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[illegible name]

1. gauss - Jordan elimation (rank of rows = rank of columns)

$$A=\begin{bmatrix}1&3&2&1\\2&0&1&-1\\-1&1&0&1\end{bmatrix}\rightarrow\begin{bmatrix}1&3&2&1\\0&2&1&1\\0&4&2&2\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1&3&2&1\\0&2&1&1\\0&0&0&0\end{bmatrix}$$ row echelon form

$$\rightarrow\begin{bmatrix}1&1&1&0\\0&2&1&1\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&\frac{1}{2}&-\frac{1}{2}\\0&1&\frac{1}{2}&\frac{1}{2}\\0&0&0&0\end{bmatrix}$$

rank = 2

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2. $\underset{m\times n\ n\times m}{AB} \Rightarrow m\times m$

$\underset{n\times m\ m\times n}{BA} \Rightarrow n\times n$

nonzero eigenvalues

$\Rightarrow \lambda \neq 0$

同理 let $\lambda$ be an non-zero eigenvalue of BA

non zero $v$

$BAv=\lambda v$

$ABAv=\lambda Av$

$\Rightarrow (AB)(Av)=\lambda(Av)$

$Av\neq 0$ 否则 $BAv=\lambda v=0 \rightarrow \lambda=0$ (不合)

$\Rightarrow$ hence $\lambda$ is a non zero eigenvalue of AB

Let $\lambda$ to be an eigenvalue of AB

eigenvalues $\det(AB-\lambda I)=0$

non-zero $v$

$ABv=\lambda v$

$BABv=\lambda Bv$

$\Rightarrow (BA)(Bv)=\lambda(Bv)$

$Bv\neq 0$ 否则 $ABv=\lambda v=0\rightarrow\lambda=0$ (不合假設)

$\Rightarrow$ hence $\lambda$ is a non-zero eigenvalue of BA

$\Rightarrow$ AB & BA have the same non-zero eigenvalues.

3.

a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 2, nullity = 1

basis for range $\rightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

basis for null space.

$$A\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 = 0$

$x_3 = 0$

$x_2 = t \in \mathbb{R}$

$\Rightarrow x_1 = -2t \in \mathbb{R}$

$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ basis of null space

b) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 4 & 0 \\ 1 & 0 \end{bmatrix} \longrightarrow \underset{4\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$ rank = 2, nullity = 0

$x_1 = 0$

$x_2 = 0$

$\text{null}(A) = \vec{0}$

↑ 僅有 zero vector

basis for range $\begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix}$

rank = 2

nullity = 2

(4-2=2)

basis for range

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underset{2\times 4}{\begin{bmatrix} 1 & 0 & 4 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

basis for null space

$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$

$-(t + 4s) = x_4$

$x_3 = s \in \mathbb{R}$

$x_1 + 4x_3 + x_4 = 0$

$-x_1 + x_2 = 0 \rightarrow x_1 = x_2 = t \in \mathbb{R}$

3. d)

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 11 & 0 & 0 & 0 \end{bmatrix}$$

rank = 3

nullity = 1

basis for range = $\left\{ \begin{bmatrix} 0 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\downarrow$

$$\underset{3\times 4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$

basis for ~~nullity~~ = $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

5. 矩陣對角化 → 用 eigenvector (basis) 來做 similiar transform

a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$\det\left(\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}\right) = 0 \Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$$

$$\lambda = 0 \text{ or } +5$$

$$A\vec{x} = \lambda\vec{x} \rightarrow (A - \lambda I)\vec{x} = 0$$

$$\lambda = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\lambda = 5 \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$2x_1 - x_2 = 0$$

2D, 2個 eigenvector ⇒ 可做 eigenbasis (diagonalizable)

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, \quad T = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 \end{matrix} \\ \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \end{array}$$

SVD

$AT = TD$

$\Rightarrow A = TDT^{-1}$

$\downarrow$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

b)

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$(A-\lambda I)\vec{x}=0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0 \qquad (3-\lambda)(-\lambda)+2=0$$

$$\lambda = 1 \vee 2$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\lambda=1 \Rightarrow \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{V_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda=2 \Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{V_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\supset D$ 且 2个 eigenvector

$\Rightarrow$ diagonalizable

$$A = TDT^{-1}$$

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

A

c)

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 3 & -1 \\ 2 & 0 \end{matrix}} \end{bmatrix}$$

