Examples Compendium

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Let Z be the MV cycle of weight α_3 with Lusztig datum $(000, 00, 1)$ and Z' the MV cycle of weight $\alpha_{1,2}$ with Lusztig datum $(010, 00, 0)$.

In terms of tableax the fusion $Z * Z'$ can be encoded as

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} * \begin{array}{|c|}\hline 3\\\hline\end{array} = \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1} 4 \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 4\\\hline 2 \\\cline{1-1} 3 \\\cline{1-1}\end{array} \tag{1}$$

$$S_3 * (1 \leftarrow 2) = (1 \leftarrow 2 \rightarrow 3) + P_3 \tag{2}$$

$$(000, 00, 1) * (010, 00, 0) = (010, 00, 1) + (001, 00, 0) \tag{3}$$

$$= (A_0, A_5, A_1A_4 - A_2A_3) \sqcup (A_0, A_1, A_3) \tag{4}$$

where the ideals in line 3 are given in coordinates on matrices of the form

$$\left[\begin{array}{c|c|c|c} 0 & A_0 & A_1 & A_2 \\ \hline 0 & 0 & A_3 & A_4 \\ \hline 0 & 0 & s & A_5 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

Check that

$$E_0 = \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_4 - \frac{1}{s}A_5\vec{\mathbf{e}}_3\} \to \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, -A_5\vec{\mathbf{e}}_3\}$$
$$E_s = \{\vec{\mathbf{e}}_3 + \frac{1}{s}A_3\vec{\mathbf{e}}_2 + \frac{1}{s}A_1\vec{\mathbf{e}}_1\} \to \{A_3\vec{\mathbf{e}}_2 + A_1\vec{\mathbf{e}}_1\}$$

The equations imposed by the factor tableaux also degenerate.

$$(A_0, sA_2 - A_1A_5, sA_4 - A_3A_5) \to (A_0, A_1A_5, A_3A_5) = (A_0, A_5) \sqcup (A_0, A_1, A_3)$$

Need to supplement $A_5 = 0$ case with $\mathrm{col}_3 \wedge \mathrm{col}_4 = 0$ or $A_1A_4 - A_2A_3 = 0$ in order that $\dim\ker A = 3$.

Remark 1. If we take $s \to \infty$ in E_0, E_s we get the spanning set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_4\} \sqcup \{\vec{\mathbf{e}}_3\}$. Does this have any significance?

Next let Z be the MV cycle of weight $\alpha_{1,3}$ having Lusztig datum $(100, 01, 0)$ and Z' the MV cycle of weight α_2 having Lusztig datum $(000, 10, 0)$.

Again, in terms of tableaux, $Z * Z'$ is given by

$$\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array} * \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \tag{5}$$
$$(1 \to 2 \leftarrow 3) * S_2 = P_2 + (2 \leftarrow 3) \oplus (1 \to 2) \tag{6}$$
$$(100, 01, 0) * (000, 10, 0) = (010, 01, 0) + (100, 11, 0) \tag{7}$$
$$= (A_5, A_0) \sqcup (A_3, A_0A_4 + A_1A_5) \tag{8}$$

Let's verify with MVy. Consider

$$A = \left[\begin{array}{c:c:c:c} s & A_0 & A_1 & A_2 \\ \hdashline 0 & 0 & A_3 & A_4 \\ \hdashline 0 & 0 & s & A_5 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \qquad A - s = \left[\begin{array}{c:c:c:c} 0 & A_0 & A_1 & A_2 \\ \hdashline 0 & -s & A_3 & A_4 \\ \hdashline 0 & 0 & 0 & A_5 \\ \hdashline 0 & 0 & 0 & -s \end{array}\right]$$

Then

$$E_0 = \{s\vec{\mathbf{e}}_2 - A_0\vec{\mathbf{e}}_1, \tag{9}$$
$$sA_0\vec{\mathbf{e}}_4 - A_0A_5\vec{\mathbf{e}}_3 + A_1A_5\vec{\mathbf{e}}_2 - A_0A_1\vec{\mathbf{e}}_1 \text{ if } sA_4 - A_3A_5 = 0\} \tag{10}$$
$$\to \{\vec{\mathbf{e}}_1, -A_0A_5\vec{\mathbf{e}}_3 + A_1A_5\vec{\mathbf{e}}_2 - A_0A_1\vec{\mathbf{e}}_1 \text{ if } A_3A_5 = 0\} \tag{11}$$
$$E_s = \{\vec{\mathbf{e}}_1, \tag{12}$$
$$s\vec{\mathbf{e}}_3 + A_3\vec{\mathbf{e}}_2 \text{ if } sA_1 + A_0A_3 = 0\} \tag{13}$$
$$\to \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2 \text{ if } A_0A_3 = 0\} \tag{14}$$

