Chapter 1

Introduction

In the course of developing a generalized factorial function, Manjul Bhargava introduced the notion of $p-$orderings of a Dedekind domain [?, ?], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [?, ?] and have also provided a natural framework to extend many classical results in analysis to a $p-$adic setting, such as polynomial approximation and mapping theorems [?, ?, ?]. In this thesis, we examine how a tool based on $p-$orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

The historical background to this work comes in two parts. On the one hand, there is the background on logarithmic capacity from potential theory, and secondly, there is the background from Bhargava's $p-$orderings. We give a brief summary of each here. A similar treatment, with slightly different perspective, is found in [?]. Jean-Luc Chabert was the first to draw a connection between the two, and many of the known results in this area stem from his work or that of his colleagues. Building on the result in [?], we extend the work by Chabert and colleagues by studying valuative capacity in a more general setting, namely that of an ultrametric space, which may or may not also be a local field. In doing so, we show many properties of capacity are in fact independent of the algebraic structure of a space, although such structure, when it exists, can act as a useful probe.

1.1 Logarithmic capacity

The theory of capacity has been developed as a topic in potential theory in a variety of settings. Classically, the notion of capacity was developed over both $\mathbb{C}$ and $\mathbb{R}^n$, although the theory has been further developed in a rather general way by Rumely for Berkovich spaces. A signficant body of work on the analytic properties of capacity can be found for a number of different contexts. For example, such a treatment of the subject over $\mathbb{C}$ can be found in [?] and [?], over Berkovich spaces in [?], and over $\mathbb{Q}_p$ in [?]. We give a brief account of capacity over $\mathbb{C}$ here, presenting only the most essential definitions and results. One advantage of tracing the historical roots of capacity back to $\mathbb{C}$ is that the theory in this setting also comes equipped with a physical interpretation. As we are about to see, capacity in the classical sense gives a mathematical model for the amount of electrostatic charge a conductor can hold. The exposition below is closely based on Ransford [?].

Even restricting ourself to the definition of capacity of subsets of $\mathbb{C}$, we find two paths, one which will give us some physical interpretation, and one which will lead more naturally to $p-$orderings. We start with the former.

Definition 1. [?] Let μ be a finite Borel measure on $\mathbb{C}$ and suppose μ has compact support. We associate to μ a function, $p_\mu : \mathbb{C} \to [-\infty, \infty)$, given by

$$p_\mu(x) = \int \log|x-y| d\mu(y)$$

called the **potential function** of μ. The **energy** of μ is

$$I(\mu) = \int\int \log|x-y| d\mu(y) d\mu(x)$$

This gives at once the physical interpretation promised above. We interpret the potential function of a measure as giving the potential energy of a point. Viewing the measure as a charge distribution, the double integral gives back the total energy in the system. Now we come upon a physical reality: charged particles in a conductor will naturally distribute themselves as to minimize the energy. This leads to the definition below:

Definition 2. [?] Let K be a compact subset of $\mathbb{C}$ and let $\mathcal{P}(K)$ be the set of Borel probability measures on K. If $\nu \in \mathcal{P}(K)$ is such that

$$I(\nu) = \sup_{\mu \in \mathcal{P}(k)} I(\mu)$$

then ν is a **equilibrium measure** for K.

comment from Keith: explain why its a sup when you're minimizing the energy; I think this is similar as to how minimizing the valuation (often) leads to maximizing the distances - the less potential energy you take up at each iteration the more total charge you can add

We state the following proposition without proof. A sketch of the proof can be found in [**?**] and the full details can be found in [**?**].

Proposition 1. *[**?**] The equilibrium measure exists for every compact set $K \in \mathbb{C}$. When finite, the equilibrium measure is unique and isometry-invariant.*

We are now ready to give our first definition of capacity.

Definition 3. [**?**] Let K be a compact subset of $\mathbb{C}$. The logarithmic capacity of E is

$$C(K) = e^{I(\nu)}$$

where ν is an equilibrium measure on K. If $I(\nu) = -\infty$, then we understand that $C(K) = 0$.

We present below a few results on capacity in $\mathbb{C}$, some of which will reappear in the remainder of this work, although the context, and the proofs (omitted here), bear little resemblence to the present case.

Proposition 2 (Randsford, 5.1.2). *Let K, K_1, K_2 be compact subsets of $\mathbb{C}$.*

1. *$K_1 \subseteq K_2$, then $C(K_1) \leq C(K_2)$.*

2. *$C(\alpha K + \beta) = |\alpha| C(K)$ for all $\alpha, \beta \in \mathbb{C}$.*

3. $C(K) = C(\delta_e K)$, *where* δ_e *is the exterior boundary*[1]

Proposition 3 (**Randsford, 5.1.4**). *Suppose* $\{B_n\}$ *is a sequence of Borel subsets of* $\mathbb{C}$*. Let* $B = \cup_n B_n$ *and* $d \geq 0$*.*

1. *If* $diam(b) \leq d$*, then* $C(B) \leq d$ *and*

$$\frac{1}{log(\frac{d}{C(B)})} \leq \sum_n \frac{1}{log(\frac{d}{C(B_n)})}$$

2. *If* $dist(B_j, B_k) \geq d$ *whenever* $j \neq k$*, then*

$$\frac{1}{log^+(\frac{d}{C(B)})} \geq \sum_n \frac{1}{log^+(\frac{d}{C(B_n)})}$$

where $log^+(x) = \max(log(x), 0)$*.*

We now know show an equivalent way of defining of capacity, still over $\mathbb{C}$, which starts with the following two definitions due to Fekete [**?**].

Definition 4. Let $K \subseteq \mathbb{C}$ be a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in K^n$, consider

$$\delta_n(z) = \prod_{j<i} |z_i - z_j|^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in K^n$ is called a **Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in K.

Note that since K is compact by assumption, a Fekete $n-$tuple exists for each n.

Definition 5. Let $K \subseteq \mathbb{C}$ be a compact subset. The **transfinite diameter** of K is

$$\lim_{n\to\infty} [\max_z \ \delta_n(z)]$$

where the maximum is taken over all $n-$tuples in K. That is, the transfinite diameter of K is $\lim_{n\to\infty} \delta_n(z)$, where z is a Fekete $n-$tuple for each n.

[1]The exterior boundary of a compact set is the boundary of the unbounded, connected component of $U = \mathbb{C} \setminus K$.

The following proposition shows the relation to capacity.

Proposition 4. ***[Fekete-Szegö Theorem]**[**?**] If K is a compact subset of $\mathbb{C}$, then the transfinite diameter of K is equal to the logarithmic capacity of K.*

We end this section with an observation about the points z_i in $\mathbb{C}$ (or some subset thereof) making up a Fekete n-tuple. For $n \geq 2$, if $(z_1, \ldots, z_n)$ is a Fekete $n-$tuple, then in general there is no z_{n+1} available such that $(z_1, \ldots, z_n, z_{n+1})$ a Fekete $(n+1)-$tuple. In physical terms, we note that the placement of a new charge in a conductor will almost always change the location of the existing charges in that conductor. Remarkably, this is not the case in ultrametric spaces. Indeed, we are able to build the analogous structure, which we call a $p-$ordering or more generally a $\rho-$ordering, *recursively*, that is by reusing the points from the previous iteration.

1.2 P-orderings

The development of $p-$ordering was motivated by the observation that the factorial function had important number-theoretic applications, yet was only defined for the set $\mathbb{Z}$. In order to generalize the factorial, Bhargava defined it via an invariant, called the $p-$sequence, which could be attached to any subset of a Dedekind domain [2] [**?**].

We cannot go much further without introducing the following definition.

