1 Settings

Head model
For simplification we start using a simple 2D 3-sphere-head-model with center at $c = (127, 127)$. We set the conductivities as shown in table 1. The head model has a diameter of 184mm.

Tissue	Outer radius (mm)	Conductivity $(1/(\Omega \cdot m))$
1	78	0.00043
2	86	0.00179
3	92	0.00001

Table 1: Parameters of 3-sphere-head-model

Dipole
In the beginning we simulate a single dipole. We choose the dipole,

$$S = ([p_x, p_y], [m_x, m_y]) = ([80, 150], [-0.898217, 0.439553])$$

which means we have a source at position $[80, 150]$ with radial orientation $[-0.898217, 0.439553]$.

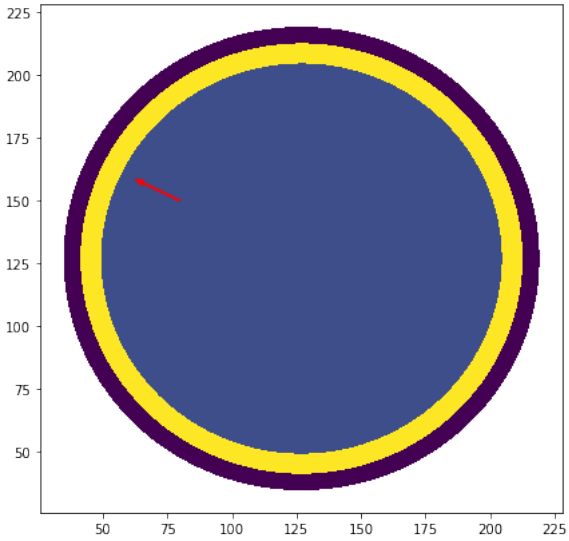
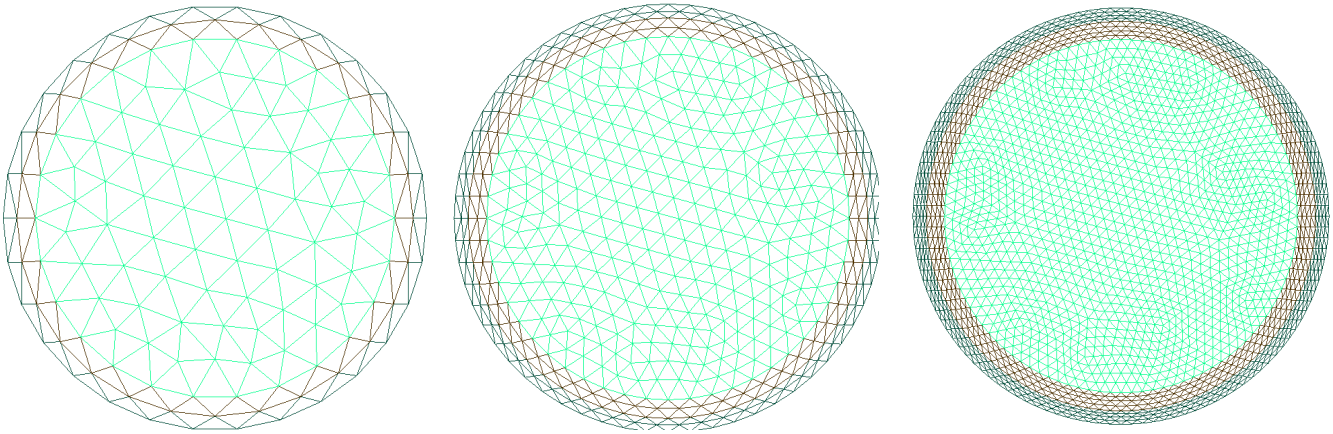


Figure 1: 3-sphere-model with radial dipole at $(80, 150)$

Meshes
We define three meshes with different coarseness. Teh different meshes will define our first hierarchy. So we create a coarse mesh for level 1. The meshes on level 2 and 3 are generated by refining the mesh on the level before by splitting. The resulting meshes are shown in figure 2.

