Simulating the Wave Equation

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Abstract

For research on gravitational waves it is necessary to simulate waves propagating through time and space, following the laws of Maxwell's equations. To get insights into the fundamental behavior, one first has to take the ideal scalar wave equation into account. For this, we simulated solutions to the wave equation by discretizing spatial dimensions, using finite difference method to approximate spatial derivatives, and using the Runge-Kutta method to integrate to resulting differential equation. Furthermore, we implemented different kinds of boundary conditions and examined the convergence of the given scheme. Finally we applied it to the potential of a black hole and show how this can be detected.
A documented version of the code can be viewed under `https://github.com/anneaux/wave-equations`.

Discretizing the Wave Equation

The ideal wave equation is one of the most fundamental partial differential equations in physics and builds the basis in analyzing theoretical fields, like electrodynamics, optics and general relativity. For a scalar field $\phi(x,t)$ defined on spacetime it reads

$$0 = \partial_{tt}\phi - c^2\partial_{xx}\phi \tag{1}$$

Spatial Differentiation using Finite Differences Method

Rewriting the wave equation using $\partial_t\phi = \Pi$ and discretizing the spatial dimension by an equidistant orthogonal grid yields a system of first order ordinary differential equations for $u = (\phi, \Pi)$:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u) \tag{2}$$

Hereby, for the approximation of the second spatial derivatives $\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$ we use the following stencil

$$D_h^2\phi(x) = \frac{\phi(x+h) + 2\phi(x) - \phi(x-h)}{h^2} \tag{3}$$

Numerical Integration using Runge-Kutta Method

By integrating both sides of (2) over t we find for timesteps $i = 0, ..., n-1$:

$$u(t_{i+1}) = u(t_i) + \int_{t_i}^{t_{i+1}} F(t, u(t))\mathrm{d}t \tag{4}$$

Given the initial values $u(t=0) = u_0$, we search for approximate values $U_i = U(t_i) \approx u(t_i) = u_i$. The Runge-Kutta scheme is a method to numerically approximate the solution of a system of ODEs. The classical method RK4 is explicit and converges to the exact solution up to an error of fourth order.

$$U_{n+1} = U_n + \Delta t\left(\tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4\right)$$

with

$$k_1 = F(U)$$
$$k_2 = F(U + \tfrac{1}{2}\Delta t k_1)$$
$$k_3 = F(U + \tfrac{1}{2}\Delta t k_2)$$
$$k_4 = F(U + \Delta t k_3).$$

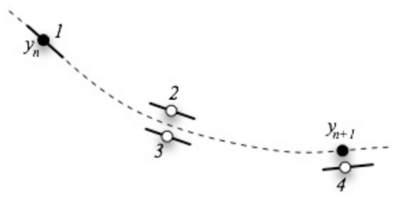

Figure 1: Included tangents to infer from y_n to y_{n+1}, from [2].

Convergence and Stability

Courant-Friedrichs-Lewy Condition

To assure the correctness of the numerical solution, errors shall decay as integration proceeds (stability), whereas the initial error should become infinitely small by adding further gridpoints (consistency) [1].
The CFL condition on convergences states that the integration timestep Δt must be less than the time the wave needs to travel to adjacent grid points $\frac{\Delta x}{v}$, i.e. the Courant number $C = \frac{v\Delta t}{\Delta x} \leq 1$. This can be interpreted as a limit on the flow of information. In our work we always used $C = 1/2$.

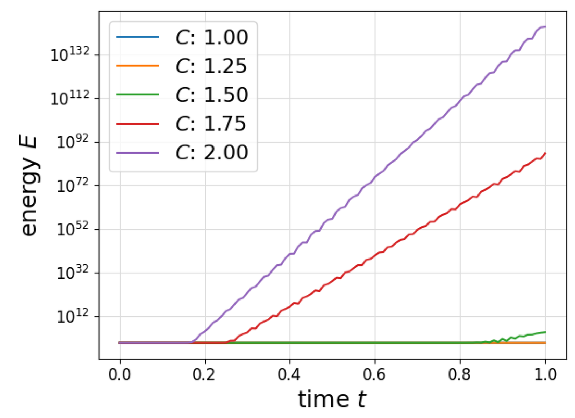

Figure 2: Energy evolution for various Courant numbers with periodic boundary conditions.

Studying Convergence

In order to verify (*absolute*) convergence, we study how a solution with grid size n_x compares to a solution with twice the grid size in error to the analytical solution.
When the analytical solution is unknown, one replaces it with a first numerical one (*self* convergence).

$$Q_{\mathrm{self}} = \frac{\|u^{n_x} - u^{2n_x}\|}{\|u^{2n_x} - u^{4n_x}\|} \tag{5}$$

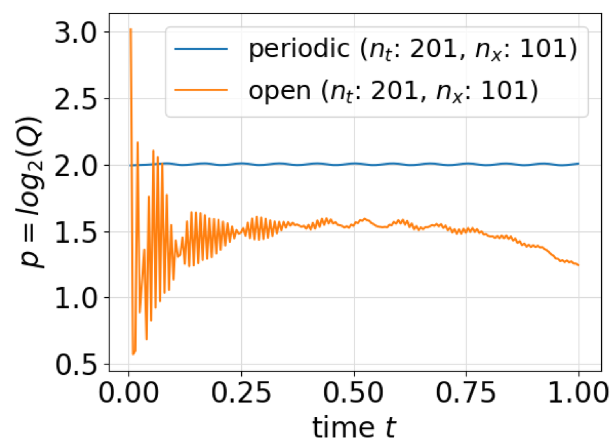

Figure 3: Order of self convergence for different boundary conditions.

