

# Introduction

In the course of developing a generalized factorial function, Bhargava introduced the notion of  $p$ -orderings of a Dedekind domain [2, 3], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [4] (also KJ) and have also provided a natural framework to extend many classical results in analysis to a  $p$ -adic setting, such as polynomial approximation and mapping theorems [2, 3, 4].

In this thesis, we examine how a tool based on  $p$ -orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

# Background

## Ultrametric basics

**Definition.** Let  $(M, \rho)$  be a metric space. If  $\rho$  satisfies the ultrametric inequality

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)), \forall x, y, z \in M$$

then  $(M, \rho)$  is an **ultrametric space**.

**Definition.** Let  $(V, N)$  be a normed vector space. Then  $N$  satisfies the **strong triangle inequality** if

$$N(x + y) \leq \max(N(x), N(y)), \forall x, y \in V$$

**Proposition.** Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle inequality. Then the metric space,  $(V, \rho_N)$ , where  $\rho_N$  is the metric induced by  $N$ , is an ultrametric space.

**Proposition.** [1] All triangles in an ultrametric space  $(M, \rho)$  are either equilateral or isosceles, with at most one short side.

**Proposition.** [1] If  $S$  is a compact subset of an ultrametric space and  $\Gamma_S$  is the set of all distances occurring between points of  $S$ , then  $\Gamma_S$  is a discrete subset of  $\mathbb{R}$ . In particular if  $|\Gamma_S| = \infty$ , then the elements of  $\Gamma_S$  can be indexed by  $\mathbb{N}$ .

Let  $(M, \rho)$  be a compact ultrametric space and let

$$B_r(a) = \{x \in M \mid \rho(x, a) < r\}$$

denote the open ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{\geq 0}$  and  $a \in (M, \rho)$ . Likewise let

$$\overline{B_r(a)} = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the closed ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{\geq 0}$  and  $a \in (M, \rho)$ .

**Proposition.** Let  $B_r(a)$  be a ball in an ultrametric space  $(M, \rho)$ . Then the diameter of  $B$ ,  $d = \text{diam}(B) = \sup_{x, y \in B} \rho(x, y)$ , is less than or equal to the radius of  $B$ .

**Proposition.** If  $(M, \rho)$  is an ultrametric space and  $B_{r_1}(x_0)$  and  $B_{r_2}(y_0)$  are balls in  $(M, \rho)$ , then either  $B_{r_1}(x_0) \cap B_{r_2}(y_0) = \emptyset$ ,  $B_{r_1}(x_0) \subseteq B_{r_2}(y_0)$ , or  $B_{r_2}(y_0) \subseteq B_{r_1}(x_0)$ . That is, in an ultrametric space, all balls are either comparable or disjoint.

**Proposition.** [1] The distance between any two balls in an ultrametric is constant. That is, if  $B_{r_1}(x_0)$  and  $B_{r_2}(y_0)$  are two balls in an ultrametric space  $(M, \rho)$ , then  $\rho(x, y) = c$  for some  $c \in \mathbb{R}$  and  $\forall x \in B_{r_1}(x_0)$  and  $\forall y \in B_{r_2}(y_0)$

**Proposition.** [1] Every point of a ball in an ultrametric is at its centre. That is, if  $B_r(x_0)$  is a ball in an ultrametric space  $(M, \rho)$ , then  $B_r(x) = B_r(x_0)$ ,  $\forall x \in B_r(x_0)$

# $\rho$ -orderings, $\rho$ -sequences, and valutive capacity

In what follows let  $S$  be a compact subset of an ultrametric space  $(M, \rho)$ .

**Definition.** [5] A  $\rho$ -ordering of  $S$  is a sequence  $\{a_i\}_{i=0}^{\infty} \subseteq S$  such that  $\forall n > 0$ ,  $a_n$  maximizes  $\prod_{i=0}^{n-1} \rho(s, a_i)$  over  $s \in S$ .

**Definition.** [5] The  $\rho$ -sequence of  $S$  is the sequence whose 0<sup>th</sup>-term is 1 and whose  $n^{\text{th}}$  term, for  $n > 0$ , is  $\prod_{i=0}^{n-1} \rho(a_n, a_i)$ .

**Proposition.** [5] The  $\rho$ -sequence of  $S$  is well-defined so long as  $S$  is compact and  $\rho$  is an ultrametric. That is, the  $\rho$ -sequence of a compact subset of an ultrametric spaces does not depend on the choice of  $\rho$ -ordering of  $S$ .

**Definition.** [5] Let  $\gamma(n)$  be the  $\rho$ -sequence of  $S$ . The **valutive capacity** of  $S$  is

$$\omega(S) := \lim_{n \rightarrow \infty} \gamma(n)^{1/n}$$

**Proposition.** [5] For  $S$  and  $\gamma(n)$  as above,  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = r < \infty$ .

