Examples Compendium

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Disjoint, non-dominant μ

Example 1. $\lambda_1 = (1,0,0)$, $\lambda_2 = (1,1,0)$, $\mu_1 = (0,1,0)$, $\mu_2 = (1,0,1)$. Joel: $\mu = \mu_1 + \mu_2$ determines the blocks we have on the RHS of the BD MVy isomorphism of Equation **??**.

In the non-BD case, MVy establish

$$\overline{\mathrm{Gr}^\lambda} \cap \mathcal{W}^\mu \to \left\{ X = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * \\ \hline 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * \\ \hline * & 0 & * & 0 & * \end{array}\right] \Big| X \in \overline{\mathbb{O}}_\lambda \right\}$$

In the BD case the RHS will consist of the same block like matrices X but now having eigenvalues $s, 0$ such that $X - s\big|_{E_s} \in \mathbb{O}_{\lambda_2}$

Some multiplicity

Example 2 (Joel's exercise). It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for $\mathbf{SL}_3$ of weights $2\alpha_1$ and $2\alpha_2$. Following the notation from the mvbasis paper, this would correspond to the following multiplication:

$$x^2y^2 = (xy - z)^2 + 2(xy - z)z + z^2 \,.$$

For this example, I think we need $\lambda_1 = (2,0,0)$, $\lambda_2 = (2,2,0)$, $\mu_1 = (0,2,0)$, $\mu_2 = (2,0,2)$.

Let's also try

$$\begin{array}{lll} \lambda_2 = (4,0,0) & \lambda_1 = (4,4,0) & \lambda = (8,4,0) \\ \mu_2 = (2,2,0) & \mu_1 = (4,2,2) & \mu = (6,4,2) \end{array}$$

Note that there is only one SSYT of shape λ_i and weight μ_i

$$\tau_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \end{array} \qquad \tau_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 & 3 \\ \hline \end{array}$$

Note also that

$$
\begin{aligned}
&t^4(t-s)^2 = t^4(t^2 - 2st + s^2) = t^6 - 2st^5 + s^2t^4 \\
&t^2(t-s)^2 = t^4 - 2st^3 + s^2t^2
\end{aligned}
$$

Elements of $\mathbb{T}^+_{\mu_1,\mu_2}$ will take the form

$$
A = \left[\begin{array}{cccccc|cccc|cc}
 & 1 & & & & & & & & & & \\
 & & 1 & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
 & & & & 1 & & & & & & & \\
 & & & & & 1 & & & & & & \\
 & & & & -s^2 & 2s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\
\hline
 & & & & & & & 1 & & & & \\
 & & & & & & & & 1 & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & -s^2 & 2s & c_1 & c_2 \\
\hline
 & & & & & & & & & & & 1 \\
 & & & & & & & & & & &
\end{array}\right]
$$

The tableau tells us for each $1 \leq i \leq 12$

$$
\text{Jordan type of } A\big|_{\mathrm{Span}(e_1,\ldots,e_i)\cap E_0} \text{ is shape of } \tau_1\big|_{\text{first } i \text{ boxes}}
$$

So take $i = 10$. Then A restricted to $\mathbb{C}^{10} \cap E_0$ should have Jordan type $(4, 2)$. Anne: How to do it box by box? s columns somehow correspond to τ_2 boxes. Therefore $a_1 = 0$. The 4-cycle is obvious. For the 2-cycle we require $a_2 = 0$.

$$
\begin{aligned}
&e_1 \leftarrow e_2 \leftarrow e_3 \leftarrow e_4 \\
&e_7 \leftarrow e_8
\end{aligned}
$$

Now looking at all of A we have continue our 2-cycle to a 4-cycle. Roger found

$$
e_{10} - \left(\frac{2s}{c_1} + \frac{s^2c_2}{c_1^2}\right) e_{11} + \frac{s^2}{c_1} e_{12}
$$

This requires $c_1^2a_4 + c_1b_2s^2 - 2sc_1b_1 - s^2c_2 = 0$ and $a_3c_1 + s^2b_1 = 0$.

