

Examples Compendium

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Last edit: approx. January 11, 2021

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1 January 11, 2021

Let Z be the MV cycle of weight α_3 with Lusztig datum $(000, 00, 1)$ and Z' the MV cycle of weight $\alpha_{1,2}$ with Lusztig datum $(010, 00, 0)$.

In terms of tableaux the fusion $Z * Z'$ can be encoded as

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} * \begin{array}{|c|} \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \quad (1)$$

$$S_3 * (1 \leftarrow 2) = (1 \leftarrow 2 \rightarrow 3) + P_3 \quad (2)$$

$$(000, 00, 1) * (010, 00, 0) = (010, 00, 1) + (001, 00, 0) \quad (3)$$

$$= (A_0, A_5, A_1A_4 - A_2A_3) \sqcup (A_0, A_1, A_3) \quad (4)$$

where the ideals in line 3 are given in coordinates on matrices of the form

$$\begin{bmatrix} 0 & A_0 & A_1 & A_2 \\ 0 & 0 & A_3 & A_4 \\ 0 & 0 & s & A_5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next let Z be the MV cycle of weight $\alpha_{1,3}$ having Lusztig datum $(100, 01, 0)$ and Z' the MV cycle of weight α_2 having Lusztig datum $(000, 10, 0)$.

Again, in terms of tableaux, $Z * Z'$ is given by

$$\begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} * \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad (5)$$

$$(1 \rightarrow 2 \leftarrow 3) * S_2 = P_2 + (2 \leftarrow 3) \oplus (1 \rightarrow 2) \quad (6)$$

$$(100, 01, 0) * (000, 10, 0) = (010, 01, 0) + (100, 11, 0) \quad (7)$$

TODO: fill in the ideals.

Now take Z to be the MV cycle of weight $\alpha_{1,3}$ with Lusztig datum $(010, 00, 1)$ and leave Z' as above. Then

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 3 & \\ \hline 4 & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & & & \\ \hline \end{array} \quad (8)$$

$$(1 \leftarrow 2 \rightarrow 3) * S_2 = (2 \rightarrow 3) \oplus (1 \leftarrow 2) + P_2 \quad (9)$$

$$(010, 00, 1) * (000, 10, 0) = (010, 10, 1) + (010, 01, 0) \quad (10)$$

TODO: fill in the ideals.

Also computed:

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 4 & & \\ \hline \end{array} \quad (11)$$

$$(000, 00, 1) * (100, 10, 0) = (100, 01, 0) + (100, 10, 1) \quad (12)$$

$$S_3 * (1 \rightarrow 2) = (1 \rightarrow 2 \leftarrow 3) + (1 \rightarrow 2 \rightarrow 3) \quad (13)$$

$$= (A_0, A_5, A_4, A_2) \sqcup (A_0, A_5, A_4, A_6) \quad (14)$$

in

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & s & A_0 & A_1 & A_2 & A_3 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & s & A_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

TODO: add explanation, double check the ideals.

Finally, we guess that the fusion $2\alpha_1 * 2\alpha_2$ is encoded by the tableau equation

$$\begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & & \\ \hline \end{array}$$

Remains to check this and make sense of the middle term.

2 Disjoint, non-dominant weight

Example 1. $\lambda_1 = (1, 0, 0)$, $\lambda_2 = (1, 1, 0)$, $\mu_1 = (0, 1, 0)$, $\mu_2 = (1, 0, 1)$. Joel: $\mu = \mu_1 + \mu_2$ determines the blocks we have on the RHS of the BD MVy isomorphism.

$$\left[\begin{array}{c|c|c} s & A_0 & A_1 \\ \hline 0 & 0 & A_2 \\ \hline 0 & 0 & s \end{array} \right]$$

3 Some multiplicity

Example 2 (Joel's exercise). It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for \mathbf{SL}_3 of weights $2\alpha_1$ and $2\alpha_2$. Following the notation from the mvbasis paper, this would correspond to the following multiplication:

$$x^2 y^2 = (xy - z)^2 + 2(xy - z)z + z^2.$$

For this example, I think we need $\lambda_1 = (2, 0, 0)$, $\lambda_2 = (2, 2, 0)$, $\mu_1 = (0, 2, 0)$, $\mu_2 = (2, 0, 2)$.