Definition 6. Let $z \in \mathbb{Z}$ and let p be any prime. The $p-$**adic valuation** of z, denoted $v_p(z)$, is the largest $n \in \mathbb{N}$ such that p^n divides $z \neq 0$ and $v_p(z) = \infty$ if $z = 0$. That is,

$$v_p(z) = \begin{cases} \max\{n \in \mathbb{N}; p^n \mid z\}, & \text{if } z \neq 0 \\ \infty, & \text{otherwise} \end{cases}$$

For $z_1, z_2 \in \mathbb{Z}$, we define the $p-$**adic metric** by

$$\rho_p(z_1, z_2) = p^{-v_p(z_1 - z_2)}$$

[2]In fact, Bhargava associated $p-$ sequences to the more general class of Dedekind rings, which are locally principal, Noetherian rings in which all nonzero primes are maximal.

where p^∞ is taken to be 0.

It is worth pausing to make a few comments about the above definitions. That the $p-$adic metric is truly a metric is easy to see. In fact, we will see in the next chapter that it is not just a metric, but also an ultrametric, since it satistfies a strengthen version of the triangle identity. The strong triangle identity is not the only interesting property enjoyed by the $p-$adic valuation though. Like the logarithm, the $p-$adic valuation also satisfies: $v_p(x \cdot y) = v_p(x) + v_p(y)$ for any prime p and x, y in $\mathbb{Z}$. Moreover, we note that the $p-$adic valuation and metric have an interesting relationship: two points which have relatively small valuation will have a relatively large distance between them and vice versa.

We are now ready to define $p-$orderings, and not long after, to give the connection to Fekete $n-$tuples.

Definition 7. [?] Let S be a subset of $\mathbb{Z}$ and let p be any prime.[3] A p**-ordering** of S is a sequence, $\{a_i\}_{i\geq 0}$ in S, such that a_0 is arbitrary and for $i > 0$, a_i minimizes

$$v_p(\prod_{j<i}(a_i - a_j))$$

over $z \in S$.

A $p-$ordering in S, like a Fekete $n-$tuple in $\mathbb{C}$, is not unique. Indeed, in most of the examples we will explore, there will be infinitely-many choices at each stage of the construction. Nonetheless, $p-$orderings give rise to $p-$sequences, which are invariants of S:

Definition 8. [?] Let S be a subset of $\mathbb{Z}$ and let p be any prime. Suppose $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S. The **p-sequence**, occasionally the **characteristic sequence**,

[3] To apply the definition to a general Dedekind domain, we replace the usual primes with the set of primes tdeals in the ring of interest.

of S is the sequence defined by $a_0 = 1$ and for $i > 0$,

$$\delta(i) = \prod_{j=0}^{i-1} v_p(a_i, a_j)$$

It is a fact, not entirely obvious, that the $p-$sequence of S is independent of the $p-$ordering used in its construction [**?**]. To define the generalized factorial, Bhargava considered the product of $p-$sequences taken over each prime p for arbitrary subsets of $\mathbb{Z}$. We will go in another direction.

Suppose we were to generalize our definition of Fekete $n-$tuple in the obvious way, as below.

Definition 9. [**?**] Let (M, ρ) be a metric space and suppose $S \subseteq M$ is a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in S^n$, consider

$$\delta_n(z) = \prod_{j<i} \rho(z_i - z_j)^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in S^n$ is called a **generalized Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in S.

What then is the connection to $p-$orderings and $p-$sequences? Suppose S is a subset of $\mathbb{Z}$ and that $\{a_i\}_{i \geq 0}$ is a $p-$ordering of S for some prime p. Then of course from the definition of $p-$orderings, we know that for $i > 0$,

$$v_p(\prod_{j<i}(a_i - a_j)) \leq v_p(\prod_{j<i}(z - a_j))$$

for $z \in S$. Something more is true though, namely,

$$v_p(\prod_{j<i}(a_i - a_j)) \leq v_p(\prod(x_i - x_j))$$

for $x_i, x_j \in S$ [**?**]. That is, when we pick a_i to minimize the $p-$adic valuation over $\prod_{j<i}(z - a_j)$, we actually achieve the minimum over *all pairs in* S. Since minimizing $v_p(x_i - x_j)$ is the same as maximizing $\rho_p(x_i, x_j)$, we have the following remarkable fact: if $\{a_i\}_{i \geq 0}$ is a $p-$ordering of S, then $\{a_i\}_{i=0}^{n}$ is a generalized Fekete $n-$tuple

for (S, ρ_p) for each n. In particular, $p-$orderings give a recursive construction for generalized Fekete $n-$tuples.

The first connection between these objects was made by Jean-Luc Chabert in [?] when he studied the limit of these sequences not just for the case $M = \mathbb{Z}$ and $\rho = \rho_p$, but in the case that M is any rank-one valuation domain [?]. We repeat his theorem 4.2 from [?] below,

Proposition 5. *Let E be a subset of V, a rank-one valuation domain with valuation v. If $\{a_i\}_{i\geq 0}$ is $v-$ordering* [4] *of E, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0,\ldots,x_n\in E} v\Big(\prod_{0\leq j<i\leq n} (x_i - x_j)\Big)$$

Chabert called this limit the valuative capacity of E, and we shall do the same. By replacing the notion of $p-$ordering (or $v-$ordering) with the more general notion of $\rho-$ordering, we are able to give a definition of valuative capacity for a general ultrametric space, without appealing to any algebraic (or measure-theoretic) structure. The following result by Johnson in [?] provides the foundation for the rest of this work:

Proposition 6. *([?], Theorem 1) If S is a compact subset of an ultrametric space (M, ρ), then the first n terms of a ρ-ordering of S always give a Fekete n-tuple of S and all Fekete n-tuples of S arise in this way.*

We explore some consequences of the above in the remainder of this thesis.

[4] A $v-$ordering of E is exactly as expected: a sequence of distinct element $\{a_i\}_{i\geq 0}$ in E is $v-$ordering of E if for $n > 0$,

$$v\Big(\prod_{k=0}^{n-1}(a_n - a_k)\Big) \leq v\Big(\prod_{k=0}^{n-1}(x - a_k)$$

for each $x \in E$.

Chapter 2

Background

The principal context for this thesis is an arbitrary ultrametric space, which is a metric space that also satisifies an additional axion, sometimes called the ultrametric inequality or (in the case of vector spaces) the strong triangle propery. We define ultrametric spaces below and for the rest of this section, we review some of their more important characteristics. The proofs offered in this section are, for the most part, standard and can be found in a number of reference texts, such as [**?**].

Definition 10. Let (M, ρ) be a metric space; that is, suppose M is a set and $\rho : M \times M \to \mathbb{R}_{\geq 0}$ is such that:

(i) $\rho(x, y) = 0$ if and only if $x = y$

(ii) $\rho(x, y) = \rho(y, x)$

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any $x, y, z \in M$. If ρ satistifies the ultrametric inequality,

$$\rho(x, z) \leq \max\left(\rho(x, y), \rho(y, z)\right)$$

for any $x, y, z \in M$, then (M, ρ) is an **ultrametric space**.

A special case of an ultrametric space, and one where much of the previous work on this topic has been completed, is one where the metric has been derived from a norm on a vector space.

Definition 11. Let (V, N) be a normed vector space; that is, suppose V is $\mathbb{F}-$vector space, for $\mathbb{F}$ some subfield of $\mathbb{C}$, and $N : V \to \mathbb{R}_{\geq 0}$ is such that:

(i) $N(x+y) \leq N(x) + N(y)$

(ii) $N(cx) = |c|\ N(X)$

(iii) $N(x) = 0$ implies $x = 0$

for any $x, y \in V$ and $c \in \mathbb{F}$. We say that N satisfies the **strong trianlge inequality** if

$$N(x+y) \leq \max(N(x), N(y))$$

for any $x, y \in V$.

Proposition 7. *Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space, (V, ρ_N), where ρ_N is the metric induced by N, that is, $\rho_N(x, y) = N(x - y)$, is an ultrametric space.*

Proof. We take for granted that (V, ρ_N) is a metric space and also note that

$$N(x+z) \leq \max(N(x), N(z))$$

implies

$$\rho_N(x, z) \leq \max(\rho_N(x, 0), \rho_N(z, 0)) \leq \max(\rho_N(x, y), \rho_N(y, z))$$

□

Notation. If (V, N) is a normed vector space, then the metric induced by N is denoted ρ_N.

