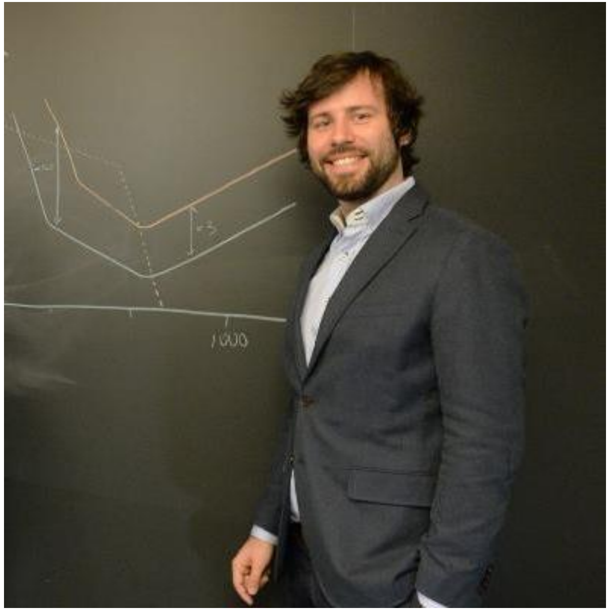
Prof. Imre Bartos



Astrophysics: black holes, gravitational waves

Office hours: Wednesdays 4-6pm

email: phy2049@phys.ufl.edu

Chapter 30: Induction and Inductance

Math: Flux of magnetic field $\Phi_B = \int \vec{B} \cdot d\vec{A}$ (same equation as for a flux of any field)

Fundamentals: Changing B-field flux induces emf: $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$ **(Faraday's Law)**

Flux can change due to one of the three reasons (Faraday's equation captures them all): $\oint \vec{E} \cdot d\vec{s} = -\dfrac{d\Phi_B}{dt}$

(1) change of flux due to a change in the area A – old physics (conductive wire moving in B)

(2) change of flux due change in the angle between B and A vectors – old physics (conductive wire moving in B)

(3) change in the magnetic field itself (fundamental physics): changing B field induces E field

Lenz's Law: "Nature's reaction" to the change in the B-field flux is to prevent the change (examples)

Applications (inductors):

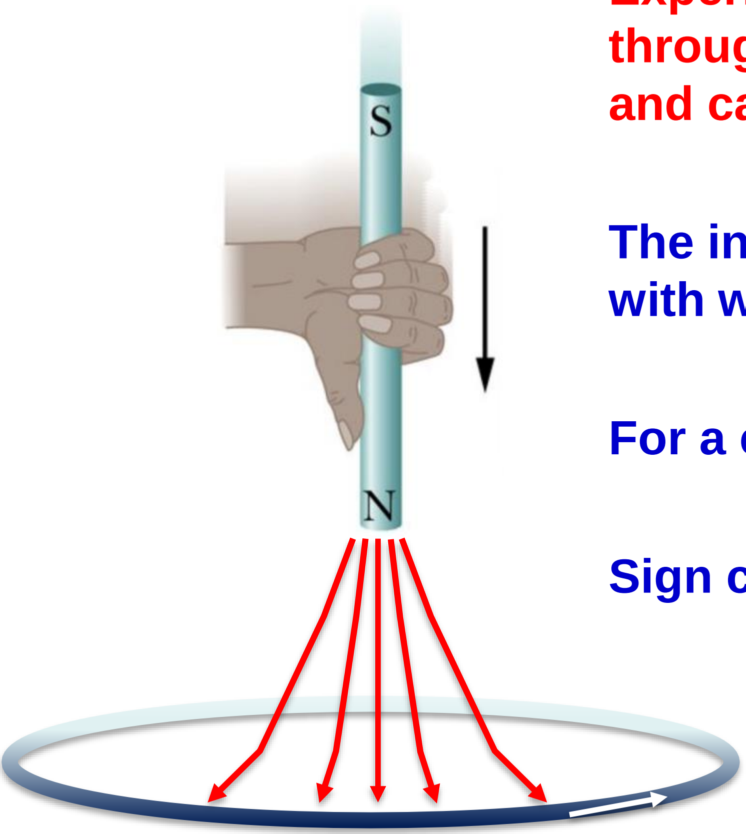
Self-induction: $\mathcal{E} = -L\dfrac{di}{dt}$ Inductance of solenoid: $L = \mu_0 n^2 A\ell$

Mutual induction $\mathcal{E}_2 = -M\dfrac{di_1}{dt}$ and $\mathcal{E}_1 = -M\dfrac{di_2}{dt}$ (mutual inductance *M* is the same in both equations!)

RL circuits: $i(t) = i_0(1 - e^{-t/\tau})$ or $i(t) = i_0 e^{-t/\tau}$, where $\tau = RL$

Fundamentals: energy density stored on B-field: $u = \dfrac{\mu_0}{2}B^2$ energy stored in B-field of an inductor $U = \dfrac{1}{2}Li^2$

Induced emf (Faraday's Law)



induced emf

Experimental fact: If one changes magnetic field going through a wire loop, an induced emf appears in the loop and causes a current

The induced emf in a loop ~ rate with which B field flux changes.

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

For a coil with N loops:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

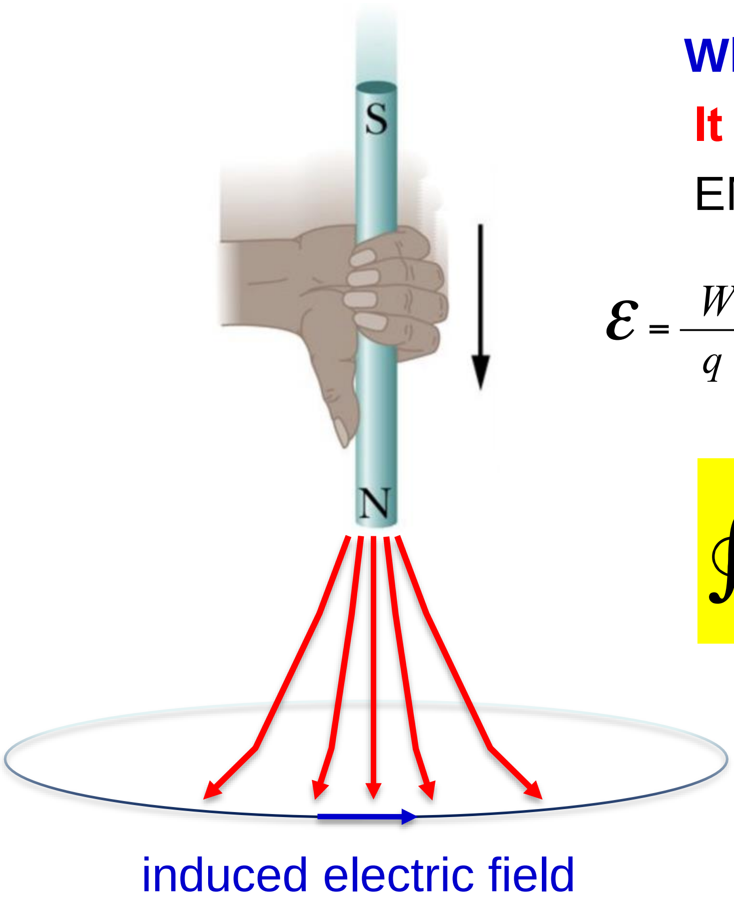
Sign convention for direction of the emf:

