Shapovalov form

Geordie suggested computing the Gram matrix of the Shapovalov form in the basis of MV cycles for the spherical Schubert varieties.
Let us try.

Useful property of the MV basis

Definition 1. B is a **perfect** basis of $\mathbb{C}[U]$ if

- $1_U \in B$
- each $b \in B$ is *homogeneous* of degree $\mathrm{wt}(b)$ wrt the Q_+ grading on $\mathbb{C}[U]$
- for each $i \in I$ and for each $b \in B$

$$e_i \cdot b - \varepsilon_i(b)\tilde{e}_i(b) \in \mathrm{Span}(\{b' \in B : \varepsilon_i(b') < \varepsilon_i(b) - 1\}) \tag{1}$$

Example 1 (The example that they like to give). Let $G = \mathbf{SL}_3$ and identify $\mathbb{C}[U] = \mathbb{C}[x, y, z]$ by coordinatizing elements of $U \leq G$ as

$$(x, y, z) \in \mathbb{A}^3 \mapsto \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} \in U$$

In this case there is just one perfect basis [**?**] of $\mathbb{C}[U]$ and it is

$$B = \{x^a z^b (xy - z)^c : a, b, c \in \mathbb{Z}_{\geq 0}\} \cup \{y^a z^b (xy - z)^c : a, b, c \in \mathbb{Z}_{\geq 0}\} \tag{2}$$

Note that $xy - z$ is homogeneous as $\mathrm{wt}(x) + \mathrm{wt}(y) = \alpha_1 + \alpha_2 = \mathrm{wt}(z)$. The Lie algebra $\mathcal{U}\mathfrak{n}$ acts on B via

$$e_1 = \partial_x \qquad e_2 = \partial_y + x\partial_z \tag{3}$$

Let's check that

$$\partial_x b = \varepsilon_1(b)\tilde{e}_1(b) + \sum_{\varepsilon_1(b') < \varepsilon_1(b) - 1} c(b')b'$$

and

$$\partial_y + x\partial_z b = \varepsilon_2(b)\tilde{e}_2(b) + \sum_{\varepsilon_2(b') < \varepsilon_2(b) - 1} c(b')b'$$

in some examples.
First, recall that a Lusztig datum $n_\bullet^{\mathbf{i}} = (n_1, n_2, \ldots, n_\ell)$ associated to a partition of $\nu \in Q_+$ defines a minor as follows.

$$\Delta(n_\bullet) = \Delta_{12..n-1, w(12..n-1)} \qquad w =$$

Fix $\underline{\mathbf{i}} = (1, 2, 1)$.
The case $b = z = \Delta_{1,3}$**.** This element has $n_\bullet = (1, 0, 1)$ since $3 = s_1 s_2(1)$. In terms of tableaux

$$\tau(b) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \in B(\omega_1 + \omega_2)$$

and

$$\varepsilon_1(\tau)\tilde{e}_1(\tau) = 0 \qquad \varepsilon_2(\tau)\tilde{e}_2(\tau) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Directly,

$$\partial_x z = 0 \qquad (\partial_y + x\partial_z)z = x$$

Indeed $n_\bullet(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}) = (1,0,0)$ agrees with the fact that $x = \Delta_{1,2}$ as $s_1(1) = 2$.

The case $b = xy - z = \Delta_{12,23}$. This element has $n_\bullet = (0,1,0)$ since $23 = s_2 s_1(12)$. In terms of tableaux

$$\tau(b) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \in B(\omega_1 + \omega_2)$$

and

$$\varepsilon_1(\tau)\tilde{e}_1(\tau) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \varepsilon_2(\tau)\tilde{e}_2(\tau) = 0$$

Directly,

$$\partial_x(xy - z) = y \qquad (\partial_y + x\partial_z)(xy - z) = x - x = 0$$

Indeed $n_\bullet(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}) = (0,0,1)$ agrees with the fact that $y = \Delta_{2,3}$ as $s_2(2) = 3$.

The spherical case

On Shapovalov forms

Let V be a representation of G and fix a highest weight vector $v \in V$. In our case the representation is (the "highest weight zero Verma" appearing as) $V = \mathbb{C}[U]$ and the highest weight vector is (probably) the weight zero vector $1_U \in U$.

Question 1. Can we get from the highest weight vector to any other MV basis vector by applying f_i (or e_i depending on the convention)? I think yes.

The Shapovalov form is a bilinear form on V normalized such that

$$(1_U, 1_U) = 1 \tag{4}$$

and respecting

$$(av, bu) = (v, a^* bu) \tag{5}$$

for any $a, b \in \{e_i, f_i\}$ and $e_i^* = f_i \ldots$?

Perhaps the easier thing (that we were meant) to consider is a "piece" of this "highest weight" form: The form of the (spherical, finite dimensional) highest weight $V(\lambda)$ irrep with highest weight vector $v_\lambda = [\{L_\lambda\}]$. Normalizing

$$(v_\lambda, v_\lambda) = 1 \tag{6}$$

In any case we are asking for a matrix A such that $x^T A y = (x, y)$ where $x, y \in V$ or V_λ are identified with their vector representations in the MV basis. In the MV basis A has (i,j) entry (b_i, b_j) i.e. $e_i^T A e_j = (b_i, b_j)$. So we'd have to figure out the path to b_i, b_j from 1_U or v_λ and apply it.

Question 2. How is this the equivalent of Mirković's question?