Project 1 – Modeling of Pandemic and Oscillator

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Section 1: Population Model

Q.1-1 Show viable solution

$$
\begin{aligned}
y' &= Ay - \mathrm{B}y^2 \\
\rightarrow y' - Ay &= -By^2
\end{aligned}
$$

Let $z = y^{1-2} = y^{-1} \Leftrightarrow y = z^{-1}$

$$
\begin{aligned}
&\rightarrow -z^{-2}\frac{dz}{dt} - Az^{-1} = -Bz^{-2} \\
&\rightarrow \frac{dz}{dt} + Az = B \\
&\rightarrow \text{integrating factor} = e^{\int Adt} = e^{At} \\
&\rightarrow z = \frac{1}{e^{At}}\int Be^{At}dt = \frac{B}{A} + ce^{-At} \\
&\rightarrow y(t) = z^{-1} = \frac{1}{ce^{-At} + \frac{B}{A}} \cdots (1)
\end{aligned}
$$

Q.1-2 Malthusian growth model

$$
\begin{aligned}
&\rightarrow y' = Ay \\
&\rightarrow \frac{dy}{dt} = Ay \\
&\rightarrow \int \frac{1}{y}dy = \int Adt \\
&\rightarrow ln|y| = At + c_0 \\
&\rightarrow y' = ce^{At}, c = e^{c_1} \cdots (2)
\end{aligned}
$$

Q.1-3 Plug in B = 0 into (1)

Plug in B = 0 into (1)

$$\rightarrow y' = \frac{1}{ce^{-At}} = ce^{At} = (2)$$

Two answers are the same.

Q.1-4 Braking term

As we observed two solutions regarding B = 0 or B > 0:

1. B = 0, we will get solutions with free exponential growth, which is Malthus model.
2. B > 0, unlike the solutions to the Malthus model, solutions to the logistic equation are bounded.
   The difference between two cases is the existence of the braking term, it effects the solutions to be bounded while growing, so that's why it might be called "braking."

Q.1-5 Initial value problem

plug in $t = 0$ into (1)

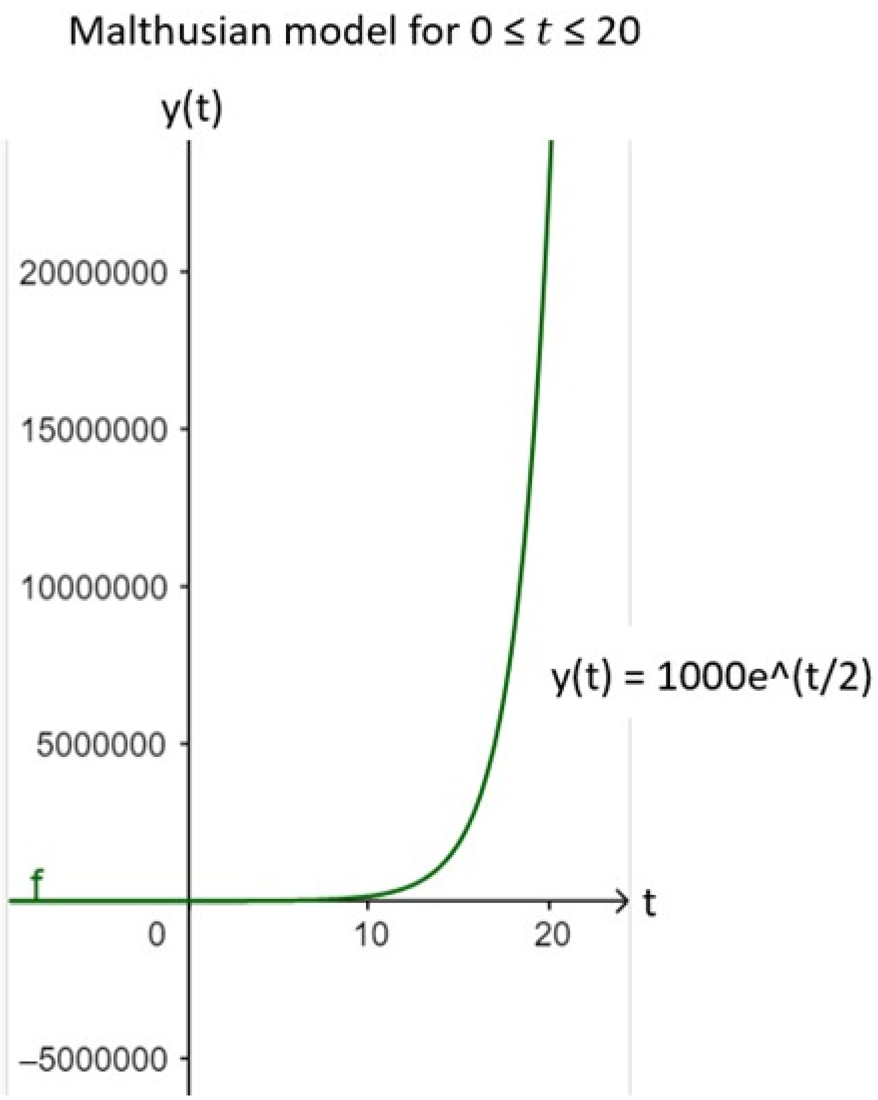
$$\rightarrow \frac{1}{ce^0} + \frac{B}{A} = \frac{1}{c + \frac{B}{A}} = y(0)$$
$$\rightarrow c = \frac{1}{y(0)} - \frac{B}{A}$$

Q.1-6 Ploting the Malthusian model

By (2)

$$\rightarrow y(0) = ce^{\frac{0}{2}} = 1000$$
$$\rightarrow c = 1000$$
$$\rightarrow y = 1000e^{\frac{t}{2}}$$

As $0 \leq t \leq 20 : 1000 \leq y(t) \leq 2.2E + 0.7$



Q.1-7

On the same plot, for the same A and $y(0)$, plot the Verhulst model for B = 0.0001.Comment on the effect of B. You might need to use log scale for y-axis.
The assumption that the rate of growth of population decreases the limited population on the environment will be supported by approached logistic model. The Verhulst population model (logistics) can be expressed by

Let K represent the carrying capacity for a particular organism in a given environment. r be a real number that represents the growth rate.

$$\frac{1}{N}\frac{\mathrm{dN}}{\mathrm{dt}} = r(1 - \frac{N}{K})$$

$$\frac{\mathrm{dN}}{\mathrm{dt}} = rN\frac{K-N}{K}$$

$$\frac{K}{N(K-N)})dN = rdt$$

$$\int \frac{\mathrm{dN}}{N} + \int \frac{\mathrm{dN}}{K-N} = \int \mathrm{r\ dt}$$

