Permutation Closed Testing with Sum-Based Statistics

1 Sum-Based Test Statistics

Given a full model $F = \{1, \dots, f\}$ of univariate hypotheses, let $S \subseteq F$ be a subset under test by closed testing with level α. Hence S is rejected if and only if all its supersets $S \cup V$ ($V \subseteq F \setminus S$) are rejected.

Let

$$g_i^\pi \quad (i = 1, \dots, f,\ \pi = 1, \dots, B)$$

be some test statistics corresponding to the f covariates and B random permutations, where the first permutation is the identity. Assume that the null hypothesis is rejected for high values of g_i (otherwise, it is sufficient to multiply all test statistics by -1), and

$$g_V^\pi = \sum_{i \in V} g_i^\pi \quad (V \subseteq F,\ \pi = 1, \dots, B).$$

Moreover, define the centered test statistics

$$d_i^\pi = g_i^\pi - g_i \quad (i = 1, \dots, f,\ \pi = 1, \dots, B),$$

so that the observed values are all $d_i = 0$, and the variability due to g_i is excluded.

For $V \subseteq F$, the vector of its statistics is

$$\mathbf{d_V} = (0, d_V^2, \dots, d_V^B)^\top.$$

Consider the sorted test statistics $d_V^{(1)} \leq d_V^{(2)} \leq \dots \leq d_V^{(B)}$, and define $k = \lceil (1-\alpha)B \rceil$. The permutation test rejects V if

$$d_V^{(k)} < 0.$$

Such a test can be slightly conservative, but it can be adapted to be exact by randomizing it.

2 Shortcut

Let $s = |S|$ and $m = f - s$, and fix a value $v \in \{0, \dots, m\}$. We will define a shortcut for the analysis of the supersets $S \cup V$ with $V \in \mathcal{V}_v$ and

$$\mathcal{V}_v = \{V \,:\, V \subseteq F \setminus S,\ |V| = v\},$$

that does not require all the $\binom{m}{v}$ vectors $\mathbf{d}_V$. It relies on the construction of a lower and an upper critical values, L_v and U_v, such that

- if $L_v \geq 0$, then at least one superset $S \cup \tilde{V}$ is not rejected, and thus S is not rejected;
- if $U_v < 0$, then all supersets are rejected, and other sizes can be explored.

Notice that if $L_v < 0 \leq U_v$, then the outcome is indecisive.

Lower critical value. In order not to reject S, it is sufficient to find a non-rejected superset. We define the lower critical value as $L_v = (d_S + l_v)^{(k)}$, where $\mathbf{l}_v = \mathbf{d}_{\tilde{V}}$ and $\tilde{V} \in \mathcal{V}_v$ is such that $S \cup \tilde{V}$ is likely to be non-rejected.

In particular, $\tilde{V}$ is defined by considering the indices of the v smallest observed statistics not in S. If $(i_1, \ldots, i_m)$ is a permutation of the indices in $F \setminus S$ such that

$$g_{i_1} \leq g_{i_2} \leq \ldots \leq g_{i_m},$$

then $\tilde{V} = \{i_1, \ldots, i_v\}$ and

$$l_v^\pi = \sum_{h=1}^{v} d_{i_h}^\pi \quad (\pi = 1, \ldots, B).$$

Upper critical value. The upper critical value is $U_v = (d_S + u_v)^{(k)}$, where

$$\mathbf{u}_v = (0, u_v^2, \ldots, u_v^B)^\top$$

is a vector such that

$$u_v^\pi \geq d_V^\pi \quad (V \in \mathcal{V}_v,\ \pi = 1, \ldots, B).$$

As a consequence,

$$U_v \geq (d_S + d_V)^{(k)} \quad (V \in \mathcal{V}_v).$$

If $U_v < 0$, then all supersets $S \cup V$ with $V \in \mathcal{V}_v$ are rejected.

For each $\pi = 1, \ldots, B$, the element u_v^π is defined by considering the v highest centered statistics not in S. If $(j_1(\pi), \ldots, j_m(\pi))$ is a permutation of the indices in $F \setminus S$ such that

$$d_{j_1(\pi)}^\pi \geq d_{j_2(\pi)}^\pi \geq \ldots d_{j_m(\pi)}^\pi,$$

then

$$u_v^\pi = \sum_{h=1}^{v} d_{j_h(\pi)}^\pi.$$

Testing. The values $v = 0, \ldots, m$ are checked in sequence.