Electrodes
Next we have to choose an electrode configuration which includes the number of electrodes and their positions. We choose these nodes of the coarsest mesh that are on the outer border of the head model as electrode positions. By this we get $m = 30$ electrodes.



(a) Mesh at Level 1 (b) Mesh at Level 2 (c) Mesh at Level 3

Figure 2: Meshes at the different levels, created by refinement by splitting

2 Reconstruction of the source

Simulate measurement values
We can compute the measurement values at the electrodes $u \in \mathbb{R}^m$ by solving the forward problem for our given dipole using DUNEuro. We use the finest mesh and choose the Venant source model.

We use these measurement values to reconstruct the source. Later we simulate noise by adding a vector of independent, normal distributed random values on U:

$$\tilde{u} = u + \mathcal{E}$$

with $\mathcal{E} \sim \mathcal{N}(0, \beta^2 I_m)$ where we choose $\beta^2 = \rho \cdot \|u\|_{L_\infty}$ with relative noise ratio ρ

Leadfield matrices
For each level we compute a leadfield matrix. As dipoles we choose radial dipoles located at the nodes of the corresponding mesh. Again we use the Venant source model.

Transfer matrices
For each level we compute a transfer matrix. Again we use the Venant source model.

Evaluate a dipole candidate
If we have a proposal for a possible dipole position p (since we only observe radial dipoles the orientation follows directly from the position) we can now evaluate this proposal in two different ways using the leadfield matrix or using the transfer matrix.

Leadfield matrix approach:

1. Find the closest mesh node using the L2-Norm.
2. Select the corresponding column of the leadfield matrix v. These are the measurement values we would expect to measure at the electrodes if the source was located at the proposal position.
3. Now we can compare these values v to the simulated measurement values u by computing the L2-Norm.

Transfer matrix approach:

1. Assemble right hand side b for the dipole p
2. Multiply transfer matrix with right hand side b

$$v = Tb$$

3. Now we can compare these values v to the simulated measurement values u by computing the L2-Norm.

3 Algorithms

3.1 Metropolis Hastings Algorithm

We use the Metropolis-Hastings algorithm to get a probability distribution of the source. We generate a new candidate ψ by performing one step of a random walk in $\mathbb{R}^2$ with variance τ^2 starting at the last sample θ_{i-1}. Then we compute the acceptance probability as

$$\alpha = \min\left(1, \exp(\pi_t(\psi) - \pi_t(\theta_{i-1}))\right)$$

with

$$\pi_t(x) = \log\left(\left(\frac{1}{2\sigma^2}\right)^{\frac{m}{2}} \exp\left(-\frac{1}{2\sigma^2}\left(\frac{\|u-v\|}{\|u\|}\right)^2\right)\right).$$

In addition we set $\pi_t(x) = \log(10^{-20})$ such that $\alpha \approx 0$ if $\|\psi - c\|_{L2} > 78$ that means ψ does not lay in the inner compartement.

The choice of σ has a strong impact on the results because it defines the variance we allow. If we add noise of relative noise ratio ρ on our measurement values we can set $\sigma = \alpha \cdot \rho \cdot \|u\|_{L_\infty}$. If we don't observe noise we have to choose σ small, say $\sigma = 0.001 \cdot \alpha \cdot \|u\|_{L_\infty}$. The factor α is the variance factor. By default it is 1 but we can increase it to allow a higher variance.

3.2 MLDA Algorithm

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4 Results

4.1 Error measures

We need an error measure to quantify the error a sample distribution contains.

The first idea is to compute the mean and compare it to the dipole position using the L2-Norm. We call this error e_1.

The second idea is to run the MH algorithm on the finest mesh for a large number of steps (50000) starting at the correct dipole. The resulting sample distribution can be used as a reference. We now compare other sample distributions to this reference by computing the L2-error. We call this error e_2.