When ultrametric spaces come from spaces with algebraic structure, such as normed vector spaces, some of this structure carries over into metric spaces structure in a rather nice way:

Proposition 8. *[**?**] Let S be a group equipped with a (right) invariant ultrametric, ρ. If $B = B(0, r)$ is a (closed) ball centred at the neutral element of S, then B is a subgroup of S.*

Proof. Let $x, y \in B$. Then

$$\rho(x - y, 0) = \rho(x, y) \leq \max(\rho(x, 0), \rho(y, 0)) \leq r,$$

so that $x - y \in B$. $\square$

The canoncial example of an ultrametric space, and one which is derived from a norm, is $p-$adic integers, $\mathbb{Z}_p$ for any prime p.

Definition 12. $p-$adic valuation

It is easy to see that the $p-$adic valuation exhibits the following properites, which makes it nice to work with:

- $v_p(ab) = v_p(a) + v_p(b)$
- $another one$

Example 1. $\mathbb{Z}_p$

We give a second example below, showing a space that does not readily come equipped with algebraic structure:

Example 2. snowflake metric

Ultrametric spaces exhibit properties much unlike traditional metric spaces, and we review of few of these below. Of particular interest to us is the behavior between (closed) balls in an ultrametric space.

Notation. Let (M, ρ) be a compact ultrametric space and let

$$B(a, r) = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the *closed* ball of radius r, centred at a for some $r \in \mathbb{R}_{>0}$ and $a \in (M, \rho)$.

In the above notation, we break from convention in that we denote a closed ball without using any decoration and in that we omit any notation for an open ball. This because, for the most part, the notion of open and closed ball in an ultrametric space overlap, although we will need a few more facts before showing this.

Definition 13. Let S be a subset of an ultrametric space. The **diameter of** S is $diam(S) = \sup_{x,y \in S} \rho(x, y)$. Note that if S is compact, $diam(S) = \max_{x,y \in S} \rho(x, y)$.

Proposition 9. *Let $B = B(a, r)$ be a (closed) ball in an ultrametric space (M, ρ). Then the diameter of B is less than or equal to the radius of B.*

Proof. Suppose $d = diam(B) > r$. This would imply there exists x, y in B such that $\rho(x, y) > r$, in particular $\rho(x, y)$ is strictly greater than $\max(\rho(x, a), \rho(y, a))$, which is a contradiction since ρ is an ultrametric. □

In the following proposition, we describe the triangles in an ultrametric space, and the result is more or less a restatement, in geometric terms, of the ultrametric inequality.

Proposition 10. *All triangles in an ultrametric space (M, ρ) are either equilateral or isosceles, with at most one short side.*

Proof. Let x, y, and z be three points in an ultrametric space (M, ρ). We show that $\rho(x, y) \neq \rho(x, z)$ and $\rho(x, y) \neq \rho(y, z)$ implies $\rho(x, y) < \rho(x, z) = \rho(y, z)$.

If $\rho(x, z) \neq \rho(y, z)$, then without loss, $\rho(x, z) > \rho(y, z)$. At the same time, the ultrametric inequality implies $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ and $\rho(y, z) \leq \max(\rho(x, y), \rho(x, z))$. The first inequality implies $\rho(x, y) < \rho(x, z)$, which means the second inequality implies $\rho(y, z) < \rho(x, z)$. This is a contradiction, so we must have $\rho(x, z) = \rho(y, z)$.

To see that $\rho(x, y) < \rho(x, z)$, simply note that $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ □

With this result in hand, we are able to quickly demonstrate some of the properties of (closed) balls, which are of fundamental importance to us. We see below that the ultrametric inequality, perhaps innocuous on the surface, quickly implies ultrametric balls are markedly different from their archimedean counterparts.

Proposition 11. *Every point of a ball in an ultrametric is at its centre. That is, if $B(x_0, r)$ is a ball in an ultrametric space (M, ρ), then $B(x, r) = B(x_0, r)$, $\forall x \in B(x_0, r)$*

Proof. Let $a \in B(x,r)$. Then $\rho(a,x) \leq r$ and since

$$\rho(a,x_0) \leq \max(\rho(a,x),\rho(x,x_0)) \leq r$$

we must have $a \in B(x_0,r)$ and $B(x,r) \subseteq B(x_0,r)$ A similar argument shows $B(x_0,r) \subseteq B(x,r)$. $\square$

Proposition 12. *If (M,ρ) is an ultrametric space and $B(x_0,r_1)$ and $B(y_0,r_2)$ are balls in (M,ρ), then either $B(x_0,r_1) \cap B(y_0,r_2) = \emptyset$, $B(x_0,r_1) \subseteq B(y_0,r_2)$, or $B(x_0,r_1) \subseteq B(x_0,r_1)$. That is, in an ultrametric space, all balls are either comparable or disjoint.*

Proof. Suppose $B(x_0,r_1) \cap B(y_0,r_2) \neq \emptyset$ and let z be a point in the intersection. We show that if there exists an $a \in B(y_0,r_2)$ such that $a \notin B(x_0,r_1)$, then $B(x_0,r_1) \subseteq B(y_0,r_2)$. Let $x \in B(x_0,r_1)$. Then we must have $\rho(x,z) < \rho(x,a)$, since $z \in B(x_0,r_1) = B(x,r_1)$ and a is not. Since the triangle with vertices (a,x,z) is isocolces with at most one short side, we must have $\rho(x,a) = \rho(a,z) \leq r_2$, since $a \in B(y_0,r_1) = B(z,r_2)$. Then $x \in B(y_0,r_1)$. $\square$

Proposition 13. *The distance between any two non-overlapping balls in an ultrametric is constant. That is, if $B(x_0,r_1)$ and $B(y_0,r_2)$ are two balls in an ultrametric space with $B(x_0,r_1) \cap B(y_0,r_2) = \emptyset$, then there exists a $c \in \mathbb{R}_{\geq 0}$ such that $\rho(x,y) = c$, $\forall x \in B(x_0,r_1)$ and $\forall y \in B(y_0,r_2)$.*

Proof. Suppose $\rho(x_0,y_0) = c$ and let $x \in B(x_0,r_1)$ and $y \in B(y_0,r_2)$ be arbitrary. Consider the triangle formed by (x_0,y_0,y). Since $\rho(x_0,y_0) = c$ and $\rho(y,y_0) \leq r_2 < c$, we must have $\rho(x_0,y) = c$ because triangles in an ultrametric space have at most one short side. Now consider the triangle formed by (x_0,x,y). Since $\rho(x_0,y) = c$ and $\rho(x,x_0) \leq r_1 < c$, we must have $\rho(x,y) = c$. $\square$

Before moving on, we give another example of an ultrametric space, showing again the quick and unusal implications of Proposition 4.

Example 3. Let S be a compact subset of an ultrametric space, (M,ρ), and then let $\mathbf{Triag}_S$ be the space whose points are the set of distinct (up to labeling) triangles in S. If t is an element of $\mathbf{Triag}_S$, denote by $long_t$ the length of the long side of t and by $short_t$ the length of the short side of t. Then define $\Delta_S(t, t') = |long_t - long_{t'}| + |short_t - short_{t'}|$, for t and $t' \in \mathbf{Triag}_S$. Now note $long_t$ and $short_t$ are in $\mathbb{R}$ and so $\Delta_S(t, t')$ is always in $\mathbb{R}_{\geq 0}$. It's also clear that $\Delta_S(t, t') = 0$ if, and only if, t and t' have the same length sides, in which case they are equal in $\mathbf{Triag}_S$. Now let t_a, t_b and t be any three points in $\mathbf{Triag}_S$. Without loss of generality, assume $\max(\Delta_S(t_a, t), \Delta_S(t_b, t)) = \Delta_S(t_b, t)$, that is,

$$|long_{t_a} - long_t| + |short_{t_a} - short_t| \leq |long_t - long_{t_b}| + |short_t - short_{t_b}|$$

We must show

$$|long_{t_a} - long_{t_b}| + |short_{t_a} - short_{t_b}| \leq |long_t - long_{t_b}| + |short_t - short_{t_b}|$$

You can easily construct a counter example with all equilateral triangles (which abouts to the fact that $\mathbb{R}$ is not normally an ultrametric space)

Note that an analogous construction would not work on the set of line segments in S.