$$\ln(N) - \ln(K-N) = rt + C$$

$$-\ln\left(\frac{K-N}{N}\right) = -rt - C$$

$$\frac{K-N}{N} = e^{-rt-c} = Ae^{-rt}$$

$$\frac{K}{N} = 1 + Ae^{-rt}$$

$$N = \frac{K}{1+Ae^{-rt}}$$

Write as the form

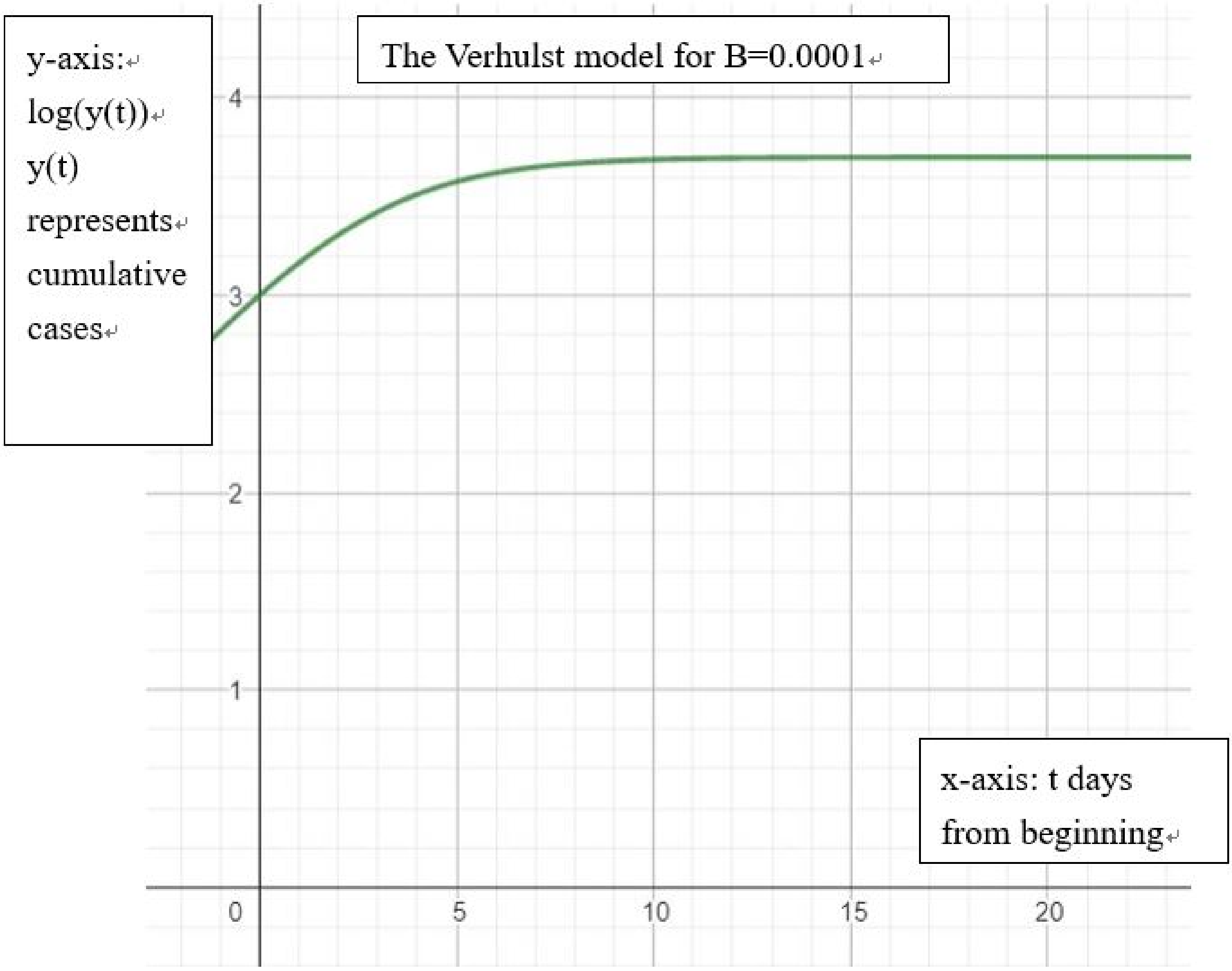
$$y(t) = \frac{1}{\frac{B}{A} + ce^{-At}}$$

$$y(0) = 1000, A = 0.5, B = 0.0001$$

$$1000 = \frac{1}{\frac{0.0001}{0.5} + c}$$

$$c = \frac{1}{1250}$$

On the same plot,B can control y(t) to approach a maximum number(such as carrying capacity of the nature environment)

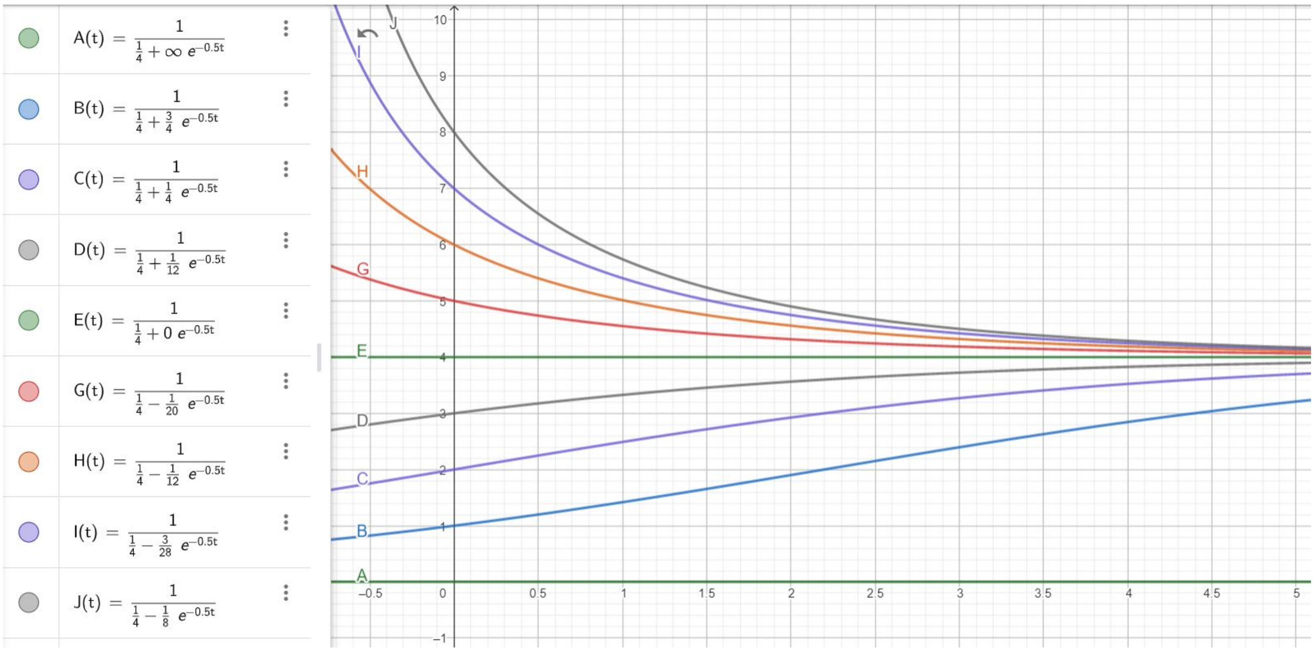


Q.1-8

On a new plot, for A/B = 4, show $y(t)$ for 0 ≤ t ≤ 5 for y(0) = 0, 1, 2, 3, 4, 5, 6, 7, 8. Comment on why all the curves seem to converge on one value given enough time.

$$\frac{A}{B} = 4\ , y(0) = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

Corresponding $c = \infty, \frac{3}{4}, \frac{1}{4}, \frac{1}{12}, 0, -\frac{1}{20}, -\frac{1}{12}, -\frac{3}{28}, -\frac{1}{8}$



When $y(0) = 4, c = 0, for\ any\ A > 0$

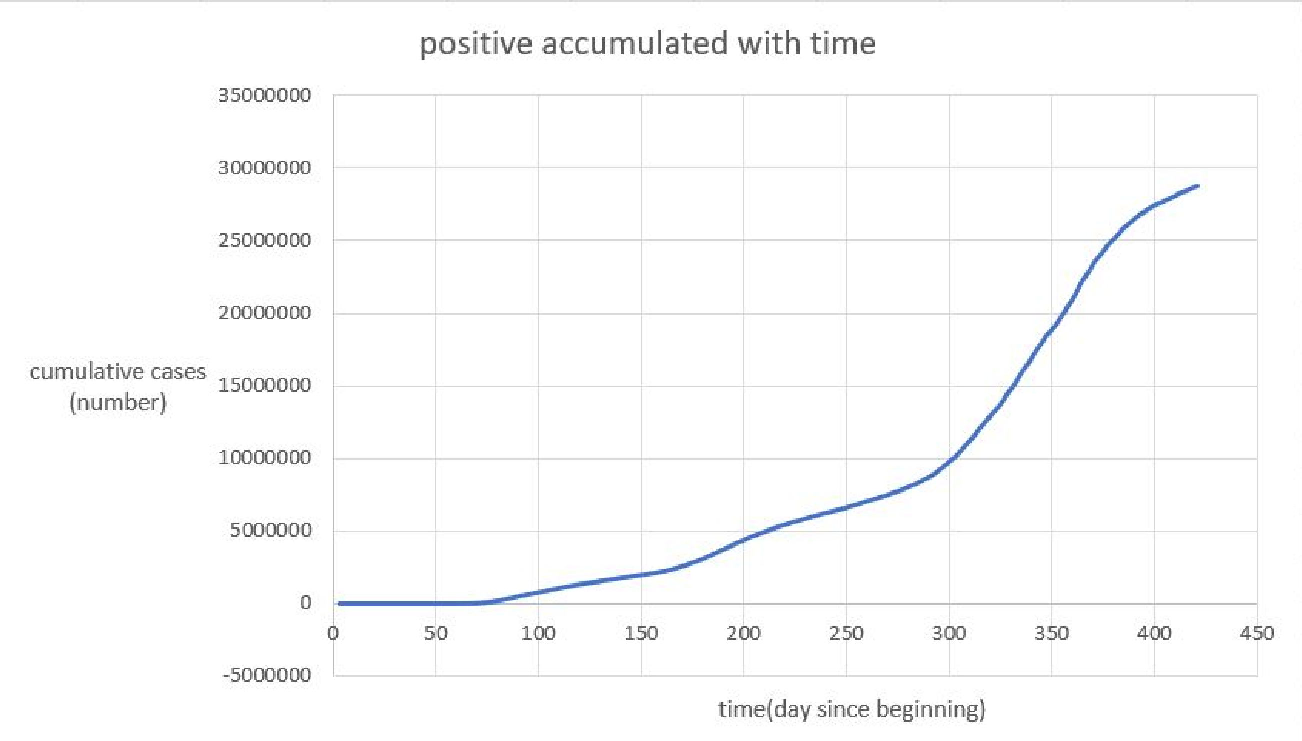
$$y(\infty) -> 4 = A/B$$

When t becomes larger, y(t) will approach 4 ,y=4 becomes one of asymptotes of y(t). Because Verhulst model(logistic equation) is autonomous ODE. If
y(t)>A/B, y' will >0,and vice versa. y=4 is a stable solution of this equation.

