Permutation Closed Testing with Sum-Based Statistics

1 Sum-Based Test Statistics

Given a full model $F = \{1, \dots, f\}$ of univariate hypotheses, let $S \subseteq F$ be a subset under test by closed testing with level α. Hence S is rejected if and only if all its supersets V $(S \subseteq V \subseteq F)$ are rejected.

Let

$$g_i^\pi \quad (i = 1, \dots, f,\ \pi = 1, \dots, B)$$

be some test statistics corresponding to the f covariates and B random permutations, where the first permutation is the identity. Assume that such test statistics are such that

$$g_V^\pi = \sum_{i \in V} g_i^\pi \quad (V \subseteq F,\ \pi = 1, \dots, B).$$

Moreover, define the centered test statistics

$$d_i^\pi = g_i^\pi - g_i \quad (i = 1, \dots, f,\ \pi = 1, \dots, B),$$

so that the observed values are all $d_i = 0$, and the variability due to g_i is excluded.

For $V \subseteq F$, the vector of its statistics is

$$\mathbf{d_V} = (0, d_V^2, \dots, d_V^B)^\top.$$

Consider $d_V^{(1)} \leq d_V^{(2)} \leq \dots \leq d_V^{(B)}$, and define $k = \lceil (1-\alpha)B \rceil$. An exact permutation test rejects V if $0 = d_V > d_V^{(k)}$, and rejects with probability

$$a = \frac{\alpha B - \#\{\pi : d_V^\pi > d_V^{(k)}\}}{\#\{\pi : d_V^\pi = d_V^{(k)}\}}$$

if $0 = d_V = d_V^{(k)}$.

2 Shortcut

Let $s = |S|$ and $m = f - s$. The possible superset sizes are $|V| = s + v$, with $v = 0, \dots, m$.

Fix a value v. We will define a shortcut for the analysis of the supersets in

$$\mathcal{V}_v = \{V : S \subseteq V \subseteq F,\ |V| = s + v\},$$

that does not require the critical values of all the $\binom{m}{v}$ vectors $\mathbf{d}_V$ $(V \in \mathcal{V}_v)$. It relies on the construction of a lower and an upper critical values, L_v and U_v, such that

- if $L_v < 0$ (and with probability a if $L_v = 0$), then at least one superset $\tilde{V} \in \mathcal{V}_v$ is not rejected, and thus S is not rejected;
- if $U_v < 0$, then all supersets $V \in \mathcal{V}_v$ are rejected, and other sizes can be explored.

Notice that if $L_v \leq 0 \leq U_v$, then the outcome is indecisive.

Lower critical value. In order not to reject S, it is sufficient to find a non-rejected superset. Hence we consider $\tilde{V} \in \mathcal{V}_v$ which is likely to be non-rejected, and define the lower critical value as $L_v = d_{\tilde{V}}^{(k)}$.

In particular, $\tilde{V}$ is defined by considering S and the indices of the remaining v smallest observed statistics. If $(i_1, \ldots, i_m)$ is a permutation of the indices in $F \setminus S$ such that

$$g_{i_1} \leq g_{i_2} \leq \ldots \leq g_{i_m},$$

then $\tilde{V} = S \cup \{i_1, \ldots, i_v\}$ and

$$d_{\tilde{V}}^{\pi} = d_S^{\pi} + \sum_{h=1}^{v} d_{i_h}^{\pi} \quad (\pi = 1, \ldots, B).$$

Upper critical value. The upper critical value is $U_v = u_v^{(k)}$, where

$$\mathbf{u}_v = (0, u_v^2, \ldots, u_v^B)^\top$$

is a vector such that

$$u_v^{\pi} \geq d_V^{\pi} \quad (V \in \mathcal{V}_v,\ \pi = 1, \ldots, B).$$

As a consequence,

$$U_v \geq d_V^{(k)} \quad (V \in \mathcal{V}_v).$$

If $U_v < 0$, then all supersets in $\mathcal{V}_v$ are rejected.