Now looking for the s-eigenspace, we expect $A - s\big|_{\mathbb{C}^6\cap E_s}$ to have Jordan type 2. The kernel is spanned by $e_1 + se_2 + s^2e_3 + s^3e_4 + s^4e_5 + s^5e_6$. It is continued to a 2-cycle by/the 2-cycle is generated by $-\frac{5}{s}e_1 - 4e_2 - 3se_3 - 2s^2e_4 - s^3e_5$. The 3-cycle is maybe $(1/s^2, -4/s, -8, 5s, 3s^2, 2s^3, -s^2/a_3, -s^3/a_3, -s^4/a_3, -s^5/a_3)$ padded with zeros.

$$A - s = \left[\begin{array}{cccccc:cccc:cc}
-s & 1 & & & & & & & & & & \\
 & -s & 1 & & & & & & & & & \\
 & & -s & 1 & & & & & & & & \\
 & & & -s & 1 & & & & & & & \\
 & & & & -s & 1 & & & & & & \\
 & & & & -s^2 & s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\
\hdashline
 & & & & & & -s & 1 & & & & \\
 & & & & & & & -s & 1 & & & \\
 & & & & & & & & -s & 1 & & \\
 & & & & & & & & -s^2 & s & c_1 & c_2 \\
\hdashline
 & & & & & & & & & & -s & 1 \\
 & & & & & & & & & & & -s
\end{array}\right]$$

Simple root weights, things working

Example 3. Let

$$\begin{array}{lll}
\mu_1 = (3,1,1) & \lambda_1 = (3,2,0) & \mu = (3,3,1) \\
\mu_2 = (0,2,0) & \lambda_2 = (2,0,0) & \lambda = (5,2,0)
\end{array}$$

and consider the companion matrices of

$$p_1(t) = t^3 \quad p_2(t) = t(t-s)^2 = t^3 - 2st^2 + s^2t \quad p_3(t) = t$$

$$X = \left[\begin{array}{ccc:ccc:c}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d & e & f & g \\
\hdashline
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -s^2 & 2s & k \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Example 4. Let

$$\begin{array}{ll} \lambda_1 = (3,2,0) & \mu_1 = (3,1,1) \\ \lambda_2 = (2,0,0) & \mu_2 = (1,1,0) \end{array}$$

so the first MV cycle $Z_1 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_1\}$ and the second MV cycle $Z_2 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_2\}$. Their fusion product corresponds to two $\mathbb{P}^2$'s intersecting along a $\mathbb{P}^1$. That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where $Z_+ \cong Z_- \cong \mathbb{P}^2$. We have

$$X = \begin{bmatrix}
0 & 1 & & & & & \\
 & 0 & 1 & & & & \\
 & & 0 & 1 & & & \\
 & & & s & a & b & c \\
 & & & & 0 & 1 & 0 \\
 & & & & 0 & s & d \\
 & & & & & & 0
\end{bmatrix}$$

The 0-generalized eigenspace E_0 of X is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, X e_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in $\ker X$, either $a = 0$ or $c = d = 0$, but the latter case cannot give a 2-cycle as $e_7 \notin \operatorname{im} X$. Then $a = 0$ and we obtain a 2-cycle

$$\left\{X\left(e_6 - \frac{s}{d}e_7\right), e_6 - \frac{s}{d}e_7\right\} = \left\{e_5, e_6 - \frac{s}{d}e_7\right\}.$$

We also obtain the equations $b \neq 0$, $d \neq 0$, and $sc - bd = 0$ from this.