From this we see that $Z * Z'$ is contained in

$$(A_3A_5, A_0A_3) = (A_3) \sqcup (A_5, A_0)$$

Expected dimension is $4+3-2-1=4$ hence the first ideal is not big enough. Indeed the case $A_3=0$ and $A_5\neq 0$ has to be supplemented with $A\big|_{s=0}^{2}=0$ since the total Jordan type is $(1,1)+(1,1)=(2,2)$. This adds the condition $A_0A_4+A_1A_5=0$.

Now take Z to be the MV cycle of weight $\alpha_{1,3}$ with Lusztig datum $(010,00,1)$ and leave Z' as above. Then

$$\begin{array}{ll} 1 & 3 \\ 2 & \\ 4 & \end{array} * \begin{array}{ll} 1 & 1 \\ 2 & 3 \end{array} = \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & \\ 4 & & & \end{array} + \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 4 & \\ 3 & & & \end{array} \tag{15}$$

$$(1\leftarrow 2\rightarrow 3)*S_2=(2\rightarrow 3)\oplus(1\leftarrow 2)+P_2 \tag{16}$$

$$(010,00,1)*(000,10,0)=(010,10,1)+(010,01,0) \tag{17}$$

Verifying. Consider

$$A=\left[\begin{array}{ccc|cc|cc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & -s^2 & 2s & A_0 & A_1 & A_2 & A_3 & A_4\\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & s & A_5 & A_6 & A_7\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & s & A_8\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

and

$$A-s=\left[\begin{array}{ccc|cc|cc|c}
-s & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -s & 1 & 0 & 0 & 0 & 0 & 0\\
0 & -s^2 & s & A_0 & A_1 & A_2 & A_3 & A_4\\
\hline
0 & 0 & 0 & -s & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & A_5 & A_6 & A_7\\
\hline
0 & 0 & 0 & 0 & 0 & -s & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_8\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -s
\end{array}\right]$$

Then

$$
\begin{aligned}
E_0 = \{&\vec{\mathbf{e}}_1, && (18)\\
&\vec{\mathbf{e}}_4 \text{ if } A_0 = 0, && (19)\\
&s\vec{\mathbf{e}}_6 - A_5\vec{\mathbf{e}}_5 + c\vec{\mathbf{e}}_2 \text{ if } A_2 = 0, && (20)\\
&A_5\vec{\mathbf{e}}_8 - A_7\vec{\mathbf{e}}_6 \text{ if } A_4, A_8 = 0\} && (21)\\
&\to \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_4, -A_5\vec{\mathbf{e}}_5 + c\vec{\mathbf{e}}_2, A_5\vec{\mathbf{e}}_8 - A_7\vec{\mathbf{e}}_6 && (22)\\
&\text{if } A_0, A_2, A_4, A_8 = 0\} && (23)\\
E_s = \{&s^2\vec{\mathbf{e}}_3 + s\vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_1, s\vec{\mathbf{e}}_3 - \frac{1}{s}\vec{\mathbf{e}}_1, && (24)\\
&s\vec{\mathbf{e}}_5 + \vec{\mathbf{e}}_4 \text{ if } A_1 = 0, && (25)\\
&s\vec{\mathbf{e}}_7 + \vec{\mathbf{e}}_6 + \vec{\mathbf{e}}_5 \text{ if } A_5 + (A_6 - 1)s = 0\} && (26)\\
&\to \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_4, \vec{\mathbf{e}}_6 + \vec{\mathbf{e}}_5 \text{ if } A_1, A_5 = 0\} && (27)
\end{aligned}
$$

In line 26 I don't think $A_3 = 0$ because we can probably add some combination $a\vec{\mathbf{e}}_1 + b\vec{\mathbf{e}}_2 + c\vec{\mathbf{e}}_3$ so that $\vec{\mathbf{e}}_7$ maps to both 1-cycles (the 2-d kernel) determined so far.

