Project 1 – Modeling of Pandemic and Oscillator

Logistics:

- This project is to be completed in a group of ⩽ 4 students. One submission per group, uploaded onto e3. Please list all members' student ID.
- Submission includes: (i) A written report, (ii) Any associated coding or Excel files.
- Report must:
  - Be written in English
  - All texts and equations must be typed, all graphs computer-generated. Hand-writing or photographed papers are not accepted.
  - Report must use the same section titles, numbering format, font and font-size as this document. It must be easily readable.
  - Questions that must be answered have been highlighted in blue below.
- This project is due by 5pm, Nov 21 2021. Late submission = zero points awarded (Don't wait until the last minute, upload 1-2 days early).
- Points will be deducted if academic dishonesty is discovered, or if it is deemed you have addressed this assignment in a manner that conflicts with the spirit of learning.
- Ask for clarification if needed.

Overview:

This project covers a very famous 1st-order ODE: the population model, which tries to model the increase/decrease of a population over time. The model is applicable to modeling number of people living in a city, the number of people contracted in a pandemic… or even the number of people having watched a viral video.

Part 2 of the project covers constant-coefficient, homogeneous 2nd-order linear ODE, which is often used to model mass-spring-damper systems and their behavior.

Section 1: Population Model:

In Topic 1.4 we introduced linear 1st-order ODE and their solution. We also mentioned a special case of nonlinear ODE called "Bernoulli Equation" that can be reduced to linear form:

$$y' + p(x)y = r(x)y^a$$

A famous version of Bernoulli Equation is the "logistics equation" (or Verhulst equation):

$$y' = Ay - By^2$$

This equation was created by Pierre Francois Verhulst (see image, source: de:Bild:P.-F. Verhulst.jpg, public domain), a Belgian mathematician and doctor in number theory, to model population dynamics.

We will investigate the behaviors of this equation.

What you need to do:

1. Show step-by-step proof that $y(t) = \frac{1}{ce^{-At} + B/A}$ is a viable solution to the logistic equation.
2. If $B = 0$, the ODE transforms from a Bernoulli equation to a normal linear 1st-order homogeneous ODE. Show step-by-step process of using the separation of variables method to derive its solution. This $B = 0$ case is actually called a "Malthusian growth model."
3. Compare the solution from (2) to the solution in (1) for $B = 0$. Describe: are they the same?
4. $-By^2$ in the ODE is called a "braking term." Just by looking at $y' = Ay - By^2$, explain why it might be called "braking." I.e. what effect does it have on $y(t)$ if $B > 0$?
5. Initial value problem: Derive an expression for determining c based on $y(0)$.

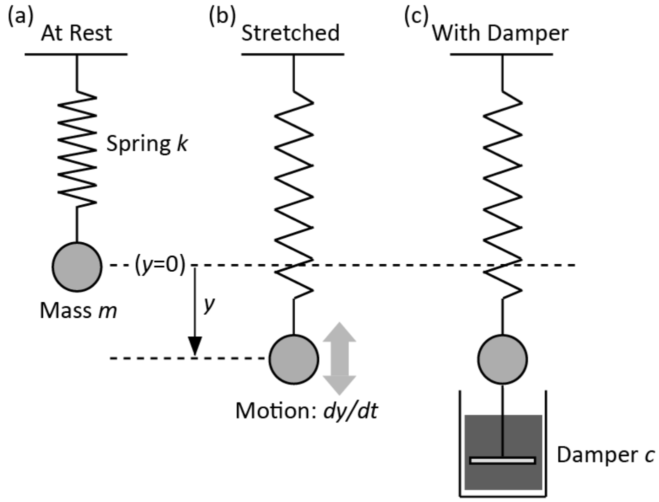
6. For $A = 0.5$ and $y(0) = 1000$, plot the Malthusian model $y(t)$ for $0 \leq t \leq 20$.
7. On the same plot, for the same A and $y(0)$, plot the Verhulst model for $B = 0.0001$. Comment on the effect of B. You might need to use log scale for *y*-axis.

8. On a new plot, for $\frac{A}{B} = 4$, show $y(t)$ for $0 \leq t \leq 5$ for $y(0) = 0, 1, 2, 3, 4, 5, 6, 7, 8$. Comment on why all the curves seem to converge on one value given enough time.

9. Application:
   i. Go to *https://covidtracking.com/data/download* and download a copy of "summary data for the United States". This file gives the daily stats and cumulative positive covid cases in the US up to Mar 7, 2021. Look at column called "positive" for cumulative cases.
   ii. You can compare against *https://www.worldometers.info/coronavirus/country/us/* to make sure you have the right data.
   iii. Plot the data for cumulative cases over time (cases vs. day since beginning).
   iv. Suggest in your report: whether the Malthusian or Verhulst model is more appropriate for this data. Why?
   v. Try to find A and B constants which best fit this data. On the same graph, plot result from your model on top of the real data.
   vi. Comment: whether the curves appear reasonably similar?
   vii. If you look at more recent data (see below) you will notice that cumulative cases are rising again. This does not follow Malthusian or Verhulst model. Explain possible reasons why.

