Linear Systems Homework #1

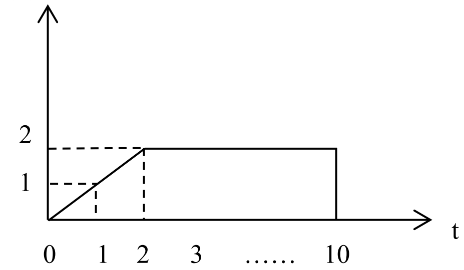
Due: 28/10/2022

1. (a) Find the Inverse Laplace Transformation (time-domain signal) for the (1), (2) signals in "S" domain.
   (b) Find the Laplace Transformation for the (3), (4), (5), (6) time-domain signals.

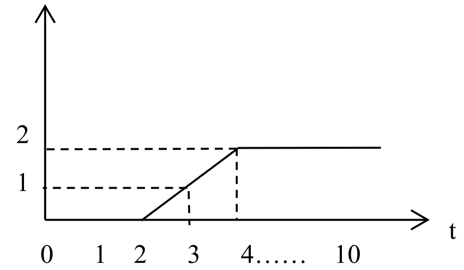
   (1) $\dfrac{2s+3}{s^2+3s+2}$ (2) $\dfrac{s+2\zeta}{s^2+2\zeta\cdot s+1}$ (3) $f(t)=\begin{cases}0 & t<0\\ 3e^{-t}+4e^{-3t} & t\geq 0\end{cases}$

   (4) $f(t)=\begin{cases}0 & t<2\\ 5e^{-3t} & t\geq 2\end{cases}$

   (5)



   (6)



2. Obtain the Laplace transformation $F(s)$ for the time function $f(t)$ or the Z transform $F(z)$ for the sequence $f(k)$ given below. Assume that $f(t)=0$ for $t<0$ and $f(k)=0$ for $k<0$.
   (a) $f(t)=10e^{-3t}\cos(5t)$
   (b) $f(t)=(1+2t+t^2)e^{-5t}$
   (c) $f(t)=\{\begin{matrix}e^{at} & for & 0\leq t<\tau\\ 0 & for & \tau\leq t<T\end{matrix}$,where T is the period
   $f(t)=f(t-nT)\quad n=1,2,3...$

3. Use Matlab function ode45 to simulate the following differential equations, and plot the trajectory of $f(t)$.

$$\ddot{f}(t)+5\dot{f}(t)+6f(t)=e^{-t},\quad f(0)=\dot{f}(0)=0.$$

4. Given a sequence $\mathrm{f(k)}=(f(0),f(1),\ldots)$, $\mathrm{F(z)}=\mathbb{Z}(f(k))$, Please show that $\mathbb{Z}(f(k+N))=z^NF(z)-\sum_{j=1}^{N}z^jf(N-j)$

5. Please derive the Z transform of the following sequence

$$e^{-a\cdot k\cdot T}, k=0,1,2,\ldots$$

6. Please write up Matlab codes to simulate the trajectory of the following discrete time system. $\mathrm{u(k)}$ is a unit step function.
   $\mathrm{x(k+2)+0.6x(k+1)+0.09x(k)=u(k)},\ \mathrm{x(0)=1},\ \mathrm{x(1)=2},\ \mathrm{k}=0,1,\ldots\ldots,50$

TABLE 2-1 TABLE OF z TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ 1 $\quad k=0$ 0 $\quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ 1 $\quad n=k$ 0 $\quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k=1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$

$x(t)=0 \quad$ for $t<0$.
$x(kT)=x(k)=0 \quad$ for $k<0$.
Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$.

1. Brief Review of Laplace Transform

Continuous time function: f : R₊ → R, $f : t \mapsto f(t)$, where t is a non-negative real. Most of the time, we use the notation $f(t)$ to denote a continuous time function. Also, $f(t) = 0$ for all $t < 0$.

1.1 Definition:

For a continuous time function $f : R_+ \to R$

$$F(s) = L\{f(t)\} \triangleq \int_{0-}^{\infty} f(t)\, e^{-st}\, dt \qquad (1.1)$$

for all $s \in C$ such that the above integral converges. If $F(s)$ exists for some s, say $s_o = \sigma_o + j\omega_o$, then it exists for all s such that $\mathrm{Re}\{s\} \geq \sigma_o$. The smallest value of σ_o, say α for which $F(s)$ exists is called the *abscissa of convergence*, and $\{s : \mathrm{Re}\{s\} \geq \alpha\}$ is called the *region of existence* of $F(s)$. Note that this in turn implies that

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

for all s in the region of existence of $F(s)$.

