Binomid Sequences

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1 Examples of Binomid Sequences

1. The sequence $(n) = (1, 2, 3, \ldots)$ is binomid. This can be shown easily by noting that the binomid coefficients generated are equal to the binomial coefficients of Pascal's triangle,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(n)} = \binom{n}{k}.$$

It is well known that the Pascal's triangle consists of integer entries, hence why this sequence is binomid.

2. For positive integer q, define sequence a_q by: $a_q(n) = \frac{q^n-1}{q-1}$. We will see below that a_q is binomid. The corresponding "q-nomial coefficients" were considered by Gauss in 1808.) See Lemma 6, with $\alpha = q$ and $\beta = 1$.

3. The Fibonacci sequence F_n, defined by

$$F_1 = F_2 = 1 \qquad \text{and} \qquad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geqslant 3,$$

is binomid. By Binet's formula, we can also write F_n as

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \widehat{\varphi}^n \right),$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \widehat{\varphi} = \frac{1-\sqrt{5}}{2}.$$

Then, using Lemma 6, with $\alpha = \phi$ and $\beta = \widehat{\phi}$, it becomes clear that F_n is binomid.

4. The sequence of triangular numbers,

$$a_n = \binom{n}{2} = \frac{n(n+1)}{2},$$

is binomid. In general, the k^{th} Pascal's triangle column, $a_k(n) = \binom{n+k-1}{k}$, is binomid. See Section 2.3.

5. The Stirling numbers of the first kind are binomid, but the Stirling numbers of the second kind are not.

Stirling numbers of the first kind satisfy the recursive relation,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

By Lemma 7, this sequence is binomid.

The Stirling numbers of the second kind do not hold this sort of recursive relationship. While this alone is not enough to prove it is not binomid, counterexamples are easy to find.

2 Properties of Binomid Sequences

2.1 Definitions

Definition 1 (Divisor chain). A sequence $\langle a \rangle$ is a divisor chain if $a_n \mid a_{n+1}$ for every n.

Definition 2 (Homomorphism). A sequence a is homomorphic if $a_{mn} = a_m a_n$ for all positive integers m and n.

Definition 3 (Divisor product). Define a sequence a to be a *divisor-product* (called *cyclotomic-like* by some students) if there exists an integer sequence b such that for every index n :

$$a_n = \prod_{d \mid n} b_d.$$

Definition 4 (Divisibility). A sequence $\langle a \rangle$ has divisibility if for all positive integers m and n, we have

$$m \mid n \implies a_m \mid a_n.$$

Remark 1. Every divisor produce has divisibility.

Divisibility does not imply binomid. As an example, consider the sequence

$$a_n = \begin{cases} 2 & \text{if } 2 \mid n \text{ or } 3 \mid n \\ 1 & \text{else} \end{cases}.$$

To show that this sequence is has divisibility, first suppose that $x \mid y$. We will show that $a_x \mid a_y$ by splitting into two cases. Firstly, if $a_x = 1$ then clearly $a_x \mid a_y$. Otherwise, $a_x = 2$ and it follows that x is a multiple of either 2 or 3. Since y is a multiple of x, we have that y must also be a multiple of either 2 or 3 so that $a_y = 2$ as well and certainly $a_x \mid a_y$. Then (a_n) has divisibility, but

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix}_a = \frac{a_7 a_6 a_5}{a_1 a_2 a_3} = \frac{1 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 2} \notin \mathbb{Z}$$

Then (a_n) is not binomid.

Definition 5 (gcd-sequence)**.** Define a sequence $\mathfrak{a}$ to be a *gcd-sequence* if $\gcd(\mathfrak{a}_{\mathfrak{m}}, \mathfrak{a}_{\mathfrak{n}}) = \mathfrak{a}_{\gcd(\mathfrak{m},\mathfrak{n})}$.

If we use the notation $\mathfrak{m} \wedge \mathfrak{n}$ for $\gcd(\mathfrak{m}, \mathfrak{n})$, a GCD sequence looks more like a homomorphism:

$$\mathfrak{a}(\mathfrak{m} \wedge \mathfrak{n}) = \mathfrak{a}(\mathfrak{m}) \wedge \mathfrak{a}(\mathfrak{n})$$

2.2 Lemmas

Lemma 1. Suppose $\langle a \rangle$ is a binomid sequence and s is a rational number such that every $sa_n \in \mathbb{Z}$. Then the sequence $\langle sa \rangle$ is also binomid. In fact, the sequences have the same binomid coefficients.

