Project 2 - System of ODE’s and Phase Plane

Group members

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Section 1: Observing ODE in Real Life

Describe and construct the model

This project will try to model the temperature variation of a system as a function of time.

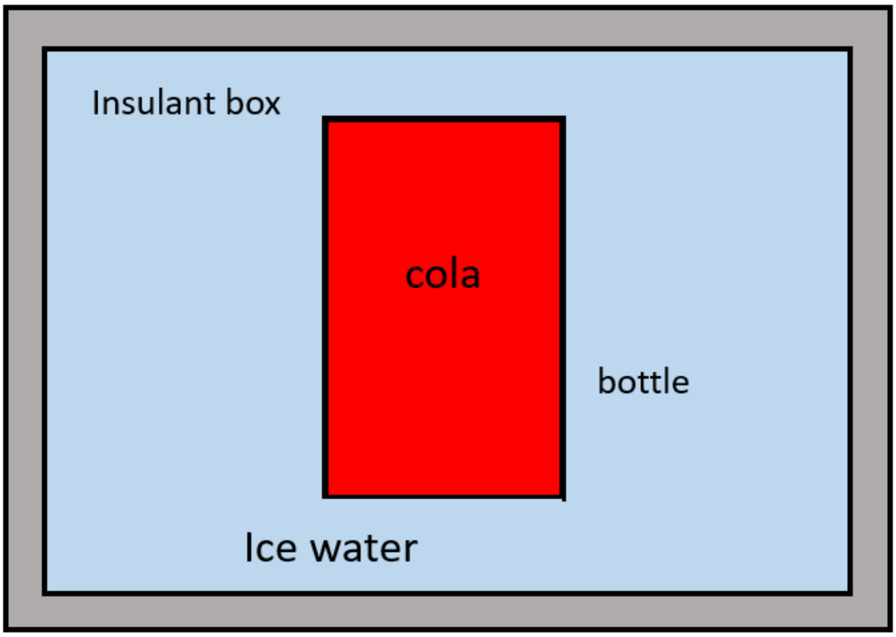

Figure 1

As shown in the Figure 1, warm cola is cooled by the cold water inside an insulant box. The equation, then used here, is called **Newton’s law of cooling**, which states that the rate of heat transfer from a surface toward fluid can be described as

$$\dot{Q}_{conv} = hA(T_s - T_f)$$

where $\dot{Q}_{conv}$ is the rate of heat transfer out of the surface, A is the surface area where convection happens, T_s is the temperature of the surface, T_f is the bulk fluid temperature *away from the surface*, and h is the **convection heat transfer coefficient**, an experimentally measured parameter. After acquiring the heat transfer rates, the combination of them with **specific heat capacity** c and **mass** m will give us the temperature with respect to time.

First, to simplify the question, we assume convection happen under these circumstances:

1. From bottle to cola
2. From bottle to water

The relation of heat transfer will therefore be

$$\dot{Q}_C = -\dot{Q}_{CB}$$
$$\dot{Q}_B = (\dot{Q}_{CB} + \dot{Q}_{BW})$$
$$\dot{Q}_W = -\dot{Q}_{BW}$$

where C, B, W stand for cola, bottle, and water. Expend the formula of the specific heat capacity, we will get

$$c = \frac{Q}{m \triangle T} \Rightarrow \triangle T = \frac{1}{mc} Q \Rightarrow \frac{dT}{dt} = \frac{1}{mc} \dot{Q}$$

Combine them together, the system can therefore be constructed as

$$\dot{C} = -h_C A_{BC}(B - C)/(m_C \times c_C)$$
$$\dot{B} = \left(h_C A_{BC}(B - C) + h_W A_{BW}(B - W)\right)/(m_B \times c_B)$$
$$\dot{W} = -h_W A_{BW}(B - W)/(m_W \times c_w)$$

In a more compact form will it be

$$\begin{bmatrix} \dot{C} \\ \dot{B} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} k_{c1} & k_{c2} & 0 \\ k_{b1} & k_{b2} & k_{b3} \\ 0 & k_{w2} & k_{w3} \end{bmatrix} \times \begin{bmatrix} C \\ B \\ W \end{bmatrix}$$

The value of each property are listed below

- h_{BW}: 200 W/m^2k , aluminum canned, ice water[1]
- h_{BC}: 200 W/m^2K, aluminum canned, a[2]
- Dimensions of cola can
  - Diameter: 65 mm; height: 120mm [3]
  - Thickness: 0.1 mm [4]
- Aluminum at 275K [5]
  - Density: 2700 kg/m^3

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1 William M. Clark, Ryan C. Shevlin 及 Tanya S. Soffen, 作者, 「Heat Transfer in Glass, Aluminum, and Plastic Beverage Bottles」, *Chemical Engineering Education* 44, 期 4 (2010 年 9 月 1 日): 253–61.