- As soon as a non-rejection is found (there exists v^* such that $L_{v^*} \geq 0$), the analysis stops and S is not rejected.
- If all values lead to rejection ($U_v < 0$ for all v), then S is rejected.
- If some values $v \in \{1, \ldots, m-1\}$ lead to indecisive outcomes ($L_v < 0 \leq U_v$), the Branch and Bound method is applied. Notice that an indecisive outcome cannot occur for $v = 0$ or $v = m$, since

$$L_0 = U_0 = d_S^{(k)} \qquad\qquad L_m = U_m = d_F^{(k)}.$$

Early stop. Assume that there exists $w \in \{1, \ldots, m\}$ such that all the elements of $\mathbf{u}_w$ are smaller than those of $\mathbf{u}_{w-1}$:

$$u_w^\pi = u_{w-1}^\pi + d_{j_w(\pi)}^\pi \leq u_{w-1}^\pi \quad (\pi = 1, \ldots, B)$$

or, equivalently,

$$d_{j_w(\pi)}^\pi \leq 0 \quad (\pi = 1, \ldots, B).$$

Then the upper critical value is non-increasing for $v \geq w$:

$$\begin{aligned} d^{\pi}_{j_m(\pi)} \leq \ldots \leq d^{\pi}_{j_w(\pi)} \leq 0 \quad (\pi = 1, \ldots, B) \\ u^{\pi}_m \leq \ldots \leq u^{\pi}_w \leq u^{\pi}_{w-1} \quad (\pi = 1, \ldots, B) \\ U_m \leq \ldots \leq U_w \leq U_{w-1}. \end{aligned}$$

In this case, it is sufficient to stop the analysis as soon as we find a value $v^* \geq w - 1$ such that $U_{v^*} < 0$. All the supersets with $v \geq v^*$ are automatically rejected.

3 Branch and Bound

Assume that some values $v \in \{1, \ldots, m-1\}$ lead to an indecisive outcome.

For a fixed index $e \in F \setminus S$, the total space $\mathbb{S} = \{V : V \subseteq F \setminus S\}$ is partitioned into two disjoint subspaces, according to the inclusion of e:

$$\mathbb{S}_{-e} = \{V : V \subseteq F \setminus (S \cup \{e\})\} \qquad \mathbb{S}_{+e} = \{V : \{e\} \subseteq V \subseteq F \setminus S\}.$$

The shortcut is applied to each subspace, in order to evaluate the indecisive values v:

- if S is not rejected in at least one subspace, it is not rejected in the total space;
- if S is rejected in both subspaces, it is rejected in the total space;
- if there is an indecisive outcome in at least one subspace, the procedure is iterated by partitioning the indecisive subspace(s).

For any choice of e, U_v does not increase in the subspaces (since it is defined by taking the maximum statistics over smaller subsets). However, the choice of e and the order in which the subspaces are explored influence L_v, and thus the efficiency of the algorithm. We wish to begin with the subspaces that are more likely to lead to a non-rejection, i.e. where L_v is more likely to be high.

We will evaluate the efficiency in four different scenarios, employing the order of the observed statistics in $F \setminus S$. The scenarios differ by the index e used for branching:

- highest statistic $e = i_m$ (L_v does not vary in $\mathbb{S}_{-e}$);
- lowest statistic $e = i_1$ (L_v does not vary in $\mathbb{S}_{+e}$).

Moreover, they differ by the branch that is explored first:

- $\mathbb{S}_{-e}$, removal;
- $\mathbb{S}_{+e}$, keeping.

Preliminary simulations suggest that removing the highest statistic could lead to the smallest number of iterations in most cases.

