Working title: Mirković–Vybornov fusion in Beilinson–Drinfeld Grassmannian

October 2020

1 Introduction

The BD Grassmannian. The convolution Grassmannian. Distinguished orbits, slices therein. Mirković–Vybornov [MV07, MV19], Cautis–Kamnitzer [CK18], Anderson–Kogan [AK05].

2 Notation

Let Gr denote the ordinary affine Grassmannian, $\mathcal{G}$ the Beilinson–Drinfeld affine Grassmannian, and $\mathfrak{G}$ the convolution affine Grassmannian.

Definition 1. The **BD Grassmannian** is the set

$$\begin{aligned}\{(V,\sigma) : V &\text{ is a rank } m \text{ vector bundle on } \mathbb{P}^1\\ &\text{and } \sigma : V \dashrightarrow \mathscr{O}^m_{\mathbb{P}^1} \text{ is a trivialization}\\ &\text{defined away from finitely many points in } \mathbb{A}^1\}\end{aligned} \tag{1}$$

More generally, one can define a BD grassmannian over any smooth curve C as the reduced ind-scheme $\mathcal{G}_C$ fibered over a finite symmetric power of C such that the fibre over the point $\vec{p}$ is a collection of vector bundles over C which are trivial away from $\vec{p}$ viewed also as a subset of C. To [MV19] the rank m of the trivial fibres $\mathscr{O}^m_C$ is the of the group $\mathbf{GL}_m\mathbb{C}$.

To quote [BGL20] the BD Grassmannian is a relative version of the affine Grassmannian where the base is the space of effective divisors on a smooth curve. The choice of $\mathbb{A}^1$ amply satisfies our needs and offers three advantages: there is a natural global coordinate it, every G-torsor on it is trivializable, and the monodromy of any local system is trivial. Formally, $\mathcal{G}$ is the functor on the category of commutative $\mathbb{C}$-algebras that assigns to an algebra R the set of isomorphism classes of triples $(\vec{p}, V, \sigma)$ where $\vec{p} \in \mathbb{A}^n(R)$, V is a $G^\vee$-torsor over $\mathbb{A}^1_R$ and σ is a trivialization of V away from $\vec{p}$. They denote by π the fibration $\mathcal{G} \to \mathbb{A}^n$ (forgetting V and σ).

Their simplified description is: it's the set of pairs $(\vec{p}, [\sigma])$ where $\vec{p} \in \mathbb{C}^n$ and $[\sigma]$ is an element of the homogeneous space

$$G^\vee(\mathbb{C}[z, (z-p_1)^{-1}, \dots, (z-p_n)^{-1}])/G^\vee(\mathbb{C}[z])$$

Their example, our setting:

Example 1. When $G = \mathbf{GL}_m\mathbb{C}$ the datum of $[\sigma]$ is equivalent to the datum of the $\mathbb{C}[z]$-lattice $\sigma(L_0)$ in $\mathbb{C}(z)^m$ with $L_0 = \mathbb{C}[z]^m$ denoting the **standard lattice**. Set $f_{\vec{p}} = (z-p_1)\cdots(z-p_n)$. Then a lattice L is of the form $\sigma(L_0)$ if and only if there exists a positive integer k such that $f_{\vec{p}}^k(L_0) \subseteq L \subseteq f_{\vec{p}}^{-k}(L_0)$ and for each k they denote by $\mathcal{G}_k$ the subset of $\mathcal{G}$ consisting of pairs $(\vec{p}, L)$ such that this sandwhich condition holds. They identify $\mathbb{C}[z]/(f_{\vec{p}}^{2k})$ with the vector space of polynomials of degree strictly less than $2kn$, and $L_0/f_{\vec{p}}^{2k}L_0$ with its Nth product. Then

$$\mathcal{G}_k \overset{\text{Zariski closed}}{\subset} \mathbb{C}^n \times \bigcup_{d=0}^{2knN} G_d(L_0/f_{\vec{p}}^{2k}L_0)$$

where $G_d(?)$ denotes the ordinary Grassmann manifold of d-planes in the argument.

Definition 2. The **deformed convolution Grassmannian** is [not needed?] pairs $(\vec{p}, [\vec{\sigma}])$ where $\vec{p} \in \mathbb{C}^n$ and $\vec{\sigma}$ is in

$$G^\vee(\mathbb{C}[z, (z-p_1)^{-1}]) \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} G^\vee(\mathbb{C}[z, (z-p_n)^{-1}])/G^\vee(\mathbb{C}[z])$$

with a map down to $\mathcal{G}$ defined by $(\vec{p}, [\vec{\sigma}]) \mapsto (\vec{p}, [\sigma_1 \cdots \sigma_n])$.

