# Working title: Mirković–Vybornov fusion in Beilinson–Drinfeld Grassmannian

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## 1 Background

The BD Grassmannian. The convolution Grassmannian. Distinguished orbits, slices therein. Mirković–Vybornov.

## 2 Notation

*Definition* 1. Say $\mu_1$ and $\mu_2$ are **disjoint** if $(\mu_1)_i \neq 0 \Rightarrow (\mu_2)_i = 0$ and $(\mu_2)_i \neq 0 \Rightarrow (\mu_1)_i = 0$.

Anne: I propose "anodyne" as another candidate for the above property after Kapranov–Shechtman.

## 3 Main results

*Claim* 1. $\widetilde{T^a_x} \to \pi^{-1}(\overline{\mathrm{Gr}^\lambda} \cap \mathrm{Gr}_\mu)$ (this does depend on $b$! we get something like a springer fibre where the action of [what] on either side has eigenvalues a permutation of $b$.)

*Claim* 2. Let $\mathcal{W}^\mu_{\mathrm{BD}} = G_1((t^{-1}))t^\mu$. Then $S^{\mu_1+\mu_2}$ is contained in $\mathcal{W}^\mu_{\mathrm{BD}}$ if $\mu$ is dominant. Joel: And $\mu_1$, $\mu_2$ are dominant also? Anne: Roger has a proof.

*Claim* 3. Let $a = (0, s)$ and suppose $\mu_1$ and $\mu_2$ are disjoint ~~"transverse"~~ Let $\mu = \mu_1 + \mu_2$. Then $X \in \widetilde{T^a_x}$ is a $\mu \times \mu$ block matrix, with $(\mu_1)_k \times (\mu_1)_k$ diagonal block conjugate to a $(\mu_1)_k$ Jordan block and $(\mu_2)_k \times (\mu_2)_k$ diagonal block conjugate to $(\mu_2)_k$ Jordan block plus $sI$.

*Question* 1. If $\mu_i$ is not a permutation of $\lambda_i$ and $\lambda_i$ are not "homogeneous" how do we proceed? E.g. if $\mu_1 = (3, 0, 2)$, $\mu_2 = (0, 2, 0)$ and $\lambda_1 = (4, 1)$, $\lambda_2 = (2, 0, 0)$.

*Question* 2. If $\mu_1$ and $\mu_2$ are not disjoint how do we proceed? E.g. if $\mu_1 = (2, 2, 0)$, $\mu_2 = (1, 0, 2)$; $\mu_1 = (2, 2, 1)$, $\mu_2 = (1, 0, 1)$.

# 4 Background on MVy

What they do. What they stop short of doing.

- Their slice $T_x$ or $T_\lambda$

- Their embedding $T_x \to \mathfrak{G}_N$

- $N$-dim $D$

- The map $\tilde{\mathbf{m}} : \tilde{\mathfrak{g}}^n \to \mathrm{End}(D)$

- The map $\mathbf{m} : \tilde{\mathcal{N}}^n \to \mathcal{N}$ sending $(x, F_\bullet)$ to $x$

- The map $\pi : \tilde{\mathfrak{G}}^n \to \mathfrak{G}$ sending $\mathcal{L}_\bullet$ to $\mathcal{L}_n$

The special case $b = \vec{0}$. In this case 0 in the affine quiver variety goes to the point $L_\lambda$ in the affine Grassmannian, and the preimage of zero in the smooth quiver variety (the core?) is identified with the preimage of $L_\lambda$ in the BD Grassmannian.

$$
\begin{array}{ccc}
\mathfrak{L}(\vec{v}, \vec{w}) & \longrightarrow & \pi^{-1}(L_\lambda) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L_\lambda
\end{array}
$$

MVy write: "we believe that one should be able to generalize this to arbitrary [$b$]" and that's where we come in!

Recall the Mirković–Vybornov immersion [MV07, Theorems 1.2 and 5.3].

**Theorem 1.** *([MV07, Theorem 1.2 and 5.3]) There exists an algebraic immersion $\tilde{\psi}$*

$$
\widetilde{\mathbf{m}}^{-1}(T_\lambda) \cap \tilde{\mathfrak{g}}^{n,a,E,\tilde{\mu}} \xrightarrow{\tilde{\psi}} \tilde{\mathfrak{G}}_b^{n,a}(P)
$$

# 5 Statements and Proofs of Results

<span style="color:blue">Anne: Maybe split for now into a Notation section and a Proofs section</span>

Define

$$
S_{\mu_1, \mu_2} = N((t^{-1}))t^{\mu_1}(t - s)^{\mu_2}
$$

and

$$
W_\mu = G_1[[t^{-1}]]t^\mu.
$$

Let $|\lambda| = |\lambda_1 + \lambda_2|$ and $|\mu| = |\mu_1 + \mu_2|$.

