

# Working title: Mirković–Vybornov fusion in Beilinson–Drinfeld Grassmannian

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## 1 Introduction

The BD Grassmannian. The convolution Grassmannian. Distinguished orbits and slices. Connections to Mirković–Vybornov [MV07, MV19], Cautis–Kamnitzer [CK18], Anderson–Kogan [AK05].

Address limitations outside of type  $A$ ?

## 2 Notation

In ordinary  $\mathrm{Gr}$  we have the following lattice descriptions valid only in type  $A$ . Given  $\mu \in X^\bullet(T)$ , write  $t^\mu$  for its image in  $G(\mathcal{K})$  and  $L_\mu$  for its image in

$$\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O}) \stackrel{A}{=} \{L \underset{\text{rank } m}{\overset{\text{free}}{\subset}} \mathcal{O}^m : tL \subset L\}$$

Example:  $L_\mu = \mathrm{Span}_{\mathcal{O}}(e_i t^j : 0 \leq j < \mu_i)$ . Fact:  $\mathrm{Gr}^T = X^\bullet(T)$  and other distinguished subsets (needed for the definition of MV cycles and later open subset thereof) are all orbits of fixed points

$$\begin{aligned} \mathrm{Gr}^\lambda &= G(\mathcal{O})L_\lambda &&= \{L \in \mathrm{Gr} : t|_{\mathcal{O}_m/L} \text{ has Jordan type } \lambda\} \\ \mathrm{Gr}_\mu &= G_1[t^{-1}]L_\mu &&= \{L \in \mathrm{Gr} : L = \mathrm{Span}_{\mathcal{O}}(v_1, \dots, v_m) \text{ such that} \\ &&&v_j = t^{\mu_j} e_j + \sum p_{ij} e_i \text{ with } \deg p_{ij} < \mu_j\} \\ S_-^\mu &= U_-(\mathcal{K})L_\mu &&= \{L \in \mathrm{Gr}_\mu : \dim(\mathcal{O}^k/L \cap \mathcal{O}^k) = \mu_1 + \dots + \mu_k\} \end{aligned}$$

Let  $\mathrm{Gr}$  denote the ordinary **affine Grassmannian**  $G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ ,  $\mathcal{G}_n^{\mathrm{BD}}$  the **Beilinson–Drinfeld affine Grassmannian**, and  $\mathrm{Gr}_n$  the **convolution affine Grassmannian**.

*Definition 1.* The **BD Grassmannian** is the set

$$\begin{aligned} \{(V, \sigma) : V \text{ is a rank } m \text{ vector bundle on } \mathbb{P}^1 \\ \text{and } \sigma : V \dashrightarrow \mathcal{O}_{\mathbb{P}^1}^m \text{ is a trivialization} \\ \text{defined away from finitely many points in } \mathbb{A}^1\} \end{aligned} \tag{1}$$

The rank of the  $m$  in the definition of  $\mathcal{G}_n^{\text{BD}}$  is the dimension of the maximal torus of  $G^\vee$ . For  $G^\vee = \mathbf{GL}_m = G$ .

More generally, one can define a BD grassmannian of  $\mathbf{GL}_m$  over any smooth curve  $C$  as the reduced ind-scheme  $\mathcal{G}_{n,C}^{\text{BD}}$  fibered over a finite symmetric power of  $C$  —  $C^{(n)}$  — such that the fibre over the point  $\vec{p} = (p_1, \dots, p_n) \in C^n$  is a collection of rank  $m$  vector bundles  $V$  over  $C$  which are trivial away from  $\vec{p}$  viewed also as a subset —  $\{p_1, \dots, p_n\}$  — of  $C$ . Trivial means  $\mathcal{O}_C^m \cong V$ .

Joel: Careful: you can define it over  $C^n$  or  $C^{(n)}$ , these are two different things.

To quote [BGL20] the BD Grassmannian is a relative version of the affine Grassmannian where the base is the space of effective divisors on a smooth curve  $C$ . The choice  $C = \mathbb{A}^1$  “amply satisfies our needs and offers three advantages: there is a natural global coordinate it, every  $G$ -torsor on it is trivializable, and the monodromy of any local system is trivial. Formally,  $\mathcal{G}_n^{\text{BD}}$  is the functor on the category of commutative  $\mathbb{C}$ -algebras that assigns to an algebra  $R$  the set of isomorphism classes of triples  $(\vec{p}, V, \sigma)$  where  $\vec{p} \in \mathbb{A}^n(R)$ ,  $V$  is a  $G^\vee$ -torsor over  $\mathbb{A}_R^1$  and  $\sigma$  is a trivialization of  $V$  away from  $\vec{p}$ .” Joel: Yes,  $G$ -torsor is the same thing as a principal  $G$ -bundle.

i.e. principal  $G$   
bundle?

