How to compute the fusion product of MV cycles in type A

Roger Bai, Anne Dranowski, Joel Kamnitzer

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1 Players

$G = \mathbf{GL}_m$.

- The ordinary affine Grassmannian Gr
- The Beilinson–Drinfeld Grassmannian $\mathcal{G}r_n^{\mathrm{BD}} \to C$
- Partitions $\mu_i \leq \lambda_i$ of N_i $(i = 1, 2)$ and $\mu = \sum \mu_i \leq \lambda = \sum \lambda_i$ of $N = \sum N_i$
- The slices Gr_μ and $\mathcal{W}_{\mu_1,\mu_2}$ to the orbits Gr^λ and $(\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2}$
- The nilpotent and semi-nilpotent cones $\mathcal{N}$ and $\mathcal{N}_s$ (of matrices with eigenvalues 0 and 0 or $s \neq 0$)
- The slices $\mathbb{T}_\mu$ and $\mathbb{T}_{\mu_1,\mu_2}$ to the orbits $\mathbb{O}_\lambda$ and $\mathbb{O}_{\lambda_1,\lambda_2}$

New (?) definitions among these are as follows.
We define

$$L_\mu = t^\mu, L_{\mu_1,\mu_2} = t^{\mu_1}(t-s)^{\mu_2} \in G((t^{-1}))/G[t]$$

$$\mathcal{W}_\mu = G_1[[t^{-1}]]L_\mu \subset G((t^{-1}))/G[t] \tag{1}$$

a subscheme of the thick affine Grassmannian

Note that $L_{\mu_1,\mu_2} = (t-s)^{\mu_2}t^{-\mu_2}L_\mu \in \mathcal{W}_\mu$ and that $L_{\mu_1,\mu_2} \in \mathcal{G}r_2^{\mathrm{BD}}$ is a $\mathbb{C}[t]$-lattice in $\mathbb{C}(t)^m$ that specializes to a $\mathbb{C}[\![t]\!]$-lattice in $\mathbb{C}(\!(t)\!)^m$ away from $t = 0$ and away from $t = s$; i.e.

$$\begin{aligned} L_{\mu_1,\mu_2} \otimes \mathbb{C}[\![t]\!] = L_{\mu_2} \otimes \mathbb{C}[\![t]\!] \text{ and} \\ L_{\mu_1,\mu_2} \otimes \mathbb{C}[\![t-s]\!] = L_{\mu_1} \otimes \mathbb{C}[\![t-s]\!] \end{aligned} \tag{2}$$

where L_{μ_i} denotes the point $t^{\mu_i}G(\mathcal{O}) \in \mathrm{Gr}$.

The family of semi-infinite orbits

$$S_{\mu_1,\mu_2} = N_-(\mathbb{C}((t^{-1})))L_{\mu_1,\mu_2}\,. \tag{3}$$

The orbit of $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^m$ whose elements specialize again to $\mathbb{C}[\![t]\!]$-lattices in $\mathbb{C}(\!(t)\!)^m$ away from $t=0$ and away from $t=s$ as follows

$$\begin{aligned}(\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2} = \{L \in \mathcal{G}r_2^{\mathrm{BD}} : L \otimes_{\mathbb{C}[t]} \mathbb{C}[\![t-s]\!] \in \mathrm{Gr}^{\lambda_2} \text{ and} \\ L \otimes_{\mathbb{C}[t]} \mathbb{C}[\![t]\!] \in \mathrm{Gr}^{\lambda_1} \text{ and} \\ L \otimes_{\mathbb{C}[t]} \mathbb{C}[\![t-a]\!] = L_0 \text{ for all } a \neq 0, s\}\,.\end{aligned} \tag{4}$$

Joel: It is important to add the condition that the lattices are trivial at a. Anne: These defining conditions are again just telling us that in the fibre over the fixed point $(0,s) \in C^{(2)}$ this set is pairs of lattices; note that we invert the indeterminates t and $(t-s)$ *after* specializing $(0,s)$. Could we give a more explicit characterization like $t^{N_1}(t-s)^{N_2}L_0 \subseteq L \subseteq t^{-N_1}(t-s)^{-N_2}$?