To steal the follow-up example in [BGL20] where the above definition is also copied from...

Example 2. When $G = \mathbf{GL}_m\mathbb{C}$ this deformation is described by the datum of $\vec{p} \in \mathbb{C}^n$ and a sequence $(L_1, \dots, L_n)$ of $\mathbb{C}[z]$-lattices in $\mathbb{C}(z)^m$ such that for some $k \in \mathbb{Z}$ and for all $j \in \{1 \dots n\}$

$$(z-p_j)^k L_{j-1} \subset L_j \subset (z-p_j)^{-k} L_{j-1}$$

where again $L_0 = \mathbb{C}[z]^m$ denotes the standard lattice, and now $L_j = (\sigma_1 \cdots \sigma_j)(L_0)$. Very nice. Very concrete. They can partition the deformation into **cells** by specifying the **relative positions** of the pairs (L_{j-1}, L_j) in terms of **invariant factors**.

To be continued: [BGL20] go on to describe the fibres of the composition deformation to $\mathcal{G}$ to $\mathbb{C}^n$ and their description maybe helpful.

For $\mu \in P$ and $p \in \mathbb{C}$ they define

$$\tilde{S}_{\mu|p} = (z-p)^\mu N^\vee(\mathbb{C}[z, (z-p)^{-1}]) = N^\vee(\mathbb{C}[z, (z-p)^{-1}])(z-p)^\mu$$

They note that $\mathbb{C}(\!(z-p)\!)$ is the completion of $\mathbb{C}(z)$ at "the place defined by p" and identify $\mathbb{C}[\![z-p]\!]$ with $\mathbb{C}[\![z]\!]$ and $\mathbb{C}(\!(z-p)\!)$ with $\mathbb{C}(\!(z)\!)$.

They claim that

$$N^\vee(\mathbb{C}[z,(z-p)^{-1}])/N^\vee(\mathbb{C}[z]) \to N^\vee(\mathbb{C}(\!(z-p)\!))/N^\vee(\mathbb{C}[\![z-p]\!]) \cong N^\vee(\mathcal{K})/N^\vee(\mathcal{O})$$

is bijective, and that mapping Gr and multiplying by $(z-p)^\mu$ one gets

$$\tilde{S}_{\mu|p}/N^\vee(\mathbb{C}[z]) \cong S_\mu$$

They go on to describe the fusion product (section 5.3) a probably worthwhile read.

Definition 3. Say μ_1 and μ_2 are **disjoint** if $(\mu_1)_i \neq 0 \Rightarrow (\mu_2)_i = 0$ and $(\mu_2)_i \neq 0 \Rightarrow (\mu_1)_i = 0$.

3 Main results

Claim 1. $\widetilde{T^a_x} \to \pi^{-1}(\overline{\mathrm{Gr}^\lambda} \cap \mathrm{Gr}_\mu)$ (this does depend on b! we get something like a springer fibre where the action of [what] on either side has eigenvalues a permutation of b.)

Claim 2. Let $\mathcal{W}^\mu_{\mathrm{BD}} = G_1(\!(t^{-1})\!)t^\mu$. Then $S^{\mu_1+\mu_2}$ is contained in $\mathcal{W}^\mu_{\mathrm{BD}}$ if μ is dominant. Joel: And μ_1, μ_2 are dominant also? Anne: Roger has a proof.

Claim 3. Let $a = (0, s)$ and suppose μ_1 and μ_2 are disjoint ~~"transverse"~~ Let $\mu = \mu_1 + \mu_2$. Then $X \in \widetilde{T^a_x}$ is a $\mu \times \mu$ block matrix, with $(\mu_1)_k \times (\mu_1)_k$ diagonal block conjugate to a $(\mu_1)_k$ Jordan block and $(\mu_2)_k \times (\mu_2)_k$ diagonal block conjugate to $(\mu_2)_k$ Jordan block plus sI.

Question 1. If μ_i is not a permutation of λ_i and λ_i are not "homogeneous" how do we proceed? E.g. if $\mu_1 = (3,0,2)$, $\mu_2 = (0,2,0)$ and $\lambda_1 = (4,1)$, $\lambda_2 = (2,0,0)$.

Question 2. If μ_1 and μ_2 are not disjoint how do we proceed? E.g. if $\mu_1 = (2,2,0)$, $\mu_2 = (1,0,2)$; $\mu_1 = (2,2,1)$, $\mu_2 = (1,0,1)$.