They denote by  $\pi$  the fibration  $\mathcal{G}_n^{\text{BD}} \rightarrow \mathbb{A}^n$  (forgetting  $V$  and  $\sigma$ ). Their simplified description is: it's the set of pairs  $(\vec{p}, [\sigma])$  where  $\vec{p} \in C^n$  and  $[\sigma]$  is an element of the homogeneous space

$$G^\vee(\mathbb{C}[z, (z - p_1)^{-1}, \dots, (z - p_n)^{-1}]) / G^\vee(\mathbb{C}[z])$$

Their example, (almost) our setting:

*Example 1.* When  $G = \mathbf{GL}_m \mathbb{C}$  the datum of  $[\sigma]$  is equivalent to the datum of the  $\mathbb{C}[z]$ -lattice  $\sigma(L_0)$  in  $\mathbb{C}(z)^m$  with  $L_0 = \mathbb{C}[z]^m$  denoting the **standard lattice**. Set  $f_{\vec{p}} = (z - p_1) \cdots (z - p_n)$ . Then a lattice  $L$  is of the form  $\sigma(L_0)$  if and only if there exists a positive integer  $k$  such that  $f_{\vec{p}}^k(L_0) \subseteq L \subseteq f_{\vec{p}}^{-k}(L_0)$  and for each  $k$  they denote by  $\mathcal{G}_{n,k}^{\text{BD}}$  the subset of  $\mathcal{G}_n^{\text{BD}}$  consisting of pairs  $(\vec{p}, L)$  such that this sandwich condition holds. They identify  $\mathbb{C}[z]/(f_{\vec{p}}^{2k})$  with the vector space of polynomials of degree strictly less than  $2kn$ , and  $L_0/f_{\vec{p}}^{2k}L_0$  with its  $N$ th product. Then

$$\mathcal{G}_{n,k}^{\text{BD}} \stackrel{\text{Zariski closed}}{\subset} \mathbb{C}^n \times \bigcup_{d=0}^{2knN} G_d(L_0/f_{\vec{p}}^{2k}L_0)$$

where  $G_d(?)$  denotes the ordinary Grassmann manifold of  $d$ -planes in the argument.

Our setting is  $G = \mathbf{GL}_m$  and  $n = 2$ .

*Definition 2.* The **deformed convolution Grassmannian** is [not needed?] pairs  $(\vec{p}, [\vec{\sigma}])$  where  $\vec{p} \in \mathbb{C}^n$  and  $\vec{\sigma}$  is in

$$G^\vee(\mathbb{C}[z, (z - p_1)^{-1}]) \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} G^\vee(\mathbb{C}[z, (z - p_n)^{-1}]) / G^\vee(\mathbb{C}[z])$$

with a map down to  $\mathcal{G}_n^{\text{BD}}$  defined by  $(\vec{p}, [\vec{\sigma}]) \mapsto (\vec{p}, [\sigma_1 \cdots \sigma_n])$ .

To steal the follow-up example in [BGL20] where the above definition is also copied from. . .

*Example 2.* When  $G = \mathbf{GL}_m \mathbb{C}$  this deformation is described by the datum of  $\vec{p} \in \mathbb{C}^n$  and a sequence  $(L_1, \dots, L_n)$  of  $\mathbb{C}[z]$ -lattices in  $\mathbb{C}(z)^m$  such that for some  $k \in \mathbb{Z}$  and for all  $j \in \{1 \dots n\}$

Why Laurent polynomials for the convolution?

$$(z - p_j)^k L_{j-1} \subset L_j \subset (z - p_j)^{-k} L_{j-1}$$

where again  $L_0 = \mathbb{C}[z]^m$  denotes the standard lattice, while  $L_j = (\sigma_1 \cdots \sigma_j)(L_0)$ . Very nice. Very concrete. They can partition the deformation into **cells** by specifying the **relative positions** of the pairs  $(L_{j-1}, L_j)$  in terms of **invariant factors**.

To be continued: [BGL20] go on to describe the fibres of the **composition**

Not to  $\mathbb{C}^{(n)}$ ? Or to  $\mathbb{C}$ ?

$$\mathrm{Gr}_n \rightarrow \mathcal{G}r_n^{\mathrm{BD}} \rightarrow \mathbb{C}^n = \mathbb{A}_{\mathbb{C}}^n$$

their description may be helpful.

For  $\mu \in P$  and  $p \in \mathbb{C}$  they define

$$\tilde{S}_{\mu|p} = (z - p)^\mu N^\vee(\mathbb{C}[z, (z - p)^{-1}]) = N^\vee(\mathbb{C}[z, (z - p)^{-1}]) (z - p)^\mu$$

They note that  $\mathbb{C}((z - p))$  is the completion of  $\mathbb{C}(z)$  at “the place defined by  $p$ ” and identify  $\mathbb{C}[[z - p]]$  with  $\mathbb{C}[[z]]$  and  $\mathbb{C}((z - p))$  with  $\mathbb{C}((z))$ .

They claim that

$$N^\vee(\mathbb{C}[z, (z - p)^{-1}]) / N^\vee(\mathbb{C}[z]) \rightarrow N^\vee(\mathbb{C}((z - p))) / N^\vee(\mathbb{C}[[z - p]]) \cong N^\vee(\mathcal{K}) / N^\vee(\mathcal{O})$$

is bijective, and that mapping  $\mathrm{Gr}$  and multiplying by  $(z - p)^\mu$  one gets

$$\tilde{S}_{\mu|p} / N^\vee(\mathbb{C}[z]) \cong S_\mu$$

They go on to describe the fusion product (section 5.3) a probably worthwhile read.

