad \text{(B2)}$$

[It was brought out in the discussions of the conference that (B2) is even more general than I had thought, in that most types of viscosity terms which might be added to (B1) involve gradients, so that if viscosity is not acting at P_1 and P_2 they cancel out.]

Second, we integrate over a very long time interval T and divide by T, thus taking a time mean value as is done in the virial theorem:

$$\int_{P_1}^{P_2} d\mathbf{l}\, T^{-1} \int_0^T \frac{\partial v}{\partial t}\, dt + \left\langle \left(\frac{v^2}{2}+\mu\right)_{P_2} \right\rangle_{\text{Av}} - \left\langle \left(\frac{v^2}{2}+\mu\right)_{P_1} \right\rangle_{\text{Av}}$$

$$= T^{-1} \int_0^T 2\int d\mathbf{l}\cdot (v \times \omega). \quad \text{(B3)}$$

We have defined ω, the vorticity, as $\frac{1}{2}\nabla \times v$ and written the time mean value at the points of steady flow in an obvious notation. The first term on the left-hand side is

$$T^{-1}\left[\left(\int_C v\cdot d\mathbf{l}\right)_T - \left(\int_C v\cdot d\mathbf{l}\right)_0\right].$$

We define a "quasi-steady" flow as one in which this difference increases less rapidly than T; almost any turbulent flow one wishes to consider, or periodic flow, etc. will satisfy this condition. Then as far as time mean values are concerned we arrive at the basic corollary of Euler's equation:

$$\langle (\tfrac{1}{2}v^2+\mu)_{P_2} \rangle_{\text{Av}} - \langle (\tfrac{1}{2}v^2+\mu)_{P_1} \rangle_{\text{Av}} = \left\langle 2\int_{P_1}^{P_2} d\mathbf{l}\cdot (v \times \omega) \right\rangle_{\text{Av}}. \quad \text{(B4)}$$

It is easy to interpret the quantity on the right-hand side. Writing v as dr/dt, the particle velocity, this is

$$2\int_{P_1}^{P_2} (dl \times dr/dt)\cdot \omega,$$

which is the *rate at which vorticity is being carried across the curve C* by the particle motion. Thus

$$\langle \Delta[(v^2/2)+\mu] \rangle_{\text{Av}} = \langle 2\text{"}(d\omega/dt)\text{"} \rangle_{\text{Av}}. \quad \text{(B5)}$$

We see immediately that this equation is far more important in a superfluid, where vorticity is conserved and quantized, than it is in ordinary fluids, where in a laminar flow, for instance, the right-hand side has little or no special significance. In helium, in fact, by turning to the integrated form of (B1) involving the potential we get the detailed Josephson equation without the special assumptions necessary here.

A number of somewhat surprising consequences immediately appear. One example is that the Pitot tube,[29] for instance, must involve transport of vorticity and thus motion of vortex lines in liquid He II. Ordinary aerodynamic lift and drag also would do so if the surface condition were $v=0$, but of course it is not; the vorticity there can be thought of as all in the surface layer outside the superfluid and thus not quantized.

I have tried at length to find a clear statement of (B4–5) in the classical literature, including the voluminous works of Rayleigh and Lamb, but have so far failed to find anything but corollaries and lemmas related to it. I will be pleased to hear from any reader who can find the theorem stated in this form; one can only assume that it was understood by the "classics" but is of no value in classical hydrodynamics so was never stated.

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[29] J. R. Pellam, Phys. Rev. **78,** 818 (1950).