The following proposition is easy to see, although the result is both unintuitive and important for our purposes.

Proposition 14. *Suppose S is a compact subset of an ultrametric space (M, ρ) and that $\cup_{i \in I} B(x_i, r_i)$ is a cover of S by (closed) balls in S. Then there exists $i_1, \ldots, i_n$, a finite subset of I, such that $\cup_{j=1}^{j=n} B(x_{i_j}, r_{i_j})$ is a partition of S.*

Proof. Since S is compact, $\cup_{i \in I} B(x_i, r_i)$ contains a finite subcover of S. Say this subcover is given by the elements $i_1, \ldots, i_{n'} \in I$, and suppose this is not a partition. That is, suppose for some i_i, i_j, $B(x_{i_i}, r_{i_i}) \cap B(x_{i_j}, r_{i_j}) \neq \emptyset$. Then, without loss of generality, we must have $B(x_{i_i}, r_{i_i}) \subseteq B(x_{i_j}, r_{i_j})$, so that the removal of $B(x_{i_i}, r_{i_i})$ is still a cover of S. We continue this process a finite number of times, since the subcover was finite to begin with, to arrive at a partition of S. $\square$

In fact, a slightly stronger statement then the above is true:

Corollary 1. *Suppose S is a compact subset of an ultrametric space (M, ρ) and that $B(x_0, r)$ is a (closed) ball in S. Then, there exists a finite partition of S having $B(x_0, r)$ as an element.*

Proof. Let $\mathcal{C}$ be the cover of S given by $\cup_{x \in S} B(x, r) \cap S$. From the proposition, we can select a finite subcover of $\mathcal{C}$ that is a partition of S. Suppose $B(y, r) \cap S$ is the element in this partition containing x_0. Then since $B(y, r)$ and $B(x_0, r)$ are equal in M, $B(y, r) \cap S = B(x_0, r) \cap S = B(x_0, r)$. □

We end this section by making a few comments about the set of distances that occur between the points of a compact ultrametric space.

Proposition 15. *[**?**] Let S is a compact subset of an ultrametric space. For $a \in S$, let $\phi_a : S \setminus \{a\} \to \mathbb{R}$ be the function defined by $\phi_a(x) = \rho(x, a)$. Then $Im(\phi_a)$ is a discrete subset of $\mathbb{R}$ for all $a \in S$.*

Proof. [**?**] □

Corollary 2. *[**?**] Let $B(a, r)$ be a closed ball in an ultrametric space. Then there exists $r' > r \in \mathbb{R}$ such that $B(a, r) = \{x \in M \mid \rho(x, a) < r'\}$; that is, every closed ball is also an open ball with the same centre and slightly larger radius.*

Proof. □

Corollary 3. *[**?**] If S is a compact subset of an ultrametric space and Γ_S is the set of all distances occurring between points of S, then Γ_S is countable; that is, there is an injective function from $\Gamma_S \mapsto \mathbb{N}$.*

Proof. □

It will become useful to write the set of distances occuring in S as a sequence, put in decreasing order.

Notation. If S is a compact (hence bounded) ultrametric space, then we denote the set of distances between points of S by

$$\Gamma_S = \{\gamma_0 = d = diam(S), \gamma_1, \gamma_2, \ldots, \gamma_\infty = 0\}$$

where $\gamma_i \in \Gamma_S$ if and only if $\exists x, y \in S$ such that $\rho(x, y) = \gamma_i$ and $\gamma_i < \gamma_j$ if and only if $i > j$.

To make use of regularity in S we should, before continuing, also put some constraints on our sequence of distances Γ_S.

Definition 14. Let S be a compact subset of an ultrametric space and Γ_S is the sequence of decreasing distances in S. Then we say S is **well-behaved** (I guess - or tame? reasonable?), if S is semi-regular and $\gamma_k = \alpha(k)^{c_k}$ for all $k \in \mathbb{N}$ and some $c_k \in \mathbb{Z}$.

Example 4. Any ultrametric space where ρ is induced from a valuation domain.

ρ-orderings, ρ-sequences, and valuative capacity

We now come to the central theme of this work. The observation that an analogous notion of $p-$ordering can be defined for a general ultrametric space, and that these structures coincide with Fekete $n-$tuples, is due to [**?**]. The exploration of this idea makes up the remainder of this work.

In what follows, we let S be a compact subset of an ultrametric space (M, ρ).

Definition 15. [**?**] A **ρ-ordering** of S is a sequence $\{a_i\}_{i=0}^{\infty}$ in S such that a_0 is arbitrary and $\forall n > 0$, a_n maximizes $\prod_{i=0}^{n-1} \rho(s, a_i)$ over $s \in S$.

Example 5. Suppose S is a finite subset of $(\mathbb{Z}, \rho_2)$, $S = \{0, 2, 8, 3\}$. Then a $\rho-$ordering of S starts (arbitrarily) with $a_0 = 0$, which forces $a_1 = 3$, since $\rho(0, 3) = 1 = diam(S)$. The sequence continues $a_2 = 2$ and $a_3 = 8$, but after this point the sequence becomes arbitrary because $\prod_{i=0}^{n-1} \rho(s, a_i)$ will contain a 0, given by the repeated term. Indeed, for any finite subset S with $|S| = n$, the $\rho-$ordering of S is arbitrary from the n^{th} point on.

Definition 16. [**?**] Let $\sigma(i)$ be a $\rho-$ordering of S. The **ρ-sequence** of S is defined by letting $a_0 = 1$ and $a_n = \prod_{i=0}^{n-1} \rho(\sigma(n), \sigma(i))$, for $n > 0$.

- this gives back the n-th diameter - the limit is the transfinite diameter and we take it as the definition of capacity
- clean up all the sequence definitions to use the same format
- check the historical background and this section to make sure you've said enough about fekete n-tuples and p-orderings corresponding
- go through the JLC chabert reference
- check that youve used the same notation for closed balls

Proposition 16. *[**?**] The ρ-sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ-sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ-ordering of S.*

Definition 17. [**?**] Let $\gamma(n)$ be the ρ-sequence of S. The **valuative capacity** of S is

$$\omega(S) := \lim_{n\to\infty} \gamma(n)^{1/n}$$

Proposition 17. *[**?**] For S and $\gamma(n)$ as above, $\lim_{n\to\infty} \gamma(n)^{1/n} = r < \infty$.*

Proposition 18. *If $S \subseteq M$ is a finite subset of an ultrametric space, then $\omega(S) = 0$.*

Proposition 19. *(upper bound) If $diam(S) = d$, then $\omega(S) < d$.*

Proof. Since d is the diameter of S, the n^{th} term of the ρ-sequence of S is bounded by d^n and so $\lim_{n\to\infty} \gamma(n)^{1/n} = d$ if, and only if, $\gamma(n) = d^n$, $\forall n$. This implies $\rho(a_n, a_i) = d$, $\forall n$ and $\forall i < n$, but then $\rho(a_i, a_j) = d$, $\forall i, j$, since the ρ-sequence is maximized at each n. This means $\omega(S) < d$ would imply that the cover of S, $\cup_{a_i} B_d(a_i)$ is in fact an infinite partition, contradicting the compactness of S. Then $\omega(S) = \lim_{n\to\infty} \gamma(n)^{1/n} < d$. $\square$

This doesn't work because $\cup_{a_i} B_d(a_i)$ could fail to be a cover - the $\rho-$ ordering will be a dense subset under mild conditions - then being compact should imply a contradiction since we should have $\overline{S} = S$ for S compact.