Thus $Z * Z'$ is contained in $(A_0, A_2, A_4, A_8, A_1, A_5)$

Also computed:

$$
\begin{ytableau} 1 \\ 2 \\ 4 \end{ytableau} * \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} = \begin{ytableau} 1 & 1 & 2 \\ 2 & 4 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 \\ 2 & 3 \\ 4 \end{ytableau} \tag{28}
$$

$$
\begin{aligned}
(000, 00, 1) * (100, 10, 0) &= (100, 01, 0) + (100, 10, 1) && (29)\\
S_3 * (1 \to 2) &= (1 \to 2 \leftarrow 3) + (1 \to 2 \to 3) && (30)\\
&= (A_0, A_5, A_4, A_2) \sqcup (A_0, A_5, A_4, A_6) && (31)
\end{aligned}
$$

in

$$
\left[\begin{array}{cc:cc:c:c}
0 & 1 & 0 & 0 & 0 & 0\\
0 & s & A_0 & A_1 & A_2 & A_3\\
\hdashline
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & s & A_4 & A_5\\
\hdashline
0 & 0 & 0 & 0 & s & A_6\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

TODO: add explanation, double check the ideals.

Finally, we guess that the fusion $2\alpha_1 * 2\alpha_2$ is encoded by the tableau equation

$$
\begin{ytableau} 2 & 2 \end{ytableau} * \begin{ytableau} 1 & 1 \\ 3 & 3 \end{ytableau} = \begin{ytableau} 1 & 1 & 2 & 2 \\ 3 & 3 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 \end{ytableau} + \begin{ytableau} 1 & 1 & 3 & 3 \\ 2 & 2 \end{ytableau}
$$

This is checked below.

2 Disjoint, non-dominant weight

Example 1. $\lambda_1 = (1,0,0)$, $\lambda_2 = (1,1,0)$, $\mu_1 = (0,1,0)$, $\mu_2 = (1,0,1)$. Joel: $\mu = \mu_1 + \mu_2$ determines the blocks we have on the RHS of the BD MVy isomorphism.

$$\left[\begin{array}{c|c|c} s & A_0 & A_1 \\ \hline 0 & 0 & A_2 \\ \hline 0 & 0 & s \end{array}\right]$$

3 Some multiplicity

Example 2 (Joel's exercise). It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for $\mathbf{SL}_3$ of weights $2\alpha_1$ and $2\alpha_2$. Following the notation from the mvbasis paper, this would correspond to the following multiplication:

$$x^2y^2 = (xy - z)^2 + 2(xy - z)z + z^2 .$$

Take $\lambda_1 = (2,0,0)$, $\lambda_2 = (2,2,0)$, $\mu_1 = (0,2,0)$, $\mu_2 = (2,0,2)$. Note that there is only one tableau of weight μ_i and type λ_i for each i.

$$\begin{array}{|c|c|} \hline & \\ \hline \end{array} = \left\{ \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \right\} \qquad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \right\}$$

Consider

$$A = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -s^2 & 2s & A_0 & A_1 & A_2 & A_3 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -s^2 & 2s \end{array}\right]$$

Then

$$
\begin{aligned}
E_s &= \{\vec{\mathbf{e}}_1 + s\vec{\mathbf{e}}_2, \vec{\mathbf{e}}_1 + (s+1)\vec{\mathbf{e}}_2 && (32)\\
&\quad \vec{\mathbf{e}}_5 + s\vec{\mathbf{e}}_6 + \frac{1}{s}(A_4 + A_5 s)\vec{\mathbf{e}}_4 + \frac{1}{s^2}(A_4 + A_5 s)\vec{\mathbf{e}}_3 && (33)\\
&\quad \text{if } (A_0 + A_1 s)(A_4 + A_5 s) + (A_2 + A_3 s)s^2 = 0, && (34)\\
&\quad \vec{\mathbf{e}}_1 + (s+a)\vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_6 + \frac{1}{s}\vec{\mathbf{e}}_5 + \frac{1}{s}A_5\vec{\mathbf{e}}_4 - \frac{1}{s^3}A_4\vec{\mathbf{e}}_3\} && (35)\\
&\quad \text{if } -2\,A_3 s^3 - A_2 s^2 + A_0 A_4 - A_5 s = 0 && (36)\\
E_0 &= \{A_0\vec{\mathbf{e}}_1 + s^2\vec{\mathbf{e}}_3, A_0\vec{\mathbf{e}}_2 + s^2\vec{\mathbf{e}}_4 + \frac{A_1}{A_0}\vec{\mathbf{e}}_1 - 2s\vec{\mathbf{e}}_3\} && (37)
\end{aligned}
$$

Taking $s \to 0$ we get

$$
\begin{aligned}
E_{s=0} &= \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3 \text{ if } A_0 A_4 = 0\} && (38)\\
E_0 &= \{\vec{\mathbf{e}}_1, A_0^2\vec{\mathbf{e}}_2 + A_1\vec{\mathbf{e}}_1\} && (39)
\end{aligned}
$$