4.2 Reconstruction of radial dipoles

Example 1: Metropolis Hastings at one Level
With this example we want to illustrate the Metropolis-Hastings Algorithm in general. For simplification we only observe radial dipoles and don't add noise on the measurement values. That means we expect a sharp reconstruction of the given dipole. We use the proposal density $\tau = 4$. We start in the center at (127,127) and try to reconstruct a radial dipole at (80,150). Figure 3 shows the positions of the generated samples. We see that in the first few steps samples close to the starting point were generated, but there is a clear shift to the dipole. After 1000 steps there is a clear concentration around the dipole. The last picture shows how a so called Burn In improves the results by rejecting the first samples. Figure 4 shows the relative number of samples per cell and figure 5 the continuous distribution of samples generated by kernel density estimation. This example shows that the Metropolis Hastings algorithm can be used to reconstruct the position of a radial dipole from the measurement values.

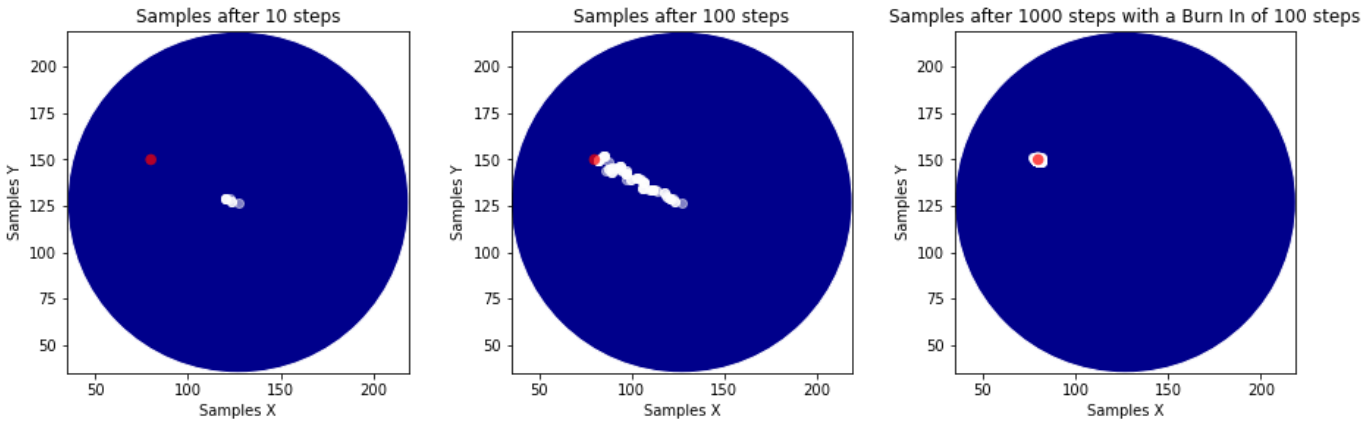

Figure 3: Samples created by the Metropolis Hastings algorithm starting in the center (127,127) without noise

Example 2: MH on different meshes
Now we want to analyse how the MH algorithm works on different meshes. We use the meshes defined in section 1. The results can be viewed in figure 6. We can see that we get good reconstructions for each mesh. To compare the results we compute the error measures defined in 4.1.

Level 1:

$$e_1 = 0.345, \quad e_2 = 0.144$$

Level 2:

$$e_1 = 0.091, \quad e_2 = 0.085$$

Level 3:

$$e_1 = 0.071, \quad e_2 = 0.028$$

This shows that we get the best result for the finest mesh.

