

# Examples Compendium

Roger Bai, Anne Dranowski

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## Disjoint, non-dominant $\mu$

*Example 1.*  $\lambda_1 = (1, 0, 0)$ ,  $\lambda_2 = (1, 1, 0)$ ,  $\mu_1 = (0, 1, 0)$ ,  $\mu_2 = (1, 0, 1)$ . **Joel:**  $\mu = \mu_1 + \mu_2$  determines the blocks we have on the RHS of the BD MVy isomorphism.

$$\left[ \begin{array}{c|c|c} s & A_0 & A_1 \\ \hline 0 & 0 & A_2 \\ \hline 0 & 0 & s \end{array} \right]$$

## Some multiplicity

*Example 2* (Joel's exercise). It would be good to do an example where we will see some multiplicity in the fusion product. I think that the simplest example would be the fusion product of the MV cycles for  $\mathbf{SL}_3$  of weights  $2\alpha_1$  and  $2\alpha_2$ . Following the notation from the mvbasis paper, this would correspond to the following multiplication:

$$x^2y^2 = (xy - z)^2 + 2(xy - z)z + z^2.$$

For this example, I think we need  $\lambda_1 = (2, 0, 0)$ ,  $\lambda_2 = (2, 2, 0)$ ,  $\mu_1 = (0, 2, 0)$ ,  $\mu_2 = (2, 0, 2)$ . Then

$$\left[ \begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -s^2 & 2s & A_0 & A_1 & A_2 & A_3 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -s^2 & 2s \end{array} \right]$$

Let's also try

$$\begin{aligned}\lambda_2 &= (4, 0, 0) & \lambda_1 &= (4, 4, 0) & \lambda &= (8, 4, 0) \\ \mu_2 &= (2, 2, 0) & \mu_1 &= (4, 2, 2) & \mu &= (6, 4, 2)\end{aligned}$$

Then

$$\left[ \begin{array}{cccccc|cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s^2 & 2s & A_6 & A_7 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Note that there is only one SSYT of shape  $\lambda_i$  and weight  $\mu_i$

$$\tau_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \end{array} \quad \tau_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ \hline \end{array}$$

Note also that

$$\begin{aligned}t^4(t-s)^2 &= t^4(t^2 - 2st + s^2) = t^6 - 2st^5 + s^2t^4 \\ t^2(t-s)^2 &= t^4 - 2st^3 + s^2t^2\end{aligned}$$

Elements of  $\mathbb{T}_{\mu_1, \mu_2}^+$  will take the form

$$A = \left[ \begin{array}{cccc|cccc|cc} & 1 & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & 1 & & & & & \\ & & & & -s^2 & 2s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\ \hline & & & & & & 1 & & & & & \\ & & & & & & & 1 & & & & \\ & & & & & & & & 1 & & & \\ & & & & & & & -s^2 & 2s & c_1 & c_2 & \\ \hline & & & & & & & & & & 1 & \end{array} \right]$$

The tableau tells us for each  $1 \leq i \leq 12$

Jordan type of  $A|_{\text{Span}(e_1, \dots, e_i) \cap E_0}$  is shape of  $\tau_1|_{\text{first } i \text{ boxes}}$

So take  $i = 10$ . Then  $A$  restricted to  $\mathbb{C}^{10} \cap E_0$  should have Jordan type  $(4, 2)$ .  
[Anne: How to do it box by box?](#)  $s$  columns somehow correspond to  $\tau_2$  boxes.  
 Therefore  $a_1 = 0$ . The 4-cycle is obvious. For the 2-cycle we require  $a_2 = 0$ .

$$\begin{aligned} e_1 &\leftarrow e_2 \leftarrow e_3 \leftarrow e_4 \\ e_7 &\leftarrow e_8 \end{aligned}$$

Now looking at all of  $A$  we have continue our 2-cycle to a 4-cycle. Roger found

$$e_{10} - \left( \frac{2s}{c_1} + \frac{s^2 c_2}{c_1^2} \right) e_{11} + \frac{s^2}{c_1} e_{12}$$

This requires  $c_1^2 a_4 + c_1 b_2 s^2 - 2s c_1 b_1 - s^2 c_2 = 0$  and  $a_3 c_1 + s^2 b_1 = 0$ .