So $Z * Z'$ is contained in $(A_0 A_4)$. If we look at $(A\big|_{s=0})^4$, which must be zero since $\lambda = (4,2)$, we pick up the additional equation $A_1 A_4 + A_0 A_5$. Therefore the ideal of $Z * Z'$ is

$$
(A_0 A_4, A_1 A_4 + A_0 A_5) \subset \mathbb{C}[A_0..A_5] \tag{40}
$$

which decomposes as $(A_4, A_5) \sqcup (A_0, A_1) \sqcup (A_0, A_4)$. Since

$$
\mathbb{C}[x,y]/(xy, x+y) \cong \mathbb{C}[z]/(z^2)
$$

the component (A_0, A_4) occurs with multiplicity 2 as expected! Localize at, or colon out (A_1, A_5).

Note that these ideals correspond to expected tableaux as follows.

$$
\begin{aligned}
\begin{ytableau} 2 & 2 \end{ytableau} * \begin{ytableau} 1 & 1 \\ 3 & 3 \end{ytableau} &= \begin{ytableau} 1 & 1 & 2 & 2 \\ 3 & 3 \end{ytableau} + 2?\begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 \end{ytableau} + \begin{ytableau} 1 & 1 & 3 & 3 \\ 2 & 2 \end{ytableau} && (41)\\
&= (A_4, A_5) + 2?(A_0, A_4) + (A_0, A_1) && (42)\\
&= (20, 2) + 2?(11, 1) + (02, 0) && (43)\\
&= (11 \to 22) + 2?((1 \to (1 \leftarrow 2)) \to 2) + (11 \leftarrow 22) && (44)\\
&= (xy - z)^2 + 2(xy - z)z + z^2 && (45)
\end{aligned}
$$

Question 1. How do we tell from the tableaux that an ideal is occuring with multiplicity?

4 Simple root weights, things working

Example 3. Let

$$
\begin{array}{lll}
\mu_1 = (3,1,1) & \lambda_1 = (3,2,0) & \mu = (3,3,1)\\
\mu_2 = (0,2,0) & \lambda_2 = (2,0,0) & \lambda = (5,2,0)
\end{array}
$$

and consider the companion matrices of

$$p_1(t) = t^3 \quad p_2(t) = t(t-s)^2 = t^3 - 2st^2 + s^2 t \quad p_3(t) = t$$

$$X = \left[\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e & f & g \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & k \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example 4. Let

$$\lambda_1 = (3,2,0) \quad \mu_1 = (3,1,1)$$
$$\lambda_2 = (2,0,0) \quad \mu_2 = (1,1,0)$$

so the first MV cycle $Z_1 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_1\}$ and the second MV cycle $Z_2 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_2\}$. Their fusion product corresponds to two $\mathbb{P}^2$'s intersecting along a $\mathbb{P}^1$. That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where $Z_+ \cong Z_- \cong \mathbb{P}^2$. We have

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & a & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix}$$

The 0-generalized eigenspace E_0 of X is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, X e_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in $\ker X$, either $a = 0$ or $c = d = 0$, but the latter case cannot give a 2-cycle as $e_7 \notin \operatorname{im} X$. Then $a = 0$ and we obtain a 2-cycle

$$\left\{X\left(e_6 - \frac{s}{d}e_7\right), e_6 - \frac{s}{d}e_7\right\} = \left\{e_5, e_6 - \frac{s}{d}e_7\right\}.$$

We also obtain the equations $b \neq 0$, $d \neq 0$, and $sc - bd = 0$ from this.

For the s-generalized eigenspace E_s, we need $a+sb \neq 0$ to obtain a 2-cycle, which can be taken as

$$\begin{aligned}
&\left\{(X-sI)\left(e_2+2se_3+3s^2e_4+\frac{s^3}{a+sb}e_5+\frac{s^4}{a+sb}e_6\right),\right. \\
&\qquad\qquad \left. e_2+2se_3+3s^2e_4+\frac{s^3}{a+sb}e_5+\frac{s^4}{a+sb}e_6\right\} \\
&=\left\{e_1+se_2+s^2e_3+s^3e_4, e_2+2se_3+3s^2e_4+\frac{s^3}{a+sb}e_5+\frac{s^4}{a+sb}e_6\right\}
\end{aligned}$$

The minimal polymonial is $X^3(X-sI)^2$, which when equated to 0 gives again the equation $cs-bd=0$. Thus the defining equations are

$$\{a=0, cs-bd=0\}.$$

When we take $s=0$, we get the equations

$$\{a=0, bd=0\}$$

which corresponds to two $\mathbb{A}^2$'s intersecting along an $\mathbb{A}^1$. This is indeed an open subset of $\mathbb{P}^2 \cup_{\mathbb{P}_1} \mathbb{P}^2$, as required.