Proof. Let $b_n := s \cdot a_n$, so that every b_n is an integer (by hypothesis). For any n, k, we have:

$$\begin{bmatrix} n \\ k \end{bmatrix}_b = \frac{(s \cdot a_n)(s \cdot a_{n-1}) \dots (s \cdot a_{n-k+1})}{(s \cdot a_1)(s \cdot a_2) \cdots (s \cdot a_k)}$$
$$= \frac{\cancel{s^k}(a_n a_{n-1} \dots a_{n-k+1})}{\cancel{s^k}(a_1 a_2 \cdots a_k)}$$
$$= \begin{bmatrix} n \\ k \end{bmatrix}_a,$$

Therefore b_n must also be binomid. ∎

Lemma 2. If $\langle a \rangle$ and $\langle b \rangle$ are binomid sequences, then $\langle ab \rangle$ is also binomid.

Proof. For any choice of n, k, notice that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{ab} = \frac{(a_n b_n)(a_{n-1} b_{n-1}) \dots (a_{n-k+1})(b_{n-k+1})}{(a_1 b_1)(a_2 b_2) \cdots (a_k b_k)}$$
$$= \left(\frac{a_n a_{n-1} \dots a_{n-k+1}}{a_1 a_2 \cdots a_k} \right) \left(\frac{b_n b_{n-1} \dots b_{n-k+1}}{b_1 b_2 \cdots b_k} \right)$$
$$= \begin{bmatrix} n \\ k \end{bmatrix}_a \begin{bmatrix} n \\ k \end{bmatrix}_b.$$

∎

Remark 2. Binomid sequences are not closed under addition.

Take the example of $a_n = (n)$ and $b_n = 1$. It was explicitly proved that a_n is binomid. However, the sequence $a_n + b_n := c_n = (2, 3, 4, 5, \dots)$ is not binomid. For example,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_c = \frac{c_2}{c_1} = \frac{3}{2} \notin \mathbb{Z}$$

Thus, binomid sequences are not necessarily closed under addition.

Lemma 3. Every divisor chain $\langle a \rangle$ is binomid.

Proof. Since $a_n \mid a_{n+1}$, it follows that $a_n \mid a_m$ whenever $m \geqslant n$. Thus, for all $n \geqslant k$ we have:

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_a &= \frac{a_n a_{n-1} \dots a_{n-k+1}}{a_k a_{k-1} \cdots a_1} \\ &= \left(\frac{a_n}{a_k}\right)\left(\frac{a_{n-1}}{a_{k-1}}\right)\dots\left(\frac{a_{n-k+1}}{a_1}\right)\end{aligned}$$

Since each $\left(\frac{a_{n-i}}{a_{k-i}}\right)$ is an integer, the sequence $\langle a \rangle$ is binomid. ■

Lemma 4. Every homomorphic sequence $\langle a \rangle$ is binomid.

Proof. First notice that a homomorphic sequence has divisibility: if $x \mid y$ (say $kx = y$), then $a_k a_x = a_y$ and therefore $a_x \mid a_y$.

From here, notice that the multiplicative property allows us to combine terms so that

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \frac{a_n a_{n-1} \dots a_{n-k+1}}{a_1 a_2 \cdots a_k} = \frac{a_{(n)_k}}{a_{k!}}$$

However, notice that $\binom{n}{k} = \frac{(n)_k}{k!}$ is an integer, meaning that $k!$ must divide $(n)_k$. Thus, it follows that $a_{k!} \mid a_{(n)_k}$ and therefore $\frac{a_{(n)_k}}{a_{k!}}$ is also an integer. ■

Lemma 5. A divisor-product sequence is *binomid*.

Proof. Suppose $\langle a \rangle$ and $\langle b \rangle$ are given as in Definition 3. Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \frac{\displaystyle\prod_{l=n-k+1}^{n} \left(\prod_{d \mid l} b_d\right)}{\displaystyle\prod_{l=1}^{k} \left(\prod_{d \mid l} b_d\right)}.$$

View the terms b_m as independent indeterminates, and count the number of occurrences of b_m in numerator and denominator.

The number of b_m terms in the denominator is $\left\lfloor \frac{k}{m} \right\rfloor$. The number in the denominator is: $\left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{n-k}{m} \right\rfloor$. Since $\lfloor x + y \rfloor \geqslant \lfloor x \rfloor + \lfloor y \rfloor$ for real numbers x and y, we have:

$$\left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{n-k}{m} \right\rfloor \geqslant \left\lfloor \frac{k}{m} \right\rfloor.$$

Therefore $\langle a \rangle$ is binomid. ■

Remark 3. Lemma 5 fails if we relax the condition for divisor-products as follows:

$$\left(\prod_{d|n} b_d\right) \Bigg| \; a_n$$

Consider the sequence $A_n = 2^n - 1$ for $n \geqslant 2$ except $A_1 = 11$. Now the same b_d for the original $a_n = 2^n - 1$ still divides A_n, but $\frac{A_2}{A_1} \notin \mathbb{Z}$.