4 Example

Considering $f = 5$ predictors and $B = 10$ permutations, we simulate a $B \times f$ matrix of global test statistics:

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(1)	(2)	(3)	(4)	(5)
28.42	16.68	9.36	6.12	9.40
0.10	0.06	1.37	0.08	0.56
0.69	3.07	4.33	0.83	0.36
1.07	30.31	1.11	8.55	0.26
0.22	7.45	2.87	0.48	1.02
1.83	0.04	2.85	0.04	0.02
17.68	1.82	6.00	1.52	1.06
1.77	26.12	0.29	0.26	4.07
2.71	0.37	8.47	5.83	4.42
1.14	0.03	24.06	8.84	2.41

Assume that we want to test $S = \{5\}$ with significance level $\alpha = 0.20$.

v							
4				$\mathbf{F = \{1, 2, 3, 4, 5\}}$			
3		$\{1,2,3,5\}$	$\{1,2,4,5\}$		$\{1,3,4,5\}$	$\mathbf{\{2,3,4,5\}}$	
2	$\{1,2,5\}$	$\{1,3,5\}$	$\{1,4,5\}$		$\{2,3,5\}$	$\{2,4,5\}$	$\mathbf{\{3,4,5\}}$
1		$\{1,5\}$	$\{2,5\}$		$\{3,5\}$	$\mathbf{\{4,5\}}$	
0				$\mathbf{S = \{5\}}$			

Table 1: Supersets of $S = \{5\}$by size. The sets in bold are used to define the lower critical value L_v.

Shortcut. We define

- $\mathbf{D}$, matrix of the centered test statistics in $F \setminus S$, where the indices appear in the order $(4, 3, 2, 1)$ (since $g_4 \leq g_3 \leq g_2 \leq g_1$);
- $\mathbf{R}$, matrix obtained from $\mathbf{D}$ by sorting the elements within each row in decreasing order;
- $\mathbf{Dsum}$ and $\mathbf{Rsum}$, matrices of the cumulative sums of $\mathbf{d}_S$ with $\mathbf{D}$ and $\mathbf{R}$, respectively.

Let $k = \lceil (1-\alpha)B \rceil = 8$. The lower and upper critical values, L_v and U_v, are the k-th ordered statistics of

$$\mathbf{d}_S + \mathbf{l}_v = \mathbf{Dsum}_{v+1} \qquad\qquad \mathbf{d}_S + \mathbf{u}_v = \mathbf{Rsum}_{v+1}.$$

The first column of $\mathbf{R}$ having no positive elements has index $w = 3$. We start by computing both bounds for $v = 0, 1, 2$. No non-rejection is found, but the values $v = 1, 2$ lead to an indecisive outcome.

We proceed by examining $v \leq 3$, until we find either a rejection or a negative U_v. Since U_v becomes negative for $v = 3$, the supersets with $v = 3, 4$ are automatically rejected.

$\mathbf{d}_S$	**D**				**R**			
(5)	(4)	(3)	(2)	(1)				
0.00	0.00	0.00	0.00	0.00	0.00 (1)	0.00 (2)	0.00 (3)	0.00 (4)
-8.84	-6.03	-7.99	-16.62	-28.32	-6.03 (4)	-7.99 (3)	-16.62 (2)	-28.32 (1)
-9.04	-5.29	-5.02	-13.61	-27.72	-5.02 (3)	-5.29 (4)	-13.61 (2)	-27.72 (1)
-9.14	2.43	-8.25	13.63	-27.34	13.63 (2)	2.43 (4)	-8.25 (3)	-27.34 (1)
-8.38	-5.63	-6.49	-9.23	-28.19	-5.63 (4)	-6.49 (3)	-9.23 (2)	-28.19 (1)
-9.38	-6.08	-6.51	-16.64	-26.59	-6.08 (4)	-6.51 (3)	-16.64 (2)	-26.59 (1)
-8.34	-4.59	-3.36	-14.86	-10.74	-3.36 (3)	-4.59 (4)	-10.74 (1)	-14.86 (2)
-5.33	-5.85	-9.07	9.44	-26.65	9.44 (2)	-5.85 (4)	-9.07 (3)	-26.65 (1)
-4.98	-0.28	-0.89	-16.31	-25.71	-0.28 (4)	-0.89 (3)	-16.31 (2)	-25.71 (1)
-6.99	2.72	14.70	-16.65	-27.27	14.70 (3)	2.72 (4)	-16.65 (2)	-27.27 (1)