Make a comment about algebraic structure here

Proposition 20. *(translation invariance) If (M, ρ) be a compact ultrametric space and s also a topological group for which ρ is (left) invariant under the group operation, then ω is also (left)-invariant. That is, if $\rho(x, y) = \rho(gx, gy)$, $\forall g, x, y \in M$, then $\omega(gS) = \omega(S)$, for $S \subseteq M$.*

Proof. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS. Then

$$\omega(gS) = \lim_{n\to\infty} \gamma(n)^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = \omega(S)$$

$\square$

Example 6. With the notation of the previous section, note that ρ_p is translation invariant since for $x, y \in (\mathbb{Z}_p, \rho_p)$, $\rho_p(x, y) = p^{-v_p(x-y)} = p^{-v_p((a+x)-(a+y))} = \rho_p(a + x, a + y)$. Then $\omega(a + S) = \omega(S)$ for $S \subseteq (\mathbb{Z}_p, \rho_p)$.

Proposition 21. *(scaling) Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity, so that (V, ρ_N) is an ultrametric space. Then if N is multiplicative, so is ω. That is, if $N(gx) = N(g)N(x), \forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.*

Proof. Let ρ be the metric induced by N, so that $\rho(x, y) = N(x-y), \forall x, y \in V$. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then since N is multiplicative, for $u, v \in gS$, $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS and

$$\omega(gS) = \lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty}[\prod_{i=0}^{n-1}N(g)\rho(a_n, a_i)]^{1/n}$$

$$= \lim_{n\to\infty}[N(g)^n\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\omega(S)$$

□

Example 7. Since ρ_p is multiplicative, $\omega(mS) = v_p(m)\cdot\omega(S)$ for $m \in \mathbb{Z}_p$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) = \frac{1}{p}\cdot\omega(\mathbb{Z})$.

This is important and you should say so. The following proposition is from [**?**], where it is given for some S written as the union of two subsets, although it is easily seen to be true for S equal to any finite union, so long as the other assumptions remain satisfied.

Proposition 22. *[**?**](subadditivity) If diam$(S) := \max_{x,y\in S}\rho(x, y) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then*

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n}\frac{1}{log(\omega(A_i)/d)}$$

Example 8. $\omega(\mathbb{Z})$

Corollary 4. *Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = diam(S)$ and also $\omega(S_i) =$*

$\omega(S_j)$, $\forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S)$, $\forall i$. Then $\omega(S) = r^{\frac{1}{n-1}} \cdot d$. In particular if $S = \mathbb{Z}$ and $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$ then $\omega(S) = (\frac{1}{p})^{1/p-1}$ for any prime p.

Example 9.

Corollary 5. *(Joins of computable sets are computable) Let $\Gamma_M = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in M. Suppose that $S = B_{\gamma_i}(x)$, for some x and i, is the union of 2 or more balls of radius γ_{i+1}, i.e., $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$ is a join in the lattice of open sets in M, then*

$$\frac{1}{log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^{n} \frac{1}{log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

Chapter 3

ρ-orderings and the structure of S

In the previous section, we defined valuative capacity for a compact subset S of an ultrametric space (M, ρ). We also got a glimpse into the way the valuative capacity of S interacts with its other properties, such as the set of distances occurring in S and the lattice of closed balls in S (or equivalently, if S has enough structure, a lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valuative capacity of S and the lattice of closed balls in S. In particular, we show how, in all cases (with S compact), the latter can be used to compute the first n terms of a $\rho-$ordering of S (for any $n < \infty$) and how, in some cases, this extends to being able to compute the valuative capacity of S.

Subspaces of S

In the section we explore the subspaces of S formed by considering closed balls of some fixed radius. We begin by letting S be, as before, a compact subset of an ultrametric space (M, ρ). Recall from the previous section that if S is compact, then the set of distances occurring in S is a discrete, bounded subset of $\mathbb{R}$ and so we may represent the set of distances by a sequence in decreasing order. As before, let the decreasing sequence of distances in S be given by $\Gamma_S = \{\gamma_0 = \text{diam}(S), \gamma_1, \ldots, \gamma_\infty = 0\}$.

Now fix some $k \in \mathbb{N}$, and consider for a moment the set of closed balls of radius γ_k in S. We could denote these alternatively by $B^M(x, \gamma_k) \cap S$ or by $B^S(x, \gamma_k)$, but when there is no risk of confusion, we will denote them simply by $B(x, \gamma_k)$. Clearly, the set $\{B(x, \gamma_k); x \in S\}$ forms a cover of S. Although we have build the cover using closed balls, since each set in an ultrametric is clopen, this gives an open cover of S

(in fact, each element is not only an open set, but also an open ball for some radius slightly bigger than γ_k). Then since S is compact, we must have some $x_1, \ldots, x_n$ such that $S = \cup_{i=1}^n B(x_i, \gamma_k)$. In fact, since ρ is an ultrametric, we can pick the x_i's so that $\cup_{i=1}^n B(x_i, \gamma_k)$ will be a disjoint union and therefore a partition of S. Note that both n and the x_i's depend on our fixed k, but that n is independent of the x_i's, since any choice of centres is equivalent. We rephrase this with following definition and lemma:

Definition 18. For S and Γ_S as above, and $k \in \mathbb{N}$, fixed, define $\sim_k$ to be the relation on S given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e., $x \sim_k y$ if and only if $B_{\gamma_k}(x) = B_{\gamma_k}(y)$.

The fact $\sim_k$ is an equivalence relation on S is equivalent to the observation that every point in a ultrametric ball is at its centre:

Lemma 1. *Let S and Γ_S be as above, then $\sim_k$ is an equivalence relation on S.*

Proof. $\sim_k$ is clearly reflexive and symmetric, since ρ is a metric. Transitivity results from the ultrametric property of ρ: if $x \sim_k y$ and $y \sim_k z$, then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(z, y)) \leq \gamma_k$$

so $x \sim_k z$. □

We denote the set of equivalence classes of $S/ \sim_k$ by S_{γ_k}. We have defined S_{γ_k} to be the set of equivalence classes in S under the relation $\sim_k$, which is equivalent to letting S_{γ_k} be the set of closed balls of fixed radius γ_k in S. We now offer a third perspective on the elements on S_{γ_k}, which is due to [**?**],

Lemma 2. *For each k, the elements of S_{γ_k}, that is, the closed balls of radius γ_k, themselves form an ultrametric space, where the metric is given by:*

$$\rho_k(B(x, \gamma_k), B(y, \gamma_k)) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } B(x, \gamma_k) = B(y, \gamma_k) \end{cases}$$

Proof. ρ_k is reflexive, symmetric and transitive since ρ is. Likewise, ρ_k satisfies the ultrametric property, since ρ does: let $B(x,\gamma_k), B(y,\gamma_k)$ and $B(z,\gamma_k)$ be any three elements of S_{γ_k} and suppose $\rho_k(B(x,\gamma_k), B(y,\gamma_k)) > 0$. Then,

$$\begin{aligned} \gamma_k &< \rho_k(B(x,\gamma_k), B(y,\gamma_k)) \\ &= \rho(x,y) \leq \max(\rho(x,z), \rho(y,z)) \\ &= \max(\rho_k(B(x,\gamma_k), B(z,\gamma_k)), \rho_k(B(y,\gamma_k), B(z,\gamma_k))) \end{aligned}$$

since $\gamma_k < \max(\rho(x,z), \rho(y,z))$ implies that at least one of $\rho_k(B(x,\gamma_k), B(z,\gamma_k))$ or $\rho_k(B(y,\gamma_k), B(z,\gamma_k))$ is greater than 0. $\square$

So now the elements of S_{γ_k} may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

Definition 19. Let S and Γ_S be as above. Define $\beta(i)_{i\geq 0}$ to be the sequence given by $\beta(i) = |S_{\gamma_i}|$, which is an invariant of S and which counts the number of connected components of S_{γ_i} (that is, the points of S_{γ_i}), when viewed as a metric space. When necessary, we use $\beta^S(i)$ to denote the β sequence for a given, compact ultrametric space S. Adapting the terminology in [**?**], we call $\beta^S(i)$ the **structure sequence** of S.