> **Lemma 6.** Suppose $r, s \in \mathbb{Z}$ and $x^2 - rx - s = (x-\alpha)(x-\beta)$ for some $\alpha, \beta \in \mathbb{C}$. Define:
>
> $$c_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$
>
> Then every c_n is an integer, and the sequence $\langle c \rangle$ is binomid.

Proof. The definition implies that $c_0 = 0, c_1 = 1$. Claim:

$$c_n = rc_{n-1} + sc_{n-2}, \quad \text{for every index } n.$$

This follows since α, β satisfy $x^2 = rx + s$. Since c_0, c_1, r, s are integers, we conclude that c_n is an integer for every $n \geqslant 0$.

Claim: $\langle c \rangle$ is a divisor product.
If that is proved, then Lemma 5 shows that $\langle c \rangle$ is binomid.

Define the *homogeneous-cyclotomic* polynomials, $\Phi_n(x, y)$ by the formula:

$$x^n - y^n = \prod_{d|n} \Phi_d(x).$$

These are two-variable versions of the usual cyclotomic polynomials $\Phi_n(x)$. The first few cases are

$$\begin{aligned}
\Phi_1(x, y) &= x - y \\
\Phi_2(x, y) &= x + y \\
\Phi_3(x, y) &= x^2 + xy + y^2 \\
\Phi_4(x, y) &= x^2 + y^2 \\
\vdots \;\; &= \;\; \vdots
\end{aligned}$$

Define sequence $\langle b \rangle$ by setting $b_1 = 1$ and $b_n = \Phi(\alpha, \beta)$ for $n > 1$. Recall that $\Phi(\alpha, \beta) = \alpha - \beta$. Definitions imply:

$$c_n \;=\; \frac{1}{\alpha - \beta} \prod_{d|n} (\alpha^n - \beta^n) \;=\; \prod_{\substack{d|n \\ d>1}} \Phi_d(\alpha, \beta) \;=\; \prod_{d|n} b_d.$$

The Claim is proved provided we can prove: $\boxed{\text{Every } b_n \text{ is an integer.}}$

For example $b_2 = \alpha+\beta = r$ and $b_3 = \alpha^2+\alpha\beta+\beta^2 = (\alpha+\beta)^2-\alpha\beta = r^2+s$. The proof for larger n is left as an exercise. A key point is that for $n > 1$ the term $b_n = \Phi_n(\alpha, \beta)$ is a symmetric polynomial in α, β with integer coefficients. You might use the theory of symmetric polynomials (or maybe Galois Theory) to conclude $b_n \in \mathbb{Z}$. Alternatively, show that every $\alpha^n + \beta^n$ is an integer (perhaps following ideas due to Isaac Newton), and use the defining formula for $\Phi_n(x, y)$ to show that each b_n is a rational number. Knowing that b_n is an algebraic integer (?) in $\mathbb{Q}$ implies that $b_n \in \mathbb{Z}$. ∎

Lemma 7. Suppose sequence f has the property that for every $m, k \in \mathbb{Z}^+$,

$$\gcd(f_m, f_k) \mid f_{m+k}.$$

Then $\langle f \rangle$ is binomid.

Proof. The hypothesis implies that $f_{m+k} = uf_m + vf_k$ for some integers u, v. Changing notations, for any k, n with $0 \leqslant k \leqslant n$ there exist integers u, v such that

$$f_{n+1} = uf_{n-k+1} + vf_k,$$

Now show that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_f = u\begin{bmatrix} n \\ k \end{bmatrix}_f + v\begin{bmatrix} n \\ k-1 \end{bmatrix}_f.$$

Conclude (via induction) that every $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_f$ is an integer, and f is binomid. ∎

Corollary 1. Every gcd-sequence is binomid.

Proof. Apply Lemma 7. ∎

2.3 Theorems

For a sequence $\langle c \rangle$, recall that $\Delta(c)$ is the binomid triangle, consisting of all the coefficients $\left[{n \atop k}\right]_c$.

Theorem 1. Suppose c is a divisor-chain. Then every column of the binomid triangle $\Delta(c)$ is also a divisor chain.

Proof. First, define the notion of a *related sequence* for a divisor chain:

Definition 6. For any divisor chain $\langle a \rangle$ with $a_0 = 1$, define its related sequence to be $\langle b \rangle = \frac{a_n}{a_{n-1}}$.