Now looking for the  $s$ -eigenspace, we expect  $A - s|_{\mathbb{C}^6 \cap E_s}$  to have Jordan type 2. The kernel is spanned by  $e_1 + s e_2 + s^2 e_3 + s^3 e_4 + s^4 e_5 + s^5 e_6$ . It is continued to a 2-cycle by/the 2-cycle is generated by  $-\frac{5}{s} e_1 - 4e_2 - 3s e_3 - 2s^2 e_4 - s^3 e_5$ . The 3-cycle is maybe  $(1/s^2, -4/s, -8, 5s, 3s^2, 2s^3, -s^2/a_3, -s^3/a_3, -s^4/a_3, -s^5/a_3)$  padded with zeros.

$$A - s = \left[ \begin{array}{cccc|cccc|cc} -s & 1 & & & & & & & & \\ & -s & 1 & & & & & & & \\ & & -s & 1 & & & & & & \\ & & & -s & 1 & & & & & \\ & & & & -s & 1 & & & & \\ & & & & & -s^2 & s & 0 & 0 & a_3 & a_4 & b_1 & b_2 \\ \hline & & & & & & & -s & 1 & & & & \\ & & & & & & & & -s & 1 & & & \\ & & & & & & & & & -s & 1 & & \\ & & & & & & & & & & -s^2 & s & c_1 & c_2 \\ \hline & & & & & & & & & & & -s & 1 \\ & & & & & & & & & & & & -s \end{array} \right]$$

## Simple root weights, things working

*Example 3.* Let

$$\begin{aligned} \mu_1 &= (3, 1, 1) & \lambda_1 &= (3, 2, 0) & \mu &= (3, 3, 1) \\ \mu_2 &= (0, 2, 0) & \lambda_2 &= (2, 0, 0) & \lambda &= (5, 2, 0) \end{aligned}$$

and consider the companion matrices of

$$p_1(t) = t^3 \quad p_2(t) = t(t-s)^2 = t^3 - 2st^2 + s^2t \quad p_3(t) = t$$

$$X = \left[ \begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e & f & g \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -s^2 & 2s & k \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

*Example 4.* Let

$$\begin{aligned}\lambda_1 &= (3, 2, 0) & \mu_1 &= (3, 1, 1) \\ \lambda_2 &= (2, 0, 0) & \mu_2 &= (1, 1, 0)\end{aligned}$$

so the first MV cycle  $Z_1 \cong \mathbb{P}^1$  has MV polytope  $\text{Conv}\{0, \alpha_1\}$  and the second MV cycle  $Z_2 \cong \mathbb{P}^1$  has MV polytope  $\text{Conv}\{0, \alpha_2\}$ . Their fusion product corresponds to two  $\mathbb{P}^2$ 's intersecting along a  $\mathbb{P}^1$ . That is, we have the fusion product

$$Z_1 * Z_2 = Z_+ + Z_-$$

where  $Z_+ \cong Z_- \cong \mathbb{P}^2$ . We have

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & a & b & c \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix}$$

The 0-generalized eigenspace  $E_0$  of  $X$  is 5-dimensional, containing a 3-cycle and a 2-cycle. The 3-cycle is

$$\{X^2 e_3, X e_3, e_3\} = \{e_1, e_2, e_3\}.$$

To obtain another vector in  $\ker X$ , either  $a = 0$  or  $c = d = 0$ , but the latter case cannot give a 2-cycle as  $e_7 \notin \text{im } X$ . Then  $a = 0$  and we obtain a 2-cycle

$$\left\{X \left(e_6 - \frac{s}{d} e_7\right), e_6 - \frac{s}{d} e_7\right\} = \left\{e_5, e_6 - \frac{s}{d} e_7\right\}.$$

We also obtain the equations  $b \neq 0$ ,  $d \neq 0$ , and  $sc - bd = 0$  from this.

