Working title: Mirković–Vybornov fusion in Beilinson–Drinfeld Grassmannian

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1 Introduction

The BD Grassmannian. The convolution Grassmannian. Distinguished orbits and slices. Connections to Mirković–Vybornov [?, ?], Cautis–Kamnitzer [?], Anderson–Kogan [?].

Address limitations outside of type A?

2 Notation

In ordinary Gr we have the following lattice descriptions valid only in type A. Given $\mu \in X^\bullet(T)$, write t^μ for its image in $G(\mathcal{K})$ and L_μ for its image in

$$\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O}) \overset{A}{=} \{L \underset{\text{rank } m}{\overset{\text{free}}{\subset}} \mathcal{O}^m : tL \subset L\}$$

Example: $L_\mu = \mathrm{Span}_{\mathcal{O}}(e_i t^j : 0 \leq j < \mu_i)$. Fact: $\mathrm{Gr}^T = X^\bullet(T)$ and other distinguished subsets (needed for the definition of MV cycles and later open subset thereof) are all orbits of fixed points

$$\begin{aligned}
\mathrm{Gr}^\lambda = G(\mathcal{O})L_\lambda \quad &= \{L \in \mathrm{Gr} : t\big|_{\mathcal{O}^m/L} \text{ has Jordan type } \lambda\} \\
\mathrm{Gr}_\mu = G_1[t^{-1}]L_\mu \quad &= \{L \in \mathrm{Gr} : L = \mathrm{Span}_{\mathcal{O}}(v_1, \ldots, v_m) \text{ such that} \\
&\qquad\qquad v_j = t^{\mu_j} e_j + \sum p_{ij} e_i \text{ with } \deg p_{ij} < \mu_j\} \\
S^\mu_- = U_-(\mathcal{K})L_\mu \quad &= \{L \in \mathrm{Gr}_\mu : \dim(\mathcal{O}^k/L \cap \mathcal{O}^k) = \mu_1 + \cdots + \mu_k\}
\end{aligned}$$

Let Gr denote the ordinary **affine Grassmannian** $G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$, $\mathcal{Gr}_n^{\mathrm{BD}}$ the **Beilinson–Drinfeld affine Grassmannian**, and Gr_n the **convolution affine Grassmannian**.

Definition 1. The **BD Grassmannian** is the set

$$\begin{aligned}
\{(V, \sigma) : \; & V \text{ is a rank } m \text{ vector bundle on } \mathbb{P}^1 \\
& \text{and } \sigma : V \dashrightarrow \mathscr{O}^m_{\mathbb{P}^1} \text{ is a trivialization} \\
& \text{defined away from finitely many points in } \mathbb{A}^1\}
\end{aligned} \tag{1}$$

The rank of the m in the definition of $\mathcal{G}r_n^{\mathrm{BD}}$ is the dimension of the maximal torus of $G^\vee$. For $G^\vee = \mathbf{GL}_m = G$.

More generally, one can define a BD grassmannian of $\mathbf{GL}_m$ over any smooth curve C as the reduced ind-scheme $\mathcal{G}r_{n,C}^{\mathrm{BD}}$ fibered over a finite symmetric power of C — $C^{(n)}$ — such that the fibre over the point $\vec{p} = (p_1, \ldots, p_n) \in C^n$ is a collection of rank m vector bundles V over C which are trivial away from $\vec{p}$ viewed also as a subset — $\{p_1, \ldots, p_n\}$ — of C. Trivial means $\mathscr{O}_C^m \cong V$.

To quote [**?**] the BD Grassmannian is a relative version of the affine Grassmannian where the base is the space of effective divisors on a smooth curve C. The choice $C = \mathbb{A}^1$ "amply satisfies our needs and offers three advantages: there is a natural global coordinate it, every G-torsor on it is trivializable, and the monodromy of any local system is trivial. Formally, $\mathcal{G}r_n^{\mathrm{BD}}$ is the functor on the category of commutative $\mathbb{C}$-algebras that assigns to an algebra R the set of isomorphism classes of triples $(\vec{p}, V, \sigma)$ where $\vec{p} \in \mathbb{A}^n(R)$, V is a $G^\vee$-torsor over $\mathbb{A}^1_R$ and σ is a trivialization of V away from $\vec{p}$." i.e. principal G bundle?