Q.1-9 Application

iii

Plot the data for cumulative cases over time (cases vs. day since beginning).



iv

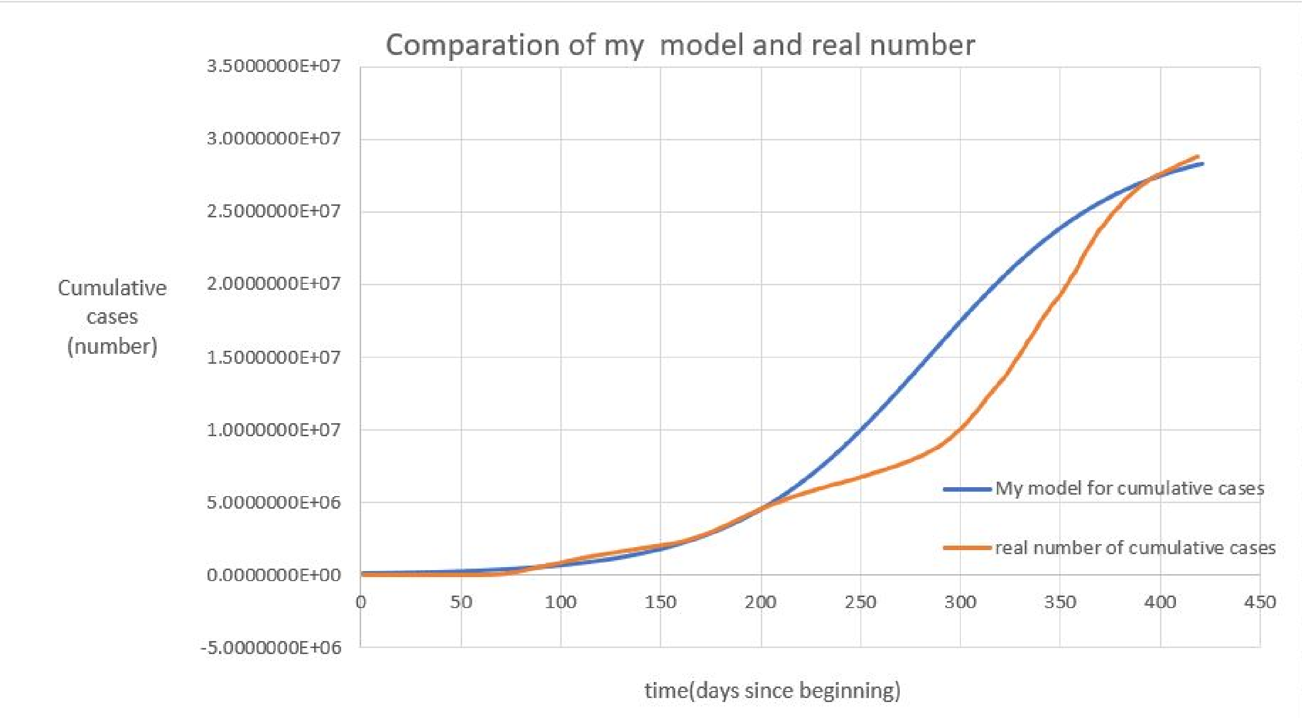
Suggest in your report: whether the Malthusian or Verhulst model is more appropriate for this data. Why?
Verhulst model.
Because when cumulative cases approach to 30000000,the increase rate becomes smaller. The solution of this question also like the form of the Logistic equation solution. I guess its A/B is 30000000. B is not zero, so it is Verhulst model.

v

Try to find A and B constants which best fit this data. On the same graph, plot result from your model on top of the real data.
Comment: whether the curves appear reasonably similar?
When t becomes larger, real data approaches to 30000000.(A/B)
Use the point t=200, y(t) is approximately 4500000.
Use another point t=400, y(t) is approximately 27500000.
Plug in $y(t) = \frac{1}{\frac{1}{3*10^7}+ce^{-At}}$, Solve c and A.

$$A = \frac{1}{200}\ln\left(\frac{\frac{1}{4500000}-\frac{1}{3*10^7}}{\frac{1}{27500000}-\frac{1}{3*10^7}}\right) = 0.02066248164$$

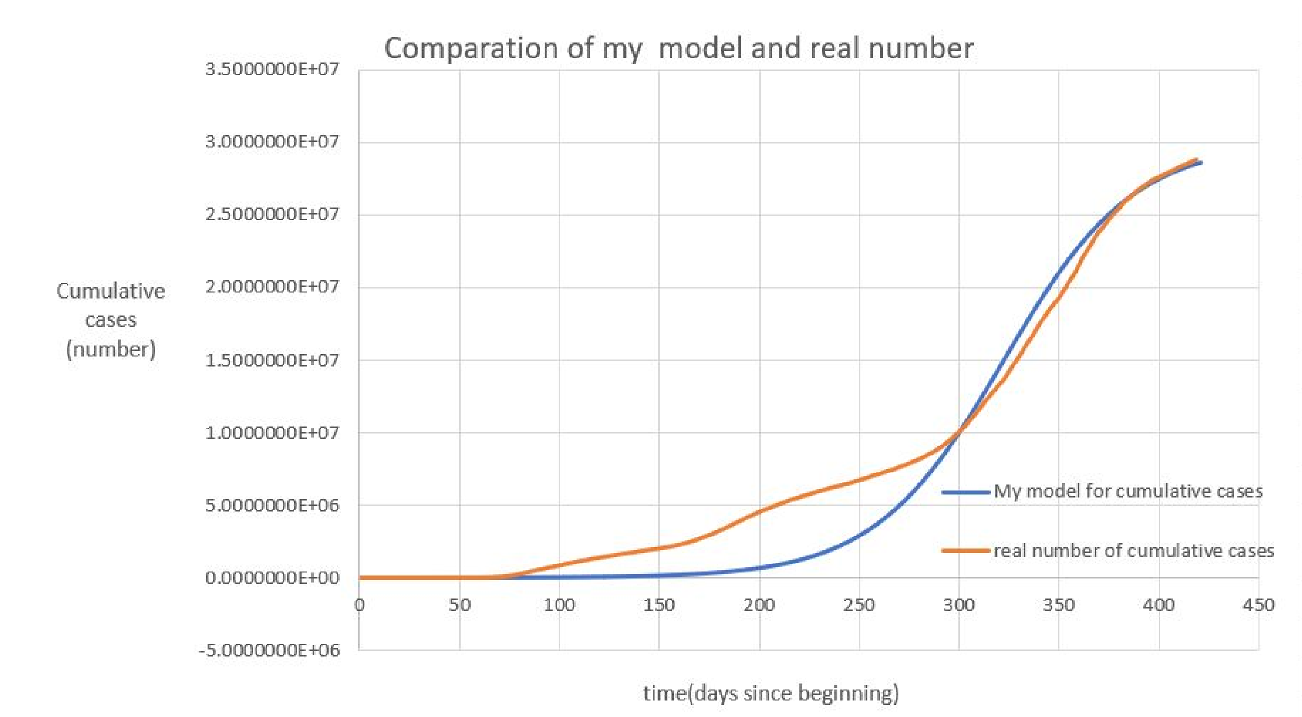
$$c = e^{200A}\left(\frac{1}{4500000}-\frac{1}{3*10^7}\right) = 1.17740741*10^{-5}$$



Use the point t=300, y(t) is approximately 10000000.
Use another point t=400, y(t) is approximately 27500000.
Solve c and A, get the graph.