It follows that we can then express a_n as

$$a_n = \prod_{i=1}^{n} b_i.$$

Conversely, note that for any sequence b_n the sequence

$$a_n = \prod_{i=m}^{n} b_i$$

is a divisor chain. Using this definition, we can prove that each column of the binomid triangle of a divisor chain also forms a divisor chain.

Suppose that a divisor chain a_n has related sequence b_n. We will show that the sequence

$$C_k(n) = \left(\begin{bmatrix} n \\ k \end{bmatrix}_a \right) \qquad \text{where} \qquad (C_k)_n = \begin{bmatrix} n \\ k \end{bmatrix}_a,$$

which is the k^{th} column of the binomid triangle $\triangle a$, is also a divisor chain.

We shall disregard the first k terms of this sequence because they are 0 (when $n < k$) and 1 (when $n = k$). Thus, for any $n > k$ we have

$$
\begin{aligned}
C_k(n) = \begin{bmatrix} n \\ k \end{bmatrix}_a &= \frac{\left(\prod_{i=1}^{n} b_i\right)\left(\prod_{i=1}^{n-1} b_i\right)\dots\left(\prod_{i=1}^{n-k+1} b_i\right)}{\left(\prod_{i=1}^{k} b_i\right)\left(\prod_{i=1}^{k-1} b_i\right)\dots\left(\prod_{i=1}^{k-k+1} b_i\right)} \\
&= \left(\frac{\prod_{i=1}^{n} b_i}{\prod_{i=1}^{k} b_i}\right)\left(\frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{k-1} b_i}\right)\dots\left(\frac{\prod_{i=1}^{n-k+1} b_i}{\prod_{i=1}^{k-k+1} b_i}\right) \\
&= \left(\prod_{i=k+1}^{n} b_i\right)\left(\prod_{i=k}^{n-1} b_i\right)\dots\left(\prod_{i=2}^{n-k+1} b_i\right) \\
&= \left(\prod_{i=k+1}^{n} b_i\right)\left(\prod_{i=k+1}^{n} b_{i-1}\right)\dots\left(\prod_{i=k+1}^{n} b_{i-k+1}\right) \\
&= \prod_{i=k+1}^{n}\left(\prod_{j=i-k+1}^{i} b_j\right)
\end{aligned}
$$

Thus, we have that $C_k(n)$ must be a divisor chain, as it has the related sequence

$$B_n := \left(\prod_{i=n-k+1}^{n} b_i\right).$$

■

Corollary 2. Any exponential sequence $a_n = (x^n)$ with related sequence $b_n = (x, x, x, \dots)$ is such that $C_k(n)$ has the related sequence

$$B_n = \left(\prod_{i=n-k+1}^{n} x\right) = (x^k, x^k, x^k, \dots).$$

Thus, $C_k(n)$ is also an exponential sequence, equal to $((x^k)^n)$.

The following is another (simpler) proof of the same theorem above.

Proof. Suppose a_n is a divisor chain. Then for $n \geqslant k$,

$$
\begin{aligned}
a_{n-k+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_a &= \frac{(a_{n+1})(a_n)(a_{n-1})\dots(a_{n-k+1})}{(a_k)(a_{k-1})\cdots(a_1)} \\
&= a_{n+1}\begin{bmatrix} n \\ k \end{bmatrix}_a \\
\begin{bmatrix} n+1 \\ k \end{bmatrix}_a &= \frac{a_{n+1}}{a_{n-k+1}}\begin{bmatrix} n \\ k \end{bmatrix}_a
\end{aligned}
$$

Since $n+1$ is greater than $n-k+1$, it follows that $\frac{a_{n+1}}{a_{n-k+1}}$ is always integral. Then the k^{th} column sequence of the binomid triangle $\Delta(a)$ is also a divisor chain. ■

A similar corollary follows as in the previous one:

Corollary 3. For $a_n = (x^n)$, we have:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_a = x^k \begin{bmatrix} n \\ k \end{bmatrix}_a .$$

Then the k^{th} column sequence of $\Delta(a)$ is $C_k(n) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_a = x^{k(n-1)}$, and: $\begin{bmatrix} n \\ k \end{bmatrix}_a = x^{k(n-k)}$.

Theorem 2. The second column of Pascal's triangle (the sequence of triangular numbers) is binomid.

Proof. This sequence is defined by $\triangle_n = \frac{n(n+1)}{2}$, but by Lemma 1, it suffices to prove $T_n = n(n+1)$ is binomid.

We need to show that for all positive integers $n \geqslant k \geqslant 1$, the T-binomid coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_T$ is a positive integer.

