Working title: Mirković–Vybornov fusion in Beilinson–Drinfeld Grassmannian

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1 Introduction

The BD Grassmannian. The convolution Grassmannian. Distinguished orbits, slices therein. Mirković–Vybornov [MV07, MV19], Cautis–Kamnitzer [CK18], Anderson–Kogan [AK05].

2 Notation

Let Gr denote the ordinary affine Grassmannian, $\mathcal{G}$ the Beilinson–Drinfeld affine Grassmannian, and $\mathfrak{G}$ the convolution affine Grassmannian.

Definition 1. The **BD Grassmannian** is the set

$$\begin{aligned}\{(V,\sigma) : V &\text{ is a rank } m \text{ vector bundle on } \mathbb{P}^1\\ &\text{and } \sigma : V \dashrightarrow \mathscr{O}_{\mathbb{P}^1}^m \text{ is a trivialization}\\ &\text{defined away from finitely many points in } \mathbb{A}^1\}\end{aligned} \tag{1}$$

More generally, one can define a BD grassmannian over any smooth curve C as the reduced ind-scheme $\mathcal{G}_C$ fibered over a finite symmetric power of C such that the fibre over the point $\vec{p}$ is a collection of vector bundles over C which are trivial away from $\vec{p}$ viewed also as a subset of C. To [MV19] the rank m of the trivial fibres $\mathscr{O}_C^m$ is the of the group $\mathbf{GL}_m\mathbb{C}$.

To quote [BGL20] the BD Grassmannian is a relative version of the affine Grassmannian where the base is the space of effective divisors on a smooth curve. The choice of $\mathbb{A}^1$ amply satisfies our needs and offers three advantages: there is a natural global coordinate it, every G-torsor on it is trivializable, and the monodromy of any local system is trivial. Formally, $\mathcal{G}$ is the functor on the category of commutative $\mathbb{C}$-algebras that assigns to an algebra R the set of isomorphism classes of triples $(\vec{p}, V, \sigma)$ where $\vec{p} \in \mathbb{A}^n(R)$, V is a $G^\vee$-torsor over $\mathbb{A}^1_R$ and σ is a trivialization of V away from $\vec{p}$. They denote by π the fibration $\mathcal{G} \to \mathbb{A}^n$ (forgetting V and σ).

Their simplified description is: it's the set of pairs $(\vec{p}, [\sigma])$ where $\vec{p} \in \mathbb{C}^n$ and $[\sigma]$ is an element of the homogeneous space

$$G^\vee(\mathbb{C}[z, (z-p_1)^{-1}, \ldots, (z-p_n)^{-1}])/G^\vee(\mathbb{C}[z])$$

Their example, our setting:

Example 1. When $G = \mathbf{GL}_m\mathbb{C}$ the datum of $[\sigma]$ is equivalent to the datum of the $\mathbb{C}[z]$-lattice $\sigma(L_0)$ in $\mathbb{C}(z)^m$ with $L_0 = \mathbb{C}[z]^m$ denoting the **standard lattice**. Set $f_{\vec{p}} = (z-p_1)\cdots(z-p_n)$. Then a lattice L is of the form $\sigma(L_0)$ if and only if there exists a positive integer k such that $f_{\vec{p}}^k(L_0) \subseteq L \subseteq f_{\vec{p}}^{-k}(L_0)$ and for each k they denote by $\mathcal{G}_k$ the subset of $\mathcal{G}$ consisting of pairs $(\vec{p}, L)$ such that this sandwhich condition holds. They identify $\mathbb{C}[z]/(f_{\vec{p}}^{2k})$ with the vector space of polynomials of degree strictly less than $2kn$, and $L_0/f_{\vec{p}}^{2k}L_0$ with its Nth product. Then

$$\mathcal{G}_k \overset{\text{Zariski closed}}{\subset} \mathbb{C}^n \times \bigcup_{d=0}^{2knN} G_d(L_0/f_{\vec{p}}^{2k}L_0)$$

where $G_d(?)$ denotes the ordinary Grassmann manifold of d-planes in the argument.