For the  $s$ -generalized eigenspace  $E_s$ , we need  $a + sb \neq 0$  to obtain a 2-cycle, which can be taken as

$$\begin{aligned} & \left\{ (X - sI) \left( e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right), \right. \\ & \quad \left. e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right\} \\ &= \left\{ e_1 + se_2 + s^2e_3 + s^3e_4, e_2 + 2se_3 + 3s^2e_4 + \frac{s^3}{a+sb}e_5 + \frac{s^4}{a+sb}e_6 \right\} \end{aligned}$$

The minimal polynomial is  $X^3(X - sI)^2$ , which when equated to 0 gives again the equation  $cs - bd = 0$ . Thus the defining equations are

$$\{a = 0, cs - bd = 0\}.$$

When we take  $s = 0$ , we get the equations

$$\{a = 0, bd = 0\}$$

which corresponds to two  $\mathbb{A}^2$ 's intersecting along an  $\mathbb{A}^1$ . This is indeed an open subset of  $\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ , as required.

*Example 5* (Continued...). The matrix  $X$  from the previous example defines (under MVy) the matrix

$$g = \begin{bmatrix} (t-s)t^3 & & \\ -bt & (t-s)t & \\ -c & -d & t \end{bmatrix}$$

in  $G(\mathcal{O})$ . Indeed the various blocks of  $X$  are in a precise sense the companion matrices of the polynomial entries of  $g$

In Gr the element  $g$  defines the lattice

$$gL_0 = \mathbb{C}[t]\langle (t-s)t^3e_1 - (a+bt)e_2 - ce_3, (t-s)te_2 - de_3, te_3 \rangle$$

and computing the matrix of the action of  $t$  on the quotient  $L_0/L$  in the basis

$$\{[e_1], [te_1], [t^2e_1], [t^3e_1], [e_2], [te_2], [e_3]\}$$

recovers  $X$  up to a transpose of course.

Now let's see what we get when we invert  $t$  and  $t-s$  respectively.

First let's invert  $t$  by considering  $L_2 = L \otimes \mathbb{C}[[t-s]]$ .

$$L_2 = \mathbb{C}[t, t^{-1}]\langle (t-s)e_1 - \frac{a+bt}{t^3}e_2 - \frac{c}{t^3}e_3, (t-s)e_2 - \frac{d}{t}e_3, e_3 \rangle$$

so in  $L_0/L_2$  we have

$$t[e_1] = s[e_1] + \frac{a+bt}{t^3}[e_2] + \frac{c}{t^3}[e_3] \quad t[e_2] = s[e_2] + \frac{d}{t}[e_3] \quad [e_3] = 0$$

and

$$\left[ t|_{L_0/L_2} \right]_{\{[e_1], [e_2]\}} = \begin{bmatrix} s & \\ \frac{a+bt}{t^3} & s \end{bmatrix}$$

which upon subtracting  $sI$  gives a matrix having block type  $\mu_2$  and Jordan type  $\lambda_2 = (2)$  assuming  $\frac{a+bt}{t^3} \neq 0$ .

Next let's invert  $t-s$  by considering  $L_1 = L \otimes \mathbb{C}[[t]]$ .

$$\begin{aligned} L_1 &= \mathbb{C}[t, (t-s)^{-1}]\langle t^3e_1 - \frac{a+bt}{t-s}e_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3 \rangle \\ &= \langle t^3e_1 - \frac{a}{t-s}e_2 - \frac{b}{t-s}te_2 - \frac{c}{t-s}e_3, te_2 - \frac{d}{t-s}e_3, te_3 \rangle \end{aligned}$$

