Examples

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1 Examples

Example 1. $\lambda_1 = (1,0,0)$, $\lambda_2 = (1,1,0)$, $\mu_1 = (0,1,0)$, $\mu_2 = (1,0,1)$. Joel: $\mu = \mu_1 + \mu_2$ determines the blocks we have on the RHS of the BD MVy isomorphism of Equation **??**.

In the non-BD case, MVy establish

$$\overline{\mathrm{Gr}^\lambda} \cap \mathcal{W}^\mu \to \left\{ X = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * \\ \hline 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * \\ \hline * & 0 & * & 0 & * \end{array}\right] \Big| X \in \overline{\mathbb{O}}_\lambda \right\}$$

In the BD case the RHS will consist of the same block like matrices X but now having eigenvalues $s, 0$ such that $X - s\big|_{E_s} \in \mathbb{O}_{\lambda_2}$

Example 2. Do Joel's exercise: It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for $\mathbf{SL}_3$ of weights $2\alpha_1$ and $2\alpha_2$. Following the notation from the mvbasis paper, this would correspond to the following multiplication: $x^2y^2 = (xy-z)^2 + 2(xy-z)z + z^2$.

For this example, I think we need $\lambda_1 = (2,0,0)$, $\lambda_2 = (2,2,0)$, $\mu_1 = (0,2,0)$, $\mu_2 = (2,0,2)$.

Adding 2020-12-30 14:03:05: Let's take

$$\begin{aligned} \lambda_2 &= (4,0,0) & \lambda_1 &= (4,4,0) & \lambda &= (8,4,0) \\ \mu_2 &= (2,2,0) & \mu_1 &= (4,2,2) & \mu &= (6,4,2) \end{aligned}$$

Note that there is only one SSYT of shape λ_i and weight μ_i

$$\tau_2 = \boxed{1|1|2|2} \qquad \tau_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 & 3 \\ \hline \end{array}$$

Note also that

$$\begin{aligned} t^4(t-s)^2 &= t^4(t^2 - 2st + s^2) = t^6 - 2st^5 + s^2t^4 \\ t^2(t-s)^2 &= t^4 - 2st^3 + s^2t^2 \end{aligned}$$

Elements of $\mathbb{T}^{+}_{\mu_1,\mu_2}$ will take the form

$$A = \left[\begin{array}{cccccc|cccc|cc}
 & 1 & & & & & & & & & & \\
 & & 1 & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
 & & & & 1 & & & & & & & \\
 & & & & & 1 & & & & & & \\
 & & & & -s^2 & 2s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\
\hline
 & & & & & & & 1 & & & & \\
 & & & & & & & & 1 & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & -s^2 & 2s & c_1 & c_2 \\
\hline
 & & & & & & & & & & & 1 \\
 & & & & & & & & & & &
\end{array}\right]$$

The tableau tells us for each $1 \le i \le 12$

$$\text{Jordan type of } A\big|_{\mathrm{Span}(e_1,\ldots,e_i)\cap E_0} \text{ is shape of } \tau_1\big|_{\text{first } i \text{ boxes}}$$

So take $i = 10$. Then A restricted to $\mathbb{C}^{10} \cap E_0$ should have Jordan type $(4, 2)$. Anne: How to do it box by box? s columns somehow correspond to τ_2 boxes. Therefore $a_1 = 0$. The 4-cycle is obvious. For the 2-cycle we require $a_2 = 0$.

$$e_1 \leftarrow e_2 \leftarrow e_3 \leftarrow e_4$$
$$e_7 \leftarrow e_8$$

Now looking at all of A we have continue our 2-cycle to a 4-cycle. Roger found

$$e_{10} - \left(\frac{2s}{c_1} + \frac{s^2 c_2}{c_1^2}\right) e_{11} + \frac{s^2}{c_1} e_{12}$$

This requires $c_1^2 a_4 + c_1 b_2 s^2 - 2sc_1 b_1 - s^2 c_2 = 0$ and $a_3 c_1 + s^2 b_1 = 0$.

