Number Theory - Day 6

Overview
- $\lfloor x \rfloor$, $\{x\}$
- terminating decimals
- repeating decimals
- decimals to fractions
- Base #s and decimal equivalents
- Adv. Problems

$\lfloor x \rfloor$ - greatest integer less than or equal to x

$\lfloor 2 \rfloor = 2$ $\quad \lfloor 2.3 \rfloor = 2$ $\quad \lfloor -3.6 \rfloor = -4$

$a, b \in \mathbb{Z}$, Prove that $\lfloor \frac{a}{b} \rfloor$ always exists.

~~$a = b$, $a \neq b$~~

Division Algorithm:

$a = q \cdot b + r$ $\quad q, r \in \mathbb{Z}$

$(-5, 2) \to -5 = 2 \cdot -3 + 1$

$a = q \cdot b + r$

$\hookrightarrow 0 \le r < b$

$4 = 2 \cdot 2 + 0$

$\lfloor \frac{a}{b} \rfloor$

$\lfloor \frac{a}{b} \rfloor = \lfloor q + \frac{r}{b} \rfloor = \lfloor q \rfloor$

$\hookrightarrow q$

$\frac{1}{13} = 0.076923076923..$

$= 0.\overline{076923}$

$\frac{1}{6} = 0.16666$

$= 0.1\overline{6}$

What is the decimal expansion of $\frac{2}{7}$?

$0.\overline{285714}$

$7 \overline{)2.00000}$

1.4

0.60

-56

40

-35

50

-49

10

-7

30

-28

$2 \ldots$

endlessly

Let $\frac{a}{b}$, $a, b \in \mathbb{Z}$

terminating: $\frac{a}{b} \cdot 10^m$

$0.236 \cdot 10^3 = 236$

$\frac{a}{2^a \cdot 5^b}$

$\frac{a \cdot 2^m \cdot 5^m}{b}$

$b = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_n^{e_n}$

$b = 2^{e_1} 5^{e_2}$

$\frac{a \cdot 10^m}{b} \to$ never an integer

While $\frac{a}{b}$ for $a, b \in \mathbb{Z}$ terminate or repeat when expressed in decimal notation,

$\sqrt{2} = 1.41421\ldots$

$\pi = 3.1415926\ldots$

$e = 2.71828\ldots$

$\frac{1}{12}$

$12 = 2^2 \cdot 3$

$\hookrightarrow 10^2$

$\frac{10^2}{2^2 \cdot 3} = \frac{5^2}{3} = \frac{25}{3} = 8\frac{1}{3}$

$8.333\ldots$

$= 8.\overline{3}$

$\to \boxed{0.08\overline{3}}$

Express $0.\overline{4}$ as a fraction?

$0.4441\ldots$

$= 0.4 + 0.04 + 0.004 \ldots$

$\frac{4}{10} + \frac{4}{100} \ldots$

$\frac{4/10}{1 - \frac{1}{10}} \to \frac{4/10}{9/10} \to \boxed{\frac{4}{9}}$

$\hookrightarrow \frac{4}{9}$

$a = 0.\overline{4}$

$10a = 4.\overline{4}$ $\to 9a = 4$

$\boxed{a = \frac{4}{9}}$

$2.0\overline{9}$ as a fraction

$a = 2.0\overline{9}$

$100a = 209.\overline{09}$

$99a = 209$

$\to a = \boxed{\frac{209}{99}}$

$x \in \mathbb{R}$

$\pi, e, -1, 12, 13.6, 1/3 \ldots$

$\varnothing$ $i, i^3, 2+i$

$x = \lfloor x \rfloor + \{x\}$

$2.3 = \lfloor 2.3 \rfloor + \{2.3\}$

$2.3 = 2 + .3$

$\lfloor x \rfloor + x = 8.75$

$x = \lfloor x \rfloor + \{x\}$ $\quad \lfloor x \rfloor \to 4$, $\{x\} \to 0.75$, $4 + 0.75 = 4.75$