Use right hand;

First, align the thumb with the **direction of the flux <u>change</u>**; then flip it (this is the minus sign in the equation);

Curl your fingers around the thumb – this is the direction of induced emf

Induced electric field



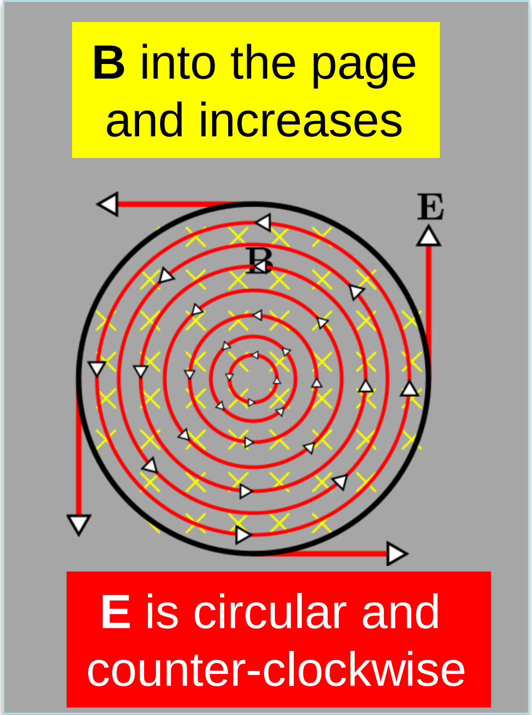
Where does induced emf come from?

It is due to an induced “circular” electric field

EMF of such field is

$$\mathcal{E} = \frac{W}{q} = \frac{\oint \vec{F}\cdot d\vec{s}}{q} = \frac{\oint q\vec{E}\cdot d\vec{s}}{q} = \oint \vec{E}\cdot d\vec{s}$$

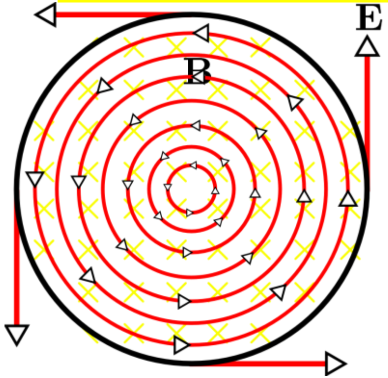
$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$



Eddy currents

- **changing flux of magnetic field → emf**
- **emf → currents,** provided the medium is conductive

B into the page and increases



E is circular and counter-clockwise

Induction stove

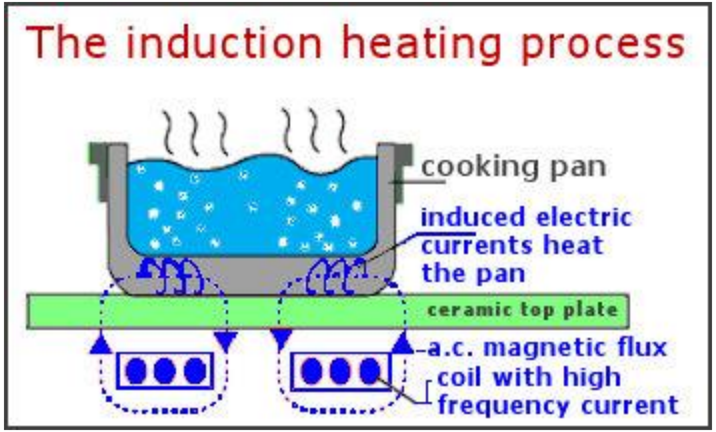
- **coils under the stove generate changing magnetic field (stove itself is not hot)**
- pots/pants must be **conductive**
- **changing B field → changing B flux → emf → Eddy currents → heat $P=(emf)^2/R$**
- and if the bottom surface of pots/pants is **ferromagnetic**,

  the magnetic field is greatly enhanced ($B = \mu B_0$),

  which results in greatly enhanced $\Delta\Phi/\Delta t$,

  which results in greatly enhanced emf,

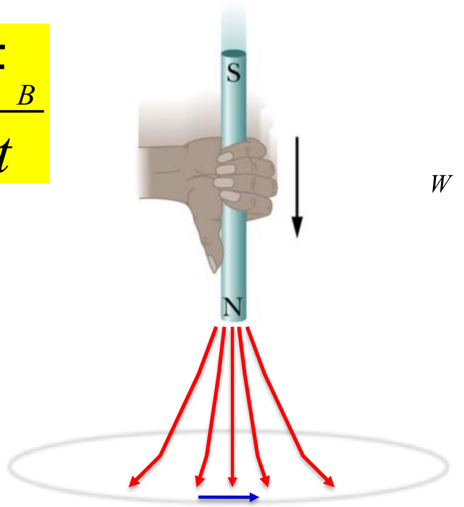
  which results in a greatly enhanced heating



Lenz's Law

- **Lenz's Law** interprets the minus sign in Faraday's Law equation (not an independent law)

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$



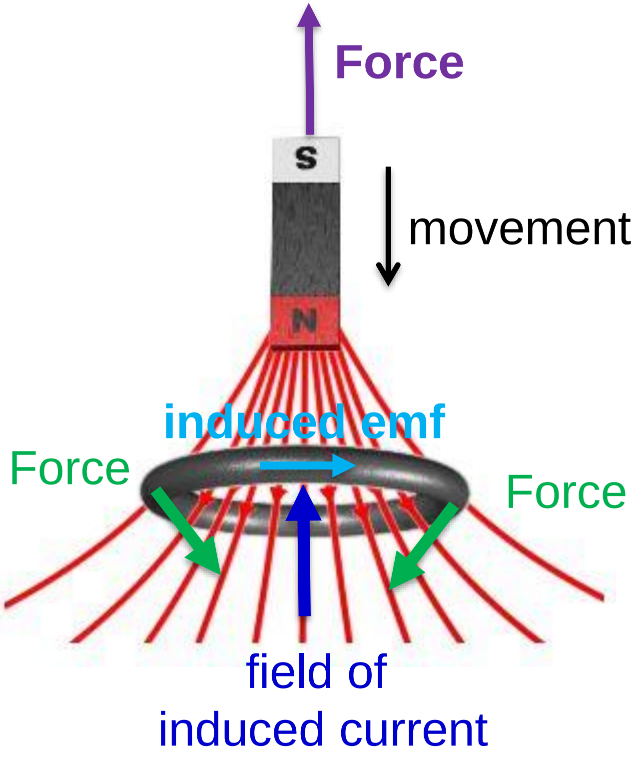
- **Lenz's Law:** If the magnetic field flux through a loop changes, <u>**induced *emf***</u> will be such *as if it was "trying" to* ***<u>counteract the change in the flux</u>***

  – **current due to induced *emf*** will have its own magnetic field in the direction compensating the change in the flux

  – **magnetic field forces on the induced current** will be "trying" to increase or decrease the loop area to compensate the change in the flux

  – **torque due to magnetic field forces on the current** will be "trying" to turn the loop in the direction compensating the change in the flux

