Valuative Capacity of some compact subsets of $\mathbb{Z}_p$

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Background: p-orderings, p-sequences

A $p-$**ordering** of an infinite set, $S \subseteq \mathbb{Z}_p$, is a sequence in S such that for $\forall n > 0$, a_n minimizes

$$v_p((x - a_{n-1}) \ldots (x - a_0))$$

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cf: A ρ-**ordering** of S, a (compact) subset of an ultrametric space (M, ρ), is a sequence in S such that $\forall n > 0$, a_n maximizes

$$\prod_{i=0}^{n-1} \rho(x, a_i)$$

Background: p-orderings, p-sequences

The p-**sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is

$$v_p((a_n - a_{n-1}) \dots (a_n - a_0))$$

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Background: valuative and logarithm capacity

The **valuative capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$L_p(S) := \lim_{n \to \infty} \frac{w_S(n,p)}{n}$$

where $w_S(n,p)$ is the $p-$sequence of S.

nb: this is the Robin's constant and can be found via the equilibrium measure:

$$L_p(S) = \inf_{\mu \in \mathcal{P}(\bar{S})} \int \int v_p(x-y) d\mu(x) d\mu(y)$$

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The **logarithm capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$V_p(E) := p^{-L_p(E)}$$

nb: this is equal to the transfinite diameter and the Chebychev constant.

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Fares and Petite, Lemma 5.1

Let $A = \{0, 1, .., d-1\}$ be a finite alphabet and $A^{\mathbb{N}}$ be the collection of infinite sequenes with values in A.

Let $p \geq d$ be a prime number and let ϕ be the canonical embedding of $A^{\mathbb{N}}$ into $\mathbb{Z}_p$ via the following continuous map:

$$\phi : A^{\mathbb{N}} \to \mathbb{Z}_p \text{ by } (x_n)_{n \geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

Fares and Petite, Lemma 5.1

Lemma

Let $w_1, w_2, \dots, w_s$ be $s \geq 2$ words with the same length l such that all the first letters are distinct. Let $X \subset A^{\mathbb{N}}$ be the set of sequences such that any factor is a factor of a concatenation of the words $w_1, w_2, \dots, w_s$. Then the set $E := \phi(X) \subset \mathbb{Z}_p$ satisfies:

$$E = \cup_{i=1}^{s} x_i + p^l E, \text{ with } x_i = \phi(w_i 0^\infty)$$

It is a regular compact set and its valuative capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

Fares and Petite, Lemma 5.1

An example:

$w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$
Then $\{x_n\}_{n\geq 0} \in X$ if each term in $\{x_n\}_{n\geq 0}$ is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Digression: projective k-space

Let k be a field that is complete with respect to a non-archimedean valuation.

Definition

The **projective line over** k, denoted $\mathbb{P}^1(k)$, is the space whose points are lines l in k^2 that intersect $(0,0)$.

Proposition

Let $\psi : k \to \mathbb{P}^1(k)$ be the map given by $\psi(\lambda_0) = [1, \lambda_0]$, where $[1, \lambda_0]$ is the line in k^2, $\{\lambda(1, \lambda_0); \lambda \in k^*\}$. Then the image of ψ is $\mathbb{P}^1(k) \setminus [0,1]$ and is isomorphic to k, so that k is identified with projective space minus a distinguished point, $[0,1]$, which is denoted by ∞.

Digression: projective k-space

Definition

We denote by $GL(2,k)$ the set of invertible 2×2 matrices over k. A **fractional linear automorphism**, ϕ, of $\mathbb{P}^1(k)$ is a map defined by $z \mapsto \frac{az+b}{cz+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,k)$. The set of fractional linear automorphisms of $\mathbb{P}^1(k)$ is denoted $PGL(2,k)$.

Note that $PGL(2,k) = GL(2,k)/\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right); \lambda \in k^*\}$. In homogeneous coordinates, we can represent the action of ϕ by $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$.

Digression: projective k-space

Definition

Suppose Γ is a subgroup of $PGL(2, k)$. A point $p \in \mathbb{P}^1(k)$ is a **limit point of** Γ, if there exists a point q in $\mathbb{P}^1(k)$ and a sequence $\{\gamma_n\}_{n \geq 1}$ in Γ such that $\lim_{n\to\infty} \gamma_n(q) = p$.

