

# 工程數學 (二) project 1

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## 1 Conversation with ChatGPT

### 1.1 the record of the conversation

**We:** can you please tell me something about the Manhattan Project in 1945

**ChatGPT:** The Manhattan Project was a top-secret, government-led research and development program during World War II that aimed to develop the first atomic bomb. It was named after the location of its main research and production site, Manhattan, New York City. The project was initiated in 1939 and lasted until 1946, involving thousands of scientists, engineers, and support personnel.

The Manhattan Project was a response to fears that Nazi Germany might be developing nuclear weapons, and it was authorized by President Franklin D. Roosevelt to harness the power of nuclear fission for military purposes. It was carried out by the United States, the United Kingdom, and Canada, with the majority of the work being done in the United States.

The project was conducted at several major research and production facilities, including Los Alamos Laboratory in New Mexico, Oak Ridge Reservation in Tennessee, and Hanford Site in Washington state. The scientists and engineers involved in

the Manhattan Project made significant advancements in nuclear physics, materials science, and engineering, leading to the successful development of the first atomic bomb.

The first successful test of an atomic bomb, code-named "Trinity," took place on July 16, 1945, at the Alamogordo Bombing and Gunnery Range in New Mexico. The successful test paved the way for the use of atomic bombs in combat.

On August 6, 1945, the United States dropped an atomic bomb code-named "Little Boy" on Hiroshima, Japan, resulting in a massive explosion and widespread destruction. Three days later, on August 9, 1945, a second atomic bomb code-named "Fat Man" was dropped on Nagasaki, Japan. The bombings led to Japan's surrender on August 15, 1945, marking the end of World War II.

The Manhattan Project had a profound and lasting impact on world history, ushering in the nuclear age and fundamentally changing the nature of warfare and international relations. It also raised ethical, moral, and humanitarian concerns about the use of atomic weapons and the potential consequences of nuclear proliferation.

**We:** what are some concepts of engineering mathematics that were used during the Manhattan Project.

**ChatGPT:** The Manhattan Project involved a wide range of engineering mathematics concepts that were crucial for the development of the atomic bomb. Some of the key concepts of engineering mathematics that were used during the Manhattan Project include:

1. Calculus: Calculus, including both differential and integral calculus, was used extensively in the Manhattan Project. It was used in the mathematical modeling of physical phenomena, such as the behavior of particles undergoing nu-

clear reactions, and in solving complex equations related to the behavior of nuclear materials.

2.Linear algebra: Linear algebra, which deals with vector spaces and linear equations, was used in various aspects of the Manhattan Project. It was used in the analysis of data from experiments and simulations, as well as in the development of mathematical models for nuclear reactions and the behavior of nuclear materials.

3.Differential equations: Differential equations, which describe the relationships between rates of change, were used in the modeling and simulation of complex physical processes involved in nuclear reactions and the behavior of nuclear materials. They were used to study the behavior of particles, the diffusion of materials, and the dynamics of nuclear reactions.

4.Probability and statistics: Probability and statistics were used in the Manhattan Project for data analysis, uncertainty quantification, and risk assessment. They were used to analyze experimental data, assess the reliability of components and systems, and estimate the probability of different outcomes in various scenarios.

5.Numerical methods: Numerical methods, including numerical approximation techniques and computational algorithms, were used in the Manhattan Project for solving complex mathematical equations and simulating physical processes. These methods were critical for performing calculations and simulations that were beyond the capabilities of manual calculations, allowing for more accurate and efficient computations.

6.Materials science and engineering: Materials science and engineering concepts were also crucial in the Manhattan Project, as the development of new materials with specific properties was essential for the design and construction of

nuclear reactors, bomb components, and other specialized equipment.

These are just some examples of the engineering mathematics concepts that were used during the Manhattan Project. The project required a deep understanding of various mathematical principles and their applications to solve complex problems related to nuclear physics, materials science, and engineering.