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \{ \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \} \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \}$$

Then

$$A = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -s^2 & 2s & A_0 & A_1 & A_2 & A_3 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -s^2 & 2s \end{array} \right]$$

needs $A-s|_{\mathbb{C}^2 \cap E_s}$ has Jordan type (2), $A|_{\mathbb{C}^4 \cap E_0}$ has Jordan type (2), and $A-s|_{E_s}$ has Jordan type (2, 2). So

$$E_0 = \{A_0 \vec{e}_1 + s^2 \vec{e}_3 \leftarrow A_0 \vec{e}_2 + s^2 \vec{e}_4 + (A_0 + A_1 + \frac{2A_0}{s}) \vec{e}_1 + s^2 \vec{e}_3\}$$

while

$$E_s^1 = \{\vec{e}_1 + s \vec{e}_2 \leftarrow \vec{e}_1 + (1+s) \vec{e}_2\} \quad (15)$$

$$E_s^2 = \{-\frac{A_2 + sA_3}{A_0 + sA_1} \vec{e}_3 - \frac{sA_2 + s^2 A_3}{A_0 + sA_1} \vec{e}_4 + \vec{e}_5 + s \vec{e}_6 \leftarrow\} \quad (16)$$

if

$$0 = A_4 + sA_5 \quad (17)$$

$$\rightarrow A_0 A_4 \text{ as } s \rightarrow 0 \quad (18)$$

and

Let's also try

$$\lambda_2 = (4, 0, 0) \quad \lambda_1 = (4, 4, 0) \quad \lambda = (8, 4, 0)$$

$$\mu_2 = (2, 2, 0) \quad \mu_1 = (4, 2, 2) \quad \mu = (6, 4, 2)$$

Then

$$\left[\begin{array}{cccccc|cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s^2 & 2s & A_6 & A_7 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Note that there is only one SSYT of shape λ_i and weight μ_i

$$\tau_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \end{array} \quad \tau_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ \hline \end{array}$$

Note also that

$$t^4(t-s)^2 = t^4(t^2 - 2st + s^2) = t^6 - 2st^5 + s^2t^4$$

$$t^2(t-s)^2 = t^4 - 2st^3 + s^2t^2$$

Elements of $\mathbb{T}_{\mu_1, \mu_2}^+$ will take the form

$$A = \left[\begin{array}{cccc|cccc|cc} & 1 & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & 1 & & & & & \\ & & & & -s^2 & 2s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\ \hline & & & & & & 1 & & & & & \\ & & & & & & & 1 & & & & \\ & & & & & & & & 1 & & & \\ & & & & & & & -s^2 & 2s & c_1 & c_2 & \\ \hline & & & & & & & & & & 1 & \end{array} \right]$$

The tableau tells us for each $1 \leq i \leq 12$

Jordan type of $A|_{\text{Span}(e_1, \dots, e_i) \cap E_0}$ is shape of $\tau_1|_{\text{first } i \text{ boxes}}$

So take $i = 10$. Then A restricted to $\mathbb{C}^{10} \cap E_0$ should have Jordan type $(4, 2)$.
[Anne: How to do it box by box?](#) s columns somehow correspond to τ_2 boxes.
 Therefore $a_1 = 0$. The 4-cycle is obvious. For the 2-cycle we require $a_2 = 0$.

$$\begin{aligned} e_1 &\leftarrow e_2 \leftarrow e_3 \leftarrow e_4 \\ e_7 &\leftarrow e_8 \end{aligned}$$

Now looking at all of A we have continue our 2-cycle to a 4-cycle. Roger found

$$e_{10} - \left(\frac{2s}{c_1} + \frac{s^2 c_2}{c_1^2} \right) e_{11} + \frac{s^2}{c_1} e_{12}$$

This requires $c_1^2 a_4 + c_1 b_2 s^2 - 2s c_1 b_1 - s^2 c_2 = 0$ and $a_3 c_1 + s^2 b_1 = 0$.