Fares and Petite, Lemma 5.1, rephrased (1/2)

Let $x_1, x_2, \ldots, x_s$ be $s \geq 2$ points in $\mathbb{Z}_p$ such that $\mid x_i - x_j \mid_p = 1$, $\forall i, j \in 1, ..., s$. Suppose also that there exists an $l \in \mathbb{N}$ such that $\forall i$,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{l} a_i p^i$$

Fares and Petite, Lemma 5.1, rephrased (2/2)

Let γ_i be the fractional linear automorphism of $\mathbb{P}^1(\mathbb{Q}_p)$ given by $\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix}$ and let Γ be the subgroup of $PGL(2, \mathbb{Q}_p)$ generated by the γ_i.

Then Γ has a subgroup H such that the limit set $\mathcal{L}$ of H has the property that $Z = \psi^{-1}(\mathcal{L})$ is equal to $\phi(X)$ in the original lemma. In particular Z is a regular, compact subset of $\mathbb{Z}_p$ satisfying

$$Z = \cup_{i=1}^{s} x_i + p^l Z = \cup_{i=1}^{s} B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

Fares and Petite, Lemma 5.1, rephrased

Sketch of proof:

- We have to show w that the set Z above is equal to $E = \phi(X)$ in the original lemma.
- That that w_i correspond to the x_i is not hard to see.
- What is the limit set of Γ?

Limit set of Γ

Let $\gamma \in \Gamma$.

- If γ is a product of the generators γ_i, then γ is represented by a matrix of the form: $\begin{pmatrix} p^{lm} & z \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and z is an element of $\mathbb{Z}_p$ whose coefficient vector is a concatenation of the coefficient vectors of the x_i (for $0 \leq i \leq ml$ and 0 for $i > ml$).
- For example,

$$\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^l & x_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^l & x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{3l} & p^{2l}x_k + p^l x_j + x_i \\ 0 & 1 \end{pmatrix}$$

- The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{lm}a_1 + za_0] \sim [1, p^{lm}\frac{a_1}{a_0} + z]$$

Limit set of Γ

Let $\gamma \in \Gamma$.

- If γ is a product of the inverses of the generators γ_i^{-1}, then γ is represented by a matrix of the form: $\begin{pmatrix} p^{-lm} & -p^{-l}z^{-1} \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and z is as above.
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$$\begin{pmatrix} p^{-l} & -p^{-l}x_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^{-l} & -p^{-l}x_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^{-l} & -p^{-l}x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{-3l} & -p^{-3l}x_k - p^{-2l}x_j - p^{-l}x_i \\ 0 & 1 \end{pmatrix}$$

- The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{-lm}a_1 - p^{-l}z^{-1}a_0] \sim [1, p^{-l}(p^{-m}\frac{a_1}{a_0} - z^{-1})]$$

Limit set of Γ

Let $\gamma \in \Gamma$.

- If γ is of the form $\gamma_j^{-1}\gamma_i$, for $i \neq j$, then γ is represented by a matrix of the form: $\left(\begin{smallmatrix} 1 & p^{-l}(x_i - x_j) \\ 0 & 1 \end{smallmatrix}\right)$
- The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, a_1 + p^{-l}(x_i - x_j)a_0] \sim [1, \frac{a_1}{a_0} + p^{-l}(x_i - x_j)]$$

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- We quotient the group Γ by the group generated by the translations to obtain H.

Discussion

In fact, all of the translations commute with each other, so we can quotient by the entire translation subgroup, ie the subgroup generated by $\{\gamma_i\gamma_j^{-1}, \gamma_i^{-1}\gamma_j; \forall i, j \in 1, \ldots, s\}$

The resulting quotient group is discontinuous, finitely generated and every element ($\neq id$) is hyperbolic, ie it is a Schottky group.

Discussion

Consider the following:

$$\begin{tikzcd} S \subseteq \mathbb{Z}_p \arrow[r] \arrow[d] & \mathbb{Q}_p \arrow[d] \\ \mathbb{P}(\mathbb{Z}_p) & \mathbb{P}(\mathbb{Q}_p) \end{tikzcd}$$

references

- Youssef Fares and Samuel Petite, The valuative capacity of subshifts of finite type.
- Keith Johnson, P-orderings and Fekete sets