How to compute the fusion product of MV cycles in type A

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1 Players

$G = \mathbf{GL}_m$.

- The ordinary affine Grassmannian Gr
- The Beilinson–Drinfeld Grassmannian $\mathcal{G}r_n^{\mathrm{BD}} \to C$
- Partitions $\mu_i \le \lambda_i$ of N_i $(i = 1, 2)$ and $\mu = \sum \mu_i \le \lambda = \sum \lambda_i$ of $N = \sum N_i$
- The slices Gr_μ and $\mathcal{W}_{\mu_1,\mu_2}$ to the orbits Gr^λ and $(\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2}$
- The nilpotent and semi-nilpotent cones $\mathcal{N}$ and $\mathcal{N}_s$ (of matrices with eigenvalues 0 and 0 or $s \neq 0$)
- The slices $\mathbb{T}_\mu$ and $\mathbb{T}_{\mu_1,\mu_2}$ to the orbits $\mathbb{O}_\lambda$ and $\mathbb{O}_{\lambda_1,\lambda_2}$

New (?) definitions among these are as follows.
We define

$$L_\mu = t^\mu, L_{\mu_1,\mu_2} = t^{\mu_1}(t-s)^{\mu_2} \in G((t^{-1}))/G[t]$$

$$\mathcal{W}_\mu = G_1[[t^{-1}]]L_\mu \subset G((t^{-1}))/G[t] \tag{1}$$

a subscheme of the thick affine Grassmannian

Note that $L_{\mu_1,\mu_2} = (t-s)^{\mu_2} t^{-\mu_2} L_\mu \in \mathcal{W}_\mu$ and that $L_{\mu_1,\mu_2} \in \mathcal{G}r_2^{\mathrm{BD}}$ is a $\mathbb{C}[t]$-lattice in $\mathbb{C}(t)^m$ that specializes to a $\mathbb{C}[\![t]\!]$-lattice in $\mathbb{C}(\!(t)\!)^m$ away from $t = 0$ and away from $t = s$; i.e.

$$\begin{aligned} L_{\mu_1,\mu_2} \otimes \mathbb{C}[\![t]\!] &= L_{\mu_2} \otimes \mathbb{C}[\![t]\!] \text{ and} \\ L_{\mu_1,\mu_2} \otimes \mathbb{C}[\![t-s]\!] &= L_{\mu_1} \otimes \mathbb{C}[\![t-s]\!] \end{aligned} \tag{2}$$

where L_{μ_i} denotes the point $t^{\mu_i}G(\mathcal{O}) \in \mathrm{Gr}$.

The family of semi-infinite orbits

$$S_{\mu_1,\mu_2} = N_-(\mathbb{C}((t^{-1})))L_{\mu_1,\mu_2}\,. \tag{3}$$

The orbit of $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^m$ whose elements specialize again to $\mathbb{C}[\![t]\!]$-lattices in $\mathbb{C}(\!(t)\!)^m$ away from $t=0$ and away from $t=s$ as follows

$$\begin{aligned}(\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2} = \{L \in \mathcal{G}r_2^{\mathrm{BD}} : L \otimes_{\mathbb{C}[t]} \mathbb{C}[\![t-s]\!] \in \mathrm{Gr}^{\lambda_2} \ \text{ and}\\ L \otimes_{\mathbb{C}[t]} \mathbb{C}[\![t]\!] \in \mathrm{Gr}^{\lambda_1} \ \text{ and}\\ L \otimes_{\mathbb{C}[t]} \mathbb{C}[\![t-a]\!] = L_0 \text{ for all } a \neq 0, s\}\,.\end{aligned} \tag{4}$$

Joel: It is important to add the condition that the lattices are trivial at a. Anne: These defining conditions are again just telling us that in the fibre over the fixed point $(0,s) \in C^{(2)}$ this set is pairs of lattices; note that we invert the indeterminates t and $(t-s)$ *after* specializing $(0,s)$. Could we give a more explicit characterization like $t^{N_1}(t-s)^{N_2}L_0 \subseteq L \subseteq t^{-N_1}(t-s)^{-N_2}$?

The [family "$\mathcal{N}\otimes\mathbb{C}[s]$? $\mathcal{N}\times\mathbb{C}$?" of] semi-nilpotent cone[s fibred over $C=\mathbb{A}^1$ with s-fibre]

$$\mathcal{N}_s = \{A \in \mathrm{Mat}(N) : \text{eigenvalues of } A \text{ are } 0 \text{ or } s\}\,. \tag{5}$$

The [family of] slice[s $\mathbb{T}_{\mu_1,\mu_2}$ fibred over $C=\mathbb{A}^1$ with s-fibre]

$$\begin{aligned}\mathbb{T}^s_{\mu_1,\mu_2} = \{B + C_s : B \text{ is a } \mu\times\mu \text{ block matrix of zeros}\\ \text{except possibly in the last } \min(\mu_i,\mu_j)\\ \text{columns of the last row of each } \mu_i\times\mu_j \text{ block}\\ \text{and } C_s \text{ is the block diagonal matrix of}\\ \text{companion matrices of } t^{\mu_{1,k}}(t-s)^{\mu_{2,k}}\}\,.\end{aligned} \tag{6}$$

The uppertriangular subfamily $\mathbb{T}^+_{\mu_1,\mu_2}$ with s-fibre

$$\mathbb{T}^{+,s}_{\mu_1,\mu_2} = \{B + C_s \in \mathbb{T}_{\mu_1,\mu_2} : B \in \mathfrak{n}\} \tag{7}$$

where $\mathfrak{n} \subset \mathrm{Mat}(N)$ is the unipotent subalgebra of uppertriangular matrices.