They denote by π the fibration $\mathcal{G}r_n^{\mathrm{BD}} \to \mathbb{A}^n$ (forgetting V and σ). Their simplified description is: it's the set of pairs $(\vec{p}, [\sigma])$ where $\vec{p} \in \mathbb{C}^n$ and $[\sigma]$ is an element of the homogeneous space

$$G^\vee(\mathbb{C}[z, (z-p_1)^{-1}, \ldots, (z-p_n)^{-1}])/G^\vee(\mathbb{C}[z])$$

Their example, (almost) our setting:

Example 1. When $G = \mathbf{GL}_m\mathbb{C}$ the datum of $[\sigma]$ is equivalent to the datum of the $\mathbb{C}[z]$-lattice $\sigma(L_0)$ in $\mathbb{C}(z)^m$ with $L_0 = \mathbb{C}[z]^m$ denoting the **standard lattice**. Set $f_{\vec{p}} = (z-p_1)\cdots(z-p_n)$. Then a lattice L is of the form $\sigma(L_0)$ if and only if there exists a positive integer k such that $f_{\vec{p}}^k(L_0) \subseteq L \subseteq f_{\vec{p}}^{-k}(L_0)$ and for each k they denote by ${\mathcal{G}r_n^{\mathrm{BD}}}_k$ the subset of $\mathcal{G}r_n^{\mathrm{BD}}$ consisting of pairs $(\vec{p}, L)$ such that this sandwhich condition holds. They identify $\mathbb{C}[z]/(f_{\vec{p}}^{2k})$ with the vector space of polynomials of degree strictly less than $2kn$, and $L_0/f_{\vec{p}}^{2k}L_0$ with its Nth product. Then

$${\mathcal{G}r_n^{\mathrm{BD}}}_k \overset{\text{Zariski closed}}{\subset} \mathbb{C}^n \times \bigcup_{d=0}^{2knN} G_d(L_0/f_{\vec{p}}^{2k}L_0)$$

where $G_d(?)$ denotes the ordinary Grassmann manifold of d-planes in the argument.

Our setting is $G = \mathbf{GL}_m$ and $n = 2$.

Definition 2. The **deformed convolution Grassmannian** is [not needed?] pairs $(\vec{p}, [\vec{\sigma}])$ where $\vec{p} \in \mathbb{C}^n$ and $\vec{\sigma}$ is in

$$G^\vee(\mathbb{C}[z, (z-p_1)^{-1}]) \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} G^\vee(\mathbb{C}[z, (z-p_n)^{-1}])/G^\vee(\mathbb{C}[z])$$

with a map down to $\mathcal{G}r_n^{\mathrm{BD}}$ defined by $(\vec{p}, [\vec{\sigma}]) \mapsto (\vec{p}, [\sigma_1 \cdots \sigma_n])$.

To steal the follow-up example in [**?**] where the above definition is also copied from. . .

Example 2. When $G = \mathbf{GL}_m\mathbb{C}$ this deformation is described by the datum of $\vec{p} \in \mathbb{C}^n$ and a sequence $(L_1, \dots, L_n)$ of $\mathbb{C}[z]$-lattices in $\mathbb{C}(z)^m$ such that for some $k \in \mathbb{Z}$ and for all $j \in \{1 \dots n\}$

Why Laurent polynomials for the convolution?

$$(z - p_j)^k L_{j-1} \subset L_j \subset (z - p_j)^{-k} L_{j-1}$$

where again $L_0 = \mathbb{C}[z]^m$ denotes the standard lattice, while $L_j = (\sigma_1 \cdots \sigma_j)(L_0)$. Very nice. Very concrete. They can partition the deformation into **cells** by specifying the **relative positions** of the pairs (L_{j-1}, L_j) in terms of **invariant factors**.

To be continued: [**?**] go on to describe the fibres of the composition

Not to $\mathbb{C}^{(n)}$? Or to $\mathbb{C}$?

$$\mathrm{Gr}_n \to \mathcal{G}r_n^{\mathrm{BD}} \to \mathbb{C}^n = \mathbb{A}^n_{\mathbb{C}}$$

their description may be helpful.

For $\mu \in P$ and $p \in \mathbb{C}$ they define

$$\tilde{S}_{\mu|p} = (z-p)^\mu N^\vee(\mathbb{C}[z,(z-p)^{-1}]) = N^\vee(\mathbb{C}[z,(z-p)^{-1}])(z-p)^\mu$$

They note that $\mathbb{C}(\!(z-p)\!)$ is the completion of $\mathbb{C}(z)$ at "the place defined by p" and identify $\mathbb{C}[\![z-p]\!]$ with $\mathbb{C}[\![z]\!]$ and $\mathbb{C}(\!(z-p)\!)$ with $\mathbb{C}(\!(z)\!)$.

They claim that

$$N^\vee(\mathbb{C}[z,(z-p)^{-1}])/N^\vee(\mathbb{C}[z]) \to N^\vee(\mathbb{C}(\!(z-p)\!))/N^\vee(\mathbb{C}[\![z-p]\!]) \cong N^\vee(\mathcal{K})/N^\vee(\mathcal{O})$$

is bijective, and that mapping Gr and multiplying by $(z-p)^\mu$ one gets

$$\tilde{S}_{\mu|p}/N^\vee(\mathbb{C}[z]) \cong S_\mu$$

They go on to describe the fusion product (section 5.3) a probably worthwhile read.