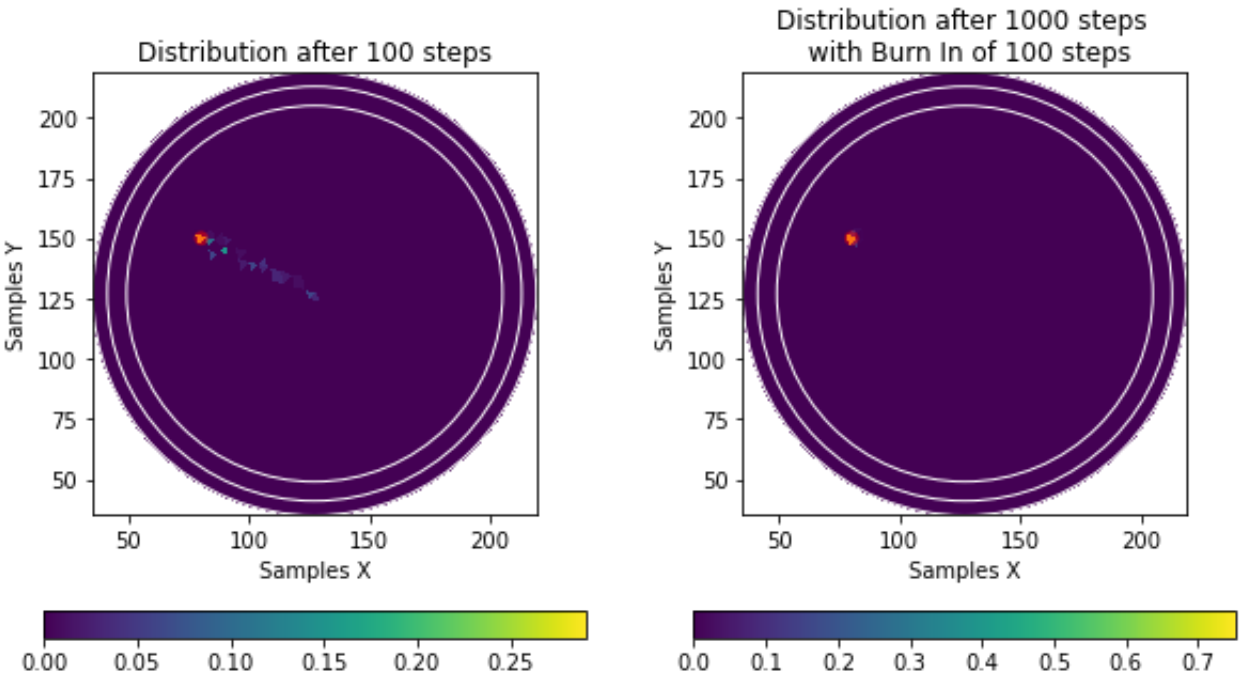

Figure 4: Relative number of samples per cell for samples created by the Metropolis Hastings algorithm starting in the center (127,127) without noise

