Number Theory - Day 1

Overview
- Look near me
- Logistics / information / etc.
- Prime factorization
- Divisibility tests
- "Factor problems"

108: 9, 12; 3, 3, 4, 3; 2, 2 $= 2^2 \cdot 3^3$

56: 7, 8; 4, 2; 2, 2 $= 7 \cdot 2^3 = 2^3 \cdot 7$

Divisibility Tests 2 – 11

2 – 0, 2, 4, 6, 8 as last digit

3 – 3 | (sum of digits)

$\frac{108}{3} = \frac{2^2 \cdot 3^3}{3} = 2^2 \cdot 3^2 = 4 \cdot 9 = 36$

↳ 36

$a \mid b$ → divides

$3 \mid 6 \Rightarrow 6/3 = 2$

$-2 \mid 6 \Rightarrow 6/2 = -3$

$4 \nmid 6 \Rightarrow 6/4 = 1.5$ ✗

108 ⇒ 1 + 0 + 8 = 9

3 | 9 ✓

1234567 → 26

3 | 28 → no

3 ∤ 1234567

4 – 4 | (last 2 digits)

$25 \cdot 4 = 100$

$26 \cdot 4 = 104$

1234726924

4 | 24

5 – if last digit is 0 or 5

6 – $6 = 2 \cdot 3$, check for 2 and 3 →

7 – /

8 – $125 \cdot 8 = 1000$

8 | 1269388008

2 – last digit, 4 – last 2 digits, 8 – last 3 digits

$2 \cdot 5 = 10$, $4 \cdot 25 = 100$, $8 \cdot 125 = 1000$

$16 \times 625 = 10000$ 2^8 → last 8 digits

9 – 9 | (sum of digits)

12366 → 1 + 2 + 3 + 6 + 6 = 18

9 | 18

10 – last digit is 0 or not

11 – $a\,b\,c\,d\,e\,f$

$\left|(a+c+e) - (b+d+f)\right|$

121 → 11^2

$|121| = |(1+1) - (2)| = |0| = 0$

$0 = 11 \cdot 0$

Theorem of Divisibility: if $a \mid n$, and $b \mid n$, then $ab \mid n$ (primes)

2 | 200, 5 | 200 → 2 · 5 = 10 | 200

2, 5, 7, 11 ...

6 | 12, 3 | 12 → 18 | 12 ✗

$k = a_1^{p_1} \cdot a_2^{p_2} \cdot a_3^{p_3} \cdots a_n^{p_n}$ →

$108 = 2^2 \cdot 3^3 \rightarrow a_1 = 2, p_1 = 2$

$a_2 = 3, p_2 = 3$

12 – 1, 2, 3, 4, 6, 12 = 6 factors of 12

$2^a \cdot 3^b$

$\underbrace{0, 1, 2, 3 \ldots a}_{(a+1)} \mid \underbrace{0, 1, 2, 3, 4 \ldots b}_{(b+1)}$

$(a+1)(b+1)$

$2^2 \cdot 3^3$

$(2+1)(3+1) = 3 \cdot 4 = 12 \Rightarrow$ 108 has 12 factors

$12 = 6 \cdot 2 = 2^2 \cdot 3$

$(2+1)(1+1) = 3 \cdot 2 = 6$

216 : 2 ✓, 3 → 2 + 1 + 6 (9) ✓

3 | 9

216 is divisible by 6

General form of # of factors

$k = a_1^{p_1} \cdot a_2^{p_2} \ldots a_n^{p_n}$

$\rightarrow (p_1 + 1)(p_2 + 1) \ldots (p_n + 1)$

How many factors does 260 have?

260 → 10, 26; 2, 5, 13, 2

$= 2^2 \cdot 5^1 \cdot 13^1$

$(2+1)(1+1)(1+1)$

$= 3 \cdot 2 \cdot 2 = \boxed{12 \text{ factors}}$

260 : How many even factors?

$2^2 \cdot 5^1 \cdot 13^1$

↳ 4 has at least 1 power of 2 in its prime factorization

1, 2

$(2)(1+1)(1+1) = \boxed{8 \text{ factors}}$

260 : How many odd factors?