**Proposition.** If  $S \subseteq M$  is a finite subset of an ultrametric space, then  $\omega(S) = 0$ .

**Proposition.** (upper bound) If  $\text{diam}(S) := \max_{x,y \in S} \rho(x, y) = d$ , then  $\omega(S) < d$ .

*Proof.* Since  $d$  is the diameter of  $S$ , the  $n^{\text{th}}$  term of the  $\rho$ -sequence of  $S$  is bounded by  $d^n$  and so  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = d$  if and only if  $\gamma(n) = d^n$ ,  $\forall n$ . This implies  $\rho(a_n, a_i) = d$ ,  $\forall n$  and  $\forall i < n$ , but then  $\rho(a_i, a_j) = d$ ,  $\forall i, j$ , since the  $\rho$ -sequence is maximized at each  $n$ . This means  $\omega(S) < d$  would imply that the cover of  $S$ ,  $\cup_{a_i} B_d(a_i)$  is in fact an infinite partition, contradicting the compactness of  $S$ . Then  $\omega(S) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} < d$ .  $\square$

**Proposition.** (translation invariance) Let  $(M, \rho)$  be a compact ultrametric space and suppose  $M$  is also a topological group. If  $\rho$  is (left) invariant under the group operation, then so is  $\omega$ . That is, if  $\rho(x, y) = \rho(gx, gy)$ ,  $\forall g, x, y \in M$ , then  $\omega(gS) = \omega(S)$ , for  $S \subseteq M$ .

*Proof.* Let  $\{a_i\}_{i=0}^\infty$  be a  $\rho$ -ordering for  $S$ . Then  $\{ga_i\}_{i=0}^\infty$  is a  $\rho$ -ordering for  $gS$ . Then

$$\omega(gS) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = \omega(S)$$

□

**Example 1.** With the notation of the previous section, note that for  $x, y \in (\mathbb{Z}_p, |\cdot|_p)$ ,  $\rho_p(x, y) = |x - y|_p = p^{-\nu_p(x-y)} = p^{-\nu_p((a+x)-(a+y))} = |(a+x) - (a+y)|_p = \rho_p(a+x, a+y)$  so that  $\omega(a+S) = \omega(S)$  for  $S \in (\mathbb{Z}_p, |\cdot|_p)$ .

**Proposition.** Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle identity. Then if  $N$  is multiplicative, so is  $\omega$ . That is, if  $N(gx) = N(g)N(x), \forall g, x \in V$ , then  $\omega(gS) = N(g)\omega(S)$ , for  $g \in V$  and  $S \subseteq M$ .

*Proof.* Let  $\rho$  be the metric induced by  $N$ , so that  $\rho(x, y) = N(x - y), \forall x, y \in V$ . Let  $\{a_i\}_{i=0}^\infty$  be a  $\rho$ -ordering for  $S$ . Then since  $N$  is multiplicative, for  $u, v \in gS$ ,  $u = gs_i$  and  $v = gs_j$  for some  $s_i, s_j \in S$ ,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then  $\{ga_i\}_{i=0}^\infty$  is a  $\rho$ -ordering for  $gS$  and

$$\begin{aligned} \omega(gS) &= \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} N(g)\rho(a_n, a_i) \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} [N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g) \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g)\omega(S) \end{aligned}$$

□

**Example 2.** Since  $|\cdot|_p$  is multiplicative,  $\omega(mS) = |m|_p \omega(S)$  for  $m \in \mathbb{Z}_p$  and  $S \subseteq \mathbb{Z}$ . In particular,  $\omega(p\mathbb{Z}) = |p|_p \omega(\mathbb{Z}) = \frac{1}{p} * p^{\frac{1}{1-p}} = p^{-p/p-1}$ .

**Proposition.** [5](subadditivity) If  $\text{diam}(S) := \max_{x,y \in S} \rho(x,y) = d$  and  $S = \cup_i^n A_i$  for  $A_i$  compact subsets of  $M$  with  $\rho(A_i, A_j) = d, \forall i, j$ , then

$$\frac{1}{\log(\omega(S)/d)} = \sum_{i=1}^n \frac{1}{\log(\omega(A_i)/d)}$$

- Put in an example here.
- Also talk about how cosets and open balls coincide and when the above is satisfied

**Corollary.** Suppose  $S = \cup_i^n S_i$  with  $\rho(S_i, S_j) = d = \text{diam}(S)$  and also  $\omega(S_i) = \omega(S_j)$ ,  $\forall i, j$ . Let  $r \in \mathbb{R}$  be such that  $\omega(S_i) = r\omega(S)$ ,  $\forall i$ . Then  $\omega(S) = r^{\frac{1}{n-1}} \cdot d$ . In particular if  $S = \mathbb{Z}$  and  $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$  then  $\omega(S) = (\frac{1}{p})^{1/p-1}$  for any prime  $p$ .

**Corollary.** (Joins of computable sets are computable) Let  $\Gamma_M = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$  be the set of distances in  $M$ . Suppose that  $S = B_{\gamma_i}(x)$ , for some  $x$  and  $i$ , is the union of 2 or more balls of radius  $\gamma_{i+1}$ , i.e.,  $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$  is a join in the lattice of open sets in  $M$ , then

$$\frac{1}{\log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^n \frac{1}{\log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

## Computing a $\rho$ -ordering

We describe an algorithm for computing the  $\rho$ -ordering of a set recursively and discuss some immediate corollaries.