Going forward, we'll use t to denote the coordinate on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$ instead of z. Then by $t^\mu \in G^\vee(\mathcal{K})$ we'll denote the point defined by the coweight $\mu \in \mathrm{Hom}(\mathbb{C}^\times, T^\vee) = T^\vee(\mathcal{K})$ and by L_μ its image $t^\mu G^\vee(\mathcal{O})$ in Gr.

Definition 3. Given μ_1, μ_2 such that $\mu = \mu_1 + \mu_2$ is a partition of N we define $T_{\mu_1,\mu_2} \subset \mathrm{Mat}_N$ to be the set of $\mu \times \mu$ block matrices that are zero everywhere except possibly in the last $\min(\mu_i, \mu_j)$ columns of the last row of the $\mu_i \times \mu_j$th block *plus* the block diagonal matrix whose $\mu_i \times \mu_i$ diagonal block is the companion matrix of $t^{\mu_{1,i}}(t-s)^{\mu_{2,i}}$ for each $i \in \{1, 2, \dots, m\}$. We call this set **name**.

Remark 1. While we limit ourselves to the case of dominant partitions, the definition above makes sense for arbitrary partitions.

Remark 2. Speak to whether or not this slice appears in [**?**]. We don't think it does. But its "lift" might.

Definition 4. Given λ_1, λ_2 such that $\lambda = \lambda_1 + \lambda_2$ is a partition of N we define $\mathbb{O}_{\lambda_1,\lambda_2}$ to be the set of $N \times N$ **semi-nilpotent** matrices X with spectrum in $\{0, s\}$ for some $s \in \mathbb{C}^\times$ such that $X\big|_{E_0} \in \mathbb{O}_{\lambda_1}$ and $(X-s)\big|_{E_S} \in \mathbb{O}_{\lambda_2}$ meaning TODO.

Cute?

Correspondingly we have

- $W_{\mu_1,\mu_2} = G_1^\vee[t^{-1},(t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}G^\vee(\mathcal{O})$ as a subset of $\mathcal{G}r_2^{\mathrm{BD}}$ where recall the subscript 2 records the fact that we are fixing two points $0, s \in \mathbb{C}$

- Does ${\mathcal{G}r_2^{\mathrm{BD}}}^{\lambda_1,\lambda_2}$ admit such an orbit description? As a fibration it is pairs (V,σ) such that is trivialized away from $(0,s)$ by σ and σ has type (λ_1,λ_2) — the data of the trivialization is equivalent to the data of pairs of lattices (L_1,L_2) such that $L_i \in \mathrm{Gr}^{\lambda_i}$ away from 0 and $L_1 = L_2 \in \mathrm{Gr}^\lambda$ at 0?

- In the deformed convolution Gr we have the subset ${\mathrm{Gr}_2}^{\lambda_1,\lambda_2}$ of pairs $((0,s),[\sigma_1,\sigma_2])$ (really want arbitrary (p_1,p_2) in place of $(0,s)$ which is the specialization that we make when we work in $\mathcal{G}r_2^{\mathrm{BD}}$ sort of?) with $\sigma_i \in G^\vee(\mathbb{C}[t])(t-p_i)^{\lambda_i}G^\vee(\mathbb{C}[z])$ but this is not important?

Question 1. Can we describe $W_{\mu_1,\mu_2} \cap \overline{\mathrm{Gr}^{\lambda_1,\lambda_2}}$ as the set of lattices L such that

$$t^{\lambda_{1,1}}(t-s)^{\lambda_{2,1}}L_0 \subset L \subset t^{-\lambda_{1,1}}(t-s)^{-\lambda_{2,1}}L_0$$

and $\lim_{s\to 0}\rho^\vee(s)\cdot L = L_\mu$

Definition 5. Say μ_1 and μ_2 are **disjoint** if $(\mu_1)_i \neq 0 \Rightarrow (\mu_2)_i = 0$ and $(\mu_2)_i \neq 0 \Rightarrow (\mu_1)_i = 0$.

3 Main results

Claim 1. $\widetilde{T_x^a} \to \pi^{-1}(\overline{\mathrm{Gr}^\lambda} \cap \mathrm{Gr}_\mu)$ (this does depend on b! we get something like a springer fibre where the action of [what] on either side has eigenvalues a permutation of b.)

Claim 2. Let $\mathcal{W}_{\mathrm{BD}}^\mu = G_1(\!(t^{-1})\!)t^\mu$. Then $S^{\mu_1+\mu_2}$ is contained in $\mathcal{W}_{\mathrm{BD}}^\mu$ if μ is dominant. Joel: And μ_1, μ_2 are dominant also? Anne: Roger has a proof.