Going forward, we’ll use  $t$  to denote the coordinate on  $\mathbb{A}_{\mathbb{C}}^1 = \mathbb{C}$  instead of  $z$ . Then by  $t^\mu \in G^\vee(\mathcal{K})$  we’ll denote the point defined by the coweight  $\mu \in \mathrm{Hom}(\mathbb{C}^\times, T^\vee) = T^\vee(\mathcal{K})$  and by  $L_\mu$  its image  $t^\mu G^\vee(\mathcal{O})$  in  $\mathrm{Gr}$ .

*Definition 3.* Given  $\mu_1, \mu_2$  such that  $\mu = \mu_1 + \mu_2$  is a partition of  $N$  we define  $T_{\mu_1, \mu_2} \subset \mathrm{Mat}_N$  to be the set of  $\mu \times \mu$  block matrices that are zero everywhere except possibly in the last  $\min(\mu_i, \mu_j)$  columns of the last row of the  $\mu_i \times \mu_j$ th block *plus* the block diagonal matrix whose  $\mu_i \times \mu_i$  diagonal block is the companion matrix of  $t^{\mu_{1,i}}(t - s)^{\mu_{2,i}}$  for each  $i \in \{1, 2, \dots, m\}$ . We call this set **name**.

*Remark 1.* While we limit ourselves to the case of dominant partitions, the definition above makes sense for arbitrary partitions.

*Remark 2.* Speak to whether or not this slice appears in [MV07]. We don’t think it does. But its “lift” might.

*Definition 4.* Given  $\lambda_1, \lambda_2$  such that  $\lambda = \lambda_1 + \lambda_2$  is a partition of  $N$  we define  $\mathbb{O}_{\lambda_1, \lambda_2}$  to be the set of  $N \times N$  **semi-nilpotent** matrices  $X$  with spectrum in  $\{0, s\}$  for some  $s \in \mathbb{C}^\times$  such that  $X|_{E_0} \in \mathbb{O}_{\lambda_1}$  and  $(X - s)|_{E_s} \in \mathbb{O}_{\lambda_2}$  meaning Cute?  
 TODO.

Correspondingly we have

- $W_{\mu_1, \mu_2} = G_1^\vee[t^{-1}, (t - s)^{-1}]t^{\mu_1}(t - s)^{\mu_2}G^\vee(\mathcal{O})$  as a subset of  $\mathcal{G}_2^{\text{BD}}$  where recall the subscript 2 records the fact that we are fixing two points  $0, s \in \mathbb{C}$
- Does  $\mathcal{G}_2^{\text{BD}, \lambda_1, \lambda_2}$  admit such an orbit description? As a fibration it is pairs  $(V, \sigma)$  such that is trivialized away from  $(0, s)$  by  $\sigma$  and  $\sigma$  has type  $(\lambda_1, \lambda_2)$  — the data of the trivialization is equivalent to the data of pairs of lattices  $(L_1, L_2)$  such that  $L_i \in \text{Gr}^{\lambda_i}$  away from 0 and  $L_1 = L_2 \in \text{Gr}^\lambda$  at 0?
- In the deformed convolution  $\text{Gr}$  we have the subset  $\text{Gr}_2^{\lambda_1, \lambda_2}$  of pairs  $((0, s), [\sigma_1, \sigma_2])$  (really want arbitrary  $(p_1, p_2)$  in place of  $(0, s)$  which is the specialization that we make when we work in  $\mathcal{G}_2^{\text{BD}}$  sort of?) with  $\sigma_i \in G^\vee(\mathbb{C}[t])(t - p_i)^{\lambda_i}G^\vee(\mathbb{C}[z])$  but this is not important?

*Question 1.* Can we describe  $W_{\mu_1, \mu_2} \cap \overline{\text{Gr}^{\lambda_1, \lambda_2}}$  as the set of lattices  $L$  such that

$$t^{\lambda_{1,1}}(t - s)^{\lambda_{2,1}}L_0 \subset L \subset t^{-\lambda_{1,1}}(t - s)^{-\lambda_{2,1}}L_0$$

and  $\lim_{s \rightarrow 0} \rho^\vee(s) \cdot L = L_\mu$

Joel: This is not correct since, you don't use all information in  $\lambda_1, \lambda_2$ . You need to describe the Jordan type of the quotients, or something equivalent.

*Definition 5.* Say  $\mu_1$  and  $\mu_2$  are **disjoint** if  $(\mu_1)_i \neq 0 \Rightarrow (\mu_2)_i = 0$  and  $(\mu_2)_i \neq 0 \Rightarrow (\mu_1)_i = 0$ .

### 3 Main results

*Claim 1.*  $\widetilde{T}_x^a \rightarrow \pi^{-1}(\overline{\text{Gr}^\lambda} \cap \text{Gr}_\mu)$  (this does depend on  $b!$  we get something like a springer fibre where the action of [what] on either side has eigenvalues a permutation of  $b$ .)

*Claim 2.* Let  $\mathcal{W}_{\text{BD}}^\mu = G_1((t^{-1}))t^\mu$ . Then  $S^{\mu_1 + \mu_2}$  is contained in  $\mathcal{W}_{\text{BD}}^\mu$  if  $\mu$  is dominant. Joel: And  $\mu_1, \mu_2$  are dominant also? Anne: Roger has a proof.