Let  $S \subseteq M$  be a compact subset of an ultrametric space  $(M, \rho)$ . Let  $\Gamma_S = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$  be the set of distances in  $S$ . Note that for each  $k \in \mathbb{N}$ , the closed balls of radius  $\gamma_k$  partition  $S$ , i.e.,  $S = S_{\gamma_k} := \cup_{i=1}^n \overline{B_{\gamma_k}(x_i)}$ , where both  $n$  and the  $x_i$ 's depend on  $k$ . In what follows, fix a  $k \in \mathbb{N}$  and let  $S_{\gamma_k} = \cup_{i=1}^n \overline{B_{\gamma_k}(x_i)}$  be such a partition. Note that we can construct  $S_{\gamma_{k+1}}$  by partitioning each of the  $\overline{B_{\gamma_k}(x_i)}$ , i.e.,

$$S = S_{\gamma_{k+1}} = \cup_{i=1}^n \cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})}$$

where  $1 \leq l_i \leq n$  and  $\cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})} = \overline{B_{\gamma_k}(x_i)}$ ,  $\forall i$ . We denote by  $x_{i,j}$  the centre of a ball of radius  $\gamma_{k+1}$  partitioning the ball  $B_{\gamma_k}(x_i)$ . Without loss of generality, when  $j = 1$ , assume  $x_{i,j} = x_i$ ,  $\forall i$ .

We now make the following observation due to [6],

**Lemma.** For each  $k$ , the elements of  $S_{\gamma_k}$ , that is, the closed balls of radius  $\gamma_k$ , themselves form an ultrametric space, where

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

We note that since  $S$  is assumed to be compact,  $S_{\gamma_k}$  is a finite metric space  $\forall k < \infty$  and  $S_{\gamma_\infty} = \cup_{x \in S} \overline{B_0(x)} = \cup_{x \in S} x = S$  and  $\rho_\infty = \rho$ . Now view  $S_{\gamma_k}$ , for fixed  $k < \infty$  as a finite ultrametric space and represent its  $n < \infty$  elements by their centres, the  $x_i$ 's. Without loss of genearlity, we can reindex the  $x_i$ 's so that they give the first  $n$  terms of

a  $\rho_k$ -ordering of  $S_{\gamma_k}$ . The following proposition is the main result of this section.

**Proposition.** Given  $S$  a compact subset of an ultrametric space  $M$  and  $\Gamma_S$ , the set of distances in  $S$ , if  $S_{\gamma_k}$  is a partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the centres of the balls are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , then a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  can be found by forming a matrix,  $A_k$ , whose  $(i, j)^{th}$ -entry is  $x_{i,j}$ , as shown below, and then concatenating the rows (where the columns are padded by \* if necessary).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \dots & x_{n,l_n} \end{pmatrix}$$

*Proof.* Note that the entries in each column are points in the ball  $B_{\gamma_k}(x_i)$  so that the pairwise distance between columns is constant and always exceeds the distance between elements within a column. Moreover, the columns are organized such that for any  $j$ ,  $x_{n,j}$  maximizes  $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j})$  since  $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j}) = \prod_{i=1}^{n-1} \rho(x_{n,1}, x_{i,1}) = \prod_{i=1}^{n-1} \rho(x_n, x_i)$  and the  $x_i$ 's are indexed according a  $\rho_k$ -ordering of  $S_{\gamma_k}$ .

Then a  $\rho_{\gamma_{k+1}}$ -ordering of  $S_{\gamma_{k+1}}$  is obtained by minimizing the number of elements from any one column and by taking the points  $x_{i,j}$  (for fixed  $j$ ) in sequence. For example, by concatenating the rows. □

**Corollary.** Interweaving the bottown row of the lattice of closed balls for a set  $S$  gives a  $\rho$ -ordering of  $S$ .

**Corollary.** Suppose  $S$  and  $T$  are compact subsets of an ultrametric space  $M$  with  $\Gamma_S = \Gamma_T$  and  $|S_{\gamma_k}| = |T_{\gamma_k}|, \forall k$ . Then  $\omega(S) = \omega(T)$ .

- i think this coincides with translation invariance when there is a group operation



**Corollary.** (regularity) Suppose that  $S$  is such that  $\forall k$ , any  $B_{\gamma_k}(x) = \cup_{j=1}^l B_{\gamma_{k+1}}(x_j)$ , that is, every ball in  $S$  breaks into exactly  $l$  smaller balls.