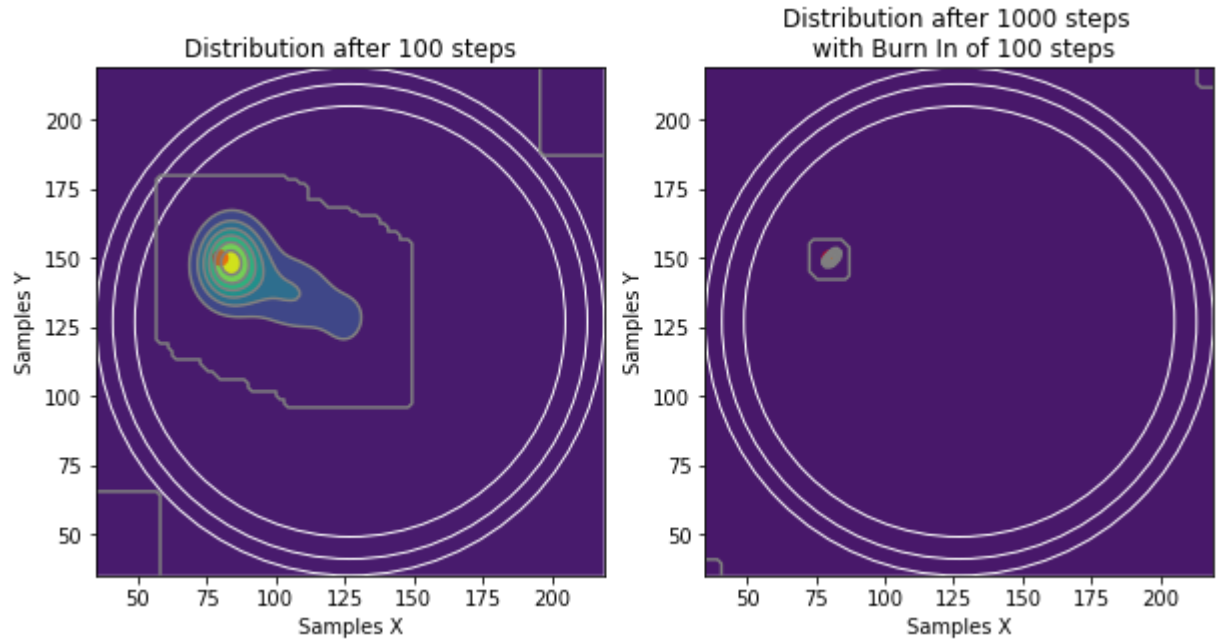

Figure 5: Distribution of samples created by the Metropolis Hastings algorithm starting in the center (127,127) without noise

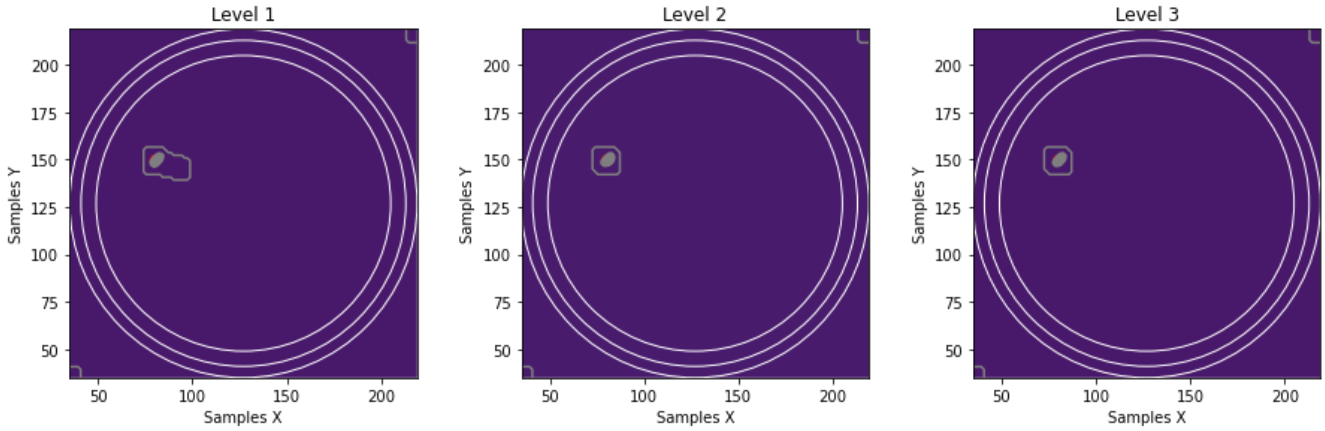

Figure 6: Distribution of samples created by MH without noise at different meshes

Example 3: MH with noise
Now we add noise an the measurement values. In figure 7 we show the impact of different noise ratios. With increasing noise ratio the variance of the resulting distribution is increasing which is exactly what we would have expected.