*Claim 3.* Let  $a = (0, s)$  and suppose  $\mu_1$  and  $\mu_2$  are disjoint “transverse”. Let  $\mu = \mu_1 + \mu_2$ . Then  $X \in \widetilde{T}_x^a$  is a  $\mu \times \mu$  block matrix, with  $(\mu_1)_k \times (\mu_1)_k$  diagonal block conjugate to a  $(\mu_1)_k$  Jordan block and  $(\mu_2)_k \times (\mu_2)_k$  diagonal block conjugate to  $(\mu_2)_k$  Jordan block plus  $sI$ .

*Question 2.* If  $\mu_i$  is not a permutation of  $\lambda_i$  and  $\lambda_i$  are not “homogeneous” how do we proceed? E.g. if  $\mu_1 = (3, 0, 2)$ ,  $\mu_2 = (0, 2, 0)$  and  $\lambda_1 = (4, 1)$ ,  $\lambda_2 = (2, 0, 0)$ .

*Question 3.* If  $\mu_1$  and  $\mu_2$  are not disjoint how do we proceed? E.g. if  $\mu_1 = (2, 2, 0)$ ,  $\mu_2 = (1, 0, 2)$ ;  $\mu_1 = (2, 2, 1)$ ,  $\mu_2 = (1, 0, 1)$ .

Joel: I think that we figured this out. We just need to take the matrix corresponding to  $z^{\mu_1}(z - s)^{\mu_2}$  under the MVy isomorphism.

## 4 Convolution vs BD

Fix  $G = \mathbf{GL}(U) \cong \mathbf{GL}_m \mathbb{C}$  and  $\{e_1, \dots, e_m\}$  a basis of  $U$ . Recall  $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$  where  $\mathcal{K}, \mathcal{O} \dots$

*Definition 6* (Beilinson–Drinfeld loop Grassmannians). Denoted  $\mathcal{G}_n^{\mathrm{BD}}_{C^{(n)}}$  with  $C$  a smooth curve (or formal neighbourhood of a finite subset thereof) and  $C^{(n)}$  its  $n$ th symmetric power. It is a reduced ind-scheme  $\mathcal{G}_n^{\mathrm{BD}}_{C^{(n)}} \rightarrow C^{(n)}$  with fibres of  $C$ -lattices  $\mathcal{G}_n^{\mathrm{BD}}_b = \{(b, \mathcal{L}) : b \in C^{(n)}\}$  made up of vector bundles such that  $\mathcal{L} \cong U \otimes \mathcal{O}_C$  off  $b$  (i.e. over  $C - \underline{b}$ ). The standard lattice is the pair  $(\emptyset, \mathcal{L}_0)$  with  $\mathcal{L}_0 = U \otimes \mathcal{O}_C$ .

Not sure what  $\mathcal{O}_C$  means  
Notation

**The case  $n = 1$ .** Fix  $b \in C$  and  $t$  a choice of formal parameter. Then  $\mathcal{G}_1^{\mathrm{BD}}_b \cong \mathrm{Gr}$ .

Why is this called “its group-theoretic realization”

Furthermore, in this case,  $C$ -lattices  $(b, \mathcal{L})$  are identified with  $\mathcal{O}$ -submodules  $L = \Gamma(\hat{b}, \mathcal{L})$  of  $U_{\mathcal{K}} = U \otimes \mathcal{K}$  such that  $L \otimes_{\mathcal{O}} \mathcal{K} \cong U_{\mathcal{K}}$ .

Under this identification, we associate to a given  $\lambda \in \mathbb{Z}^m$  the lattice (a priori a  $\mathcal{O}$ -submodule)  $L_{\lambda} = \bigoplus_1^m t^{\lambda_i} e_i \mathcal{O}$ . Nb. our lattices will be contained in the standard lattice  $L_0$  whereas MVy’s lattices contain.

Connected components of  $\mathrm{Gr}$  are

$G(\mathcal{O})$ -orbits are indexed by coweights  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$  of  $G$ . In terms of lattices

$$\mathrm{Gr}^{\lambda} = \left\{ L \supset L_0 \mid t|_{L/L_0} \in \mathbb{O}_{\lambda} \right\} \quad (2)$$

in the connected component  $\mathrm{Gr}_N$  are indexed

[MV07] define a map

$$\mathcal{G}_n^{\mathrm{BD}} \rightarrow \mathrm{Gr}_n \quad (3)$$

- Their slice  $T_x$  or  $T_{\lambda}$
- Their embedding  $T_x \rightarrow \mathfrak{G}_N$
- $N$ -dim  $D$
- The map  $\tilde{\mathbf{m}} : \tilde{\mathfrak{g}}^n \rightarrow \mathrm{End}(D)$
- The map  $\mathbf{m} : \tilde{\mathcal{N}}^n \rightarrow \mathcal{N}$  sending  $(x, F_{\bullet})$  to  $x$
- The map  $\pi : \tilde{\mathfrak{G}}^n \rightarrow \mathfrak{G}$  sending  $\mathcal{L}_{\bullet}$  to  $\mathcal{L}_n$

The special case  $b = \vec{0}$ . In this case  $0$  in the affine quiver variety goes to the point  $L_{\lambda}$  in the affine Grassmannian, and the preimage of zero in the smooth quiver variety (= the core?) is identified with the preimage of  $L_{\lambda}$  in the BD Grassmannian.

$$\begin{array}{ccc} \mathfrak{L}(\vec{v}, \vec{w}) & \longrightarrow & \pi^{-1}(L_{\lambda}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & L_{\lambda} \end{array}$$

MVy write: “we believe that one should be able to generalize this to arbitrary  $b$ ” and that’s where we come in!