# Product space

As a first point of departure, a natural space to consider is the product space of ultrametric spaces, for example  $\mathbb{Z}^n$  (or  $\mathbb{Z}_p^n$  or  $\mathbb{Q}_p^n$ ), for some  $n > 1$ . A natural candidate for an ultrametric on the space is the  $L_\infty$  metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sup_i \{\rho(x_i, y_i)\}$$

where  $\rho$  is the metric from the base space. Then  $\rho_\infty$  is easily seen to be an ultrametric, so long as  $\rho$  is. Indeed, let  $(M, \rho)$  be an ultrametric space and consider two points,  $x$  and  $y$ , in the product space,  $M^n$  for some  $n > 1$ . Clearly,  $\rho_\infty(x, y) \geq 0$  since each  $\rho(x_i, y_i) \geq 0$ , and  $\rho_\infty(x, y) = 0 \iff \rho(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$ . The fact that  $\rho_\infty$  is symmetric is also an easy consequence of the fact that  $\rho$  is symmetric since  $\rho(x_i, y_i) = \rho(y_i, x_i)$  implies  $\sup_i \{\rho(x_i, y_i)\} = \sup_i \{\rho(y_i, x_i)\}$ . To see that  $\rho_\infty$  is an ultrametric, note that if  $z = z_i$  is any other point of  $M$ , then

$$\begin{aligned} \rho_\infty(x, y) &= \sup_i \{\rho(x_i, y_i)\} \\ &\leq \sup_i \{\max(\rho(x_i, z_i), \rho(y_i, z_i))\} && \text{since } \rho \text{ is an ultrametric} \\ &= \max(\sup_i \{\rho(x_i, z_i)\}, \sup_i \{\rho(y_i, z_i)\}) \\ &= \max(\rho_\infty(x, z), \rho_\infty(y, z)) \end{aligned}$$

In fact, in the proof that  $(M^n, \rho_\infty)$  is an ultrametric, we see that a more general statement is true, given below.

**Proposition.** Let  $(M_i, \rho_i)$  for  $i$  in some finite or countably infinite index set  $I$  be a collection of ultrametric spaces. Then  $(M, \rho_\infty)$  is an ultrametric space, where  $M = M_1 \times M_2 \times M_3 \times \dots$  and  $\rho_\infty$  is the  $L_\infty$  metric described above.

*Proof.* Above. □

We show a few quick results ultrametric spaces formed as product spaces, which allows us to quickly calculate the valutive capacity of a few subsets.

**Proposition.** Suppose  $(M, \rho_\infty)$  is the product of ultrametric spaces  $(M_i, \rho_i)$ . Then  $\rho_\infty$  is (left) translation invariant if each  $\rho_i$  is, in which case valutive capacity is also (left) translation invariant.

*Proof.* Suppose  $(M, \rho_\infty)$  is the product of ultrametric spaces  $(M_i, \rho_i)$  and each  $M_i$  is a topological group with operation  $+$ . Suppose also that

$$\rho(x_i, y_i) = \rho(s_i + x_i, s_i + y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each  $\rho_i$  is (left) translation invariant. Then,

$$\rho_\infty(s + x, s + y) = \sup_i \{\rho(s_i + x_i, s_i + y_i)\} = \sup_i \{\rho(x_i, y_i)\} = \rho_\infty(x, y).$$

so that  $\rho_\infty$  is translation invariant. Proposition *xyz* implies valutive capacity is as well. □

**Proposition.** Suppose  $(M, \rho_\infty)$  is the product of ultrametric spaces  $(M_i, \rho_i)$ . Then  $\rho_\infty$  is multiplicative if each  $\rho_i$  is, in which valutive capacity is also multiplicative.

*Proof.* Similar. □

**Example 3.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  be the metric space with elements  $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$  and metric  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2))$ , where  $\rho_p$  is the p-adic

metric for some fixed prime  $p$ . Since  $\rho_p$  is translation invariant and multiplicative in  $\mathbb{Z}_p$ , valutive capacity is also translation invariant and multiplicative in  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ .

**Example 4.** Let  $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \dots, \rho_{P,\infty})$  be the metric space formed by taking the product of  $(\mathbb{Z}_p, \rho_p)$  for every prime  $p$ . Since  $\rho_p$  is translation invariant and multiplicative in each  $\mathbb{Z}_p$ , valutive capacity is also translation invariant and multiplicative in  $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \dots, \rho_{P,\infty})$ .

## 2-fold products

What is the valutive capacity of  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  from the example above? Suppose  $p = 2$ . Using translation invariance, scaling and subadditivity, we can compute the result by first noting that we can write  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always  $1 = \text{diam}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ , subadditivity implies that

$$\begin{aligned} & \frac{1}{\log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} \\ = & \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))} \\ = & \frac{4}{\log(\|2\|_2 * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{\log(\frac{1}{2} * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{\log(\frac{1}{2}) + \log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} \end{aligned}$$

Taking logs base 2, we have that

$$\omega(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^{\frac{-1 + \log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} 2^{\frac{\log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} (2^{\log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))})^{\frac{1}{4}} = 2^{\frac{-1}{4}} \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\frac{1}{4}}$$

so that  $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$  is a solution of the equation  $x^4 - \frac{x}{2}$ , for which there is a single real

positive root, given by  $2^{-1/3}$ .