$\hookrightarrow$ 2个 block 分别做 左上 同 a) $\Rightarrow D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ 為 b) 之 transpose，而計算後 characteristic eqn 不變 $\Rightarrow D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$\Rightarrow J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

d) $\begin{bmatrix} 2 & 11 \\ 0 & 2 \end{bmatrix}$ 已為上三角 matrix $\rightarrow \lambda = 2, 2$ algebraic multiplicity $= 2$

$$\begin{bmatrix} 0 & 11 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0 \quad \vec{V_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\leftarrow$ NOT diagonalizable

$\rightarrow$ Jordan form

geometric multiplicity $= 1$

即 $\text{null}(A-\lambda I) = 1$

[illegible] eigenvec

$$J = \begin{bmatrix} \overset{\lambda}{2} & 1 \\ 0 & \underset{\lambda}{2} \end{bmatrix}$$

無法对角化

generalized eigenvector

$$\begin{bmatrix} 0 & 11 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x_2 = \frac{1}{11} \Rightarrow \underset{\text{choose}}{\vec{V_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

6. $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ max column sum

a) $\|M\|_{1/1}$ column sum 看哪个 column 有和之 max.

$\Rightarrow$ col 1: $1+4=5$ col 2: $2+5=7$ col 3: $3+6=9$ $\Rightarrow \|M\|_{1/1} = 9$.

b) 贴在附件.

d)

c) $\|M\|_{\infty/\infty}$ max row sum

rol. 1 $\Rightarrow 1+2+3=6$

row 2 $\Rightarrow 4+5+6=15$

$\Rightarrow \|M\|_{\infty/\infty} = 15$

7. Show $\frac{1}{n}\|V\|_1 \le \|V\|_2 \le \|V_1\|$

1) upper bound

$\|V\|_2 \le \|V_1\|$

由科西不等式 $\|\vec{a}\cdot\vec{b}\| \le \|\vec{a}\|\|\vec{b}\|$

$\sum_i^n |V_i \cdot V_i^T| \le \left(\sum_i |V_i|\right)\left(\sum_i |V_i|\right)$

$\Downarrow$

$$\|V\|_2 = \sqrt{\sum_{i=1}^n |V_i|^2} = \sqrt{\sum_{i=1}^n |V_i \cdot V_i^T|} \le \sum_{i=1}^n |V_i| = \|V\|_1$$

$\Rightarrow \|V\|_2 \le \|V\|_1$

2) lower bound $\frac{1}{n}\|V\|_1 \le \underbrace{\frac{1}{\sqrt{n}}\|V\|_1 \le \|V\|_2}_{\text{show}}$

AM-QM mean inquality

$$\Big(\sum_{i=1}^n x_i \cdot 1\Big)^2 \underset{\text{科西}}{\le} \sum_{i=1}^n x_i^2 \sum_{i=1}^n 1^2 = n\sum_{i=1}^n x_i^2$$

$$\Rightarrow \left(\frac{\sum_{i=1}^n x_i}{n}\right)^2 \le \frac{\sum_{i=1}^n x_i^2}{n}$$

$\Rightarrow$ 则 $\frac{1}{n}\|V\|_1 \le \|V\|_2$ 成立

$$\frac{1}{n}\sum_{i=1}^n |V_i|^2 \ge \left(\frac{1}{n}\sum_{i=1}^n |V_i|\right)^2$$

$\Rightarrow$ 故命题成立

$$\underset{\text{同乘}n}{\Rightarrow} \sum_{i=1}^n |V_i|^2 \ge \frac{1}{n}\left(\sum_{i=1}^n |V_i|\right)^2 \underset{\text{2边}\sqrt{\ }}{\longrightarrow} \overset{\|V\|_2}{\overset{\shortparallel}{\sqrt{\sum_{i=1}^n |V_i|^2}}} \ge \frac{1}{\sqrt{n}}\sum_{i=1}^n |V_i| = \frac{1}{\sqrt{n}}\|V\|_1$$

$\Rightarrow \|V\|_2 \ge \frac{1}{\sqrt{n}}\|V\|_1 \ge \frac{1}{n}\|V\|_1$

8.