Now looking for the s-eigenspace, we expect $A - s\big|_{\mathbb{C}^6 \cap E_s}$ to have Jordan type 2. The kernel is spanned by $e_1 + se_2 + s^2 e_3 + s^3 e_4 + s^4 e_5 + s^5 e_6$. It is continued to a 2-cycle by/the 2-cycle is generated by $-\frac{5}{s} e_1 - 4e_2 - 3se_3 - 2s^2 e_4 - s^3 e_5$. The 3-cycle is maybe $(1/s^2, -4/s, -8, 5s, 3s^2, 2s^3, -s^2/a_3, -s^3/a_3, -s^4/a_3, -s^5/a_3)$ padded with zeros.

$$A - s = \left[\begin{array}{cccccc:cccc:cc}
-s & 1 & & & & & & & & & & \\
 & -s & 1 & & & & & & & & & \\
 & & -s & 1 & & & & & & & & \\
 & & & -s & 1 & & & & & & & \\
 & & & & -s & 1 & & & & & & \\
 & & & & -s^2 & s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\
\hdashline
 & & & & & & -s & 1 & & & & \\
 & & & & & & & -s & 1 & & & \\
 & & & & & & & & -s & 1 & & \\
 & & & & & & & & -s^2 & s & c_1 & c_2 \\
\hdashline
 & & & & & & & & & & -s & 1 \\
 & & & & & & & & & & & -s
\end{array}\right]$$

Example 3. Let

$$\begin{array}{lll}
\mu_1 = (2) & \lambda_1 = & \mu = (5) \\
\mu_2 = (3) & \lambda_2 = & \lambda =
\end{array}$$

Consider the companion matrix $C(p)$ of

$$p(t) = (t-s)^3 t^2 = (t^3 - 3t^2 s + 3ts^2 - s^3)t^2 = t^5 - 3t^4 s + 3t^3 s^2 - t^2 s^3$$

Let $X = C(p)^T$ so

$$X = \begin{bmatrix}
0 & 1 & & & \\
 & 0 & 1 & & \\
 & & 0 & 1 & \\
 & & & 0 & 1 \\
0 & 0 & s^3 & -3s^2 & 3s
\end{bmatrix}$$

Ask that $X\big|_{E_0}$ has Jordan type λ_1 and $X - s\big|_{E_s}$ has Jordan type λ_2. In this rank 1 case we are forced to take $\lambda_i = \mu_i$.

So what are the generalized eigenspaces E_i $(i = 1, 2)$? Note $\dim E_0 = 2$ and $\dim E_s = 3$.

Anne: The basis

$$[1], [t], [\mathbf{1}], [\mathbf{t}], [\mathbf{t^2}]$$

with $t[t] = 0$ and $t[\mathbf{t^2}] = s^3[\mathbf{1}] - 3s^2[\mathbf{t}] + 3s[\mathbf{t^2}]$ *is not the correct basis to consider.* Hence my confusion of yore: what we would like is $t[t] = 0$ no? what the matrix is telling us is that $t[t] = [\mathbf{1}]$. Can we still speak of *two* generalized eigenspaces?

Rather, take B to be the basis $b_1 = e_1$, $b_2 = e_2$, $b_5 = e_5$, $b_4 = Xe_5$, $b_3 = X^2e_5$). In this basis

$$X_B = [X(b_i)] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 1 & 0 \\ 0 & 0 & 0 & s & 1 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

Example 4. Let

$$\mu_1 = (3, 1, 1) \quad \lambda_1 = (3, 2, 0) \quad \mu = (3, 3, 1)$$
$$\mu_2 = (0, 2, 0) \quad \lambda_2 = (2, 0, 0) \quad \lambda = (5, 2, 0)$$

and consider the companion matrices of

$$p_1(t) = t^3 \quad p_2(t) = t(t - s)^2 = t^3 - 2st^2 + s^2t \quad p_3(t) = t$$

$$X = \left[\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e & f & g \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & k \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example 5. Let

$$\lambda_1 = (3, 2, 0) \quad \mu_1 = (3, 1, 1)$$
$$\lambda_2 = (2, 0, 0) \quad \mu_2 = (1, 1, 0)$$

so the first MV cycle $Z_1 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_1\}$ and the second MV cycle $Z_2 \cong \mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0, \alpha_2\}$. Their fusion product corresponds to two $\mathbb{P}^2$'s intersecting along a $\mathbb{P}^1$. That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where $Z_+ \cong Z_- \cong \mathbb{P}^2$. We have

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & a & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix}$$

The 0-generalized eigenspace E_0 of X is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, X e_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in $\ker X$, either $a = 0$ or $c = d = 0$, but the latter case cannot give a 2-cycle as $e_7 \notin \operatorname{im} X$. Then $a = 0$ and we obtain a 2-cycle

$$\left\{X\left(e_6 - \frac{s}{d}e_7\right), e_6 - \frac{s}{d}e_7\right\} = \left\{e_5, e_6 - \frac{s}{d}e_7\right\}.$$

We also obtain the equations $b \neq 0$, $d \neq 0$, and $sc - bd = 0$ from this.

