# $\rho$-orderings and valuative capacity in ultrametric spaces

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# Ultrametric basics

### Definition

A $p-$**ordering** of an infinite set, $S \subseteq \mathbb{Z}_p$, is a sequence in $S$ such that for $n > 0$, $a_n$ minimizes

$$v_p((x - a_{n-1}) \ldots (x - a_0))$$

### Definition-Proposition

The $p$-**sequence** of $S$ is the sequence whose $0^{th}$-term is 1 and whose $n^{th}$ term, for $n > 0$, is $v_p((a_n - a_{n-1}) \ldots (a_n - a_0)$.

# $\rho$-orderings, $\rho$-sequences, and valuative capacity

In what follows, let $S$ be a compact subset of an ultrametric space $(M, \rho)$.

## Definition

[2] A $\rho$-**ordering** of $S$ is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ such that $\forall n > 0$, $a_n$ maximizes $\prod_{i=0}^{n-1} \rho(s, a_i)$ over $s \in S$.

# $\rho$-orderings, $\rho$-sequences, and valuative capacity

## Definition-Proposition

[2] Let $\gamma(n)$ be the $\rho$-sequence of $S$. The **valuative capacity** of $S$ is

$$\omega(S) := \lim_{n \to \infty} \gamma(n)^{1/n}$$

# valuative capacity: quick results

## Proposition

(translation invariance) Let $(M, \rho)$ be a compact ultrametric space and suppose $M$ is also a topological group. If $\rho$ is (left) invariant under the group operation, then so is $\omega$. That is, if $\rho(x, y) = \rho(gx, gy)$, $\forall g, x, y \in M$, then $\omega(gS) = \omega(S)$, for $S \subseteq M$.

## Proposition

(scaling) Let $(V, N)$ be a normed vector space and suppose $N$ satisfies the strong triangle identity. Then if $N$ is multiplicative, so is $\omega$. That is, if $N(gx) = N(g)N(x)$,$\forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.

# valuative capacity: subadditivity

## Proposition

[2](subadditivity) If $diam(S) := \max_{x,y\in S} \rho(x,y) = d$ and $S = \cup_i^n A_i$ for $A_i$ compact subsets of $M$ with $\rho(x_i, x_j) = d$, $\forall x_i \in A_i$, $\forall x_j \in A_j$ and $\forall i, j$, then

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n} \frac{1}{log(\omega(A_i)/d)}$$

## Corollary

*Suppose* $S = \cup_i^n S_i$ *with* $\rho(S_i, S_j) = d = diam(S)$ *and also* $\omega(S_i) = \omega(S_j)$, $\forall i, j$ . *Let* $r \in \mathbb{R}$ *be such that* $\omega(S_i) = r\omega(S)$, $\forall i$. *Then* $\omega(S) = r^{\frac{1}{n-1}} \cdot d$. *In particular if* $S = \mathbb{Z}$ *and* $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$ *then* $\omega(S) = (\frac{1}{p})^{1/p-1}$ *for any prime* $p$.

# Constructing a $\rho$-ordering

Setup:

- Let $S \subseteq M$ be a compact subset of an ultrametric space $(M, \rho)$.
- Let $\Gamma_S = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in $S$.
- Note that for each $k \in \mathbb{N}$, the closed balls of radius $\gamma_k$ partition $S$. That is,

$$S = S_{\gamma_k} := \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$$

  where both $n$ and the $x_i$'s depend on $k$.

# Constructing a $\rho$-ordering

Setup, continuted:
Fix a $k \in \mathbb{N}$.

- Let $S_{\gamma_k} = \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$ be a partition of $S$, as above.
- Note that we can construct $S_{\gamma_{k+1}}$ by partitioning each of the $\overline{B_{\gamma_k}(x_i)}$ , i.e.,

$$S = S_{\gamma_{k+1}} = \cup_{i=1}^{n} \cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})}$$

  where $1 \leq l_i \leq n$ and $\cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})} = \overline{B_{\gamma_k}(x_i)}$, $\forall i$.
- We denote by $x_{i,j}$ the centre of a ball of radius $\gamma_{k+1}$ partitioning the ball $B_{\gamma_k}(x_i)$.
- Without loss of generality, when $j = 1$, assume $x_{i,j} = x_i$, $\forall i$.

# Constructing a $\rho$-ordering

We now make the following observation due to [3],

**Lemma**

*For each $k \in \mathbb{N}$, the elements of $S_{\gamma_k}$, that is, the closed balls of radius $\gamma_k$, themselves form an ultrametric space, where*

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

# Constructing a $\rho$-ordering

We make the following observations:

- Since $S$ is compact, $S_{\gamma_k}$ is a finite metric space $\forall k < \infty$ and $S_{\gamma_\infty} = \cup_{x \in S} \overline{B_0(x)} = \cup_{x \in S} x = S$ and $\rho_\infty = \rho$.
- View $S_{\gamma_k}$, for fixed $k < \infty$ as a finite ultrametric space with $n$ elements. Let us denote an element of $S_{\gamma_k}$, that is a $\overline{B_{\gamma_k}(x_i)}$, by its centre, $x_i$.
- Without loss of genearlity, we can reindex the $x_i$'s so that they give the first $n$ terms of a $\rho_k$-ordering of $S_{\gamma_k}$.

# Constructing a $\rho$-ordering

Setup, revisited:

- Let $S_{\gamma_k} = \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$ be the finite metric space describe above, and suppose the $x_i$ are indexed according to a $\rho_k$-ordering of $S_{\gamma_k}$.
- Let $S_{\gamma_{k+1}}$ be the finite metric space formed by partitioning each of the $B_{\gamma_k}(x_i)$, so that $S_{\gamma_{k+1}} = \cup_{i=1}^{n} \cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})}$ and $x_{i,j}$ is a point in the ball $B_{\gamma_k}(x_i)$ with the convention that $x_{i,1} = x_i$, $\forall i$.

# Constructing a $\rho$-ordering

Consider the matrix $A_k$, whose $(i,j)^{th}$-entry is $x_{i,j}$ (or * if $l_i < j$).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \cdots & x_{n,l_n} \end{pmatrix}$$

# Constructing a $\rho$-ordering

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A $\rho_{k+1}$-ordering of $S_{\gamma_{k+1}}$ can be found by concatenating the rows of $A_k$ (and ignoring *'s).

# Some corollaries

### Corollary

*Interweaving the bottown row of the lattice of closed balls for a set $S$ gives a $\rho$-ordering of $S$. In particular, the natural ordering on the integers gives a $\rho_p$-ordering for every prime $p$.*

### Corollary

*Suppose $S$ and $T$ are compact subsets of an ultrametric space $M$ with $\Gamma_S = \Gamma_T$ and $\mid S_{\gamma_k} \mid = \mid T_{\gamma_k} \mid, \forall k$. Then $\omega(S) = \omega(T)$.*

# references


- Alain M. Robert, A course in p-adic analysis.
- Keith Johnson, P-orderings and Fekete sets
- Nate Ackerman, Completeness in Generalized Ultrametric Spaces