Exchange Relations Examples

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Last edit: approx. February 13, 2021

Contents

Preliminaries

Let $G = SL_4$. Each example will have a fusion product of MV cycles where each cycle is represented by its corresponding generic module. Two tableau are equivalent if we can pad one to get the other. In determining how many restrictions we have at each step, we have the formula:

$$\#\text{ of free variables in the column} = \#\text{ of box} - \#\text{ of boxes in the column.}$$

We first list out the tableau of each MV cycle:

$$
\begin{array}{llllll}
S_1 & \rightsquigarrow \begin{ytableau} 1 & 2 \end{ytableau} & S_2 & \rightsquigarrow \begin{ytableau} 1 & 1 \\ 2 & 3 \end{ytableau} & S_3 & \rightsquigarrow \begin{ytableau} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{ytableau} \\
1 \to 2 & \rightsquigarrow \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} & 1 \leftarrow 2 & \rightsquigarrow \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} & 1 \to 2 \leftarrow 3 & \rightsquigarrow \begin{ytableau} 1 & 1 & 2 \\ 2 & 4 \\ 3 \end{ytableau} \\
2 \to 3 & \rightsquigarrow \begin{ytableau} 1 & 1 \\ 2 & 3 \\ 4 \end{ytableau} & 2 \leftarrow 3 & \rightsquigarrow \begin{ytableau} 1 & 1 \\ 2 & 4 \\ 3 \end{ytableau} & 1 \leftarrow 2 \to 3 & \rightsquigarrow \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau} \\
P_1 & \rightsquigarrow \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} & P_2 & \rightsquigarrow \begin{ytableau} 1 & 1 & 3 \\ 2 & 4 \end{ytableau} & P_3 & \rightsquigarrow \begin{ytableau} 1 & 1 & 4 \\ 2 & 2 \\ 3 \end{ytableau}
\end{array}
$$

1 Example 1: $S_1 * S_2 = 1 \leftarrow 2 + 1 \to 2$

Take

$$
\begin{aligned}
\lambda_1 &= (2,0,0,0) & \mu_1 &= (1,1,0,0) \\
\lambda_2 &= (2,2,0,0) & \mu_2 &= (2,1,1,0) \\
\lambda &= (4,2,0,0) & \mu &= (3,2,1,0)
\end{aligned}
$$

and the tableaux are

$$\begin{ytableau} 1 & 2 \end{ytableau} \text{ and } \begin{ytableau} 1 & 1 \\ 2 & 3 \end{ytableau}.$$

The matrix in consideration is:

$$
X = \left[\begin{array}{ccc:cc:c}
0 & 1 & & & & \\
 & 0 & 1 & & & \\
 & -s^2 & 2s & a_1 & a_2 & b_1 \\
\hdashline
 & & & 0 & 1 & \\
 & & & & s & b_2 \\
\hdashline
 & & & & & s
\end{array}\right]
$$

- $\boxed{1}$: There are no restrictions here, but we do have $e_1 \in \ker X$ and $e_1 + se_2 + s^2 e_3 \in \ker(X - s)$.

- $\boxed{2}$: There are no restrictions from X, but we do expect 1 restriction from $X - s$. Indeed, we require $a_1 + sa_2 = 0$ so that $e_4 + se_5 \in \ker(X - s)$.

- $\boxed{3}$: We expect 1 restriction from $X - s$. Indeed, we have $a_2 b_2 + sb_1 = 0$ so that $\ker(X - s) \subset \operatorname{col}(X - s)$.

The minimal polynomial $X^2(X-s)^2$ does not give any new equations so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, b_1, b_2]}{\langle a_1, a_2 b_2 \rangle} \cong \frac{\mathbb{C}[a_2, b_1, b_2]}{\langle a_2 \rangle \cap \langle b_2 \rangle}$$

The corresponding tableau for each ideal is:

$$\begin{array}{lll}
\langle a_2 \rangle & \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} & \rightsquigarrow 1 \leftarrow 2 \\
\langle b_2 \rangle & \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} & \rightsquigarrow 1 \rightarrow 2
\end{array}$$