Definition 2. The **deformed convolution Grassmannian** is [not needed?] pairs $(\vec{p}, [\vec{\sigma}])$ where $\vec{p} \in \mathbb{C}^n$ and $\vec{\sigma}$ is in

$$G^\vee(\mathbb{C}[z, (z-p_1)^{-1}]) \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} G^\vee(\mathbb{C}[z, (z-p_n)^{-1}])/G^\vee(\mathbb{C}[z])$$

with a map down to $\mathcal{G}$ defined by $(\vec{p}, [\vec{\sigma}]) \mapsto (\vec{p}, [\sigma_1 \cdots \sigma_n])$.

To steal the follow-up example in [BGL20] where the above definition is also copied from...

Example 2. When $G = \mathbf{GL}_m\mathbb{C}$ this deformation is described by the datum of $\vec{p} \in \mathbb{C}^n$ and a sequence $(L_1, \ldots, L_n)$ of $\mathbb{C}[z]$-lattices in $\mathbb{C}(z)^m$ such that for some $k \in \mathbb{Z}$ and for all $j \in \{1 \ldots n\}$

$$(z-p_j)^k L_{j-1} \subset L_j \subset (z-p_j)^{-k} L_{j-1}$$

where again $L_0 = \mathbb{C}[z]^m$ denotes the standard lattice, and now $L_j = (\sigma_1 \cdots \sigma_j)(L_0)$. Very nice. Very concrete. They can partition the deformation into **cells** by specifying the **relative positions** of the pairs (L_{j-1}, L_j) in terms of **invariant factors**.

To be continued: [BGL20] go on to describe the fibres of the composition deformation to $\mathcal{G}$ to $\mathbb{C}^n$ and their description maybe helpful.

For $\mu \in P$ and $p \in \mathbb{C}$ they define

$$\tilde{S}_{\mu|p} = (z-p)^\mu N^\vee(\mathbb{C}[z, (z-p)^{-1}]) = N^\vee(\mathbb{C}[z, (z-p)^{-1}])(z-p)^\mu$$

They note that $\mathbb{C}(\!(z-p)\!)$ is the completion of $\mathbb{C}(z)$ at "the place defined by p" and identify $\mathbb{C}[\![z-p]\!]$ with $\mathbb{C}[\![z]\!]$ and $\mathbb{C}(\!(z-p)\!)$ with $\mathbb{C}(\!(z)\!)$.

They claim that

$$N^\vee(\mathbb{C}[z,(z-p)^{-1}])/N^\vee(\mathbb{C}[z]) \to N^\vee(\mathbb{C}(\!(z-p)\!))/N^\vee(\mathbb{C}[\![z-p]\!]) \cong N^\vee(\mathcal{K})/N^\vee(\mathcal{O})$$

is bijective, and that mapping Gr and multiplying by $(z-p)^\mu$ one gets

$$\tilde{S}_{\mu|p}/N^\vee(\mathbb{C}[z]) \cong S_\mu$$

They go on to describe the fusion product (section 5.3) a probably worthwhile read.

Definition 3. Say μ_1 and μ_2 are **disjoint** if $(\mu_1)_i \neq 0 \Rightarrow (\mu_2)_i = 0$ and $(\mu_2)_i \neq 0 \Rightarrow (\mu_1)_i = 0$.

3 Main results

Claim 1. $\widetilde{T_x^a} \to \pi^{-1}(\overline{\mathrm{Gr}^\lambda} \cap \mathrm{Gr}_\mu)$ (this does depend on b! we get something like a springer fibre where the action of [what] on either side has eigenvalues a permutation of b.)

Claim 2. Let $\mathcal{W}^\mu_{\mathrm{BD}} = G_1(\!(t^{-1})\!)t^\mu$. Then $S^{\mu_1+\mu_2}$ is contained in $\mathcal{W}^\mu_{\mathrm{BD}}$ if μ is dominant. Joel: And μ_1, μ_2 are dominant also? Anne: Roger has a proof.