Anne: or—as Joel pointed out, may be ok with: The slice $\mathbb{T}_\mu$ as defined in MVy, no change, and the family of slices

$$\mathbb{T}^{+,s}_{\mu_1,\mu_2} = \mathbb{T}_\mu \cap \mathfrak{n} + C^s_{\mu_1,\mu_2}$$

where

$$C_s \text{ is the block diagonal matrix of companion matrices of } t^{\mu_{1,k}}(t-s)^{\mu_{2,k}} \tag{8}$$

The [family of] orbit[s $\mathbb{O}_{\lambda_1,\lambda_2}$ fibred over $C=\mathbb{A}^1$ with s-fibre]

$$\mathbb{O}^s_{\lambda_1,\lambda_2} = \{A \in \mathcal{N}_s : A \text{ is conjugate to } J_{\lambda_1} \oplus (sI_{N_2} + J_{\lambda_2})\} \tag{9}$$

where J_{λ_i} is the Jordan normal form of block type λ_i and I_{N_2} is the identity matrix in $\mathrm{Mat}(N_2)$.

2 Exposition

3 Rising Action

4 Climax

5 Falling Action

Theorem 1. *Let $\lambda_i \geq \mu_i$ be dominant $(i = 1, 2)$, $\mu = \mu_1 + \mu_2$, and $\lambda = \lambda_1 + \lambda_2$. There is an isomorphism*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_{\mu_1,\mu_2} \to \overline{(\mathcal{G}r_2^{BD})^{\lambda_1,\lambda_2}} \cap \mathcal{W}_{\mu_1,\mu_2} \tag{10}$$

got by taking a $\mu \times \mu$ block matrix A in the s-fibre $\overline{\mathbb{O}^s_{\lambda_1,\lambda_2}} \cap \mathbb{T}^s_{\mu_1,\mu_2}$ on the left to the representative of the s-fibre on the right defined by

$$\begin{aligned} g &= t^{\mu_1}(t-s)^{\mu_2} + a(t) \\ a_{ij}(t) &= -\sum_{k=1}^{\mu_i} A^k_{ji} t^{k-1} \end{aligned} \tag{11}$$

where A^k_{ji} is the kth entry from the left of the last row of the $\mu_j \times \mu_i$ block of A.

Let's call this the MVyBD isomorphism.

Proof. The proof is fibre by fibre, so fix $s \neq 0$. Anne: Emphasize in the intro later (because this always confuses me) that by the s-fibre we really mean the $(0, s)$-fibre; i.e. its the BD Grassmannian over the second symmetric power of $C = \mathbb{A}^1$; better just replace s-fibre by $(0, s)$-fibre everywhere it occurs.

1. The map is well defined. In particular, it defines $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^m$. Moreover, these lattices break down to give pairs of lattices upon inverting t or $t - s$ that have the right properties. [Copy Roger's proof]

2. The inverse map is got by taking the matrix of multiplication by t. More precisely, let $L \in Gr^{BD} \cap \mathcal{W}_\mu$. We work with the quotient $\mathbb{C}[t]^m/L$ just as in the ordinary MVy isomorphism—the only difference being $\mathbb{C}[\![t]\!]$ is replaced by $\mathbb{C}[t]$.

   (a) We claim that

   $$\{[e_i], [te_i], \ldots, [t^{\mu_i - 1}e_i] : 1 \leq i \leq m\} \tag{12}$$

   is a $\mathbb{C}$-basis of $\mathbb{C}[t]^m/L$.

   To see this, we use that L has a $\mathbb{C}[t]$-basis of the form

   $$v_i = t^{\mu_i} + \sum_{j>i} p_{ij}(t) e_j \tag{13}$$

with $\deg p_{ij}(t) < \mu_i = \mu_{1,i} + \mu_{2,i}$ $(1 \le i \le m)$. Anne: I don't know why this should be true. We might have to just define fibres of $\mathcal{W}_{\mu_1,\mu_2}$ in this way?