Example 5 (Continued...). The matrix X from the previous example defines (under MVy) the matrix

$$g=\begin{bmatrix} (t-s)t^3 & & \\ -bt & (t-s)t & \\ -c & -d & t \end{bmatrix}$$

in $G(\mathcal{O})$. Indeed the various blocks of X are in a precise sense the companion matrices of the polynomial entries of g

In Gr the element g defines the lattice

$$gL_0=\mathbb{C}[t]\langle (t-s)t^3e_1-(a+bt)e_2-ce_3, (t-s)te_2-de_3, te_3\rangle$$

and computing the matrix of the action of t on the quotient L_0/L in the basis

$$\{[e_1],[te_1],[t^2e_1],[t^3e_1],[e_2],[te_2],[e_3]\}$$

recovers X up to a transpose of course.

Now let's see what we get when we invert t and $t-s$ respectively.

First let's invert t by considering $L_2=L\otimes \mathbb{C}[\![t-s]\!]$.

$$L_2=\mathbb{C}[t,t^{-1}]\langle (t-s)e_1-\frac{a+bt}{t^3}e_2-\frac{c}{t^3}e_3, (t-s)e_2-\frac{d}{t}e_3, e_3\rangle$$

so in L_0/L_2 we have

$$t[e_1]=s[e_1]+\frac{a+bt}{t^3}[e_2]+\frac{c}{t^3}[e_3] \qquad t[e_2]=s[e_2]+\frac{d}{t}[e_3] \qquad [e_3]=0$$

and

$$\left[t\big|_{L_0/L_2}\right]_{\{[e_1],[e_2]\}} = \begin{bmatrix} s & \\ \frac{a+bt}{t^3} & s \end{bmatrix}$$

which upon subtracting sI gives a matrix having block type μ_2 and Jordan type $\lambda_2 = (2)$ assuming $\frac{a+bt}{t^3} \neq 0$.

Next let's invert $t - s$ by considering $L_1 = L \otimes \mathbb{C}[\![t]\!]$.

$$\begin{aligned} L_1 &= \mathbb{C}[t,(t-s)^{-1}]\langle t^3 e_1 - \frac{a+bt}{t-s}e_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle \\ &= \langle t^3 e_1 - \frac{a}{t-s}e_2 - \frac{b}{t-s}te_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle \end{aligned}$$

so in L_0/L_1 we have

$$\begin{aligned} t[e_1] &= [te_1] \\ t[te_1] &= [t^2 e_1] \\ t[t^2 e_1] &= \frac{a}{t-s}[e_2] + \frac{b}{t-s}t[e_2] + \frac{c}{t-s}[e_3] \\ &= \frac{b}{t-s}\frac{d}{t-s}[e_3] + \frac{c}{t-s}[e_3] \\ &= \frac{bd + (t-s)c}{(t-s)^2}[e_3] \\ &= \frac{bd - sc}{(t-s)^2}[e_3] + \frac{c}{(t-s)^2}t[e_3] = 0 \\ t[e_2] &= \frac{d}{t-s}[e_3] \\ t[e_3] &= 0 \end{aligned}$$

and

$$\left[t\big|_{L_0/L_1}\right]_{\{[e_1],[te_1],[t^2e_1],[e_2],[e_3]\}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & & 0 & 0 & \\ & & 0 & \frac{d}{t-s} & 0 \end{bmatrix}$$

taking transpose

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}$$

which has block type μ_1 and Jordan type $\lambda_1 = (3,2)$ assuming $d \neq 0$.

I have used the relations Roger found (and I checked) $a = 0$ and $cs - bd = 0$ in the calculations above.

To sum up, the pair of matrices above should contain the same information as the matrix from the previous example

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix} \Leftrightarrow \left(\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} s & \frac{b}{t^2} \\ & s \end{bmatrix} \right)$$

Wish. Given L in $\mathcal{G}r_n^{\mathrm{BD}}$ define a map to T_μ just like MVy by taking $[t\big|_{L_0/L}]$ and use the fact that $[t\big|_{L_0/L_i}]$ for $i = 1, 2$ are companion matrices of the right type, piece together two MVy isomorphisms to make a BD MVy iso.

Equivalent linear algebra question(?): If $p(t, t-s) = p_1(t)p_2(t-s)$ then how are $C(p_1)$, $C(p_2)$, and $C(p)$ related?