$$G(s) = \frac{\overset{\downarrow b_1}{2}s + \overset{\downarrow b_0}{3}}{s^2 + \underset{\uparrow a_1}{5}s + \underset{\uparrow a_0}{6}} + \underset{\substack{\uparrow \\ D=0}}{0}$$

find state eqns (a) controllable canonical form

a)

$$= \frac{Y(s)}{U(s)}$$

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [b_0 \; b_1], \; D = 0$$

$$\Rightarrow Y(s)(s^2+5s+6) = (2s+3)U(s)$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} y'' + 5y' + 6y(t) = 2u'(t) + 3u(t)$$

let $x_1 = y$

$x_2 = y'$

$\Rightarrow y'' = \dot{x}_2$

$$y'' = 2u' + 3u - 5y' - 6y$$

$$= 2u' + 3u - 5x_2 - 6x_1$$

$$= \dot{x}_2$$

$$\dot{x}_1 = y' = x_2$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -6x_1 - 5x_2 + 2u' + 3u \end{cases}$$

代入

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{[3 \;\; 2]}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \ddot{D}$$

$D = 0$

↑ no direct feedthrough term

$$sX_1(s) = \overbrace{(2U - 5X_1(s))}^{} + X_2(s)$$

$$sX_2(s) = 3U - 6Y \rightarrow X_1(s)$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} \begin{cases} \dot{x}_1(t) = 2u - 5x_1(t) + x_2(t) \\ \dot{x}_2(t) = 3u - 6x_1(t) \end{cases}$$

$$\begin{cases} X_1(s) = Y(s) \\ X_2(s) = \frac{1}{s}(3U - 6Y) \end{cases}$$

$\Rightarrow x_1(t) = y(t)$

$\dot{x}_2(t) = 3u(t) - 6y(t)$

b) observable form

$$\frac{Y(s)}{U(s)} = G(s) = \frac{\frac{2}{s} + \frac{3}{s^2}}{1 + \frac{5}{s} + \frac{6}{s^2}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u$$

$$\vec{y} = [1 \; 0] \vec{x}$$

$$\underbrace{= \frac{1}{s}\left((2U - 5Y) + \overbrace{\frac{1}{s}(3U - 6Y)}^{X_2(s)}\right)}_{X_1(s)}$$

$$\Rightarrow Y(s)\left(1 + \frac{5}{s} + \frac{6}{s^2}\right) = U(s)\left(\frac{2}{s} + \frac{3}{s^2}\right)$$

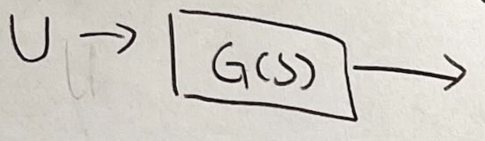
$$\Rightarrow Y(s) = -\frac{5}{s}Y(s) - \frac{6}{s^2}Y + \frac{2}{s}U + \frac{3}{s^2}U = \frac{1}{s}(2U - 5Y) + \frac{1}{s^2}(3U - 6Y)$$

# 8. c) diagonal canonical form

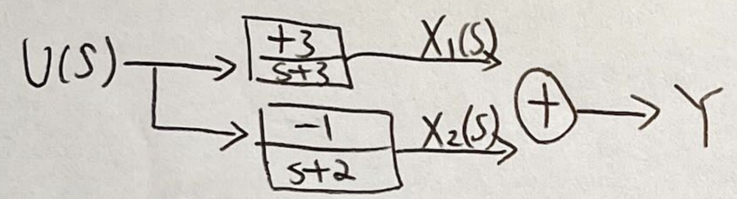
$$G(s) = \frac{2s+3}{s^2+5s+6} = \frac{2s+3}{(s+3)(s+2)}$$ ← poles at $-3$ & $-2$ not repeat

$$= \frac{\overset{k_1}{+3}}{\underbrace{s+3}_{\to p_1}} + \frac{\overset{k_2}{-1}}{\underbrace{s+2}_{\to p_2}}$$

$$A = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$$




$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$




$$C = [\underset{\downarrow \, 3}{k_1} \ \ \underset{\downarrow \, -1}{k_2}] \qquad D = 0.$$

$$\frac{X_1}{U} = \frac{+3}{s+3} \Rightarrow (s+3)X_1 = +3U \Rightarrow sX_1 = +3U - 3X_1$$

$$\frac{X_2}{U} = \frac{-1}{s+2} \Rightarrow (s+2)X_2 = -U \Rightarrow sX_2 = -U - 2X_2$$

$$\mathcal{L}^{-1} \downarrow \quad \dot{x}_1 = +3u(t) - 3x_1$$

$$\dot{x}_2 = -u - 2x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [3 \ \ -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

⟺ 和公式解等效

(有两种表示法)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} +3 \\ -1 \end{bmatrix} u$$

$$y = [1 \ \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$