so in  $L_0/L_1$  we have

$$\begin{aligned}
t[e_1] &= [te_1] \\
t[te_1] &= [t^2e_1] \\
t[t^2e_1] &= \frac{a}{t-s}[e_2] + \frac{b}{t-s}t[e_2] + \frac{c}{t-s}[e_3] \\
&= \frac{b}{t-s}\frac{d}{t-s}[e_3] + \frac{c}{t-s}[e_3] \\
&= \frac{bd + (t-s)c}{(t-s)^2}[e_3] \\
&= \frac{bd - sc}{(t-s)^2}[e_3] + \frac{c}{(t-s)^2}t[e_3] = 0 \\
t[e_2] &= \frac{d}{t-s}[e_3] \\
t[e_3] &= 0
\end{aligned}$$

and

$$[t|_{L_0/L_1}]_{\{[e_1], [te_1], [t^2e_1], [e_2], [e_3]\}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & & 0 & 0 & \\ & & 0 & \frac{d}{t-s} & 0 \end{bmatrix}$$

taking transpose

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}$$

which has block type  $\mu_1$  and Jordan type  $\lambda_1 = (3, 2)$  assuming  $d \neq 0$ .

I have used the relations Roger found (and I checked)  $a = 0$  and  $cs - bd = 0$  in the calculations above.

To sum up, the pair of matrices above should contain the same information as the matrix from the previous example

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & s & b & c & \\ & & & & 0 & 1 & 0 \\ & & & & 0 & s & d \\ & & & & & & 0 \end{bmatrix} \Leftrightarrow \left( \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \frac{d}{t-s} \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} s & \frac{b}{t^2} \\ & s \end{bmatrix} \right)$$

**Wish.** Given  $L$  in  $\mathcal{G}_n^{\text{BD}}$  define a map to  $T_\mu$  just like MVy by taking  $[t|_{L_0/L}]$  and use the fact that  $[t|_{L_0/L_i}]$  for  $i = 1, 2$  are companion matrices of the right type, piece together two MVy isomorphisms to make a BD MVy iso.

Equivalent linear algebra question(?): If  $p(t, t-s) = p_1(t)p_2(t-s)$  then how are  $C(p_1)$ ,  $C(p_2)$ , and  $C(p)$  related?



## Non simple root weights

*Example 6* (Anne). Let  $G = \mathbf{SL}_3$  and  $\mathbf{i} = 121$ .

Take  $n_{\bullet}^1 = (1, 0, 0)$ , and  $n_{\bullet}^2 = (1, 0, 1)$  or  $(0, 1, 0)$ . So

$$\begin{aligned} \mu_1 &= (2, 2, 1) & \mu_2 &= (1, 1, 1) & \mu &= (3, 3, 2) \\ \lambda_1 &= (3, 1, 1) & \lambda_2 &= (2, 1, 0) & \lambda &= (5, 2, 1) \end{aligned}$$

Note

$$\tau(1, 0, 0) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array} \quad \tau(1, 0, 1) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \tau(0, 1, 0) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$$

Anne: We should show that order does not matter; i.e. swapping indices on  $\lambda$ 's and  $\mu$ 's produces the same result.

$\mathbb{T}_{\mu_1, \mu_2}^+ \cap \mathbb{O}_{\lambda_1, \lambda_2}$  is made up of elements of the form

$$A = \left[ \begin{array}{ccc|ccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & A_0 & A_1 & A_2 & A_3 & A_4 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & A_5 & A_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \end{array} \right]$$

As usual, denote by  $E_e$  the generalized  $e$ -eigenspace of  $A$ .  $A|_{\mathbb{C}^3 \cap E_0}$  should have Jordan type (2). The obvious 2-cycle is generated by  $e_2$ :  $\{e_2, Ae_2\}$ .  $A|_{\mathbb{C}^3 \cap E_s}$  should have Jordan type (1). We take  $e_1 + se_2 + s^2e_3 \in \text{Ker}(A - s)$ . Next  $A|_{\mathbb{C}^6 \cap E_0}$  should have Jordan type (3, 1) while  $A|_{\mathbb{C}^6 \cap E_s}$  will have Jordan type (2) or (1, 1). Anne: This example breaks. Why? How should we choose weights?