4 Convolution vs BD

Fix $G = \mathbf{GL}(U) \cong \mathbf{GL}_m\mathbb{C}$ and $\{e_1, \dots, e_m\}$ a basis of U. Recall $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ where $\mathcal{K}, \mathcal{O}\dots$

Definition 4 (Beilinson–Drinfeld loop Grassmannians). Denoted $\mathcal{G}_{C^{(n)}}$ with C a smooth curve (or formal neighbourhood of a finite subset thereof) and $C^{(n)}$ its nth symmetric power. It is a reduced ind-scheme $\mathcal{G}_{C^{(n)}} \to C^{(n)}$ with fibres of C-lattices $\mathcal{G}_b = \{(b, \mathcal{L}) : b \in C^{(n)}\}$ made up of vector bundles such that $\mathcal{L} \cong U \otimes \mathcal{O}_C$ off b (i.e. over $C - \underline{b}$). The standard lattice is the pair $(\varnothing, \mathcal{L}_0)$ with $\mathcal{L}_0 = U \otimes \mathcal{O}_C$.

Not sure what $\mathcal{O}_C$ means

Notation

The case $n = 1$**.** Fix $b \in C$ and t a choice of formal parameter. Then $\mathcal{G}_b \cong \mathrm{Gr}$.

Why is this called "its group-theoretic realization"

Furthermore, in this case, C-lattices $(b, \mathcal{L})$ are identified with $\mathcal{O}$-submodules $L = \Gamma(\hat{b}, \mathcal{L})$ of $U_\mathcal{K} = U \otimes \mathcal{K}$ such that $L \otimes_\mathcal{O} \mathcal{K} \cong U_\mathcal{K}$.

Under this identification, we associate to a given $\lambda \in \mathbb{Z}^m$ the lattice (a priori a $\mathcal{O}$-submodule) $L_\lambda = \oplus_1^m t^{\lambda_i} e_i \mathcal{O}$. Nb. our lattices will be contained in the standard lattice L_0 whereas MVy's lattices contain.

Connected components of Gr are

$G(\mathcal{O})$-orbits are indexed by coweights $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ of G. In terms of lattices

$$\mathrm{Gr}^\lambda = \left\{ L \supset L_0 \,\middle|\, t\big|_{L/L_0} \in \mathbb{O}_\lambda \right\} \tag{2}$$

in the connected component Gr_N are indexed

[MV07] define a map

$$\mathcal{G} \to \mathfrak{G} \tag{3}$$

- Their slice T_x or T_λ
- Their embedding $T_x \to \mathfrak{G}_N$
- N-dim D
- The map $\tilde{\mathbf{m}} : \tilde{\mathfrak{g}}^n \to \mathrm{End}(D)$
- The map $\mathbf{m} : \tilde{\mathcal{N}}^n \to \mathcal{N}$ sending $(x, F_\bullet)$ to x
- The map $\pi : \tilde{\mathfrak{G}}^n \to \mathfrak{G}$ sending $\mathcal{L}_\bullet$ to $\mathcal{L}_n$

The special case $b = \vec{0}$. In this case 0 in the affine quiver variety goes to the point L_λ in the affine Grassmannian, and the preimage of zero in the smooth quiver variety (= the core?) is identified with the preimage of L_λ in the BD Grassmannian.

$$\begin{array}{ccc} \mathfrak{L}(\vec{v}, \vec{w}) & \longrightarrow & \pi^{-1}(L_\lambda) \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & L_\lambda \end{array}$$

MVy write: "we believe that one should be able to generalize this to arbitrary b" and that's where we come in!

Recall the Mirković–Vybornov immersion [MV07, Theorems 1.2 and 5.3].

Theorem 1. *([MV07, Theorem 1.2 and 5.3]) There exists an algebraic immersion* $\tilde{\psi}$

$$\tilde{\mathbf{m}}^{-1}(T_\lambda) \cap \tilde{\mathfrak{g}}^{n,a,E,\tilde{\mu}} \xrightarrow{\tilde{\psi}} \tilde{\mathfrak{G}}_b^{n,a}(P)$$

5 Statements and Proofs of Results

Anne: Maybe split for now into a Notation section and a Proofs section

Define

$$S_{\mu_1,\mu_2} = N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2}$$

and

$$W_\mu = G_1[[t^{-1}]]t^\mu.$$

Let $|\lambda| = |\lambda_1 + \lambda_2|$ and $|\mu| = |\mu_1 + \mu_2|$.

Anne: Why not $\lambda = \lambda_1 + \lambda_2$ and recall $|\nu|$ in general.

