Solution of the Initial Boundary Value Problem with the Wave Equation

Report on Research Labworks

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1 Theoretical Background

1.1 General Analytical Solution

The wave equation (WE) for the scalar field $\phi(t,x)$ in one spatial and one time dimension is

$$0 = \frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = \partial_{tt}\phi - c^2\partial_{xx}\phi \tag{1}$$

with $c \in \mathbb{R}$. According to the symmetry of second derivatives this can be divided into two factors

$$0 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)\,\phi\ . \tag{2}$$

With introduction of two variables, $\xi = x - ct$ and $\nu = x + ct$, now we can write:

$$0 = \partial_\xi\partial_\nu\phi\ . \tag{3}$$

Therefor the wave equation has the general solution

$$\phi(t,x) = f(\xi) + g(\nu) = f(x-ct) + g(x+ct) \tag{4}$$

which is a composition of a forward and a backward propagating elementary wave. As boundary condition we assume the initial time value and its derivative:

$$\phi_0(x) = \phi(0,x) = f(x) + g(x) =: A(x) \tag{5}$$

$$\Big(\partial_t\phi(0,x)\Big) = cf'(x) - cg'(x) =: B(x)\ . \tag{6}$$

Integrating eq. 6 results in

$$f(x) - g(x) = \int_{x_0}^{x} B(s)\,\mathrm{d}s \tag{7}$$

so that the separated solution is given by (here $x_0 = x(t=0)$)

$$f(x) = \frac{1}{2}\left(A(x) + \frac{1}{c}\int_{x_0}^{x} B(s)\,\mathrm{d}s\right) \tag{8}$$

$$g(x) = \frac{1}{2}\left(A(x) - \frac{1}{c}\int_{x_0}^{x} B(s)\,\mathrm{d}s\right)\ . \tag{9}$$

Putting these together yields the general solution in dependence of the initial values, the d'Alembert's formula:

$$\phi(t,x) = \frac{1}{2}\left(A(x-ct) + A(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} B(s)\,\mathrm{d}s\right) \tag{10}$$

This means, for given initial parameters the wave's evolution in time is clearly defined.

1.2 Rewriting the WE as two differential equations out of one

Now we introduce the time derivative of the field as a new variable: $\Pi(t,x) = \partial_t\phi$. Thereby the initial wave equation eq. 1 can be rewritten as a wave equation including the first-order derivative in time and second-order derivative in space:

$$\partial_t\Pi = \partial_{tt}\phi = c^2\partial_{xx}\phi\ . \tag{11}$$

Now one can, in accordance to the notation we introduced above, regard the field's derivative at the initial time $\Pi(0,x)$ as $B(x)$. Typical values for this initial velocity $\Pi(0,x)$ are either 0 or a constant.

Example 1: $\Pi(0, x) = 0$ If we consider the field's initial velocity to be zero, $\Pi(0, x) = 0$ almost everywhere, this means the wave roughly is a pulse. Furthermore, if we assume no spatial boundaries the pulse will dominate the shape of the wave as time moves forward.

Example 2: $\Pi(0, x) =$ **const.** If we consider the initial velocity to be a constant, the wave function is at rest in a suitable inertial frame of reference. Hence, there will be no wave propagation.

1.3 Rewriting the WE as a fully first order system / eigenvalue problem / Matrix form of the wave equation

Similar to the definition of the velocity variable $\Pi(t, x)$ we now introduce the slope variable as the spatial derivative of the field: $\chi = \partial_x \phi$. Thereby we can write the waveequation 1 as a first order system

$$\partial_t \vec{u} + \hat{M} \partial_x \vec{u} = \vec{S} \ . \tag{12}$$

Herein u is the state vector $\vec{u} = \{\phi, \Pi, \chi\}$, S is a source term and the matrix $\hat{M}$ can be found by simple coefficient comparision to be

$$\hat{M} = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{pmatrix} \ .$$

The source the then is of type $S = \{\Pi + c\chi, 0, 0\}$.

1.3.1 Choices for Initial Values

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1.3.2 Eigenvalue Problem

The matrix formulation 12 in fact is an eigenvalue problem, whichs eigenvalues can be easily found to be $\lambda_{1/2} = \pm c$. For $\lambda_1 = +c$, eigenvectors are $v_1 = \{1, 0, 0\}$ and $v_2 = \{0, c, 1\}$, for $\lambda_2 = -c$ the eigenvector is $v_3 = \{0, -c, 1\}$. Horizontally concatenated these yield the transformation matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ -c & c & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

by which we can diagonalize matrix $\hat{M}$:

$$\begin{aligned} \Lambda = \mathrm{diag}\hat{M} = R^{-1} \hat{M} R &= \begin{pmatrix} 0 & -1/2c & 1/2 \\ 0 & 1/2c & 1/2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -c & c & 0 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \end{aligned}$$