Notation 1. Let S_{γ_k} be as above. We denote the elements of S_{γ_k} by $B_1^k, \ldots, B_{\beta(k)}^k$ or by $B_1^{S,k}, \ldots, B_{\beta(k)}^{S,k}$, when necessary.

We return to the sequence $\beta(i)$ at the end of this section. For now, we show how a $\rho-$ordering of S can be built recursively from the spaces S_{γ_k}. This begins by noting that the spaces themselves can be built recursively:

Observation 1. Let S, Γ_S, and S_{γ_k} be as above. Then S_{γ_k+1} can be constructed by partitioning each of the closed balls in S_{γ_k} into closed balls of radius γ_{k+1} and taking their union: Let $B(x_i,\gamma_k)$ be an element of S_{γ_k}, denoted by B_i^k. Then, there exists $x_{i,1}, \ldots, x_{i,l_i} \in B_i^k$ such that,

$$B_i^k = \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1})$$

and

$$B(x_{i,j}, \gamma_{k+1}) \cap B(x_{i,j'}, \gamma_{k+1}) = \emptyset, \forall j, j' \in 1 : l_i$$

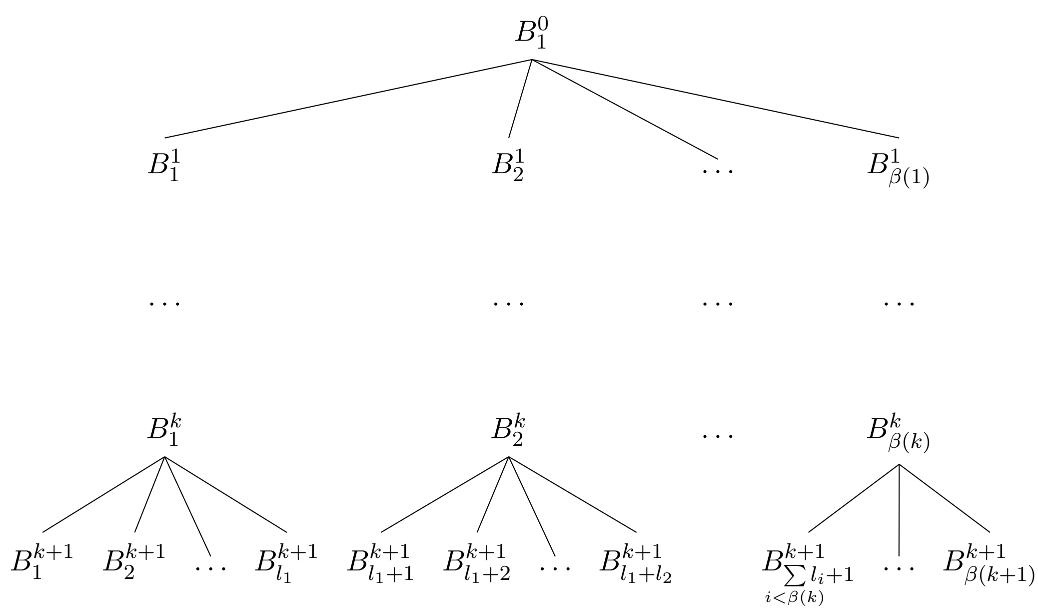
and so

$$S_{\gamma_{k+1}} = \bigcup_{i=1}^{\beta(k)} \cup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1}) = \bigcup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where $\cup_{j=1}^{l_i} B(x_{i,j}, \gamma_k) = B(x_i, \gamma_{k+1}) = B_i^k$, $\forall i$.

Since S is compact, hence bounded, if we represent this process schematically we obtain a tree, where the root node is $B_1^0 = B(x, \gamma_0)$, for any choice of $x \in S$, and the children of any given B_n^m are such that they form a partition of their join. Since we will often refer to this schematic representation, we define it below.

Definition 20. If S is a compact subset of an ultrametric space, then T_s is the tree whose vertices are B_i^k, that is the elements of S_{γ_k}, and whose edgeset, E, is given by $(B_k^i, B_l^j) \in E$ if, and only if, $j = i+1$ and $B_l^j \subseteq B_k^i$ for some choice of representatives $B(x_k, \gamma_i)$ and $B(x_l, \gamma_j)$, as shown below:



Before going on, first note that we have drawn T_S such that leftmost child of some B_i^k is B_j^{k+1} where j is minimal among the children of B_i^k, and then continued in increasing order. In general, if we draw T_S so that the children of a given vertex are depicted in increasing order according to their index, then each choice of indexing for the elements of S_{γ_k} produces a different graphical representation of T_S. The structures produced by different choices of indices are clearly isomorphic as trees, and as we will see by the end of the section, each choice of indexing will be valid for our purposes as well.

Of central importance to us is the distance between two vertices in T_s. Since each vertex represents an element of S_{γ_k}, that is a closed ball in an ultrametric space, it is well-defined to let the distance between vertices be equal to the distance between a choice of centres for those balls. Note that if the distance between B_i^k and B_j^l is taken to be $\rho(x_i, x_j)$, for some choice of $x_i \in B_i^k$ and $x_j \in B_j^l$, say $\rho(x_i, x_j) = \gamma_n$, then the join of B_i^k and B_j^l is some B_x^n.

Lemma 3. *If B_i^k and B_j^l are two vertices in T_S, then $\rho(x_i, x_j)$, for any choice of $x_i \in B_i^k$ and $x_j \in B_j^l$, is equal to the diameter of the join of B_i^k and B_j^l.*

Proof. Let B_i^k and B_j^l be two (distinct) vertices in T_S and let B_x^n be their join. The diameter of B_x^n is γ_n since $B_x^n = B(x_0, \gamma_n)$ for some x_0. Since ρ is an ultrametric the distance between any $x_i \in B_i^k$ and $x_j \in B_j^l$ is constant, and must be equal to the diameter of the smallest ball containing both of them, that is γ_n. □

In particular, we have that for any k and any $i < \beta(k)$, the distances between the children of B_i^k will be γ_k and for any $i \neq j$ the distance between the children of B_i^k and B_j^k will be equal to the distance between B_i^k and B_j^k (which will be some $\gamma_m, m < k$).

Recusive ρ-orderings

In this section, we show how the recursive partioning of S into the spaces S_{γ_k} gives rise to a $\rho-$ordering of S. We first note that without loss of generality, for any $k \in \mathbb{N}$, we can reindex the B_i^k's so that they give the first $\beta(k)$ terms of a ρ_k-ordering of S_{γ_k}, when the latter is viewed as a (finite) metric space. In the first proposition below, we note that if the B_i^k's are so indexed, then finding a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ is straightforward: select a B_j^{k+1} from each of the B_i^k's in order and then start over.

Proposition 23. *Let be S a compact subset of an ultrametric space (M, ρ) and Γ_S, the set of distances in S. If S_{γ_k} is the partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k-ordering of S_{γ_k}, then the first $\beta(k+1)$ terms in a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ can be found by selecting at each stage n, a child from $B^k_{\overline{n}}$, where $\overline{n} = n \mod \beta(k) + r$ and r is minimal in $\{0, \dots, \beta(k)-1\}$ such that $B^k_{n \mod \beta(k)+r}$ still has unused children.*

Proof. Let S, S_{γ_K}, and $S_{\gamma_{k+1}}$ be as above. In particular, suppose the elements of S_{γ_k} are indexed according to a ρ_k-ordering. Denote the elements of $S_{\gamma_{k+1}}$ by $B^{k+1}_{i,j}$ where the first subscript indicates that the elements is a child of B^k_i. To form a ρ_{k+1} ordering of $S_{\gamma_{k+1}}$, we must maximize the product of distances at each step n.