For the s-generalized eigenspace E_s, we need $a + sb \neq 0$ to obtain a 2-cycle, which can be taken as

$$\begin{aligned} &\left\{(X - sI)\left(e_2 + 2se_3 + 3s^2 e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right),\right. \\ &\qquad\left. e_2 + 2se_3 + 3s^2 e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right\} \\ = &\left\{e_1 + se_2 + s^2 e_3 + s^3 e_4, e_2 + 2se_3 + 3s^2 e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right\} \end{aligned}$$

The minimal polymonial is $X^3(X - sI)^2$, which when equated to 0 gives again the equation $cs - bd = 0$. Thus the defining equations are

$$\{a = 0, cs - bd = 0\}.$$

When we take $s = 0$, we get the equations

$$\{a = 0, bd = 0\}$$

which corresponds to two $\mathbb{A}^2$'s intersecting along an $\mathbb{A}^1$. This is indeed an open subset of $\mathbb{P}^2 \cup_{\mathbb{P}_1} \mathbb{P}^2$, as required.

Example 6 (Example 7 continued…). The matrix X from the previous example defines (under MVy) the matrix

$$g = \begin{bmatrix} (t-s)t^3 & & \\ -bt & (t-s)t & \\ -c & -d & t \end{bmatrix}$$

in $G(\mathcal{O})$. Indeed the various blocks of X are in a precise sense the companion matrices of the polynomial entries of g

In Gr the element g defines the lattice

$$gL_0 = \mathbb{C}[t]\langle (t-s)t^3 e_1 - (a+bt)e_2 - ce_3, (t-s)te_2 - de_3, te_3\rangle$$

and computing the matrix of the action of t on the quotient L_0/L in the basis

$$\{[e_1], [te_1], [t^2e_1], [t^3e_1], [e_2], [te_2], [e_3]\}$$

recovers X up to a transpose of course.

Now let's see what we get when we invert t and $t-s$ respectively.

First let's invert t by considering $L_2 = L \otimes \mathbb{C}[\![t-s]\!]$.

$$L_2 = \mathbb{C}[t, t^{-1}]\langle (t-s)e_1 - \frac{a+bt}{t^3}e_2 - \frac{c}{t^3}e_3, (t-s)e_2 - \frac{d}{t}e_3, e_3\rangle$$

so in L_0/L_2 we have

$$t[e_1] = s[e_1] + \frac{a+bt}{t^3}[e_2] + \frac{c}{t^3}[e_3] \qquad t[e_2] = s[e_2] + \frac{d}{t}[e_3] \qquad [e_3] = 0$$

and

$$\left[t\big|_{L_0/L_2}\right]_{\{[e_1],[e_2]\}} = \begin{bmatrix} s & \\ \frac{a+bt}{t^3} & s \end{bmatrix}$$

which upon subtracting sI gives a matrix having block type μ_2 and Jordan type $\lambda_2 = (2)$ assuming $\frac{a+bt}{t^3} \neq 0$.

Next let's invert $t-s$ by considering $L_1 = L \otimes \mathbb{C}[\![t]\!]$.

$$\begin{aligned} L_1 &= \mathbb{C}[t, (t-s)^{-1}]\langle t^3e_1 - \frac{a+bt}{t-s}e_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle \\ &= \langle t^3e_1 - \frac{a}{t-s}e_2 - \frac{b}{t-s}te_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle \end{aligned}$$

so in L_0/L_1 we have

$$\begin{aligned} t[e_1] &= [te_1] \\ t[te_1] &= [t^2e_1] \\ t[t^2e_1] &= \frac{a}{t-s}[e_2] + \frac{b}{t-s}t[e_2] + \frac{c}{t-s}[e_3] \\ &= \frac{b}{t-s}\frac{d}{t-s}[e_3] + \frac{c}{t-s}[e_3] \\ &= \frac{bd + (t-s)c}{(t-s)^2}[e_3] \\ &= \frac{bd - sc}{(t-s)^2}[e_3] + \frac{c}{(t-s)^2}t[e_3] = 0 \\ t[e_2] &= \frac{d}{t-s}[e_3] \\ t[e_3] &= 0 \end{aligned}$$

and

$$\left[t\big|_{L_0/L_1}\right]_{\{[e_1],[te_1],[t^2e_1],[e_2],[e_3]\}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & & 0 & 0 & \\ & & 0 & \frac{d}{t-s} & 0 \end{bmatrix}$$

taking transpose

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}$$

which has block type μ_1 and Jordan type $\lambda_1 = (3,2)$ assuming $d \neq 0$.

