CTRP - A Toy Example

Data. We simulate a covariate matrix $\mathbf{X}$ having $n = 20$ observations of $f = 5$ variables from a standard normal distribution. Then we simulate a response variable $\mathbf{y} = (y_1, \ldots, y_n)^\top$ according to the logistic regression model

$$y_j \sim \text{Ber}(p_j), \qquad \text{logit}(p_j) = \beta_0 + \mathbf{X}_j \boldsymbol{\beta} \qquad (j = 1, \ldots, n)$$

with $\beta_0 = 0$ and $\beta = (20, 10, 5, 0, 0)$.

Finally, we consider B random permutations of $\mathbf{y}$ (where the first is the identity) and compute the global test statistics

$$g_i^\pi = \mathbf{y}^{\pi\top} \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^\top \right) \mathbf{X}_i \mathbf{X}_i^\top \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^\top \right) \mathbf{y}^\pi \qquad (i = 1, \ldots, f, \quad \pi = 1, \ldots, B),$$

as well as the centered test statistics

$$d_i^\pi = g_i^\pi - g_i \qquad (i = 1, \ldots, f, \quad \pi = 1, \ldots, B).$$

Analysis. Assume that we want to test $S = \{3\}$. Then the subsets that need to be tested are the following:

|S|+4				$\mathbf{F = \{1, 2, 3, 4, 5\}}$			
|S|+3		$\{1, 2, 3, 4\}$	$\{1, 2, 3, 5\}$		$\{1, 3, 4, 5\}$	$\mathbf{\{2, 3, 4, 5\}}$	
|S|+2	$\{1, 2, 3\}$	$\{1, 3, 4\}$	$\{1, 3, 5\}$		$\{2, 3, 4\}$	$\mathbf{\{2, 3, 5\}}$	$\{3, 4, 5\}$
|S|+1		$\{1, 3\}$	$\{2, 3\}$		$\{3, 4\}$	$\mathbf{\{3, 5\}}$	
|S|				$\mathbf{S = \{3\}}$			

We define the following quantities:

- M_S, vector of the centered test statistics d_S^π $(\pi = 1, \ldots, B)$;
- M, matrix of the individual centered test statistics d_i^π corresponding to the indices in $F \setminus S$. The rows are sorted so that the observed values are in descending order;
- D, matrix obtained by sorting the elements of M within each row in descending order;
- I, matrix of indices such that $I_{\pi h} = i$ if the element $D_{\pi h}$ corresponds to predictor i.

$\mathbf{M_S}$	$\mathbf{M}$				$\mathbf{D}$				$\mathbf{I}$			
d_2	d_1	d_4	d_2	d_5								
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1	4	2	5
-8.84	-28.32	-16.62	-7.99	-6.03	-6.03	-7.99	-16.62	-28.32	5	2	4	1
-9.04	-27.72	-13.61	-5.02	-5.29	-5.02	-5.29	-13.61	-27.72	2	5	4	1
-9.14	-27.34	13.63	-8.25	2.43	13.63	2.43	-8.25	-27.34	4	5	2	1
-8.38	-28.19	-9.23	-6.49	-5.63	-5.63	-6.49	-9.23	-28.19	5	2	4	1
-9.38	-26.59	-16.64	-6.51	-6.08	-6.08	-6.51	-16.64	-26.59	5	2	4	1
-8.34	-10.74	-14.86	-3.36	-4.59	-3.36	-4.59	-10.74	-14.86	2	5	1	4
-5.33	-26.65	9.44	-9.07	-5.85	9.44	-5.85	-9.07	-26.65	4	5	2	1
-4.98	-25.71	-16.31	-0.89	-0.28	-0.28	-0.89	-16.31	-25.71	5	2	4	1
-6.99	-27.27	-16.65	14.70	2.72	14.70	2.72	-16.65	-27.27	2	4	4	1

Consider each possible superset size $|S| + v$ (with $v = 0, \ldots, 4$). The lower bound

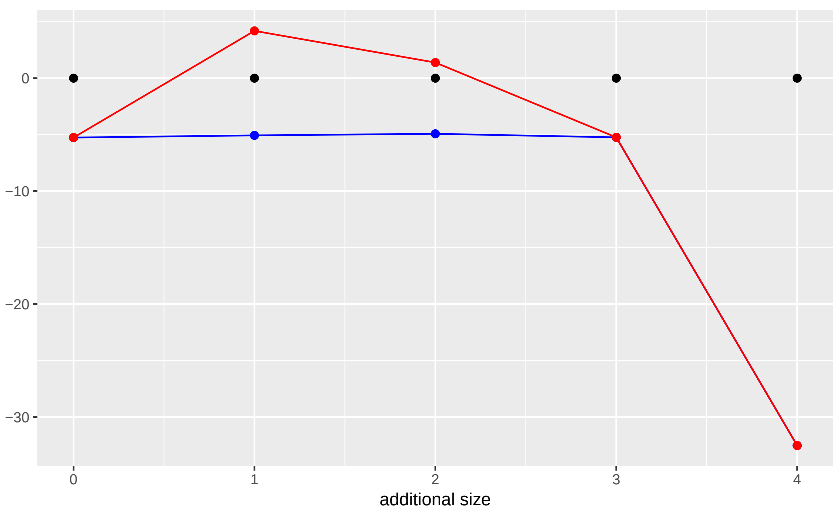
$$L_v^\pi = M_S^\pi + \sum_{i=5-v}^{4} M_i^\pi$$

is determined by the v predictors with the lowest observed value, so that only the supersets $S \subset \{3,5\} \subset \{2,3,5\} \subset \{2,3,4,5\} \subset F$ are considered. The upper bound

$$U(v)^\pi = M_S^\pi + \sum_{i=1}^{v} D_i^\pi$$

is determined by the v highest values for each permutation, i.e. the first v columns of D. The $(1-\alpha)$-quantiles of the bounds (here $\alpha = 0.20$) are:

v	0	1	2	3	4
$c_v(U)$	-5.26	4.19	1.38	-5.24	-32.53
$c_v(L)$	-5.26	-5.06	-4.93	-5.24	-32.53
rejection	T	?	?	T	T



Branch and Bound method. The first index, $i = 1$, determines the branching rule:

	remove			keep		
	$S \subseteq V \subseteq F \setminus \{1\}$			$S \cup \{1\} \subseteq V \subseteq F$		
|S|+2	$\{2,3,4\}$	$\{\mathbf{2},\mathbf{3},\mathbf{5}\}$	$\{3,4,5\}$	$\{1,2,3\}$	$\{1,3,4\}$	$\{1,3,5\}$
|S|+1	$\{2,3\}$	$\{3,4\}$	$\{\mathbf{3},\mathbf{5}\}$		$\{1,3\}$	

In the "remove" branch, the lower bound remains the same, as the sum of the last v columns of M is not affected by the removal of the first column. However, the upper bound changes in both branches, as the elements corresponding to $i = 1$ (in I) are removed.