Dsum					**Rsum**				
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-8.84	-14.87	-22.86	-39.48	-67.80	-8.84	-14.87	-22.86	-39.48	-67.80
-9.04	-14.33	-19.35	-32.97	-60.69	-9.04	-14.07	-19.35	-32.97	-60.69
-9.14	-6.71	-14.97	-1.33	-28.68	-9.14	4.49	6.92	-1.33	-28.68
-8.38	-14.02	-20.51	-29.74	-57.94	-8.38	-14.02	-20.51	-29.74	-57.94
-9.38	-15.46	-21.96	-38.61	-65.20	-9.38	-15.46	-21.96	-38.61	-65.20
-8.34	-12.93	-16.29	-31.15	-41.89	-8.34	-11.69	-16.29	-27.03	-41.89
-5.33	-11.18	-20.26	-10.81	-37.46	-5.33	4.11	-1.74	-10.81	-37.46
-4.98	-5.26	-6.16	-22.47	-48.18	-4.98	-5.26	-6.16	-22.47	-48.18
-6.99	-4.27	10.43	-6.22	-33.49	-6.99	7.71	10.43	-6.22	-33.49

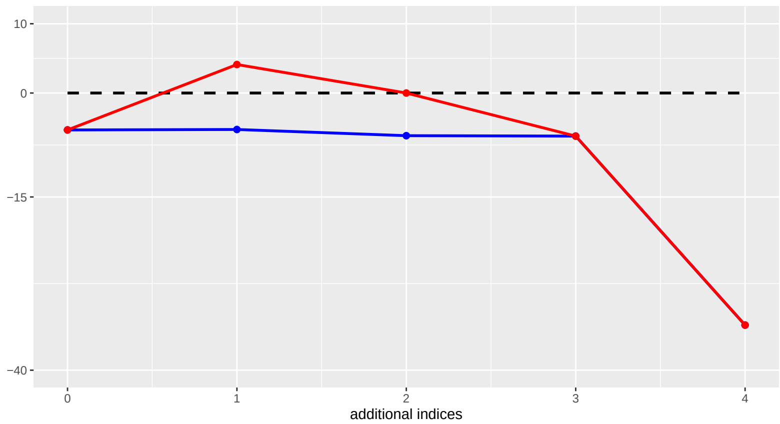

Figure 1: Upper (red) and lower (blue) critical values by additional superset size v. The bounds for $v = 4$ have not been computed in the analysis.

v	0	1	2	3	4
U_v	-5.33	4.11	0.00	-6.22	(-33.49)
L_v	-5.33	-5.26	-6.16	-6.22	(-33.49)
rej	T	?	?	T	T

Branch and Bound (removal of the highest statistic). The index of the highest test statistic, $e = 1$, determines the branching rule. We start by studying U_v for the indecisive sizes in the subspace $\mathbb{S}_{-1}$ (since L_v does not vary). As shown in figure 2, the outcome is still indecisive for both sizes $v = 1, 2$.

Then we remove the second highest statistic ($e = 2$). In $\mathbb{S}_{-1,-2}$, where we examine only U_v, the null hypothesis is rejected.

Finally, we analyze both bounds in $\mathbb{S}_{-1,+2}$, where a non-rejection is found for $v = 0$. In conclusion, the null hypothesis is not rejected after 3 steps.