Now note that $\Gamma_{S_{\gamma_k}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$ and $\Gamma_{S_{\gamma_{k+1}}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}, \gamma_k\}$. That is, the distances in $S_{\gamma_{k+1}}$ are the same as the distances in S_{γ_k}, although they also include the smaller distance γ_k. Since we know that the elements $B^k_1, \dots, B^k_{\beta(k)}$ already maximizes the product of distances in $\{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$, the first $\beta(k)$ terms of a ρ_{k+1}-ordering of S_{k+1} can be found by taking $B^k_{1,j_1}, \dots, B^k_{1,j_{\beta(k)}}$ for any choice of j's. At this point, any choice of next element will produce a copy of γ_k in the ρ_{k+1}-sequence; however, if we chose another child of B^k_1, we are able to keep building the ordering in a canonical fashion, since we know that we will then be able to maximize the product at the next step by chosing another child of B^k_2.

We see then that a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ is found by minimizing the number of times γ_k is introduced into the ρ_{k+1}-sequence and maximizing the product among the $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$, and the latter is already known to be achieved by taking the B^k_i in order. If the B^k_i's all have the same number of children, then we can always select a child of $B^k_{\overline{n}}$, where $\overline{n} = n \mod \beta(k)$ at each stage n, $n < \beta(k+1)$, since there will always be one available. On the other hand, suppose the B^k_i have an unequal number of children and n is the first step at which all the children of $B^k_{\overline{n}}$ have been exhausted. What element will maximize the $\rho_{k+1}-$sequence?

Consider the space $(S_{\gamma_k} \setminus B^k_{\overline{n}})$. Removal of $B^k_{\overline{n}}$ will not effect the first m terms of a ρ_k-ordering of this space, for $m < \overline{n}$, since if a sequence of elements maximizes a function over a set X, they will also maximize that function of a subset of X (provided they themselves remain in the subset). Then the ρ_k-sequence of $(S_{\gamma_k} \setminus B^k_{\overline{n}})$ begins $\{B^k_1, \ldots, B^k_{\overline{n}-1}\}$.

Moreover, if $B^k_{\overline{n}+1}$ maximizes $\prod_{i=1}^{\overline{n}} \rho_k(x, B^k_i)$ over S_{γ_k}, then it also maximizes $\prod_{i=1}^{\overline{n}-1} \rho_k(x, B^k_i)$ over $(S_{\gamma_k} \setminus B^k_{\overline{n}})$, since $\prod_{i=1}^{\overline{n}} \rho_k(x, B^k_i) = (\prod_{i=1}^{\overline{n}-1} \rho_k(x, B^k_i)) \cdot \rho_k(x, B^k_{\overline{n}})$.

Then the ρ_k-sequence of $(S_{\gamma_k} \setminus B^k_{\overline{n}})$ is simply $\{B^k_1, \ldots, B^k_{\overline{n}-1}, B^k_{\overline{n}+1}, \ldots, B^k_{\beta(k)}\}$.

Now we see that $\rho_{k+1}-$sequence of $S_{\gamma_{k+1}}$ is maximized by simply skipping over $B^k_{\overline{n}}$, should all its children be exhausted, and selecting a child from $B^k_{\overline{n}+1}$. Then a $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$ is found by selecting elements of each B^k_i in order as much as possible, and skipping to B^k_{i+1}, when it is not possible.

$\square$

Note that in building the $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$ we selected, at each step, a child of some B^k_i, but we did not concern ourselves over which child was selected. This is because the distances between any two children of some B^k_i is γ_k, and the distance between any one of them and a child of some B^k_j, $i \neq j$, is the same. We can now see, as claimed above, that any of the isomorphic versions of T_S are valid for producing $\rho-$orderings. Suppose then that we have created T_s and (arbitrarily) indexed the children of each vertex. Then, there is no loss of genearlity in assuming that at each stage, we select a child with smallest index among its siblings, that is, that we select the leftmost available child in T_s. Since for ease of indexing, we will assume a $\rho-$ordering has been built by this convention, we introduce the following definition.

Definition 21. The $\rho-$ordering of S formed by pulling elements from left to right in (a choice of) T_s is call the **canonical ρ-ordering** of S (with respect to T_s).

The above proposition quickly leds to a recursive contruction for a $\rho-$ordering of S. Indeed, to build a $\rho-$ordering of S from the above, it suffices only to make a choice of centres for each of B^k_i's.

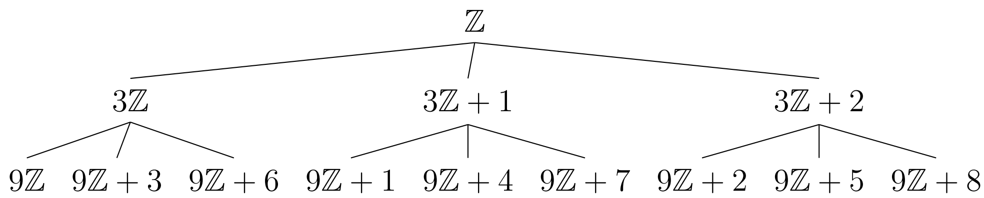
Proposition 24. *Let be S a compact subset of an ultrametric space (M, ρ) and let Γ_S be the set of distances in S. Let S_{γ_k} be the partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k-ordering of S_{γ_k}. Suppose each of the element of S_{γ_k} have also been partitioned into closed balls of radius γ_{k+1}, $B_i^k = \cup_{j=1}^{l_i} B_{i,j}^{k+1}, \forall i$.*

*Let $x_{i,j}$ denote a choice of centre for the element $B_{i,j}^{k+1}$. Then the first $\beta(k+1)$ elements of a $\rho-$ordering of S can be found by forming a matrix, A_k, whose $(i,j)^{th}$-entry is $x_{i,j}$, if $j \leq l_i$ and * otherwise, and then concatenating the rows.*

Proof. The matrix A_k is a representation of the k^{th} and $(k+1)^{th}$ levesl of T_S where the B_i^k's (and $B_{i,j}^{k+1}$'s) have been replaced by a choice of centres. Since matrices must be rectangluar, the case where some B_i^k and B_j^k have an unequal number children is handled by inserting a placeholder, *, into A_k. Moreover, since the ρ_{k+1} distance between distinct closed balls is just the ρ distance between a choice of centres of those balls, a choice of centres in a ρ_{k+1}-ordering gives the beginning of a ρ-ordering. By the above proposition, we must select elements from each B_i^K one after the other, which is achieved by selecting one element from each column in order, for example by concatenating the rows (and then deleting *'s if necessary). □

We get the most use out of the construction above if, in selecting a choice of centres for the $B_{i,j}^{k+1}$'s, we reuse the previous the choices as much as possible. Suppose for example we have made a choice of centres for the balls of radius γ_k and constructed the matrix A_{k-1}. At the next iteration, we will need a choice of centres for the balls of radius γ_{k+1}. If x_i was our choice of representative for B_i^k and $x_i \in B_{i,j}^{k+1}$, we may as well let x_i be our choice of representative for $B_{i,j}^{k+1}$. If we make our choice of centres in this way, then when we concatenate the rows of some A_{k-1}, we obtain (without loss) the first row of A_k. We follow this convention in the two examples below.