**We:** can you discuss more about how linear algebra and vector calculus were used

**ChatGPT:** Sure! Linear algebra and vector calculus were fundamental mathematical tools used in the Manhattan Project in several ways:

1.Mathematical modeling of physical phenomena: Linear algebra and vector calculus were used to develop mathematical models that described the behavior of particles, such as neutrons and protons, undergoing nuclear reactions. These models helped scientists and engineers understand the interactions between particles and the properties of materials used in nuclear reactors and bomb components. Linear algebra was used to represent and manipulate vectors and matrices that describe the state and behavior of these particles, while vector calculus was used to describe the rates of change of quantities, such as velocities and concentrations, as particles moved and interacted

2.Design and analysis of experiments: Linear algebra and vector calculus were used in the design and analysis of experiments conducted during the Manhattan Project. These experiments involved the measurement and analysis of data related to nuclear reactions, radiation, and other physical phenomena. Linear algebra was used to analyze and process experimental data, and vector calculus was used to

describe the behavior of quantities such as energy, momentum, and flux.

3.Simulation and prediction of physical processes: Linear algebra and vector calculus were used in computer simulations that aimed to predict the behavior of complex physical processes relevant to the development of the atomic bomb. These simulations involved solving systems of linear equations, using vector calculus to describe the behavior of quantities over time, and making predictions about the outcomes of nuclear reactions and other processes.

4.Design and analysis of nuclear systems: Linear algebra and vector calculus were used in the design and analysis of nuclear systems, such as nuclear reactors and bomb components. These applications included the calculation of criticality, which is a measure of how a nuclear system can sustain a chain reaction, and the determination of optimal configurations of nuclear materials to achieve desired outcomes. Linear algebra was used in the analysis of matrices that represent the properties of nuclear systems, and vector calculus was used to describe the behavior of quantities such as neutron flux and energy distribution.

5.Control and optimization of nuclear systems: Linear algebra and vector calculus were used in the development of control and optimization algorithms for nuclear systems. These algorithms aimed to control the behavior of nuclear systems, such as maintaining safe and efficient operation of nuclear reactors, and optimizing the performance of bomb components. Linear algebra and vector calculus were used to develop mathematical models and algorithms that described the optimal control and optimization of these systems.

Overall, linear algebra and vector calculus were essential tools in the mathematical analysis, modeling, simulation, and opti-

mization of the physical processes involved in the Manhattan Project, contributing to the successful development of the first atomic bomb.

Linear algebra can describe the state and behavior of particles in the context of the Manhattan Project. It is the concept of **state vectors** that used to represent the behavior of particles undergoing nuclear reactions.

## 1.2 An example of mentioned concepts

Linear algebra concept	Eigenvalues and eigenvectors
Equation	Schrödinger's equation
Modeling variable	Energy levels
Phenomenon	Quantum mechanical behavior of particles in a potential well

The use of linear algebra allows for the mathematical representation and manipulation of the energy levels of particles. This provides insights into the quantum mechanical behavior of particles in a potential well, and is crucial in understanding and predicting the behavior of particles in quantum systems, such as atoms, molecules, and solid-state materials.

In the field of quantum mechanics, linear algebra is used to model the energy levels of particles in a potential well, which is a fundamental concept for understanding the behavior of particles in quantum systems. Schrödinger's equation, a partial differential equation used in quantum mechanics, can be used to describe the behavior of particles in a potential well. Eigenvalues and eigenvectors from linear algebra can be used to solve this equation and obtain the energy levels of the particles.

The eigenvalues and eigenvectors of Schrödinger's equation represent the energy levels and corresponding wave functions of the particles in the potential well, respectively. The eigenvalues represent the allowed energy levels that a particle can possess in the potential well, while the eigenvectors represent the corresponding wave functions that describe the spatial distribution of the particle's probability density.

By solving Schrödinger's equation and finding the eigenvalues and eigenvectors, scientists and engineers can determine the quantized energy

levels that particles can occupy in the potential well. This directly affects their behavior, such as their stability, probability of being found in certain regions of space, and interactions with other particles.