$$A = \frac{1}{100}\ln\left(\frac{\frac{2}{3*10^7}}{\frac{1}{27500000}-\frac{1}{3*10^7}}\right) = 0.03091042453$$

$$c = e^{300A}\left(\frac{2}{3*10^7}\right) = 7.09866666*10^{-4}$$

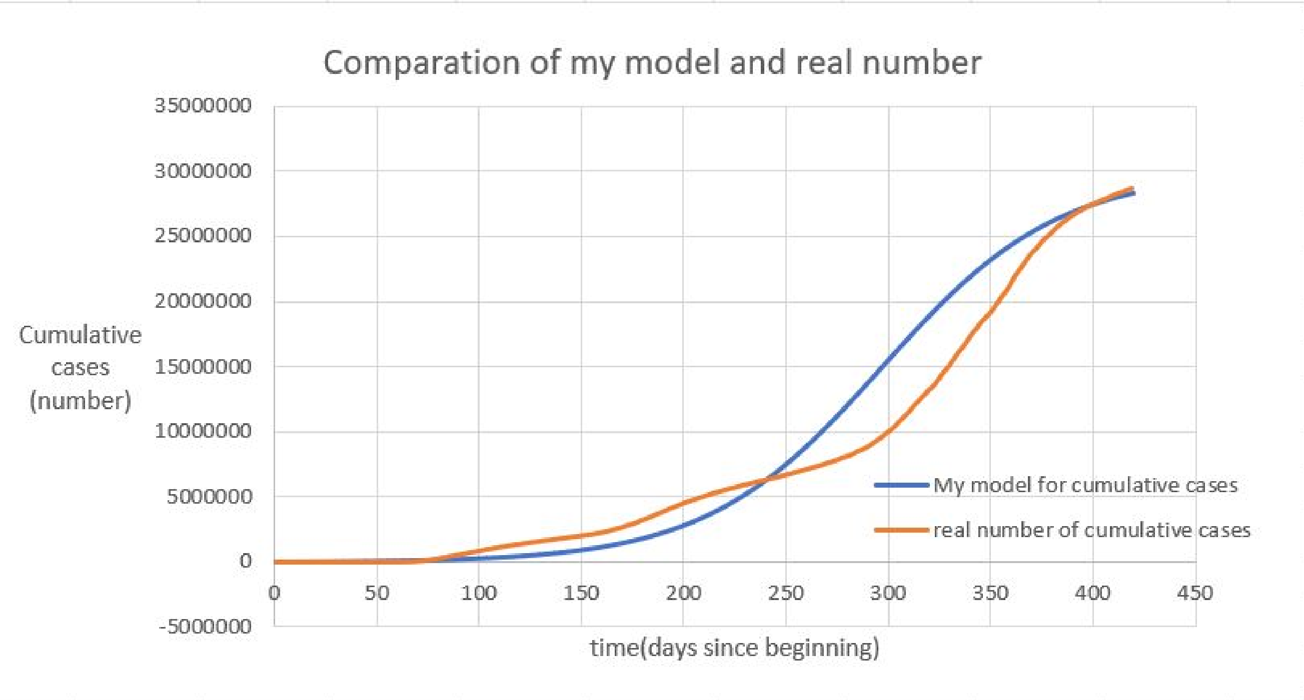


Use the point t=250, y(t) is approximately 6800000.

Use the point t=400, y(t) is approximately 27500000.

$$A = \frac{1}{150}\ln\left(\frac{\frac{1}{6800000}-\frac{1}{3*10^7}}{\frac{1}{27500000}-\frac{1}{3*10^7}}\right) = 0.0241674996$$

$$c = e^{250A}\left(\frac{1}{6800000}-\frac{1}{3*10^7}\right) = 4.78421564*10^{-5}$$



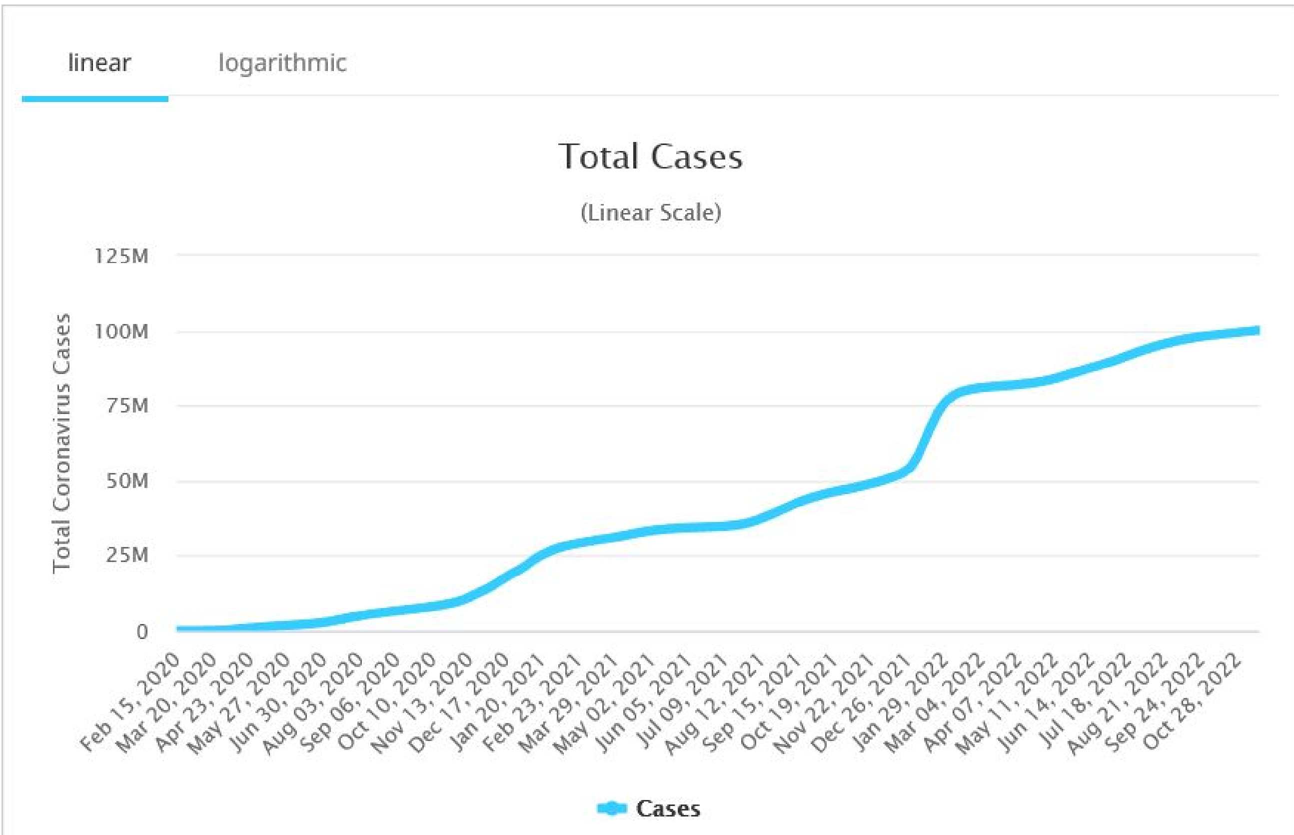
This case (case 3) fit the data best. A/B=30000000.