Claim 3. Let $a = (0,s)$ and suppose μ_1 and μ_2 are disjoint ~~"transverse"~~ Let $\mu = \mu_1 + \mu_2$. Then $X \in \widetilde{T_x^a}$ is a $\mu \times \mu$ block matrix, with $(\mu_1)_k \times (\mu_1)_k$ diagonal block conjugate to a $(\mu_1)_k$ Jordan block and $(\mu_2)_k \times (\mu_2)_k$ diagonal block conjugate to $(\mu_2)_k$ Jordan block plus sI.

Question 1. If μ_i is not a permutation of λ_i and λ_i are not "homogeneous" how do we proceed? E.g. if $\mu_1 = (3,0,2)$, $\mu_2 = (0,2,0)$ and $\lambda_1 = (4,1)$, $\lambda_2 = (2,0,0)$.

Question 2. If μ_1 and μ_2 are not disjoint how do we proceed? E.g. if $\mu_1 = (2,2,0)$, $\mu_2 = (1,0,2)$; $\mu_1 = (2,2,1)$, $\mu_2 = (1,0,1)$.

4 Convolution vs BD

Fix $G = \mathbf{GL}(U) \cong \mathbf{GL}_m\mathbb{C}$ and $\{e_1,\dots,e_m\}$ a basis of U. Recall $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ where $\mathcal{K},\mathcal{O}\dots$

Definition 4 (Beilinson–Drinfeld loop Grassmannians). Denoted $\mathcal{G}_{C^{(n)}}$ with C a smooth curve (or formal neighbourhood of a finite subset thereof) and $C^{(n)}$ its nth symmetric power. It is a reduced ind-scheme $\mathcal{G}_{C^{(n)}} \to C^{(n)}$ with fibres of C-lattices $\mathcal{G}_b = \{(b,\mathcal{L}) : b \in C^{(n)}\}$ made up of vector bundles such that $\mathcal{L} \cong U \otimes \mathcal{O}_C$ off b (i.e. over $C - \underline{b}$). The standard lattice is the pair $(\varnothing, \mathcal{L}_0)$ with $\mathcal{L}_0 = U \otimes \mathcal{O}_C$.

Not sure what $\mathcal{O}_C$ means

Notation

The case $n = 1$**.** Fix $b \in C$ and t a choice of formal parameter. Then $\mathcal{G}_b \cong \mathrm{Gr}$.

Why is this called "its group-theoretic realization"

Furthermore, in this case, C-lattices $(b,\mathcal{L})$ are identified with $\mathcal{O}$-submodules $L = \Gamma(\hat{b},\mathcal{L})$ of $U_\mathcal{K} = U \otimes \mathcal{K}$ such that $L \otimes_\mathcal{O} \mathcal{K} \cong U_\mathcal{K}$.

Under this identification, we associate to a given $\lambda \in \mathbb{Z}^m$ the lattice (a priori a $\mathcal{O}$-submodule) $L_\lambda = \oplus_1^m t^{\lambda_i} e_i \mathcal{O}$. Nb. our lattices will be contained in the standard lattice L_0 whereas MVy's lattices contain.

Connected components of Gr are

$G(\mathcal{O})$-orbits are indexed by coweights $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ of G. In terms of lattices

$$\mathrm{Gr}^\lambda = \left\{ L \supset L_0 \,\middle|\, t\big|_{L/L_0} \in \mathbb{O}_\lambda \right\} \tag{2}$$

in the connected component Gr_N are indexed

[MV07] define a map

$$\mathcal{G} \to \mathfrak{G} \tag{3}$$

- Their slice T_x or T_λ
- Their embedding $T_x \to \mathfrak{G}_N$
- N-dim D
- The map $\tilde{\mathbf{m}} : \tilde{\mathfrak{g}}^n \to \mathrm{End}(D)$
- The map $\mathbf{m} : \tilde{\mathcal{N}}^n \to \mathcal{N}$ sending $(x, F_\bullet)$ to x
- The map $\pi : \tilde{\mathfrak{G}}^n \to \mathfrak{G}$ sending $\mathcal{L}_\bullet$ to $\mathcal{L}_n$

