CTRP - A Toy Example

1 Data

Considering $f = 5$ predictors and $B = 10$ permutations, we simulate a $B \times f$ matrix of global test statistics:

G

(1)	(2)	(3)	(4)	(5)
28.42	16.68	9.36	6.12	9.40
0.10	0.06	1.37	0.08	0.56
0.69	3.07	4.33	0.83	0.36
1.07	30.31	1.11	8.55	0.26
0.22	7.45	2.87	0.48	1.02
1.83	0.04	2.85	0.04	0.02
17.68	1.82	6.00	1.52	1.06
1.77	26.12	0.29	0.26	4.07
2.71	0.37	8.47	5.83	4.42
1.14	0.03	24.06	8.84	2.41

Assume that we want to test $S = \{5\}$ with significance level $\alpha = 0.20$.

2 Shortcut

v							
4				$\mathbf{F} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$			
3		$\{1,2,3,5\}$	$\{1,2,4,5\}$		$\{1,3,4,5\}$	$\{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$	
2	$\{1,2,5\}$	$\{1,3,5\}$	$\{1,4,5\}$		$\{2,3,5\}$	$\{2,4,5\}$	$\{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$
1		$\{1,5\}$	$\{2,5\}$		$\{3,5\}$	$\{\mathbf{4}, \mathbf{5}\}$	
0				$\mathbf{S} = \{\mathbf{5}\}$			

Table 1: Supersets of $S = \{5\}$, having sizes $|V| = 1 + v$ with $v = 0, \ldots, 4$. The sets in bold are used to define the lower critical value L_v.

We define

- $\mathbf{D}$, matrix of the centered test statistics in $F \setminus S$, where the indices appear in the order $(4, 3, 2, 1)$ (since $g_4 \leq g_3 \leq g_2 \leq g_1$);
- $\mathbf{R}$, matrix obtained from $\mathbf{D}$ by sorting the elements within each row in decreasing order.

$\mathbf{d}_S$	$\mathbf{D}$				$\mathbf{R}$			
(5)	(4)	(3)	(2)	(1)				
0.00	0.00	0.00	0.00	0.00	0.00 (1)	0.00 (2)	0.00 (3)	0.00 (4)
-8.84	-6.03	-7.99	-16.62	-28.32	-6.03 (4)	-7.99 (3)	-16.62 (2)	-28.32 (1)
-9.04	-5.29	-5.02	-13.61	-27.72	-5.02 (3)	-5.29 (4)	-13.61 (2)	-27.72 (1)
-9.14	2.43	-8.25	13.63	-27.34	13.63 (2)	2.43 (4)	-8.25 (3)	-27.34 (1)
-8.38	-5.63	-6.49	-9.23	-28.19	-5.63 (4)	-6.49 (3)	-9.23 (2)	-28.19 (1)
-9.38	-6.08	-6.51	-16.64	-26.59	-6.08 (4)	-6.51 (3)	-16.64 (2)	-26.59 (1)
-8.34	-4.59	-3.36	-14.86	-10.74	-3.36 (3)	-4.59 (4)	-10.74 (1)	-14.86 (2)
-5.33	-5.85	-9.07	9.44	-26.65	9.44 (2)	-5.85 (4)	-9.07 (3)	-26.65 (1)
-4.98	-0.28	-0.89	-16.31	-25.71	-0.28 (4)	-0.89 (3)	-16.31 (2)	-25.71 (1)
-6.99	2.72	14.70	-16.65	-27.27	14.70 (3)	2.72 (4)	-16.65 (2)	-27.27 (1)

The lower and upper critical values, L_v and U_v, are the 8-th ordered statistics of

$$\mathbf{d}_{\tilde{V}} = \mathbf{d}_3 + \sum_{i=1}^{v} \mathbf{D}_i \qquad\qquad \mathbf{u}_v = \mathbf{d}_3 + \sum_{i=1}^{v} \mathbf{R}_i.$$

Since the first column of $\mathbf{R}$ having no positive elements has index $w = 3$, we start by computing L_v for $v = 0, 1, 2$.