$$B = 8.0558332*10^{-10}$$

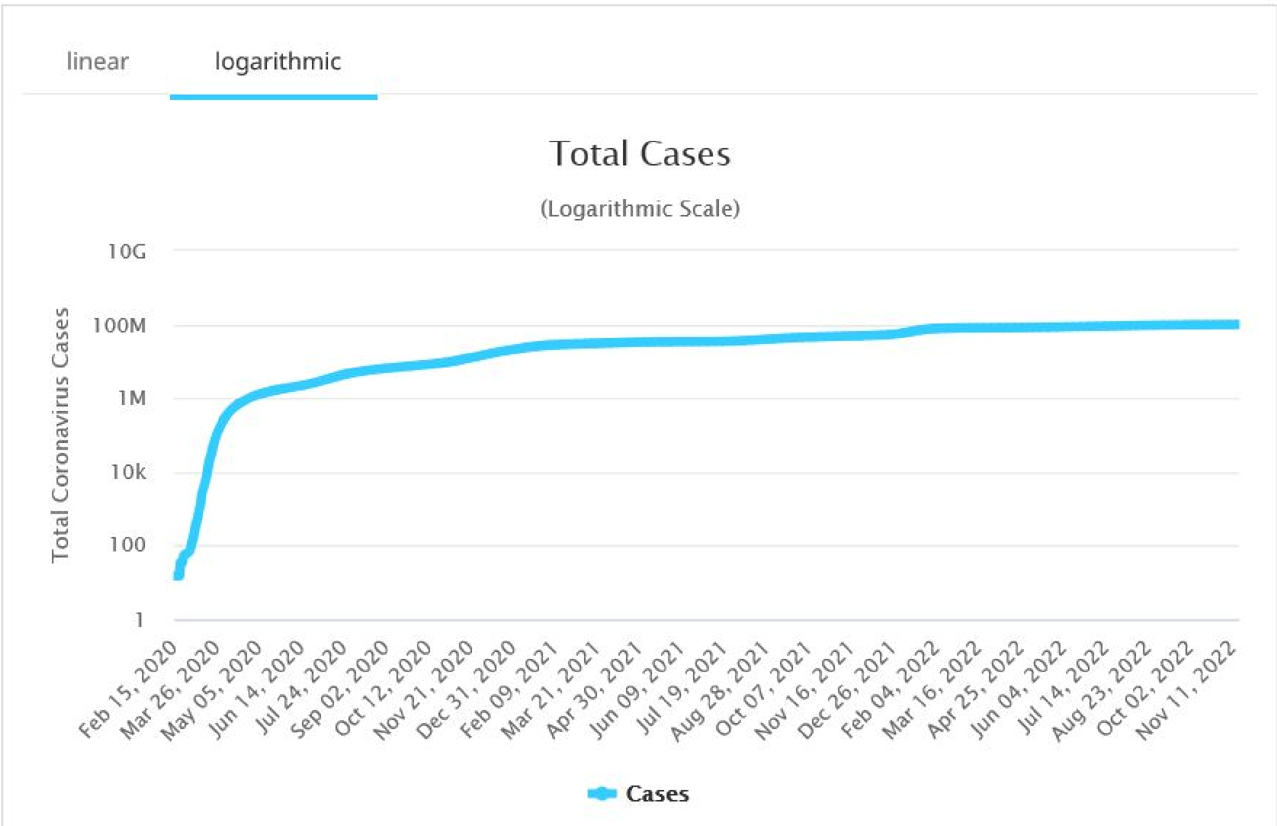
vi.

If you look at more recent data (see below) you will notice that cumulative cases are rising again. This does not follow Malthusian or Verhulst model. Explain possible reasons why.
Use the graph from https://www.worldometers.info/coronavirus/country/us/

Total Coronavirus Cases in the United States



Total Coronavirus Cases in the United States



Because coronavirus variants occur continually. Virus Mutants such as alpha, delta, omicron , separated all over the world. Otherwise,many people may be infected two times or more. The enforcement of quarantine policy also affect the coronavirus cases. On the other hand, the number of people who get screened every day affects the increase of cumulative cases.

Section 2: Mass-Spring-Damper Systems

Q.2-1 model for mass-spring system

Here is Newton's second law :

$$F = my''$$

The force term in mass-spring system is the force from the spring.

$$\begin{cases} F_{spring} = my'' \\ F_{spring} = -ky \end{cases} \Rightarrow -ky = my''$$

Q.2-2 model for mass-spring-damper system

The model of mass-spring-damper system looks like the previous one, but it requires one more term to describe the effect from the damper.

$$\begin{cases} F_{spring} + F_{damper} = my'' \\ F_{spring} = -ky \\ F_{damper} = -cy' \end{cases} \Rightarrow -ky - cy' = my''$$

Q.2-3 The undamped system

Code in Q.2-3 is written in matlab

i.ii.

Set $\omega = \sqrt{k/m}$. The equation cam then be written as $y'' + \omega^2 y = 0$. Substitute $y = A\cos\omega t + B\sin\omega t$ in :

$$\begin{aligned} y'' + \omega^2 y &= (A\cos\omega t + B\sin\omega t)'' + \omega^2(A\cos\omega t + B\sin\omega t) \\ &= A\omega^2(-\cos\omega t) + B\omega^2(-\sin\omega t) + A\omega^2\cos(\omega t) + B\omega^2\sin\omega t \\ &= 0 \quad \blacksquare \end{aligned}$$

Therefore, $y = A\cos\omega t + B\sin\omega t$ is able to be the genaral solution of this ODE.

iii.

The first derivative of $y(t)$ is

$$y' = A\omega(-\sin\omega t) + B\omega(\cos\omega t)$$

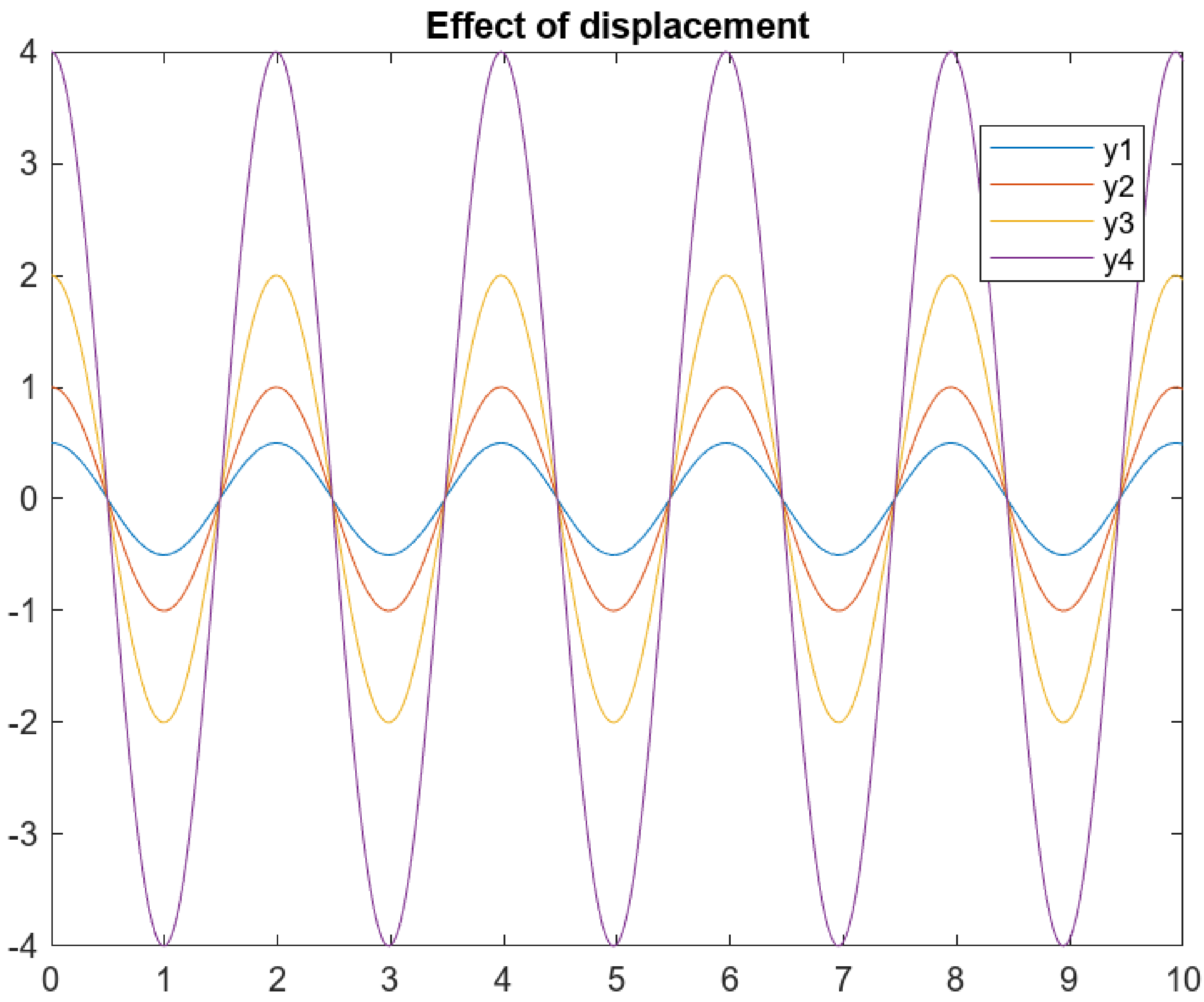
At $t = 0$, y and y' will be $y(0) = A$, $y'(0) = B\omega$, so A and B in terms of $y(0)$, $y'(0)$ will be :