** Note: all graphs in this report ***must*** have labelled axis! Labels must include name of variable, symbol and unit when applicable. Example axis label: velocity, *V* (*m/s*).
Graph should also have major grid-lines.
Font sizes, line thickness and other visual elements on the graph must be formatted for easy reading.
Examples about how to create graphs: https://www.clips.edu.au/displaying-data/
Badly-made graphs will have points deducted.
It does not matter if your graph is created in Excel, Matlab or other software.

<u>Section 2: Mass-Spring-Damper Systems:</u>



Consider the system shown above:

(a) A metal ball with mass m (*kg*) hangs from a spring with spring-constant k (N/m). When the system is at rest and the spring and mass are in equilibrium, we denote the position of the ball as $y = 0$.

(b) I pull the ball downward by some distance $y > 0$. The spring is now stretched and is exerting additional force F_{spring} on the ball. If I let go, the ball will be pulled upwards, and will have some motion with speed $\frac{dy}{dt}$ (m/s).

(c) Now I also connect a damper to the ball. The damper, like your car's shock-absorber, is usually a cylinder of viscous oil. A flat-plate connected to the ball moves in the oil and dissipates kinetic energy through friction against the oil ("fluid viscous loss"). I.e. damper will slow down motion.

What you need to do:

1. <u>Show:</u> step-by-step creation of a 2nd-order ODE model for the mass-spring system (a)(b). Hint:
   i. Start with Newton's 2nd law: $F = my''$.
   ii. Replace F with F_{spring}.
   iii. Replace F_{spring} with appropriate combination of k and y.
   iv. Pay attention to direction of motion and forces (i.e. negative vs. positive directions).

2. <u>Show:</u> step-by-step creation of an ODE model for the mass-spring-damper system (c). Hint:
   i. The damper also exert a force F_{damper}.
   ii. The force is proportional to speed: $F_{damper} = cy'$.

3. The system (a)(b) in the provided image is called an undamped system. We want to investigate how it behaves.
   i. <u>Proof</u> in *step-by-step derivation* that for positive k and m, the general solution is: $y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$
   ii. $\omega_0 = \sqrt{k/m}$ is called the "natural frequency" of the system.
   iii. <u>Initial value problem:</u> <u>write</u> A and B in terms of initial conditions $y(0)$ and $y'(0)$.
   iv. <u>Effect of displacement:</u> for $m = 1kg$, $k = 10\frac{N}{m}$, <u>show</u> on the same plot: $y(t)$ curves for $y(0) = 0.5, 1, 2, 4m$, all at $y'(0) = 0$. Let time domain be $0 \leq t \leq 10s$. <u>Comment</u> on whether the period

of oscillation on the plot matches $\omega_0 = \sqrt{k/m}$ (i.e. you need to calculated ω_0 by hand for each case, and then also estimate ω_0 from looking at and labeling the plot).

v. Effect of speed: for the same mass and spring, and for $y(0) = 0.5m$, show what happens when there are initial speeds: $y'(0) = -0.5, 0, 0.5, 1, 2m/s$. Does the oscillation frequency change?

vi. Effect of mass: for $y(0) = 0.5m$, $y'(0) = 0$ and spring $k = 10\frac{N}{m}$. Show what happens when $m = 1, 2, 4, 8kg$. Calculate and comment: does the graph's frequencies match $\omega_0 = \sqrt{k/m}$?

vii. Effect of spring stiffness: $y(0) = 0.5m$, $y'(0) = 0$ and mass $m = 1kg$. Show what happens when $k = 10, 20, 40, 80\frac{N}{m}$. Calculate and comment: does the graph's frequencies match $\omega_0 = \sqrt{k/m}$?

4. Next, we want to investigate the effects of damper; i.e. image (c):
    i. Proof that with appropriate value of c (i.e. $c^2 > 4mk$, $c^2 = 4mk$, $c^2 < 4mk$) you can get distinct roots, repeated roots and complex roots from the model ODE.
    ii. Write the general solution for all three cases above (show step-by-step).
    iii. For $m = 1kg$, $k = 10N/m$, find a value of c that will result in repeated roots. (Include the correct units).
    iv. The case of repeated roots is called "Critical Damping", while the distinct roots case is called "Overdamped" and the complex roots "Underdamped". To find out why, make a plot of $y(t)$ for $y(0) = 0.5m$, $y'(0) = 0m/s$ for $c = c_{critical}$ to achieve critical damping. Show $y(t)$ for $0 \leq t \leq 20s$.
    v. On the same plot, for the same mass, spring and initial conditions, show $y(t)$ for $c = 0.25c_{critical}, 0.5c_{critical}, 2c_{critical}, 4c_{critical}$.
    vi. Comment on the physical meaning and behavior of Underdamping, Critical Damping and Overdamping.
    vii. Proof and comment: if I use a stiffer spring, will I need more or less damper force to maintain critical damping?
    viii. Proof and comment: if I use a heavier mass, will I need more or less damper force to maintain critical damping?
5. Compare a train and a car. Which one should use a bigger damper. Why?
6. Compare a sports car and normal family car. Which one should use a stiffer spring. Why?