Examples:

1) $f(t) = e^{-st}, a \in \mathrm{R}, F(s) = \frac{1}{s+a}$

2) Step function:

$$f(t) = 1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad F(s) = L\{1(t)\} = \frac{1}{s}.$$

3) $f(t) = \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j},\ F(s) = \frac{\omega}{s^2+\omega^2}$

The Laplace transform of $\mathrm{f(t)} = e^{t^2}$ does not exist since $\int_{0-}^{\infty} e^{t^2 - st}\, dt$ does not converge for any $s \in$ C.

1.2 Properties:

Linearity:

For $f(t)$ and $g(t)$ and α, $\beta \in$ R, $L\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$

Differentiation:

For $\dot{f}(t) = {df(t)}/{dt}$,

$$L\{\dot{f}(t)\} = \int_{0-}^{\infty} \dot{f}(t)\, e^{-st}\, dt \;=\; f(t)e^{-st}\,\big|_{0-}^{\infty} - (-s)\int_{0-}^{\infty} f(t)e^{-st}dt \;=\; sF(s) - f(0-)\,.$$

$$\mathcal{L}\left\{\frac{d^p f(t)}{dt^p}\right\} = s^p F(s) - s^{p-1} f(0-) \cdots - \frac{d^{p-1} f}{dt^{p-1}}(0-)$$

Integration:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Convolution:

For *f*(*t*) and *g*(*t*), define the convolution integral

$$(f * g)(t) = (g * f)(t) \triangleq \int_0^t f(t-\tau)g(\tau)\, d\tau$$

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

Proof:

$$\mathcal{L}\{(f\ *\ g)(t)\} = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)\, d\tau dt$$

$$= \int_0^\infty \int_0^t (e^{-s(t-\tau)} f(t-\tau))(e^{-s\tau} g(\tau))d\tau dt$$

$$= \int_0^\infty \int_0^\infty (e^{-s(t-\tau)} f(t-\tau))(e^{-s\tau} g(\tau))d\tau dt, \qquad (\mathrm{f}(\mathrm{t}\ -\ \tau)\ =\ 0 \text{ for } \tau > \mathrm{t})$$

$$= \int_0^\infty \left\{\int_0^\infty (e^{-s(t-\tau)} f(t-\tau)dt\right\} e^{-s\tau} g(\tau)d\tau$$

$$= \int_0^\infty \left\{\int_{-\tau}^\infty (e^{-s\gamma} f(\gamma)d\gamma\right\} e^{-s\tau} g(\tau)d\tau \qquad (\mathrm{f}(\gamma)\ =\ 0 \text{ for } \gamma < 0)$$

$$= \left\{\int_0^\infty (e^{-s\gamma} f(\gamma)d\gamma\right\}\left\{\int_0^\infty (e^{-s\tau} f(\tau)d\tau\right\} = F(s)G(s)$$

Example:

Consider the first order linear time invariant (LTI) system

$$\dot{y(t)} = -ay(t) + bu(t)\,, \quad y(0) = 0$$

Given an arbitrary function of time $u_{[0,t]} \triangleq u\colon [0,t] \to \mathrm{R}$, the solution $y_{[0,t]}$ (i.e. $\mathrm{y(t)}$ for $t \geq 0$) for this system is

$$y(t) = \int_0^t e^{-a(t-\tau)} bu(\tau)\, d\tau = e^{-at}b * u(t)$$

Defining $\mathrm{Y(s)} = \mathcal{L}\{y(t)\}, \mathrm{U(s)} = \mathcal{L}\{u(t)\}$ and noticing that $\mathcal{L}\{e^{-at}b\} = \frac{b}{s+a}$, we obtain

$$\mathrm{Y(s)} = \mathcal{L}\{e^{-at}b * u(t)\} = \frac{b}{s+a}U(s)$$

We can verify this result by taking Laplace transforms of both sides of above equation and solving for $\mathrm{Y(s)}$:

$$sY(s) = -aY(s) + bU(s) \Rightarrow Y(s) = \frac{b}{s+a}U(s)$$

The above results can be generalized for higher order LTI differential equations.