Now looking for the s -eigenspace, we expect $A - s|_{\mathbb{C}^6 \cap E_s}$ to have Jordan type 2. The kernel is spanned by $e_1 + s e_2 + s^2 e_3 + s^3 e_4 + s^4 e_5 + s^5 e_6$. It is continued to a 2-cycle by/the 2-cycle is generated by $-\frac{5}{s} e_1 - 4e_2 - 3s e_3 - 2s^2 e_4 - s^3 e_5$. The 3-cycle is maybe $(1/s^2, -4/s, -8, 5s, 3s^2, 2s^3, -s^2/a_3, -s^3/a_3, -s^4/a_3, -s^5/a_3)$ padded with zeros.

$$A - s = \left[\begin{array}{cccc|cccc|cc} -s & 1 & & & & & & & & \\ & -s & 1 & & & & & & & \\ & & -s & 1 & & & & & & \\ & & & -s & 1 & & & & & \\ & & & & -s & 1 & & & & \\ & & & & & -s^2 & s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\ \hline & & & & & & & -s & 1 & & & & \\ & & & & & & & & -s & 1 & & & \\ & & & & & & & & & -s & 1 & & \\ & & & & & & & & & & -s^2 & s & c_1 & c_2 \\ \hline & & & & & & & & & & & -s & 1 \\ & & & & & & & & & & & & -s \end{array} \right]$$

4 Simple root weights, things working

Example 3. Let

$$\begin{aligned} \mu_1 &= (3, 1, 1) & \lambda_1 &= (3, 2, 0) & \mu &= (3, 3, 1) \\ \mu_2 &= (0, 2, 0) & \lambda_2 &= (2, 0, 0) & \lambda &= (5, 2, 0) \end{aligned}$$

and consider the companion matrices of

$$p_1(t) = t^3 \quad p_2(t) = t(t-s)^2 = t^3 - 2st^2 + s^2t \quad p_3(t) = t$$

$$X = \left[\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e & f & g \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & k \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Example 4. Let

$$\begin{aligned}\lambda_1 &= (3, 2, 0) & \mu_1 &= (3, 1, 1) \\ \lambda_2 &= (2, 0, 0) & \mu_2 &= (1, 1, 0)\end{aligned}$$

so the first MV cycle $Z_1 \cong \mathbb{P}^1$ has MV polytope $\text{Conv}\{0, \alpha_1\}$ and the second MV cycle $Z_2 \cong \mathbb{P}^1$ has MV polytope $\text{Conv}\{0, \alpha_2\}$. Their fusion product corresponds to two \mathbb{P}^2 's intersecting along a \mathbb{P}^1 . That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where $Z_+ \cong Z_- \cong \mathbb{P}^2$. We have

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & a & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix}$$

The 0-generalized eigenspace E_0 of X is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, X e_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in $\ker X$, either $a = 0$ or $c = d = 0$, but the latter case cannot give a 2-cycle as $e_7 \notin \text{im } X$. Then $a = 0$ and we obtain a 2-cycle

$$\left\{X \left(e_6 - \frac{s}{d} e_7\right), e_6 - \frac{s}{d} e_7\right\} = \left\{e_5, e_6 - \frac{s}{d} e_7\right\}.$$

We also obtain the equations $b \neq 0$, $d \neq 0$, and $sc - bd = 0$ from this.

For the s -generalized eigenspace E_s , we need $a + sb \neq 0$ to obtain a 2-cycle, which can be taken as

$$\begin{aligned} & \left\{ (X - sI) \left(e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right), \right. \\ & \quad \left. e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right\} \\ &= \left\{ e_1 + se_2 + s^2e_3 + s^3e_4, e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right\} \end{aligned}$$

The minimal polynomial is $X^3(X - sI)^2$, which when equated to 0 gives again the equation $cs - bd = 0$. Thus the defining equations are

$$\{a = 0, cs - bd = 0\}.$$

When we take $s = 0$, we get the equations

$$\{a = 0, bd = 0\}$$

which corresponds to two \mathbb{A}^2 's intersecting along an \mathbb{A}^1 . This is indeed an open subset of $\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$, as required.

Example 5 (Continued...). The matrix X from the previous example defines (under MVy) the matrix

$$g = \begin{bmatrix} (t-s)t^3 & & \\ -bt & (t-s)t & \\ -c & -d & t \end{bmatrix}$$

in $G(\mathcal{O})$. Indeed the various blocks of X are in a precise sense the companion matrices of the polynomial entries of g

In Gr the element g defines the lattice

$$gL_0 = \mathbb{C}[t]\langle (t-s)t^3e_1 - (a+bt)e_2 - ce_3, (t-s)te_2 - de_3, te_3 \rangle$$

and computing the matrix of the action of t on the quotient L_0/L in the basis

$$\{[e_1], [te_1], [t^2e_1], [t^3e_1], [e_2], [te_2], [e_3]\}$$

recovers X up to a transpose of course.