The [family "$\mathcal{N}\otimes\mathbb{C}[s]$? $\mathcal{N}\times\mathbb{C}$?" of] semi-nilpotent cone[s fibred over $C=\mathbb{A}^1$ with s-fibre]

$$\mathcal{N}_s = \{A \in \mathrm{Mat}(N) : \text{eigenvalues of } A \text{ are } 0 \text{ or } s\}\,. \tag{5}$$

The [family of] slice[s $\mathbb{T}_{\mu_1,\mu_2}$ fibred over $C=\mathbb{A}^1$ with s-fibre]

$$\begin{aligned}\mathbb{T}^s_{\mu_1,\mu_2} = \{B + C_s : B \text{ is a } \mu\times\mu \text{ block matrix of zeros} \\ \text{except possibly in the last } \min(\mu_i,\mu_j) \\ \text{columns of the last row of each } \mu_i\times\mu_j \text{ block} \\ \text{and } C_s \text{ is the block diagonal matrix of} \\ \text{companion matrices of } t^{\mu_{1,k}}(t-s)^{\mu_{2,k}}\}\,.\end{aligned} \tag{6}$$

The uppertriangular subfamily $\mathbb{T}^+_{\mu_1,\mu_2}$ with s-fibre

$$\mathbb{T}^{+,s}_{\mu_1,\mu_2} = \{B + C_s \in \mathbb{T}_{\mu_1,\mu_2} : B \in \mathfrak{n}\} \tag{7}$$

where $\mathfrak{n} \subset \mathrm{Mat}(N)$ is the unipotent subalgebra of uppertriangular matrices.

Anne: or—as Joel pointed out, may be ok with: The slice $\mathbb{T}_\mu$ as defined in MVy, no change, and the family of slices

$$\mathbb{T}^{+,s}_{\mu_1,\mu_2} = \mathbb{T}_\mu \cap \mathfrak{n} + C^s_{\mu_1,\mu_2}$$

where

$$C_s \text{ is the block diagonal matrix of companion matrices of } t^{\mu_{1,k}}(t-s)^{\mu_{2,k}} \tag{8}$$

The [family of] orbit[s $\mathbb{O}_{\lambda_1,\lambda_2}$ fibred over $C=\mathbb{A}^1$ with s-fibre]

$$\mathbb{O}^s_{\lambda_1,\lambda_2} = \{A \in \mathcal{N}_s : A \text{ is conjugate to } J_{\lambda_1} \oplus (sI_{N_2} + J_{\lambda_2})\} \tag{9}$$

where J_{λ_i} is the Jordan normal form of block type λ_i and I_{N_2} is the identity matrix in $\mathrm{Mat}(N_2)$.

2 Exposition

In Anderson and Kogan conjectured in [AK05] Anne: or earlier? that those MV polynomials which are cluster monomials for a Fomin–Zelevinsky cluster algebra structure on $\mathbb{C}[N]$ are naturally expressible as determinants. . . and they conjecture a formula for many of them.

It's not clear how this work helps/relates to AK/their conjectures.

The generalized MVy is interesting in its own right.

Computing fusion still hard but at least boiled down to linear algebra. Cf. fusion product as it appears in BD, FL, MV, AK, BFM.

> However, we make crucial use of an idea of Drinfeld, going back to around 1990. He discovered an elegant way of obtaining the commutativity constraint by interpreting the convolution product of sheaves as a "fusion" product.

Exchange relations only work on cluster modules where one is a mutation of the other (i.e. those corresponding to cluster monomials which are related by mutation). Of course this gives me everything. Up to A_4 as in type A_5 there exist indecomposable modules which are not cluster, so exchange relations do not apply. The hope (conjecture) is that this paper gives a way to compute on such modules. Cf. counterexample satisfying $\overline{D}(c_Y) = \overline{D}(b_Z) + 2\overline{D}(b)$, where b is (possibly) cluster, and suggesting that $c_Y = b_Z + 2b$.