For the s-generalized eigenspace E_s, we need $a + sb \neq 0$ to obtain a 2-cycle, which can be taken as

$$
\begin{aligned}
&\left\{(X-sI)\left(e_2+2se_3+3s^2e_4+\frac{s^3}{a+sb}e_5+\frac{s^4}{a+sb}e_6\right),\right.\\
&\qquad\qquad \left. e_2+2se_3+3s^2e_4+\frac{s^3}{a+sb}e_5+\frac{s^4}{a+sb}e_6\right\}\\
&=\left\{e_1+se_2+s^2e_3+s^3e_4, e_2+2se_3+3s^2e_4+\frac{s^3}{a+sb}e_5+\frac{s^4}{a+sb}e_6\right\}
\end{aligned}
$$

The minimal polymonial is $X^3(X-sI)^2$, which when equated to 0 gives again the equation $cs-bd=0$. Thus the defining equations are

$$\{a=0, cs-bd=0\}.$$

When we take $s=0$, we get the equations

$$\{a=0, bd=0\}$$

which corresponds to two $\mathbb{A}^2$'s intersecting along an $\mathbb{A}^1$. This is indeed an open subset of $\mathbb{P}^2\cup_{\mathbb{P}_1}\mathbb{P}^2$, as required.

Example 5 (Continued...). The matrix X from the previous example defines (under MVy) the matrix

$$
g=\begin{bmatrix}(t-s)t^3 & & \\ -bt & (t-s)t & \\ -c & -d & t\end{bmatrix}
$$

in $G(\mathcal{O})$. Indeed the various blocks of X are in a precise sense the companion matrices of the polynomial entries of g

In Gr the element g defines the lattice

$$gL_0=\mathbb{C}[t]\langle (t-s)t^3e_1-(a+bt)e_2-ce_3, (t-s)te_2-de_3, te_3\rangle$$

and computing the matrix of the action of t on the quotient L_0/L in the basis

$$\{[e_1],[te_1],[t^2e_1],[t^3e_1],[e_2],[te_2],[e_3]\}$$

recovers X up to a transpose of course.

Now let's see what we get when we invert t and $t-s$ respectively.

First let's invert t by considering $L_2=L\otimes\mathbb{C}[\![t-s]\!]$.

$$L_2=\mathbb{C}[t,t^{-1}]\langle (t-s)e_1-\frac{a+bt}{t^3}e_2-\frac{c}{t^3}e_3, (t-s)e_2-\frac{d}{t}e_3, e_3\rangle$$

so in L_0/L_2 we have

$$t[e_1]=s[e_1]+\frac{a+bt}{t^3}[e_2]+\frac{c}{t^3}[e_3]\qquad t[e_2]=s[e_2]+\frac{d}{t}[e_3]\qquad [e_3]=0$$

and

$$\left[t\big|_{L_0/L_2}\right]_{\{[e_1],[e_2]\}} = \begin{bmatrix} s & \\ \frac{a+bt}{t^3} & s \end{bmatrix}$$

which upon subtracting sI gives a matrix having block type μ_2 and Jordan type $\lambda_2 = (2)$ assuming $\frac{a+bt}{t^3} \neq 0$.

Next let's invert $t - s$ by considering $L_1 = L \otimes \mathbb{C}[\![t]\!]$.

$$\begin{aligned} L_1 &= \mathbb{C}[t,(t-s)^{-1}]\langle t^3 e_1 - \frac{a+bt}{t-s}e_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle \\ &= \langle t^3 e_1 - \frac{a}{t-s}e_2 - \frac{b}{t-s}te_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle \end{aligned}$$

so in L_0/L_1 we have

$$\begin{aligned} t[e_1] &= [te_1] \\ t[te_1] &= [t^2 e_1] \\ t[t^2 e_1] &= \frac{a}{t-s}[e_2] + \frac{b}{t-s}t[e_2] + \frac{c}{t-s}[e_3] \\ &= \frac{b}{t-s}\frac{d}{t-s}[e_3] + \frac{c}{t-s}[e_3] \\ &= \frac{bd + (t-s)c}{(t-s)^2}[e_3] \\ &= \frac{bd - sc}{(t-s)^2}[e_3] + \frac{c}{(t-s)^2}t[e_3] = 0 \\ t[e_2] &= \frac{d}{t-s}[e_3] \\ t[e_3] &= 0 \end{aligned}$$

and

$$\left[t\big|_{L_0/L_1}\right]_{\{[e_1],[te_1],[t^2e_1],[e_2],[e_3]\}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & & 0 & 0 & \\ & & 0 & \frac{d}{t-s} & 0 \end{bmatrix}$$

taking transpose

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}$$

which has block type μ_1 and Jordan type $\lambda_1 = (3,2)$ assuming $d \neq 0$.

