NOTES ON SCHUBERT CALCULUS AND QUANTUM INTEGRABILITY

Abstract.

Contents

1. Introduction

Here is a template for a simple commutative diagram in tikz:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Here is a template for an elaborate commutative diagram in tikz:

$$\begin{tikzcd}
 & X'_\alpha \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd, "\psi'"'] & & Y'_\alpha \arrow[dl] \arrow[dd, "\psi"] \\
X' \arrow[rr, "f'"] \arrow[dd] & & Y' \arrow[dd, "h"] & \\
 & X_\alpha \arrow[rr, "\phi"] \arrow[dl] & & Y_\alpha \arrow[dl] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd}$$

2. Lecture 1 (Allen Knutson)

Let V be a k-plane in $\mathbb{C}^n$ with basis represented as a $k \times n$ matrix with basis elements as row vectors. Put this matrix into Reduced Row Echelon Form and consider left action by $GL_k(\mathbb{C})$ and right action by upper triangular matrices to get an open subgroup of *right word row operations.*

Theorem 2.1. *A matrix $Mat_n(\mathbb{C})$ acted on the left by $GL_k(\mathbb{C})$ downward row operations and on the right by upper triangular matrices rightward column operations, that is,*

$$\begin{bmatrix} \ddots & 0 \\ * & \ddots \end{bmatrix} \circlearrowright Mat_n(\mathbb{C}) \circlearrowleft \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$$

*has one orbit for each partial permutation matrix. This is the **Bruhet decomposition** of the matrix.*

Definition 2.2. A matrix **Schubert Variety** is $\overline{X_\pi} := \overline{B_-\pi B_+}$ where π is the permutation matrix.

Theorem 2.3. *$\overline{X_\pi}$ is the set of $n \times n$ matrices, M, such that for all $i, j \in [n]$, the $i \times j$ submatrix of M is less than or equal to the $i \times j$ submatrix of π. That is, the determinants summarized by these conditions generate a prime ideal whose vanishing set is $\overline{X_\pi}$.*

As an example, consider $\pi = 3142$ pictured below:

$$\begin{array}{c|cccc|}
\hline
3 & 0 & 0 & 1 & \\
1 & 1 & & & \\
4 & & * & & 1 \\
2 & & 1 & & \\
\hline
\end{array}$$

The arrows are referred to as **death rays** since each leading one eliminates the entries to the right of and below it. In this example we have that

$$m_{11} = m_{12} = 0 = \det \begin{pmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{pmatrix}$$

and the associated **Rothe diagram** is

$$\begin{bmatrix} 0 & 0 \\ & * \end{bmatrix}$$

Two natural questions arise. First, how big is $\overline{X_\pi}$? We see that

$$\begin{aligned} \dim \overline{X_\pi} &= \dim(B_-\pi B_+) \\ &= \dim(B_- \times B_+) - \dim(\mathrm{stab}(\pi)) \\ &= \text{the number of entries crossed out in the death ray diagram} \\ &= \dim T_\pi(B_-\pi B_+) \\ &= \dim(b_-\pi + \pi b_+) \end{aligned}$$

Where b_- and b_+ are lie algebras. Hence the codimension of $\overline{X_\pi}$ is the number of entries in the Rothe diagram.

Theorem 2.4. *The only essential rank conditions are at the southeast corners of the Rothe diagram.*

The second question is what is the volumn of $\mathbb{P}(\overline{X_\pi})$? Considering the degree as a projective variety, we get the following axioms for $Y \subseteq \mathbb{P}(V)$ defined by some homogeneous ideal.

(0) The degree of p^1 is 1.
(1) If Y is reducible, that is the ideal is not prime, such that $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_k$ and $I \subseteq P_i$ is minimal, then the degree of Y is the sum over the top dimensional components of the product of the multiplicity of Y_i and the degree of Y_i.
(2) If $W \subseteq V$ is a hyperplane and $\mathbb{P}(W) \supseteq Y$ then the degree of Y in $\mathbb{P}(V)$ is equal to the degree of Y in $\mathbb{P}(W)$.
(3) If $W \subseteq V$ is a hyperplane and Y is reduced and irreducible, that is I is prime and $Y \subseteq \mathbb{P}(W)$, then the degree of Y in $\mathbb{P}(V)$ is equal to the degree of $Y \cap \mathbb{P}(W)$ in $\mathbb{P}(W)$.