3 Rising Action

4 Climax

5 Falling Action

Theorem 1. *Let $\lambda_i \geq \mu_i$ be dominant ($i = 1, 2$), $\mu = \mu_1 + \mu_2$, and $\lambda = \lambda_1 + \lambda_2$. There is an isomorphism*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_{\mu_1,\mu_2} \to \overline{(\mathcal{G}r_2^{BD})^{\lambda_1,\lambda_2}} \cap \mathcal{W}_{\mu_1,\mu_2} \tag{10}$$

got by taking a $\mu \times \mu$ block matrix A in the s-fibre $\overline{\mathbb{O}^s_{\lambda_1,\lambda_2}} \cap \mathbb{T}^s_{\mu_1,\mu_2}$ on the left to the representative of the s-fibre on the right defined by

$$\begin{aligned} g &= t^{\mu_1}(t-s)^{\mu_2} + a(t) \\ a_{ij}(t) &= -\sum_{k=1}^{\mu_i} A^k_{ji} t^{k-1} \end{aligned} \tag{11}$$

where A^k_{ji} is the kth entry from the left of the last row of the $\mu_j \times \mu_i$ block of A.

Let's call this the MVyBD isomorphism.

Proof. The proof is fibre by fibre, so fix $s \neq 0$. Anne: Emphasize in the intro later (because this always confuses me) that by the s-fibre we really mean the $(0, s)$-fibre; i.e. its the BD Grassmannian over the second symmetric power of $C = \mathbb{A}^1$; better just replace s-fibre by $(0, s)$-fibre everywhere it occurs.

1. The map is well defined. In particular, it defines $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^m$. Moreover, these lattices break down to give pairs of lattices upon inverting t or $t - s$ that have the right properties. [Copy Roger's proof]

2. The inverse map is got by taking the matrix of multiplication by t. More precisely, let $L \in Gr^{BD} \cap \mathcal{W}_\mu$. We work with the quotient $\mathbb{C}[t]^m/L$ just as in the ordinary MVy isomorphism—the only difference being $\mathbb{C}[\![t]\!]$ is replaced by $\mathbb{C}[t]$.

   (a) We claim that

   $$\{[e_i], [te_i], \ldots, [t^{\mu_i - 1}e_i] : 1 \leq i \leq m\} \tag{12}$$

   is a $\mathbb{C}$-basis of $\mathbb{C}[t]^m/L$.
   To see this, we use that L has a $\mathbb{C}[t]$-basis of the form

   $$v_i = t^{\mu_i} + \sum_{j>i} p_{ij}(t)e_j \tag{13}$$

   with $\deg p_{ij}(t) < \mu_i = \mu_{1,i} + \mu_{2,i}$ $(1 \leq i \leq m)$. Anne: I don't know why this should be true. We might have to just define fibres of $\mathcal{W}_{\mu_1,\mu_2}$ in this way?

   (b) $t\big|_{\mathbb{C}[t]^m/L}$ will have two eigenvalues, 0 and s, and its generalized 0-eigenspace will have block type $\leq \lambda_1$ while its generalized s-eigenspace will have block type $\leq \lambda_2$. To see this, note that there is a natural isomorphism

   $$\mathbb{C}[[t]]^m/(L \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]) = \text{generalized 0 eigenspace of } t \text{ on } \mathbb{C}[t]^m/L$$

   carrying the action of t to the action of t.
   The left hand side is the same thing as

   $$\mathbb{C}[[t]]^m/(L \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]) = (\mathbb{C}[t]^m/L) \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$$

   the defining fact that lattices satisfying Equation 4 equivalently satisfy

   $$\begin{aligned} t\big|_{\mathbb{C}[\![t]\!]^m/L_1} &\text{ has Jordan type} \leq \lambda_1 \\ t\big|_{\mathbb{C}[\![t]\!]^m/L_2} &\text{ has Jordan type} \leq \lambda_2 \end{aligned} \tag{14}$$

   where recall $L_i = L \otimes \mathbb{C}[\![t]\!]$??? and $p_1 = s$ while $p_2 = 0$. Anne: Somehow, restricting to an eigenspace is like inverting/forgetting the action of t by any other generalized eigenvalue? Basic linear algebra? Joel?

$\square$

Theorem 2 (Theorem 1 version 2)**.** *Let λ_1, λ_2 and μ be arbitrary, such that $\lambda = \lambda_1 + \lambda_2 \geq \mu$. Then there is an isomorphism*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_\mu \to \overline{\mathcal{G}r_2^{BD\lambda_2,\lambda_2}} \cap \mathcal{W}_\mu \tag{15}$$

defined by the same map as in Theorem 1.