$$A = y(0)$$
$$B = y'(0)/\omega$$

iv. Effect of displacement

Let $y(0) = [0.5, 1, 2, 4]$, $y'(0) = [0, 0, 0, 0]$, $k = 10\,(N/m)$, $m = 10kg$

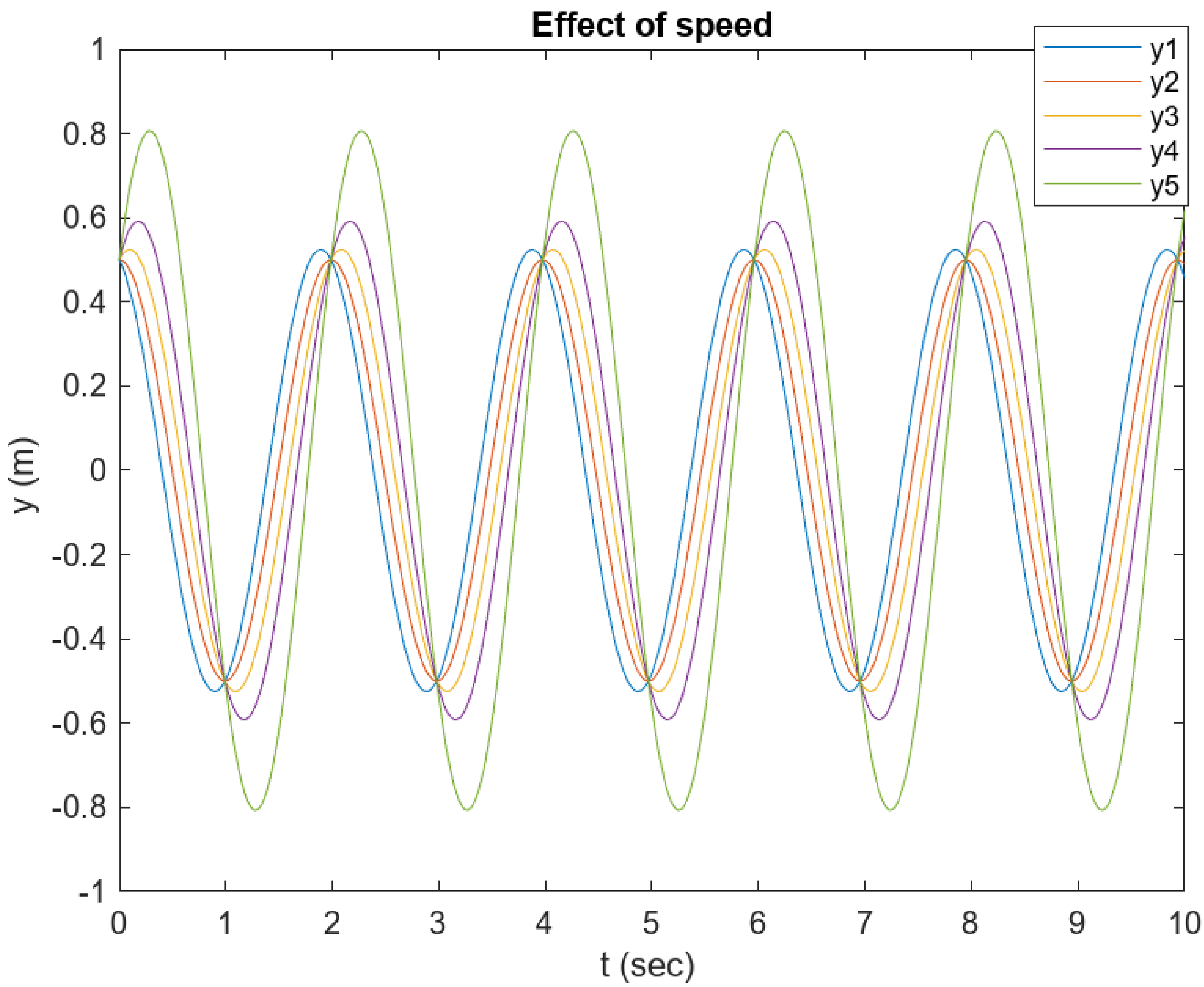
```
t= linspace(0,10,1000);
w = sqrt(10/1);
y1 = 0.5*cos(t*w) + 0*sin(w*t);
y2 = 1*cos(t*w) + 0*sin(w*t);
y3 = 2*cos(t*w) + 0*sin(w*t);
y4 = 4*cos(t*w) + 0*sin(w*t);

plot(t,y1,t,y2,t,y3,t,y4);
title('Effect of displacement');
legend('y1','y2','y3','y4');
```



The observed period for each solution is $T = 10/5 = 2$(sec), so $\omega_{observed} = 2\pi/2 = \pi$. Since the error is 0.0065 , we can say that the period of matches $\omega = \sqrt{k/m}$.

v. Effect of speed

```
t= linspace(0,10,1000);
w = sqrt(10/1);
y1 = 0.5*cos(t*w) + (-0.5/w)*sin(w*t);
y2 = 0.5*cos(t*w) + (0/w)*sin(w*t);
y3 = 0.5*cos(t*w) + (0.5/w)*sin(w*t);
y4 = 0.5*cos(t*w) + (1/w)*sin(w*t);
y5 = 0.5*cos(t*w) + (2/w)*sin(w*t);

plot(t,y1,t,y2,t,y3,t,y4,t,y5);
title('Effect of speed');
xlabel('t (sec) ');
ylabel('y (m) ');
legend('y1','y2','y3','y4','y5');
```

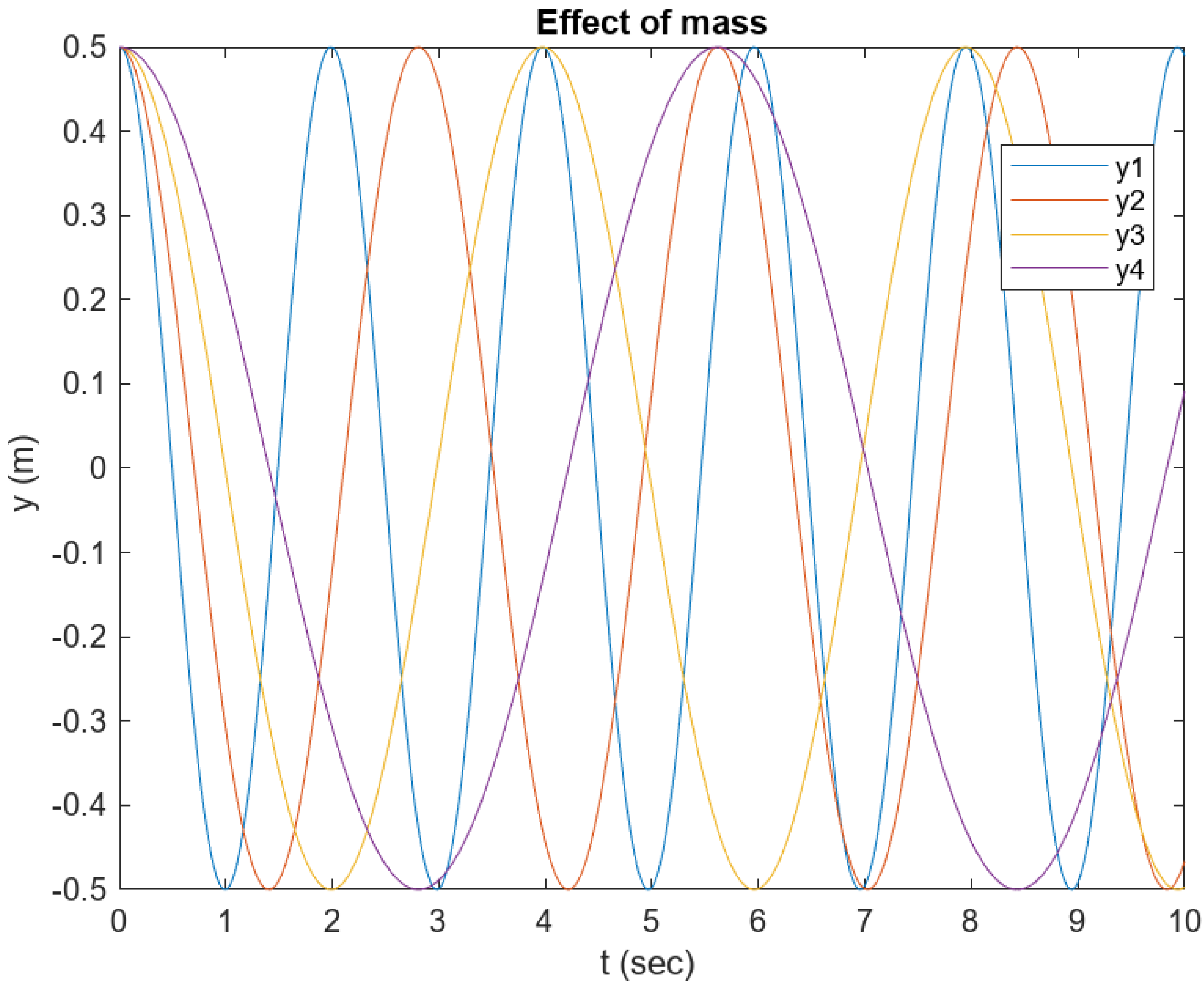


The initial velocity , as shown in the equation, effects both the amplitude and the starting phase of the oscillation. However, as observed in the graph, all modeling functions still behave almost 5 cycles of oscillatons, meaning that the oscillation freqency didn't change.