(b) $t\big|_{\mathbb{C}[t]^m/L}$ will have two eigenvalues, 0 and s, and its generalized 0-eigenspace will have block type $\le \lambda_1$ while its generalized s-eigenspace will have block type $\le \lambda_2$. To see this, note that there is a natural isomorphism

$$\mathbb{C}[[t]]^m/(L \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]) = \text{generalized 0 eigenspace of } t \text{ on } \mathbb{C}[t]^m/L$$

carrying the action of t to the action of t.
The left hand side is the same thing as

$$\mathbb{C}[[t]]^m/(L \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]) = (\mathbb{C}[t]^m/L) \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$$

the defining fact that lattices satisfying Equation 4 equivalently satisfy

$$\begin{aligned} t\big|_{\mathbb{C}[\![t]\!]^m/L_1} &\text{ has Jordan type} \le \lambda_1 \\ t\big|_{\mathbb{C}[\![t]\!]^m/L_2} &\text{ has Jordan type} \le \lambda_2 \end{aligned} \tag{14}$$

where recall $L_i = L \otimes \mathbb{C}[\![t]\!]$??? and $p_1 = s$ while $p_2 = 0$. Anne: Somehow, restricting to an eigenspace is like inverting/forgetting the action of t by any other generalized eigenvalue? Basic linear algebra? Joel?

□

Theorem 2 (Theorem 1 version 2)**.** *Let λ_1, λ_2 and μ be arbitrary, such that $\lambda = \lambda_1 + \lambda_2 \ge \mu$. Then there is an isomorphism*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_\mu \to \overline{\mathcal{G}r_2^{BD^{\lambda_2,\lambda_2}}} \cap \mathcal{W}_\mu \tag{15}$$

defined by the same map as in Theorem 1.

Joel: This is true as stated with the "larger" definition of $\mathcal{W}_\mu$. In fact, for any λ_1, λ_2, it is true $\overline{\mathcal{G}r_2^{\mathrm{BD}^{\lambda_2,\lambda_2}}} \cap \mathcal{W}_\mu$ is contained in a subset that we could call $\mathcal{W}_\mu^s$ which we could define as

$$\mathcal{W}_\mu^s = G_1[[t^{-1}]]t^\mu \cap G[t, t^{-1}, (t-s)^{-1}]/G[t]$$

where we regard $G[t, t^{-1}, (t-s)^{-1}]/G[t] \subset G((t^{-1}))/G[t]$

The way to think about this is as follows: inside the thick affine Grassmannian we can consider the G-bundles trivialized away from just 0, s, or equivalently those lattices which become the standard lattice after tensoring with $\mathbb{C}[[t-a]]$ for any $a \ne 0, s$.

Corollary 1. *The MVyBD isomorphism restricts to an isomorphism of subfamilies*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}^+_{\mu_1,\mu_2} \to \overline{(\mathcal{G}r_2^{BD})^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2}\,. \tag{16}$$

Proof. Let $A \in \overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}^+_{\mu_1,\mu_2}$ and let g be the polynomial matrix formed by the Mirkovic-Vybornov isomorphism. Then the diagonal entries of g are $t^{\mu_{1,k}}(t-s)^{\mu_{2,k}}$ and we can factor

$$g = (gt^{-\mu_1}(t-s)^{-\mu_2})t^{\mu_1}(t-s)^{\mu_2} \in N[t, t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$$

So we get containment in one direction.

For the reverse containment, we choose $[g] \in G\overline{(\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2}$. By the lemma below, $[g] \in \mathcal{W}_\mu$ and thus it lies in the image of our map and we are done. □

Anne: is it a fibre of S_{μ_1,μ_2} defined above?

We could also make the following claim.

Lemma 1 (KWWY14)**.** *Let μ be dominant. Then*

$$N_-(\!(t^{-1})\!)L_\mu = N_1[\![t^{-1}]\!]L_\mu \tag{17}$$

Anne: where I am not sure about the double brackets.

Lemma 2. *Let μ_1, μ_2 be dominant and let $s \in \mathbb{A}^1 - \{0\}$. Then*

$$S^s_{\mu_1,\mu_2} \subset \mathcal{W}_\mu \tag{18}$$

where $\mu = \mu_1 + \mu_2$.

Proof. We have

$$\begin{aligned}
S_{\mu_1,\mu_2} &= N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&\subset T_1[[t^{-1}]]N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&= T_1[[t^{-1}]]N_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&\subset G_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&= W_{\mu_1+\mu_2}
\end{aligned}$$

where $B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} = B_1[[t^{-1}]]t^{\mu_1+\mu_2}$ since

$$\frac{t}{t-s} = 1 + \frac{s}{t} + \frac{s^2}{t^2} + \cdots \in B_1[[t^{-1}]].$$

□

6 Denouement

As an application we can compute fusion of stable MV cycles of type α_i for any $i \in I$. What about more general weights? Having kpf > 1.

Proposition 1. *Given two MV cycles Z_τ and Z_σ of type...*

$$Z_\tau * Z_\sigma \tag{19}$$

is found by???

Conjecture 1. *Let $Z_i \subset \overline{S^{\nu_i} \cap S^0_-}$ be an MV cycle of weight ν_i and put $\nu = \nu_1 + \nu_2$ $(i = 1, 2)$.*