KLR modules in type $A_l^{(1)}$

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1 Type $A_1^{(1)}$

Consider the unipotent cell of $\mathbb{C}[N]$ associated with the reduced word $(1,0,1,0)$. Then the $\bar{D}$ of the simple modules of the initial seed (i.e. the standard seed corresponding to the same reduced word) are given by

$$\frac{1}{\alpha_1} \quad \frac{1}{\alpha_1^2(\delta+\alpha_1)} \quad \frac{1}{\alpha_1^3(\delta+\alpha_1)^2(2\delta+\alpha_1)} \quad \frac{1}{\alpha_1^4(\delta+\alpha_1)^3(2\delta+\alpha_1)^2(3\delta+\alpha_1)}.$$

Conjecturally, when considering the reduced word $(1,0,1,0,\ldots,1,0)$ with n occurrences of 0 and 1 we will get fractions of the form

$$\frac{1}{\alpha_1^n(\delta+\alpha_1)^{n-1}\cdots((n-1)\delta+\alpha_1)}.$$

2 Type $A_2^{(1)}$

The Weyl group of type $A_2^{(1)}$ is generated by s_0, s_1, s_2 satisfying braid relations. Consider the unipotent cell of $\mathbb{C}[N]$ associated with the reduced word $(1,2,0,1,2,0)$. Let $M_1,\ldots,M_6$ denote the corresponding KLR modules. Assuming things go as in finite simply-laced type, i.e. $\bar{D}(M_i) = 1/P_i$ with the P_i satisfying the relations

$$P_j P_{j_-} = \beta_j \prod_{k<j<k_+} P_k$$

we obtain the following:

$$\begin{aligned}
&P_1 = \alpha_1, \quad P_2 = \alpha_1(\alpha_1+\alpha_2), \quad P_3 = (\delta+\alpha_1)(\alpha_1+\alpha_2)\alpha_1^2, \\
&P_4 = (\delta+\alpha_1+\alpha_2)(\delta+\alpha_1)(\alpha_1+\alpha_2)^2\alpha_1^2, \\
&P_5 = (2\delta+\alpha_1)(\delta+\alpha_1+\alpha_2)(\delta+\alpha_1)^2(\alpha_1+\alpha_2)^2\alpha_1^3, \\
&P_6 = (2\delta+\alpha_1)(2\delta+\alpha_1+\alpha_2)(\delta+\alpha_1+\alpha_2)^2(\delta+\alpha_1)^2(\alpha_1+\alpha_2)^3\alpha_1^3
\end{aligned}$$

Probably for the reduced word $(s_1 s_2 s_0)^n$ we shall obtain two families of polynomials given by

$$Q_n = ((n-1)\delta+\alpha_1)((n-1)\delta+\alpha_1+\alpha_2)\cdots(\delta+\alpha_1+\alpha_2)^{n-1}(\delta+\alpha_1)^{n-1}(\alpha_1+\alpha_2)^n\alpha_1^n$$

and

$$R_n = ((n-1)\delta + \alpha_1)((n-2)\delta + \alpha_1 + \alpha_2) \cdots (\delta + \alpha_1)^{n-1}(\alpha_1 + \alpha_2)^{n-1}\alpha_1^n.$$

On the example $n = 2$, we have $P_1 = R_1, P_2 = Q_1, P_3 = R_2, P_4 = Q_2, P_5 = R_3, P_6 = Q_3$. First note that

$$R_{n+1}/Q_n = \alpha_1(\alpha_1 + \delta) \cdots (\alpha_1 + n\delta).$$

We also observe the following:

$$\begin{aligned} &P_3/P_1 = \alpha_1(\alpha_1 + \alpha_2)(\delta + \alpha_1), \quad P_3/P_2 = \alpha_1(\delta + \alpha_1), \\ &P_5/P_4 = \alpha_1(\delta + \alpha_1)(2\delta + \alpha_1), \quad P_5/P_3 = \alpha_1(\alpha_1 + \alpha_2)(\delta + \alpha_1)(\delta + \alpha_1 + \alpha_2)(2\delta + \alpha_1) \end{aligned}$$

So P_3/P_1 recovers the first eqm of weight $(2, 1)$ you computed in type A_2. It seems that one can actually recover the whole first family for this weight, by performing $P_2/P_1 \times R_n/Q_{n-1}$. I do not know how to obtain the second family.

Another question: do the polynomials

$$Q_{n+1}/Q_n = \alpha_1 \cdots (\alpha_1 + n\delta)(\alpha_1 + \alpha_2) \cdots (\alpha_1 + \alpha_2 + n\delta)$$

Anne: Yes I think this one coincedes with Lusztig datum $(n, 0, n)$. Have to think about the one below.

and

$$R_{n+1}/R_n = \alpha_1 \cdots (\alpha_1 + n\delta)(\alpha_1 + \alpha_2) \cdots ((n-1)\delta + \alpha_1 + \alpha_2)$$

coincide with the denominators of certain type A_2 eqms?