2 Example 2: $S_3 * S_2 = 2 \to 3 + 2 \leftarrow 3$

$S_2 * S_3 = 2 \to 3 + 2 \leftarrow 3$

Take

$$\begin{array}{rr}
\lambda_1 = (2,2,2,0) & \mu_1 = (2,2,1,1) \\
\lambda_2 = (2,2,0,0) & \mu_2 = (2,1,1,0) \\
\lambda = (4,4,2,0) & \mu = (4,3,2,1)
\end{array}$$

and the tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{cccc|ccc|cc|c}
0 & 1 & & & & & & & & \\
 & 0 & 1 & & & & & & & \\
 & & 0 & 1 & & & & & & \\
 & & -s^2 & 2s & a_1 & a_2 & a_3 & b_1 & b_2 & c_1 \\
\hline
 & & & & 0 & 1 & & & & \\
 & & & & & 0 & 1 & & & \\
 & & & & & & s & b_3 & b_4 & c_2 \\
\hline
 & & & & & & & 0 & 1 & \\
 & & & & & & & & s & c_3 \\
\hline
 & & & & & & & & & 0
\end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we do have $e_1 \in \ker X$ and $e_1 + se_2 + s^2 e_3 + s^3 e_4 \in \ker(X - s)$.

$\boxed{2}$: We expect 2 restrictions from X and 1 restriction from $X - s$. Indeed, we require $a_1 = 0$ so that $e_5 \in \ker X$ and $a_2 = 0$ for $\mathrm{Span}\{e_1, e_5\} \subset \mathrm{col}\, X$. Furthermore, we need $a_3 = 0$ for $e_5 + se_6 + s^2 e_7 \in \ker(X - s)$.

$\boxed{3}$: We expect 2 restrictions from X and 1 restriction from $X - s$. Indeed, we have $b_1 = b_3 = 0$ so that $e_8 \in \ker X$ and $b_2 = 0$ for $\ker(X - s) \subset \mathrm{col}(X - s)$.

$\boxed{4}$: We expect 2 restrictions from X. We require $c_1 = 0$ and $b_4 c_3 - sc_2 = 0$ so that $\ker X \subset \mathrm{col}\, X$.

The minimal polynomial $X^2(X - s)^2$ does not give any new equations so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2, c_3]}{\langle a_1, a_2, a_3, b_1, b_2, b_3, c_1, b_4 c_3\rangle} \cong \frac{\mathbb{C}[b_4, c_2, c_3]}{\langle b_4\rangle \cap \langle c_3\rangle}$$

The corresponding tableau for each ideal is:

$$\langle b_4\rangle \quad \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 4 \\ \hline 3 & 3 \\ \cline{1-2}\end{array} \quad \rightsquigarrow 2 \leftarrow 3$$

$$\langle c_3\rangle \quad \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \rightsquigarrow 2 \rightarrow 3$$

3 Example 3: $S_1 * 2 \to 3 = P_1 + 1 \leftarrow 2 \to 3$

Take

$$\begin{aligned} \lambda_1 &= (2, 0, 0, 0) & \mu_1 &= (1, 1, 0, 0) \\ \lambda_2 &= (2, 2, 1, 0) & \mu_2 &= (2, 1, 1, 1) \\ \lambda &= (4, 2, 1, 0) & \mu &= (3, 2, 1, 1) \end{aligned}$$

and the tableaux are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{ccc:cc:c:c} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & -s^2 & 2s & a_1 & a_2 & b_1 & c_1 \\ \hdashline & & & 0 & 1 & & \\ & & & & s & b_2 & c_2 \\ \hdashline & & & & & s & c_3 \\ \hdashline & & & & & & s \end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 + s^2 e_3 \in \ker(X - s)$.

$\boxed{2}$: There are no restrictions coming from X, but we expect 1 coming from $X - s$. Indeed, we require $a_1 + sa_2$ for $e_4 + se_5 \in \ker(X - s)$.

$\boxed{3}$: There is 1 restriction from $X - s$. We need $a_1 b_2 - s^2 b_1 = a_2 b_2 + s b_1 = 0$ so that $\mathrm{Span}\{e_1 + se_2 + s^2 e_3, e_4 + se_5\} \subset \mathrm{col}(X - s)$.

$\boxed{4}$: There are 2 restrictions from $X - s$. We need $c_3 = b_1 c_2 - b_2 c_1 = 0$ for $\dim \ker(X - s) = 3$.