Applying Lenz’s Law (example)



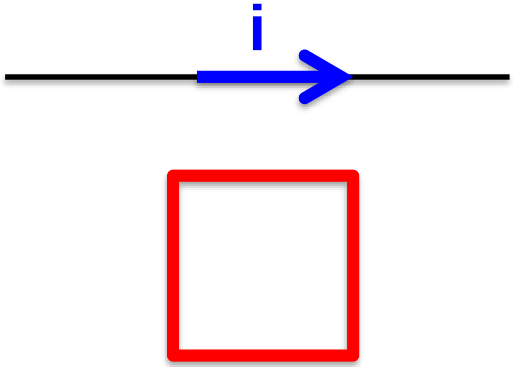
In the figure, **the flux of external field is increasing.**

The induced emf causes a current in the loop, with the following consequences “trying to prevent the change of the flux”:

- Flux of the induced current is in opposite direction
- Force on the loop: contraction and repulsion
- Force on the magnet: repulsion
- If loop was tilted, there would be torque

Sample problem

A square loop (side a = 2 cm) with the total resistance R is placed at a distance b = 1 cm from a wire carrying current as shown. The current in the wire is $i(t)=3t^2$.



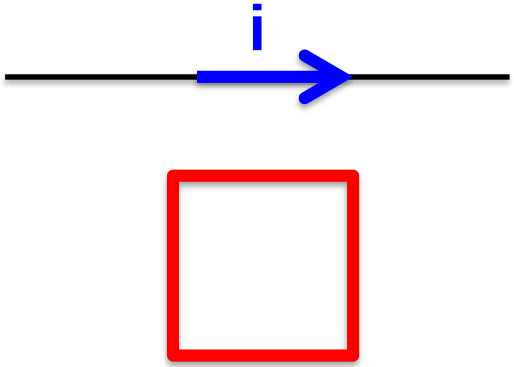
1) Find the induced current in the loop at t = 2s.
What is its direction?

2) Find the force on the loop at t = 2s.
What is its direction?

3) Find the torque on the loop at t = 2s.
What is its direction?

Sample problem

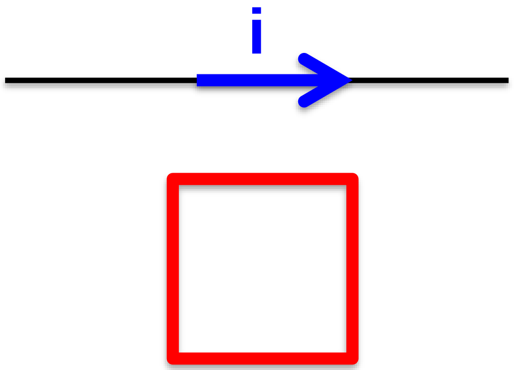
A square loop (side a = 2 cm) with the total resistance R is placed at a distance b = 1 cm from a wire carrying current as shown. The current in the wire is $i(t)=3t^2$.



1) Find the induced current in the loop at t = 2s.
What is its direction?

Sample problem

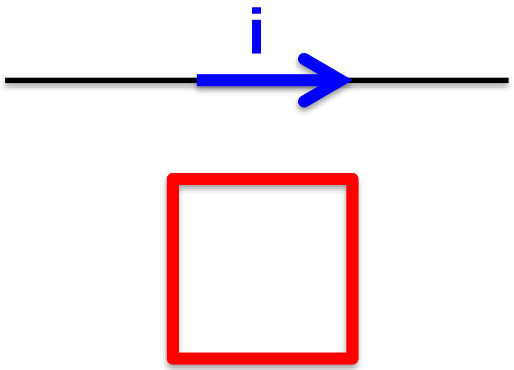
A square loop (side a = 2 cm) with the total resistance R is placed at a distance b = 1 cm from a wire carrying current as shown. The current in the wire is $i(t)=3t^2$.



2) Find the force on the loop at t = 2s.
What is its direction?

Sample problem

A square loop (side a = 2 cm) with the total resistance R is placed at a distance b = 1 cm from a wire carrying current as shown. The current in the wire is $i(t)=3t^2$.

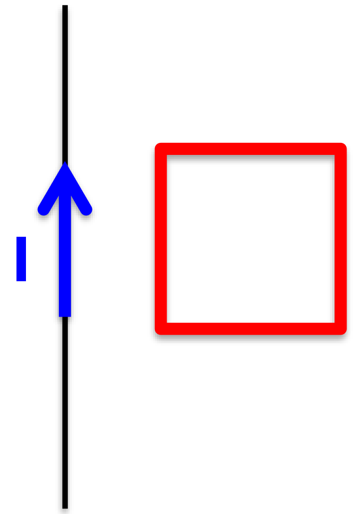


3) Find the torque on the loop at t = 2s.
What is its direction?

iClicker

A conductive loop is placed next to a wire carrying current as shown. If the current is increasing, what is the direction of the force on the loop?

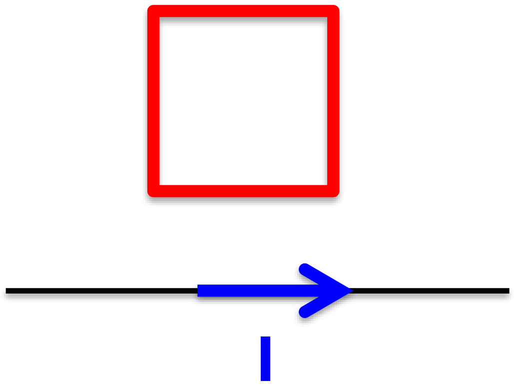
(a) up
(b) down
(c) left
(d) right
(e) net force is zero



iClicker

A conductive loop is placed next to a wire carrying current as shown. If the current is <u>decreasing</u>, what is the direction of the force on the loop?

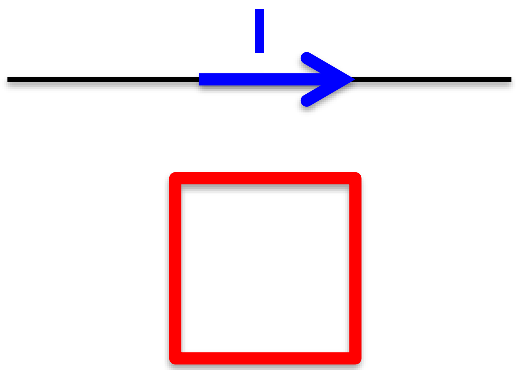
(a) up
(b) down
(c) left
(d) right
(e) net force is zero



iClicker

A conductive loop is placed next to a wire carrying current as shown. If the current is increasing, what is the direction of the force on the loop?

(a) up
(b) down
(c) left
(d) right
(e) net force is zero

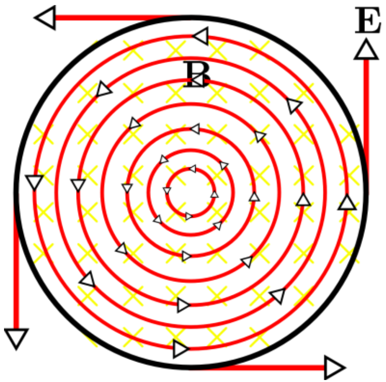


Sample problem

A solenoid of radius R and length l has N turns of wire. If the current through solenoid changes as $i(t)=i_0 sin(\omega t)$, find the strength of electric field $E(r,t)$ as a function of distance from the axis of the solenoid, r, and time t.