Now let's see what we get when we invert t and $t-s$ respectively.

First let's invert t by considering $L_2 = L \otimes \mathbb{C}[[t-s]]$.

$$L_2 = \mathbb{C}[t, t^{-1}]\langle (t-s)e_1 - \frac{a+bt}{t^3}e_2 - \frac{c}{t^3}e_3, (t-s)e_2 - \frac{d}{t}e_3, e_3 \rangle$$

so in L_0/L_2 we have

$$t[e_1] = s[e_1] + \frac{a+bt}{t^3}[e_2] + \frac{c}{t^3}[e_3] \quad t[e_2] = s[e_2] + \frac{d}{t}[e_3] \quad [e_3] = 0$$

and

$$\left[t|_{L_0/L_2} \right]_{\{[e_1], [e_2]\}} = \begin{bmatrix} s & \\ \frac{a+bt}{t^3} & s \end{bmatrix}$$

which upon subtracting sI gives a matrix having block type μ_2 and Jordan type $\lambda_2 = (2)$ assuming $\frac{a+bt}{t^3} \neq 0$.

Next let's invert $t-s$ by considering $L_1 = L \otimes \mathbb{C}[[t]]$.

$$\begin{aligned} L_1 &= \mathbb{C}[t, (t-s)^{-1}]\langle t^3e_1 - \frac{a+bt}{t-s}e_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3 \rangle \\ &= \langle t^3e_1 - \frac{a}{t-s}e_2 - \frac{b}{t-s}te_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3 \rangle \end{aligned}$$

so in L_0/L_1 we have

$$\begin{aligned}
t[e_1] &= [te_1] \\
t[te_1] &= [t^2e_1] \\
t[t^2e_1] &= \frac{a}{t-s}[e_2] + \frac{b}{t-s}t[e_2] + \frac{c}{t-s}[e_3] \\
&= \frac{b}{t-s}\frac{d}{t-s}[e_3] + \frac{c}{t-s}[e_3] \\
&= \frac{bd + (t-s)c}{(t-s)^2}[e_3] \\
&= \frac{bd - sc}{(t-s)^2}[e_3] + \frac{c}{(t-s)^2}t[e_3] = 0 \\
t[e_2] &= \frac{d}{t-s}[e_3] \\
t[e_3] &= 0
\end{aligned}$$

and

$$[t|_{L_0/L_1}]_{\{[e_1], [te_1], [t^2e_1], [e_2], [e_3]\}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & & 0 & 0 & \\ & & 0 & \frac{d}{t-s} & 0 \end{bmatrix}$$

taking transpose

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}$$

which has block type μ_1 and Jordan type $\lambda_1 = (3, 2)$ assuming $d \neq 0$.

I have used the relations Roger found (and I checked) $a = 0$ and $cs - bd = 0$ in the calculations above.

To sum up, the pair of matrices above should contain the same information as the matrix from the previous example

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & b & c & \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix} \Leftrightarrow \left(\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} s & \frac{b}{t^2} \\ & s \end{bmatrix} \right)$$

Wish. Given L in $\mathcal{G}_n^{\text{BD}}$ define a map to T_μ just like MVy by taking $[t|_{L_0/L}]$ and use the fact that $[t|_{L_0/L_i}]$ for $i = 1, 2$ are companion matrices of the right type, piece together two MVy isomorphisms to make a BD MVy iso.

Equivalent linear algebra question(?): If $p(t, t-s) = p_1(t)p_2(t-s)$ then how are $C(p_1)$, $C(p_2)$, and $C(p)$ related?

5 Non simple root weights

Example 6 (Anne). Let $G = \mathbf{SL}_3$ and $\mathbf{i} = 121$.

Take $n_{\bullet}^1 = (1, 0, 0)$, and $n_{\bullet}^2 = (1, 0, 1)$ or $(0, 1, 0)$. So

$$\begin{aligned} \mu_1 &= (2, 2, 1) & \mu_2 &= (1, 1, 1) & \mu &= (3, 3, 2) \\ \lambda_1 &= (3, 1, 1) & \lambda_2 &= (2, 1, 0) & \lambda &= (5, 2, 1) \end{aligned}$$

Note

$$\tau(1, 0, 0) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array} \quad \tau(1, 0, 1) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \tau(0, 1, 0) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$$

Anne: We should show that order does not matter; i.e. swapping indices on λ 's and μ 's produces the same result.

