NOTES ON SCHUBERT CALCULUS AND QUANTUM INTEGRABILITY

Abstract.

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1. Introduction

Here is a template for a simple commutative diagram in tikz:

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Here is a template for an elaborate commutative diagram in tikz:

$$\begin{tikzcd}
 & X'_\alpha \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd, "\psi'" near end] & & Y'_\alpha \arrow[dl] \arrow[dd, "\psi"] \\
X' \arrow[rr, "f'" near end] \arrow[dd] & & Y' \arrow[dd, "h" near start] & \\
 & X_\alpha \arrow[rr, "\phi" near start] \arrow[dl] & & Y_\alpha \arrow[dl] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd}$$

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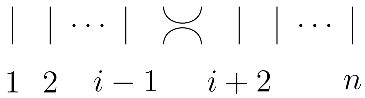
15.1. **Basic Definitions.**

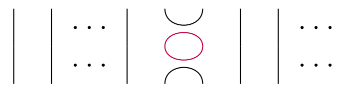
Definition 15.1 (Temperley-Lieb Algebra)**.** Let τ be a complex number. The **Temperley-Lieb algebra** $\mathrm{TL}_n(\tau)$ is generated by an identity 1 and generators $e_1, \dots e_{n-1}$ satisfying the relations

- $e_i^2 = \tau e_i$
- $e_i e_{i+1} e_i = e_i$
- $e_i e_j = e_j e_i, |i-j| > 1$

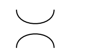
We can give a pictorial description of this algebra: we regard each generator e_i as corresponding to a diagram

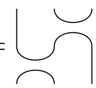


and regard τ as corresponding to the "fugacity of a bubble." We then multiply by composing vertically (with the rightmost element at the top), and "popping" bubbles to obtain a factor of τ:

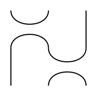
$e_i^2 \quad =$ 

$= \quad \cdot$ 

Example 15.2 ($\mathrm{TL}_3(\tau)$)**.** When $n = 3$, we have generators $1 =$ , $e_1 =$ , and $e_2 =$ . We also obtain

$e_1 e_2 =$ 

and

$e_2 e_1 =$ ,

so that $\mathrm{TL}_3(\tau)$ is 5-dimensional.

Proposition 15.3. *The dimension of* TL_n *is*

$$\dim \mathrm{TL}_n = C_n = \frac{(2n)!}{n!(n+1)!},$$

the nth Catalan number.

15.2. **Temperley-Lieb Algebras and the Yang-Baxter Equation.** We can reinterpret the Yang-Baxter equation through the lens of Temperley-Lieb algebras. Let $\tau = -(q + q^{-1}), a(u) = qu - q^{-1}u^{-1}$, and $b(u) = u - u^{-1}$. We can then regard $\check{R}_i$ as an element of the algebra obtained by adding $u^{\pm}$ and $v^{\pm}$ to $\mathrm{TL}_n(\tau)$:

$$\check{R}_i(u) = a(u)1 + b(u)e_i \in \mathrm{TL}_n(\tau)[u^{\pm}, v^{\pm}].$$

One can check that the equation

$$\check{R}_i(u)\check{R}_{i+1}(uv)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(uv)\check{R}_{i+1}(u)$$

holds; we may thus interpret the Yang-Baxter equation as an identity in $\mathrm{TL}_n(\tau)[u^{\pm}, v^{\pm}]$.

Via this identification, we can regard systems satisfying the Yang-Baxter equation as representations of this algebra $\mathrm{TL}_n(-(q+q^{-1}))[u^{\pm}, v^{\pm}]$. We may obtain one such representation via the map

$$\phi : \mathrm{TL}_n(-(q + q^{-1})) \to \mathrm{End}((\mathbb{C}^2)^{\otimes n})$$

sending

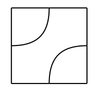
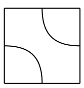
$$e_i \mapsto I \otimes I \otimes \ldots \otimes I \otimes \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \otimes I \otimes \ldots \otimes I,$$

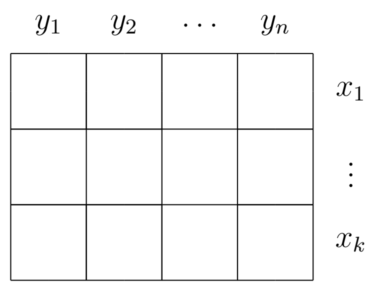
where the matrix is located in the i and $i+1$ coordinates.