We know that;

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_T &= \frac{T_n T_{n-1} \cdots T_{n-k+1}}{T_k T_{k-1} \cdots T_1} \\
&= \frac{(n(n+1))((n-1)n) \cdots ((n-k+1)(n-k+2))}{(k(k+1))((k-1)k) \cdots (1 \cdot 2)} \\
&= \left(\frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots (1)} \right) \left(\frac{(n+1)n \cdots (n-k+2)}{(k+1)k \cdots 2} \right) \\
&= \left(\frac{n!}{k!(n-k)!} \right) \left(\frac{(n+1)!}{(k+1)!(n-k+1)!} \right) \\
&= \left(\frac{(n+1)!}{(n+1)k!(n-k)!} \right) \left(\frac{(n+1)!}{(k+1)!(n-k+1)!} \right) \\
&= \left(\frac{(n+1)!}{(n+1)k!(n-k+1)!} \right) \left(\frac{(n+1)!}{(k+1)!(n-k)!} \right) \\
&= \frac{1}{n+1} \left(\frac{(n+1)!}{k!(n-k+1)!} \right) \left(\frac{(n+1)!}{(k+1)!(n-k)!} \right).
\end{aligned}$$

We know that $\frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$, and $\frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}$, thus we can rewrite the T-binomid coefficient as

$$\begin{bmatrix} n \\ k \end{bmatrix}_T = \frac{\binom{n+1}{k} \cdot \binom{n+1}{k+1}}{n+1}.$$

Hence, it suffices to prove that the above expression is integral for all positive integers n and k satisfying $0 \leqslant k \leqslant n$.

Lemma 8. For all positive integers $0 < k \leqslant n$, we have

$$\frac{\binom{n+1}{k} \cdot \binom{n+1}{k+1}}{n+1} = \binom{n}{k}\binom{n+2}{k+1} - \binom{n+1}{k}\binom{n+1}{k+1}.$$

Proof. We have;

$$
\begin{aligned}
\frac{\binom{n+1}{k+1}\binom{n+1}{k}}{(n+1)} &= \binom{n+1}{k+1} \cdot \frac{\binom{n+1}{k}}{(n+1)} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \cdot \frac{(n+1)!}{k!(n-k+1)!(n+1)} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \cdot \frac{n!}{k!(n-k+1)!} \\
&= \frac{n!(n+1)!}{k!(k+1)!(n-k+1)!(n-k)!} \\
&= \frac{(n+1)!^2 \cdot \frac{1}{n+1}}{k!(k+1)!(n-k+1)!(n-k)!} \\
&= \frac{(n+1)!^2 \left(\frac{n+2}{n+1} - 1\right)}{k!(k+1)!(n-k+1)!(n-k)!} \\
&= \frac{n!(n+2)! - (n+1)!^2}{k!(k+1)!(n-k+1)!(n-k)!} \\
&= \frac{n!(n+2)!}{k!(k+1)!(n-k)!(n-k+1)!} - \frac{(n+1)!^2}{k!(k+1)!(n-k+1)!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \cdot \frac{(n+2)!}{(k+1)!(n-k+1)!} - \frac{(n+1)!}{k!(n-k+1)!} \cdot \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \binom{n}{k}\binom{n+2}{k+1} - \binom{n+1}{k}\binom{n+1}{k+1},
\end{aligned}
$$

as desired. ∎

Since all numbers of the above form are integers, it follows that the T-binomid ceofficients are integral and therefore T (and the triangular numbers) is binomid. [1] ∎

[1] Remark. "Narayana Numbers" are related these T-binomid coefficients.

Theorem 3. A divisor product is binomid at every level.

Proof. Define

$$C_d(n) = \begin{bmatrix} n+d-1 \\ d \end{bmatrix}_c$$

to be the d-th column of the binomid triangle of some binomid sequence c.

Note that

$$C_d(n) = \binom{n+d-1}{d} = \frac{\langle n+d-1\rangle!_c}{\langle d\rangle!_c \, \langle n-1\rangle!_c}$$

is an integer sequence. For a fixed d, $\langle d\rangle!_c$ is constant, so by Lemma 1, $C_d(n)$ is binomid if and only if

$$C'_d(n) := \langle d\rangle!_c \cdot C_d(n) = \frac{\langle n+d-1\rangle!_c}{\langle n-1\rangle!_c} = \prod_{j=n}^{n+d-1} c(j)$$

is binomid.