Recall the Mirković–Vybornov immersion [MV07, Theorems 1.2 and 5.3].

**Theorem 1.** ([MV07, Theorem 1.2 and 5.3]) *There exists an algebraic immersion  $\tilde{\psi}$*

$$\tilde{\mathfrak{m}}^{-1}(T_\lambda) \cap \tilde{\mathfrak{g}}^{n,a,E,\tilde{\mu}} \xrightarrow{\tilde{\psi}} \tilde{\mathfrak{G}}_b^{n,a}(P)$$

## 5 Statements and Proofs of Results

Anne: Maybe split for now into a Notation section and a Proofs section

Define

$$S_{\mu_1, \mu_2} = N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2}$$

and

$$W_\mu = G_1[[t^{-1}]]t^\mu.$$

Let  $|\lambda| = |\lambda_1 + \lambda_2|$  and  $|\mu| = |\mu_1 + \mu_2|$ .

Anne: Why not  $\lambda = \lambda_1 + \lambda_2$  and recall  $|\nu|$  in general.

**Lemma 1** (Proof in Proposition 2.6 of KWWY). *Suppose  $\mu$  is dominant. Then*

$$N((t^{-1}))t^\mu = N_1[[t^{-1}]]t^\mu.$$

**Lemma 2.** *For dominant  $\mu_1, \mu_2$ , we have*

$$S_{\mu_1, \mu_2} \subset W_{\mu_1 + \mu_2}.$$

*Proof.* We have

$$\begin{aligned} S_{\mu_1, \mu_2} &= N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\ &\subset T_1[[t^{-1}]]N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\ &= T_1[[t^{-1}]]N_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\ &= B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\ &= B_1[[t^{-1}]]t^{\mu_1 + \mu_2} \\ &\subset G_1[[t^{-1}]]t^{\mu_1 + \mu_2} \\ &= W_{\mu_1 + \mu_2} \end{aligned}$$

where  $B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} = B_1[[t^{-1}]]t^{\mu_1 + \mu_2}$  since

$$\frac{t}{t-s} = 1 + \frac{s}{t} + \frac{s^2}{t^2} + \cdots \in B_1[[t^{-1}]].$$

□

Define  $\text{Gr}^{\lambda_1, \lambda_2} \subset \text{Gr}_{BD}$  to be the family with generic fibre  $\text{Gr}^{\lambda_1} \times \text{Gr}^{\lambda_2}$  and 0-fibre  $\text{Gr}^{\lambda_1 + \lambda_2}$ .

Define  $\mathbb{O}_{\lambda_1, \lambda_2}$  to be matrices  $X$  of size  $|\lambda| \times |\lambda|$  such that

$$X|_{E_0} \in \mathbb{O}_{\lambda_1} \text{ and } (X - sI)|_{E_s} \in \mathbb{O}_{\lambda_2}$$

Let

$$\mu = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n)}).$$

Define  $\mathbb{T}_{\mu_1, \mu_2}$  to be  $|\mu| \times |\mu|$  matrices  $X$  such that  $X$  consists of block matrices where the size of the  $i$ -th diagonal block is  $|\mu^{(i)}| \times |\mu^{(i)}|$ , for  $1 \leq i \leq n$ . Each diagonal block is the companion matrix for  $t^{\mu_1}(t-s)^{\mu_2}$ . Each off-diagonal block is zero everywhere except possibly in the last  $\min(\mu_i, \mu_j)$  columns of the last row.

**Theorem 2.** *We have an isomorphism*

$$\overline{\text{Gr}^{\lambda_1, \lambda_2}} \cap S_{\mu_1, \mu_2} \cong \overline{\mathbb{O}_{\lambda_1, \lambda_2}} \cap \mathbb{T}_{\mu_1, \mu_2} \cap \mathfrak{n}.$$

Anne: Rather, corollary?

*Proof.* We will prove this similarly to how the usual Mirković–Vybornov isomorphism is proven.

Step 1: Define a map  $\mathbb{T}_{\mu_1, \mu_2} \cap \mathcal{N} \rightarrow G_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$ .

$$A \mapsto t^{\mu_1}(t-s)^{\mu_2} + a(t, t-s) \mapsto (L_1 \subset L_2) : (t-s)|_{L_2/L_1} = A|_{E_s}, t|_{L_1/L_0} = A|_{E_0}$$

Question: 1. is the middle matrix similar to a block matrix? 2. is the composition of these maps some intermediate level of MVy's  $\psi$ 's

BD Gr as lattices?  $(L_1, L_2) \in \text{Gr} \times \text{Gr}$  corresponds to  $L$  such that  $L \otimes \mathbb{C}[[t]] \cong L_1 \otimes \mathbb{C}[[t]]$  and  $L \otimes \mathbb{C}[[t-s]] \cong L_2 \otimes \mathbb{C}[[t-s]]$  where  $\otimes = \otimes_{\mathbb{C}[t]}$  or  $\otimes_{\mathbb{C}[t-s]}$  respectively even though Roger believes  $\mathbb{C}[t] = \mathbb{C}[t-s]$ .