To compute the valutive capacity for a 2-fold product for an arbitrary prime  $p$ , note that we can always decompose  $\mathbb{Z}_p \times \mathbb{Z}_p$  into a union of  $p^2$  sets each of the form  $\{p\mathbb{Z}_p + s \times p\mathbb{Z}_p + t\}$  for  $s, t \in (0, \dots, p-1)$ , and the pairwise distance between these sets will always be  $1 = \text{diam}(\mathbb{Z}_p \times \mathbb{Z}_p)$  (to see this, either note that we can always find co-prime elements, or note that each set is a ball of radius  $1/p$  centred at  $(s, t)$  and so the distance between them must be greater than  $1/p$ , and 1 is the only possible distance greater than  $1/p$  in  $\mathbb{Z}_p \times \mathbb{Z}_p$ ). Then, we combine our tools as before to obtain the equation,

$$\frac{1}{\log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(\|p\|_p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(1/p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, taking logs base  $p$ , we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p) = p^{\frac{-1}{p^2}} \omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{\frac{1}{p^2}}$$

So that  $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$  is a solution of the equation  $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p})$  over  $\mathbb{R}$  and since  $\mathbb{R}$  is a division ring, this means the positive solutions are given by solving  $x^{p^2-1} - \frac{1}{p}$ . Solutions of this equation are of the form  $p^{\frac{-1}{p^2-1}}$  times a  $p^2 - 1$  root of unity, and so there is exactly one positive, real solution, namely  $p^{\frac{-1}{p^2-1}}$  itself. Then the valutive capacity of the entire product space  $\mathbb{Z}_p \times \mathbb{Z}_p$  is  $p^{\frac{-1}{p^2-1}}$ . In fact, from here it is not hard to see that by taking the  $n$ -fold product, we would end up with the same equation except that the exponent of  $p$  would become  $n$  rather than 2. We arrive at the following result:

**Proposition.** Let  $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$  be the ultrametric space with points equal to the  $n$ -fold product of  $\mathbb{Z}_p$  (for  $n < \infty$ ) for some fixed prime  $p$ . The valutive capacity of  $M$  is  $(\frac{1}{p})^{\frac{1}{p^n-1}}$ .

*Proof.* Above. □

Taking  $n = 1$ , we see that this agrees with the valutive capacity of  $\mathbb{Z}_p$  computed in the

last chapter.

We end this section with a few observations on the results above. First we observe the asymptotic behavior of capacity in these spaces. For a fixed prime  $p$ ,  $(\frac{1}{p})^{\frac{1}{p^n-1}}$  is a monotone, increasing sequence in  $n$  with  $\lim_{n \rightarrow \infty} (\frac{1}{p})^{\frac{1}{p^n-1}} = 1$ . For fixed  $n$ , the sequence in  $p$  is also monotone, increasing, again with  $\lim_{p \rightarrow \infty} (\frac{1}{p})^{\frac{1}{p^n-1}} = 1$ . In both cases, the limiting value is equal to the diameter of space.

Recall that computing the valutive capacity of these spaces, we were ultimately reduced to finding solutions to polynomials of the form  $x^{p^n} - \frac{x}{p}$  for some  $n$  and for some  $p$ . The second observation is that these polynomials are  $p\mathbb{Z}$ -valued on  $p\mathbb{Z}$ .

## Infinite products

Consider the infinite product spaces  $(\mathbb{Z}_p \times \mathbb{Z}_p \times \dots, \rho_{p,\infty})$  for fixed prime  $p$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \dots, \rho_{P,\infty})$  for each prime  $p$ . Both spaces are compact by Tychonoff's theorem, and we know then that the diameter of each space must give a strict upper bound on its valutive capacity.

## Inclusions

Start considering the behavior of capacity under maps

In this section, we considered the notion of valutive capacity in product spaces, that is, spaces formed by taking copies of ultrametric spaces. In the following sections, we consider valutive capacity in spaces formed by adding points, that is extension fields, or by both taking copies and adding (a distinguished) point, as in projective spaces. For these purposes, it will be more productive to start working over the field  $\mathbb{Q}_p$ , instead of  $\mathbb{Z}_p$ .

# projective space

## Background from Gerritzen and van der Put

Background results from [7]. Let  $k$  be a field that is complete with respect to a non-archimedean valuation and let  $K$  be a complete and algebraically closed field containing  $k$ .

**Definition.** [7] The set  $\{\lambda \in k; |\lambda| \leq 1\}$ , denoted  $k^0$ , is the **valuation ring** of  $k$ . It has a unique maximal ideal, denoted  $k^{00}$ , given by  $\{\lambda \in k; |\lambda| < 1\}$ . The **residue field** of  $k$  is  $\bar{k} := k^0/k^{00}$ .

**Definition.** [7] The **projective line over  $k$** , denoted  $\mathbb{P}^1(k)$ , is the space whose points are lines  $l$  in  $k^2$  that intersect  $(0, 0)$  and whose topology and field structure are inherited from  $k$ .

