How to compute the fusion product of MV cycles in type A

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1 Players

$G = \mathbf{GL}_m$.

- The ordinary affine Grassmannian Gr
- The Beilinson–Drinfeld Grassmannian $\mathcal{G}r_n^{\mathrm{BD}} \to C$
- Partitions $\mu_i \le \lambda_i$ of N_i $(i = 1, 2)$ and $\mu = \sum \mu_i \le \lambda = \sum \lambda_i$ of $N = \sum N_i$
- The slices Gr_μ and $\mathcal{W}_{\mu_1,\mu_2}$ to the orbits Gr^λ and $(\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2}$
- The nilpotent and semi-nilpotent cones $\mathcal{N}$ and $\mathcal{N}_s$ (of matrices with eigenvalues 0 and 0 or $s \neq 0$)
- The slices $\mathbb{T}_\mu$ and $\mathbb{T}_{\mu_1,\mu_2}$ to the orbits $\mathbb{O}_\lambda$ and $\mathbb{O}_{\lambda_1,\lambda_2}$

New (?) definitions among these are as follows.

The [family of] slices [with s-fibre?]

$$\mathcal{W}_{\mu_1,\mu_2} = G_1[t^{-1}, (t-s)^{-1}] L_{\mu_1,\mu_2} \tag{1}$$

where $L_{\mu_1,\mu_2} \in \mathcal{G}r_2^{\mathrm{BD}}$ is a $\mathbb{C}[t]$-lattice in $\mathbb{C}(t)^m$ that specializes to a $\mathbb{C}[\![t]\!]$-lattice in $\mathbb{C}(\!(t)\!)^m$ away from $t = 0$ and away from $t = s$; i.e.

$$\begin{gathered} L_{\mu_1,\mu_2} \otimes \mathbb{C}[(t-s)^{-1}] = L_{\mu_1,\mu_2} \otimes \mathbb{C}[\![t]\!] = L_{\mu_2} \text{ and} \\ L_{\mu_1,\mu_2} \otimes \mathbb{C}[t^{-1}] = L_{\mu_1,\mu_2} \otimes \mathbb{C}[\![t-s]\!] = L_{\mu_1} \end{gathered} \tag{2}$$

where L_{μ_i} denotes the point $t^{\mu_i} G(\mathcal{O}) \in \mathrm{Gr}$.

The family of semi-infinite orbits Anne: Maybe doesn't make sense before specialization?

$$S_{\mu_1,\mu_2} = N_-(\mathcal{K}) L_{\mu_1,\mu_2}\,. \tag{3}$$

The orbit of $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^m$ whose elements specialize again to $\mathbb{C}[\![t]\!]$-lattices in $\mathbb{C}(\!(t)\!)^m$ away from $t = 0$ and away from $t = s$ as follows

$$\begin{aligned} (\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2} = \{L \in \mathcal{G}r_2^{\mathrm{BD}} : L \otimes \mathbb{C}[t^{-1}] \in \mathrm{Gr}^{\lambda_2} \text{ and} \\ L \otimes \mathbb{C}[(s-t)^{-1}] \in \mathrm{Gr}^{\lambda_1}\}\,. \end{aligned} \tag{4}$$

Anne: These defining conditions are again just telling us that in the fibre over the fixed point $(0, s) \in C^{(2)}$ this set is pairs of lattices; note that we invert the indeterminates t and $(t - s)$ *after* specializing $(0, s)$. Could we give a more explicit characterization like $t^{N_1}(t-s)^{N_2}L_0 \subseteq L \subseteq t^{-N_1}(t-s)^{-N_2}$?