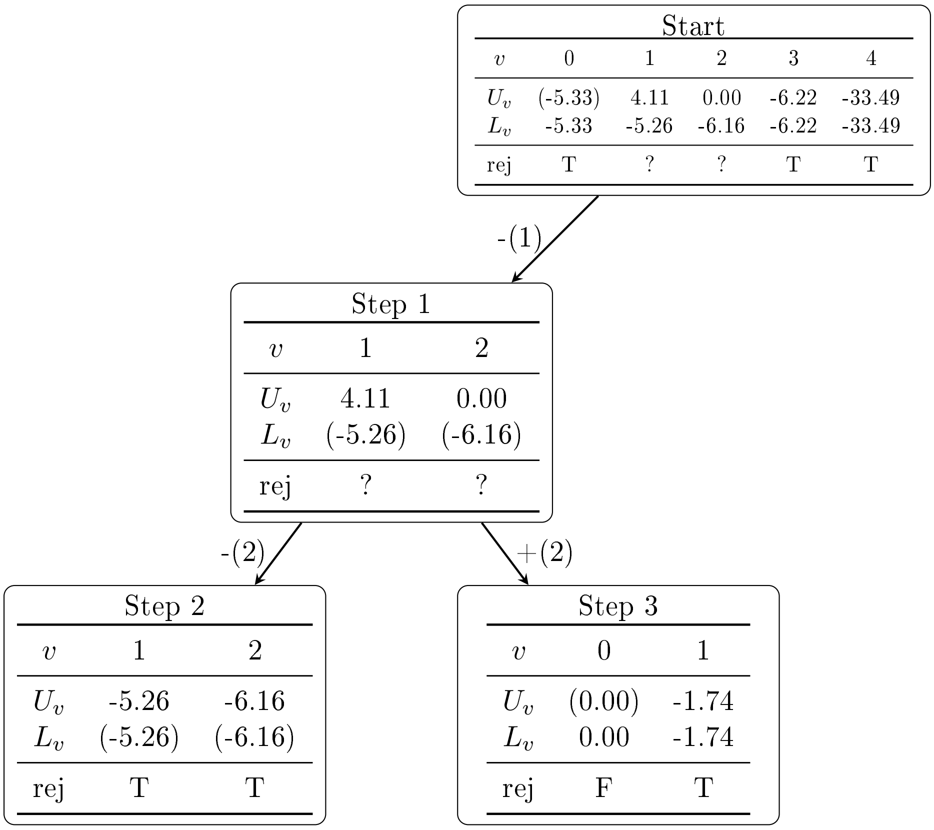

Figure 2: Branch and Bound procedure, carried out by removing the highest test statistic.

5 True Discovery Proportion

Assume that S is rejected by closed testing. The true discoveries can be estimated by analysing the subsets

$$\mathcal{Z}_z = \{Z \subseteq S \, : \, |Z| = s - z\}$$

for $z = 0, \ldots, s-1$. Indeed, if all elements of $\mathcal{Z}_z$ are rejected by closed testing, then $TD \geq z+1$. The highest bound corresponds to the maximum z value such that the condition is true.

$\mathcal{Z}_0 = \{S\}$ has already been rejected. The supersets of $Z \in \mathcal{Z}_1$ are

- S and its supersets (already rejected by the local tests);
- supersets with the form $Z \cup V$, with $V \subseteq F \setminus S$.

Analogously, consider $\mathcal{Z}_z$, after rejecting all the elements of $\mathcal{Z}_0, \ldots, \mathcal{Z}_{z-1}$. In order to reject $Z \in \mathcal{Z}_z$, the only supersets that need to be studied have the form $Z \cup V$, with $V \subseteq F \setminus S$. We will define bounds $L_{z,v}$ and $U_{z,v}$ for the critical values of such sets.

Lower critical value. Let $\mathbf{l}_{0,v} = \mathbf{l}_v$ as defined in section 2. Let $(x_1, \ldots, x_s)$ be a permutation of the indices in S such that

$$g_{x_1} \leq g_{x_2} \leq \ldots \leq g_{x_s}.$$

For $z = 1, \ldots, s$, the lower critical value is $L_{z,v} = (d_S + l_{z,v})^{(k)}$, where

$$l^\pi_{z,v} = l^\pi_{z-1,v} - d^\pi_{x_{s-z+1}}.$$

Notice that

$$d^\pi_S + l^\pi_{z,v} = \sum_{h=1}^{z} d^\pi_{x_h} + \sum_{h=1}^{v} d^\pi_{i_h}$$

is the test statistic corresponding to the set $\tilde{Z} \cup \tilde{V} = \{x_1, \ldots, x_z\} \cup \{i_1, \ldots, i_v\}$, likely to be rejected.

Upper critical value. Let $\mathbf{u}_{0,v} = \mathbf{u}_v$ as defined in section 2. For each $\pi = 1, \ldots, B$, let $(y_1(\pi), \ldots, y_s(\pi))$ be a permutation of the indices in S such that

$$d^\pi_{y_1(\pi)} \geq d^\pi_{y_2(\pi)} \geq \ldots d^\pi_{y_s(\pi)}.$$

Then $U_{z,v} = (d_S + u_{z,v})^{(k)}$ with

$$u^\pi_{z,v} = u^\pi_{z-1,v} - d^\pi_{y_{s-z+1}(\pi)}.$$

Notice that

$$d^\pi_S + u^\pi_{z,v} = \sum_{h=1}^{z} d^\pi_{y_h(\pi)} + \sum_{h=1}^{v} d^\pi_{j_h(\pi)}.$$