Sample problem

A solenoid of radius R and length l has N turns of wire. If the current through solenoid changes as $i(t)=i_0 sin(\omega t)$, find the strength of electric field $E(r,t)$ as a function of distance from the axis of the solenoid, r, and time t.



Generalizing: changing Φ_B induces *emf*

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

Flux of magnetic field through a loop of area A:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int B \cdot dA \cdot \cos\theta$$

The flux can change due to:

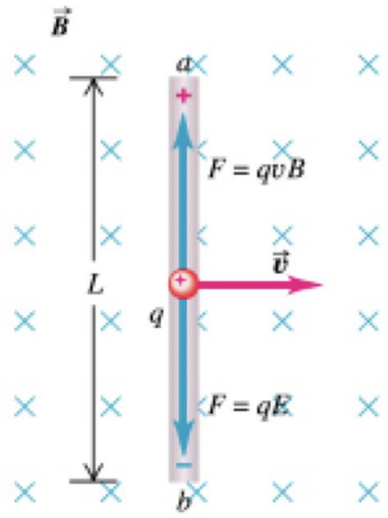
1) change of B

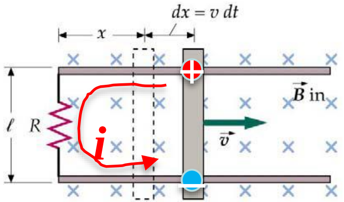
3) change of orientation (e.g, turning loop)

2) change of area (e.g., moving rod, expanding loop, etc.)

one size fits all, i.e. *one equation works in all situations*!

emf induced in a rod moving across B





$$\vec{F}_B = q\vec{v} \times \vec{B}$$

$$F_B = qvB\sin\theta = qvB_\perp$$

$$F_E = qE$$

$$F_B = F_E \quad \Rightarrow \quad E = qB_\perp v$$

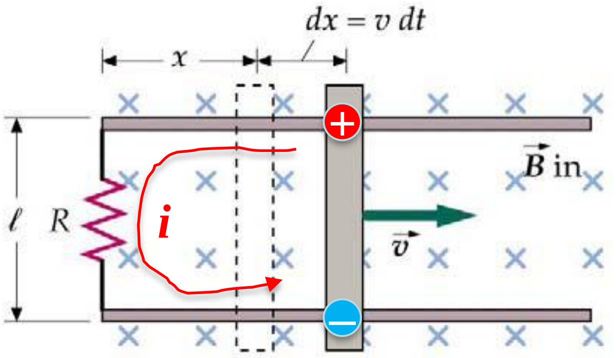
$$\Delta V(emf) = EL = vB_\perp L$$

$$\boxed{emf = vB_\perp L}$$

Charges in the wire move across magnetic field -> experience force -> pulled sideways -> emf

Crosscheck: change of area

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$



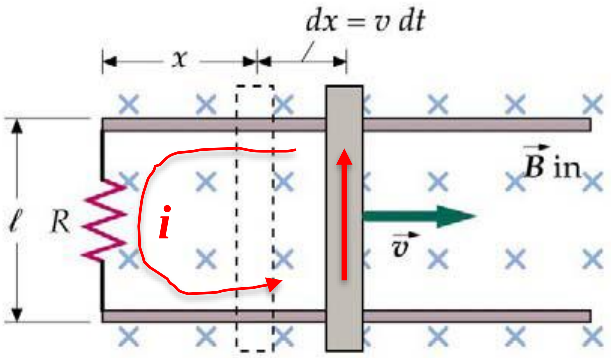
$$\Phi = Bx\ell$$

$$\frac{d\Phi}{dt} = \frac{d(Bx\ell)}{dt} = B\ell\frac{dx}{dt} = B\ell v$$

$$\mathcal{E} = vB\ell$$ **(absolute value)**

Direction: Align your thumb with the change of the flux (into the page).
Fingers – flip the thumb
Curl your fingers – this is the direction of EMF

Where does the energy come from?



Current through R: $i = \dfrac{V}{R} = \dfrac{vB\ell}{R}$

Power dissipated in R: $P = i^2 R = \dfrac{v^2 B^2 \ell^2}{R}$

Where does all this energy come from? From you!

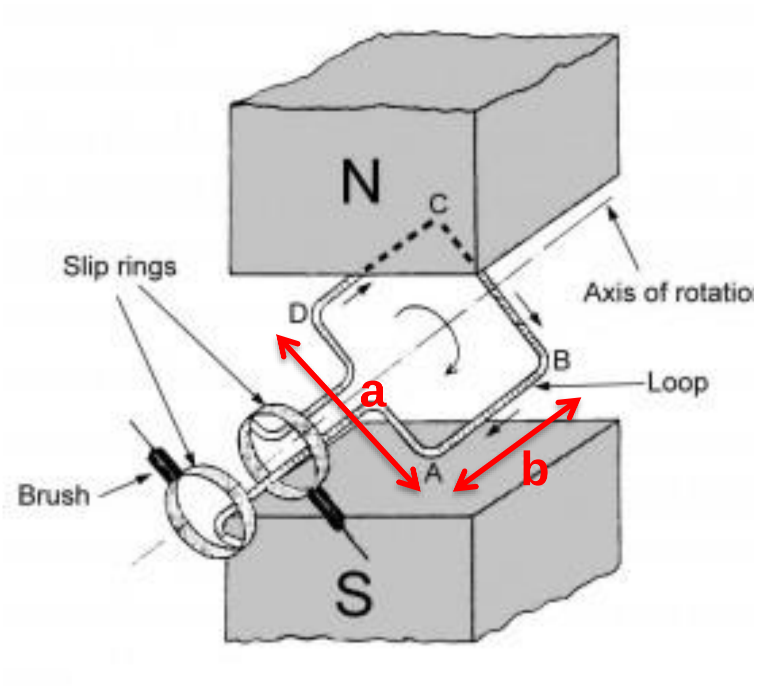
Force on the bar due to current in B: $\vec{F} = i\vec{\ell} \times \vec{B}$

The force point left: you have to pull the bar with the force at least as larger!