We give two equivalent representations for the points of  $\mathbb{P}^1(k)$ . A point  $p \in \mathbb{P}^1(k)$  is an equivalence class of  $k^2 \setminus (0, 0)$  under the relation  $(x, y) \sim (x', y')$  if there exists a  $\lambda \in k \setminus 0$  such that  $(x, y) = \lambda(x', y')$ . Equivalently, suppose that  $l$  is a line in  $k^2$  intersecting the origin, that is a point in  $\mathbb{P}^1(k)$ . We denote  $l$  by a representative  $[x_0, x_1] \in k^2$  such that  $l = \{\lambda(x_0, x_1); \lambda \in k\}$ , called homogeneous coordinates of  $l$ .

**Proposition.** [7] Let  $\psi : k \rightarrow \mathbb{P}^1(k)$  be the map given by  $\psi(\lambda_0) = [1, \lambda_0]$ , where  $[1, \lambda_0]$  is the line in  $k^2$ ,  $\{\lambda(1, \lambda_0); \lambda \in k\}$ . Then the image of  $\psi$  is  $\mathbb{P}^1(k) \setminus [0, 1]$  and is isomorphic to  $k$ , so that  $k$  is identified with projective space minus a distinguished point,  $[0, 1]$ , which is denoted by  $\infty$ .

**Definition.** [7]  $k$  is called a **local field** if  $k$  is locally compact.

**Proposition.** [7] The following are equivalent:

1.  $k$  is a local field.
2.  $|k^*| \cong \mathbb{Z}$  and  $\bar{k}$  is finite, where  $k^*$  is the set of units in  $k$ , ie  $k^* = \{\lambda \in k, \lambda \neq 0\}$ .
3.  $k$  is a finite extension of either  $\mathbb{Q}_p$  or  $\mathbb{F}_p((t))$ .
4.  $\mathbb{P}^1(k)$  is compact

**Definition.** [7] We denote by  $GL(2, k)$  the set of invertible  $2 \times 2$  matrices over  $k$ . A **fractional linear automorphism**,  $\phi$ , of  $\mathbb{P}^1(k)$  is a map defined by  $z \mapsto \frac{az+b}{cz+d}$  for some  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, k)$ . The set of fractional linear automorphisms of  $\mathbb{P}^1(k)$  is denoted  $PGL(2, k)$ . Note that  $PGL(2, k) = GL(2, k) / \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \lambda \in k^* \right\}$ . In homogeneous coordinates, we can represent the action of  $\phi$  by  $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$ .

**Definition.** [7] Suppose  $\Gamma$  is a subgroup of  $PGL(2, k)$ . A point  $p \in \mathbb{P}^1(k)$  is a **limit point** of  $\Gamma$ , if there exists a point  $q$  in  $\mathbb{P}^1(k)$  and a sequence  $\{\gamma_n\}_{n \geq 1}$  in  $\Gamma$  such that  $\lim_{n \rightarrow \infty} \gamma_n(q) = p$ .

**Proposition.** [7] If  $\Gamma$  is not a discrete subgroup of  $PGL(2, k)$  then every point of  $\mathbb{P}^1(k)$  is a limit point of  $\Gamma$ .

*Proof.* Since  $\Gamma$  is not discrete, the sequence  $\{\gamma_n\}_{n \geq 1}$  has a limit  $\gamma$  in  $\Gamma$ . Let  $p$  be any point of  $\mathbb{P}^1(k)$  and let  $q = \gamma^{-1}(p)$ . Then  $\lim_{n \rightarrow \infty} \gamma_n(q) = \lim_{n \rightarrow \infty} \gamma_n(\gamma^{-1}(p)) = p$ .  $\square$

**Definition.** [7] A subgroup  $\Gamma$  of  $PGL(2, k)$  is **discontinuous** if the closure of every orbit of  $\Gamma$  in  $\mathbb{P}^1(k)$  is compact and the set of all limit points of  $\Gamma$  is not equal to  $\mathbb{P}^1(k)$ .

**Proposition.** [7] If  $\Gamma$  is a discontinuous subgroup of  $PGL(2, k)$  and  $\mathcal{L}$  is the set of limit points of  $\Gamma$ , then  $\mathcal{L}$  is compact, nowhere dense and if  $\mathcal{L}$  contains more than two points,  $\mathcal{L}$  is perfect.



**Definition.** [7] Let  $A$  be an element of  $GL(2, k)$  and let  $a_1$  and  $a_2$  be the eigenvalues of  $A$ . Then  $A$  is called **elliptic** if  $a_1 \neq a_2$ , but  $|a_1| = |a_2|$ .  $A$  is called **parabolic** if  $a_1 = a_2$ , and  $A$  is called **hyperbolic** if  $|a_1| \neq |a_2|$ .