Consider for $n, m \in Z_+$, and $n \geq m$

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_o y(t) = b_m \frac{d^m u(t)}{dt^{n-1}} + \cdots + b_o u(t)$$

and assume that

$$y(0-) = \dot{y}(0-) = \cdot \frac{d^{n-1}y}{dt^{n-1}} \mid_{t\to 0-} = 0$$
$$u(0-) = \dot{u}(0-) = \cdot \frac{d^{m-1}u}{dt^{m-1}} \mid_{t\to 0-} = 0$$

Taking Laplace transformation,

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_0\right) Y(s) = \left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0\right) U(s)$$
$$\Rightarrow Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} U(s) .$$

Let *Y* (*s*) = *G*(*s*)*U*(*s*), then the *nth* order *transfer function G*(*s*) is given by

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

- *A*(*s*) = 0 is the *characteristic equation*. Its roots are the *poles* of *G*(*s*).
- The roots of *B*(*s*) = 0 are the *zeros* of *G*(*s*).

Initial value theorem

If $\lim_{t\to 0+} f(t)$ exists,

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{t\to\infty} sF(s)$$

Proof: Taking $\lim_{s\to\infty}$ of Eq. (1.2),

$$\lim_{s\to\infty}\left\{\int_{0-}^{\infty} \dot{f}(t)e^{-st}dt\right\} = \lim_{s\to\infty}\{sF(s)\} - f(0-)$$

Evaluating the left hand side of Eq. (1.7) we obtain

$$\lim_{s\to\infty}\left\{\int_{0-}^{\infty} \dot{f}(t)e^{-st}dt\right\} = \lim_{s\to\infty}\left\{\int_{0-}^{0+} \dot{f}(t)e^{-st}dt + \int_{0+}^{\infty} \dot{f}(t)e^{-st}dt\right\} .$$

We now consider two cases:

1. *f*(0−) = *f*(0+): In this case $\lim_{s\to\infty}\{\int_{0-}^{\infty} \dot{f}(t)e^{-st}dt\} = 0$ and the result of the theorem follows.

2. *f*(0−) $\neq$ *f*(0+): In this case

$$\lim_{s\to\infty}\int_{0-}^{0+} \dot{f}(t)e^{-st}dt = f(0+) - f(0-) \qquad \textit{and} \qquad \lim_{s\to\infty}\int_{0+}^{\infty} \dot{f}(t)e^{-st}dt = 0$$

and the result of the theorem also follows.

The second case arises when we have an impulsive input function at the origin.

Definition [Dirac delta function $\delta(t)$]

$$\delta(t-T) = \lim_{\Delta\to 0}\begin{cases} 1/\Delta & T \leq t \leq T+\Delta \\ 0 & t > T+\Delta \end{cases}, \qquad \int_0^{\infty} \delta(t-T)f(t)dt = f(T).$$

$$\mathcal{L}\{\delta(t)\} = \int_{0-}^{\infty} \delta(t)e^{-st}dt = e^0 = 1.$$

Final value theorem:

Theorem *if* $\lim_{t\to\infty} f(t)$ exists,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Note: The assumptions regarding the existence of the limits are necessary. For example, $f(t) = e^t$ and $\lim_{t\to\infty} e^t$ does not exist. However, $F(s) = \mathcal{L}\{e^t\} = \frac{1}{s-1}$ and $\lim_{s\to 0} sF(s) = 0$

1.3 Inverse Laplace Transform

Definition Given f : R₊ → R, the smallest value $\sigma_0 \in R$ for which the Laplace transform converges is the abscissa of convergence.

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

The region $\{s \in C \mid Re\{s\} \geq \sigma_o\}$ is the region of existence of the Laplace transform.

Example: For $f(t) = e^{3t}$, the region of existence is $\{s \in C \mid Re\{s\} > 3\}$.

The inverse Laplace transform of $F(s)$ is define by the contour integral

$$f(t) = \mathcal{L}^{-1} = \{F(s)\} \triangleq \int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \qquad t \geq 0-$$

for any $c \geq \sigma_o$.

In general, we will calculate inverse Laplace transforms using partial fraction expansions and the look-up table approach.