Power required from you: $P_{you} = Fv = (i\ell B)v = \left(\dfrac{vB\ell}{R}\ell B\right)v = \dfrac{v^2 B^2 \ell^2}{R}$

Power spent by you = power dissipated in R

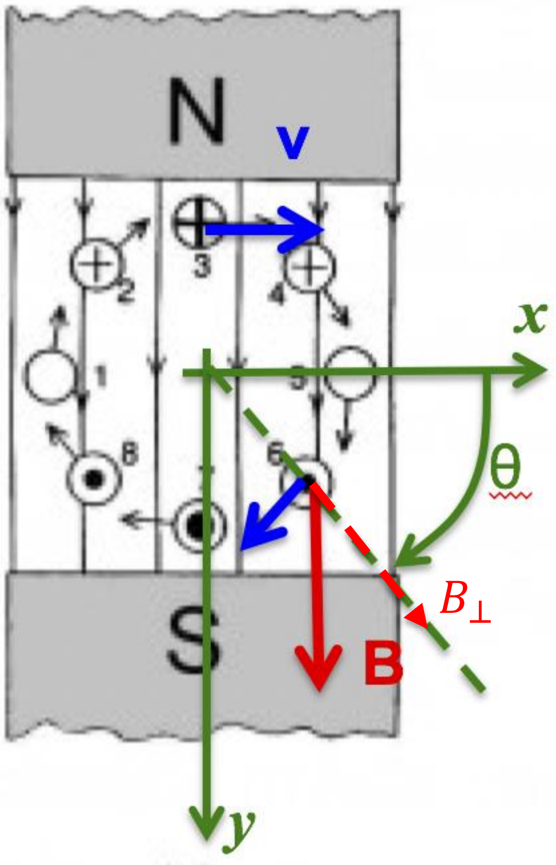
emf induced in a loop turning in B (electric generator)



$$q = wt$$

$$v = wr = w\frac{a}{2}$$

emf induced in a loop turning in B (electric generator)



$$\theta = \omega t$$

$$v = \omega r = \omega \frac{a}{2}$$

Only sides AB and CD generate *emf*:

- Position 1-5: $emf = vB_\perp b + vB_\perp b = 0 + 0 = 0$
- Position 3-7: $emf = vB_\perp b + vB_\perp b = 2vB_\perp b = 2\left(\omega \frac{a}{2}\right) \cdot Bb = \omega Bab = \omega BA$
- Position 2-6 (and in general): $emf = vB_\perp b + vB_\perp b = 2\left(\omega \frac{a}{2} \cdot B\sin\theta \cdot b\right) = \omega BA \cdot \sin\omega t$
- A coil of N loops: $\boxed{emf = \omega NBA \cdot \sin\omega t}$

Crosscheck: change of orientation

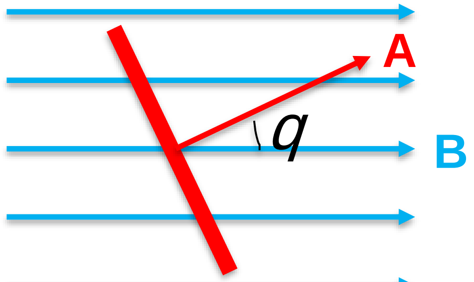
$$e = -\frac{dF_B}{dt}$$

$$q = wt$$

$$F = BA\cos q = BA\cos wt$$

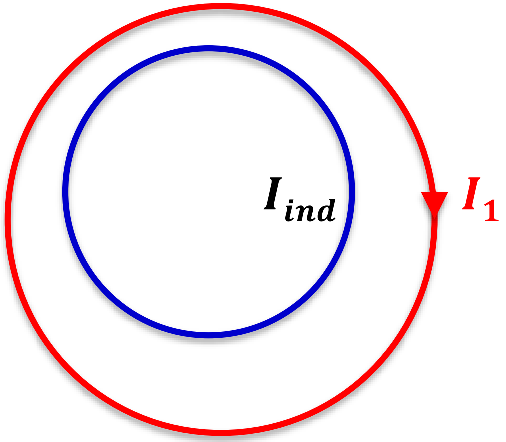
$$e = -\frac{dF_B}{dt} = wBA\sin wt$$

For N turns in a coil: $e = wBNA\sin wt$



iClicker

There are two loops placed on the plane as shown. What is the direction of I_{ind} when clockwise current I_1 increases



A: zero

B: clockwise

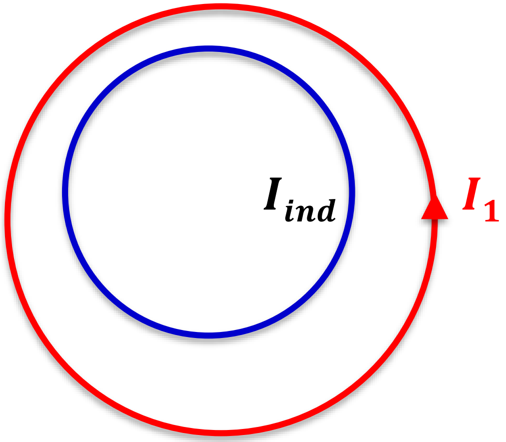
C: counterclockwise

D: oscillating

E: to answer,
one needs to know the sign of dI_1/dt

iClicker

There are two loops placed on the plane as shown. What is the direction of I_{ind} when counter-clockwise current I_1 increases



A: zero

B: clockwise

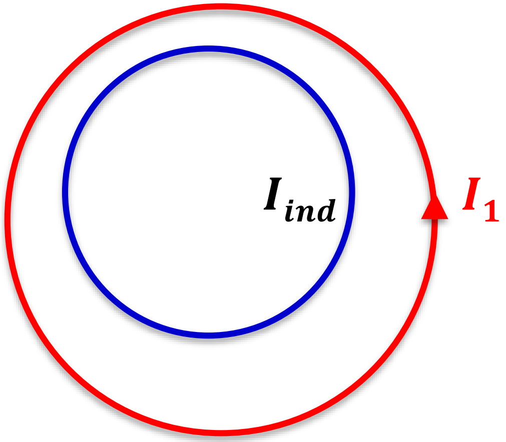
C: counterclockwise

D: oscillating

E: to answer,
one needs to know the sign of $\mathrm{d}I_1/\mathrm{dt}$

iClicker

There are two loops placed on the plane as shown. What is the direction of I_{ind} when counter-clockwise current I_1 is constant



A: zero

B: clockwise

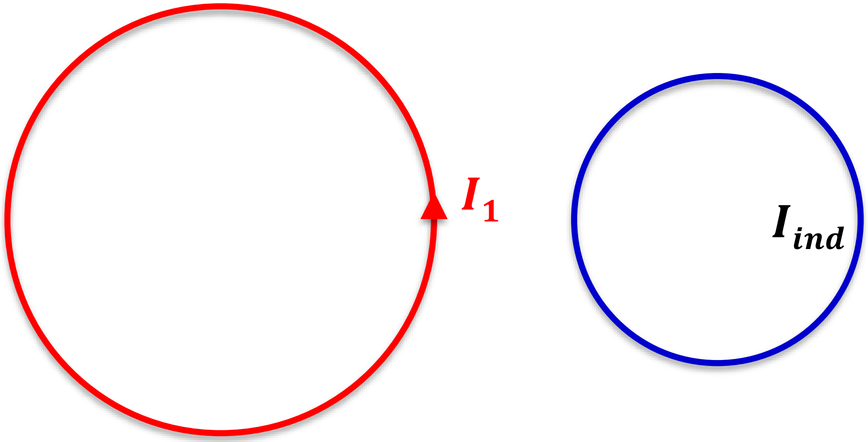
C: counterclockwise

D: oscillating

E: none of the above

iClicker

There are two loops placed on the plane as shown. What is the direction of I_{ind} when counter-clockwise current I_1 increases