Claim 3. Let $a = (0,s)$ and suppose μ_1 and μ_2 are disjoint ~~“transverse”~~ Let $\mu = \mu_1 + \mu_2$. Then $X \in \widetilde{T_x^a}$ is a $\mu \times \mu$ block matrix, with $(\mu_1)_k \times (\mu_1)_k$ diagonal block conjugate to a $(\mu_1)_k$ Jordan block and $(\mu_2)_k \times (\mu_2)_k$ diagonal block conjugate to $(\mu_2)_k$ Jordan block plus sI.

Question 2. If μ_i is not a permutation of λ_i and λ_i are not “homogeneous” how do we proceed? E.g. if $\mu_1 = (3,0,2)$, $\mu_2 = (0,2,0)$ and $\lambda_1 = (4,1)$, $\lambda_2 = (2,0,0)$.

Question 3. If μ_1 and μ_2 are not disjoint how do we proceed? E.g. if $\mu_1 = (2,2,0)$, $\mu_2 = (1,0,2)$; $\mu_1 = (2,2,1)$, $\mu_2 = (1,0,1)$.

4 Convolution vs BD

Fix $G = \mathbf{GL}(U) \cong \mathbf{GL}_m\mathbb{C}$ and $\{e_1,\ldots,e_m\}$ a basis of U. Recall $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ where $\mathcal{K},\mathcal{O}\ldots$

Definition 6 (Beilinson–Drinfeld loop Grassmannians). Denoted ${\mathscr{G}r_n^{\mathrm{BD}}}_{C^{(n)}}$ with C a smooth curve (or formal neighbourhood of a finite subset thereof) and $C^{(n)}$ its nth symmetric power. It is a reduced ind-scheme ${\mathscr{G}r_n^{\mathrm{BD}}}_{C^{(n)}} \to C^{(n)}$ with fibres of C-lattices ${\mathscr{G}r_n^{\mathrm{BD}}}_b = \{(b, \mathcal{L}) : b \in C^{(n)}\}$ made up of vector bundles such that $\mathcal{L} \cong U \otimes \mathcal{O}_C$ off b (i.e. over $C - \underline{b}$). The standard lattice is the pair $(\varnothing, \mathcal{L}_0)$ with $\mathcal{L}_0 = U \otimes \mathcal{O}_C$.

Not sure what $\mathcal{O}_C$ means

Notation

The case $n = 1$**.** Fix $b \in C$ and t a choice of formal parameter. Then ${\mathscr{G}r_n^{\mathrm{BD}}}_b \cong \mathrm{Gr}$.

Why is this called "its group-theoretic realization"

Furthermore, in this case, C-lattices $(b, \mathcal{L})$ are identified with $\mathcal{O}$-submodules $L = \Gamma(\hat{b}, \mathcal{L})$ of $U_{\mathcal{K}} = U \otimes \mathcal{K}$ such that $L \otimes_{\mathcal{O}} \mathcal{K} \cong U_{\mathcal{K}}$.

Under this identification, we associate to a given $\lambda \in \mathbb{Z}^m$ the lattice (a priori a $\mathcal{O}$-submodule) $L_\lambda = \oplus_1^m t^{\lambda_i} e_i \mathcal{O}$. Nb. our lattices will be contained in the standard lattice L_0 whereas MVy's lattices contain.

Connected components of Gr are

$G(\mathcal{O})$-orbits are indexed by coweights $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ of G. In terms of lattices

$$\mathrm{Gr}^\lambda = \left\{ L \supset L_0 \,\middle|\, t\big|_{L/L_0} \in \mathbb{O}_\lambda \right\} \tag{2}$$

in the connected component Gr_N are indexed

[**?**] define a map

$$\mathscr{G}r_n^{\mathrm{BD}} \to \mathrm{Gr}_n \tag{3}$$

- Their slice T_x or T_λ
- Their embedding $T_x \to \mathfrak{G}_N$
- N-dim D
- The map $\tilde{\mathbf{m}} : \tilde{\mathfrak{g}}^n \to \mathrm{End}(D)$
- The map $\mathbf{m} : \tilde{\mathcal{N}}^n \to \mathcal{N}$ sending $(x, F_\bullet)$ to x
- The map $\pi : \tilde{\mathfrak{G}}^n \to \mathfrak{G}$ sending $\mathcal{L}_\bullet$ to $\mathcal{L}_n$

The special case $b = \vec{0}$. In this case 0 in the affine quiver variety goes to the point L_λ in the affine Grassmannian, and the preimage of zero in the smooth quiver variety (= the core?) is identified with the preimage of L_λ in the BD Grassmannian.

$$\begin{array}{ccc} \mathfrak{L}(\vec{v}, \vec{w}) & \longrightarrow & \pi^{-1}(L_\lambda) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & L_\lambda \end{array}$$

MVy write: "we believe that one should be able to generalize this to arbitrary b" and that's where we come in!