Proof of Step 1: Let  $\mu_1 = (\mu_{1,1}, \mu_{1,2}, \dots, \mu_{1,n})$ ,  $\mu_2 = (\mu_{2,1}, \dots, \mu_{2,n})$ , and  $\mu = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n)})$ . Consider the matrix where  $C_i$  is the  $\mu^{(i)} \times \mu^{(i)}$  companion matrix of  $t^{\mu_{1,i}}(t-s)^{\mu_{2,i}}$ . We send this matrix to the matrix

$$\begin{bmatrix} t^{\mu_{1,1}}(t-s)^{\mu_{2,1}} & & & \\ -\sum_{k=1}^{\mu^{(2)}} a_{12}^k t^{k-1} & t^{\mu_{1,2}}(t-s)^{\mu_{2,2}} & & \\ \vdots & \ddots & \ddots & \\ -\sum_{k=1}^{\mu^{(n)}} a_{1n}^k t^{k-1} & \dots & -\sum_{k=1}^{\mu^{(n)}} a_{n-1,n}^k t^{k-1} & t^{\mu_{1,n}}(t-s)^{\mu_{2,n}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ p_{1,2}(t) & 1 & & \\ \vdots & \ddots & \ddots & \\ p_{1,n}(t) & \dots & p_{n-1,n}(t) & 1 \end{bmatrix} \begin{bmatrix} t^{\mu_{1,1}}(t-s)^{\mu_{2,1}} & & & \\ & \ddots & & \\ & & t^{\mu_{1,n}}(t-s)^{\mu_{2,n}} & \end{bmatrix}$$

where

$$p_{i,j}(t) = \frac{-\sum_{k=1}^{\mu(j)} a_{i,j}^k t^{k-1}}{t^{\mu_1,i}(t-s)^{\mu_2,i}}$$

As  $\mu_1$  and  $\mu_2$  are dominant, we have  $p_{i,j}(t) \rightarrow 0$  as  $t \rightarrow \infty$  so this matrix is indeed in  $G_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$ .

Step 2: If  $A \in \mathbb{T}_{\mu_1, \mu_2} \cap \mathfrak{n}$  then  $A$  is sent to  $(N_-)_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$ .  
[Anne: Requires MVyBD!](#)

Step 3: Conversely, given  $L \in W_{\mu_1 + \mu_2}$ , want to show surjectivity.

□

Last meeting's todos:

- make sure that the image of our map is in the  $G_1$  orbit
- more generally, define the map, check that the map is well-defined
- Anne: say what little  $a$  is, i.e. insert the MVy theorem as stated in CK, or thesis
- Roger: check it



## 6 Examples

*Example 3.*  $\lambda_1 = (1, 0, 0)$ ,  $\lambda_2 = (1, 1, 0)$ ,  $\mu_1 = (0, 1, 0)$ ,  $\mu_2 = (1, 0, 1)$ . **Joel:**  $\mu = \mu_1 + \mu_2$  determines the blocks we have on the RHS of the BD MVy isomorphism of Equation ??.

In the non-BD case, MVy establish

$$\overline{\text{Gr}}^\lambda \cap \mathcal{W}^\mu \rightarrow \left\{ X = \left[ \begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * \\ * & 0 & * & 0 & * \end{array} \right] \mid X \in \overline{\mathbb{O}}_\lambda \right\}$$

In the BD case the RHS will consist of the same block like matrices  $X$  but now having eigenvalues  $s, 0$  such that  $X - s|_{E_s} \in \mathbb{O}_{\lambda_2}$

*Example 4.* Do Joel's exercise: It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for  $\mathbf{SL}_3$  of weights  $2\alpha_1$  and  $2\alpha_2$ . Following the notation from the mvbasis paper, this would correspond to the following multiplication:  $x^2y^2 = (xy - z)^2 + 2(xy - z)z + z^2$ . For this example, I think we need  $\lambda_1 = (2, 0, 0)$ ,  $\lambda_2 = (2, 2, 0)$ ,  $\mu_1 = (0, 2, 0)$ ,  $\mu_2 = (2, 0, 2)$ .

*Example 5.* Let

$$\begin{array}{lll} \mu_1 = (2) & \lambda_1 = & \mu = (5) \\ \mu_2 = (3) & \lambda_2 = & \lambda = \end{array}$$

Consider the companion matrix  $C(p)$  of

$$p(t) = (t - s)^3 t^2 = (t^3 - 3t^2s + 3ts^2 - s^3)t^2 = t^5 - 3t^4s + 3t^3s^2 - t^2s^3$$

Let  $X = C(p)^T$  so

$$X = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 0 & 0 & s^3 & -3s^2 & 3s \end{bmatrix}$$

Ask that  $X|_{E_0}$  has Jordan type  $\lambda_1$  and  $X - s|_{E_s}$  has Jordan type  $\lambda_2$ . In this rank 1 case we are forced to take  $\lambda_i = \mu_i$ .

So what are the generalized eigenspaces  $E_i$  ( $i = 1, 2$ )? Note  $\dim E_0 = 2$  and  $\dim E_s = 3$ .