The minimal polynomial $X^2(X-s)^2$ gives an additional equation of $a_2 c_2 + b_1 c_3 = 0$ so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2, c_3]}{\langle a_1, c_3, a_2 b_2, a_2 c_2, b_1 c_2 - b_2 c_1 \rangle} \cong \frac{\mathbb{C}[a_2, b_1, b_2, c_1, c_2]}{\langle b_2, c_2 \rangle \cap \langle a_2, b_1 c_2 - b_2 c_1 \rangle}$$

The corresponding tableau for each ideal is:

$$\begin{array}{ccccc} \langle b_2, c_2 \rangle & \rightsquigarrow & \ytableaushort{1112,23,4} & \rightsquigarrow & P_1 \\ \langle a_2, b_1 c_2 - b_2 c_1 \rangle & \rightsquigarrow & \ytableaushort{1113,22,4} & \rightsquigarrow & 1 \leftarrow 2 \rightarrow 3 \end{array}$$

4 Example 4: $S_1 * 2 \leftarrow 3 = P_3 + 1 \rightarrow 2 \leftarrow 3$

Take

$$\begin{aligned} \lambda_1 &= (2,0,0,0) & \mu_1 &= (1,1,0,0) \\ \lambda_2 &= (2,2,1,0) & \mu_2 &= (2,1,1,1) \\ \lambda &= (4,2,1,0) & \mu &= (3,2,1,1) \end{aligned}$$

and the tableaux are

$$\ytableaushort{12} \text{ and } \ytableaushort{11,24,3}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{ccc:cc:c:c}
0 & 1 & & & & & \\
 & 0 & 1 & & & & \\
 & -s^2 & 2s & a_1 & a_2 & b_1 & c_1 \\
\hdashline
 & & & 0 & 1 & & \\
 & & & & s & b_2 & c_2 \\
\hdashline
 & & & & & s & c_3 \\
\hdashline
 & & & & & & s
\end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 + s^2 e_3 \in \ker(X - s)$.

$\boxed{2}$: There are no restrictions coming from X, but we expect 1 coming from $X - s$. Indeed, we require $a_1 + sa_2$ for $e_4 + se_5 \in \ker(X - s)$.

$\boxed{3}$: There are 2 restrictionsfrom $X - s$. We need $b_1 = b_2 = 0$ so that $e_6 \in \ker(X - s)$.

$\boxed{4}$: There is 1 restriction from $X - s$. We need $a_2 c_2 + sc_1 = 0$ so that $\ker(X - s) \subset \operatorname{col}(X - s)$.

The equations from the minimal polynomial $X^2(X - s)^2$ all go away so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2, c_3]}{\langle a_1, b_1, b_2, a_2 c_2 \rangle} \cong \frac{\mathbb{C}[a_2, c_1, c_2, c_3]}{\langle a_2 \rangle \cap \langle c_2 \rangle}$$

The corresponding tableau for each ideal is:

$$\langle a_2 \rangle \quad \rightsquigarrow \begin{ytableau} 1 & 1 & 1 & 4 \\ 2 & 2 \\ 3 \end{ytableau} \qquad \rightsquigarrow P_3$$

$$\langle c_2 \rangle \quad \rightsquigarrow \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 4 \\ 3 \end{ytableau} \qquad \rightsquigarrow 1 \to 2 \leftarrow 3$$

5 Example 5: $S_3 * 1 \to 2 = 1 \to 2 \leftarrow 3 + P_1$

Take

$$\begin{aligned}
\lambda_1 &= (2, 2, 2, 0) & \mu_1 &= (2, 2, 1, 1) \\
\lambda_2 &= (2, 1, 0, 0) & \mu_2 &= (1, 1, 1, 0) \\
\lambda &= (4, 3, 2, 0) & \mu &= (3, 3, 2, 1)
\end{aligned}$$

and the tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{ccc:ccc:cc:c}
0 & 1 & & & & & & & \\
 & 0 & 1 & & & & & & \\
 & & s & a_1 & a_2 & a_3 & b_1 & b_2 & c_1 \\
\hdashline
 & & & 0 & 1 & & & & \\
 & & & & 0 & 1 & & & \\
 & & & & & s & b_3 & b_4 & c_2 \\
\hdashline
 & & & & & & 0 & 1 & \\
 & & & & & & & s & c_3 \\
\hdashline
 & & & & & & & & 0
\end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 + s^2e_3 \in \ker(X - s)$.

$\boxed{2}$: There are 2 restrictions coming from X and no restrictions from $X - s$. Indeed, we require $a_1 = 0$ for $e_4 \in \ker X$ and $a_2 = 0$ for $\mathrm{Span}\{e_1, e_4\} \subset \operatorname{col} X$.