Recall the Mirković–Vybornov immersion [**?**, Theorems 1.2 and 5.3].

Theorem 1. *([**?**, Theorem 1.2 and 5.3]) There exists an algebraic immersion* $\tilde{\psi}$

$$\widetilde{\mathbf{m}}^{-1}(T_\lambda) \cap \tilde{\mathfrak{g}}^{n,a,E,\tilde{\mu}} \xrightarrow{\tilde{\psi}} \tilde{\mathfrak{G}}_b^{n,a}(P)$$

5 Statements and Proofs of Results

Anne: Maybe split for now into a Notation section and a Proofs section

Define

$$S_{\mu_1,\mu_2} = N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2}$$

and

$$W_\mu = G_1[[t^{-1}]]t^\mu.$$

Let $|\lambda| = |\lambda_1 + \lambda_2|$ and $|\mu| = |\mu_1 + \mu_2|$.

Anne: Why not $\lambda = \lambda_1 + \lambda_2$ and recall $|\nu|$ in general.

Lemma 1 (Proof in Proposition 2.6 of KWWY)**.** *Suppose μ is dominant. Then*

$$N((t^{-1}))t^\mu = N_1[[t^{-1}]]t^\mu.$$

Lemma 2. *For dominant μ_1, μ_2, we have*

$$S_{\mu_1,\mu_2} \subset W_{\mu_1+\mu_2}.$$

Proof. We have

$$\begin{aligned}
S_{\mu_1,\mu_2} &= N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&\subset T_1[[t^{-1}]]N((t^{-1}))t^{\mu_1}(t-s)^{\mu_2} \\
&= T_1[[t^{-1}]]N_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} \\
&= B_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&\subset G_1[[t^{-1}]]t^{\mu_1+\mu_2} \\
&= W_{\mu_1+\mu_2}
\end{aligned}$$

where $B_1[[t^{-1}]]t^{\mu_1}(t-s)^{\mu_2} = B_1[[t^{-1}]]t^{\mu_1+\mu_2}$ since

$$\frac{t}{t-s} = 1 + \frac{s}{t} + \frac{s^2}{t^2} + \cdots \in B_1[[t^{-1}]].$$

□

Define $\mathrm{Gr}^{\lambda_1,\lambda_2} \subset \mathrm{Gr}_{BD}$ to be the family with generic fibre $\mathrm{Gr}^{\lambda_1} \times \mathrm{Gr}^{\lambda_2}$ and 0-fibre $\mathrm{Gr}^{\lambda_1+\lambda_2}$.

Define $\mathbb{O}_{\lambda_1,\lambda_2}$ to be matrices X of size $|\lambda| \times |\lambda|$ such that

$$X\big|_{E_0} \in \mathbb{O}_{\lambda_1} \text{ and } (X - sI)\big|_{E_s} \in \mathbb{O}_{\lambda_2}$$

Let

$$\mu = (\mu^{(1)}, \mu^{(2)}, ..., \mu^{(n)}).$$

Define $\mathbb{T}_{\mu_1,\mu_2}$ to be $|\mu| \times |\mu|$ matrices X such that X consists of block matrices where the size of the i-th diagonal block is $|\mu^{(i)}| \times |\mu^{(i)}|$, for $1 \leq i \leq n$. Each diagonal block is the companion matrix for $t^{\mu_1}(t-s)^{\mu_2}$. Each off-diagonal block is zero everywhere except possibly in the last $\min(\mu_i, \mu_j)$ columns of the last row.

Theorem 2. *We have an isomorphism*

$$\overline{\mathrm{Gr}^{\lambda_1,\lambda_2}} \cap S_{\mu_1,\mu_2} \cong \overline{\mathbb{O}_{\lambda_1,\lambda_2}} \cap \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}.$$

Anne: Rather, corollary?

Proof. We will prove this similarly to how the usual Mirković–Vybornov isomorphism is proven.

Step 1: Define a map $\mathbb{T}_{\mu_1,\mu_2} \cap \mathcal{N} \to G_1[t^{-1},(t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$.

$$A \mapsto t^{\mu_1}(t-s)^{\mu_2} + a(t,t-s) \mapsto (L_1 \subset L_2) : (t-s)\big|_{L_2/L_1} = A\big|_{E_s}, t\big|_{L_1/L_0} = A\big|_{E_0}$$

Question: 1. is the middle matrix similar to a block matrix? 2. is the composition of these maps some intermediate level of MVy's ψ's

BD Gr as lattices? $(L_1, L_2) \in \mathrm{Gr} \times \mathrm{Gr}$ corresponds to L such that $L \otimes \mathbb{C}[\![t]\!] \cong L_1 \otimes \mathbb{C}[\![t]\!]$ and $L \otimes \mathbb{C}[\![t-s]\!] \cong L_2 \otimes \mathbb{C}[\![t-s]\!]$ where $\otimes = \otimes_{\mathbb{C}[t]}$ or $\otimes_{\mathbb{C}[t-s]}$ respectively even though Roger believes $\mathbb{C}[t] = \mathbb{C}[t-s]$.