Anne: The basis

$$[1], [t], [1], [t], [t^2]$$

with  $t[t] = 0$  and  $t[t^2] = s^3[1] - 3s^2[t] + 3s[t^2]$  is not the correct basis to consider. Hence my confusion of yore: what we would like is  $t[t] = 0$  no? what the matrix is telling us is that  $t[t] = [1]$ . Can we still speak of *two* generalized eigenspaces?

Rather, take  $B$  to be the basis  $b_1 = e_1$ ,  $b_2 = e_2$ ,  $b_5 = e_5$ ,  $b_4 = Xe_5$ ,  $b_3 = X^2e_5$ . In this basis

$$X_B = [X(b_i)] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 1 & 0 \\ 0 & 0 & 0 & s & 1 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

*Example 6.* Let

$$\begin{aligned} \mu_1 &= (3, 1, 1) & \lambda_1 &= (3, 2, 0) & \mu &= (3, 3, 1) \\ \mu_2 &= (0, 2, 0) & \lambda_2 &= (2, 0, 0) & \lambda &= (5, 2, 0) \end{aligned}$$

and consider the companion matrices of

$$p_1(t) = t^3 \quad p_2(t) = t(t-s)^2 = t^3 - 2st^2 + s^2t \quad p_3(t) = t$$

$$X = \left[ \begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e & f & g \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & k \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

*Example 7.* Let

$$\begin{aligned} \lambda_1 &= (3, 2, 0) & \mu_1 &= (3, 1, 1) \\ \lambda_2 &= (2, 0, 0) & \mu_2 &= (1, 1, 0) \end{aligned}$$

so the first MV cycle  $Z_1 \cong \mathbb{P}^1$  has MV polytope  $\text{Conv}\{0, \alpha_1\}$  and the second MV cycle  $Z_2 \cong \mathbb{P}^1$  has MV polytope  $\text{Conv}\{0, \alpha_2\}$ . Their fusion product corresponds to two  $\mathbb{P}^2$ 's intersecting along a  $\mathbb{P}^1$ . That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where  $Z_+ \cong Z_- \cong \mathbb{P}^2$ . We have

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & a & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix}$$

The 0-generalized eigenspace  $E_0$  of  $X$  is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, X e_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in  $\ker X$ , either  $a = 0$  or  $c = d = 0$ , but the latter case cannot give a 2-cycle as  $e_7 \notin \text{im } X$ . Then  $a = 0$  and we obtain a 2-cycle

$$\left\{X \left(e_6 - \frac{s}{d} e_7\right), e_6 - \frac{s}{d} e_7\right\} = \left\{e_5, e_6 - \frac{s}{d} e_7\right\}.$$

We also obtain the equations  $b \neq 0$ ,  $d \neq 0$ , and  $sc - bd = 0$  from this.

For the  $s$ -generalized eigenspace  $E_s$ , we need  $a + sb \neq 0$  to obtain a 2-cycle, which can be taken as

$$\begin{aligned} & \left\{ (X - sI) \left( e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right), \right. \\ & \quad \left. e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right\} \\ &= \left\{ e_1 + se_2 + s^2e_3 + s^3e_4, e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right\} \end{aligned}$$

The minimal polynomial is  $X^3(X - sI)^2$ , which when equated to 0 gives again the equation  $cs - bd = 0$ . Thus the defining equations are

$$\{a = 0, cs - bd = 0\}.$$

When we take  $s = 0$ , we get the equations

$$\{a = 0, bd = 0\}$$

which corresponds to two  $\mathbb{A}^2$ 's intersecting along an  $\mathbb{A}^1$ . This is indeed an open subset of  $\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ , as required.

*Example 8* (Example 7 continued...). The matrix  $X$  from the previous example defines (under MVy) the matrix

$$g = \begin{bmatrix} (t-s)t^3 & & & \\ -bt & (t-s)t & & \\ -c & -d & t & \end{bmatrix}$$

in  $G(\mathcal{O})$ . Indeed the various blocks of  $X$  are in a precise sense the companion matrices of the polynomial entries of  $g$

In  $\text{Gr}$  the element  $g$  defines the lattice

$$gL_0 = \mathbb{C}[t]\langle (t-s)t^3e_1 - (a+bt)e_2 - ce_3, (t-s)te_2 - de_3, te_3 \rangle$$

and computing the matrix of the action of  $t$  on the quotient  $L_0/L$  in the basis

$$\{[e_1], [te_1], [t^2e_1], [t^3e_1], [e_2], [te_2], [e_3]\}$$

recovers  $X$  up to a transpose of course.

Now let's see what we get when we invert  $t$  and  $t-s$  respectively.

First let's invert  $t$  by considering  $L_2 = L \otimes \mathbb{C}[[t-s]]$ .

$$L_2 = \mathbb{C}[t, t^{-1}]\langle (t-s)e_1 - \frac{a+bt}{t^3}e_2 - \frac{c}{t^3}e_3, (t-s)e_2 - \frac{d}{t}e_3, e_3 \rangle$$

so in  $L_0/L_2$  we have

$$t[e_1] = s[e_1] + \frac{a+bt}{t^3}[e_2] + \frac{c}{t^3}[e_3] \quad t[e_2] = s[e_2] + \frac{d}{t}[e_3] \quad [e_3] = 0$$

and

$$\left[ t|_{L_0/L_2} \right]_{\{[e_1], [e_2]\}} = \begin{bmatrix} s & \\ \frac{a+bt}{t^3} & s \end{bmatrix}$$

which upon subtracting  $sI$  gives a matrix having block type  $\mu_2$  and Jordan type  $\lambda_2 = (2)$  assuming  $\frac{a+bt}{t^3} \neq 0$ .