$\boxed{3}$: There are 2 restrictions from X and 1 restriction from $X - s$. We need $b_1 = b_3 = 0$ so that $e_7 \in \ker X$. Furthermore, we need $b_4 = 0$ so that $\dim\ker(X - s) = 2$.

$\boxed{4}$: There are 2 restrictions from X. We need $c_2 = b_2c_3 - sc_1 = 0$ for $\ker X \subset \operatorname{col} X$.

The minimal polynomial $X^2(X - s)^2$ does not give any new equations so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2, c_3]}{\langle a_1, a_2, b_1, b_3, b_4, c_2, b_2c_3\rangle} \cong \frac{\mathbb{C}[a_3, b_2, c_1, c_3]}{\langle b_2\rangle \cap \langle c_3\rangle}$$

The corresponding tableau for each ideal is:

$$\langle b_2\rangle \quad \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array} \quad \rightsquigarrow 1 \to 2 \leftarrow 3$$

$$\langle c_3\rangle \quad \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} \end{array} \quad \rightsquigarrow P_1$$

6 Example 6: $S_3 * 1 \leftarrow 2 = P_3 + 1 \leftarrow 2 \rightarrow 3$

Take

$$\begin{aligned} \lambda_1 &= (2,2,2,0) & \mu_1 &= (2,2,1,1) \\ \lambda_2 &= (2,1,0,0) & \mu_2 &= (1,1,1,0) \\ \lambda &= (4,3,2,0) & \mu &= (3,3,2,1) \end{aligned}$$

and the tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{ccc|ccc|cc|c} 0 & 1 & & & & & & & \\ & 0 & 1 & & & & & & \\ & & s & a_1 & a_2 & a_3 & b_1 & b_2 & c_1 \\ \hline & & & 0 & 1 & & & & \\ & & & & 0 & 1 & & & \\ & & & & & s & b_3 & b_4 & c_2 \\ \hline & & & & & & 0 & 1 & \\ & & & & & & & s & c_3 \\ \hline & & & & & & & & 0 \end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 + s^2 e_3 \in \ker(X - s)$.

$\boxed{2}$: There are 2 restrictions coming from X and 1 restriction from $X - s$. Indeed, we require $a_1 = 0$ for $e_4 \in \ker X$ and $a_2 = 0$ for $\operatorname{Span}\{e_1, e_4\} \subset \operatorname{col} X$. Furthermore, we require $a_3 = 0$ for $e_4 + se_5 + s^2 e_6 \in \ker(X - s)$.

$\boxed{3}$: There are 2 restrictions from X but no restrictions from $X - s$. We need $b_1 = b_3 = 0$ so that $e_7 \in \ker X$.

$\boxed{4}$: There are 2 restrictions from X. We need $b_4 c_3 - sc_2 = b_2 c_3 - sc_1 = 0$ for $\ker X \subset \operatorname{col} X$.

The minimal polynomial $X^2(X - s)^2$ does not give any new equations so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2, c_3]}{\langle a_1, a_2, a_3, b_1, b_3, b_2 c_3, b_4 c_3 \rangle} \cong \frac{\mathbb{C}[b_2, b_4, c_1, c_2, c_3]}{\langle b_2, b_4 \rangle \cap \langle c_3 \rangle}$$

However, for the ideal $\langle c_3 \rangle$, we require $\dim \ker(X|_{s=0}) = 3$ so we also need to include the equation $b_2 c_2 - b_4 c_1 = 0$. Then the coordinate ring is really

$$\frac{\mathbb{C}[b_2, b_4, c_1, c_2, c_3]}{\langle b_2, b_4 \rangle \cap \langle c_3, b_2 c_2 - b_4 c_1 \rangle}$$

The corresponding tableau for each ideal is:

$$\langle b_2, b_4 \rangle \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array} \quad \rightsquigarrow P_3$$

$$\langle c_3, b_2c_2 - b_4c_1 \rangle \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 4 \\ \cline{1-2}\end{array} \quad \rightsquigarrow 1 \leftarrow 2 \rightarrow 3$$

7 Example 7: $1 \leftarrow 2 * 2 \leftarrow 3 = P_2 + S_2 \oplus P_3$

Take

$$\begin{aligned} \lambda_1 &= (2,1,0,0) & \mu_1 &= (1,1,1,0) \\ \lambda_2 &= (2,2,1,0) & \mu_2 &= (2,1,1,1) \\ \lambda &= (4,3,1,0) & \mu &= (3,2,2,1) \end{aligned}$$

and the tableaux are

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{ccc:cc:cc:c} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & -s^2 & 2s & a_1 & a_2 & b_1 & b_2 & c_1 \\ \hdashline & & & 0 & 1 & & & \\ & & & & s & b_3 & b_4 & c_2 \\ \hdashline & & & & & 0 & 1 & \\ & & & & & & s & c_3 \\ \hdashline & & & & & & & s \end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 + s^2e_3 \in \ker(X - s)$.