We note that

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_{C'_d} &= \frac{\langle n\rangle!_{C'_d}}{\langle k\rangle!_{C'_d} \langle n-k\rangle!_{C'_d}} \\
&= \frac{\left(\prod_{i=n-k+1}^{n} C'_d(i)\right)}{\left(\prod_{i=1}^{k} C'_d(i)\right)} \\
&= \frac{\left(\prod_{i=n-k+1}^{n} \prod_{j=i}^{i+d-1} c(j)\right)}{\left(\prod_{i=1}^{k} \prod_{j=i}^{i+d-1} c(j)\right)} \\
&= \frac{\prod_{j=0}^{d-1} \left(\prod_{i=n-k+j+1}^{n+j} c(i)\right)}{\prod_{j=0}^{d-1} \prod_{i=j+1}^{j+k} c(i)} \\
&= \prod_{j=0}^{d-1} \frac{\langle n+j\rangle!_c \langle j\rangle!_c}{\langle n-k+j\rangle!_c \langle j+k\rangle!_c}
\end{aligned}$$

Thus, $C_d(n)$ is binomid if and only if

$$\prod_{j=0}^{d-1} \frac{\langle n+j\rangle!_c \langle j\rangle!_c}{\langle n-k+j\rangle!_c \langle j+k\rangle!_c}$$

is integral for all n and k.

If we assume that c is a divisor product arising from sequence g, then we count the number of times a $g(\delta)$ term appears in that big fraction. We find $g(\delta)$ appears in $\langle\omega\rangle!_c$ a total of $\lfloor\frac{\omega}{\delta}\rfloor$ times. If we consider the number of times $g(\delta)$ divides into

$$\prod_{j=0}^{d-1}\frac{\langle n+j\rangle!_c\ \langle j\rangle!_c}{\langle n-k+j\rangle!_c\ \langle j+k\rangle!_c},$$

we get

$$\begin{aligned}
&\sum_{j=0}^{d-1}\left(\left\lfloor\frac{n+j}{\delta}\right\rfloor+\left\lfloor\frac{j}{\delta}\right\rfloor-\left\lfloor\frac{n+j-k}{\delta}\right\rfloor-\left\lfloor\frac{j+k}{\delta}\right\rfloor\right)\\
&=\sum_{j=0}^{\delta\lfloor\frac{d-1}{\delta}\rfloor}\left(\left\lfloor\frac{n+j}{\delta}\right\rfloor+\left\lfloor\frac{j}{\delta}\right\rfloor-\left\lfloor\frac{n+j-k}{\delta}\right\rfloor-\left\lfloor\frac{j+k}{\delta}\right\rfloor\right)\\
&+\sum_{j=0}^{\delta\{\frac{d-1}{\delta}\}}\left(\left\lfloor\frac{n+j+\delta\lfloor\frac{d-1}{\delta}\rfloor}{\delta}\right\rfloor+\left\lfloor\frac{j+\delta\lfloor\frac{d-1}{\delta}\rfloor}{\delta}\right\rfloor-\left\lfloor\frac{n+j+\delta\lfloor\frac{d-1}{\delta}\rfloor-k}{\delta}\right\rfloor-\left\lfloor\frac{j+\delta\lfloor\frac{d-1}{\delta}\rfloor+k}{\delta}\right\rfloor\right)\\
&=\delta\sum_{j=0}^{\lfloor\frac{d-1}{\delta}\rfloor}\left(\left\lfloor\frac{n+j}{\delta}\right\rfloor+\left\lfloor\frac{j}{\delta}\right\rfloor-\left\lfloor\frac{n+j-k}{\delta}\right\rfloor-\left\lfloor\frac{j+k}{\delta}\right\rfloor\right)\\
&+\sum_{j=0}^{\delta\{\frac{d-1}{\delta}\}}\left(\left\lfloor\frac{n+j}{\delta}\right\rfloor+\left\lfloor\frac{j}{\delta}\right\rfloor-\left\lfloor\frac{n+j-k}{\delta}\right\rfloor-\left\lfloor\frac{j+k}{\delta}\right\rfloor\right).
\end{aligned}$$

Thus, we can use the well ordering principle to restrict our focus the cases when $d-1\leqslant\delta-1$, or $d\leqslant\delta$, since the first term of the last expression above will be recurring. Note that because

$$\left\lfloor\frac{x+j}{\delta}\right\rfloor=\begin{cases}\left\lfloor\frac{x}{\delta}\right\rfloor, & \frac{j}{\delta}<1-\left\{\frac{x}{\delta}\right\}\\[2ex] \left\lfloor\frac{x}{\delta}\right\rfloor+1, & \frac{j}{\delta}\geqslant 1-\left\{\frac{x}{\delta}\right\}\end{cases},$$

