

Kamnitzer–Knutson–Mirkovic Conjecture

Zhijie asks about the following example.

In A_3 ($G = \mathbf{PGL}_4$) fix the usual reduced expression $\mathbf{i} = (1, 2, 3, 1, 2, 1)$ and consider the Lusztig datum $n_\bullet = (1, 1, 0, 0, 1, 1)$.

The smallest λ and μ for the associated stable MV cycle satisfying Proposition XYZ in Section 2.2 of my thesis are

$$\lambda = (5, 3, 2, 0) \quad \mu = (3, 3, 2, 2)$$

and the associated tableau is

1	1	1	2	3
2	2	4		
3	4			

By applying Conjecture 4.6.2 of my thesis we can say what the free entries in each column of a matrix A in the MVy slice to \mathbb{O}_λ associated to this tableau. Let us fix notation for the coordinates of A .

$$A = \left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{c_1} & 0 & \textcircled{b_2} & \textcircled{a_3} & \textcircled{b_3} \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{b_4} & a_5 & \textcircled{b_5} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Our conjecture tells us that the circled entries will be the free entries, i.e. the nonzero variable entries. We went ahead and used this information to deduce and omit the entries that must be zero.

With the help of M2 and speeding things up by substituting the first six prime numbers for the circled entries we find that the ideal of X_τ is

$$I = (b_1, a_6, a_4, a_2, a_1, -a_5b_2 + a_3b_4, b_2b_6 + a_3, b_4b_6 + a_5)$$

in $R = \mathbb{C}[a_1, \dots, a_6, b_1, \dots, b_6, c_1]$.

We compute its multidegree

$$\text{mdeg } I = (\alpha_2 + \alpha_1)(\alpha_2 + \alpha_3)(\alpha_2 + \alpha_3 + \alpha_1)(\hbar + \alpha_1)(\alpha_1\alpha_2\alpha_3)$$

and Hilbert series

$$\text{ch } I = \frac{1}{(1 - e^{\alpha_1 + \alpha_2 + \alpha_3 + \hbar})(1 - e^{\alpha_1 + \alpha_2 + \hbar})(1 - e^{\alpha_2 + \alpha_3 + \hbar})(1 - e^{\alpha_1 + 2\hbar})(1 - e^{\alpha_3 + \hbar})(1 - e^{\alpha_2 + \hbar})}$$

Now Zhijie would like to know what the Plucker coordinates are.

In my thesis I claim that the Plucker embedding is given by the minors $\Delta_C B$ of a certain matrix B using columns $C \in \binom{S}{mp-N}$. Here $m = 4$, $p = 5$, $N = 10$, and

$$S = ((k, 0), (k, 1), (k, 2), (k, 3), (k, 4))_{k=1}^4$$

Recall $mp - N = \dim L/t^p L_0$ and S indexes the basis $v_{(i,j)} = [e_i t^j]$ of $V = L_0/t^p L_0$. The matrix B is the matrix whose row vectors are the basis vectors

$$[ge_1], [tge_1], [ge_2], [tge_2], [ge_3], [tge_3], [t^2ge_3], [ge_4], [tge_4], [t^2ge_4]$$

of $L/t^p L_0 \subset L_0/t^p L_0$.

In this case

$$g = g_A = \begin{bmatrix} t^3 & & & & \\ -c_1 t^2 & t^3 & & & \\ -b_2 t & -b_4 t & t^2 & & \\ -a_3 - b_3 t & -a_5 - b_5 t & -b_6 t & t^2 & \end{bmatrix}$$

so

$$\begin{aligned} [ge_1] &= [t^3 e_1] - c_1 [t^2 e_2] - b_2 [te_3] - a_3 [e_4] - b_3 [te_4] \\ [tge_1] &= [t^4 e_1] - c_1 [t^3 e_2] - b_2 [t^2 e_3] - a_3 [te_4] - b_3 [t^2 e_4] \\ [ge_2] &= [t^3 e_2] - b_4 [te_3] - a_5 [e_4] - b_5 [te_4] \\ [tge_2] &= [t^4 e_2] - b_4 [t^2 e_3] - a_5 [te_4] - b_5 [t^2 e_4] \\ [ge_3] &= [t^2 e_3] - b_6 [te_4] \\ [tge_3] &= [t^3 e_3] - b_6 [t^2 e_4] \\ [t^2ge_3] &= [t^4 e_3] - b_6 [t^3 e_4] \\ [ge_4] &= [t^2 e_4] \\ [tge_4] &= [t^3 e_4] \\ [t^2ge_4] &= [t^4 e_4] \end{aligned}$$

and, omitting zero columns, we get

$$\begin{array}{c} [ge_1] \\ [tge_1] \\ [ge_2] \\ [tge_2] \\ [ge_3] \\ [tge_3] \\ [t^2ge_3] \\ [ge_4] \\ [tge_4] \\ [t^2ge_4] \end{array} \begin{pmatrix} [t^3e_1] & [t^4e_1] & [t^2e_2] & [t^3e_2] & [t^4e_2] & [te_3] & [t^2e_3] & [t^3e_3] & [t^4e_3] & [e_4] & [te_4] & [t^2e_4] & [t^3e_4] & [t^4e_4] \\ 1 & & -c_1 & & & -b_2 & & & & -a_3 & -b_3 & & & \\ & 1 & & -c_1 & & & -b_2 & & & & -a_3 & -b_3 & & \\ & & 1 & & & -b_4 & & & & -a_5 & -b_5 & & & \\ & & & 1 & & & -b_4 & & & & -a_5 & -b_5 & & \\ & & & & 1 & & 1 & & & & -b_6 & & & \\ & & & & & 1 & & 1 & & & & -b_6 & & \\ & & & & & & & & 1 & & & & -b_6 & \\ & & & & & & & & & 1 & & & & 1 \\ & & & & & & & & & & 1 & & & & 1 \end{pmatrix}$$

for B so $C_0 = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$.