Proof of Step 1: Let $\mu_1 = (\mu_{1,1}, \mu_{1,2}, ..., \mu_{1,n})$, $\mu_2 = (\mu_{2,1}, ..., \mu_{2,n})$, and $\mu = (\mu^{(1)}, \mu^{(2)}, ..., \mu^{(n)})$. Consider the matrix

$$A = \left[\begin{array}{c|c|c|c|c} C_1 & a_{12}^1 \ \ldots \ a_{12}^{\mu^{(2)}} & \cdots & \cdots & a_{1n}^1 \ \ldots \ a_{1n}^{\mu^{(n)}} \\ \hline & C_2 & a_{23}^1 \ \ldots \ a_{23}^{\mu^{(3)}} & \cdots & \vdots \\ \hline & & \ddots & \ddots & \vdots \\ \hline & & & C_{n-1} & a_{n-1,n}^1 \ \ldots \ a_{n-1,n}^{\mu^{(n)}} \\ \hline & & & & C_n \end{array}\right]$$

where C_i is the $\mu^{(i)} \times \mu^{(i)}$ companion matrix of $t^{\mu_{1,i}}(t-s)^{\mu_{2,i}}$. We send this matrix to the matrix

$$\begin{bmatrix} t^{\mu_{1,1}}(t-s)^{\mu_{2,1}} & & & \\ -\sum_{k=1}^{\mu^{(2)}} a_{12}^k t^{k-1} & t^{\mu_{1,2}}(t-s)^{\mu_{2,2}} & & \\ \vdots & \ddots & \ddots & \\ -\sum_{k=1}^{\mu^{(n)}} a_{1n}^k t^{k-1} & \cdots & -\sum_{k=1}^{\mu^{(n)}} a_{n-1,n}^k t^{k-1} & t^{\mu_{1,n}}(t-s)^{\mu_{2,n}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ p_{1,2}(t) & 1 & & \\ \vdots & \ddots & \ddots & \\ p_{1,n}(t) & \cdots & p_{n-1,n}(t) & 1 \end{bmatrix} \begin{bmatrix} t^{\mu_{1,1}}(t-s)^{\mu_{2,1}} & & \\ & \ddots & \\ & & t^{\mu_{1,n}}(t-s)^{\mu_{2,n}} \end{bmatrix}$$

where

$$p_{i,j}(t) = \frac{-\sum_{k=1}^{\mu^{(j)}} a_{i,j}^k t^{k-1}}{t^{\mu_{1,i}}(t-s)^{\mu_{2,i}}}$$

As μ_1 and μ_2 are dominant, we have $p_{i,j}(t) \to 0$ as $t \to \infty$ so this matrix is indeed in $G_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$.

Step 2: If $A \in \mathbb{T}_{\mu_1,\mu_2} \cap \mathfrak{n}$ then A is sent to $(N_-)_1[t^{-1}, (t-s)^{-1}]t^{\mu_1}(t-s)^{\mu_2}$. Anne: Requires MVyBD!

Step 3: Conversely, given $L \in W_{\mu_1+\mu_2}$, want to show surjectivity.

$\square$

Last meeting's todos:

- make sure that the image of our map is in the G_1 orbit
- more generally, define the map, check that the map is well-defined
- Anne: say what little a is, i.e. insert the MVy theorem as stated in CK, or thesis
- Roger: check it

6 Examples

Example 3. $\lambda_1 = (1,0,0)$, $\lambda_2 = (1,1,0)$, $\mu_1 = (0,1,0)$, $\mu_2 = (1,0,1)$. Joel: $\mu = \mu_1 + \mu_2$ determines the blocks we have on the RHS of the BD MVy isomorphism of Equation **??**.

In the non-BD case, MVy establish

$$\overline{\mathrm{Gr}^{\lambda}} \cap \mathcal{W}^{\mu} \to \left\{ X = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * \\ \hline 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * \\ \hline * & 0 & * & 0 & * \end{array}\right] \middle| X \in \overline{\mathbb{O}}_{\lambda} \right\}$$

In the BD case the RHS will consist of the same block like matrices X but now having eigenvalues $s, 0$ such that $X - s\big|_{E_s} \in \mathbb{O}_{\lambda_2}$

Example 4. Do Joel's exercise: It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for $\mathbf{SL}_3$ of weights $2\alpha_1$ and $2\alpha_2$. Following the notation from the mvbasis paper, this would correspond to the following multiplication: $x^2y^2 = (xy - z)^2 + 2(xy - z)z + z^2$. For this example, I think we need $\lambda_1 = (2,0,0)$, $\lambda_2 = (2,2,0)$, $\mu_1 = (0,2,0)$, $\mu_2 = (2,0,2)$.