Boundary Conditions

In order to solve the wave equation at the end points of the computational domain, additional points have to be included so that the finite differences can still be calculated. Implementing different boundary conditions then translates to filling the these "ghost points" accordingly.

We simulated a gaussian pulse

$$\phi(x) = a\exp(-(x-\mu)^2/(2\sigma^2))$$

with $a = 1$, $\sigma = 0.05$ and $\mu = 0.5$ as initial state, together with the velocity $\Pi(x) = -\frac{\partial\phi(x)}{\partial x}$.

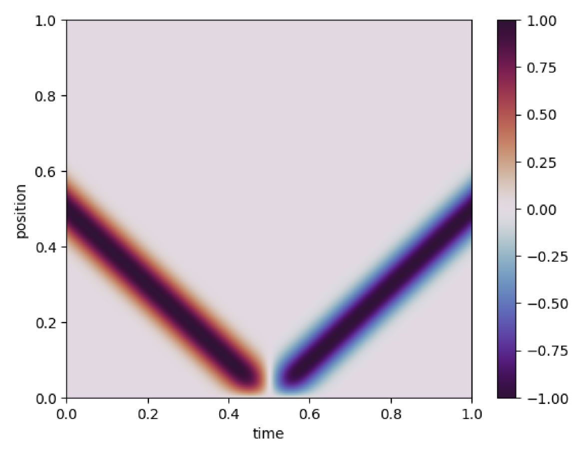

Figure 4: Fixed boundaries.

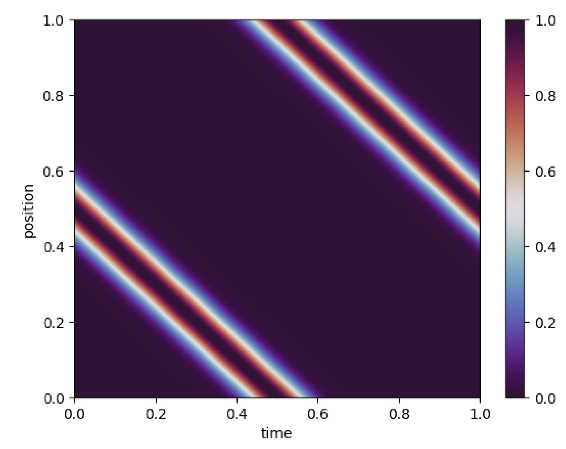

Figure 5: Periodic boundaries.

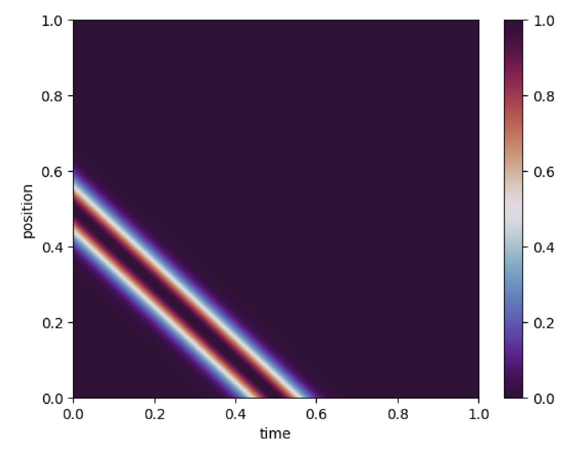

Figure 6: Open boundaries.

Application: Regge-Wheeler Equation

The Regge-Wheeler equation

$$(\partial_{tt} - c^2\partial_{rr} + V(r))\phi = 0 \tag{6}$$

is used to describe the radial perturbation of the Schwarzschild metric by a spherically symmetric potential. It therefor models the response of Schwarzschild black holes to approaching waves (incident matter or radiation) [3]. Qualitatively this process can be studied by employing the spherically symmetric Pöschl-Teller (PT) potential

$$V(r) = -V_0\,\mathrm{sech}^2(\kappa r) \tag{7}$$

where the width κ and the depth V_0 are usually small parameters. We simulated a gaussian pulse ($\sigma = 10$, $a = 50$) approaching such a potential with $\kappa = 0.1$ and $V_0 = 0.15$ in both one and two dimensions.