I have used the relations Roger found (and I checked) $a = 0$ and $cs - bd = 0$ in the calculations above.

To sum up, the pair of matrices above should contain the same information as the matrix from the previous example

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix} \Leftrightarrow \left(\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} s & \frac{b}{t^2} \\ & s \end{bmatrix} \right)$$

Wish. Given L in $\mathcal{G}r_n^{\mathrm{BD}}$ define a map to T_μ just like MVy by taking $[t\big|_{L_0/L}]$ and use the fact that $[t\big|_{L_0/L_i}]$ for $i = 1, 2$ are companion matrices of the right type, piece together two MVy isomorphisms to make a BD MVy iso.

Equivalent linear algebra question(?): If $p(t, t-s) = p_1(t)p_2(t-s)$ then how are $C(p_1)$, $C(p_2)$, and $C(p)$ related?

Non simple root weights

Example 6 (Anne). Let $G = \mathbf{SL}_3$ and $\underline{\mathbf{i}} = 121$. Take $n_\bullet^1 = (1,0,0)$, and $n_\bullet^2 = (1,0,1)$ or $(0,1,0)$. So $\mu_1, = (2,2,1)$, $\lambda_1 = (3,1,1)$ and $\mu_2 = (1,1,1)$, $\lambda_2 = (2,1,0)$. Anne: We should show that order does not matter. Then $\mathbb{T}^+_{\mu_1,\mu_2} \cap \mathbb{O}_{\lambda_1,\lambda_2}$ is made up of elements

$$\begin{bmatrix} 0 & 1 & 0 & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & s & 0 & a_2 & a_3 & b_1 & b_2 \\ & & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 1 & & \\ & & & 0 & 0 & s & c_1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & 0 & s \end{bmatrix}$$

and zero eigenspace conforming to the shape

$$\lambda_1 = \ydiagram{3,1,1}$$

is made of three cycles

$$\begin{gathered} a_2 e_1 \leftarrow a_2 e_2 - s e_4 \leftarrow a_2 e_3 - s e_5 \\ e_4 \\ e_7 \end{gathered}$$

while the s-eigenspace confirming to the shape

$$\lambda_2 = \ydiagram{2,1}$$

is made of the two cycles

$$
\begin{aligned}
&e_1 + se_2 + s^2 e_3 \leftarrow e_2 + 2se_3 + e_7 + se_8 \\
&e_4 + se_5 + s^2 e_6
\end{aligned}
$$

assuming $b_2 = s$ and $a_2 + sa_3 = 0$.

Example 7 (Roger). Let

$$
\begin{aligned}
\lambda_1 &= (2,0,0,0) \quad & \mu_1 &= (1,1,0,0) \\
\lambda_2 &= (2,2,1,0) \quad & \mu_2 &= (3,2,1,1)
\end{aligned}
$$

so $\lambda_1 - \mu_1 = \alpha_1$ and $\lambda_2 - \mu_2 = \alpha_2 + \alpha_3$. We have the following young tableaux:

$$
\tau_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \tau_2 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \tau_2' = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}
$$

where τ_1 corresponds to the module S_1, τ_2 corresponds to the module $2 \to 3$, and τ_2' corresponds to the module $2 \leftarrow 3$.

The matrix we are considering is

$$
X = \begin{bmatrix}
0 & 1 & & & & & \\
 & 0 & 1 & & & & \\
 & -s^2 & 2s & a & b & c & d \\
 & & & 0 & 1 & & \\
 & & & & s & e & f \\
 & & & & & s & g \\
 & & & & & & s
\end{bmatrix}
$$

such that $\dim E_0 = 2$, $\dim \ker X = 1$, $\dim E_s = 5$, and $\dim \ker(X - sI) = 3$ where E_0 and E_s are the 0- and s-generalized eigenspaces.