A: zero

B: clockwise

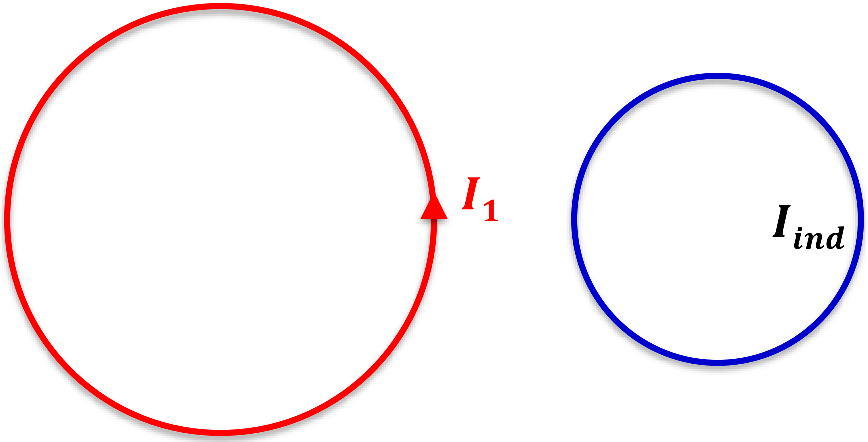
C: counterclockwise

D: oscillating

E: none of the above

iClicker

There are two loops placed on the plane as shown. What is the direction of I_{ind} when counter-clockwise current I_1 is constant



A: zero

B: clockwise

C: counterclockwise

D: oscillating

E: I do not really care

Mutual induction

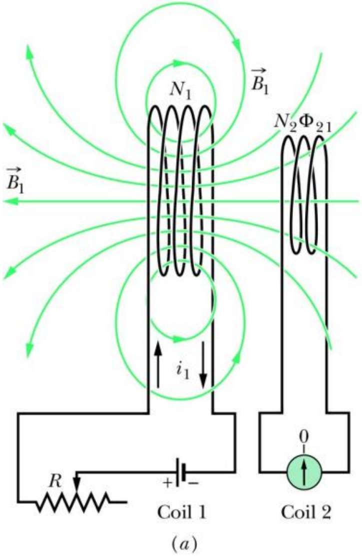
i_1 in the first coil $\rightarrow$
B_1 in place of second coil

$$B_1 \sim i_1$$

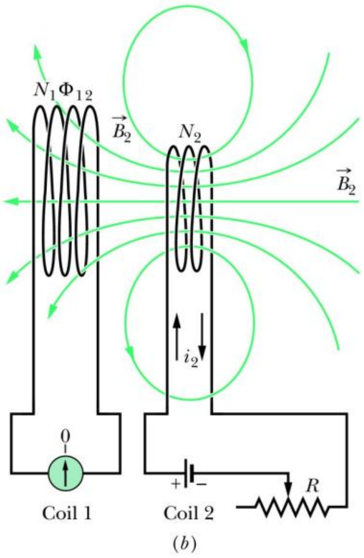
$$\Phi_2 \sim B_1 \sim i_1$$

$$\frac{d\Phi_2}{dt} \sim \frac{di_1}{dt}$$

$$\mathcal{E}_2 = -M_{21}\frac{di_1}{dt}$$



Mutual induction



i_2 in the second coil → B_2 in place of first coil

$$B_2 \sim i_2$$

$$F_1 \sim B_2 \sim i_2$$

$$\frac{dF_1}{dt} \sim \frac{di_2}{dt}$$

$$e_1 = -M_{12}\frac{di_2}{dt}$$

Mutual induction

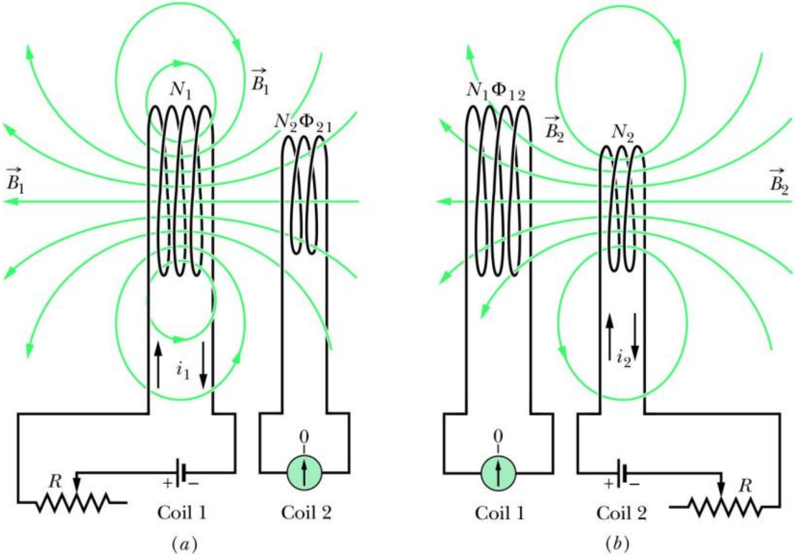
i_1 in the first coil →
B_1 in place of second coil

$$B_1 \sim i_1$$

$$F_2 \sim B_1 \sim i_1$$

$$\frac{dF_2}{dt} \sim \frac{di_1}{dt}$$

$$e_2 = -M_{21}\frac{di_1}{dt}$$



i_2 in the second coil →
B_2 in place of first coil

$$B_2 \sim i_2$$

$$F_1 \sim B_2 \sim i_2$$

$$\frac{dF_1}{dt} \sim \frac{di_2}{dt}$$

$$e_1 = -M_{12}\frac{di_2}{dt}$$

Without proof: $M_{21} = M_{12} = M$

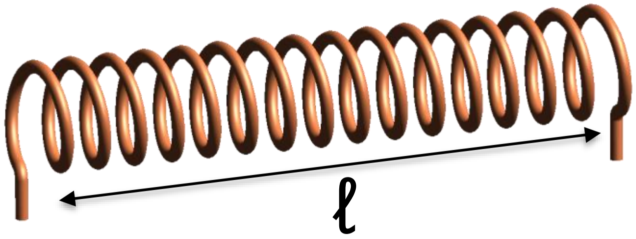
M – mutual inductance (units: henry)

Inductance

- **Inductor:** any conductor, e.g. a wire
- **Self-induction:** A **changing current** through a conductor will result in a **changing magnetic field** around it, which in its turn will give rise to an **induced an electric field**, which will result in an **induced emf** in the conductor

$$\boxed{\varepsilon = -L\frac{di}{dt}}$$

- **Inductance = coefficient L**
  - property of the conductor, the value depends on its physical arrangement (e.g., is a wire straight or coiled? how tightly coiled? etc.)
  - units: **henry (H)**; 1 H = 1 Vs/A = 1 Tm^{2}A
  - negative sign in the equation is as expected (Lenz's law)
- **Example of inductor: Solenoid (** ***A*** **– cross section,** ***l*** **– length,** ***N*** **turns,** ***n=N/l*** **)**



$$B = \mu_0 i n$$

$$\Phi_B = BA = \mu_0 i n A$$

$$\varepsilon = -\frac{d\Phi_B}{dt}N = -\mu_0 n A N\frac{di}{dt} = -\mu_0 n^2 A\ell\frac{di}{dt}$$

$$\boxed{L = \mu_0 n^2 A\ell}$$

Sample problem

You have a wire of length H and diameter d, and a long cylinder of diameter D. If you wind the wire neatly on the cylinder, all wire turns being neatly pressed next to each other and having no overlaps, what is the inductance of this-way made solenoid?