## 2 Modeling of Temperature Field

### 2.1 Define the scalar field

Denote the temperature at any point in 3D space as  $T(x, y, z)$ . Based on the given constraints, we can use a Gaussian distribution centered at  $(0, 0, 0)$  to describe the temperature field. The equation for the 3D temperature field can be written as:

$$T(x, y, z) = T0 * \exp\left(-\frac{x^2 + y^2 + z^2}{2\sigma^2}\right)$$

where:

- $T0$  is the temperature at the center (given as 1000,000K)
- $(x, y, z)$  are the coordinates in 3D space
- $\sigma$  is the standard deviation of the Gaussian distribution, controlling the spread of the temperature field.
- $r$  is the radius of the sphere of temperature field, where  $x^2 + y^2 + z^2 = r^2$

The Gaussian distribution ensures that the temperature field is spherically symmetric, with the maximum temperature at the center and decreasing exponentially as we move away from the center along any direction.

The standard deviation  $\sigma$  controls how quickly the temperature decreases with distance from the center, with larger  $\sigma$  values resulting in slower temperature decay and a broader temperature field, and smaller  $\sigma$  values resulting in faster temperature decay and a narrower temperature field.

After using the trial-and-error method to fit the data, we have determined that  $\sigma = 330$  is the appropriate value. This allows us to approximate the temperature at  $r = 1000m$  as  $T(r = 1000m) = 10138K$ .

## 2.2 Finding the level surface

To find the radius  $R_{15000}$  where  $T = 15,000$  K, we plug in  $T = 15,000$  K into the equation  $T(x, y, z) = T_{\text{centre}} + (T_r - T_{\text{centre}}) \times e^{-r^2/\sigma^2}$  and solve for  $r$ .

$$15,000 \text{ K} = T_{\text{centre}} + (T_r - T_{\text{centre}}) \times e^{-r^2/\sigma^2}$$

Substituting  $T_{\text{centre}} = 1,000,000$  K and  $T_r = 10,000$  K from the given constraints, we get:

$$15,000 \text{ K} = 1,000,000 \text{ K} + (10,000 \text{ K} - 1,000,000 \text{ K}) \times e^{-r^2/\sigma^2}$$

Simplifying further, we have:

$$15,000 \text{ K} - 1,000,000 \text{ K} = -990,000 \text{ K} \times e^{-r^2/\sigma^2}$$

Dividing both sides by  $-990,000$  K, we get:

$$\frac{15,000 \text{ K} - 1,000,000 \text{ K}}{-990,000 \text{ K}} = e^{-r^2/\sigma^2}$$

Taking the natural logarithm of both sides, we obtain:

$$\ln \left[ \frac{15,000 \text{ K} - 1,000,000 \text{ K}}{-990,000 \text{ K}} \right] = -\frac{r^2}{\sigma^2}$$

Multiplying both sides by  $-\sigma^2$ , we get:

$$-\sigma^2 \times \ln \left[ \frac{15,000 \text{ K} - 1,000,000 \text{ K}}{-990,000 \text{ K}} \right] = r^2$$

Finally, taking the square root of both sides, we can find  $r$ :

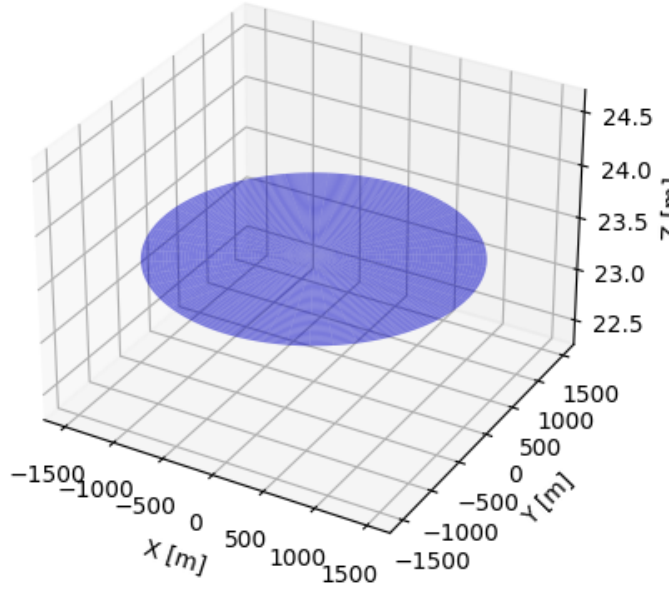
$$r = \sqrt{-\sigma^2 \times \ln \left[ \frac{15,000 \text{ K} - 1,000,000 \text{ K}}{-990,000 \text{ K}} \right]}$$