I have used the relations Roger found (and I checked) $a = 0$ and $cs - bd = 0$ in the calculations above.

To sum up, the pair of matrices above should contain the same information as the matrix from the previous example

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix} \Leftrightarrow \left(\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} s & \frac{b}{t^2} \\ & s \end{bmatrix} \right)$$

Wish. Given L in $\mathcal{G}r_n^{\mathrm{BD}}$ define a map to T_μ just like MVy by taking $[t\big|_{L_0/L}]$ and use the fact that $[t\big|_{L_0/L_i}]$ for $i = 1,2$ are companion matrices of the right type, piece together two MVy isomorphisms to make a BD MVy iso.

Equivalent lin alg question: If $p(t, t-s) = p_1(t)p_2(t-s)$ then how are $C(p_1)$, $C(p_2)$, and $C(p)$ related? I think it's basically this theorem `https://en.wikipedia.org/wiki/Structure_theorem_for_finitely_generated_modules_over_a_principal_ideal_domain`

Example 7. Let $G = \mathbf{SL}_3$ and $\underline{\mathbf{i}} = 121$. Take $n^1_\bullet = (1,0,0)$, and $n^2_\bullet = (1,0,1)$ or $(0,1,0)$. So $\mu_1, = (2,2,1)$, $\lambda_1 = (3,1,1)$ and $\mu_2 = (1,1,1)$, $\lambda_2 = (2,1,0)$. Anne: We should show that order does not matter. Then $\mathbb{T}^+_{\mu_1,\mu_2} \cap \mathbb{O}_{\lambda_1,\lambda_2}$ is made up of elements

$$\begin{bmatrix} 0 & 1 & 0 & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & s & 0 & a_2 & a_3 & b_1 & b_2 \\ & & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 1 & & \\ & & & 0 & 0 & s & c_1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & 0 & s \end{bmatrix}$$

and zero eigenspace conforming to the shape

$$\lambda_1 = \ydiagram{3,1,1}$$

is made of three cycles

$$\begin{aligned}
&a_2e_1 \leftarrow a_2e_2 - se_4 \leftarrow a_2e_3 - se_5\\
&\quad e_4\\
&\quad e_7
\end{aligned}$$

while the s-eigenspace confirming to the shape

$$\lambda_2 = \ydiagram{2,1}$$

is made of the two cycles

$$\begin{aligned}
&e_1 + se_2 + s^2e_3 \leftarrow e_2 + 2se_3 + e_7 + se_8\\
&e_4 + se_5 + s^2e_6
\end{aligned}$$

assuming $b_2 = s$ and $a_2 + sa_3 = 0$.

Example 8. Let

$$\begin{aligned}
\lambda_1 &= (2, 0, 0, 0) \quad \mu_1 = (1, 1, 0, 0)\\
\lambda_2 &= (2, 2, 1, 0) \quad \mu_2 = (3, 2, 1, 1)
\end{aligned}$$

so $\lambda_1 - \mu_1 = \alpha_1$ and $\lambda_2 - \mu_2 = \alpha_2 + \alpha_3$. We have the following young tableaux:

$$\tau_1 = \begin{ytableau} 1 & 2 \end{ytableau} \qquad \tau_2 = \begin{ytableau} 1 & 1 \\ 2 & 3 \\ 4 \end{ytableau} \qquad \tau_2' = \begin{ytableau} 1 & 1 \\ 2 & 4 \\ 3 \end{ytableau}$$

where τ_1 corresponds to the module S_1, τ_2 corresponds to the module $2 \to 3$, and τ_2' corresponds to the module $2 \leftarrow 3$.