Sample problem

You have a wire of length H and diameter d, and a long cylinder of diameter D. If you wind the wire neatly on the cylinder, all wire turns being neatly pressed next to each other and having no overlaps, what is the inductance of this-way made solenoid?

$$L = \mu_0 n^2 A \ell$$

Where

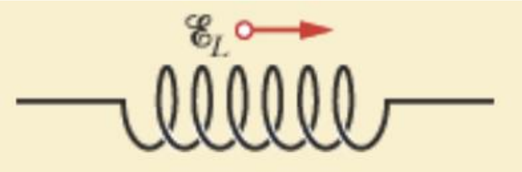
$$n = \frac{1}{d}$$

$$\ell = \left(\frac{H}{\pi D}\right)/n$$

HITT

The figure shows an emf induced in a coil.

Which of the following can describe the current through the coil:



(a) constant and rightward,

(b) constant and leftward,

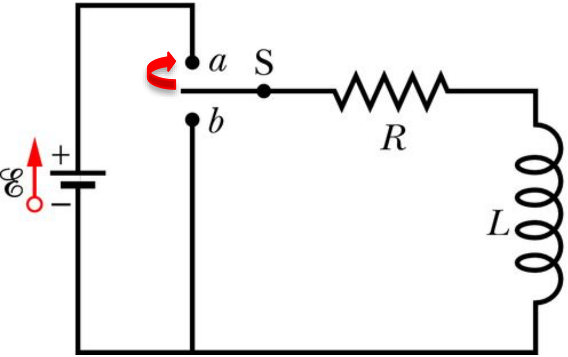
(c) increasing and rightward,

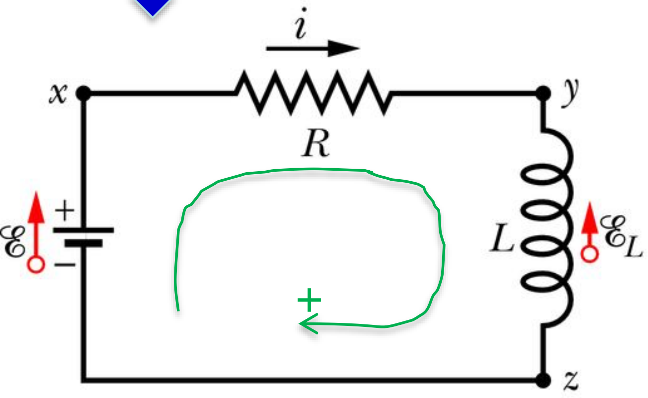
(d) decreasing and rightward,

(e) increasing and leftward,

(f) decreasing and leftward?

RL circuits (1)





The switch is flipped to position (a)

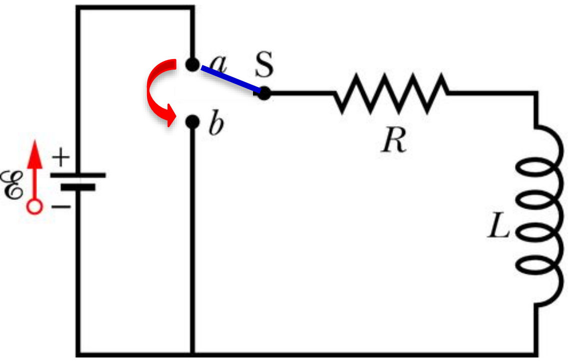
- **At t=0+, the current cannot onset instantly**, as this would create an infinite induced emf in the inductor opposing such a change in the current; hence, $i_{initial} = 0$
- The current will start increasing at not too high pace $i(t)$
- **After a long time, the current has settled at** some final value and its time derivative must be zero

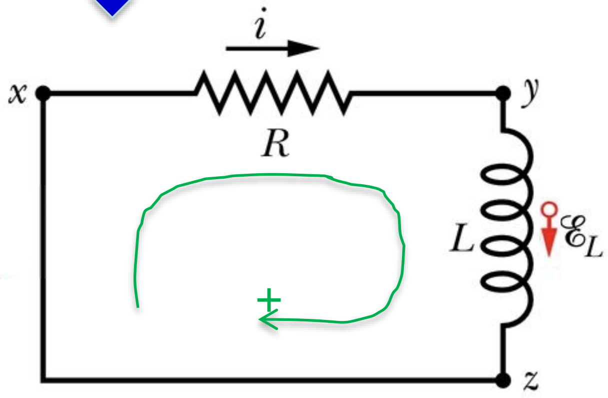
$$\mathcal{E} - iR = 0 \quad \Rightarrow \quad i_{final} = \frac{\mathcal{E}}{R}$$

- **At any time**, the differences of potentials across the three circuit elements should add up to zero

$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$

RL circuits (2)





After being in position (a) for a long time, the switch is flipped to position (b)

- **At t=0+, the current cannot change instantly**, as this would create an infinite induced emf in the inductor opposing such a change in the current; hence,

$$i_{initial} = \frac{e}{R}$$

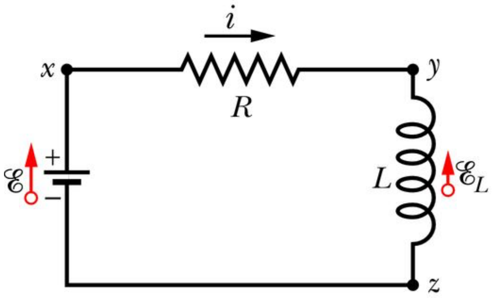
- The current will start decreasing at not too high pace $i(t)$
- **After a long time, the current has settled** and it must be 0

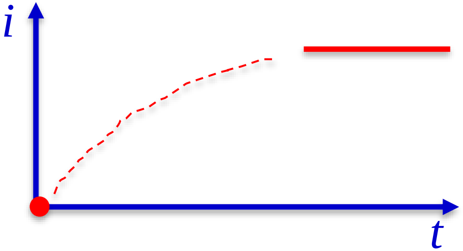
$$i_{final} = 0$$

- At any time, the differences of potentials across the three circuit elements should add up to zero

$$-iR - L\frac{di}{dt} = 0$$

RL circuits (3)



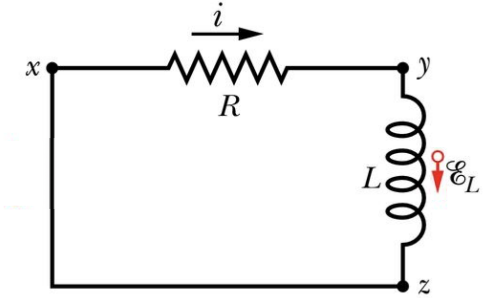


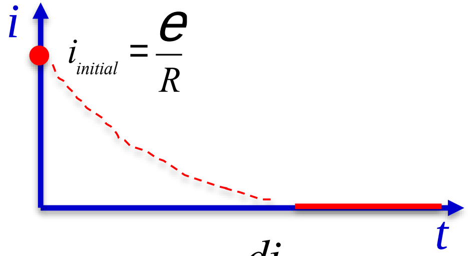
$$i_{final} = \frac{\varepsilon}{R}$$

$$\varepsilon - iR - L\frac{di}{dt} = 0$$

Solution: $$i(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$$

$$\tau = \frac{L}{R}$$





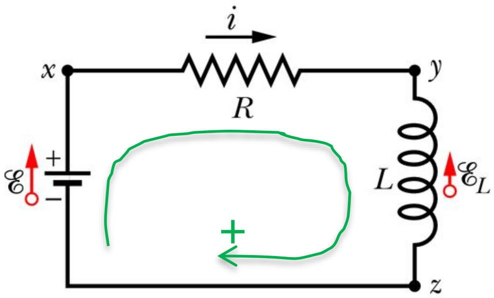
$$i_{initial} = \frac{\varepsilon}{R}$$

$$-iR - L\frac{di}{dt} = 0$$

Solution: $$i(t) = \frac{\varepsilon}{R}e^{-t/\tau}$$

Similar to equations describing charging/discharging a capacitor (where time constant was $\tau = RC$)