Take 2: Let's try different weights.

$$\begin{aligned} \mu_1 &= (1, 1, 0) & \lambda_1 &= (2, 0, 0) \\ \mu_2 &= (1, 1, 1) & \lambda_2 &= (2, 1, 0) \\ \mu &= (2, 2, 1) & \lambda &= (4, 1, 0) \end{aligned}$$

and

$$\tau(1, 0, 0) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad \tau(1, 0, 1) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \tau(0, 1, 0) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$$

Then

$$A = \left[ \begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & s & A_0 & A_1 & A_2 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s & A_3 \\ \hline 0 & 0 & 0 & 0 & s \end{array} \right]$$

We have  $E_0 \cap \mathbb{C}^2 = \text{Span}(e_1)$ ,  $E_s \cap \mathbb{C}^2 = \text{Span}(e_1 + se_2)$ . Next  $E_0 \cap \mathbb{C}^4$  is spanned by a 2-cycle generated by  $-\frac{A_0}{s}e_2 + e_3$  and  $E_s \cap \mathbb{C}^4$  is spanned by a 2-cycle generated by

$$\frac{s}{A_1 + \frac{A_0}{s}}e_4 + \frac{1}{A_1 + \frac{A_0}{s}}e_3 - \frac{1}{s}e_1$$

or the additional 1-cycle  $e_4 - \frac{A_1}{A_0}e_3$  assuming  $A_0 + sA_1 = 0$ . Finally  $E_s$  is spanned by an additional 1-cycle

$$e_5 - \frac{A_2}{A_1 + \frac{A_0}{s}}e_4 + \frac{1}{s} \frac{A_2}{A_1 + \frac{A_0}{s}}e_3$$

assuming  $A_3 = 0$  and  $A_2 = 0$ . Or the two 1-cycles are extended to a 2-cycle and a 1-cycle, the 2-cycle generated by  $-\frac{1}{s}e_1 + \frac{s}{A_2}e_5$  and again assuming  $A_3 = 0$  in addition to  $A_0 + sA_1 = 0$ . This gives us

$$\boxed{1 \mid 2} * \boxed{\frac{1}{3} \mid 2} = (A_2, A_3) \quad \boxed{1 \mid 2} * \boxed{\frac{1}{2} \mid 3} = (A_0 + sA_1, A_3) \rightarrow (A_0, A_3)$$

Does it agree with what is expected on the module/cluster side?

$$S_1 * (1 \rightarrow 2) = S_1 \oplus (1 \rightarrow 2) \quad S_1 * (1 \leftarrow 2) = S_1 \oplus (1 \leftarrow 2)$$

The MV cycle of  $\boxed{1 \mid 2}$  is a  $\mathbb{P}^1$ : via MVy it has an open subset comprised of matrices

$$\left[ \begin{array}{c|c|c} 0 & A_0 & \\ \hline - & - & - \\ 0 & 0 & \end{array} \right] : A_0 \neq 0$$

The MV cycles of the other two tableaux are made up of matrices of the form

$$\left[ \begin{array}{c|c|c} 0 & A_0 & A_1 \\ \hline 0 & 0 & A_2 \\ \hline - & - & - \\ 0 & 0 & 0 \end{array} \right] : \begin{cases} A_0 \neq 0 \text{ and } A_2 = 0 & \tau = \boxed{\frac{1}{3} \mid 2} \\ A_0 = 0 \text{ and } A_2 \neq 0 & \tau = \boxed{\frac{1}{2} \mid 3} \end{cases}$$

both  $\mathbb{C}^2$ 's. [Anne: How do the coordinates relate?](#)