vi. Effect of mass

```
t= linspace(0,10,1000);
w1 = sqrt(10/1);
w2 = sqrt(10/2);
w3 = sqrt(10/4);
w4 = sqrt(10/8);
y1 = 0.5*cos(t*w1) + (0/w1)*sin(w1*t);
y2 = 0.5*cos(t*w2) + (0/w2)*sin(w2*t);
y3 = 0.5*cos(t*w3) + (0/w3)*sin(w3*t);
y4 = 0.5*cos(t*w4) + (0/w4)*sin(w4*t);


plot(t,y1,t,y2,t,y3,t,y4);
title('Effect of mass');
xlabel('t (sec) ');
```

```
ylabel('y (m) ');
legend('y1','y2','y3','y4');
```



From the graph above, assume that the amount of cycles behaved by each modeling function are

$$\begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3.5 \\ 2.5 \\ 1.75 \end{bmatrix}$$

The error will then be

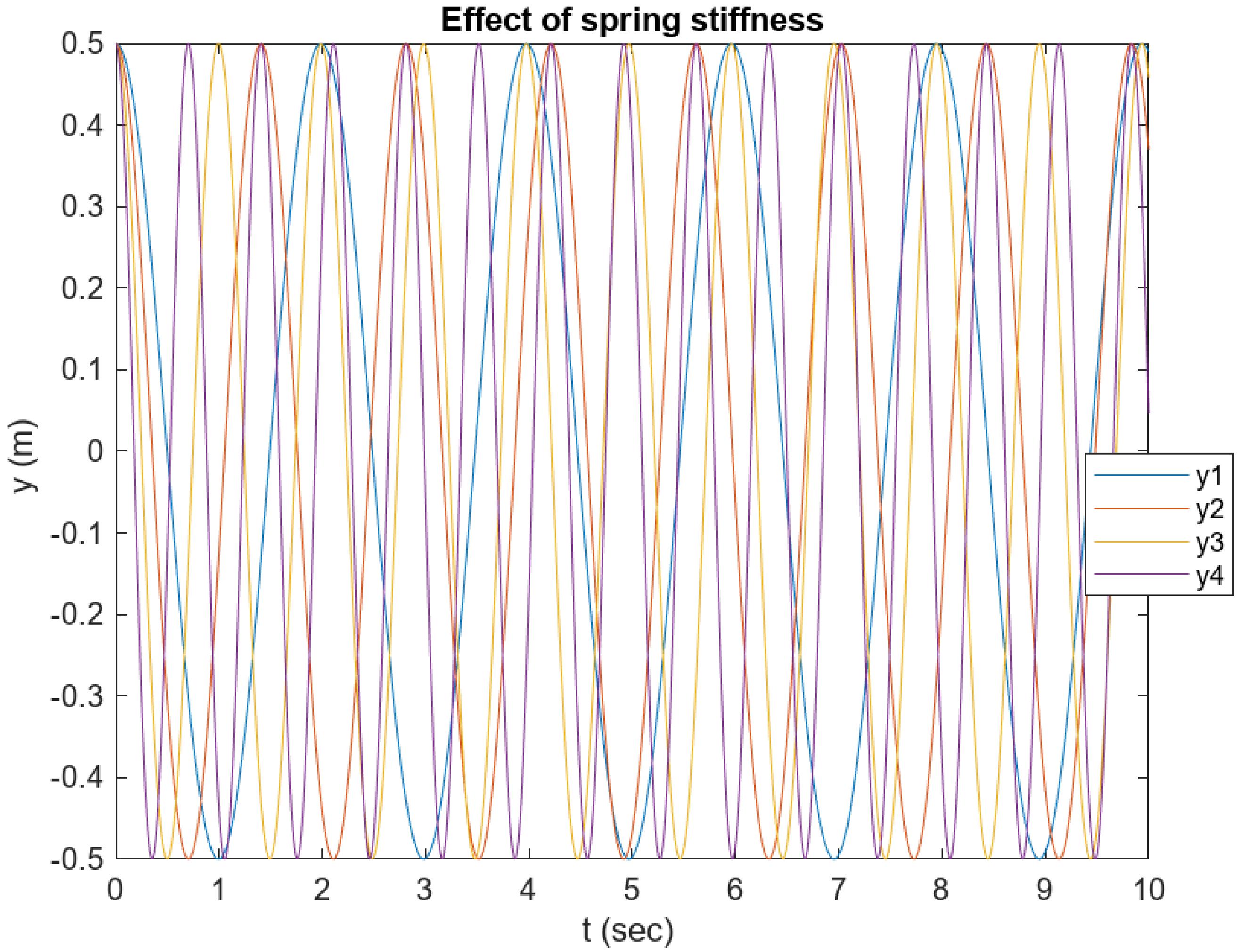
```
f = [5 3.5 2.5 1.75]/10;
w_observe = 2*pi*f;

w = [w1 w2 w3 w4];
error = (w -w_observe) ./w
```

$$\begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \end{bmatrix} = \begin{bmatrix} 0.0065 \\ 0.0165 \\ 0.0065 \\ 0.0165 \end{bmatrix}$$

So we can say that the frequency will still match $\omega = \sqrt{k/m}$.

vii. Effect of spring stiffness

```
t= linspace(0,10,1000);
w1 = sqrt(10/1);
w2 = sqrt(20/1);
w3 = sqrt(40/1);
w4 = sqrt(80/1);
y1 = 0.5*cos(t*w1) + (0/w1)*sin(w1*t);
y2 = 0.5*cos(t*w2) + (0/w2)*sin(w2*t);
y3 = 0.5*cos(t*w3) + (0/w3)*sin(w3*t);
y4 = 0.5*cos(t*w4) + (0/w4)*sin(w4*t);


plot(t,y1,t,y2,t,y3,t,y4);
title('Effect of spring stiffness');
xlabel('t (sec) ');
ylabel('y (m) ');
legend('y1','y2','y3','y4');
```



From the graph above, assume that the amount of cycles behaved by each modeling function are

$$\begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 10 \\ 14 \end{bmatrix}$$

The error will then be

```
f = [5 7 10 14]/10;
w_observe = 2*pi*f;

w = [w1 w2 w3 w4];
error = (w -w_observe) ./w
```

$$\begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \end{bmatrix} = \begin{bmatrix} 0.0065 \\ 0.0165 \\ 0.0065 \\ 0.0165 \end{bmatrix}$$

So we can say that the frequency will still match $\omega = \sqrt{k/m}$.

Q.2-4

Code in Q.2-4 is written in Python

i.

ODE :

$$-cy'(x) - ky(x) = my''(x)$$

rearrange :

$$y''(x) + \frac{c}{m}y'(x) + \frac{k}{m}y = 0$$

which is a homogeneous second order linear differential equation where x represents the time passing by for the damping system
assume

$$y = e^{\lambda x}$$

the characteristic equation is

$$\lambda^2 + \frac{c}{m} + \frac{k}{m} = 0 \implies = \frac{\frac{-c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

ii.

1. when $\sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}} > 0$, that is , $c^2 > 4mk$, we get two distinct roots. The general solution is

$$y = c_1 e^{\frac{\frac{-c}{m}+\sqrt{\left(\frac{c}{m}\right)^2-4\frac{k}{m}}}{2}x} + c_2 e^{\frac{\frac{-c}{m}-\sqrt{\left(\frac{c}{m}\right)^2-4\frac{k}{m}}}{2}x}$$

2. when $\sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}} = 0$, that is , $c^2 = 4mk$, we get two repeated roots. The general solution is

$$y = (c_1 + c_2 x)e^{\frac{-c}{2m}x}$$

3. when $\sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}} < 0$, that is , $c^2 < 4mk$, we get two complex roots. By substituting Euler's equation $e^{ix} = \cos x + i\sin x$ into it, the general solution becomes $y = e^{\frac{-c}{2m}x}(c_1 \cos w_d x + c_2 \sin w_d x)$, where

$$w_n = \sqrt{\frac{k}{m}}$$

$$w_d = \sqrt{w_n^2 - \left(\frac{b}{2m}\right)^2}$$

iii.