5 Non simple root weights

Example 6 (Anne). Let $G = \mathbf{SL}_3$ and $\underline{\mathbf{i}} = 121$.

Take $n^1_\bullet = (1,0,0)$, and $n^2_\bullet = (1,0,1)$ or $(0,1,0)$. So

$$\begin{array}{lll} \mu_1 = (2,2,1) & \mu_2 = (1,1,1) & \mu = (3,3,2) \\ \lambda_1 = (3,1,1) & \lambda_2 = (2,1,0) & \lambda = (5,2,1) \end{array}$$

Note

$$\tau(1,0,0) = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \qquad \tau(1,0,1) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \tau(0,1,0) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Anne: We should show that order does not matter; i.e. swapping indices on λ's and μ's produces the same result.

$\mathbb{T}^+_{\mu_1,\mu_2} \cap \mathbb{O}_{\lambda_1,\lambda_2}$ is made up of elements of the form

$$A = \left[\begin{array}{ccc|ccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & A_0 & A_1 & A_2 & A_3 & A_4 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & A_5 & A_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \end{array}\right]$$

As usual, denote by E_e the generalized e-eigenspace of A. $A\big|_{\mathbb{C}^3 \cap E_0}$ should have Jordan type (2). The obvious 2-cycle is generated by e_2: $\{e_2, Ae_2\}$. $A\big|_{\mathbb{C}^3 \cap E_s}$ should have Jordan type (1). We take $e_1 + se_2 + s^2e_3 \in \operatorname{Ker}(A - s)$. Next $A\big|_{\mathbb{C}^6 \cap E_0}$ should have Jordan type $(3,1)$ while $A\big|_{\mathbb{C}^6 \cap E_s}$ will have Jordan type (2) or $(1,1)$. Anne: This example breaks. Why? How should we choose weights?

Take 2: Let's try different weights.

$$\begin{array}{ll} \mu_1 = (1,1,0) & \lambda_1 = (2,0,0) \\ \mu_2 = (1,1,1) & \lambda_2 = (2,1,0) \\ \mu = (2,2,1) & \lambda = (4,1,0) \end{array}$$

and

$$\tau(1,0,0) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \tau(1,0,1) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \tau(0,1,0) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Then

$$A = \left[\begin{array}{cc:cc:c} 0 & 1 & 0 & 0 & 0 \\ 0 & s & A_0 & A_1 & A_2 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s & A_3 \\ \hdashline 0 & 0 & 0 & 0 & s \end{array}\right]$$

We have $E_0 \cap \mathbb{C}^2 = \mathrm{Span}(e_1)$, $E_s \cap \mathbb{C}^2 = \mathrm{Span}(e_1 + se_2)$. Next $E_0 \cap \mathbb{C}^4$ is spanned by a 2-cycle generated by $-\frac{A_0}{s}e_2 + e_3$ and $E_s \cap \mathbb{C}^4$ is spanned by a 2-cycle generated by

$$\frac{s}{A_1 + \frac{A_0}{s}} e_4 + \frac{1}{A_1 + \frac{A_0}{s}} e_3 - \frac{1}{s} e_1$$

or the additional 1-cycle $e_4 - \frac{A_1}{A_0} e_3$ assuming $A_0 + sA_1 = 0$. Finally E_s is spanned by an additional 1-cycle

$$e_5 - \frac{A_2}{A_1 + \frac{A_0}{s}} e_4 + \frac{1}{s} \frac{A_2}{A_1 + \frac{A_0}{s}} e_3$$

assuming $A_3 = 0$. Or the two 1-cycles are extended to a 2-cycle and a 1-cycle, the 2-cycle generated by $-\frac{1}{s}e_1 + \frac{s}{A_2}e_5$. This gives us

$$\begin{ytableau} 1 & 2 \end{ytableau} * \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} = (A_3) \qquad \begin{ytableau} 1 & 2 \end{ytableau} * \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} = (A_0 + sA_1) \to (A_0)$$

Does it agree with what is expected on the module/cluster side?

$$S_1 * (1 \to 2) = S_1 \oplus (1 \to 2) \qquad S_1 * (1 \leftarrow 2) = S_1 \oplus (1 \leftarrow 2)$$

The MV cycle of $\begin{ytableau} 1 & 2 \end{ytableau}$ is a $\mathbb{P}^1$: via MVy it has an open subset comprised of matrices

$$\left[\begin{array}{c:c} 0 & A_0 \\ \hdashline 0 & 0 \end{array}\right] : A_0 \neq 0$$

The MV cycles of the other two tableaux are made up of matrices of the form

$$\left[\begin{array}{c:c:c} 0 & A_0 & A_1 \\ \hdashline 0 & 0 & A_2 \\ \hdashline 0 & 0 & 0 \end{array}\right] : \begin{cases} A_0 \neq 0 \text{ and } A_2 = 0 & \tau = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \\ A_0 = 0 \text{ and } A_2 \neq 0 & \tau = \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \end{cases}$$

both $\mathbb{C}^2$'s. Anne: How do the coordinates relate?

