Deformed DH measure?

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1 Goal

In [BKK19] the authors introduce a measure $\overline{D} : \mathbb{C}[N] \to \mathbb{C}(\mathfrak{t})$ defined by

$$\overline{D}(f) = \sum_{\mathbf{i} \in \mathrm{Seq}(\nu)} \langle e_{\mathbf{i}}, f \rangle \overline{D}_{\mathbf{i}} \qquad f \in \mathbb{C}[N]_{-\nu} \tag{1}$$

where

$$\overline{D}_{\mathbf{i}} = \prod_{k=1}^{p} \frac{1}{\alpha_{i_1} + \cdots + \alpha_{i_k}} \qquad p = \mathrm{ht}\,\nu \tag{2}$$

By BKK Proposition 8.4, or Thesis Theorem 3.4.2, it can be realized more invariantly as

$$\overline{D}(f)(x) = f(n_x) \tag{3}$$

They (BKK, Proposition A.5) and we (Thesis, Proposition 5.4.4) show that when $f = b_Z$ is an element of the MV basis indexed by a stable MV cycle of weight ν, i.e. $Z \subset \overline{S_+^\nu \cap S_-^0}$

$$\overline{D}(b_Z) = \varepsilon_{L_0}^T(Z) \tag{4}$$

When $f = c_Y$ is an element of the dual semicanonical basis indexed by an irreducible component Y of $\Lambda(\nu)$

$$\overline{D}(c_Y) = \sum_{\mathbf{i} \in \mathrm{Seq}(\nu)} \chi(F_{\mathbf{i}}(M)) \overline{D}_{\mathbf{i}} \qquad M \in Y \tag{5}$$

We would like an “asymptotic” $\hbar$-version of $\overline{D}$ which manifests as a $T \times \mathbb{C}^\times$-equivariant multiplicity on elements of the MV basis:

$$\overline{D}_\hbar(b_Z) = \varepsilon_{L_0}^{T \times \mathbb{C}^\times}(Z) \in \mathbb{C}(\mathfrak{t}, \hbar)$$

Recall the MVy isomorphism $\tilde{\Phi} : \mathbb{T}_\mu \cap \mathcal{N} \to G_1[t^{-1}]t^\mu$ defined by

$$\tilde{\Phi}(A) = t^\mu + a(t) \qquad a(t)_{ij} = -\sum A_{ij}^k t^{k-1} \tag{6}$$

where A_{ij}^k denotes the kth entry from the left of the $\mu_j \times \mu_i$ block. We will use it to see what is the $T \times \mathbb{C}^\times$ multidegree of $Z \subset \mathrm{Gr}_\mu$. Let $s \in \mathbb{C}^\times$ act by loop rotation, and $g = \mathrm{diag}(t_1, \ldots, t_m) \in T$ act by conjugation. Note that if we allowed $g \in T(\mathcal{O})$ then these actions would not commute. As it is we set

$$(g, s) \cdot \tilde{\Phi}(A) = s^\mu((s^{-1}t)^\mu + g \cdot a(s^{-1}t)) = t^\mu + g \cdot s^\mu a(s^{-1}t)$$

where

$$\begin{aligned}(g \cdot s^\mu a(st))_{ij} &= -t_j t_i^{-1} \sum_{k=1}^{\mu_i} A_{ij}^k s^{k-1+\mu_j} t^{k-1} \\ &= -t_j t_i^{-1}(A_{ij}^1 s^{\mu_j} + A_{ij}^2 s^{1+\mu_j} t + \cdots + A_{ij}^{\mu_i} s^{\mu_i+\mu_j-1} t^{\mu_i-1}) \\ &= -t_j t_i^{-1} s^{\mu_i+\mu_j-1}(A_{ij}^1 s^{-\mu_i+1} + A_{ij}^2 s^{-\mu_i+2} t + \cdots + A_{ij}^{\mu_i} t^{\mu_i-1})\end{aligned}$$

In the limit $s \to \infty$ the $A_{ij}^{\mu_i}$ term dominates, so the multidegree of $a(t)_{ij} = 0$ is

$$z_j - z_i + (\mu_i + \mu_j - 1)\hbar$$

In particular, the multidegree of zero in $\mathfrak{n}$ will be

$$\prod_{\beta \in \Delta_+} (\beta + \hbar)$$

because all $\mu_i = 1$. More generally, the multidegree of zero in $\mathbb{T}_\mu \cap \mathfrak{n}$ will be

$$\prod_{1 \le i < j \le m} (z_i - z_j + (\mu_i + \mu_j - 1)\hbar)$$

Now in order to define the asymptotic analogue of $\overline{D}$ on $\mathbb{C}[N]$ we probably have to deform $\mathbb{C}[N]$. Why? Let's recall how the geometric Satake works. The class of $Z \in \mathrm{Irr}\,\overline{\mathrm{Gr}^\lambda \cap S_-^\mu}$ in the Borel–Moore homology of $\overline{\mathrm{Gr}^\lambda \cap S_-^\mu}$ is identified with a vector in $L(\lambda)_\mu$ such that the class of the fixed point L_λ is sent to the highest weight vector v_λ. Then $v \in L(\lambda)$ is sent to $f \in \mathbb{C}[N]$ such that

$$f(n) = v_\lambda^*(n \cdot v)$$

But what happens to the $\mathbb{C}^\times$ action under this map? What happens to the T action for that matter?

Joel says: what do you mean by the $\mathbb{C}^\times$ action and the T action? Careful that you are not getting confused with the actioin on the MV cycles (the action used for the multidegrees). There is a T action on $L(\lambda)$ and one on $\mathbb{C}[N]$, and the map $L(\lambda) \to \mathbb{C}[N]$ is equivariant, up to a character.

It may be helpful to recall that if Z is stable of type ν then b_Z is unique such that whenever $\nu + \mu \in P_+$

$$t^\mu Z \subset \overline{\mathrm{Gr}^{\nu+\mu}} \Rightarrow b_Z = \Psi_{\nu+\mu}([t^\mu Z])$$

2 Two $\mathbb{C}^\times$ actions

3 Ben's suggestion

To investigate the special case of Gelfand–Tsetlin modules and W-algebras discussed at the end of [KTW+19]. I have yet to do any examples.

Further references include papers of Losev ([Los10a, Los10b, L+11]) and Brundan and Kleschev ([BK06, BK09]) on W-algebras.

References

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