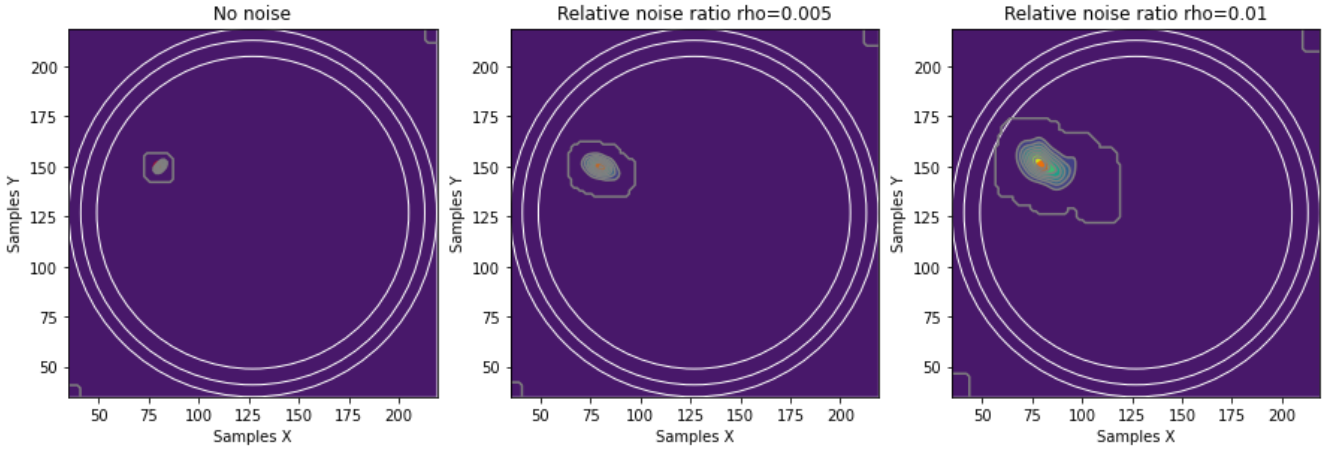

Figure 7: Distribution of samples created by MH algorithm starting in the center (127,127) with different noise ratios

Example 4: Comparison of MH and MLDA
The comparison of MH and MLDA depends highly on the choice of parameters for each algorithm. In table 2 the chosen parameters (which seem to be a good choice) can be found.

	MH	MLDA Level 1	MLDA Level 2	MLDA Level 3
Proposal variance position	2	1	1	1
Number of subsamples	-	5	3	-
Target variance factor	1	8	2	1

Table 2: Parameters of MH and MLDA algorithm in example 4

A first result is shown in figure 8 where we can see that MLDA can reach similar results as the MH algorithm.

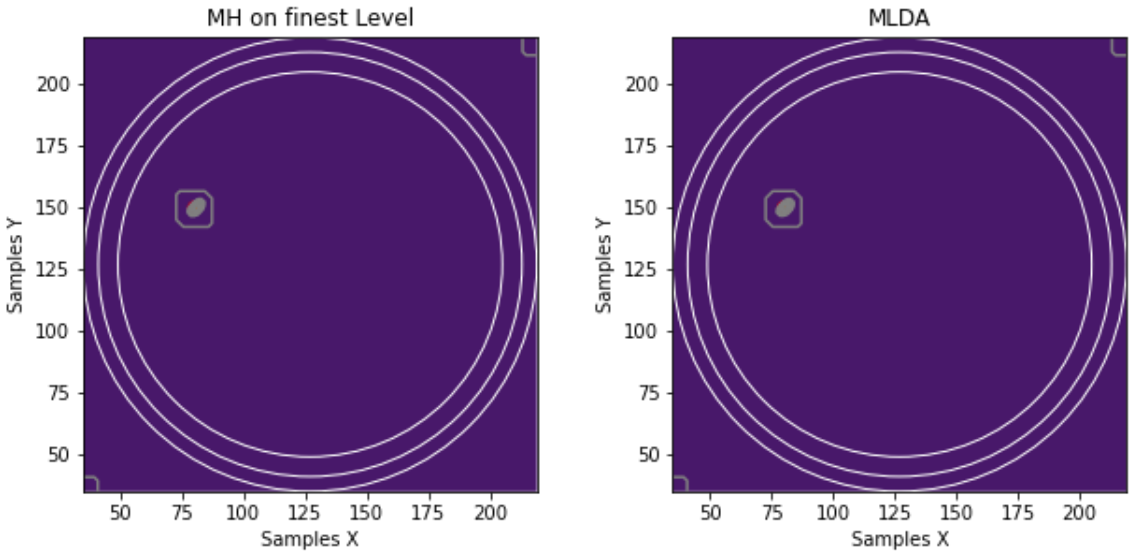

Figure 8: Comparison of the resulting distribution of samples generated by MH on the finest level and by MLDA for a radial dipole and no noise on the measurement values

If we compute the error measures we see that MLDA is performing better than MH.