The [family "$\mathcal{N} \otimes \mathbb{C}[s]$? $\mathcal{N} \times \mathbb{C}$?" of] semi-nilpotent cone[s fibred over $C = \mathbb{A}^1$ with s-fibre]

$$\mathcal{N}_s = \{A \in \mathrm{Mat}(N) : \text{eigenvalues of } A \text{ are } 0 \text{ or } s\}\,. \tag{5}$$

The [family of] slice[s $\mathbb{T}_{\mu_1,\mu_2}$ fibred over $C = \mathbb{A}^1$ with s-fibre]

$$\begin{aligned}\mathbb{T}^s_{\mu_1,\mu_2} = \{B + C_s : B &\text{ is a } \mu \times \mu \text{ block matrix of zeros} \\ &\text{except possibly in the last } \min(\mu_i, \mu_j) \\ &\text{columns of the last row of each } \mu_i \times \mu_j \text{ block} \\ &\text{and } C_s \text{ is the block diagonal matrix of} \\ &\text{companion matrices of } t^{\mu_{1,k}}(t-s)^{\mu_{2,k}}\}\,.\end{aligned} \tag{6}$$

The uppertriangular subfamily $\mathbb{T}^+_{\mu_1,\mu_2}$ with s-fibre

$$\mathbb{T}^{+,s}_{\mu_1,\mu_2} = \{B + C_s \in \mathbb{T}_{\mu_1,\mu_2} : B \in \mathfrak{n}\} \tag{7}$$

where $\mathfrak{n} \subset \mathrm{Mat}(N)$ is the unipotent subalgebra of uppertriangular matrices.

Anne: or—as Joel pointed out, may be ok with: The slice $\mathbb{T}_\mu$ as defined in MVy, no change, and the family of slices

$$\mathbb{T}^{+,s}_{\mu_1,\mu_2} = \mathbb{T}_\mu \cap \mathfrak{n} + C^s_{\mu_1,\mu_2}$$

where

$$C_s \text{ is the block diagonal matrix of companion matrices of } t^{\mu_{1,k}}(t-s)^{\mu_{2,k}} \tag{8}$$

The [family of] orbit[s $\mathbb{O}_{\lambda_1,\lambda_2}$ fibred over $C = \mathbb{A}^1$ with s-fibre]

$$\mathbb{O}^s_{\lambda_1,\lambda_2} = \{A \in \mathcal{N}_s : A \text{ is conjugate to } J_{\lambda_1} \oplus (sI_{N_2} + J_{\lambda_2})\} \tag{9}$$

where J_{λ_i} is the Jordan normal form of block type λ_i and I_{N_2} is the identity matrix in $\mathrm{Mat}(N_2)$.

2 Exposition

3 Rising Action

4 Climax

5 Falling Action

Theorem 1. *Let $\lambda_i \geq \mu_i$ be dominant $(i = 1, 2)$, $\mu = \mu_1 + \mu_2$, and $\lambda = \lambda_1 + \lambda_2$. There is an isomorphism*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_{\mu_1,\mu_2} \to \overline{(\mathcal{G}r_2^{BD})^{\lambda_1,\lambda_2}} \cap \mathcal{W}_{\mu_1,\mu_2} \tag{10}$$

got by taking a $\mu \times \mu$ block matrix A in the s-fibre $\overline{\mathbb{O}^s_{\lambda_1,\lambda_2}} \cap \mathbb{T}^s_{\mu_1,\mu_2}$ on the left to the representative of the s-fibre on the right defined by

$$\begin{aligned} g &= t^{\mu_1}(t-s)^{\mu_2} + a(t) \\ a_{ij}(t) &= -\sum_{k=1}^{\mu_i} A^k_{ji} t^{k-1} \end{aligned} \tag{11}$$

where A^k_{ji} is the kth entry from the left of the last row of the $\mu_j \times \mu_i$ block of A.

Let's call this the MVyBD isomorphism.

Proof. The proof is fibre by fibre, so fix $s \neq 0$. Anne: Emphasize in the intro later (because this always confuses me) that by the s-fibre we really mean the $(0, s)$-fibre; i.e. its the BD Grassmannian over the second symmetric power of $C = \mathbb{A}^1$; better just replace s-fibre by $(0, s)$-fibre everywhere it occurs.