Here, one can see that the diagonalized matrix simply has the eigenvalues on the diagonal. For the eigenvalue problem $\hat{M}R = \Lambda R$ holds, so that $\hat{M} = R\Lambda R^{-1}$, which we can plug in 12:

$$\partial_t \vec{u} + R\Lambda R^{-1}\partial_x \vec{u} = \vec{S} \ . \tag{13}$$

Multiplying both sides by R^{-1} yields

$$R^{-1}\partial_t \vec{u} + \Lambda R^{-1}\partial_x \vec{u} = R^{-1}\vec{S} \ . \tag{14}$$

By introducing the new variable $w := R^{-1}\vec{u}$, this becomes

$$\partial_t w + \Lambda \partial_x w = R^{-1}\vec{S} \tag{15}$$

and we arrive at a decoupled system.

2 Numerical Differentiation

2.1 Forward, Backward and Central Difference Quotients

In order to numerically solve a wave equation one typically approximates all the derivatives by finite differences with regard to discrete grid points. Hence, the first step is to choose a discretization. For brevity we consider a uniform partition in the spatial dimension, so that

$$x_i = ih \quad \text{with } i = 0, ..., n-1$$

and stepsize $h = 1/(n-1)$ specifies the position on the domain $\Omega = [0,1]$.

Now we want to find a formula for the approximation of the derivative of the field function $u(x)$ (we neglect the temporal depency for a moment and will tackle that later). We consider the Taylor expansion of the field at "the next position", i.e. $u(x+h)$:

$$u(x+h) = \sum_{j=0}^{\infty} \frac{h^j}{j!} \left(\frac{\mathrm{d}^n u}{\mathrm{d}x^n}\right)_{x+h} = u(x) + h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2!}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{h^3}{3!}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + ... \tag{16}$$

Solving this for the first derivative yields

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u(x+h) - u(x)}{h} - \frac{h}{2!}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{h^2}{3!}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} - ... \tag{17}$$

Taking into account only the first term of the right-hand side, this is called the *forward difference quotient*:

$$\mathrm{D}_h^+ u(x) = \frac{u(x+h) - u(x)}{h} \tag{18}$$

Similarly one can approximate the derivatve by expanding the value $u(x-h)$ in a Taylor series. This results in the *backward difference quotient*

$$\mathrm{D}_h^- u(x) = \frac{u(x) - u(x-h)}{h} \ . \tag{19}$$

In order two minimize errors we can combine the forward and backward difference to a *central difference quotient*

$$\mathrm{D}_h u(x) = \frac{1}{2}[\mathrm{D}_h^+ + \mathrm{D}_h^-]u(x) = \frac{u(x+h) - u(x-h)}{2h} \ . \tag{20}$$

2.2 Error Estimations

The discretization error is the deviation of the numerical approximation compared to the analytical value

$$\mathrm{E}_h = |\frac{\mathrm{d}u}{\mathrm{d}x} - \mathrm{D}_h u(x)| \ . \tag{21}$$

It stems from the finiteness of step size h, as well as from rounding errors arising from the computer accuracy when conducting numerical calculations. In the following we want to find expressions for the former error to compare them for the various difference quotients. In general both, forward and backward difference quotient, are consistent, i.e. for $h \to 0$ the approximated derivative approaches the analytical derivative. From the comparison of 16 with 18 we find that the discretization error for the forward differentiation method is

$$\begin{aligned} \mathrm{E}_h^+ &= \frac{h}{2!}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{h^2}{3!}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + \dots && (22) \\ &= \mathcal{O}(h) \ . && (23) \end{aligned}$$

The same holds for the backward differentiation method. However, for the central difference quotient we get

$$\begin{aligned} \mathrm{E}_h &= \frac{h^2}{3!}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + \frac{h^3}{4!}\frac{\mathrm{d}^4u}{\mathrm{d}x^4}\dots && (24) \\ &= \mathcal{O}(h^2) \ . && (25) \end{aligned}$$

2.3 Approximating the Second Derivative

To approximate the second derivative of function $u(x)$ we recall the Taylor expansion of $u(x+h)$ and $u(x-h)$:

$$\begin{aligned} u(x+h) &= u(x) + h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2!}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{h^3}{3!}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + \dots && (26) \\ u(x-h) &= u(x) - h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2!}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{h^3}{3!}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + \dots && (27) \end{aligned}$$

Adding up both sides respectively and solving for the second derivative yields

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} - \frac{h^2}{3!}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} - \dots \qquad (28)$$

which results in the differential operator

$$\mathrm{D}_h^2 u(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} \qquad (29)$$

with the error of $\mathcal{O}(h^2)$.