NOTES ON SCHUBERT CALCULUS AND QUANTUM INTEGRABILITY

Abstract.

Contents

1. Introduction

Here is a template for a simple commutative diagram in tikz:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Here is a template for an elaborate commutative diagram in tikz:

$$
\begin{tikzcd}
& X'_\alpha \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd, "\psi'"'] & & Y'_\alpha \arrow[dl] \arrow[dd, "\psi"] \\
X' \arrow[rr, "f'"] \arrow[dd] & & Y' \arrow[dd, "h"] & \\
& X_\alpha \arrow[rr, "\phi"] \arrow[dl] & & Y_\alpha \arrow[dl] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd}
$$

2. Lecture 1 (Allen Knutson)

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4. Lecture 3 (Paul Zinn-Justin)

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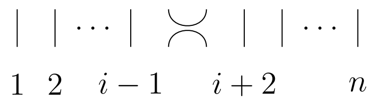
15. Lecture 14 (Paul Zinn-Justin)

15.1. Basic Definitions.

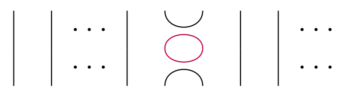
Definition 15.1 (Temperley-Lieb Algebra)**.** Let τ be a complex number. The Temperley-Lieb Algebra $\mathrm{TL}_n(\tau)$ is generated by an identity 1 and generators $e_1, \dots e_{n-1}$ satisfying the relations

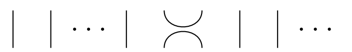
- $e_i^2 = \tau e_i$
- $e_i e_{i+1} e_i = e_i$
- $e_i e_j = e_j e_i, |i-j| > 1$

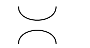
We can give a pictorial description of this algebra: we regard each generator e_i as corresponding to a diagram

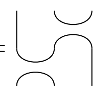


and regard τ as corresponding to the "fugacity of a bubble." We then multiply by composing vertically (with the rightmost element at the top), and "popping" bubbles to obtain a factor of τ:

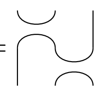
$e_i^2 \quad =$ 

$= \quad \cdot$ 

Example 15.2 ($\mathrm{TL}_3(\tau)$)**.** When $n = 3$, we have generators $1 =$ , $e_1 =$ , and $e_2 =$ . We also obtain

$e_1 e_2 =$ 

and

$e_2 e_1 =$ ,

so that $\mathrm{TL}_3(\tau)$ is 5-dimensional.

Proposition 15.3. *The dimension of* TL_n *is*

$$\dim \mathrm{TL}_n = C_n = \frac{(2n)!}{n!(n+1)!},$$

the nth Catalan number.

15.2. **Temperley-Lieb Algebras and the Yang-Baxter Equation.** We can reinterpret the Yang-Baxter equation through the lens of Temperley-Lieb algebras. Let $\tau = -(q + q^{-1}), a(u) = qu - q^{-1}u^{-1}$, and $b(u) = u - u^{-1}$. We can then regard $\check{R}_i$ as an element of the algebra obtained by adding $u^{\pm}$ and $v^{\pm}$ to $\mathrm{TL}_n(\tau)$:

$$\check{R}_i(u) = a(u)1 + b(u)e_i \in \mathrm{TL}_n(\tau)[u^{\pm}, v^{\pm}].$$

One can check that the equation

$$\check{R}_i(u)\check{R}_{i+1}(uv)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(uv)\check{R}_{i+1}(u)$$

holds; we may thus interpret the Yang-Baxter equation as an identity in $\mathrm{TL}_n(\tau)[u^{\pm}, v^{\pm}]$.

Via this identification, we can regard systems satisfying the Yang-Baxter equation as representations of this algebra $\mathrm{TL}_n(-(q+q^{-1}))[u^{\pm}, v^{\pm}]$. We may obtain one such representation via the map

$$\phi : \mathrm{TL}_n(-(q + q^{-1})) \to \mathrm{End}((\mathbf{C}^2)^{\otimes n})$$

sending

$$e_i \mapsto I \otimes I \otimes \ldots \otimes I \otimes \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \otimes I \otimes \ldots \otimes I,$$

where the matrix is located in the i and $i+1$ coordinates.

One can check that this map respects the Temperley-Lieb relations, and that it induces a representation $\rho : \mathrm{TL}_n(-(q+q^{-1}))[u^{\pm}, v^{\pm}] \to \mathrm{End}((\mathbf{C}^2)^{\otimes n})$ sending

$$\check{R}_i(u) \mapsto I \otimes \ldots \otimes I \otimes \begin{bmatrix} qu - q^{-1}u^{-1} & 0 & 0 & 0 \\ 0 & u(q-q^{-1}) & u - u^{-1} & 0 \\ 0 & u - u^{-1} & u^{-1}(q-q^{-1}) & 0 \\ 0 & 0 & 0 & qu - q^{-1}u^{-1} \end{bmatrix} \otimes I \otimes \ldots \otimes I$$

15.3. **Loop Models.**

16. Lecture 15 (Allen Knutson)

17. Lecture 16 (Allen Knutson)

18. Lecture 17 (Paul Zinn-Justin)

19. Lecture 18 (Paul Zinn-Justin)