$2^2 \cdot 5 \cdot 13$

3 has no power of 2 in its prime factorization

0

$(0+1)(1+1)(1+1)$

$1 \cdot 2 \cdot 2 = \boxed{4 \text{ factors}}$

Factorials

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{120}$

$\frac{(n+1)!}{n!}$ $(n+1)! = (n+1) \cdot n \cdot (n-1) \ldots = (n+1) \cdot n!$

$\frac{(n+1)\cancel{n!}}{\cancel{n!}} = (n+1)$ $\frac{6!}{5!} = 6$

19. Find the largest prime divisor of 5!+6!+7!+8!.

$5! + 6! + 7! + 8!$

$5! + 6 \cdot 5! + 7 \cdot 6 \cdot 5! + 8 \cdot 7 \cdot 6 \cdot 5!$

$= 5!(1 + 6 + 7 \cdot 6 + 8 \cdot 7 \cdot 6)$

$= 5!(1 + 6(\underbrace{1 + 7 + 8 \cdot 7}_{56 + 8 = 64}))$

$= 5!(\underbrace{1 + 6(64)}_{384 + 1 = 385})$

$= 5!(385)$

$= 5! \cdot 5 \cdot 11 \cdot 7$

→ 11

$385 = 5 \cdot 77$, $77 = 11 \cdot 7$

$5 \overline{)385}$, -35, 35

21. Find the largest prime less than 100 that can be written as the difference of 2 squares.
22. Find the smallest prime divisor of $5^{23} + 7^{17}$.
23. How many positive cubes divide 3! · 5! · 7!?

21) $a^2, b^2 \Rightarrow n = a^2 - b^2$

$= \underbrace{(a-b)}_{1}(a+b)$

$n = a + b$

$a = 50, b = 49 \rightarrow n = 99 \rightarrow$ not a prime

$a = 49, b = 48 \rightarrow \boxed{n = 97}$ ✓

22) $5^{23} + 7^{17}$

odd + odd = even

2 | even → $\boxed{2}$ must a factor of $5^{23} + 7^{17}$

23) $k^3 \mid \underbrace{(3! \cdot 5! \cdot 7!)}$

$3 \cdot 2 \cdot 1$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$\rightarrow 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$

m^2, m^3

$36 = 6^2 = 4 \cdot 9 = 2^2 \cdot 3^2 = (2 \cdot 3)^2$

$k^3 \rightarrow a^3 b^3 c^3 \ldots$

$\underbrace{0, 3, 6}_{3}, \underbrace{0, 3}_{2}, 0, 0$

$\rightarrow 3 \cdot 2 \cdot 1 \cdot 1 = 6$

Find the # of trailing 0s at the end of 64!.

100 → 2 trailing 0s

1000 → 3 zeroes

10100 → 2 zeroes

5! → how many powers of 10 are in 5!.

$10 = 5 \cdot 2$

2 4 6 8 10 (5) 5 10 (2)

→ how many factors of 5 are in 5!

Legendre's Formula

n! → number of powers of # k

$\rightarrow \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n}{k^2} \right\rfloor + \left\lfloor \frac{n}{k^3} \right\rfloor \ldots \left\lfloor \frac{n}{k^m} \right\rfloor$

powers of five in 5!

n = 5, k = 5

$\left\lfloor \frac{5}{5} \right\rfloor + \underbrace{\left\lfloor \frac{5}{25} \right\rfloor \ldots}_{0}$

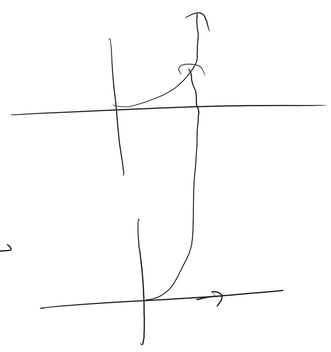
1

$\lfloor 200.1 \rfloor = 200$

$\lfloor 36 \rfloor = 36$

$\lfloor -36 \rfloor = -36$

$\lfloor -36.4 \rfloor = -37$



# of powers of 5 in 5! = 1

# of trailing 0s = 1

5! = 120

64! → $\left\lfloor \frac{64}{5} \right\rfloor + \left\lfloor \frac{64}{25} \right\rfloor + \underbrace{\left\lfloor \frac{64}{125} \right\rfloor \ldots}_{0}$

$\underbrace{12 + 2}_{14}$

64! has 14 zeroes at the end of it