2. Brief Review of Z Transform

Discrete time function (sequence): $f : Z_+ \to R$, $f : k \to f(k)$, where k is a nonnegative integer. Most of the time, we use the notation $f(k)$ to denote an infinite sequence or discrete time function.

2.1 Definition:

For a discrete time function $f : Z_+ \to R$

$$F(z) = \mathcal{Z}\{f(k)\} \triangleq \sum_{k=0}^{\infty} f(k)z^{-k} = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

for all $z \in C$ such that the above series converges.

If $F(z)$ exists for some z, say $z_e = r_e e^{j\varphi_e}$, then it exists for all z such that $|z| \geq r_e$. The smallest value of r_e, say r_o, for which $F(z)$ exists is called the *radius of convergence*, and

$\{z : |z| \geq r_o\}$ is called the *region of existence* of $F(z)$. Notice that this in turn implies that

$$\lim_{k\to\infty} z^{-k} f(k) = 0$$

for all z in the region of existence of $F(z)$.

Examples:

1) Unit step sequence:

$$f(t) = 1(t) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$\begin{aligned} F(z) &= 1 + z^{-1} + z^{-2} + \cdots \\ &= 1 + z^{-1}\left[1 + z^{-1} + z^{-2} + \cdots\right] = 1 + z^{-1}F(z) \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}. \end{aligned}$$

2) $\mathrm{f(k)} = p^k, \mathrm{p} \in \mathrm{R}$ (geometric sequence), $F(z) = \frac{1}{1-pz^{-1}} = \frac{z}{z-p}$.

3) For a periodic sequence $f(k + N) = f(k)$

$$\begin{aligned} F(z) &= f(0) + f(1)z^{-1} + \cdots + f(N-1)z^{-(N-1)} \\ &\quad + z^{-N}\left[f(0) + f(1)z^{-1} + \cdots + f(N-1)z^{-(N-1)}\right] \\ &\quad + z^{-2N}\left[f(0) + f(1)z^{-1} + \cdots + f(N-1)z^{-(N-1)}\right] + \cdots \\ &= \frac{1}{1 - z^{-N}}\left[f(0) + f(1)z^{-1} + \cdots + f(N-1)z^{-(N-1)}\right] \end{aligned}$$

2.2 Properties:

Linearity:

For $f(k)$ and $g(k)$ and $\alpha, \beta \in \mathrm{R}$,

$$\mathrm{Z}\{\alpha f(k) + \beta g(k)\} = \alpha F(z) + \beta G(z)$$

Advance:

Given the sequence $f(k) = (f(0), f(1), \cdots)$, we define the N steps advance sequence

$$f(k+N) \triangleq (f(N), f(N+1), \cdots)$$

$$\begin{aligned} \mathcal{Z}\{f(k+N)\} &= \sum_{k=0}^{\infty} f(k+N) z^{-k} \\ &= z^N F(z) - \sum_{j=1}^{N} z^j f(N-j) \end{aligned}.$$

$$Z\{f(k+1)\} = zF(z) - zf(0)$$

Delay:

Given the sequence $f(k) = (f(0), f(1), \cdots)$, we define the N steps delayed sequence

$$f(k-N) = \begin{cases} 0 & 0 \le k < N \\ f(k-N) & k \ge N \end{cases}$$

i.e.

$$f(k-N) = (\underbrace{0, 0, \cdots 0}_{N-1}, f(0), f(1), \cdots).$$

$Z\{f(k - N)\} = z - NF(z)$.

Convolution:

For f(k) and g(k), define the convolution

$$(\mathrm{f} * \mathrm{g})(\mathrm{k}) = (\mathrm{g} * \mathrm{f})(\mathrm{k}) \triangleq \sum_{j=0}^{k} f(k-j) g(j)$$

$$Z\{(f * g)(k)\} = F(z)G(z)$$

Proof:

$$Z\{(f * g)(k)\} = \sum_{k=0}^{\infty} z^{-k} \left\{ \sum_{j=0}^{k} f(k-j) g(j) \right\} = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{\infty} z^{-(k-j)} f(k-j) z^{-j} g(j) \right\}$$