and plugged in  $\sigma = 330$ , we get  $r = 23.48m$  that is,

$$R_{15000} = 23.48m$$

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
```



图 1: The level surface at  $T = 15000\text{K}$ 

```

T_centre = 1000000
T_r = 10000 #temperature @ r=1000
sigma = 330 # degree of dispersion
T = 15000 # level surface

def get_radius(T, T_centre, T_r, sigma):
    return np.sqrt(-sigma**2 * np.log((T - T_centre) / (T_r - T_centre)))

r = np.linspace(0, 1500, 330)
theta = np.linspace(0, 2*np.pi, 330)
r, theta = np.meshgrid(r, theta)
x = r * np.cos(theta)
y = r * np.sin(theta)
z = get_radius(T, T_centre, T_r, sigma) * np.ones_like(x) # z constant

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(x, y, z, color='blue', alpha=0.5)
ax.set_xlabel('X□[m]')
ax.set_ylabel('Y□[m]')
ax.set_zlabel('Z□[m]')

```

```
ax.set_title("Level Surface for T=15000 K")
plt.show()
```

### 2.3 Surface normal via tangents of curves

Our first equation that is tangent to the level surface is

$$x^2 + y^2 = 23.48^2, z = 0$$

and the second one is

$$y^2 + z^2 = 23.48^2, x = 0$$

Circle 1 and circle 2 are intersect at point  $(0, 23.48, 0)$  and  $(0, -23.48, 0)$

#### 2.3.1 Circle 1

Assume that

$$r_1(t) = \begin{bmatrix} 23.48 \cos(t) \\ 23.48 \sin(t) \\ 0 \end{bmatrix}$$

then

$$r'_1(t) = \begin{bmatrix} -23.48 \sin(t) \\ 23.48 \cos(t) \\ 0 \end{bmatrix}$$

when  $t = \frac{1}{2}\pi$

$$r_1\left(\frac{1}{2}\pi\right) = \begin{bmatrix} 0 \\ 23.48 \\ 0 \end{bmatrix}$$

and the tangent vector of the intersect point is

$$\begin{bmatrix} -23.48 \\ 0 \\ 0 \end{bmatrix}$$

**2.3.2 Circle 2**

Assume that

$$r_2(t) = \begin{bmatrix} 0 \\ 23.48 \sin(t) \\ 23.48 \cos(t) \end{bmatrix}$$

then

$$r_2'(t) = \begin{bmatrix} 0 \\ 23.48 \cos(t) \\ -23.48 \sin(t) \end{bmatrix}$$

when  $t = \frac{1}{2}\pi$

$$r_2(\frac{1}{2}\pi) = \begin{bmatrix} 0 \\ 23.48 \\ 0 \end{bmatrix}$$

and the tangent vector of the intersect point is

$$\begin{bmatrix} 0 \\ 0 \\ -23.48 \end{bmatrix}$$

**2.3.3 Normal vector**

Assume  $\vec{N} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ . Since the normal vector should be perpendicular

to both tangent vector derived from  $\overrightarrow{r_1(t)}$  and  $\overrightarrow{r_2(t)}$

$$\vec{N} \cdot \overrightarrow{r_1'(t)} = \vec{N} \cdot \overrightarrow{r_2'(t)} = 0$$

Therefore, the only possible form for  $\vec{N}$  will be