*Example 7* (Roger). Let

$$\begin{aligned}\lambda_1 &= (2, 0, 0, 0) & \mu_1 &= (1, 1, 0, 0) \\ \lambda_2 &= (2, 2, 1, 0) & \mu_2 &= (3, 2, 1, 1)\end{aligned}$$

so  $\lambda_1 - \mu_1 = \alpha_1$  and  $\lambda_2 - \mu_2 = \alpha_2 + \alpha_3$ . We have the following young tableaux:

$$\tau_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad \tau_2 = \begin{array}{|c|c|} \hline 1 & 1 \\ 2 & 3 \\ \hline 4 \\ \hline \end{array} \quad \tau'_2 = \begin{array}{|c|c|} \hline 1 & 1 \\ 2 & 4 \\ \hline 3 \\ \hline \end{array}$$

where  $\tau_1$  corresponds to the module  $S_1$ ,  $\tau_2$  corresponds to the module  $2 \rightarrow 3$ , and  $\tau'_2$  corresponds to the module  $2 \leftarrow 3$ .

The matrix we are considering is

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & -s^2 & 2s & a & b & c & d \\ & & & 0 & 1 & & \\ & & & & s & e & f \\ & & & & & s & g \\ & & & & & & s \end{bmatrix}$$

such that  $\dim E_0 = 2$ ,  $\dim \ker X = 1$ ,  $\dim E_s = 5$ , and  $\dim \ker(X - sI) = 3$  where  $E_0$  and  $E_s$  are the 0- and  $s$ -generalized eigenspaces.

We see that the two-cycle in  $E_0$  is

$$\left\{ X \left( e_2 + \frac{s^2}{a} e_4 \right), e_2 + \frac{s^2}{a} e_4 \right\} = \left\{ e_1, e_2 + \frac{s^2}{a} e_4 \right\}.$$

As  $\tau_2$  and  $\tau'_2$  both share  $\begin{array}{|c|c|} \hline 1 & 1 \\ 2 & \\ \hline \end{array}$ , we can find a 2-cycle from just the upper-left  $3 \times 3$  block, and an additional vector in  $\ker(X - sI)$  from the upper-left  $5 \times 5$ -submatrix. The 2-cycle from the  $3 \times 3$  block is

$$\left\{ e_1 + se_2 + s^2 e_3, -\frac{2}{s} e_1 - e_2 \right\}.$$

The additional vector in  $\ker(X - sI)$  is  $e_4 + se_5$  and this requires  $a + sb = 0$ .

Now consider the case that the young diagram we are working with is  $\tau_2$ . Then we have  $e_4 + se_5$  part of a 2-cycle that can be found by looking at the upper-left  $6 \times 6$ -submatrix. We find that the 2-cycle is

$$\left\{ e_4 + se_5, -\frac{1}{s} e_4 + \frac{s}{e} e_6 \right\}$$

and this requires that  $ae - s^2 c = 0$ .

The last vector in  $\ker(X - sI)$  comes from the entire  $X - sI$  and we see it is  $-fe_6 + ee_7$ , which requires  $g = 0$  and  $ed - cf = 0$ .

For the case  $\tau'_2$ , we start with find the third vector in  $\ker(X - sI)$  from the upper-left  $6 \times 6$ -submatrix. We see that it is  $e_6$ , which requires  $c = 0$  and  $e = 0$ .

For the remaining 2-cycle, we note that  $e_4 + se_5 \notin \text{col}(X - sI)$  but  $e_6 \in \text{col}(X - sI)$  so our 2-cycle is

$$\left\{ e_6, \frac{1}{g}e_7 \right\}$$

which requires  $d = 0$  and  $f = 0$ .

From the minimal polynomial, we have  $X^2(X - sI)^2 = 0$  which gives us the equations

$$a + sb = cs + eb = bf + cg + ds = esg = 0.$$

Taking  $s \rightarrow 0$ , we have the following equations for our two cases of  $\tau_2$  and  $\tau'_2$ :

$\tau_2$	$\tau'_2$
$a = 0$	$a = 0$
$g = 0$	$c = 0$
$eb = 0$	$d = 0$
$bf = 0$	$e = 0$
$ed - cf = 0$	$f = 0$

For the  $\tau_2$  case, the coordinate ring is

$$\frac{\mathbb{C}[a, b, c, d, e, f, g]}{\langle a, g, eb, bf, ed - cf \rangle} \cong \frac{\mathbb{C}[b, c, d, e, f]}{\langle eb, bf, ed - cf \rangle} = \frac{\mathbb{C}[b, c, d, e, f]}{\langle e, f \rangle \cap \langle b, ed - cf \rangle}$$

Hence the associated algebraic set is reducible with two irreducible components. The component corresponding to the ideal  $\langle e, f \rangle$  is  $\mathbb{A}^3$ , which corresponds to  $\mathbb{P}^3$ , while the ideal  $\langle b, ed - cf \rangle$  corresponds to the toric variety whose toric polytope is a square-based pyramid.

