CTRP - A Toy Example

1 Data.

We simulate a covariate matrix $\mathbf{X}$ having $n = 20$ observations of $f = 5$ variables from a standard normal distribution. Then we simulate a response variable $\mathbf{y} = (y_1, \ldots, y_n)^\top$ according to the logistic regression model

$$y_j \sim \mathrm{Ber}(p_j), \qquad \mathrm{logit}(p_j) = \beta_0 + \mathbf{X}_j \boldsymbol{\beta} \qquad (j = 1, \ldots, n)$$

with $\beta_0 = 0$ and $\beta = (20, 10, 5, 0, 0)$.

Finally, we consider B random permutations of $\mathbf{y}$ (where the first is the identity) and compute the global test statistics

$$g_i^\pi = \mathbf{y}^{\pi\top} \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^\top \right) \mathbf{X}_i \mathbf{X}_i^\top \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^\top \right) \mathbf{y}^\pi \qquad (i = 1, \ldots, f, \quad \pi = 1, \ldots, B),$$

as well as the centered test statistics

$$d_i^\pi = g_i^\pi - g_i \qquad (i = 1, \ldots, f, \quad \pi = 1, \ldots, B).$$

We define the matrix M of the individual centered test statistics d_i^π, where the rows are sorted so that the observed values g_i are in increasing order, as well as the matrix D, obtained by sorting the elements of M within each row in descending order:

M					**D**				
d_5	d_2	d_3	d_4	d_1					
0.00	0.00	0.00	0.00	0.00	0.00 (1)	0.00 (4)	0.00 (3)	0.00 (2)	0.00 (5)
-6.03	-7.99	-8.84	-16.62	-28.32	-6.03 (5)	-7.99 (2)	-8.84 (3)	-16.62 (4)	-28.32 (1)
-5.29	-5.02	-9.04	-13.61	-27.72	-5.02 (2)	-5.29 (5)	-9.04 (3)	-13.61 (4)	-27.72 (1)
2.43	-8.25	-9.14	13.63	-27.34	13.63 (4)	2.43 (5)	-8.25 (2)	-9.14 (3)	-27.34 (1)
-5.63	-6.49	-8.38	-9.23	-28.19	-5.63 (5)	-6.49 (2)	-8.38 (3)	-9.23 (4)	-28.19 (1)
-6.08	-6.51	-9.38	-16.64	-26.59	-6.08 (5)	-6.51 (2)	-9.38 (3)	-16.64 (4)	-26.59 (1)
-4.59	-3.36	-8.34	-14.86	-10.74	-3.36 (2)	-4.59 (5)	-8.34 (3)	-10.74 (1)	-14.86 (4)
-5.85	-9.07	-5.33	9.44	-26.65	9.44 (4)	-5.33 (3)	-5.85 (5)	-9.07 (2)	-26.65 (1)
-0.28	-0.89	-4.98	-16.31	-25.71	-0.28 (5)	-0.89 (2)	-4.98 (3)	-16.31 (4)	-25.71 (1)
2.72	14.70	-6.99	-16.65	-27.27	14.70 (2)	2.72 (5)	-6.99 (3)	-16.65 (4)	-27.27 (1)

2 Test.

Assume that we want to test $S = \{3\}$. Then the subsets that need to be tested are the following:

size							
5				$\mathbf{F = \{1,2,3,4,5\}}$			
4		$\{1,2,3,4\}$	$\{1,2,3,5\}$		$\{1,3,4,5\}$	$\mathbf{\{2,3,4,5\}}$	
3	$\{1,2,3\}$	$\{1,3,4\}$	$\{1,3,5\}$		$\{2,3,4\}$	$\mathbf{\{2,3,5\}}$	$\{3,4,5\}$
2		$\{1,3\}$	$\{2,3\}$		$\{3,4\}$	$\mathbf{\{3,5\}}$	
1				$\mathbf{S = \{3\}}$			

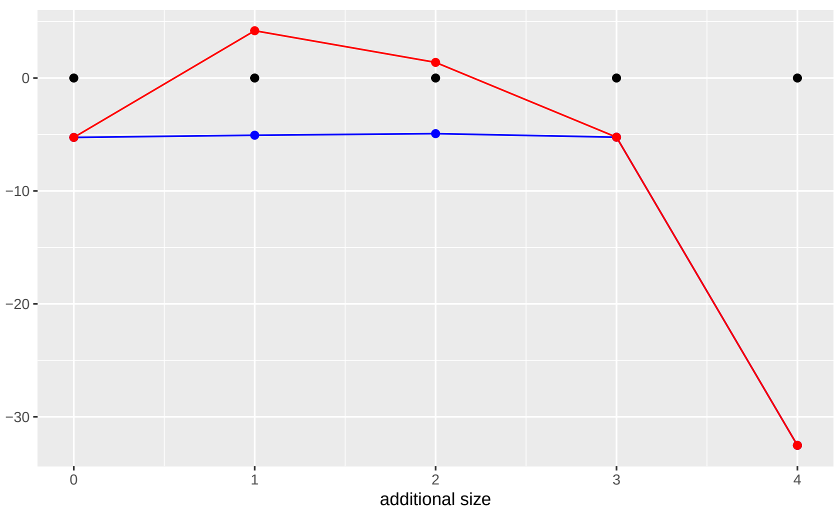
We define the matrices $\tilde{M}$ and $\tilde{D}$ by moving the elements corresponding to S to the first column of M and D:

		$\tilde{\mathbf{M}}$					$\tilde{\mathbf{D}}$		
d_3	d_5	d_2	d_4	d_1	d_3				
0.00	0.00	0.00	0.00	0.00	0.00 (3)	0.00 (1)	0.00 (4)	0.00 (2)	0.00 (5)
-8.84	-6.03	-7.99	-16.62	-28.32	-8.84 (3)	-6.03 (5)	-7.99 (2)	-16.62 (4)	-28.32 (1)
-9.04	-5.29	-5.02	-13.61	-27.72	-9.04 (3)	-5.02 (2)	-5.29 (5)	-13.61 (4)	-27.72 (1)
-9.14	2.43	-8.25	13.63	-27.34	-9.14 (3)	13.63 (4)	2.43 (5)	-8.25 (2)	-27.34 (1)
-8.38	-5.63	-6.49	-9.23	-28.19	-8.38 (3)	-5.63 (5)	-6.49 (2)	-9.23 (4)	-28.19 (1)
-9.38	-6.08	-6.51	-16.64	-26.59	-9.38 (3)	-6.08 (5)	-6.51 (2)	-16.64 (4)	-26.59 (1)
-8.34	-4.59	-3.36	-14.86	-10.74	-8.34 (3)	-3.36 (2)	-4.59 (5)	-10.74 (1)	-14.86 (4)
-5.33	-5.85	-9.07	9.44	-26.65	-5.33 (3)	9.44 (4)	-5.85 (5)	-9.07 (2)	-26.65 (1)
-4.98	-0.28	-0.89	-16.31	-25.71	-4.98 (3)	-0.28 (5)	-0.89 (2)	-16.31 (4)	-25.71 (1)
-6.99	2.72	14.70	-16.65	-27.27	-6.99 (3)	14.70 (2)	2.72 (5)	-16.65 (4)	-27.27 (1)

For each possible superset size $v = 0, \ldots, f$, the lower and upper bounds are determined by the cumulative sums

$$L_v^\pi = \sum_{i=1}^{v} \tilde{M}_i^\pi \qquad U_v^\pi = \sum_{i=1}^{v} \tilde{D}_i^\pi,$$

and the critical values are the corresponding $(1-\alpha)$-quantiles (here $\alpha = 0.20$).



v	1	2	3	4	5
$c_v(U)$	-5.26	4.19	1.38	-5.24	-32.53
$c_v(L)$	-5.26	-5.06	-4.93	-5.24	-32.53
rejection	T	?	?	T	T

However, if there is a column $\tilde{\mathbf{D}}_j$ ($j > 1$) such that all the elements are non-positive, then also the following columns will be non-positive, and from that point on the upper critical value $c_v(U)$ will not increase. In particular, when $c_v(U)$ becomes negative, it will always remain negative, and thus all supersets from that size to f will be rejected. Hence:

- we determine the column index j, if it exists;
- we compute the lower bound up to column $j-1$ (size $|S|+j-2$), to check for non-rejections;
- if needed, we continue by computing both bounds until $c_v(U)$ becomes negative. We know that higher sizes are all rejected;
- finally, $c_v(U)$ is computed for columns from 1 to $j-1$, in order to determine indecisive sizes.

If at any point a non-rejection is found, the search stops.

In this case, $j = 4$. The lower bounds up to size 3 are all negative, hence there are no non-rejections.

Branch and Bound method. The first index, $i = 1$, determines the branching rule:

	remove			keep		
	$S \subseteq V \subseteq F \setminus \{1\}$			$S \cup \{1\} \subseteq V \subseteq F$		
|S|+2	$\{2,3,4\}$	$\{\mathbf{2},\mathbf{3},\mathbf{5}\}$	$\{3,4,5\}$	$\{1,2,3\}$	$\{1,3,4\}$	$\{1,3,5\}$
|S|+1	$\{2,3\}$	$\{3,4\}$	$\{\mathbf{3},\mathbf{5}\}$		$\{1,3\}$	

In the "remove" branch, the lower bound remains the same, as the sum of the last v columns of M is not affected by the removal of the first column. However, the upper bound changes in both branches, as the elements corresponding to $i = 1$ (in I) are removed.

To do. Check common addenda between keep and remove.

Check the shape of the upper bound curve. Does it always decrease from some point on? (Such point being when a column of D has all negative elements). If it is true, we can compute the points where the curve crosses zero: we automatically reject for all the sizes such that the curve is below zero.

Consider two vectors

$$X = (x_1, \dots, x_B)^\top \in \mathbb{R}^B \qquad\qquad Y = (y_1, \dots, y_B)^\top \in [0, +\infty)^B$$

as well as their difference $Z = X - Y$. Let q_X and q_Z be the quantiles of X and Z, such that

$$P(x_i \leq q_X) = 1 - \alpha \qquad\qquad P(z_i \leq q_Z) = 1 - \alpha.$$

Since $z_i \leq x_i$ for each i, then $P(z_i \leq q_X) \geq P(x_i \leq q_X) = 1-\alpha = P(z_i \leq q_Z)$. As a consequence, $q_X \geq q_Z$.

Check the behavior of different test statistics, e.g. harmonic mean p-value, Fisher combination etc.