Lemma 1 (Proof in Proposition 2.6 of KWWY)**.** *Suppose μ is dominant. Then*

$$N((t^{-1}))t^\mu = N_1[[t^{-1}]]t^\mu.$$

Lemma 2. *For dominant μ_1, μ_2, we have*

$$S_{\mu_1,\mu_2} \subset W_{\mu_1+\mu_2}.$$

Proof. We have

$$\begin{aligned}
S_{\mu_1,\mu_2} &= N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&\subset T_1[[t^{-1}]]N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&= T_1[[t^{-1}]]N_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&\subset G_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&= W_{\mu_1+\mu_2}
\end{aligned}$$

where $B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} = B_1[[t^{-1}]]t^{\mu_1+\mu_2}$ since

$$\frac{t}{t-s} = 1 + \frac{s}{t} + \frac{s^2}{t^2} + \cdots \in B_1[[t^{-1}]].$$

□

Define $\mathrm{Gr}^{\lambda_1,\lambda_2} \subset \mathrm{Gr}_{BD}$ to be the family with generic fibre $\mathrm{Gr}^{\lambda_1} \times \mathrm{Gr}^{\lambda_2}$ and 0-fibre $\mathrm{Gr}^{\lambda_1+\lambda_2}$.

Define $\mathbb{O}_{\lambda_1,\lambda_2}$ to be matrices X of size $|\lambda| \times |\lambda|$ such that

$$X\big|_{E_0} \in \mathbb{O}_{\lambda_1} \text{ and } (X - sI)\big|_{E_s} \in \mathbb{O}_{\lambda_2}$$

Let

$$\mu = (\mu^{(1)}, \mu^{(2)}, ..., \mu^{(n)}).$$

Define $\mathbb{T}_{\mu_1,\mu_2}$ to be $|\mu| \times |\mu|$ matrices X such that X consists of block matrices where the size of the i-th diagonal block is $|\mu^{(i)}| \times |\mu^{(i)}|$, for $1 \leq i \leq n$. Each diagonal block is the companion matrix for $t^{\mu_1}(t-s)^{\mu_2}$.

Theorem 2. *We have an isomorphism*

$$\overline{\mathrm{Gr}^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2} \cong \overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}.$$

Anne: Rather, corollary?

Proof. We will prove this similarly to how the usual Mirković–Vybornov isomorphism is proven.

Step 1: Define a map $\mathbb{T}_{\mu_1,\mu_2} \cap \mathcal{N} \to G_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$.

$A \mapsto t^{\mu_1}(t-s)^{\mu_2} + a(t, t-s) \mapsto (L_1 \subset L_2) : (t-s)\big|_{L_2/L_1} = A\big|_{E_s}, t\big|_{L_1/L_0} = A\big|_{E_0}$

Question: 1. is the middle matrix similar to a block matrix? 2. is the composition of these maps some intermediate level of MVy's ψ's

BD Gr as lattices? $(L_1, L_2) \in \mathrm{Gr} \times \mathrm{Gr}$ corresponds to L such that $L \otimes \mathbb{C}[\![t]\!] \cong L_1 \otimes \mathbb{C}[\![t]\!]$ and $L \otimes \mathbb{C}[\![t-s]\!] \cong L_2 \otimes \mathbb{C}[\![t-s]\!]$ where $\otimes = \otimes_{\mathbb{C}[t]}$ or $\otimes_{\mathbb{C}[t-s]}$ respectively even though Roger believes $\mathbb{C}[t] = \mathbb{C}[t-s]$.

Step 2: If $A \in \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}$ then A is sent to $(N_-)_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$. Anne: Requires MVyBD!

Step 3: Conversely, given $L \in W_{\mu_1+\mu_2}$, want to show surjectivity.

□

Last meeting's todos:

- make sure that the image of our map is in the G_1 orbit
- more generally, define the map, check that the map is well-defined
- Anne: say what little a is, i.e. insert the MVy theorem as stated in CK, or thesis
- Roger: check it

References

[AK05] Jared E Anderson and Mikhail Kogan. The algebra of mirkovic-vilonen cycles in type a. arXiv preprint math/0505100, 2005. 1

[BGL20] Pierre Baumann, Stéphane Gaussent, and Peter Littelmann. Bases of tensor products and geometric satake correspondence. arXiv preprint arXiv:2009.00042, 2020. 1, 2

[CK18] Sabin Cautis and Joel Kamnitzer. Categorical geometric symmetric howe duality. Selecta Mathematica, 24(2):1593–1631, 2018. 1

[MV07] Ivan Mirković and Maxim Vybornov. Quiver varieties and beilinson-drinfeld grassmannians of type a. arXiv preprint arXiv:0712.4160, 2007. 1, 4

[MV19] Ivan Mirkovic and Maxim Vybornov. Comparison of quiver varieties, loop grassmannians and nilpotent cones in type a. arXiv preprint arXiv:1905.01810, 2019. 1