we must also have

$$\begin{aligned}
\sum_{j=0}^{d-1}\left\lfloor\frac{x+j}{\delta}\right\rfloor &= \sum_{j=0}^{\delta-\delta\{x/\delta\}-1}\left\lfloor\frac{x+j}{\delta}\right\rfloor + \sum_{j=\delta-\delta\{x/\delta\}}^{d-1}\left\lfloor\frac{x+j}{\delta}\right\rfloor \\
&= \left(\delta-\delta\left\{\frac{x}{\delta}\right\}\right)\left\lfloor\frac{x}{\delta}\right\rfloor + \left(d-\left(\delta-\delta\left\{\frac{x}{\delta}\right\}\right)\right)\left(\left\lfloor\frac{x}{\delta}\right\rfloor+1\right) \\
&= d\left\lfloor\frac{x}{\delta}\right\rfloor + \left(d-\delta+\delta\left\{\frac{x}{\delta}\right\}\right)\sum_{j=0}^{d-1}\left(\left\lfloor\frac{n+j}{\delta}\right\rfloor+\left\lfloor\frac{j}{\delta}\right\rfloor-\left\lfloor\frac{n+j-k}{\delta}\right\rfloor-\left\lfloor\frac{j+k}{\delta}\right\rfloor\right) \\
&= \left(d\left\lfloor\frac{n}{\delta}\right\rfloor+\left(d-\delta+\delta\left\{\frac{n}{\delta}\right\}\right)\right)+\left(d\left\lfloor\frac{0}{\delta}\right\rfloor+\left(d-\delta+\delta\left\{\frac{0}{\delta}\right\}\right)\right) \\
&\quad -\left(d\left\lfloor\frac{n-k}{\delta}\right\rfloor+\left(d-\delta+\delta\left\{\frac{n-k}{\delta}\right\}\right)\right)-\left(d\left\lfloor\frac{k}{\delta}\right\rfloor+\left(d-\delta+\delta\left\{\frac{k}{\delta}\right\}\right)\right) \\
&= d\left(\left\lfloor\frac{n}{\delta}\right\rfloor+\left\lfloor\frac{0}{\delta}\right\rfloor-\left\lfloor\frac{n-k}{\delta}\right\rfloor-\left\lfloor\frac{k}{\delta}\right\rfloor\right)+\delta\left(\left\{\frac{n}{\delta}\right\}+\left\{\frac{0}{\delta}\right\}-\left\{\frac{n-k}{\delta}\right\}-\left\{\frac{k}{\delta}\right\}\right) \\
&= \delta\left(\frac{n}{\delta}-\frac{n-k}{\delta}-\frac{k}{\delta}\right)+(d-\delta)\left(\left\lfloor\frac{n}{\delta}\right\rfloor-\left\lfloor\frac{n-k}{\delta}\right\rfloor-\left\lfloor\frac{k}{\delta}\right\rfloor\right)
\end{aligned}$$

Since, $\lfloor a+b\rfloor \geqslant \lfloor a\rfloor+\lfloor b\rfloor$, we must have

$$\left\lfloor\frac{n}{\delta}\right\rfloor-\left\lfloor\frac{n-k}{\delta}\right\rfloor-\left\lfloor\frac{k}{\delta}\right\rfloor \geqslant 0,$$

and therefore

$$\begin{aligned}
&\sum_{j=0}^{d-1}\left(\left\lfloor\frac{n+j}{\delta}\right\rfloor+\left\lfloor\frac{j}{\delta}\right\rfloor-\left\lfloor\frac{n+j-k}{\delta}\right\rfloor-\left\lfloor\frac{j+k}{\delta}\right\rfloor\right) \\
&\quad = \delta\left(\frac{n}{\delta}-\frac{n-k}{\delta}-\frac{k}{\delta}\right)+(d-\delta)\left(\left\lfloor\frac{n}{\delta}\right\rfloor-\left\lfloor\frac{n-k}{\delta}\right\rfloor-\left\lfloor\frac{k}{\delta}\right\rfloor\right) \\
&\quad = (d-\delta)\left(\left\lfloor\frac{n}{\delta}\right\rfloor-\left\lfloor\frac{n-k}{\delta}\right\rfloor-\left\lfloor\frac{k}{\delta}\right\rfloor\right) \geqslant 0.
\end{aligned}$$

This proves that for all positive integers δ, $g(\delta)$ is not a factor in the denominator of $C_d(n)$, so the only possible denominator is 1, and therefore we are done. ■

Remark 4. This proves that all columns Pascal's triangle are binomid because it is the binomid triangle of $c = (1, 2, 3, 4, \dots)$, and $c(n) = n$ is the divisor-product sequence $c(n) = \prod_{d|n} g(d)$, where

$$g(n) = \begin{cases} p & \text{if } n = p^k > 1 \text{ is a prime power} \\ 1 & \text{otherwise} \end{cases}.$$

A Useful Lemmas

Lemma 9. For any real numbers α and β,

$$\lfloor \alpha+\beta \rfloor = \begin{cases} \lfloor \alpha \rfloor + \lfloor \beta \rfloor & \{\alpha\} \leqslant \{\alpha+\beta\} \\ \lfloor \alpha \rfloor + \lfloor \beta \rfloor + 1 & \{\alpha\} > \{\alpha+\beta\} \end{cases},$$

where $\{x\}$ is the fractional part of x.