No non-rejection is found, and thus we proceed by examining $v > 2$, until we find either a rejection or a negative U_v. Here U_v becomes negative for $v = 3$, hence the supersets with $v = 3, 4$ are automatically rejected.

Finally, we determine the indecisive values by looking at U_v. Here they are $v = 1, 2$.

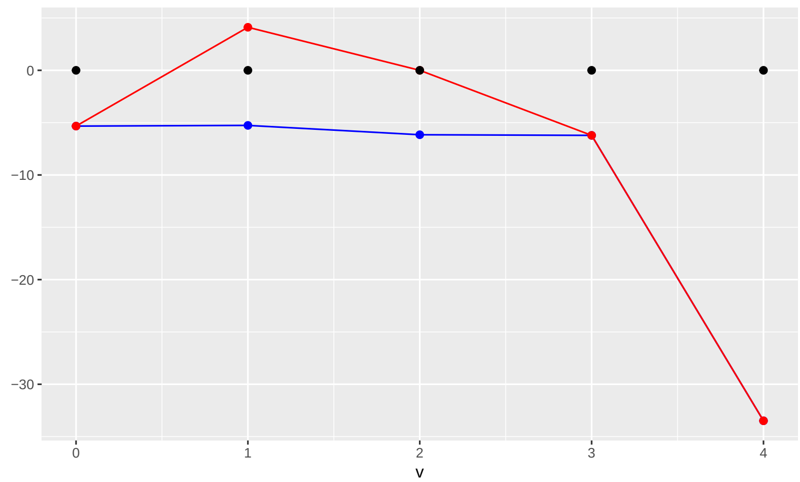


Figure 1: Upper (red) and lower (blue) critical values and observed values (zero, black) by additional superset size v. The bounds for $v = 4$ have not been computed in the analysis.

v	0	1	2	3	4
U_v	-5.33	4.11	0.00	-6.22	(-33.49)
L_v	-5.33	-5.26	-6.16	-6.22	(-33.49)
rej	T	?	?	T	T

3 Branch and Bound

3.1 Removal of the highest statistic.

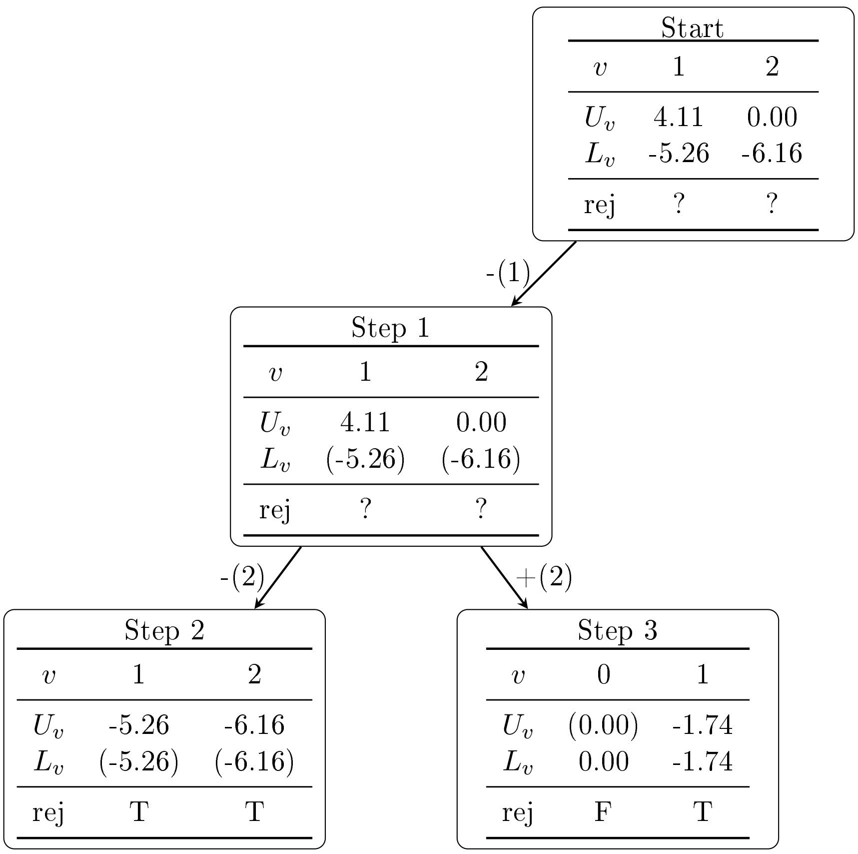
The last index in $\mathbf{D}$, $e = 1$, determines the branching rule. We explore first the subspace $\mathbb{S}_{-1}$, where we examine only U_v.