1. The map is well defined. In particular, it defines $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^m$. Moreover, these lattices break down to give pairs of lattices upon inverting t or $t - s$ that have the right properties. [Copy Roger's proof]

2. The inverse map is got by taking the matrix of multiplication by t on the quotient $\mathbb{C}[t]^m/L$ just as in the ordinary MVy isomorphism—the only difference being $\mathbb{C}[\![t]\!]$ is replaced by $\mathbb{C}[t]$.

   (a) The matrix of t will have the right block type with respect to the basis
   $$\{[e_i], [te_i], \ldots, [t^{\mu_i-1}e_i] : 1 \leq i \leq m\} \tag{12}$$
   of $\mathbb{C}[t]^m/t^{\mu_1}(t-s)^{\mu_2}\mathbb{C}[t]^m$. We can show this if we can show that $\mathbb{C}[t]$-lattices satisfying Equation 2 have a basis of the form
   $$v_i = t^{\mu_{1,i}}(t-s)^{\mu_{2,i}} + \sum_{j>i} p_{ij}(t)e_j \tag{13}$$
   with $\deg p_{ij}(t) < \mu_i = \mu_{1,i} + \mu_{2,i}$ $(1 \leq i \leq m)$. Anne: I don't know why this should be true. We might have to just define fibres of $\mathcal{W}_{\mu_1,\mu_2}$ in this way?

   (b) $t\big|_{\mathbb{C}[t]^m/L}$ will have two eigenvalues, 0 and s, and its generalized 0-eigenspace will have block type $\leq \lambda_1$ while its generalized s-eigenspace will have block type $\leq \lambda_2$. This should follow from the defining fact that lattices satisfying Equation 4 equivalently satisfy
   $$\begin{aligned} t\big|_{\mathbb{C}[\![t]\!]^m/L_1} &\text{ has Jordan type} \leq \lambda_1 \\ t\big|_{\mathbb{C}[\![t]\!]^m/L_2} &\text{ has Jordan type} \leq \lambda_2 \end{aligned} \tag{14}$$
   where recall $L_i = L \otimes \mathbb{C}[(t-p_i)^{-1}]$ and $p_1 = s$ while $p_2 = 0$. Anne: Somehow, restricting to an eigenspace is like inverting/forgetting the action of t by any other generalized eigenvalue? Basic linear algebra? Joel?

$\square$

Corollary 1. *The MVyBD isomorphism restricts to an isomorphism of subfamilies*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}^+_{\mu_1,\mu_2} \to \overline{(\mathcal{G}r_2^{BD})^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2}\,. \tag{15}$$

Define $S^s_{\mu_1,\mu_2} = N_-(\!(t^{-1})\!)t^{\mu_1}(t-s)^{\mu_2}$.

Anne: is it a fibre of S_{μ_1,μ_2} defined above?

We could also make the following claim.

Theorem 2 (Theorem 1 version 2)**.** *Let λ_1, λ_2 and μ be dominant, such that $\lambda = \lambda_1 + \lambda_2 \geq \mu$. Then there is an isomorphism*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_\mu \to \overline{\mathcal{G}r_2^{BD\lambda_2,\lambda_2}} \cap \mathcal{W}_\mu \tag{16}$$

defined by the same map as in Theorem 1. Note $\mathcal{W}_\mu = \mathrm{Gr}_\mu$.

Lemma 1 (KWWY14)**.** *Let μ be dominant. Then*

$$N_-(\!(t^{-1})\!)L_\mu = N_1[\![t^{-1}]\!]L_\mu \tag{17}$$

Anne: where I am not sure about the double brackets.

Lemma 2. *Let μ_1, μ_2 be dominant and let $s \in \mathbb{A}^1 - \{0\}$. Then*

$$S^s_{\mu_1,\mu_2} \subset \mathrm{Gr}_\mu \tag{18}$$

where $\mu = \mu_1 + \mu_2$.

Proof. Copy Roger's proof. $\square$

6 Denouement