$(\mathrm{f}(\mathrm{k} - \mathrm{j}) = 0$ for $\mathrm{j} > \mathrm{k})$

$$= \sum_{j=0}^{\infty}\left\{\sum_{k=0}^{\infty} z^{-(k-j)} f(k-j)\right\} z^{-j} g(j)$$

$$= \sum_{j=0}^{\infty}\left\{\sum_{m=-j}^{\infty} z^{-m} f(m)\right\} z^{-j} g(j) \quad (\mathrm{f(m)} = 0 \text{ for } \mathrm{m} < 0)$$

$$= \left\{\sum_{m=0}^{\infty} z^{-m} f(m)\right\}\left\{\sum_{j=0}^{\infty} z^{-j} f(j)\right\} = F(z)G(z)$$

Example:

Consider the first order discrete time linear time invariant (LTI) system

$$y(k) = ay(k-1) + bu(k), \qquad y(-1) = 0,$$

Given an arbitrary sequence ($\mathrm{u(j)}|_{j=0}^{k}$, the solution ($\mathrm{y(j)}|_{j=0}^{k}$, i.e. $\mathrm{y(j)}$ for $\mathrm{j} \in [0, k]$) for this system is

$$y(k) = \sum_{j=0}^{\infty} a^{(k-j)} bu(j) = a^k b * u(k)$$

Defining $(z) \triangleq Z\{y(k)\}$, $U(z) \triangleq Z\{u(k)\}$ and noticing that $Z\{(a^k b\} = \frac{b}{1-az^{-1}}$,

we obtain

$$Y(z) = \mathcal{Z}\left\{a^k b \star u(k)\right\} = \frac{b}{1-az^{-1}}U(z) = \frac{zb}{z-a}U(z).$$

We can verify this result by taking Z transforms of both sides and solving for $Y(z)$:

$$z^{-1}Y(z) = aY(z) + bU(z) \Rightarrow Y(z) = \frac{zb}{z-a}U(z).$$

The result shown above can be generalized for higher order LTI difference equations.

Consider for $\mathrm{n, m} \in Z_+$, and $n \geq m$

$$y(k) + a_{n-1}y(k-1) + \cdots + a_o y(k-n) = b_m u(k+m-n) + \cdots + b_o u(k-n)$$

and assume that

$y(-1) = y(-2) = \cdots = y(-n) = 0$

$u(-n+m) = u(-n+m-1) = \cdots = u(-n) = 0$

Utilizing "delay" theorem, we obtain

$$\left(1+a_{n-1}z^{-1}+\cdots+a_0z^{-n}\right)Y(z)=z^{-n}\left(b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0\right)U(z)$$

$$\Rightarrow Y(z)=\frac{b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_0}U(z)\,.$$

Let, $Y(z) = G(z)U(z)$, where the *nth* order *transfer function* $G(z)$ is given by

$$G(z)=\frac{B(z)}{A(z)}=\frac{b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_0}$$

- $A(z) = 0$ is the *characteristic equation*. Its roots are the *poles* of $G(z)$.
- The roots of $B(z) = 0$ are the zeros of $G(z)$.

Initial value theorem:

$$f(0)=\lim_{z\to\infty}F(z)$$

Final value theorem:

Theorem If $\lim_{k\to\infty} f(k)$ exists,

$$\lim_{k\to\infty} f(k)=\lim_{z\to 1}(z-1)F(z)$$

Note: The assumptions regarding the existence of the limit is necessary. For example, $f(k) = 2^k$ and $\lim_{k\to\infty} 2^k$ does not exist. However, $F(z) = \frac{z}{z-2}$, and $\lim_{z\to 1}(z-1)F(z)=0$

2.3 Inverse Z Transform

Definition Given $f: Z_+ \rightarrow R$, the smallest value $r_o \in R$ for which the Z transform converges when $z = r_o e^{j\varphi}$ is the radius of convergence.

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

The region $\{z \in C \mid |z| \geq r_o\}$ is the region of existence of the Z transform.

Example: For $f(k) = 3^k$, the region of existence is $\{ z \in C \mid |z| > 3\}$.

The inverse Z transform of $F(z)$ is define by the contour integral

$$f(k) \triangleq \frac{1}{2\pi j} \oint_{\Gamma} F(z) z^{k-1} dz$$

where Γ is a simple, closed rectifiable curve enclosing the origin and lying outside of the closed disk $\{ z \in C \mid |z| = r_o\}$.

In general we will calculate inverse Z transforms using partial fraction expansions and the look-up table approach.