Example 7 (Roger). Let

$$\begin{aligned}\lambda_1 &= (2,0,0,0) \quad \mu_1 = (1,1,0,0)\\ \lambda_2 &= (2,2,1,0) \quad \mu_2 = (2,1,1,1)\end{aligned}$$

so $\lambda_1 - \mu_1 = \alpha_1$ and $\lambda_2 - \mu_2 = \alpha_2 + \alpha_3$. We have the following young tableaux:

$$\tau_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad \tau_2 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \tau_2' = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

where τ_1 corresponds to the module S_1, τ_2 corresponds to the module $2 \to 3$, and τ_2' corresponds to the module $2 \leftarrow 3$.

The matrix we are considering is

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & -s^2 & 2s & a & b & c & d \\ & & & 0 & 1 & & \\ & & & & s & e & f \\ & & & & & s & g \\ & & & & & & s \end{bmatrix}$$

such that $\dim E_0 = 2$, $\dim\ker X = 1$, $\dim E_s = 5$, and $\dim\ker(X - sI) = 3$ where E_0 and E_s are the 0- and s-generalized eigenspaces.

We see that the two-cycle in E_0 is

$$\left\{X\left(e_2 + \frac{s^2}{a}e_4\right), e_2 + \frac{s^2}{a}e_4\right\} = \left\{e_1, e_2 + \frac{s^2}{a}e_4\right\}.$$

As τ_2 and τ_2' both share $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, we can find a 2-cycle from just the upper-left 3×3 block, and an additional vector in $\ker(X - sI)$ from the upper-left 5×5-submatrix. The 2-cycle from the 3×3 block is

$$\left\{e_1 + se_2 + s^2e_3, -\frac{2}{s}e_1 - e_2\right\}.$$

The additional vector in $\ker(X - sI)$ is $e_4 + se_5$ and this requires $a + sb = 0$.

Now consider the case that the young diagram we are working with is τ_2. Then we have $e_4 + se_5 + x(e_1 + se_2 + s^2e_3)$ part of a 2-cycle that can be found by looking at the upper-left 6×6-submatrix. We find that the 2-cycle is

$$\left\{e_4 + se_5 + x(e_1 + se_2 + s^2e_3), -\frac{2x}{s}e_1 - xe_2 - \frac{1}{s}e_4 + \frac{s}{e}e_6\right\}$$

and this requires that $ae - s^2c = 0$.

The last vector in $\ker(X - sI)$ comes from the entire $X - sI$ and we see it is $-fe_6 + ee_7$, which requires $g = 0$ and $ed - cf = 0$.

For the case τ_2', we start with find the third vector in $\ker(X - sI)$ from the upper-left 6×6-submatrix. We see that it is e_6, which requires $c = 0$ and $e = 0$.

For the remaining 2-cycle, we want it to end with $x(e_1 + se_2 + s^2e_3) + (e_4 + se_5) + ye_6$ so our 2-cycle is

$$\left\{ x(e_1 + se_2 + s^2e_3) + (e_4 + se_5) + ye_6, -\frac{2x}{s}e_1 - xe_2 - \frac{1}{s}e_4 + \frac{s}{f}e_7 \right\}$$

which requires $af - ds^2 = 0$ and $fy - sg = 0$. As y is free, the last equation is not really a restriction on f and g.

From the minimal polynomial, we have $X^2(X - sI)^2 = 0$ which gives us the equations

$$a + sb = cs + eb = bf + cg + ds = esg = 0.$$

Taking $s \to 0$, we have the following equations for our two cases of τ_2 and τ_2':

τ_2	τ_2'
$a = 0$	$a = 0$
$g = 0$	$c = 0$
$eb = 0$	$e = 0$
$bf = 0$	$bf = 0$
$ed - cf = 0$	

For the τ_2 case, the coordinate ring is

$$\frac{\mathbb{C}[a,b,c,d,e,f,g]}{\langle a, g, eb, bf, ed - cf \rangle} \cong \frac{\mathbb{C}[b,c,d,e,f]}{\langle eb, bf, ed - cf \rangle} = \frac{\mathbb{C}[b,c,d,e,f]}{\langle e, f \rangle \cap \langle b, ed - cf \rangle}$$

Hence the associated algebraic set is reducible with two irreducible components. The component corresponding to the ideal $\langle e, f \rangle$ is $\mathbb{A}^3$, which corresponds to $\mathbb{P}^3$, while the ideal $\langle b, ed - cf \rangle$ corresponds to the toric variety whose toric polytope is a square-based pyramid.