Finding Plucker coordinates in M2

```

R = QQ[a_1,b_1,c_1,a_2,b_2,a_3,b_3,a_4,b_4,a_5,b_5,a_6,b_6,Degrees
    =>{{1,0,0,0},{1,0,0,1},{1,0,0,2},{1,1,0,0},{1,1,0,1},{1,1,1,0},{1,1,1,1},{0,1,0,0},

J = ideal(b_1,a_6,a_4,a_2,a_1,-a_5*b_2+a_3*b_4,b_2*b_6+a_3,b_4*b_6+a_5)

plm = matrix{{1,0,-c_1,0,0,-b_2,0,0,0,-a_3,-b_3,0,0,0},{0,1,0,-c_1
    ,0,0,-b_2,0,0,0,-a_3,-b_3,0,0},{0,0,0,1,0,-b_4,0,0,0,-a_5,-b_5
    ,0,0,0},{0,0,0,0,1,0,-b_4,0,0,0,-a_5,-b_5
    ,0,0},{0,0,0,0,0,0,1,0,0,0,-b_6,0,0,0},{0,0,0,0,0,0,0,1,0,0,0,-b_6
    ,0,0},{0,0,0,0,0,0,0,0,1,0,0,0,-b_6
    ,0},{0,0,0,0,0,0,0,0,0,0,0,0,1,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,1,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},

s = subsets(14,10);

e0 = {0..9}

mins := {}

for e in s do if ( det(plm^e0_e) != 0 ) then mins = append(mins,det(plm
    ^e0_e))

inds := {}

for e in s do if ( det(plm^e0_e) != 0 ) then inds = append(inds,e)

% check that #inds == #mins

wts := {}

for e in s do if ( det(plm^e0_e) != 0 ) then wts = append(wts,degree(
    det(plm^e0_e)))

pluck = QQ[apply(inds,i->p_i)]

f = map(R,pluck,mins)

Q = R/J

fbar = map(Q,pluck,mins)

K = kernel fbar

Kh = homogenize(K,p_{0,1,3,4,6,7,8,11,12,13})

indsnu := {}

for i in 0..#wts-1 do if take(wts#i,3)=={1,1,1} then indsnu = append(

```


[illegible]

$$\begin{aligned}
& \mathcal{P}\{2, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 1, 4, 5, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{1, 2, 3, 4, 7, 8, 10, 11, 12, 13\} + \mathcal{P}\{0, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{2, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{0, 1, 3, 4, 7, 8, 10, 11, 12, 13\} \\
& \mathcal{P}\{2, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{1, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{0, 1, 3, 4, 7, 8, 10, 11, 12, 13\} - \mathcal{P}\{1, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{0, 1, 3, 4, 7, 8, 10, 11, 12, 13\} \\
& \mathcal{P}\{0, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 1, 4, 5, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{0, 1, 3, 4, 7, 8, 10, 11, 12, 13\} - \mathcal{P}\{0, 1, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{1, 2, 3, 4, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{0, 1, 3, 4, 7, 8, 10, 11, 12, 13\} \\
& \mathcal{P}\{1, 2, 3, 4, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}_{\{0, 3, 4, 5, 6, 7, 8, 11, 12, 13\}}^2 - \mathcal{P}\{0, 1, 4, 5, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{2, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \cdot \mathcal{P}\{1, 2, 3, 4, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{0, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13\} \\
& \mathcal{P}\{2, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \cdot \mathcal{P}\{1, 2, 3, 4, 6, 7, 8, 11, 12, 13\} \mathcal{P}\{0, 1, 4, 5, 6, 7, 8, 11, 12, 13\} + \mathcal{P}\{0, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \Big) \\
& \Big(\mathcal{P}\{0, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 1, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 7, 8, 9, 10, 11, 12, 13\} \cdot \mathcal{P}\{2, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \\
& \mathcal{P}\{0, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 1, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 7, 8, 9, 10, 11, 12, 13\} \cdot \mathcal{P}\{2, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \\
& \Big(\mathcal{P}\{0, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 5, 6, 7, 8, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 7, 8, 9, 10, 11, 12, 13\} \cdot \mathcal{P}\{2, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 6, 7, 8, 10, 11, 12, 13\} \\
& \Big(\mathcal{P}\{0, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 1, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 7, 8, 9, 10, 11, 12, 13\} \cdot \mathcal{P}\{2, 3, 4, 5, 7, 8, 10, 11, 12, 13\} \cdot \mathcal{P}\{0, 3, 4, 6, 7, 8, 10, 11, 12, 13\} \Big)
\end{aligned}$$

Do Zhijie's smaller example

to make sure...