The special case $b = \vec{0}$. In this case 0 in the affine quiver variety goes to the point L_λ in the affine Grassmannian, and the preimage of zero in the smooth quiver variety (= the core?) is identified with the preimage of L_λ in the BD Grassmannian.

$$\begin{tikzcd}
\mathfrak{L}(\vec{v}, \vec{w}) \arrow[r] \arrow[d] & \pi^{-1}(L_\lambda) \arrow[d] \\
0 \arrow[r] & L_\lambda
\end{tikzcd}$$

MVy write: "we believe that one should be able to generalize this to arbitrary b" and that's where we come in!

Recall the Mirković–Vybornov immersion [MV07, Theorems 1.2 and 5.3].

Theorem 1. *([MV07, Theorem 1.2 and 5.3]) There exists an algebraic immersion $\tilde{\psi}$*

$$\tilde{\mathbf{m}}^{-1}(T_\lambda) \cap \tilde{\mathfrak{g}}^{n,a,E,\tilde{\mu}} \xrightarrow{\tilde{\psi}} \tilde{\mathfrak{G}}_b^{n,a}(P)$$

5 Statements and Proofs of Results

Anne: Maybe split for now into a Notation section and a Proofs section

Define

$$S_{\mu_1,\mu_2} = N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2}$$

and

$$W_\mu = G_1[[t^{-1}]]t^\mu.$$

Let $|\lambda| = |\lambda_1 + \lambda_2|$ and $|\mu| = |\mu_1 + \mu_2|$.

Anne: Why not $\lambda = \lambda_1 + \lambda_2$ and recall $|\nu|$ in general.

Lemma 1 (Proof in Proposition 2.6 of KWWY)**.** *Suppose μ is dominant. Then*

$$N((t^{-1}))t^{\mu} = N_1[[t^{-1}]]t^{\mu}.$$

Lemma 2. *For dominant μ_1, μ_2, we have*

$$S_{\mu_1,\mu_2} \subset W_{\mu_1+\mu_2}.$$

Proof. We have

$$\begin{aligned}
S_{\mu_1,\mu_2} &= N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&\subset T_1[[t^{-1}]]N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&= T_1[[t^{-1}]]N_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&\subset G_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&= W_{\mu_1+\mu_2}
\end{aligned}$$

where $B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} = B_1[[t^{-1}]]t^{\mu_1+\mu_2}$ since

$$\frac{t}{t-s} = 1 + \frac{s}{t} + \frac{s^2}{t^2} + \cdots \in B_1[[t^{-1}]].$$

□

Define $\mathrm{Gr}^{\lambda_1,\lambda_2} \subset \mathrm{Gr}_{BD}$ to be the family with generic fibre $\mathrm{Gr}^{\lambda_1} \times \mathrm{Gr}^{\lambda_2}$ and 0-fibre $\mathrm{Gr}^{\lambda_1+\lambda_2}$.

Define $\mathbb{O}_{\lambda_1,\lambda_2}$ to be matrices X of size $|\lambda| \times |\lambda|$ such that

$$X\big|_{E_0} \in \mathbb{O}_{\lambda_1} \text{ and } (X - sI)\big|_{E_s} \in \mathbb{O}_{\lambda_2}$$

Let

$$\mu = (\mu^{(1)}, \mu^{(2)}, ..., \mu^{(n)}).$$

Define $\mathbb{T}_{\mu_1,\mu_2}$ to be $|\mu| \times |\mu|$ matrices X such that X consists of block matrices where the size of the i-th diagonal block is $|\mu^{(i)}| \times |\mu^{(i)}|$, for $1 \leq i \leq n$. Each diagonal block is the companion matrix for $t^{\mu_1}(t-s)^{\mu_2}$. Each off-diagonal block is zero everywhere except possibly in the last $\min(\mu_i, \mu_j)$ columns of the last row.