As  $\tau_2$  corresponds to the module  $2 \rightarrow 3$ , the irreducible components should correspond to the modules  $P_1 = 1 \rightarrow 2 \rightarrow 3$  and  $1 \leftarrow 2 \rightarrow 3$ . Indeed, the MV cycle corresponding to  $P_1$  is the Grassmannian  $Gr(1, 4) \cong \mathbb{P}^3$  and for  $1 \leftarrow 2 \rightarrow 3$ , we do get a toric variety with polytope the square-based pyramid.

However for the  $\tau'_2$  case, the coordinate ring is

$$\frac{\mathbb{C}[a, b, c, d, e, f, g]}{\langle a, c, d, e, f \rangle} \cong \mathbb{C}[b, g]$$

which corresponds to  $\mathbb{A}^2$ . Since  $\tau'_2$  corresponds to the module  $2 \leftarrow 3$ , we expect two irreducible components corresponding to the modules  $P_3 = 1 \leftarrow 2 \leftarrow 3$  and  $1 \rightarrow 2 \leftarrow 3$ .  $P_3$  corresponds to the variety  $Gr(3, 4) \cong \mathbb{P}^3$  and  $1 \rightarrow 2 \leftarrow 3$  also corresponds to a toric variety whose polytope is a square-based pyramid (?).

Example 8 (Above example redone). In  $\mathbb{C}[A_0..A_6]$  where

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s^2 & 2s & A_0 & A_1 & A_2 & A_3 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & A_4 & A_5 \\ \hline 0 & 0 & 0 & 0 & 0 & s & A_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & s \end{bmatrix}$$

we have

$$\begin{aligned} \begin{bmatrix} 1 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 \end{bmatrix} &= (A_6, A_2A_5 - A_3A_4, A_1A_4 + A_2s, A_1s + A_0) \\ &= (A_6, A_2A_5 - A_3A_4, A_1A_4, A_0) \text{ at } s = 0 \\ &= (A_0, A_4, A_5, A_6) \sqcup (A_0, A_1, A_6, A_2A_5 - A_3A_4) \sqcup (A_0, A_2, A_4, A_6) \end{aligned}$$

while

$$\begin{aligned} \begin{bmatrix} 1 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 \end{bmatrix} &= (sA_0A_5 + (1 + s^2)A_1A_5 + sA_3, A_2, A_4, A_1s + A_0) \\ &= (A_1A_5, A_2, A_4, A_0) \text{ at } s = 0 \\ &= (A_0, A_2, A_4, A_5) \sqcup (A_0, A_1, A_2, A_4) \end{aligned}$$

Cycles are

$$\frac{A_0}{s^2}e_2 + e_4 \xrightarrow{A} e_1$$

and, in the  $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$  case,

$$\begin{aligned} e_2 + 2se_3 &\xrightarrow{A-s} e_1 + se_2 + s^2e_3 \\ (sA_4e_4 + (1 + s^2)A_4e_5 + se_6)/(sA_4) &\xrightarrow{A-s} (e_4 + se_5)/s \end{aligned}$$

while in the  $\begin{bmatrix} 1 & 1 \\ 2 \\ 3 \end{bmatrix}$  case,

$$\begin{aligned} e_6 &\xrightarrow{A-s} 0 \\ e_4 + (1/s + s)e_5 + (1/A_5)e_7 &\xrightarrow{A-s} (1/s)e_4 + e_5 + *e_6 \end{aligned}$$