$\mathbb{T}_{\mu_1, \mu_2}^+ \cap \mathbb{O}_{\lambda_1, \lambda_2}$ is made up of elements of the form

$$A = \left[\begin{array}{ccc|ccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & A_0 & A_1 & A_2 & A_3 & A_4 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & A_5 & A_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \end{array} \right]$$

As usual, denote by E_e the generalized e -eigenspace of A . $A|_{\mathbb{C}^3 \cap E_0}$ should have Jordan type (2). The obvious 2-cycle is generated by e_2 : $\{e_2, Ae_2\}$. $A|_{\mathbb{C}^3 \cap E_s}$ should have Jordan type (1). We take $e_1 + se_2 + s^2e_3 \in \text{Ker}(A - s)$. Next $A|_{\mathbb{C}^6 \cap E_0}$ should have Jordan type (3, 1) while $A|_{\mathbb{C}^6 \cap E_s}$ will have Jordan type (2) or (1, 1). Anne: This example breaks. Why? How should we choose weights?

Take 2: Let's try different weights.

$$\begin{aligned} \mu_1 &= (1, 1, 0) & \lambda_1 &= (2, 0, 0) \\ \mu_2 &= (1, 1, 1) & \lambda_2 &= (2, 1, 0) \\ \mu &= (2, 2, 1) & \lambda &= (4, 1, 0) \end{aligned}$$

and

$$\tau(1, 0, 0) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad \tau(1, 0, 1) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \tau(0, 1, 0) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$$

Then

$$A = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & s & A_0 & A_1 & A_2 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s & A_3 \\ \hline 0 & 0 & 0 & 0 & s \end{array} \right]$$

We have $E_0 \cap \mathbb{C}^2 = \text{Span}(e_1)$, $E_s \cap \mathbb{C}^2 = \text{Span}(e_1 + se_2)$. Next $E_0 \cap \mathbb{C}^4$ is spanned by a 2-cycle generated by $-\frac{A_0}{s}e_2 + e_3$ and $E_s \cap \mathbb{C}^4$ is spanned by a 2-cycle generated by

$$\frac{s}{A_1 + \frac{A_0}{s}}e_4 + \frac{1}{A_1 + \frac{A_0}{s}}e_3 - \frac{1}{s}e_1$$

or the additional 1-cycle $e_4 - \frac{A_1}{A_0}e_3$ assuming $A_0 + sA_1 = 0$. Finally E_s is spanned by an additional 1-cycle

$$e_5 - \frac{A_2}{A_1 + \frac{A_0}{s}}e_4 + \frac{1}{s} \frac{A_2}{A_1 + \frac{A_0}{s}}e_3$$

assuming $A_3 = 0$. Or the two 1-cycles are extended to a 2-cycle and a 1-cycle, the 2-cycle generated by $-\frac{1}{s}e_1 + \frac{s}{A_2}e_5$. This gives us

$$\boxed{1 \mid 2} * \boxed{\frac{1}{3} \mid 2} = (A_3) \quad \boxed{1 \mid 2} * \boxed{\frac{1}{2} \mid 3} = (A_0 + sA_1) \rightarrow (A_0)$$

Does it agree with what is expected on the module/cluster side?

$$S_1 * (1 \rightarrow 2) = S_1 \oplus (1 \rightarrow 2) \quad S_1 * (1 \leftarrow 2) = S_1 \oplus (1 \leftarrow 2)$$

The MV cycle of $\boxed{1 \mid 2}$ is a \mathbb{P}^1 : via MVy it has an open subset comprised of matrices

$$\left[\begin{array}{c|c} 0 & A_0 \\ \hline 0 & 0 \end{array} \right] : A_0 \neq 0$$

The MV cycles of the other two tableaux are made up of matrices of the form

$$\left[\begin{array}{c|c|c} 0 & A_0 & A_1 \\ \hline 0 & 0 & A_2 \\ \hline 0 & 0 & 0 \end{array} \right] : \begin{cases} A_0 \neq 0 \text{ and } A_2 = 0 & \tau = \boxed{\frac{1}{3} \mid 2} \\ A_0 = 0 \text{ and } A_2 \neq 0 & \tau = \boxed{\frac{1}{2} \mid 3} \end{cases}$$

both \mathbb{C}^2 's. [Anne: How do the coordinates relate?](#)