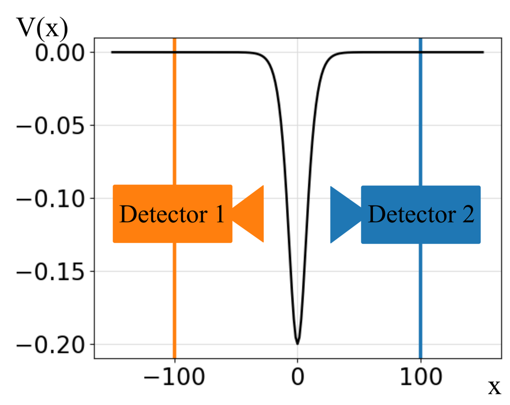

Figure 7: PT potential (black) and detector positions.

For the 1D case the wave's magnitude is detected over time at two fixed points on one side of the potential each (cf. fig. 7 for schematic setup). This yields damped waves of notably different order of magnitude, see fig. 8, which can be interpreted as reflected and transmitted parts of a scattering process. Additionally, we present the 2D evolution of the gaussian pulse in the vicinity of the PT potential (see fig. 9). These results show, that the interaction of a perturbation with a black hole resembles a scattering process of a wave packet at a compact potential (almost) barrier. The response of black holes is dominated by damped oscillations which are usually then called gravitational waves.

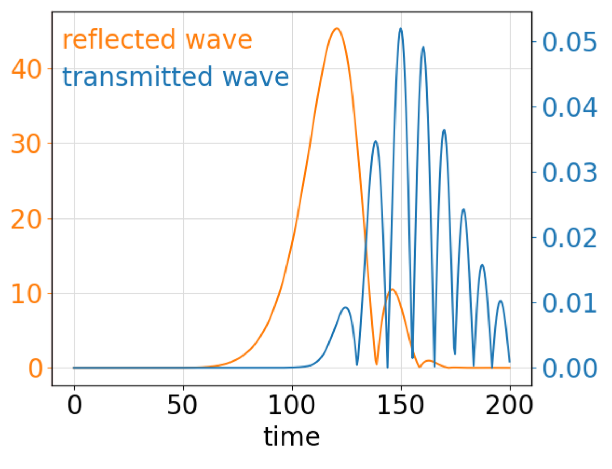

Figure 8: Absolute values of detected signals.

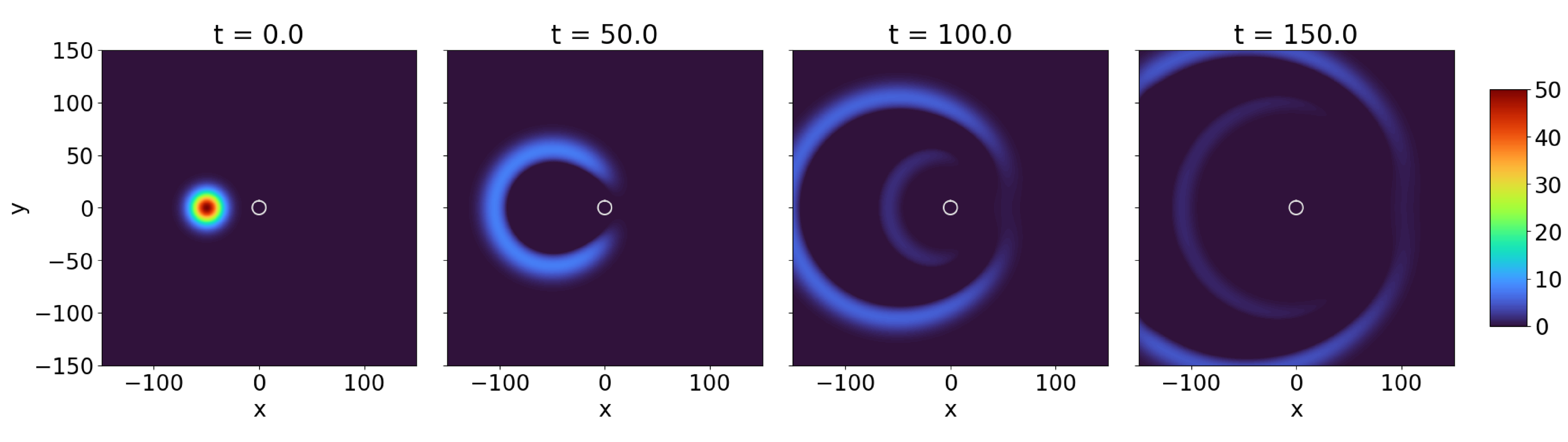

Figure 9: Evolution of a gaussian pulse as it is scattered by a PT potential (the grey circle denotes its FWHM).

Conclusion

We simulated the 1D and 2D wave propagation for different boundary conditions and also included a perturbative potential. Open boundaries were achieved by preventing "information" to flow from outside the computational domain to the inside. For the 2D case, this could be done more efficiently, i.e. not only for three chosen directions of escape. Future work could make use of different discretization methods, e.g. finite element methods to deal with non-orthogonal spatial grids.

References

[1] Rider Love. "On the convergence of finite difference methods for PDE under temporal refinement". In: *Computers & Mathematics with Applications* (2013).
[2] T. Pertsch. "Computational Physics I (Lecture Notes)". FSU Jena. 2019.
[3] Kip S Thorne. "Introduction to Regge and Wheeler "Stability of a Schwarzschild Singularity"". 2020.

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