6 Example 2

Considering $f = 7$ predictors and $B = 10$ permutations, we simulate a $B \times f$ matrix of global test statistics:

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(1)	(2)	(3)	(4)	(5)	(6)	(7)
9.45	8.26	7.91	0.76	35.34	16.42	13.72
5.86	0.57	0.37	29.85	0.98	0.44	0.00
0.02	2.03	3.33	1.42	0.00	2.40	0.15
4.07	33.21	1.24	0.28	0.41	7.93	3.97
0.52	24.95	13.23	0.06	1.48	0.96	2.81
0.28	0.34	0.56	0.25	11.41	3.19	5.41
5.55	2.79	1.01	6.44	3.71	1.73	2.07
0.14	5.84	4.76	1.12	0.92	0.10	0.09
22.72	10.99	7.75	18.32	18.87	0.09	0.06
7.92	0.49	3.18	3.05	20.55	14.36	4.97

After rejecting $S = \{5, 6, 7\}$ with significance level $\alpha = 0.20$, we want to estimate the TDP. We examine the subsets of size $s - z$, with $z = 1, 2$.

We define

- **Ds** and **D**, matrices of the centered test statistics in S and $F \setminus S$, respectively;
- **Rs** and **R**, matrices obtained from **Ds** and **D** by sorting the elements within each row in decreasing order;
- **Dsum** and **Rsum**, matrices of the cumulative sums of $\mathbf{d}_S$ with **D** and **R**, respectively.

Let $k = \lceil (1 - \alpha)B \rceil = 8$. The lower and upper critical values $L_{z,v}$ and $U_{z,v}$ are the k-th ordered statistics of $\mathbf{Dsum}(\mathbf{z})_{v+1}$ and $\mathbf{Rsum}(\mathbf{z})_{v+1}$, where

$$\mathbf{Dsum}(\mathbf{1}) = \mathbf{Dsum} - \mathbf{Ds}_s \qquad \mathbf{Rsum}(\mathbf{1}) = \mathbf{Rsum} - \mathbf{Rs}_s$$
$$\mathbf{Dsum}(\mathbf{2}) = \mathbf{Dsum}(\mathbf{1}) - \mathbf{Ds}_{s-1} \qquad \mathbf{Rsum}(\mathbf{2}) = \mathbf{Rsum}(\mathbf{1}) - \mathbf{Rs}_{s-1}$$

The first column of **R** having no positive elements has index $w = 4$. Hence we compute both bounds for $v < 4$ and we examine $v = 2$ only if $U_{z,3} \geq 0$.

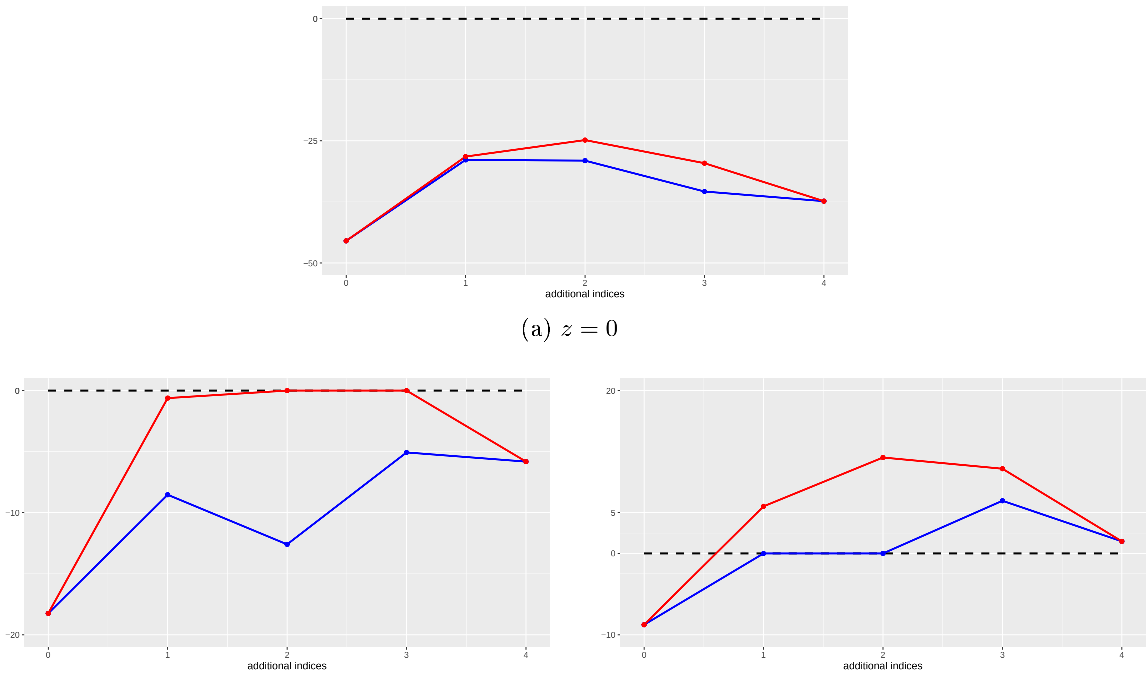
- $z = 1$: no non-rejection, but values $v = 2, 3$ lead to an indecisive outcome. The Branch and Bound method is needed.
- $z = 2$: a non-rejection is found for $v = 1$, hence not all subsets of size $s-2$ can be rejected.