$\boxed{2}$: There is 1 restriction coming from X and 1 restriction from $X - s$. We require $a_1 = 0$ for $e_4 \in \ker X$ and $a_2 = 0$ for $e_4 + se_5 \in \ker(X - s)$.

$\boxed{3}$: There are no restrictions from X but 2 restrictions from $X - s$. We need $b_1 + sb_2 = b_3 + sb_4 = 0$ so that $e_6 + se_7 \in \ker(X - s)$.

$\boxed{4}$: There is 1 restriction from $X - s$. We need $b_2c_3 - sc_1 = 0$ for $\ker(X - s) \subset \operatorname{col}(X - s)$.

The minimal polynomial $X^2(X-s)^2$ does not give any new equations so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, b_1, b_2, b_3, b_4, c_1, c_2, c_3]}{\langle a_1, a_2, b_1, b_3, b_2 c_3 \rangle} \cong \frac{\mathbb{C}[b_2, b_4, c_1, c_2, c_3]}{\langle b_2 \rangle \cap \langle c_3 \rangle}$$

The corresponding tableau for each ideal is:

$$\langle b_2 \rangle \quad \rightsquigarrow \quad \begin{ytableau} 1 & 1 & 1 & 4 \\ 2 & 2 & 3 \\ 3 \end{ytableau} \quad \rightsquigarrow S_2 \oplus P_3$$

$$\langle c_3 \rangle \quad \rightsquigarrow \quad \begin{ytableau} 1 & 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 \end{ytableau} \quad \rightsquigarrow P_2$$

8 Example 8: $1 \to 2 * 2 \to 3 = P_2 + S_2 \oplus P_1$

Take

$$\begin{aligned} \lambda_1 &= (2,1,0,0) & \mu_1 &= (1,1,1,0) \\ \lambda_2 &= (2,2,1,0) & \mu_2 &= (2,1,1,1) \\ \lambda &= (4,3,1,0) & \mu &= (3,2,2,1) \end{aligned}$$

and the tableaux are

$$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \text{ and } \begin{ytableau} 1 & 1 \\ 2 & 3 \\ 4 \end{ytableau}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{ccc:cc:cc:c} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & -s^2 & 2s & a_1 & a_2 & b_1 & b_2 & c_1 \\ \hdashline & & & 0 & 1 & & & \\ & & & & s & b_3 & b_4 & c_2 \\ \hdashline & & & & & 0 & 1 & \\ & & & & & & s & c_3 \\ \hdashline & & & & & & & s \end{array}\right]$$

- $\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 + s^2 e_3 \in \ker(X - s)$.

- $\boxed{2}$: There is no restriction coming from X and 1 restriction from $X - s$. We require $a_1 + sa_2 = 0$ for $e_4 + se_5 \in \ker(X - s)$.

$\boxed{3}$: There is 1 restriction from X and 1 restriction from $X-s$. We need $b_3 = 0$ so that $\dim\ker X = 2$. Furthermore, we have $b_1 + a_2b_4 + sb_2 = 0$ so that $\mathrm{Span}\{e_1 + se_2 + s^2e_3, e_4 + se_5\} \subset \mathrm{col}(X - s)$.

$\boxed{4}$: There are 2 restrictions from $X-s$. We need $c_3 = b_1c_2 + s(b_2c_2 - b_4c_1) = 0$ for $\dim\ker(X - s) = 3$.