The matrix we are considering is

$$X = \begin{bmatrix}
0 & 1 & & & & & \\
 & 0 & 1 & & & & \\
 & -s^2 & 2s & a & b & c & d\\
 & & & 0 & 1 & & \\
 & & & & s & e & f\\
 & & & & & s & g\\
 & & & & & & s
\end{bmatrix}$$

such that $\dim E_0 = 2$, $\dim\ker X = 1$, $\dim E_s = 5$, and $\dim\ker(X - sI) = 3$ where E_0 and E_s are the 0- and s-generalized eigenspaces.

We see that the two-cycle in E_0 is

$$\left\{X\left(e_2+\frac{s^2}{a}e_4\right), e_2+\frac{s^2}{a}e_4\right\}=\left\{e_1, e_2+\frac{s^2}{a}e_4\right\}.$$

As τ_2 and τ_2' both share $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$, we can find a 2-cycle from just the upper-left 3×3 block, and an additional vector in $\ker(X-sI)$ from the upper-left 5×5-submatrix. The 2-cycle from the 3×3 block is

$$\left\{e_1+se_2+s^2e_3, -\frac{2}{s}e_1-e_2\right\}.$$

The additional vector in $\ker(X-sI)$ is e_4+se_5 and this requires $a+sb=0$.

Now consider the case that the young diagram we are working with is τ_2. Then we have e_4+se_5 part of a 2-cycle that can be found by looking at the upper-left 6×6-submatrix. We find that the 2-cycle is

$$\left\{e_4+se_5, -\frac{1}{s}e_4+\frac{s}{e}e_6\right\}$$

and this requires that $ae-s^2c=0$.

The last vector in $\ker(X-sI)$ comes from the entire $X-sI$ and we see it is $-fe_6+ee_7$, which requires $g=0$ and $ed-cf=0$.

For the case τ_2', we start with find the third vector in $\ker(X-sI)$ from the upper-left 6×6-submatrix. We see that it is e_6, which requires $c=0$ and $e=0$.

For the remaining 2-cycle, we note that $e_4+se_5\notin \mathrm{col}(X-sI)$ but $e_6\in \mathrm{col}(X-sI)$ so our 2-cycle is

$$\left\{e_6, \frac{1}{g}e_7\right\}$$

which requires $d=0$ and $f=0$.

From the minimal polynomial, we have $X^2(X-sI)^2=0$ which gives us the equations

$$a+sb=cs+eb=bf+cg+ds=esg=0.$$

Taking $s\to 0$, we have the following equations for our two cases of τ_2 and τ_2':

τ_2	τ_2'
$a=0$	$a=0$
$g=0$	$c=0$
$eb=0$	$d=0$
$bf=0$	$e=0$
$ed-cf=0$	$f=0$

For the τ_2 case, the coordinate ring is

$$\frac{\mathbb{C}[a,b,c,d,e,f,g]}{\langle a,g,eb,bf,ed-cf\rangle}\cong\frac{\mathbb{C}[b,c,d,e,f]}{\langle eb,bf,ed-cf\rangle}=\frac{\mathbb{C}[b,c,d,e,f]}{\langle e,f\rangle\cap\langle b,ed-cf\rangle}$$

Hence the associated algebraic set is reducible with two irreducible components. The component corresponding to the ideal $\langle e, f\rangle$ is $\mathbb{A}^3$, which corresponds to $\mathbb{P}^3$, while the ideal $\langle b, ed - cf\rangle$ corresponds to the toric variety whose toric polytope is a square-based pyramid.

As τ_2 corresponds to the module $2 \to 3$, the irreducible components should correspond to the modules $P_1 = 1 \to 2 \to 3$ and $1 \leftarrow 2 \to 3$. Indeed, the MV cycle corresponding to P_1 is the Grassmannian $Gr(1, 4) \cong \mathbb{P}^3$ and for $1 \leftarrow 2 \to 3$, we do get a toric variety with polytope the square-based pyramid.

However for the τ_2' case, the coordinate ring is

$$\frac{\mathbb{C}[a, b, c, d, e, f, g]}{\langle a, c, d, e, f\rangle} \cong \mathbb{C}[b, g]$$

which corresponds to $\mathbb{A}^2$. Since τ_2' corresponds to the module $2 \leftarrow 3$, we expect two irreducible components corresponding to the modules $P_3 = 1 \leftarrow 2 \leftarrow 3$ and $1 \to 2 \leftarrow 3$. P_3 corresponds to the variety $Gr(3, 4) \cong \mathbb{P}^3$ and $1 \to 2 \leftarrow 3$ also corresponds to a toric variety whose polytope is a square-based pyramid (?).