For each $\pi = 1, \ldots, B$, the element u_v^{π} is defined by considering d_S^{π} and the remaining v highest centered statistics. If $(j_1(\pi), \ldots, j_m(\pi))$ is a permutation of the indices in $F \setminus S$ such that

$$d_{j_1(\pi)}^{\pi} \geq d_{j_2(\pi)}^{\pi} \geq \ldots d_{j_m(\pi)}^{\pi},$$

then

$$u_v^{\pi} = d_S^{\pi} + \sum_{h=1}^{v} d_{j_h(\pi)}^{\pi}.$$

Testing. The values $v = 0, \ldots, m$ are checked in sequence.

- As soon as a non-rejection is found ($\tilde{v}$ with $L_{\tilde{v}} < 0$), the analysis stops and S is not rejected.
- If all values lead to rejection ($U_v < 0$ for all v), then S is rejected.
- If some values $v \in \{1, \ldots, m-1\}$ lead to indecisive outcomes ($L_v \leq 0 \leq U_v$), the Branch and Bound method is applied. Notice that an indecisive outcome cannot occur for $v = 0$ or $v = m$, since

$$L_0 = U_0 = d_S^{(k)} \qquad\qquad L_m = U_m = d_F^{(k)}.$$

Early stop. Assume that there exists $w \in \{1, \dots, m\}$ such that

$$d^{\pi}_{j_w(\pi)} \leq 0 \quad (\pi = 1, \dots, B)$$

or, equivalently,

$$u^{\pi}_{w} = u^{\pi}_{w-1} + d^{\pi}_{j_w(\pi)} \leq u^{\pi}_{w-1} \quad (\pi = 1, \dots, B).$$

Then the upper critical value is non-increasing for $v \geq w$:

$$\begin{aligned} d^{\pi}_{j_m(\pi)} \leq \dots \leq d^{\pi}_{j_w(\pi)} \leq 0 \quad (\pi = 1, \dots, B) \\ u^{\pi}_{m} \leq \dots \leq u^{\pi}_{w} \leq u^{\pi}_{w-1} \quad (\pi = 1, \dots, B) \\ U_m \leq \dots \leq U_w \leq U_{w-1}. \end{aligned}$$

In this case, it is sufficient to stop the analysis as soon as we find a value $v^* \geq w - 1$ such that $U_{v^*} < 0$. All the supersets with $v \geq v^*$ are automatically rejected.

3 Branch and Bound

Assume that some values $v \in \{1, \dots, m-1\}$ lead to an indecisive outcome.

For a fixed index $e \in F \setminus S$, the total space $\mathbb{S} = \{V : S \subseteq V \subseteq F\}$ is partitioned into two disjoint subspaces, according to the inclusion of e:

$$\mathbb{S}_- = \{V : S \subseteq V \subseteq F \setminus \{e\}\} \qquad \mathbb{S}_+ = \{V : S \cup \{e\} \subseteq V \subseteq F\}.$$

The shortcut is applied to each subspace, in order to evaluate the indecisive values v:

- if S is not rejected in at least one subspace, it is not rejected in the total space;
- if S is rejected in both subspaces, it is rejected in the total space;
- if there is an indecisive outcome in at least one subspace, the procedure is iterated by partitioning the indecisive subspace(s).

For any choice of e, U_v does not increase in the subspaces (since it is defined by taking the maximum statistics over smaller subsets). However, the choice of e and the order in which the subspaces are explored influence L_v, and thus the efficiency of the algorithm. We wish to begin with the subspaces that are more likely to lead to a rejection, i.e. where L_v is more likely to be high.

I will evaluate the efficiency in three different scenarios, employing the order of the observed statistics in $F \setminus S$.

- Removal of the highest statistic: $e = i_m$, and $\mathbb{S}_-$ is explored first. In this case, L_v does not vary in $\mathbb{S}_-$, and is likely to decrease in $\mathbb{S}_+$.
- Keeping of the lowest statistic: $e = i_1$, and $\mathbb{S}_+$ is explored first. In this case, L_v may decrease in $\mathbb{S}_-$, and does not vary in $\mathbb{S}_+$.
- Removal of the lowest statistic: $e = i_m$, and $\mathbb{S}_-$ is explored first.