<span style="color:blue">Anne: Why not $\lambda = \lambda_1 + \lambda_2$ and recall $|\nu|$ in general.</span>

**Lemma 1** (Proof in Proposition 2.6 of KWWY)**.** *Suppose $\mu$ is dominant. Then*

$$
N((t^{-1}))t^\mu = N_1[[t^{-1}]]t^\mu.
$$

**Lemma 2.** *For dominant $\mu_1, \mu_2$, we have*

$$S_{\mu_1,\mu_2} \subset W_{\mu_1+\mu_2}.$$

*Proof.* We have

$$
\begin{aligned}
S_{\mu_1,\mu_2} &= N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&\subset T_1[[t^{-1}]]N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&= T_1[[t^{-1}]]N_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&\subset G_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&= W_{\mu_1+\mu_2}
\end{aligned}
$$

where $B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} = B_1[[t^{-1}]]t^{\mu_1+\mu_2}$ since

$$\frac{t}{t-s} = 1 + \frac{s}{t} + \frac{s^2}{t^2} + \cdots \in B_1[[t^{-1}]].$$

$\square$

Define $\mathrm{Gr}^{\lambda_1,\lambda_2} \subset \mathrm{Gr}_{BD}$ to be the family with generic fibre $\mathrm{Gr}^{\lambda_1} \times \mathrm{Gr}^{\lambda_2}$ and 0-fibre $\mathrm{Gr}^{\lambda_1+\lambda_2}$.

Define $\mathbb{O}_{\lambda_1,\lambda_2}$ to be matrices $X$ of size $|\lambda| \times |\lambda|$ such that

$$X\big|_{E_0} \in \mathbb{O}_{\lambda_1} \text{ and } (X - sI)\big|_{E_s} \in \mathbb{O}_{\lambda_2}$$

Let

$$\mu = (\mu^{(1)}, \mu^{(2)}, ..., \mu^{(n)}).$$

Define $\mathbb{T}_{\mu_1,\mu_2}$ to be $|\mu| \times |\mu|$ matrices $X$ such that $X$ consists of block matrices where the size of the $i$-th diagonal block is $|\mu^{(i)}| \times |\mu^{(i)}|$, for $1 \le i \le n$.

**Theorem 2.** *We have an isomorphism*

$$\overline{\mathrm{Gr}^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2} \cong \overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}.$$

Anne: Rather, corollary?

*Proof.* We will prove this similarly to how the usual Mirković–Vybornov isomorphism is proven.

Step 1: Define a map $\mathbb{T}_{\mu_1,\mu_2} \cap \mathcal{N} \to G_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$.

$$A \mapsto t^{\mu_1}(t-s)^{\mu_2}+a(t,t-s) \mapsto (L_1 \subset L_2) : (t-s)\big|_{L_2/L_1} = A\big|_{E_s}, t\big|_{L_1/L_0} = A\big|_{E_0}$$

Question: 1. is the middle matrix similar to a block matrix? 2. is the composition of these maps some intermediate level of MVy's $\psi$'s

BD Gr as lattices? $(L_1, L_2) \in \mathrm{Gr} \times \mathrm{Gr}$ corresponds to $L$ such that $L \otimes \mathbb{C}[[t]] \cong L_1 \otimes \mathbb{C}[[t]]$ and $L \otimes \mathbb{C}[[t-s]] \cong L_2 \otimes \mathbb{C}[[t-s]]$ where $\otimes = \otimes_{\mathbb{C}[t]}$ or $\otimes_{\mathbb{C}[t-s]}$ respectively even though Roger believes $\mathbb{C}[t] = \mathbb{C}[t-s]$.

Step 2: If $A \in \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}$ then $A$ is sent to $(N_-)_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$.
Anne: Requires MVyBD!

Step 3: Conversely, given $L \in W_{\mu_1+\mu_2}$, want to show surjectivity.

$\square$

# References

[MV07] Ivan Mirković and Maxim Vybornov. Quiver varieties and beilinson-drinfeld grassmannians of type a. arXiv preprint arXiv:0712.4160, 2007. 2