Proof. First note that for any $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} \lfloor \alpha \rfloor + \{\alpha\} + \lfloor \beta \rfloor \{\beta\} &= \alpha + \beta \\ &= \lfloor \alpha+\beta \rfloor + \{\alpha+\beta\} \\ \lfloor \alpha+\beta \rfloor &= (\alpha+\beta) + (\{\alpha\} + \{\beta\} - \{\alpha+\beta\}). \end{aligned}$$

Therefore, we need only prove that

$$\{\alpha\} + \{\beta\} - \{\alpha+\beta\} = \begin{cases} 0 & \{\alpha\} \leqslant \{\alpha+\beta\} \\ 1 & \{\alpha\} > \{\alpha+\beta\}. \end{cases}$$

We have that

$$alpha + \beta = \lfloor \alpha+\beta \rfloor + \{\alpha+\beta\},$$

where $\{\alpha\} + \{\beta\} = \{\alpha+\beta\} + n$ for some integer $n \in \mathbb{Z}_{\geqslant 0}$. Because $\{\alpha\} + \{\beta\} < 2$ and $\{\alpha+\beta\} < 1$, by definition of $\{x\}$, it follows that $n = 0$ or $n = 1$.

Suppose first that

$$\{\alpha\} \leqslant \{\alpha+\beta\} = \{\alpha\} + \{\beta\} - n,$$

so that $n \leqslant \{\beta\} < 1$. In this case, it follows that we must have $n = 0$. Now suppose that

$$\{\alpha\} > \{\alpha+\beta\} = \{\alpha\} + \{\beta\} - n.$$

By the same logic as before, we have that $n > \{\beta\} > 0$. Notice that $\{\beta\} = 0$ implies that $\{\alpha\} = \{\alpha+\beta\}$ which contradicts our assumption that $\{\alpha\} > \{\alpha+\beta\}$. In this case, it follows that we must have $n = 1$. ■

Lemma 10. For all integers a and p, $\left\lfloor \frac{a}{p} \right\rfloor = \left\lfloor \frac{\lfloor a \rfloor}{p} \right\rfloor$.

Proof. Firstly, note that since $a \geqslant \lfloor a \rfloor$, we must have $\left\lfloor \frac{a}{p} \right\rfloor \geqslant \left\lfloor \frac{\lfloor a \rfloor}{p} \right\rfloor$.

Let $k = \left\lfloor \frac{a}{p} \right\rfloor$. Then, we have $kp \leqslant a < (k+1)p$. Since $kp \in \mathbb{Z}$, we have $\lfloor a \rfloor \geqslant kp$, so $\left\lfloor \frac{\lfloor a \rfloor}{p} \right\rfloor \geqslant \left\lfloor \frac{a}{p} \right\rfloor$.

As we have $\left\lfloor \frac{a}{p} \right\rfloor \geqslant \left\lfloor \frac{\lfloor a \rfloor}{p} \right\rfloor$ and $\left\lfloor \frac{\lfloor a \rfloor}{p} \right\rfloor \geqslant \left\lfloor \frac{a}{p} \right\rfloor$, it is implied that $\left\lfloor \frac{a}{p} \right\rfloor = \left\lfloor \frac{\lfloor a \rfloor}{p} \right\rfloor$. ■

Lemma 11. For all integers a, b, c, and p,

$$\lfloor a \rfloor = \lfloor b \rfloor + \lfloor c \rfloor \implies \left\lfloor \frac{a}{p} \right\rfloor \geqslant \left\lfloor \frac{b}{p} \right\rfloor + \left\lfloor \frac{c}{p} \right\rfloor .$$

Proof. Using Lemma 10, we have

$$\begin{aligned} \left\lfloor \frac{a}{p} \right\rfloor &= \left\lfloor \frac{\lfloor a \rfloor}{p} \right\rfloor \\ &= \left\lfloor \frac{\lfloor b \rfloor + \lfloor c \rfloor}{p} \right\rfloor \\ &\geqslant \left\lfloor \frac{\lfloor b \rfloor}{p} \right\rfloor + \left\lfloor \frac{\lfloor c \rfloor}{p} \right\rfloor \\ &= \left\lfloor \frac{b}{p} \right\rfloor + \left\lfloor \frac{c}{p} \right\rfloor . \end{aligned}$$

■