Theorem 2. *We have an isomorphism*

$$\overline{\mathrm{Gr}^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2} \cong \overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}.$$

Anne: Rather, corollary?

Proof. We will prove this similarly to how the usual Mirković–Vybornov isomorphism is proven.

Step 1: Define a map $\mathbb{T}_{\mu_1,\mu_2} \cap \mathcal{N} \to G_1[t^{-1},(t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$.

$$A \mapsto t^{\mu_1}(t-s)^{\mu_2}+a(t,t-s) \mapsto (L_1 \subset L_2) : (t-s)\big|_{L_2/L_1} = A\big|_{E_s}, t\big|_{L_1/L_0} = A\big|_{E_0}$$

Question: 1. is the middle matrix similar to a block matrix? 2. is the composition of these maps some intermediate level of MVy's ψ's

BD Gr as lattices? $(L_1,L_2) \in \mathrm{Gr}\times\mathrm{Gr}$ corresponds to L such that $L \otimes \mathbb{C}[\![t]\!] \cong L_1 \otimes \mathbb{C}[\![t]\!]$ and $L \otimes \mathbb{C}[\![t-s]\!] \cong L_2 \otimes \mathbb{C}[\![t-s]\!]$ where $\otimes = \otimes_{\mathbb{C}[t]}$ or $\otimes_{\mathbb{C}[t-s]}$ respectively even though Roger believes $\mathbb{C}[t] = \mathbb{C}[t-s]$.

Step 2: If $A \in \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}$ then A is sent to $(N_-)_1[t^{-1},(t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$. Anne: Requires MVyBD!

Step 3: Conversely, given $L \in W_{\mu_1+\mu_2}$, want to show surjectivity.

□

Last meeting's todos:

- make sure that the image of our map is in the G_1 orbit
- more generally, define the map, check that the map is well-defined
- Anne: say what little a is, i.e. insert the MVy theorem as stated in CK, or thesis
- Roger: check it

6 Examples

Example 3. $\lambda_1 = (1,0,0)$, $\lambda_2 = (1,1,0)$, $\mu_1 = (0,1,0)$, $\mu_2 = (1,0,1)$. Joel: $\mu = \mu_1 + \mu_2$ determines the blocks we have on the RHS of the BD MVy isomorphism of Equation **??**.

In the non-BD case, MVy establish

$$\overline{\mathrm{Gr}^\lambda} \cap \mathcal{W}^\mu \to \left\{ X = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * \\ \hline 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * \\ \hline * & 0 & * & 0 & * \end{array}\right] \big| X \in \overline{\mathbb{O}}_\lambda \right\}$$

In the BD case the RHS will consist of the same block like matrices X but now having eigenvalues $s, 0$ such that $X - s\big|_{E_s} \in \mathbb{O}_{\lambda_2}$

Example 4. Do Joel's exercise: It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for $\mathbf{SL}_3$ of weights $2\alpha_1$ and $2\alpha_2$. Following the notation from the mvbasis paper, this would correspond to the following multiplication: $x^2y^2 = (xy - z)^2 + 2(xy - z)z + z^2$. For this example, I think we need $\lambda_1 = (2,0,0)$, $\lambda_2 = (2,2,0)$, $\mu_1 = (0,2,0)$, $\mu_2 = (2,0,2)$.

Example 5. Let

$$\begin{aligned} \mu_1 &= (2) \quad \lambda_1 = \quad \mu = (5) \\ \mu_2 &= (3) \quad \lambda_2 = \quad \lambda = \end{aligned}$$

Consider the companion matrix $C(p)$ of

$$p(t) = (t-s)^3 t^2 = (t^3 - 3t^2 s + 3ts^2 - s^3)t^2 = t^5 - 3t^4 s + 3t^3 s^2 - t^2 s^3$$

Let $X = C(p)^T$ so

$$X = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 0 & 0 & s^3 & -3s^2 & 3s \end{bmatrix}$$

Ask that $X\big|_{E_0}$ has Jordan type λ_1 and $X - s\big|_{E_s}$ has Jordan type λ_2. In this rank 1 case we are forced to take $\lambda_i = \mu_i$.