In conclusion, the true discoveries are $1 \leq TD \leq 2$.

Ds			D			
(7)	(6)	(5)	(4)	(3)	(2)	(1)
0.00	0.00	0.00	0.00	0.00	0.00	0.00
-13.72	-15.98	-34.36	29.08	-7.54	-7.68	-3.58
-13.58	-14.02	-35.34	0.65	-4.58	-6.23	-9.42
-9.75	-8.49	-34.93	-0.49	-6.67	24.95	-5.37
-10.91	-15.47	-33.86	-0.70	5.32	16.69	-8.92
-8.32	-13.23	-23.92	-0.52	-7.34	-7.92	-9.16
-11.66	-14.69	-31.62	5.68	-6.89	-5.47	-3.89
-13.64	-16.33	-34.42	0.35	-3.14	-2.41	-9.30
-13.66	-16.33	-16.46	17.56	-0.16	2.73	13.27
-8.75	-2.06	-14.79	2.29	-4.73	-7.77	-1.53

Rs			R			
0.00 (5)	0.00 (6)	0.00 (7)	0.00 (1)	0.00 (2)	0.00 (3)	0.00 (4)
-13.72 (7)	-15.98 (6)	-34.36 (5)	29.08 (4)	-3.58 (1)	-7.54 (3)	-7.68 (2)
-13.58 (7)	-14.02 (6)	-35.34 (5)	0.65 (4)	-4.58 (3)	-6.23 (2)	-9.42 (1)
-8.49 (6)	-9.75 (7)	-34.93 (5)	24.95 (2)	-0.49 (4)	-5.37 (1)	-6.67 (3)
-10.91 (7)	-15.47 (6)	-33.86 (5)	16.69 (2)	5.32 (3)	-0.70 (4)	-8.92 (1)
-8.32 (7)	-13.23 (6)	-23.92 (5)	-0.52 (4)	-7.34 (3)	-7.92 (2)	-9.16 (1)
-11.66 (7)	-14.69 (6)	-31.62 (5)	5.68 (4)	-3.89 (1)	-5.47 (2)	-6.89 (3)
-13.64 (7)	-16.33 (6)	-34.42 (5)	0.35 (4)	-2.41 (2)	-3.14 (3)	-9.30 (1)
-13.66 (7)	-16.33 (6)	-16.46 (5)	17.56 (4)	13.27 (1)	2.73 (2)	-0.16 (3)
-2.06 (6)	-8.75 (7)	-14.79 (5)	2.29 (4)	-1.53 (1)	-4.73 (3)	-7.77 (2)

Dsum					Rsum				
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-64.06	-34.97	-42.51	-50.20	-53.78	-64.06	-34.97	-38.56	-46.10	-53.78
-62.93	-62.28	-66.86	-73.09	-82.51	-62.93	-62.28	-66.86	-73.09	-82.51
-53.17	-53.66	-60.33	-35.37	-40.75	-53.17	-28.22	-28.71	-34.08	-40.75
-60.24	-60.93	-55.62	-38.92	-47.84	-60.24	-43.54	-38.22	-38.92	-47.84
-45.47	-45.99	-53.33	-61.25	-70.41	-45.47	-45.99	-53.33	-61.25	-70.41
-57.97	-52.29	-59.18	-64.65	-68.54	-57.97	-52.29	-56.18	-61.65	-68.54
-64.39	-64.03	-67.18	-69.59	-78.89	-64.39	-64.03	-66.44	-69.59	-78.89
-46.45	-28.90	-29.05	-26.32	-13.05	-46.45	-28.90	-15.63	-12.90	-13.05
-25.60	-23.32	-28.05	-35.82	-37.35	-25.60	-23.32	-24.85	-29.58	-37.35



(a) $z = 0$

(b) $z = 1$

(c) $z = 2$

Figure 3: Upper (red) and lower (blue) critical values by additional superset size v for the subsets of size $s - z$.