**Example 5.** Consider the matrix  $T_s = \begin{pmatrix} p & s \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q}_p)$  for some  $s$  in  $(0, \dots, p-1)$  (note that  $\det(T_s) = p$  is invertible in  $\mathbb{Q}_p$ , so that  $T_s \in GL(2, \mathbb{Q}_p)$ , although it is not in  $GL(2, \mathbb{Z}_p)$ ).  $T_s$  has eigenvalues  $p$  and  $1$  and so  $T_s$  is hyperbolic for any choice of  $s$  or  $p$ . Consider the action of  $T_s$  on  $\mathbb{Z}_p \subset \mathbb{Q}_p$ , where  $\mathbb{Z}_p$  is identified with the subspace  $\{[1, \lambda]; \lambda \in \mathbb{Z}_p\}$  of  $\mathbb{P}^1(\mathbb{Q}_p)$ . In homogeneous coordinates, this action is given by  $[1, \lambda] \mapsto [1, p\lambda + s]$ . Since  $|p\lambda + s - s| = |p\lambda| \leq \frac{1}{p}$ ,  $T_s$  sends  $\lambda$  to  $B_{\frac{1}{p}}(s)$ . Also note that for  $s = 0$ ,  $T_s$  has the effect of shifting the index of  $\lambda$  by 1, that is, if  $\lambda = \sum_{i=n}^{\infty} a_i p^i$ , where  $n = \text{ord}(\lambda)$ , then  $T_0([1, \lambda]) = [1, p\lambda] \rightsquigarrow p\lambda = \sum_{i=n+1}^{\infty} a_{(i-1)} p^i$ .

## Computation of the capacity of some sets

### (F&P, section 5)

#### Setup

Let  $A = \{0, 1, \dots, d-1\}$  be a finite alphabet and  $A^{\mathbb{N}}$  be the collection of infinite sequences with values in  $A$ . Note  $A^{\mathbb{N}}$  is a Cantor set, so it is perfect, nowhere dense, and compact.

A basis for the topology is given by the cylinder set: take countably many copies of  $\{0, 1, \dots, d-1\}$  where each copy has the discrete topology.

Let  $p \geq d$  be a prime number and let  $\phi$  be the canonical embedding of  $A^{\mathbb{N}}$  into  $\mathbb{Z}_p$  via the following continuous (under the above topology) map:

$$\phi : A^{\mathbb{N}} \rightarrow \mathbb{Z}_p \text{ by } (x_n)_{n \geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

**Lemma.** (F&P Lemma 5.1)

Let  $w_1, w_2, \dots, w_s$  be  $s \geq 2$  words with the same length  $l$  such that all the first letters are distinct. Let  $X \subset A^{\mathbb{N}}$  be the set of sequences such that any factor is a factor of a concatenation of the words  $w_1, w_2, \dots, w_s$ . Then the set  $E := \phi(X) \subset \mathbb{Z}_p$  satisfies:

$$E = \cup_{i=1}^s x_i + p^l E, \text{ with } x_i = \phi(w_i 0^\infty)$$

It is a regular compact set and its valiative capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

**Example 6.**  $w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$

Then  $\{x_n\}_{n \geq 0} \in X$  if each term in  $\{x_n\}_{n \geq 0}$  is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Note that we can rephrase the lemma as follows:

Let  $x_1, x_2, \dots, x_s$  be  $s \geq 2$  points in  $\mathbb{Z}_p$  such that  $|x_i - x_j|_p = 1, \forall i, j \in 1, \dots, s$ . Suppose also that there exists an  $l \in \mathbb{N}$  such that  $\forall i$ ,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^l a_i p^i$$

Let  $\gamma_i$  be the fractional linear automorphism of  $\mathbb{P}^1(\mathbb{Q}_p)$  given by  $\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix}$  and let  $\Gamma$  be the subgroup of  $PGL(2, \mathbb{Q}_p)$  generated by the  $\gamma_i$ .

Then  $\Gamma$  has a subgroup  $H$  such that the limit set  $\mathcal{L}$  of  $H$  has the property that  $Z = \psi^{-1}(\mathcal{L})$  is equal to  $\phi(X)$  in the original lemma. In particular  $Z$  is a regular, compact subset of  $\mathbb{Z}_p$  satisfying

$$Z = \cup_{i=1}^s x_i + p^l Z = \cup_{i=1}^s B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

*Proof.* We must show that the set  $Z$  above is equal to  $E = \phi(X)$  in the original lemma. First note that if  $w_1, w_2, \dots, w_s$  are words in  $A^{\mathbb{N}}$ , then the first letter of each  $w_i$  is distinct if and only if  $|\phi(w_i) - \phi(w_j)|_p = 1, \forall i, j$  (since the pairwise distance is 1 if and only if  $\text{ord}(\phi(w_i) - \phi(w_j)) = 0$  for any  $i$  and  $j$ , if and only if the coefficient of  $p^0$  (i.e., the first letter each  $w_i$ ) is different  $\forall i, j$ ). So then the  $x_i$  are just the  $\phi(w_i)$ .