Example 7 (Roger). Let

$$\begin{aligned}\lambda_1 &= (2, 0, 0, 0) & \mu_1 &= (1, 1, 0, 0) \\ \lambda_2 &= (2, 2, 1, 0) & \mu_2 &= (3, 2, 1, 1)\end{aligned}$$

so $\lambda_1 - \mu_1 = \alpha_1$ and $\lambda_2 - \mu_2 = \alpha_2 + \alpha_3$. We have the following young tableaux:

$$\tau_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad \tau_2 = \begin{array}{|c|c|} \hline 1 & 1 \\ 2 & 3 \\ \hline 4 \\ \hline \end{array} \quad \tau'_2 = \begin{array}{|c|c|} \hline 1 & 1 \\ 2 & 4 \\ \hline 3 \\ \hline \end{array}$$

where τ_1 corresponds to the module S_1 , τ_2 corresponds to the module $2 \rightarrow 3$, and τ'_2 corresponds to the module $2 \leftarrow 3$.

The matrix we are considering is

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & -s^2 & 2s & a & b & c & d \\ & & & 0 & 1 & & \\ & & & & s & e & f \\ & & & & & s & g \\ & & & & & & s \end{bmatrix}$$

such that $\dim E_0 = 2$, $\dim \ker X = 1$, $\dim E_s = 5$, and $\dim \ker(X - sI) = 3$ where E_0 and E_s are the 0- and s -generalized eigenspaces.

We see that the two-cycle in E_0 is

$$\left\{ X \left(e_2 + \frac{s^2}{a} e_4 \right), e_2 + \frac{s^2}{a} e_4 \right\} = \left\{ e_1, e_2 + \frac{s^2}{a} e_4 \right\}.$$

As τ_2 and τ'_2 both share $\begin{array}{|c|c|} \hline 1 & 1 \\ 2 & \\ \hline \end{array}$, we can find a 2-cycle from just the upper-left 3×3 block, and an additional vector in $\ker(X - sI)$ from the upper-left 5×5 -submatrix. The 2-cycle from the 3×3 block is

$$\left\{ e_1 + se_2 + s^2 e_3, -\frac{2}{s} e_1 - e_2 \right\}.$$

The additional vector in $\ker(X - sI)$ is $e_4 + se_5$ and this requires $a + sb = 0$.

Now consider the case that the young diagram we are working with is τ_2 . Then we have $e_4 + se_5 + x(e_1 + se_2 + s^2 e_3)$ part of a 2-cycle that can be found by looking at the upper-left 6×6 -submatrix. We find that the 2-cycle is

$$\left\{ e_4 + se_5 + x(e_1 + se_2 + s^2 e_3), -\frac{2x}{s} e_1 - xe_2 - \frac{1}{s} e_4 + \frac{s}{e} e_6 \right\}$$

and this requires that $ae - s^2 c = 0$.

The last vector in $\ker(X - sI)$ comes from the entire $X - sI$ and we see it is $-fe_6 + ee_7$, which requires $g = 0$ and $ed - cf = 0$.

For the case τ'_2 , we start with find the third vector in $\ker(X - sI)$ from the upper-left 6×6 -submatrix. We see that it is e_6 , which requires $c = 0$ and $e = 0$.

For the remaining 2-cycle, we want it to end with $x(e_1 + se_2 + s^2e_3) + (e_4 + se_5) + ye_6$ so our 2-cycle is

$$\left\{ x(e_1 + se_2 + s^2e_3) + (e_4 + se_5) + ye_6, -\frac{2x}{s}e_1 - xe_2 - \frac{1}{s}e_4 + \frac{s}{f}e_7 \right\}$$

which requires $af - ds^2 = 0$ and $fy - sg = 0$. As y is free, the last equation is not really a restriction on f and g .

From the minimal polynomial, we have $X^2(X - sI)^2 = 0$ which gives us the equations

$$a + sb = cs + eb = bf + cg + ds = esg = 0.$$

Taking $s \rightarrow 0$, we have the following equations for our two cases of τ_2 and τ'_2 :

τ_2	τ'_2
$a = 0$	$a = 0$
$g = 0$	$c = 0$
$eb = 0$	$e = 0$
$bf = 0$	$bf = 0$
$ed - cf = 0$	

For the τ_2 case, the coordinate ring is

$$\frac{\mathbb{C}[a, b, c, d, e, f, g]}{\langle a, g, eb, bf, ed - cf \rangle} \cong \frac{\mathbb{C}[b, c, d, e, f]}{\langle eb, bf, ed - cf \rangle} = \frac{\mathbb{C}[b, c, d, e, f]}{\langle e, f \rangle \cap \langle b, ed - cf \rangle}$$

Hence the associated algebraic set is reducible with two irreducible components. The component corresponding to the ideal $\langle e, f \rangle$ is \mathbb{A}^3 , which corresponds to \mathbb{P}^3 , while the ideal $\langle b, ed - cf \rangle$ corresponds to the toric variety whose toric polytope is a square-based pyramid.