$\lfloor x \rfloor + \lfloor x \rfloor + \{x\} = 8.75$

$2\lfloor x \rfloor + \{x\} = 8.75$

$.75$

$2\lfloor x \rfloor = 8$

$\hookrightarrow \lfloor x \rfloor = 4$

$8.75 = \lfloor 4.75 \rfloor + 4.75$

$4 + 4.75$ ✓

0.364, $\frac{1}{2}$, 2.0, 12.0, $13.123469 7\ldots_{12}$

$n \in \mathbb{Z}^+$ $\quad \frac{2.6}{10^n} \to 10^{-1} \cdot n$

$n = 1$

$\frac{2.6}{10} = 0.26$

$n \in \mathbb{Z}^+$, how many digits least the decimal pt do we need to write $\frac{1}{2^n}$?

$n = 1 \to \frac{1}{2} = 0.5 \to 1$

$n = 2 \to \frac{1}{4} = 0.25 \to 2$

$n = 3 \to \frac{1}{8} = 0.125 \to 3$

$\frac{1}{5^n}$

$n = 1 \to \frac{1}{5} = 0.2 \to \underline{1}$

$n = 2 \to \frac{1 \cdot 4}{25 \cdot 4} = 0.\underline{04} \to \underline{2}$

$\frac{4}{100}$

$n = 3 \to \frac{1}{5^3} = 0.008 \to 3$ digits

8.0

$\frac{1}{6^n} \to n = 1 \to \frac{1}{6} = 0.1666\ldots$

$a \in \mathbb{Z}^+$

$\frac{a}{2^n}$, $\frac{a}{5^n}$

n places after decimal point to represent it

$\frac{a}{2^n} \cdot 10^n \to \frac{5^n a}{1}$

$\frac{1}{5^7} \cdot 10^7 \to \frac{2^7}{1}$

$\frac{n}{d} \to d = 2^a \cdot 5^b \to a, b \in \mathbb{Z}^+$

$\frac{1}{2^3 \cdot 5} \cdot 10^{③} \to 5^2 = 25$

$\boxed{\max(a, b)}$

Find # of dig past decimal pt to express

~~917~~ $\quad \frac{1013624 \, 31}{500}$

$\boxed{500} \to 100 \cdot 5$

$5^2 \cdot 4 \cdot 5 \to 5^3 \cdot 2^2$

$\max(3, 2) = 3$

Base #s in decimals

0.2362

$10^{-1} \leftarrow \frac{2}{10} + \frac{3}{100} + \frac{6}{1000} + \frac{2}{10000}$

$7^{-1} \leftarrow 0.5_7 \to \frac{5}{7}$

β_7

$0.101_2 \to \frac{1}{2} + \frac{0}{4} + \frac{1}{8}$

$\to \frac{5}{8}$

2^{-1}

Radix point 0.5

$\frac{1}{8}$ as a base 9 numeral

$\frac{1}{2^3} \cdot 10^3 \to \frac{5^3}{1} = 125$

$\boxed{0.125_{10}}$

$\frac{1}{8} - \frac{1}{9^1} = \frac{1}{8 \cdot 9^1}$

$\frac{1}{8 \cdot 9^1} - \frac{1}{9^2} = \frac{1}{8 \cdot 9^2}$

$\ldots$

$\frac{1}{8 \cdot 9^{n-1}} - \frac{1}{9^n} \quad \frac{1}{8 \cdot 9^n}$

$\frac{1}{8} = \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} \ldots$

$\frac{\frac{1}{9}}{\frac{8}{9}} = \frac{1}{8}$

$\frac{1}{8} = 1 \cdot 9^{-1} + 1 \cdot 9^{-2} \ldots$

$= 0.1111\ldots_9$

$= \boxed{0.\overline{1}_9}$

$0.24_7 \to$ Base 10 and find the first 4 digits

$0.24_7 = \frac{2}{7} + \frac{4}{7^2} = \frac{2}{7} + \frac{4}{49}$

$\frac{14}{49} + \frac{4}{49} = \frac{18}{49}$

$= 0.\boxed{3673}\ldots$