Next let's invert  $t-s$  by considering  $L_1 = L \otimes \mathbb{C}[[t]]$ .

$$\begin{aligned} L_1 &= \mathbb{C}[t, (t-s)^{-1}]\langle t^3e_1 - \frac{a+bt}{t-s}e_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3 \rangle \\ &= \langle t^3e_1 - \frac{a}{t-s}e_2 - \frac{b}{t-s}te_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3 \rangle \end{aligned}$$

so in  $L_0/L_1$  we have

$$\begin{aligned} t[e_1] &= [te_1] \\ t[te_1] &= [t^2e_1] \\ t[t^2e_1] &= \frac{a}{t-s}[e_2] + \frac{b}{t-s}t[e_2] + \frac{c}{t-s}[e_3] \\ &= \frac{b}{t-s}\frac{d}{t-s}[e_3] + \frac{c}{t-s}[e_3] \\ &= \frac{bd + (t-s)c}{(t-s)^2}[e_3] \\ &= \frac{bd - sc}{(t-s)^2}[e_3] + \frac{c}{(t-s)^2}t[e_3] = 0 \\ t[e_2] &= \frac{d}{t-s}[e_3] \\ t[e_3] &= 0 \end{aligned}$$

and

$$[t|_{L_0/L_1}]_{\{[e_1],[te_1],[t^2e_1],[e_2],[e_3]\}} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ & 0 & 0 & \\ & 0 & \frac{d}{t-s} & 0 \end{bmatrix}$$

taking transpose

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}$$

which has block type  $\mu_1$  and Jordan type  $\lambda_1 = (3, 2)$  assuming  $d \neq 0$ .

I have used the relations Roger found (and I checked)  $a = 0$  and  $cs - bd = 0$  in the calculations above.

To sum up, the pair of matrices above should contain the same information as the matrix from the previous example

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & b & c & \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix} \Leftrightarrow \left( \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \frac{d}{t-s} & \\ & & & 0 & \end{bmatrix}, \begin{bmatrix} s & \frac{b}{t^2} \\ & s \end{bmatrix} \right)$$

**Wish.** Given  $L$  in  $\mathcal{G}_n^{\text{BD}}$  define a map to  $T_\mu$  just like MVy by taking  $[t|_{L_0/L}]$  and use the fact that  $[t|_{L_0/L_i}]$  for  $i = 1, 2$  are companion matrices of the right type, piece together two MVy isomorphisms to make a BD MVy iso.

Equivalent lin alg question: If  $p(t, t-s) = p_1(t)p_2(t-s)$  then how are  $C(p_1)$ ,  $C(p_2)$ , and  $C(p)$  related? I think it's basically this theorem [https://en.wikipedia.org/wiki/Structure\\_theorem\\_for\\_finitely\\_generated\\_modules\\_over\\_a\\_principal\\_ideal\\_domain](https://en.wikipedia.org/wiki/Structure_theorem_for_finitely_generated_modules_over_a_principal_ideal_domain)

*Example 9.* Let  $G = \mathbf{SL}_3$  and  $\mathbf{i} = 121$ . Take  $n_\bullet^1 = (1, 0, 0)$ , and  $n_\bullet^2 = (1, 0, 1)$  or  $(0, 1, 0)$ . So  $\mu_1 = (2, 2, 1)$ ,  $\lambda_1 = (3, 1, 1)$  and  $\mu_2 = (1, 1, 1)$ ,  $\lambda_2 = (2, 1, 0)$ . *Anne:* We should show that order does not matter. Then  $\mathbb{T}_{\mu_1, \mu_2}^+ \cap \mathbb{O}_{\lambda_1, \lambda_2}$  is made up of elements

$$\begin{bmatrix} 0 & 1 & 0 & & & & & & \\ 0 & 0 & 1 & & & & & & \\ 0 & 0 & s & 0 & a_2 & a_3 & b_1 & b_2 & \\ & & & 0 & 1 & 0 & & & \\ & & & 0 & 0 & 1 & & & \\ & & & 0 & 0 & s & c_1 & 0 & \\ & & & & & & 0 & 1 & \\ & & & & & & & 0 & s \end{bmatrix}$$

and zero eigenspace conforming to the shape

$$\lambda_1 = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

is made of three cycles

$$\begin{array}{c} a_2 e_1 \leftarrow a_2 e_2 - s e_4 \leftarrow a_2 e_3 - s e_5 \\ e_4 \\ e_7 \end{array}$$

while the  $s$ -eigenspace confirming to the shape

$$\lambda_2 = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

is made of the two cycles

$$\begin{array}{c} e_1 + s e_2 + s^2 e_3 \leftarrow e_2 + 2 s e_3 + e_7 + s e_8 \\ e_4 + s e_5 + s^2 e_6 \end{array}$$

assuming  $b_2 = s$  and  $a_2 + s a_3 = 0$ .

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