MH on finest level (2000 samples, Burn In 200):

$$e_1 = 0.053, \quad e_2 = 0.020$$

MLDA (2000 samples, Burn In 50):

$$e_1 = 0.022, \quad e_2 = 0.018$$

There are two benefits of the MLDA algorithm we want to show:

1. It is possible to reconstruct a source that is far away from the starting point without loosing accuracy or strongly increasing number of samples.
2. We need less computation time as we need less samples on the finest level to reach the same accuracy as the MH algorithm.

The first point is visible in figure 9. While MH algorithm needs a Burn In of 200 samples, MLDA generates good samples after a Burn In of 50 samples.

To show the second point, we compute the error measures from section 4.1.

MH on finest level (2000 samples, Burn In 200):

$$e_1 = 0.053, \quad e_2 = 0.023$$

MLDA (2000 samples, Burn In 50):

$$e_1 = 0.022, \quad e_2 = 0.017$$

Now we can compute the number of samples we need to reach a given accuracy, see table 3.

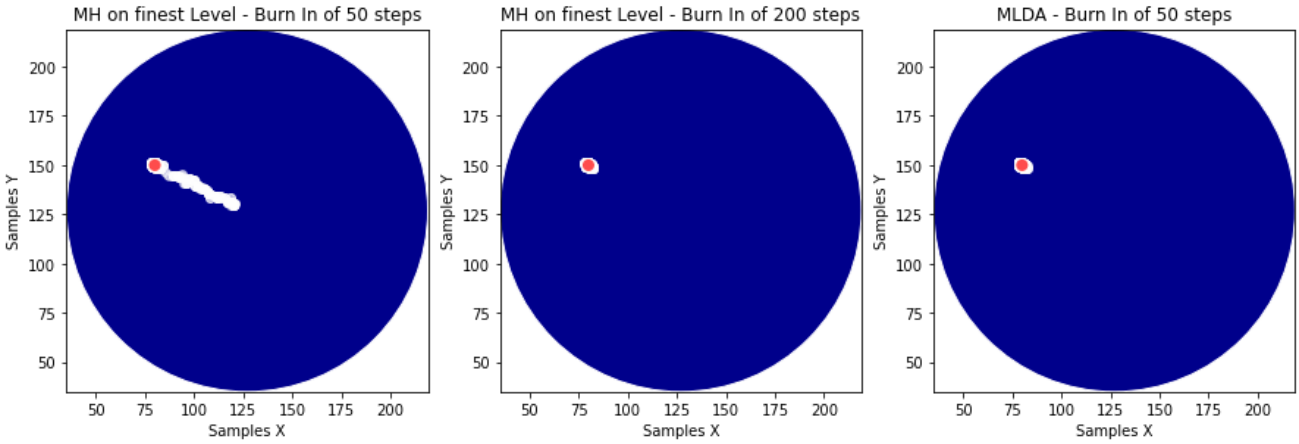

Figure 9: Samples of MH and MLDA with different Burn In

Max e_2	Samples MH	Samples MLDA
0.1	500	200
0.075	1100	200
0.05	1200	500
0.025	2300	900

Table 3: Number of samples to reach a given accuracy

4.3 Reconstruction of dipoles with arbitrary orientation

Example 5: MH for reconstruction of position and orientation
We now observe no longer only radial dipoles but dipoles with an arbitrary orientation which is defined by the angle ϕ, which means we have a 3-dimensional model now.
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Examples 6: MLDA to reconstruct position and orientation
One advantage of the MLDA algorithm is that we are able to make large steps on the coarse mesh to move fast from the start close to our dipole and small steps on the fine mesh to get a high accuracy.
...

Level	1	2	3
Proposal variance position	16	8	1
Proposal variance orientation	0.2	0.1	0.05
Variance factor	8	4	1

Table 4: Parameters of MLDA in example 6

5 Open points and further ideas

- Optimize parameter choice for MH and MLDA
- Find a good representation of generated dipole candidates including position and orientation.
- Find a good measure to quantify the error of the generated samples. This can be used to compare the number of samples MH and MLDA need to reach a given error.
- Results are random and might depend on the chosen dipole - how to get reliable results?
- Measure running time of MH and MLDA