We see that the two-cycle in E_0 is

$$
\left\{ X\left(e_2 + \frac{s^2}{a} e_4\right), e_2 + \frac{s^2}{a} e_4 \right\} = \left\{ e_1, e_2 + \frac{s^2}{a} e_4 \right\}.
$$

As τ_2 and τ_2' both share $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$, we can find a 2-cycle from just the upper-left 3×3 block, and an additional vector in $\ker(X - sI)$ from the upper-left 5×5-submatrix. The 2-cycle from the 3×3 block is

$$
\left\{ e_1 + se_2 + s^2 e_3, -\frac{2}{s} e_1 - e_2 \right\}.
$$

The additional vector in $\ker(X - sI)$ is $e_4 + se_5$ and this requires $a + sb = 0$.

Now consider the case that the young diagram we are working with is τ_2. Then we have $e_4 + se_5$ part of a 2-cycle that can be found by looking at the upper-left 6×6-submatrix. We find that the 2-cycle is

$$
\left\{ e_4 + se_5, -\frac{1}{s} e_4 + \frac{s}{e} e_6 \right\}
$$

and this requires that $ae - s^2c = 0$.

The last vector in $\ker(X - sI)$ comes from the entire $X - sI$ and we see it is $-fe_6 + ee_7$, which requires $g = 0$ and $ed - cf = 0$.

For the case τ_2', we start with find the third vector in $\ker(X - sI)$ from the upper-left 6×6-submatrix. We see that it is e_6, which requires $c = 0$ and $e = 0$.

For the remaining 2-cycle, we note that $e_4 + se_5 \notin \operatorname{col}(X - sI)$ but $e_6 \in \operatorname{col}(X - sI)$ so our 2-cycle is

$$\left\{ e_6, \frac{1}{g} e_7 \right\}$$

which requires $d = 0$ and $f = 0$.

From the minimal polynomial, we have $X^2(X - sI)^2 = 0$ which gives us the equations

$$a + sb = cs + eb = bf + cg + ds = esg = 0.$$

Taking $s \to 0$, we have the following equations for our two cases of τ_2 and τ_2':

τ_2	τ_2'
$a = 0$	$a = 0$
$g = 0$	$c = 0$
$eb = 0$	$d = 0$
$bf = 0$	$e = 0$
$ed - cf = 0$	$f = 0$

For the τ_2 case, the coordinate ring is

$$\frac{\mathbb{C}[a, b, c, d, e, f, g]}{\langle a, g, eb, bf, ed - cf \rangle} \cong \frac{\mathbb{C}[b, c, d, e, f]}{\langle eb, bf, ed - cf \rangle} = \frac{\mathbb{C}[b, c, d, e, f]}{\langle e, f \rangle \cap \langle b, ed - cf \rangle}$$

Hence the associated algebraic set is reducible with two irreducible components. The component corresponding to the ideal $\langle e, f \rangle$ is $\mathbb{A}^3$, which corresponds to $\mathbb{P}^3$, while the ideal $\langle b, ed - cf \rangle$ corresponds to the toric variety whose toric polytope is a square-based pyramid.

As τ_2 corresponds to the module $2 \to 3$, the irreducible components should correspond to the modules $P_1 = 1 \to 2 \to 3$ and $1 \leftarrow 2 \to 3$. Indeed, the MV cycle corresponding to P_1 is the Grassmannian $Gr(1, 4) \cong \mathbb{P}^3$ and for $1 \leftarrow 2 \to 3$, we do get a toric variety with polytope the square-based pyramid.

However for the τ_2' case, the coordinate ring is

$$\frac{\mathbb{C}[a, b, c, d, e, f, g]}{\langle a, c, d, e, f \rangle} \cong \mathbb{C}[b, g]$$

which corresponds to $\mathbb{A}^2$. Since τ_2' corresponds to the module $2 \leftarrow 3$, we expect two irreducible components corresponding to the modules $P_3 = 1 \leftarrow 2 \leftarrow 3$ and $1 \to 2 \leftarrow 3$. P_3 corresponds to the variety $Gr(3, 4) \cong \mathbb{P}^3$ and $1 \to 2 \leftarrow 3$ also corresponds to a toric variety whose polytope is a square-based pyramid (?).