As τ_2 corresponds to the module $2 \to 3$, the irreducible components should correspond to the modules $P_1 = 1 \to 2 \to 3$ and $1 \leftarrow 2 \to 3$. Indeed, the MV cycle corresponding to P_1 is the Grassmannian $Gr(1,4) \cong \mathbb{P}^3$ and for $1 \leftarrow 2 \to 3$, we do get a toric variety with polytope the square-based pyramid.

However for the τ_2' case, the coordinate ring is

$$\frac{\mathbb{C}[a,b,c,d,e,f,g]}{\langle a, c, e, df \rangle} \cong \frac{\mathbb{C}[b,d,f,g]}{\langle df \rangle}$$

which corresponds to $\mathbb{A}^3 \cup \mathbb{A}^3$. Since τ_2' corresponds to the module $2 \leftarrow 3$, we expect two irreducible components corresponding to the modules $P_3 = 1 \leftarrow 2 \leftarrow 3$ and $1 \to 2 \leftarrow 3$. P_3 corresponds to the variety $Gr(3,4) \cong \mathbb{P}^3$ and $1 \to 2 \leftarrow 3$ also corresponds to a toric variety whose polytope is a square-based pyramid (?).

Example 8 (Above example redone). In $\mathbb{C}[A_0..A_6]$ where

$$\left[\begin{array}{ccc|cc|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s^2 & 2s & A_0 & A_1 & A_2 & A_3 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & 0 & s & A_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & s \end{array}\right]$$

we have

$$\begin{aligned} \boxed{1}\boxed{2} * \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} &= (A_6, A_2A_5 - A_3A_4, A_1A_4 + A_2s, A_1s + A_0) \\ &= (A_6, A_2A_5 - A_3A_4, A_1A_4, A_0) \text{ at } s = 0 \\ &= (A_0, A_4, A_5, A_6) \sqcup (A_0, A_1, A_6, A_2A_5 - A_3A_4) \sqcup (A_0, A_2, A_4, A_6) \end{aligned}$$

while

$$\begin{aligned} \boxed{1}\boxed{2} * \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} &= (sA_0A_5 + (1 + s^2)A_1A_5 + sA_3, A_2, A_4, A_1s + A_0) \\ &= (A_1A_5, A_2, A_4, A_0) \text{ at } s = 0 \\ &= (A_0, A_2, A_4, A_5) \sqcup (A_0, A_1, A_2, A_4) \end{aligned}$$

Cycles are

$$\frac{A_0}{s^2}e_2 + e_4 \xrightarrow{A} e_1$$

and, in the $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}$ case,

$$\begin{aligned} e_2 + 2se_3 &\xrightarrow{A-s} e_1 + se_2 + s^2e_3 \\ (sA_4e_4 + (1 + s^2)A_4e_5 + se_6)/(sA_4) &\xrightarrow{A-s} (e_4 + se_5)/s \end{aligned}$$

while in the $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ case,

$$\begin{aligned} e_6 &\xrightarrow{A-s} 0 \\ e_4 + (1/s + s)e_5 + (1/A_5)e_7 &\xrightarrow{A-s} (1/s)e_4 + e_5 + *e_6 \end{aligned}$$

Example 9. Try the following tableaux:

$$1 \to 2 \leftarrow 3 = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \end{array} \qquad 1 \leftarrow 2 \to 3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

which won't work because $\mu_1 + \mu_2$ is not dominant. Can try anyway to see what fails.

Next best guess:

$$1 \to 2 \leftarrow 3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Question: can we do better?

6 February 1, 2021

Example 10. Let $G = \mathbf{SL}_3(\mathbb{C})$ with the standard choices of B, T, and N. Then $\mathbb{C}[N] = \mathbb{C}[x, y, z]$ where

$$\begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} \tag{46}$$

and the unique biperfect basis (also cluster basis in that it's generated by cluster monomials) is

$$\{x^a z^b (xy - z)^c\} \cup \{y^a z^b (xy - z)^c\} \tag{47}$$

where a, b, c are non-negative integers.