So what are the generalized eigenspaces E_i $(i = 1, 2)$? Note $\dim E_0 = 2$ and $\dim E_s = 3$.

Anne: The basis

$$[1], [t], [\mathbf{1}], [\mathbf{t}], [\mathbf{t^2}]$$

with $t[t] = 0$ and $t[\mathbf{t^2}] = s^3[\mathbf{1}] - 3s^2[\mathbf{t}] + 3s[\mathbf{t^2}]$ *is not the correct basis to consider.* Hence my confusion of yore: what we would like is $t[t] = 0$ no? what the matrix is telling us is that $t[t] = [\mathbf{1}]$. Can we still speak of *two* generalized eigenspaces?

Rather, take B to be the basis $b_1 = e_1$, $b_2 = e_2$, $b_5 = e_5$, $b_4 = Xe_5$, $b_3 = X^2e_5$). In this basis

$$X_B = [X(b_i)] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 1 & 0 \\ 0 & 0 & 0 & s & 1 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

Example 6. Let

$$\begin{aligned} \mu_1 &= (3,1,1) \quad & \lambda_1 &= (3,2,0) \quad & \mu &= (3,3,1) \\ \mu_2 &= (0,2,0) \quad & \lambda_2 &= (2,0,0) \quad & \lambda &= (5,2,0) \end{aligned}$$

and consider the companion matrices of

$$p_1(t) = t^3 \quad p_2(t) = t(t-s)^2 = t^3 - 2st^2 + s^2t \quad p_3(t) = t$$

$$X = \left[\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e & f & g \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & k \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example 7. Let

$$\begin{aligned} \lambda_1 &= (3,2,0) \quad & \mu_1 &= (3,1,1) \\ \lambda_2 &= (2,0,0) \quad & \mu_2 &= (1,1,0) \end{aligned}$$

so the first MV cycle $Z_1 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_1\}$ and the second MV cycle $Z_2 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_2\}$. Their fusion product corresponds to two $\mathbb{P}^2$'s intersecting along a $\mathbb{P}^1$. That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where $Z_+ \cong Z_- \cong \mathbb{P}^2$. We have

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & a & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix}$$

The 0-generalized eigenspace E_0 of X is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, Xe_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in $\ker X$, either $a = 0$ or $c = d = 0$, but the latter case cannot give a 2-cycle as $e_7 \notin \operatorname{im} X$. Then $a = 0$ and we obtain a 2-cycle

$$\left\{X\left(e_6 - \frac{s}{d}e_7\right), e_6 - \frac{s}{d}e_7\right\} = \left\{e_5, e_6 - \frac{s}{d}e_7\right\}.$$

We also obtain the equations $b \neq 0$, $d \neq 0$, and $sc - bd = 0$ from this.

For the s-generalized eigenspace E_s, we need $a + sb \neq 0$ to obtain a 2-cycle, which can be taken as

$$\begin{aligned}&\left\{(X - sI)\left(e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right), e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right\}\\ &= \left\{e_1 + se_2 + s^2e_3 + s^3e_4, e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right\}\end{aligned}$$

The minimal polymonial is $X^3(X - sI)^2$, which when equated to 0 gives again the equation $cs - bd = 0$. Thus the defining equations are

$$\{a = 0, cs - bd = 0\}.$$

When we take $s = 0$, we get the equations

$$\{a = 0, bd = 0\}$$

which corresponds to two $\mathbb{A}^2$'s intersecting along an $\mathbb{A}^1$. This is indeed an open subset of $\mathbb{P}^2 \cup_{\mathbb{P}_1} \mathbb{P}^2$, as required.

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