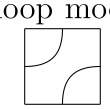
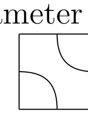
One can check that this map respects the Temperley-Lieb relations, and that it induces a representation $\rho : \mathrm{TL}_n(-(q+q^{-1}))[u^{\pm}, v^{\pm}] \to \mathrm{End}((\mathbb{C}^2)^{\otimes n})$ sending

$$\check{R}_i(u) \mapsto I \otimes \ldots \otimes I \otimes \begin{bmatrix} qu - q^{-1}u^{-1} & 0 & 0 & 0 \\ 0 & u(q-q^{-1}) & u-u^{-1} & 0 \\ 0 & u-u^{-1} & u^{-1}(q-q^{-1}) & 0 \\ 0 & 0 & 0 & qu-q^{-1}u^{-1} \end{bmatrix} \otimes I \otimes \ldots \otimes I$$

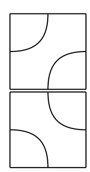
15.3. **Loop Models.**

Definition 15.4 (Loop Model)**.** A **loop model** is an assignment of the tiles  and  to the $k \times n$ rectangle



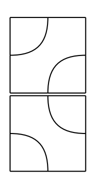
We can compute the **fugacity** of a loop model with respect to a parameter $q \neq \pm 1$ by assigning a fugacity of $a(x_i/y_j)$ to the  tile, and $b(x_i/y_j)$ to the  tile, and multiplying by $\tau^{\#\text{ closed loops}}$, where $\tau = -(q+q^{-1})$. That is, the fugacity is

$$\tau^{\#\text{ closed loops}} \cdot \prod_{i=1}^{k}\prod_{j=1}^{n} \begin{cases} a(x_i/y_j) \\ b(x_i/y_j) \end{cases}$$



When $q = \pm 1$, we instead assign weights of $a(x_i - y_j)$ and $b(x_i - y_j)$ respectively, so that the fugacity of the loop model is

$$\tau^{\#\text{ closed loops}} \cdot \prod_{i=1}^{k}\prod_{j=1}^{n} \begin{cases} a(x_i - y_j) \\ b(x_i - y_j) \end{cases}$$



15.4. **Loop Models and Grassmannians.** Let $X = T^* \mathrm{Gr}(k,n)$ be the cotangent bundle of the Grassmannian. We wish to consider its $T \times \mathbb{C}^\times$-equivariant cohomology. Observe that

$$H^*_{T\times\mathbb{C}^\times}(X) \cong \mathbb{Z}[x_1,\dots,x_k,y_1,\dots,y_n,h]^{S_k}/\mathcal{I},$$

where:

- The ring $\mathbb{Z}[x_1,\dots,x_k,y_1,\dots,y_n,h]^{S_k}$ consists of polynomials symmetric in the x_i, i.e., invariant under the action of S_k on $x_1,\dots,x_k$.
- $\mathcal{I}$ is the ideal generated by polynomials which vanish when each x_i is replaced by some (distinct) y_j. That is, for any $I \in \binom{[n]}{k}$, we can define a map

  $$\phi_I : \mathbb{Z}[x_1,\dots,x_k,y_1,\dots,y_n,h]^{S_k} \to \mathbb{Z}[y_1,\dots y_n,h]$$

  which sends $p(x_1,x_2,\dots,x_k,y_1,\dots,y_n,h)$ to $p(y_{I_1},y_{I_2},\dots,y_{I_k},y_1,\dots,y_n,h)$. Note that the order on the elements of I does not matter, since p is symmetric in the x_i. Then $\mathcal{I}$ is the ideal generated by the kernels of the ϕ_I.