The minimal polynomial $X^2(X-s)^2$ gives the additional equation $a_2c_2 + sc_1 = 0$ so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, b_1, b_2, b_3, b_4, c_1, c_2, c_3]}{\langle a_1, b_3, c_3, b_1 + a_2b_4, b_1c_2, a_2c_2\rangle} \cong \frac{\mathbb{C}[a_2, b_1, b_2, b_4, c_1, c_2]}{\langle a_2, b_1\rangle \cap \langle b_1 + a_2b_4, c_2\rangle}$$

The corresponding tableau for each ideal is:

$$\begin{array}{ccc}
\langle a_2, b_1\rangle & \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\ \hline 2&2&4\\ \cline{1-3} 3\\ \cline{1-1}\end{array} & \rightsquigarrow P_2 \\
\langle b_1 + a_2b_4, c_2\rangle & \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&3&3\\ \cline{1-3} 4\\ \cline{1-1}\end{array} & \rightsquigarrow S_2 \oplus P_1
\end{array}$$

9 Example 9: $S_2 * 1 \leftarrow 2 \rightarrow 3 = P_2 + 1 \leftarrow 2 \oplus 2 \rightarrow 3$

Take

$$\begin{aligned}
\lambda_1 &= (2, 2, 0, 0) & \mu_1 &= (2, 1, 1, 0)\\
\lambda_2 &= (2, 1, 1, 0) & \mu_2 &= (1, 1, 1, 1)\\
\lambda &= (4, 3, 1, 0) & \mu &= (3, 2, 2, 1)
\end{aligned}$$

and the tableaux are

$$\begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline\end{array} \text{ and } \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1} 4\\ \cline{1-1}\end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{ccc|cc|cc|c}
0 & 1 & & & & & & \\
 & 0 & 1 & & & & & \\
 & & s & a_1 & a_2 & b_1 & b_2 & c_1\\
\hline
 & & & 0 & 1 & & & \\
 & & & & s & b_3 & b_4 & c_2\\
\hline
 & & & & & 0 & 1 & \\
 & & & & & & s & c_3\\
\hline
 & & & & & & & s
\end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 + s^2e_3 \in \ker(X - s)$.

$\boxed{2}$: There is 1 restriction coming from X and 1 restriction from $X - s$. We require $a_1 = 0$ for $e_4 \in \ker X$ and $a_2 = 0$ for $e_4 + se_5 \in \ker(X - s)$.

$\boxed{3}$: There is 1 restriction from X but no restrictions from $X - s$. We need $b_1 = 0$ so that $\ker X \subset \operatorname{col} X$.

$\boxed{4}$: There are 2 restrictions from $X - s$. We need $c_3 = b_3c_1 + s(b_2c_2 - b_4c_1) = 0$ for $\dim \ker(X - s) = 3$.

The minimal polynomial $X^2(X - s)^2$ does not give any new equations so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, b_1, b_2, b_3, b_4, c_1, c_2, c_3]}{\langle a_1, a_2, b_1, c_3, b_3c_1\rangle} \cong \frac{\mathbb{C}[b_2, b_3, b_4, c_1, c_2]}{\langle b_3\rangle \cap \langle c_1\rangle}$$

The corresponding tableau for each ideal is:

$$\langle b_3\rangle \quad \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \qquad \rightsquigarrow P_2$$

$$\langle c_1\rangle \quad \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \quad \rightsquigarrow 1 \leftarrow 2 \oplus 2 \rightarrow 3$$

10 Example 10: $S_2 * 1 \to 2 \leftarrow 3 = P_2 + 1 \to 2 \oplus 2 \leftarrow 3$

Take

$$\begin{aligned} \lambda_1 &= (2, 2, 0, 0) & \mu_1 &= (2, 1, 1, 0) \\ \lambda_2 &= (3, 2, 1, 0) & \mu_2 &= (2, 2, 1, 1) \\ \lambda &= (5, 4, 1, 0) & \mu &= (4, 3, 2, 1) \end{aligned}$$

and the tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array} \text{ and } \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{cccc:ccc:cc:c}
0 & 1 & & & & & & & & \\
 & 0 & 1 & & & & & & & \\
 & & 0 & 1 & & & & & & \\
 & & -s^2 & 2s & a_1 & a_2 & a_3 & b_1 & b_2 & c_1 \\
\hdashline
 & & & & 0 & 1 & & & & \\
 & & & & & 0 & 1 & & & \\
 & & & & & s^2 & 2s & b_3 & b_4 & c_2 \\
\hdashline
 & & & & & & & 0 & 1 & \\
 & & & & & & & & s & c_3 \\
\hdashline
 & & & & & & & & & s
\end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1+se_2+s^2e_3 \in \ker(X-s)$.

$\boxed{2}$: There is 1 restriction coming from X and 1 restriction from $X-s$. We require $a_1 = 0$ for $e_5 \in \ker X$ and $a_2 + sa_3 = 0$ for $e_5 + se_6 + s^2e_7 \in \ker(X-s)$.