Now consider the limit set of  $\Gamma$ . Let  $\gamma \in \Gamma$ . If  $\gamma$  is a product of the generators  $\gamma_i$ , then  $\gamma$  is represented by a matrix of the form:  $\begin{pmatrix} p^{lm} & z_m \\ 0 & 1 \end{pmatrix}$ , where  $m \in \mathbb{N}$  and  $z_m$  is an element of  $\mathbb{Z}_p$  whose coefficient vector is a concatenation of the coefficient vectors of the  $x_i$  (for  $0 \leq i \leq ml$  and 0 for  $i > ml$ ). For example,  $\gamma_i \gamma_j \gamma_k =$

$$\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^l & x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^l & x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{3l} & p^{2l}x_k + p^l x_j + x_i \\ 0 & 1 \end{pmatrix}$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{lm}a_1 + za_0] \sim [1, p^{lm}\frac{a_1}{a_0} + z_m]$$

As  $m$  tends to infinity, this point tends to  $[1, 0 + z]$ , where  $z$  is an element of  $\mathbb{Z}_p$  whose entire coefficient vector is a concatenation of the  $x_i$ 's. The set  $\psi^{-1}([1, z])$  for all such  $z$  is exactly the set  $E = \phi(X)$ .

Now suppose  $\gamma$  is a product of the inverses of the generators  $\gamma_i^{-1}$ , then  $\gamma$  is represented by a matrix of the form:  $\begin{pmatrix} p^{-lm} & -p^{-l}z^{-1} \\ 0 & 1 \end{pmatrix}$ , where  $m \in \mathbb{N}$  and  $z$  is as above. For example,

$$\begin{pmatrix} p^{-l} & -p^{-l}x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{-3l} & -p^{-3l}x_k - p^{-2l}x_j - p^{-l}x_i \\ 0 & 1 \end{pmatrix}$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{-lm}a_1 - p^{-lm}za_0] \sim [1, p^{-lm}(\frac{a_1}{a_0} - z)]$$

This time as  $m$  grows, the image approaches infinity and so has empty preimage under  $\psi$ .

Lastly, we consider elements of  $\Gamma$  made up of both generators and the inverse of generators. These elements will produce translations, either of the form  $\begin{pmatrix} 1 & p^{-l}(x_i - x_j) \\ 0 & 1 \end{pmatrix}$  if  $\gamma = \gamma_j^{-1}\gamma_i$  or  $\begin{pmatrix} 1 & x_j - x_i \\ 0 & 1 \end{pmatrix}$  if  $\gamma = \gamma_j\gamma_i^{-1}$ . These elements commute with each other and so the subgroup which they generate is normal. We can quotient by the entire translation subgroup, ie the subgroup generated by  $\{\gamma_i\gamma_j^{-1}, \gamma_i^{-1}\gamma_j; \forall i, j \in 1, \dots, s\}$  to obtain  $H$ . Then  $\mathcal{L} = \infty \cup \{[1, z] \mid z \in \mathbb{Z}_p \text{ and the coefficient vector of } z \text{ is a concatenation of the coefficient vectors of the } x_i\text{'s}\}$  and  $\psi^{-1}(\mathcal{L}) = E = \phi(X)$ , as required.

□

$H$  is not a subgroup of  $PGL(2, k)$  - the torsion means its not closed, although it is a group (in general) and also a subset of  $PGL(2, k)$ .

$\Gamma$  is not discrete in  $PGL(2, k)$  - is  $H$  or  $\{\gamma_i\}$  or  $\{\gamma_i^{-1}\}$  a maximal discrete subset?

Does  $H$  have a maximal (fg) subgroup and is it interesting?

More background from [7].

**Definition.** [7] A **Schottky group** is a finitely-generated, free and discontinuous subgroup of  $PGL(2, k)$

Notation: Let  $X$  be a subset of  $\mathbb{P}^1(k)$  and denote by  $X^{(3)}$  the set  $X \times X \times X \setminus \Delta$ , where  $\Delta$  is the fat diagonal, that is the set  $\{x_i = x_j; \text{ for some } i \neq j, i, j \in 1, 2, 3\}$

Notation: Let  $a = (a_0, a_1, a_\infty)$  be an element of  $X^{(3)}$  for some  $X \subseteq \mathbb{P}^1(k)$ . We denote by  $\gamma_a$  the element of  $PGL(2, k)$  such that  $\gamma_a(a_0) = 0$ ,  $\gamma_a(a_1) = 1$ ,  $\gamma_a(a_\infty) = \infty$ .

Note that given such a point  $a$ , we can calculate the map  $\gamma_a$  explicitly, as follows: Suppose  $a = (a_0, a_1, a_\infty)$  and  $a_i \neq \infty = [0, 1], \forall i$ . Then  $\gamma_a(a_0) = 0 = [1, 0]$ ,  $\gamma_a(a_1) = 1 = [1, 1]$ , and  $\gamma_a(a_\infty) = \infty = [0, 1]$ , so that if  $\gamma_a$  is represented by some  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , we have:

**Definition.** Let  $x_0, x_1$  be two elements of  $k$  such that  $\max(|x_0|, |x_1|)$ , ie both  $x_0$  and  $x_1$  are in the valuation ring of  $k$  and at least one is in its residue field. The map  $R$  that sends  $[x_0, x_1] \mapsto [\bar{x}_0, \bar{x}_1]$ , that is that sends each component to its residue class in  $\bar{k}$  is called the **standard reduction**.

# algebraic extensions

Type up the example Keith gave you.

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