Example 5. Let

$$\begin{aligned} \mu_1 &= (2) \quad \lambda_1 = \quad \mu = (5) \\ \mu_2 &= (3) \quad \lambda_2 = \quad \lambda = \end{aligned}$$

Consider the companion matrix $C(p)$ of

$$p(t) = (t-s)^3 t^2 = (t^3 - 3t^2 s + 3ts^2 - s^3)t^2 = t^5 - 3t^4 s + 3t^3 s^2 - t^2 s^3$$

Let $X = C(p)^T$ so

$$X = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 0 & 0 & s^3 & -3s^2 & 3s \end{bmatrix}$$

Ask that $X\big|_{E_0}$ has Jordan type λ_1 and $X - s\big|_{E_s}$ has Jordan type λ_2. In this rank 1 case we are forced to take $\lambda_i = \mu_i$.

So what are the generalized eigenspaces E_i $(i = 1, 2)$? Note $\dim E_0 = 2$ and $\dim E_s = 3$.

Anne: The basis

$$[1], [t], [\mathbf{1}], [\mathbf{t}], [\mathbf{t^2}]$$

with $t[t]=0$ and $t[\mathbf{t^2}]=s^3[\mathbf{1}]-3s^2[\mathbf{t}]+3s[\mathbf{t^2}]$ *is not the correct basis to consider.* Hence my confusion of yore: what we would like is $t[t]=0$ no? what the matrix is telling us is that $t[t]=[\mathbf{1}]$. Can we still speak of *two* generalized eigenspaces?

Rather, take B to be the basis $b_1=e_1$, $b_2=e_2$, $b_5=e_5$, $b_4=Xe_5$, $b_3=X^2e_5$). In this basis

$$X_B=[X(b_i)]=\begin{bmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&s&1&0\\0&0&0&s&1\\0&0&0&0&s\end{bmatrix}$$

Example 6. Let

$$\begin{matrix}\mu_1=(3,1,1) & \lambda_1=(3,2,0) & \mu=(3,3,1)\\ \mu_2=(0,2,0) & \lambda_2=(2,0,0) & \lambda=(5,2,0)\end{matrix}$$

and consider the companion matrices of

$$p_1(t)=t^3 \quad p_2(t)=t(t-s)^2=t^3-2st^2+s^2t \quad p_3(t)=t$$

$$X=\left[\begin{array}{ccc:ccc:c}0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&d&e&f&g\\\hdashline 0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&-s^2&2s&k\\\hdashline 0&0&0&0&0&0&0\end{array}\right]$$

Example 7. Let

$$\begin{matrix}\lambda_1=(3,2,0) & \mu_1=(3,1,1)\\ \lambda_2=(2,0,0) & \mu_2=(1,1,0)\end{matrix}$$

so the first MV cycle $Z_1\cong\mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0,\alpha_1\}$ and the second MV cycle $Z_2\cong\mathbb{P}^1$ has MV polytope $\mathrm{Conv}\{0,\alpha_2\}$. Their fusion product corresponds to two $\mathbb{P}^2$'s intersecting along a $\mathbb{P}^1$. That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where $Z_+ \cong Z_- \cong \mathbb{P}^2$. We have

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & a & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix}$$

The 0-generalized eigenspace E_0 of X is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, Xe_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in $\ker X$, either $a = 0$ or $c = d = 0$, but the latter case cannot give a 2-cycle as $e_7 \notin \operatorname{im} X$. Then $a = 0$ and we obtain a 2-cycle

$$\left\{X\left(e_6 - \frac{s}{d}e_7\right), e_6 - \frac{s}{d}e_7\right\} = \left\{e_5, e_6 - \frac{s}{d}e_7\right\}.$$

We also obtain the equations $b \neq 0$, $d \neq 0$, and $sc - bd = 0$ from this.

For the s-generalized eigenspace E_s, we need $a + sb \neq 0$ to obtain a 2-cycle, which can be taken as

$$\begin{aligned} &\left\{(X - sI)\left(e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right),\right. \\ &\qquad\qquad \left. e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right\} \\ = &\left\{e_1 + se_2 + s^2e_3 + s^3e_4, e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6\right\} \end{aligned}$$

The minimal polymonial is $X^3(X - sI)^2$, which when equated to 0 gives again the equation $cs - bd = 0$. Thus the defining equations are

$$\{a = 0, cs - bd = 0\}.$$

When we take $s = 0$, we get the equations

$$\{a = 0, bd = 0\}$$

which corresponds to two $\mathbb{A}^2$'s intersecting along an $\mathbb{A}^1$. This is indeed an open subset of $\mathbb{P}^2 \cup_{\mathbb{P}_1} \mathbb{P}^2$, as required.