Branch and Bound (removal of the highest statistic not in S). The method is applied to examine $z = 1$ and $v = 2, 3$. The index of the highest test statistic, $e = 1$, determines the branching rule. We start by studying the subspace $\mathbb{S}_{-1}$.

As shown in figure 4, after 4 steps the subsets of size $s - 1$ are not rejected. Hence the true discoveries are $TD = 1$.

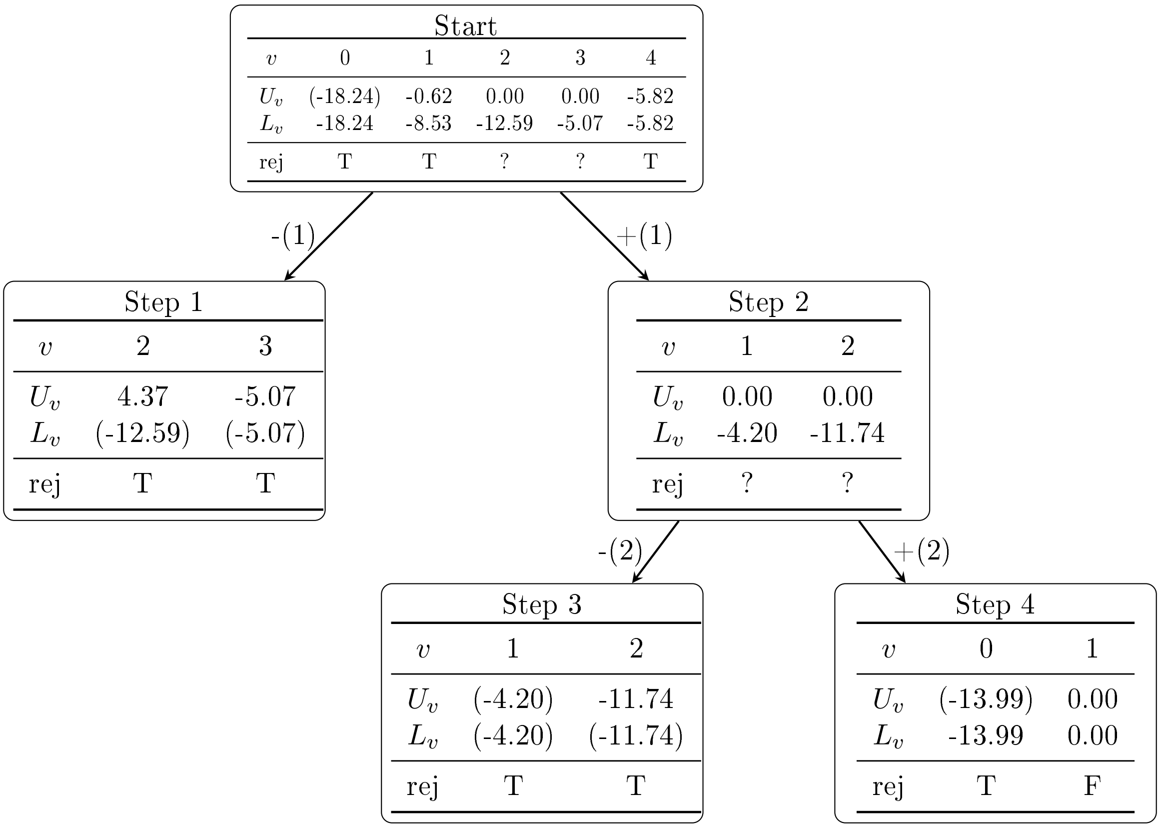


Figure 4: Branch and Bound procedure for the subsets of size $s-1$, carried out by removing the highest test statistic not in S.