2 Clark, Shevlin 及 Soffen.

3 「所有的易拉罐裝可樂啤酒罐尺寸都一樣嗎」, 引見於 2023 年 1 月 4 日, https://www.doyouknow.wiki/a/202103/134291.html.

4 「The surprising science behind the aluminum soda can」, *Christian Science Monitor*, 引見於 2023 年 1 月 4 日, https://www.csmonitor.com/Science/Science-Notebook/2015/0414/The-surprising-science-behind-the-aluminum-soda-can.

5 Yunus A. Cengel 及 Michael A. Boles, 作者, *Thermodynamics: An Engineering Approach*, 9 本 (McGrawHill, 2020).

- specific heat capacity c_B : 0.8805 kJ/kg-K
- mass m_B 8.408*10^(-3) kg

- Cola
  - Volume: 330ml
  - Density: 1.042g/mL[6]
  - specific heat capacity c_C : 3.90 kJ/kg*K[7]
  - mass m_c 0.343386 kg[8]
- Dimension of the insulant box
  - 10 times the dimension of the cola can
- Water at 0℃[9]
  - Density 1000 kg/m^3
  - specific heat capacity c_W 4.22 kJ/kg-K
  - mass m_W 126.75 kg

Section 2: Solving the ODE via Eigenvalue Techinque

1.Solve system of ODE using the eigenvalue technique.

We can see that this is an eigenvalue problem, so we can solve it with a standard technique. The matrix above can also be written into a vector form as

$$\overrightarrow{y'}(t) = [\mathrm{A}]\ \vec{y}(t)$$

assume

$$\vec{y} = \vec{k}e^{\lambda t}$$

and we get

$$\overrightarrow{y'} = \lambda\vec{k}e^{\lambda t}$$

plug into the equation gives

$$e^{\lambda t}([\mathrm{A}] - \lambda \mathrm{I})\vec{k} = 0$$

Exponential term is not zero, and we don't want $\vec{k}$ to be a zero vector, which gives no new information, thus det([A] − λI) has to be zero, which means the dimension of

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6 「Density Column of Coke and Diet Coke | Department of Chemistry | University of Washington」, 引見於 2023 年 1 月 4 日, https://chem.washington.edu/lecture-demos/density-column-coke-and-diet-coke.

7 「R09 SI:Thermal Properties of Foods.pdf」, 引見於 2023 年 1 月 4 日, https://www.cae.tntech.edu/~jbiernacki/CHE%204410%202016/Thermal%20Properties%20of%20Foods.pdf.

8 {Citation}

9 Yunus A. Cengel 及 Michael A. Boles, *Thermodynamics: An Engineering Approach*.

vector $\vec{k}$ has been changed due to the transformation. So there's no inverse matrix for this problem. Instead, we use augmented matrix method.

$$\begin{vmatrix} k_{c1} - \lambda & k_{c2} & 0 \\ k_{b1} & k_{b2} - \lambda & k_{b3} \\ 0 & k_{w2} & k_{w3} - \lambda \end{vmatrix} = 0$$

With the given data, we can calculate that:

$$A_{BC} = 0.031141 \ \mathrm{m}^2$$
$$A_{BW} = 0.030984 \ \mathrm{m}^2$$

$$\begin{bmatrix} k_{c1} & k_{c2} & k_{c3} \\ k_{b1} & k_{b2} & k_{b3} \\ k_{w1} & k_{w2} & k_{w3} \end{bmatrix} = \begin{bmatrix} -4.65 & 4.65 & 0 \\ 842 & -1680 & 838 \\ 0 & 0.0116 & -0.0116 \end{bmatrix}$$

$$unit: [\ \frac{1}{s}\ ]$$

Plug into the determinant gives:

$$-\lambda^3 - 1684.66\lambda^2 - 3906.5\lambda = 0$$

Use computer to calculate λ gives:

$$\lambda_1 = 0.0000000$$
$$\lambda_2 = -2.3220763$$
$$\lambda_3 = -1682.3395$$

Then, we can use augmented matrix method to get the corresponding eigenvector to each eigenvalue.