What is the degree of $\mathbb{P}(\overline{X_\pi})$? Consider the base case where

$$\pi = w_0^{(n)} = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

.

After eliminating entries via death rays we see that $\overline{X_\pi} = \left\{ M : \begin{bmatrix} 0 & \\ \diagup & \\ & * \end{bmatrix} \right\}$ where the 0's in the upper left of the matrix aligns with axiom 2 and the free variables in the lower left align with axiom 3. This gives us that $\deg(\overline{X_\pi}) = 1$.

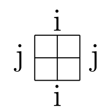
If $\pi \neq w_0^{(n)}$ pick $i \in [n]$ least such that $\pi(i) \neq w_0^{(n)}(i) = n+1-i$. Then $W := \{M : m_i\pi(i) = 0\}$ and

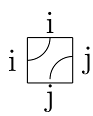
$$\overline{X_\pi} \cap W = \overline{X_\pi} \cap \left\{ \begin{array}{cc} & \pi(i) \\ i & \begin{bmatrix} 0 & \\ & \end{bmatrix} \end{array} \right\} \circlearrowleft B_- \times B_+ \text{ invariant}$$

which is the union of $\overline{X_{\pi'}}$ over certain π' none of which are strictly partial permutations and all multiplicities are 1.

Definition 2.5. A **pipe dream** for π is a diagram

$$\begin{array}{c|c|c|c|c|} & 1 & 2 & \cdots & n \\ \hline \pi(1) & & & & \\ \hline \pi(2) & & & & \\ \hline \cdots & & & & \\ \hline \pi(n) & & & & \\ \hline \end{array}$$

where each box is filed in with one of two tiles: crosses  when $i \leq j$ and elbows

 when $i \neq j$.

Theorem 2.6. *The degree of $\overline{X_\pi}$ is the number of pipe dreams of π.*

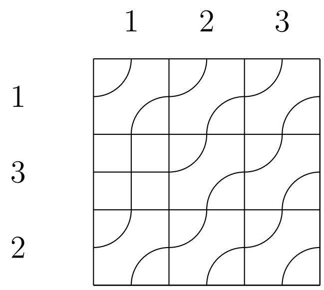
For a torus $T \cong (\mathbb{C}^x)^N$ and $Y \subseteq V) \circlearrowleft T$ where Y is a T-invariant subvariety and V is a T-representation can soup up the degree to a T-equivariant cohomology class.

Our above axioms hold from above with the exception of axiom 2 which can be rewritten at follows:

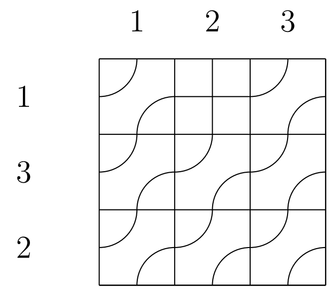
(2) $[Y \subseteq V]$ equals the weight of $T(V/W)[Y \subseteq W]$ for $T \in T^*$. The T^* weight lattice is given by $\mathrm{Hom}(T, \mathbb{C}^x) \cong \mathbb{Z}^N$ such that for $\Lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ and $t = (t_1, \ldots, t_N) \in (\mathbb{C}^x)^N$, $\Lambda \cdot t = \prod_{i=1}^{N} t_i^{\lambda_i}$. Then $[Y \subseteq V] \in \mathrm{Sum}(T^*) \cong \mathbb{Z}[Y_1, \ldots, Y_n]$.

Definition 2.7. The **Double Schubert Polynomial** is $S_\pi(X, Y) = [\overline{X_\pi} \subseteq \mathrm{Mat}_n]$ with respect to $T \times T \circlearrowleft \mathrm{Mat}_n$ such that $\overline{S_\pi}(X, Y) = \sum_{\mathrm{p}} \prod_{\mathrm{c}} (X_{\mathrm{row}} - Y_{\mathrm{col}})$ where p stands for pipe dreams for π and c represents crosses.

Consider the possible pipe dreams for $\pi = 132$.



and



Frome these pipe dreams we see that the Double Schubert Polynomial is $S_{132} = (X_2 - Y_1) + (X_1 - Y_2)$.

3. Lecture 2 (Allen Knutson)

4. Lecture 3 (Paul Zinn-Justin)

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