Example 8 (Example 7 continued...). The matrix X from the previous example defines (under MVy) the matrix

$$g = \begin{bmatrix} (t-s)t^3 & & \\ -bt & (t-s)t & \\ -c & -d & t \end{bmatrix}$$

in $G(\mathcal{O})$. Indeed the various blocks of X are in a precise sense the companion matrices of the polynomial entries of g

In Gr the element g defines the lattice

$$gL_0 = \mathbb{C}[t]\langle (t-s)t^3e_1 - (a+bt)e_2 - ce_3, (t-s)te_2 - de_3, te_3\rangle$$

and computing the matrix of the action of t on the quotient L_0/L in the basis

$$\{[e_1], [te_1], [t^2e_1], [t^3e_1], [e_2], [te_2], [e_3]\}$$

recovers X up to a transpose of course.

Now let's see what we get when we invert t and $t-s$ respectively.

First let's invert t by considering $L_2 = L \otimes \mathbb{C}[\![t-s]\!]$.

$$L_2 = \mathbb{C}[t, t^{-1}]\langle (t-s)e_1 - \frac{a+bt}{t^3}e_2 - \frac{c}{t^3}e_3, (t-s)e_2 - \frac{d}{t}e_3, e_3\rangle$$

so in L_0/L_2 we have

$$t[e_1] = s[e_1] + \frac{a+bt}{t^3}[e_2] + \frac{c}{t^3}[e_3] \qquad t[e_2] = s[e_2] + \frac{d}{t}[e_3] \qquad [e_3] = 0$$

and

$$\left[t\big|_{L_0/L_2}\right]_{\{[e_1],[e_2]\}} = \begin{bmatrix} s & \\ \frac{a+bt}{t^3} & s \end{bmatrix}$$

which upon subtracting sI gives a matrix having block type μ_2 and Jordan type $\lambda_2 = (2)$ assuming $\frac{a+bt}{t^3} \neq 0$.

Next let's invert $t-s$ by considering $L_1 = L \otimes \mathbb{C}[\![t]\!]$.

$$\begin{aligned}
L_1 &= \mathbb{C}[t, (t-s)^{-1}]\langle t^3e_1 - \frac{a+bt}{t-s}e_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle \\
&= \langle t^3e_1 - \frac{a}{t-s}e_2 - \frac{b}{t-s}te_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3\rangle
\end{aligned}$$

so in L_0/L_1 we have

$$\begin{aligned}
t[e_1] &= [te_1] \\
t[te_1] &= [t^2e_1] \\
t[t^2e_1] &= \frac{a}{t-s}[e_2] + \frac{b}{t-s}t[e_2] + \frac{c}{t-s}[e_3] \\
&= \frac{b}{t-s}\frac{d}{t-s}[e_3] + \frac{c}{t-s}[e_3] \\
&= \frac{bd + (t-s)c}{(t-s)^2}[e_3] \\
&= \frac{bd - sc}{(t-s)^2}[e_3] + \frac{c}{(t-s)^2}t[e_3] = 0 \\
t[e_2] &= \frac{d}{t-s}[e_3] \\
t[e_3] &= 0
\end{aligned}$$

and

$$\left[t\big|_{L_0/L_1}\right]_{\{[e_1],[te_1],[t^2e_1],[e_2],[e_3]\}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & & 0 & 0 & \\ & & 0 & \frac{d}{t-s} & 0 \end{bmatrix}$$

taking transpose

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}$$

which has block type μ_1 and Jordan type $\lambda_1 = (3,2)$ assuming $d \neq 0$.

I have used the relations Roger found (and I checked) $a = 0$ and $cs - bd = 0$ in the calculations above.

To sum up, the pair of matrices above should contain the same information as the matrix from the previous example

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix} \Leftrightarrow \left(\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} s & \frac{b}{t^2} \\ & s \end{bmatrix} \right)$$

Wish. Given L in $\mathcal{G}r_n^{\mathrm{BD}}$ define a map to T_μ just like MVy by taking $[t\big|_{L_0/L}]$ and use the fact that $[t\big|_{L_0/L_i}]$ for $i = 1,2$ are companion matrices of the right type, piece together two MVy isomorphisms to make a BD MVy iso.

Equivalent lin alg question: If $p(t, t-s) = p_1(t)p_2(t-s)$ then how are $C(p_1)$, $C(p_2)$, and $C(p)$ related? I think it's basically this theorem `https://en.wikipedia.org/wiki/Structure_theorem_for_finitely_generated_modules_over_a_principal_ideal_domain`