As τ_2 corresponds to the module $2 \rightarrow 3$, the irreducible components should correspond to the modules $P_1 = 1 \rightarrow 2 \rightarrow 3$ and $1 \leftarrow 2 \rightarrow 3$. Indeed, the MV cycle corresponding to P_1 is the Grassmannian $Gr(1, 4) \cong \mathbb{P}^3$ and for $1 \leftarrow 2 \rightarrow 3$, we do get a toric variety with polytope the square-based pyramid.

However for the τ'_2 case, the coordinate ring is

$$\frac{\mathbb{C}[a, b, c, d, e, f, g]}{\langle a, c, e, df \rangle} \cong \frac{\mathbb{C}[b, d, f, g]}{\langle df \rangle}$$

which corresponds to $\mathbb{A}^3 \cup \mathbb{A}^3$. Since τ'_2 corresponds to the module $2 \leftarrow 3$, we expect two irreducible components corresponding to the modules $P_3 = 1 \leftarrow 2 \leftarrow 3$ and $1 \rightarrow 2 \leftarrow 3$. P_3 corresponds to the variety $Gr(3, 4) \cong \mathbb{P}^3$ and $1 \rightarrow 2 \leftarrow 3$ also corresponds to a toric variety whose polytope is a square-based pyramid (?).

Example 8 (Above example redone). In $\mathbb{C}[A_0..A_6]$ where

$$\begin{bmatrix} 0 & 1 & 0 & | & 0 & 0 & | & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 & 0 & | & 0 & | & 0 \\ 0 & -s^2 & 2s & | & A_0 & A_1 & | & A_2 & | & A_3 \\ \hline 0 & 0 & 0 & | & 0 & 1 & | & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 & s & | & A_4 & | & A_5 \\ \hline 0 & 0 & 0 & | & 0 & 0 & | & s & | & A_6 \\ \hline 0 & 0 & 0 & | & 0 & 0 & | & 0 & | & s \end{bmatrix}$$

we have

$$\begin{aligned} \begin{bmatrix} 1 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 \end{bmatrix} &= (A_6, A_2A_5 - A_3A_4, A_1A_4 + A_2s, A_1s + A_0) \\ &= (A_6, A_2A_5 - A_3A_4, A_1A_4, A_0) \text{ at } s = 0 \\ &= (A_0, A_4, A_5, A_6) \sqcup (A_0, A_1, A_6, A_2A_5 - A_3A_4) \sqcup (A_0, A_2, A_4, A_6) \end{aligned}$$

while

$$\begin{aligned} \begin{bmatrix} 1 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 \end{bmatrix} &= (sA_0A_5 + (1 + s^2)A_1A_5 + sA_3, A_2, A_4, A_1s + A_0) \\ &= (A_1A_5, A_2, A_4, A_0) \text{ at } s = 0 \\ &= (A_0, A_2, A_4, A_5) \sqcup (A_0, A_1, A_2, A_4) \end{aligned}$$

Cycles are

$$\frac{A_0}{s^2}e_2 + e_4 \xrightarrow{A} e_1$$

and, in the $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ case,

$$\begin{aligned} e_2 + 2se_3 &\xrightarrow{A-s} e_1 + se_2 + s^2e_3 \\ (sA_4e_4 + (1 + s^2)A_4e_5 + se_6)/(sA_4) &\xrightarrow{A-s} (e_4 + se_5)/s \end{aligned}$$

while in the $\begin{bmatrix} 1 & 1 \\ 2 \\ 3 \end{bmatrix}$ case,

$$\begin{aligned} e_6 &\xrightarrow{A-s} 0 \\ e_4 + (1/s + s)e_5 + (1/A_5)e_7 &\xrightarrow{A-s} (1/s)e_4 + e_5 + *e_6 \end{aligned}$$