for $\lambda_1 = 0$

$$\vec{k}_1 = < 1, 1, 1 >$$

for $\lambda_2 = -2.3220763$

$$\vec{k}_2 = < -398, -199, 1 >$$

for $\lambda_3 = -1682.3395$

$$\vec{k}_3 = < 402, -145028, 1 >$$

Put all the pieces together we get

$$\vec{y}(t) \ = \ c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-398\\-199\\1\end{bmatrix}e^{-2.3220763t} + c_3\begin{bmatrix}402\\-145028\\1\end{bmatrix}e^{-1682.3395t}$$

When time t=0, the initial conditional will be

$$\vec{y}(0) = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-398\\-199\\1\end{bmatrix} + c_3\begin{bmatrix}402\\-145028\\1\end{bmatrix}$$

2. Explore the particular solutions.

i.

Cola in the can be 0~100 degree Celsius before it is boiled (283.15K~373.15K)

Cola's boiling point is slightly higher than 373.15K

Cola can (made by Aluminum) has the identical temperature with Cola.

Iced Water is at 0℃(273.15K)

For Cola is at 283.15K

$$\vec{y}(0) = \begin{bmatrix}283.15\\283.15\\273.15\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-398\\-199\\1\end{bmatrix} + c_3\begin{bmatrix}402\\-145028\\1\end{bmatrix}$$

$$\mathrm{D} = \begin{vmatrix}1 & -398 & 402\\1 & -199 & -145028\\1 & 1 & 1\end{vmatrix} = 57946771$$

$$D_x = \begin{vmatrix}283.15 & -398 & 402\\283.15 & -199 & -145028\\273.15 & 1 & 1\end{vmatrix} = 15829616789$$

$$D_y = \begin{vmatrix}1 & 283.15 & 402\\1 & 283.15 & -145028\\1 & 273.15 & 1\end{vmatrix} = -1454300$$

$$D_z = \begin{vmatrix}1 & -398 & 283.15\\1 & -199 & 283.15\\1 & 1 & 273.15\end{vmatrix} = -1990$$

$$c_1 \approx 273.2 = \frac{D_x}{D}$$

$$c_2 \approx -0.0251 = \frac{D_y}{D}$$

$$c_3 \approx 0.00003434 = \frac{D_z}{D}$$

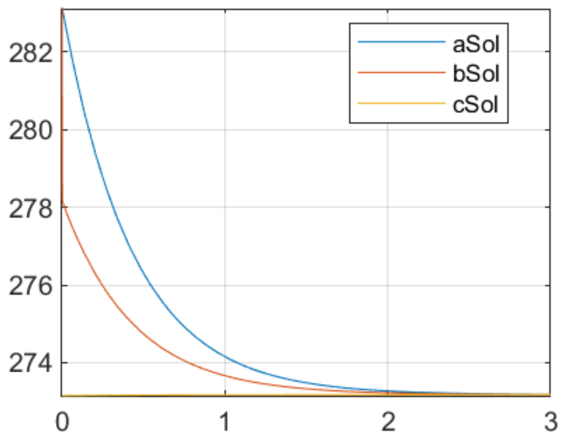
$$\vec{y}(t) = 273.2\begin{bmatrix}1\\1\\1\end{bmatrix} - 0.0251\begin{bmatrix}-398\\-199\\1\end{bmatrix}e^{-2.3221t}$$

$$+\,0.00003434\begin{bmatrix}402\\-145028\\1\end{bmatrix}e^{-1682.3395t}$$

ii.

When system's action approach equilibrium, it spends Δt.

$\vec{y_1}(\Delta t) \approx \vec{y_2}(\Delta t) \approx \vec{y_3}(\Delta t)$ (temperature)