	$\mathbb{S}_{-1}$			$\mathbb{S}_{+1}$		
2	$\{2,3,5\}$	$\{2,4,5\}$	$\{3,4,5\}$	$\{1,2,5\}$	$\{1,3,5\}$	$\{1,4,5\}$
1	$\{2,5\}$	$\{3,5\}$	$\{4,5\}$		$\{1,5\}$	

Since the outcome is still indecisive for both sizes $v = 1, 2$, the subspace is partitioned again according to the inclusion of $e = 2$ (corresponding to the second highest statistic g_i).

	$\mathbb{S}_{-1,-2}$			$\mathbb{S}_{-1,+2}$		
2		$\{3,4,5\}$		$\{2,3,5\}$		$\{2,4,5\}$
1	$\{2,5\}$		$\{3,5\}$		$\{4,5\}$	

Since S is not rejected in $\mathbb{S}_{-1,+2}$, it is not rejected in the total space after 3 steps.



3.2 Keeping of the highest statistic.

The last index in $\mathbf{D}$, $e = 1$, determines the branching rule. We explore first the subspace $\mathbb{S}_{+1}$, where we examine both critical values.

S is not rejected after 3 steps: it is rejected in $\mathbb{S}_{+1}$, but it is not rejected in $\mathbb{S}_{-1,+2}$.

Start

v	1	2
U_v	4.11	0.00
L_v	-5.26	-6.16
rej	?	?

-(1)

+(1)

Step 2

v	1	2
U_v	4.11	0.00
L_v	(-5.26)	(-6.16)
rej	?	?

Step 1

v	0	1
U_v	(-30.69)	-30.97
L_v	-30.69	-22.43
rej	T	T

+(2)

Step 3

v	0	1
U_v	(0.00)	-1.74
L_v	0.00	-1.74
rej	F	T

3.3 Removal of the lowest statistic.

The first index in $\mathbf{D}$, $e = 4$, determines the branching rule. We explore first the subspace $\mathbb{S}_{-4}$, where we examine both critical values.

S is not rejected after 2 steps, where the subspaces $\mathbb{S}_{-4}$ and $\mathbb{S}_{-4,-3}$ are explored.

Start

v	1	2
U_v	4.11	0.00
L_v	-5.26	-6.16
rej	?	?

-(4)

Step 1

v	1	2
U_v	4.11	-4.96
L_v	-5.87	-4.96
rej	?	T

-(3)

Step 2

v	1
U_v	0.00
L_v	0.00
rej	F

3.4 Keeping of the lowest statistic.

The first index in $\mathbf{D}$, $e = 4$, determines the branching rule. We explore first the subspace $\mathbb{S}_{+4}$, where we examine only U_v.

After 2 steps, S is rejected in the subspace $\mathbb{S}_{+4}$. Subsequently, after 2 other steps, it is not rejected in $\mathbb{S}_{-4}$. In conclusion, it is not rejected after 4 steps.