Joel: This is true as stated with the "larger" definition of $\mathcal{W}_\mu$. In fact, for any λ_1, λ_2, it is true $\overline{\mathcal{G}r_2^{\mathrm{BD}\lambda_2,\lambda_2}} \cap \mathcal{W}_\mu$ is contained in a subset that we could call $\mathcal{W}_\mu^s$ which we could define as

$$\mathcal{W}_\mu^s = G_1[[t^{-1}]]t^\mu \cap G[t, t^{-1}, (t-s)^{-1}]/G[t]$$

where we regard $G[t, t^{-1}, (t-s)^{-1}]/G[t] \subset G((t^{-1}))/G[t]$

The way to think about this is as follows: inside the thick affine Grassmannian we can consider the G-bundles trivialized away from just 0, s, or equivalently those lattices which become the standard lattice after tensoring with $\mathbb{C}[[t-a]]$ for any $a \neq 0, s$.

Corollary 1. *The MVyBD isomorphism restricts to an isomorphism of subfamilies*

$$\overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}^+_{\mu_1,\mu_2} \to \overline{(\mathcal{G}r_2^{BD})^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2}\,. \tag{16}$$

Proof. Let $A \in \overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}^+_{\mu_1,\mu_2}$ and let g be the polynomial matrix formed by the Mirkovic-Vybornov isomorphism. Then the diagonal entries of g are $t^{\mu_{1,k}}(t-s)^{\mu_{2,k}}$ and we can factor

$$g = (gt^{-\mu_1}(t-s)^{-\mu_2})t^{\mu_1}(t-s)^{\mu_2} \in N[t, t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$$

So we get containment in one direction.

For the reverse containment, we choose $[g] \in G\overline{(\mathcal{G}r_2^{\mathrm{BD}})^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2}$. By the lemma below, $[g] \in \mathcal{W}_\mu$ and thus it lies in the image of our map and we are done. $\square$

Anne: is it a fibre of S_{μ_1,μ_2} defined above?

We could also make the following claim.

Lemma 1 (KWWY14)**.** *Let μ be dominant. Then*

$$N_-((t^{-1}))L_\mu = N_1[\![t^{-1}]\!]L_\mu \tag{17}$$

Anne: where I am not sure about the double brackets.

Lemma 2. *Let μ_1, μ_2 be dominant and let $s \in \mathbb{A}^1 - \{0\}$. Then*

$$S^s_{\mu_1,\mu_2} \subset \mathcal{W}_\mu \tag{18}$$

where $\mu = \mu_1 + \mu_2$.

Proof. We have

$$\begin{aligned}
S_{\mu_1,\mu_2} &= N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&\subset T_1[[t^{-1}]]N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&= T_1[[t^{-1}]]N_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \qquad \text{[KWWY14, Lemma 2.3]} \\
&= B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&\subset G_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&= W_{\mu_1+\mu_2}
\end{aligned}$$

where $B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} = B_1[[t^{-1}]]t^{\mu_1+\mu_2}$ since

$$\frac{t}{t-s} = 1 + \frac{s}{t} + \frac{s^2}{t^2} + \cdots \in B_1[[t^{-1}]].$$

□

6 Denouement

As an application we can compute fusion of stable MV cycles of type α_i for any $i \in I$. What about more general weights? Having kpf > 1.

Proposition 1. *Given two MV cycles Z_τ and Z_σ of type...*

$$Z_\tau * Z_\sigma \tag{19}$$

is found by???

Conjecture 1. *Let $Z_i \subset \overline{S^{\nu_i} \cap S^0_-}$ be an MV cycle of weight ν_i $(i = 1, 2)$ and put $\nu = \nu_1 + \nu_2$.*

References

[AK05] Jared E Anderson and Mikhail Kogan. The algebra of mirkovic-vilonen cycles in type a. arXiv preprint math/0505100, 2005. 3

[KWWY14] Joel Kamnitzer, Ben Webster, Alex Weekes, and Oded Yacobi. Yangians and quantizations of slices in the affine Grassmannian. Algebra & Number Theory, 8(4):857–893, 2014. 6