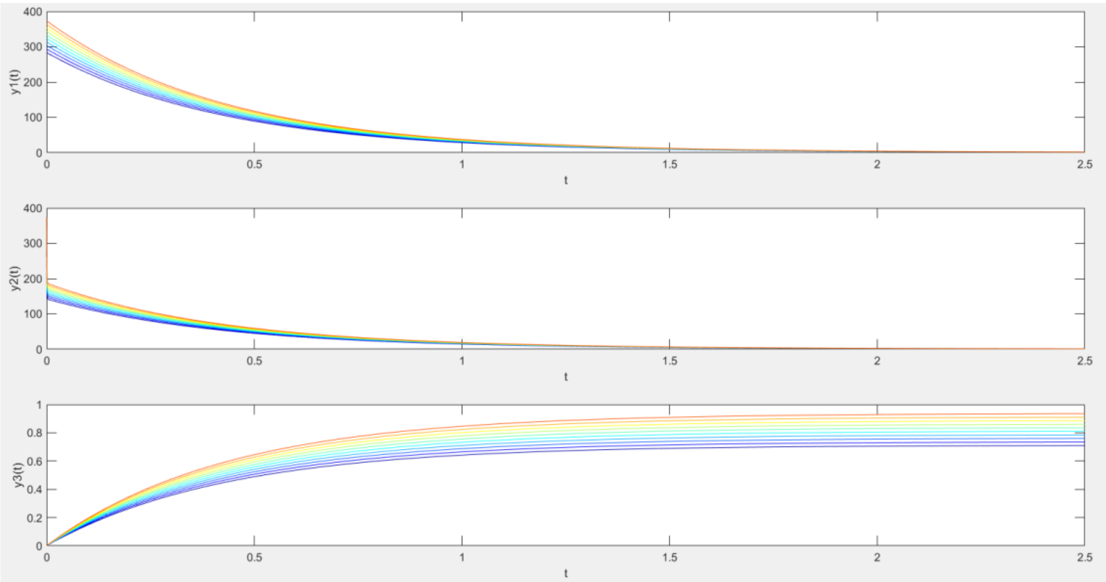
Matlab code:

```
syms a(t) b(t) c(t)

A = [-4.65 , 4.65 , 0 ; 842 , -1680 , 838 ;  0 , 0.0116 , -0.0116 ];
Y = [a;b;c];
odes = diff(Y) == A*Y;

cond1 = a(0) == 283.15;
cond2 = b(0) == 283.15;
cond3 = c(0) == 273.15;
conds = [cond1, cond2, cond3];

[aSolt(t),bSolt(t),cSolt(t)] = dsolve(odes, conds);

%%plot
fplot(aSolt,[0,3])
hold on
fplot(bSolt,[0,3])
hold on
fplot(cSolt,[0,3])
grid on
legend('aSol','bSol','cSol','Location','best')
```

For the initial condition $\vec{y}(0) = \begin{bmatrix} 283.15 \\ 283.15 \\ 273.15 \end{bmatrix}$, system spend approximately 2.4 second to approach equilibrium.

iii.

Cola in the can be 0~100 degree Celsius before it is boiled (283.15K~373.15K)
Cola can (made by Aluminum) has the identical temperature with Cola. (283.15K~373.15K)
Iced Water reamin at 0℃(273.15K)
$\vec{y_1}(t)$ represent the temperature of cola that is varied with time
$\vec{y_2}(t)$ represent the temperature of cola that is varied with time
$\vec{y_3}(t)$ represent the temperature of cola that is varied with time
t(second) represent the system's time from cola is in the box



Matlab code:

```
% Set the initial condition ranges and step size

tic =0;
tfinal = 2.5;

% Create a color map for the initial conditions
cmap = jet(10);

% Solve the ODE system for 10 different initial conditions
for i = 1:10;length(i)
x0 = 283.15:10:373.15;
y0 = 283.15:10:373.15;

    % Solve the ODE using the ode45 function
    [t,y] = ode45(@ODE, [tic,tfinal],[x0(i), y0(i), 0]);
    % Plot the solution for each variable in a separate subplot
    subplot(3, 1, 1);
    xlabel("t")
    plot(t, y(:,1), 'Color', cmap(i,:));
    ylabel("y1(t)")
    hold on
```

```
xlabel("t")
    subplot(3, 1, 2);
    plot(t, y(:,2), 'Color', cmap(i,:));
    ylabel("y2(t)")
    hold on

    subplot(3, 1, 3);
     xlabel("t")
    plot(t, y(:,3), 'Color', cmap(i,:));
    ylabel("y3(t)")
    hold on
end

% Define the ODE system
```

```
function dydt = ODE(~, y)
   dydt = [y(1).*(-4.65)+y(2).*(4.65);y(1).*(842)+y(2).*(-1680)+y(3).*(838);y(2).*(0.0116)+y(3).*(-0.0116)];
end
```