$$\vec{N} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$$

With further normalization, the unit normal vector of the  $R_{15000}$  level surface at the intersection is

$$\begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}$$

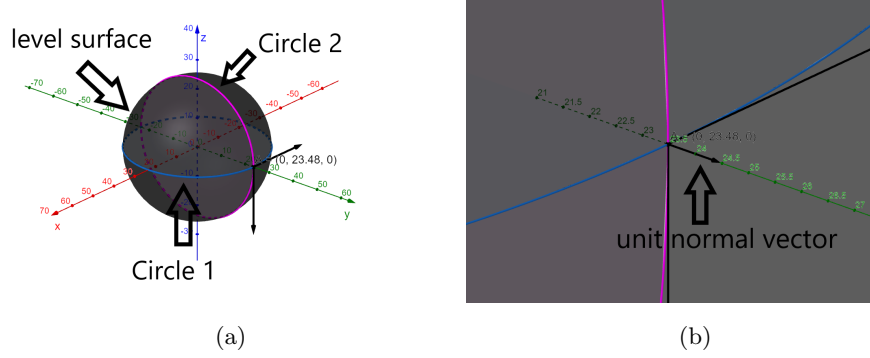


图 2: (a)The level surface, Circles 1 and 2, the intersection point, the two tangent vectors. (b)The unit normal vector.

## 2.4 Surface normal via gradient

Since the level surface is given by

$$f(\vec{r}(t)) = f(x(t), y(t), z(t)) = c$$

where  $c$  is a constant. Taking the derivative of this equation on either side, we get

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} \\ &= \frac{\partial f}{\partial x} * x' + \frac{\partial f}{\partial y} * y' + \frac{\partial f}{\partial z} * z' \\ &= \nabla f \cdot \vec{r}'(t) \\ &= 0 \end{aligned}$$

This demonstrates that  $\nabla f$  is orthogonal to  $\vec{r}'(t)$ . Furthermore, since this result holds for any possible curve  $\vec{r}_i(t)$  passing through the point of interest, the  $\nabla f$  must also be orthogonal to all the other tangent vectors derived from those curve. Therefore, it indicates that  $\nabla f$  can be seen as a normal vector at that point.

With  $f = T_0 * \exp(-\frac{x^2+y^2+z^2}{2*330^2})$ , and  $r^2 = x^2 + y^2 + z^2$ , the gradient of it should be

$$\nabla f = T_0 * \left( -\frac{x e^{-r^2/217800}}{108900} \right) \hat{i} + T_0 * \left( -\frac{y e^{-r^2/217800}}{108900} \right) \hat{j} + T_0 * \left( -\frac{z e^{-r^2/217800}}{108900} \right) \hat{k}$$

Inserting the intersection point  $(0, 23.48, 0)$ , the outcome will be

$$\nabla f\left(\begin{bmatrix} 0 \\ 23.48 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -215.0655736 \\ 0 \end{bmatrix}$$

which can be normalized into

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

## 2.5 Conclusion

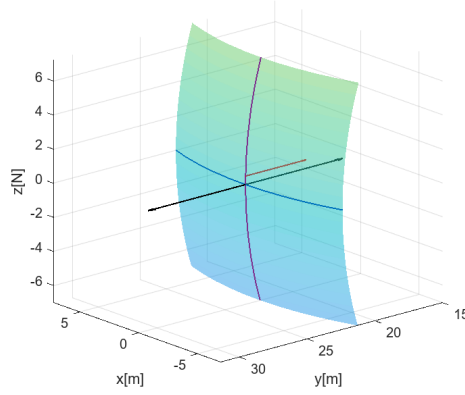


图 3: Normal vectors derived from tangents(black) and gradient(red)

Above is a figure showing two normal vectors that were obtained from different method. The black vectors were created by constraining normal vector to be orthogonal to tangent vectors from two circles. On the other hand, the red vector is the gradient at the intersection point. (We move it slightly upward, so that it could be seen more clearly.) It points inward, because that is the main direction for the level surface to increase in *height*.