For example, when $k=1$, we have that $\mathcal{I} = \langle (x_1-y_1),(x_1-y_2),\dots,(x_1-y_n)\rangle$, so that

$$H^*_{T\times\mathbb{C}^\times}(\mathrm{Gr}(1,n)) \cong \mathbb{Z}[x_1,y_1,\dots,y_n,h]^{S_k}/\mathcal{I}$$

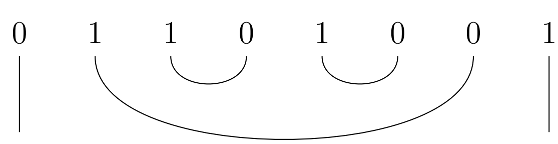
Now, for each $I \in \binom{[n]}{k}$, let $X_I^\circ = B_-\mathbb{C}^I$, where $\mathbb{C}^I$ is the coordinate subspace spanned by $e_{I_1},\dots,e_{I_k}$, denote the Schubert cell corresponding to I. One can check that $X_I^\circ \cong \mathbb{C}^{k(n-k)-\mathrm{inv}(I)}$, where $\mathrm{inv}(I)$ is the inversion number of the binary string corresponding to I. In particular, X_I° is smooth.

Let $C_I = \overline{\mathbb{C}_{X_I^\circ}\,\mathrm{Gr}(k,n)}$ denote the closure of the conormal bundle of X_I° in $T^*(\mathrm{Gr}(k,n))$. Let $\tilde{S}_I = [C_I] \in H^*_{T\times\mathbb{C}^\times}(X)$ be the fundamental class of this conormal bundle.

Recall that when examining $H^*_T(\mathrm{Gr}(\cdot,n))$, the T-equivariant cohomology of $\bigsqcup_{k=1}^n \mathrm{Gr}(k,n)$, we assigned

$$S_I \mapsto \bigotimes_{i=1}^{n} \begin{cases} \begin{pmatrix}1 & 0\end{pmatrix}^{Tr} & i \notin I \\ \begin{pmatrix}0 & 1\end{pmatrix}^{Tr} & i \in I \end{cases}$$

In the case of the cotangent bundle, we attach an element of $(\mathbb{C}^2)^{\otimes n}$ to each $\tilde{S}_I$ via **link patterns**. We begin with the binary string associated to I, and repeatedly attach links between each substring of 10, ignoring previously linked letters. This is perhaps most easily illustrated with an example: if our binary string is 01101001, the associated link pattern would be



To the link $\underset{i \quad\quad j}{\smile}$ between positions i and j, we associate $\begin{pmatrix}1\\0\end{pmatrix}_i \otimes \begin{pmatrix}0\\1\end{pmatrix}_j - q^{-1}\begin{pmatrix}0\\1\end{pmatrix}_i \otimes \begin{pmatrix}1\\0\end{pmatrix}_j$, while to a "lonely" 0 or 1 in position i we associate $\begin{pmatrix}1\\0\end{pmatrix}_i$ and $\begin{pmatrix}0\\1\end{pmatrix}_i$ respectively. We then take the tensor product over the vectors associated to all links/lonely elements.

Theorem 15.5 (Z-J, Loop Models and K-Theory, 2016)**.** *$\tilde{S}_I = [C_I]$ is equal to the loop model partition function on a $k \times n$ rectangle where the connectivity of the external points satisfy:*

- *No bottom points are connected to other bottom points*
- *No top points are connected to right points*
- *The top lines reproduce the link pattern of the associated binary string.*

16. Lecture 15 (Allen Knutson)

17. Lecture 16 (Allen Knutson)

18. Lecture 17 (Paul Zinn-Justin)

19. Lecture 18 (Paul Zinn-Justin)