Energy stored in inductor



$$U = \frac{1}{2} L i^2$$

Note the analogy with a capacitor: $U = \frac{1}{2}\frac{Q^2}{C}$

$$\varepsilon - iR - L\frac{di}{dt} = 0$$

$$\varepsilon = iR + L\frac{di}{dt}$$

$$\varepsilon i = i^2 R + L\frac{di}{dt} i$$

power provided by battery

power dissipated by resistor

Where else does the battery's energy go?
It goes into the inductor…

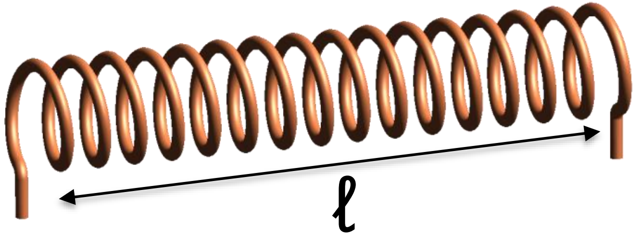
$$\frac{dU}{dt} = Li\frac{di}{dt}$$

$$dU = Li \cdot di$$

$$U = \int dU = \int Li \cdot dt = \frac{1}{2} L i^2$$

Energy is stored in magnetic field

Consider solenoid:



$$L = \mu_0 n^2 A\ell$$

$$B = \mu_0 n i$$

$$U = \frac{1}{2} L i^2 = \frac{1}{2}\left[\mu_0 n^2 (A\ell)\right]\left[\frac{B}{\mu_0 n}\right]^2 = \frac{B^2}{2\mu_0}(A\ell)$$

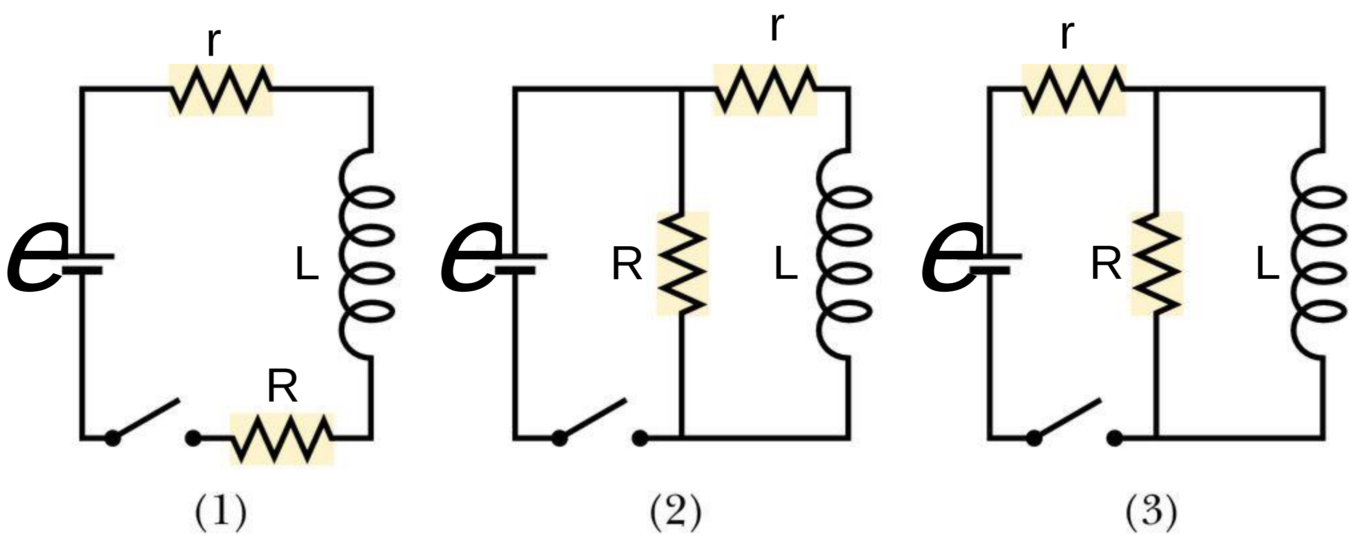
volume

energy density (J/m³)

$$\boxed{u_B = \frac{B^2}{2\mu_0}}$$

Compare to energy stored in electric field $u_B = \frac{1}{2}\epsilon_0 E^2$

Three sample problems

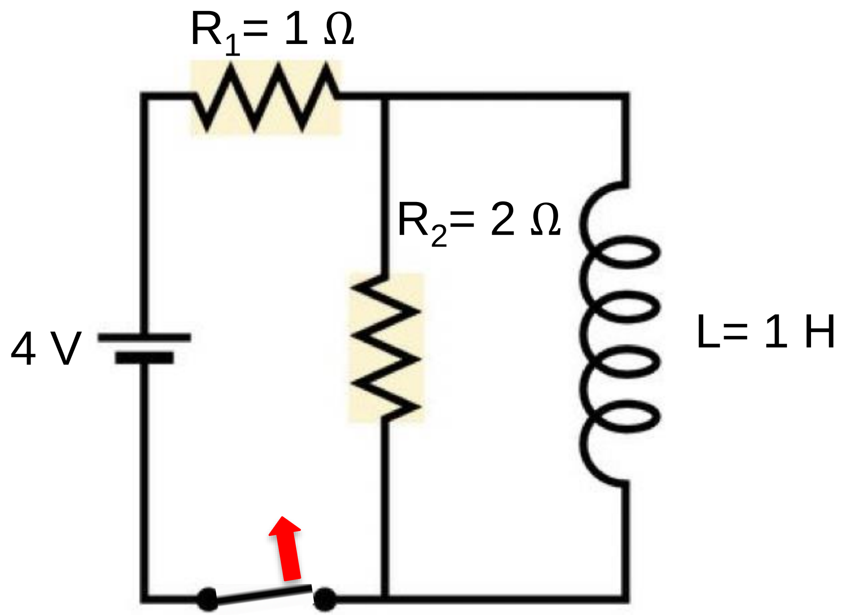


First, in all three circuits the switches are closed for long time.

What are the currents through resistor r and R

- while the switches remain closed?
- right after the switches are opened?
- right after switches are closed again, after having been open for a long time?

HITT



The switch in the shown circuit has been closed for a long time.

What is the current in resistor R_2 at the first instance after someone opens the switch?

(A) 0 A **(B) 1 A** **(C) 2 A** **(D) 3 A** **(E) 4 A**