$\boxed{3}$: There is 1 restriction from X and 2 restrictions from $X-s$. We need $a_2b_3 + s^2b_1 = a_3b_3 - sb_1 = 0$ so that $\ker X \subset \operatorname{col} X$. Furthermore, we require $b_1 + sb_2 = b_3 + sb_4 = 0$ so that $e_8 + se_9 \in \ker(X-s)$.

$\boxed{4}$: There is 1 restriction from $X-s$. We need $b_3c_3 - s^2c_2 = b_4c_3 + sc_2 = 0$ for $\ker(X-s) \subset \operatorname{col}(X-s)$.

The minimal polynomial $X^2(X-s)^3$ gives the additional equations $a_3b_4 + b_1 = a_3c_2 + b_2c_3 = 0$ so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1,a_2,a_3,b_1,b_2,b_3,b_4,c_1,c_2,c_3]}{\langle a_1,a_2,b_1,b_3,a_3b_4,b_4c_3,a_3c_2+b_2c_3\rangle} \cong \frac{\mathbb{C}[a_3,b_2,b_4,c_1,c_2,c_3]}{\langle b_4, a_3c_2+b_2c_3\rangle \cap \langle a_3,c_3\rangle}$$

The corresponding tableau for each ideal is:

$$\langle b_4, a_3c_2+b_2c_3\rangle \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\\hline 2&2&3&4\\\cline{1-4} 3\\\cline{1-1}\end{array} \rightsquigarrow 1 \to 2 \oplus 2 \leftarrow 3$$

$$\langle a_3, c_3\rangle \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&3\\\hline 2&2&2&4\\\cline{1-4} 3\\\cline{1-1}\end{array} \rightsquigarrow P_2$$

11 Example 11: $1 \leftarrow 2 \rightarrow 3 * 1 \rightarrow 2 = 1 \leftarrow 2 \oplus P_1 + S_1 \oplus P_2$

Take

$$\begin{aligned} \lambda_1 &= (2,1,1,0) & \mu_1 &= (1,1,1,1) \\ \lambda_2 &= (2,1,0,0) & \mu_2 &= (1,1,1,0) \\ \lambda &= (4,2,1,0) & \mu &= (2,2,2,1) \end{aligned}$$

and the tableaux are

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \text{ and } \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

The matrix in consideration is:

$$X = \left[\begin{array}{cc:cc:cc:c} 0 & 1 & & & & & \\ & s & a_1 & a_2 & b_1 & b_2 & c_1 \\ \hdashline & & 0 & 1 & & & \\ & & & s & b_3 & b_4 & c_2 \\ \hdashline & & & & 0 & 1 & \\ & & & & & s & c_3 \\ \hdashline & & & & & & 0 \end{array}\right]$$

$\boxed{1}$: There are no restrictions here, but we have $e_1 \in \ker X$ and $e_1 + se_2 \in \ker(X - s)$.

$\boxed{2}$: There is 1 restriction coming from X but no restrictions from $X - s$. We require $a_1 = 0$ for $e_3 \in \ker X$.

$\boxed{3}$: There are no restrictions from X and 1 restriction from $X - s$. We need $b_3 + sb_4 = 0$ so that $e_5 + se_6 \in \ker(X - s)$.

$\boxed{4}$: There are 2 restrictions from $X - s$. We need $c_3 = b_1c_2 - b_3c_1 = 0$ for $\ker(X - s) \subset \operatorname{col}(X - s)$.

The minimal polynomial $X^2(X - s)^3$ does not give any new equations so our coordinate ring after taking $s \to 0$ is

$$\frac{\mathbb{C}[a_1, a_2, b_1, b_2, b_3, b_4, c_1, c_2, c_3]}{\langle a_1, b_3, c_3, b_1c_2 \rangle} \cong \frac{\mathbb{C}[a_2, b_1, b_2, b_4, c_1, c_2]}{\langle b_1 \rangle \cap \langle c_2 \rangle}$$

The corresponding tableau for each ideal is:

$$\begin{array}{ccc} \langle b_1 \rangle & \rightsquigarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} & \rightsquigarrow S_1 \oplus P_2 \\ \langle c_2 \rangle & \rightsquigarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} & \rightsquigarrow 1 \leftarrow 2 \oplus P_1 \end{array}$$