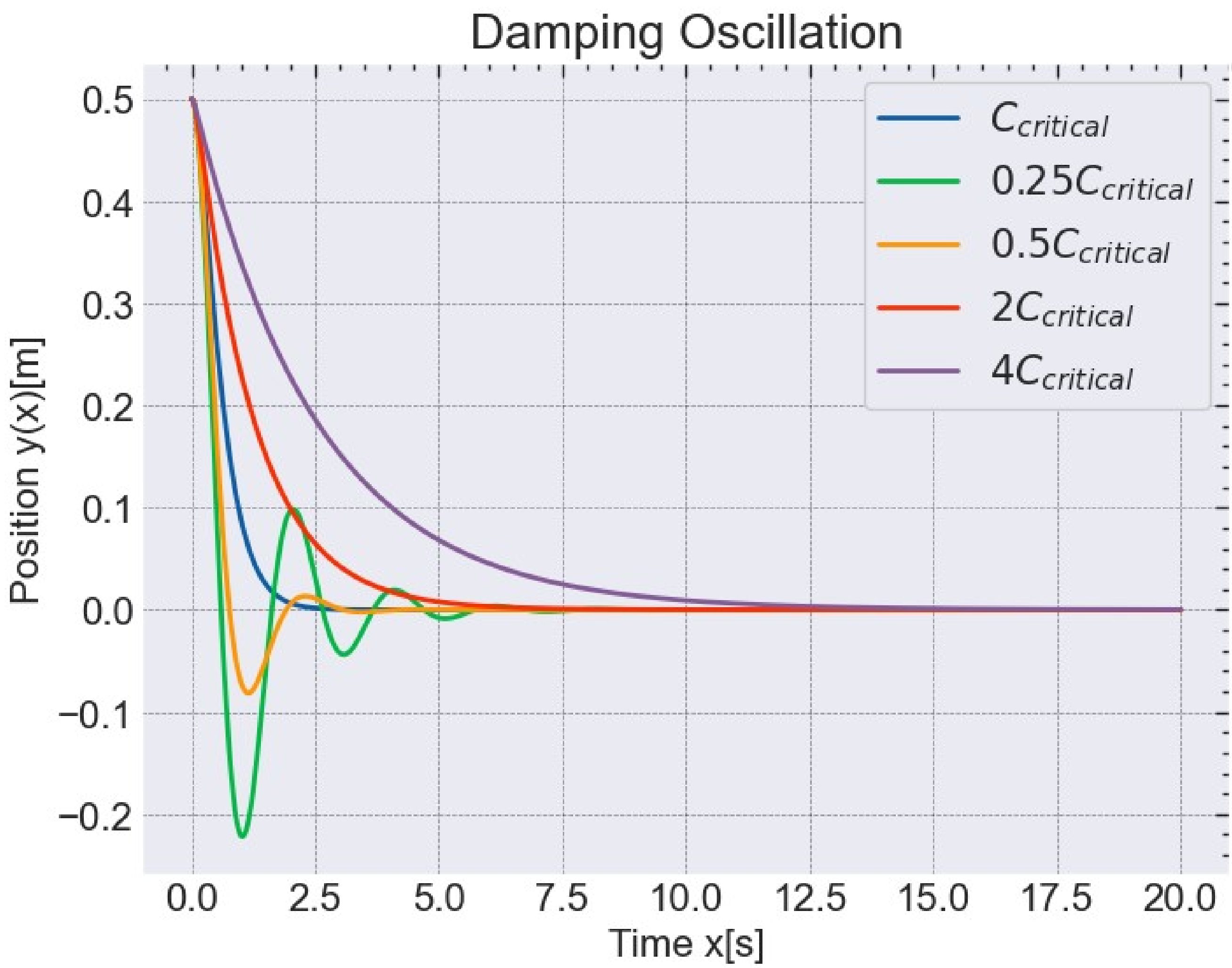
For m=$1kg$, $k = 10\frac{N}{m}$, if we want to have repeated roots, then

$$c^2 = 40 \implies c = \pm 2\sqrt{10}\frac{kgN}{m}$$

iv.

```
import sympy as sp
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt
import math
from scipy.integrate import solve_ivp
import seaborn as sns
sns.set()

plt.style.use(['science','notebook','grid'])

t = np.linspace(0,20,1000)
y = [0.5,0]
c_1=2*math.sqrt(10)
c_2=0.25*2*math.sqrt(10)

c_3=0.5*2*math.sqrt(10)
c_4=2*2*math.sqrt(10)
c_5=4*2*math.sqrt(10)
m=1
k=10
gamma_list={c_1/m,c_2/m,c_3/m,c_4/m,c_5/m}

omega_sqr = k/m
def sho(t,y):
        solution1 = (y[1],(-c_1*y[1]-omega_sqr*y[0]))
        return solution1
solution1 = solve_ivp(sho, [0,1000], y0 = y, t_eval = t)
def sho(t,y):
        solution2 = (y[1],(-c_2*y[1]-omega_sqr*y[0]))
        return solution2
solution2 = solve_ivp(sho, [0,1000], y0 = y, t_eval = t)
def sho(t,y):
```

```
        solution3 = (y[1],(-c_3*y[1]-omega_sqr*y[0]))
        return solution3
solution3 = solve_ivp(sho, [0,1000], y0 = y, t_eval = t)
def sho(t,y):
        solution4 = (y[1],(-c_4*y[1]-omega_sqr*y[0]))
        return solution4
solution4 = solve_ivp(sho, [0,1000], y0 = y, t_eval = t)
def sho(t,y):
        solution5 = (y[1],(-c_5*y[1]-omega_sqr*y[0]))
        return solution5
solution5 = solve_ivp(sho, [0,1000], y0 = y, t_eval = t)
plt.plot(t,solution1.y[0],label='$C_{critical}$')
plt.plot(t,solution2.y[0],label='$0.25C_{critical}$')
plt.plot(t,solution3.y[0],label='$0.5C_{critical}$')
plt.plot(t,solution4.y[0],label='$2C_{critical}$')
plt.plot(t,solution5.y[0],label='$4C_{critical}$')
plt.ylabel("Position y(x)[m]")
plt.xlabel("Time x[s]")
plt.title('Damping Oscillation', fontsize = 20)
plt.legend()
```



vi.

Physical meaning and behavior :
x represents the time passing by for the damping system

Underdamping:

when $c^2 < 4mk$, we get two complex roots ,which can produce two parts includes

1. $e^{\frac{-c}{2m}x}$ which makes the amplitude decrease over time (exponential decay)
2. $(c_1 \cos w_d x + c_2 \sin w_d x)$,as a function of sinx and cosx, makes the system oscillate overtime .

Critical damping:

when $c^2 = 4mk$, we get $y = (c_1 + c_2 x)e^{\frac{-c}{2m}x}$. The system does not oscillate and goes back to the equilibrium in the shortest time among all damping system.

Overdamping:

when $c^2 > 4mk$, we get

$$y = c_1 e^{\frac{\frac{-c}{m} + \sqrt{(\frac{c}{m})^2 - 4\frac{k}{m}}}{2}x} + c_2 e^{\frac{\frac{-c}{m} - \sqrt{(\frac{c}{m})^2 - 4\frac{k}{m}}}{2}x}$$

since λ_1 and λ_2 are both negative value , y(x) decreases overtime without oscillation , also , the larger c is , the longer it takes to reach equilibrium

vii.

Damper Force=$-cy'(x)$. The minus sign means the force is always in the opposite direction with velocity. If I use a stiffer string(a bigger k), then , in order to maintain critical damping $c^2 = 4mk$,c must get larger as k increases ,thus I need more damper force. A stiffer spring needs more damper force to maintain stability.

viii.

Damper Force=$-cy'(x)$. The minus sign means the force is always in the opposite direction with velocity
. If I use a heavier mass(a bigger m), then , in order to maintain critical damping $c^2 = 4mk$, c must get larger as m increases (assume k does not change),thus I need more damper force.

Q.2-5 comparing train and car

Dampers don't support the weight of the car. Instead, that's the job of the springs.
What dampers do is control the bouncing movement of the spring by providing resistance, dissipating the stored energy and controling the speed at which your springs move up and down by stopping them from compressing or rebounding too quickly to ensure the tyres are kept in contact with the road.
A train should use a bigger damper than a car due to its larger mass, so that when there're oscillations caused by uneven road, it can reach equilibrium in the shortest amount of time (critical damping must fulfill $c^2 = 4mk$) (assume $k_{train} \approx k_{car}$)

Q.2-6 sports car and normal family car

Assume the mass of a sports car is approximately equal to the mass of a normal car. A sports car should use a stiffer string than a normal family car. Stiffer spring will give you a firmer feel , many sports car drivers tend to prefer stiffer springs to reduce body roll and body lean. Looser spring will provide you with a softer feel. When going over a bump or curb, softer springs will maintain contact with the road better than stiffer springs , so it's favored for family car.