Example 10. Let us use the above to start a ρ-ordering of $S = (\mathbb{Z}, |\cdot|_3)$. We have that $\Gamma_S = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\}$ and T_s begins:



We start by finding a ρ_0-ordering of S_{γ_0}, but this is trival since S_{γ_0} has only a single element. Let us pick 0 to be our choice on centre for $B_1^0 = B(0,1) = \mathbb{Z}$. As we see from T_S, S_{γ_0} is partitioned into 3 closed balls of radius $\gamma_1 = \frac{1}{3}$, namely $3\mathbb{Z}, 3\mathbb{Z}+1$, and $3\mathbb{Z}+2$. A choice of centres is given by $0, 1$, and 2, so that A_0 becomes:

$$A_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

To start the $\rho-$ordering, concatenate the rows to obtain $\{0, 1, 2\}$, and to continue it, make a choice of centres for each of the closed balls of radius $\gamma_2 = \frac{1}{9}$ partitioning the sets $3\mathbb{Z}+i$, $i \in 0, 1, 2$. For example, $3\mathbb{Z} = 9\mathbb{Z} \cup 9\mathbb{Z}+3 \cup 9\mathbb{Z}+6$, so a choice of centres for B_1^1 is given by $\{0, 3, 6\}$. Making choices for the remaining elements, we obtain:

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

To continue the $\rho-$ordering we concatenate the rows, $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, which also gives the first row of A_2. The remaining rows are found by partitioning each of the closed balls of radius $\frac{1}{9}$ and again making a choice of centres:

$$A_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{pmatrix}$$

And so on.

We are able to make two statements following this example. The first is that in starting the $\rho-$ordering, the fact that S_{γ_0} had only a single element allowed us to get started for free. In fact, all compact ultrametric spaces are bounded, so this is always the case.

The second takeaway is that we found the start of a $\rho-$ordering of $S = (\mathbb{Z}, |\cdot|_3)$ was given by taking the integers starting at 0 in their natural order. If we had continued building the ordering, we would have continued to find this. The fact that the natural ordering on the integers is a ρ_p-ordering, where ρ_p is the $p-$adic metric for any prime p, is well known (cf. ...), but we give an alternate proof of it here:

Corollary 6. *Let S be the ultrametric space $(\mathbb{Z}, \rho_p)$, where ρ_p is $p-$adic metric for any prime p. The a ρ_p-ordering of S can be found by taking the integers, starting at 0, in their natural order.*

Proof. We prove the above by induction on k. First note that for any choice of prime, the elements of S_{γ_1} are the cosets of $\mathbb{Z}$ modulo p, so that A_0 has p columns. Since $\{0, 1, 2 \ldots, p-1\}$ are distributed among each of these cosets, without loss of generality the first row of A_0 is given by $[0, 1, 2, \ldots, p-1]$ in order.

Now suppose that the first row of A_k is given by $[0, 1, 2, \ldots, n]$ for $0 \leq k < k+1$. We show the first row of A_{k+1}, and therefore the first n' elements in a ρ_p-ordering of S, where n' is the column dimension of A_{k+1}, can be obtained as $[0, 1, 2, \ldots, n, n+1, \ldots, n']$. First note that each closed ball of radius $p^k = \gamma_k$ is in fact a coset of $\mathbb{Z}$ modulo p^k, of which there are p. Then for any k, A_k is a matrix with p^k columns and p rows. In particular, $n = p^k - 1$. Let $i \in \{0, 1, \ldots, p^k - 1\}$ be arbitrary. Then i is in exactly one of the cosets of $\mathbb{Z}$ modulo p^k and since the first row of A_k is $[0, 1, 2, \ldots, p^k - 1]$, it must have been chosen as our representative of this coset. If we split $p^k\mathbb{Z} + i$ into balls of radius p^{k+1}, we have

$$p^k\mathbb{Z} + i = \bigcup_{j=0}^{p-1} p^{k+1}\mathbb{Z} + (p^k j + i)$$

since there will be p elements in the partition, each of which will be equal to i modulo p^k and distinct modulo p^{k+1}. Then, there is a choice of centres such that the i^{th} column of A_k is

$$[i, p^k + i, 2p^k + i, \ldots, (p-1)p^k + i]^T$$

filling this in for each i, we see that A_k can be obtained as:

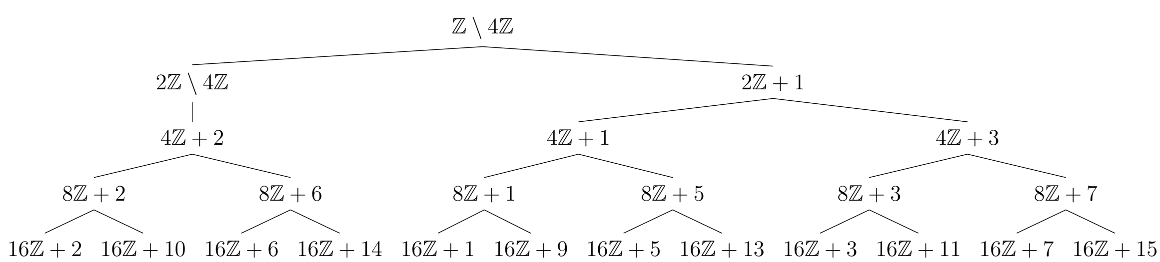
$$A_k = \begin{pmatrix} 0 & 1 & 2 & \dots & p^k - 1 \\ p^k & p^k + 1 & p^k + 2 & \dots & p^k + (p^k - 1) \\ 2p^k & 2p^k + 1 & 2p^k + 2 & \dots & 2p^k + (p^k - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)p^k & (p-1)p^k + 1 & (p-1)p^k + 2 & \dots & (p-1)p^k + (p^k - 1) \end{pmatrix}$$

Concatenating the rows, we see the first row of A_{k+1} will be

$$[0, 1, 2, \dots, p^k - 1, p^k, \dots, p^{k+1} - 1]$$

as required. $\square$

Example 11. Let us now see an example where there is an uneven number of children between the vertices on a given level. Suppose $S = \mathbb{Z}_{(2)} \setminus 4\mathbb{Z}$. In this case, we have that $\Gamma_S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ and T_s begins:



Choosing centres for the partition of $\mathbb{Z}$ into closed balls of radius $\frac{1}{2}$, we have:

$$A_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have taken S to be the complement of $4\mathbb{Z}$ in $\mathbb{Z}_{(2)}$, so $B(0, \gamma_1)$ has only one child, since $2\mathbb{Z} \setminus 4\mathbb{Z} = 4\mathbb{Z} + 2$, while $B(1, \gamma_1)$ has two. Making a choice of centres, we have:

$$A_1 = \begin{pmatrix} 2 & 1 \\ * & 3 \end{pmatrix}$$

We concatenate the rows, skipping over *, and again make a choice of centres for the closed balls of radius $\frac{1}{8}$:

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \end{pmatrix}$$

One more iteration yields:

$$A_2 = \begin{pmatrix} 2 & 1 & 3 & 6 & 5 & 7 \\ 10 & 9 & 11 & 14 & 13 & 15 \end{pmatrix}$$

So that a ρ_2-ordering of $S = \mathbb{Z}_{(2)} \backslash 4\mathbb{Z}$ starts: $\{2, 1, 3, 6, 5, 7, 10, 9, 11, 14, 13, 15, \ldots\}$.

In the two propositions above, there was notational difficulty that arose when there was an unequal number of children between the vertices on a given level of T_s. This difficulty is, in fact, more than a notational inconvenience, and the situation simplifies considerably when it is not the case. We are far from the first to observe this. Amice noted this as far back as her 1964 paper, and it has been observed more recently in several papers by Fares and colleagues. The following section discusses this in more detail, first by supplying some preliminary lemmas and then showing how calculations are simplified in this setting.

Semi-regularity

In this section, we restrict to the case where in the tree T_s, for S some compact subset of an ultrametric space, every vertex on a given level has the same number of children. In this case, we can attach another sequence to S, which we call the $\alpha-$sequence of S and which describes, for each level $k \in \mathbb{N}$, the size of the partitions on that level. We develop some preliminary lemmas, which we then use to derive formulae for this special case.

Definition 22. Let S be as above, a compact subset of an ultrametric space (M, ρ). We say that S is **semi-regular** if $T_{B_i^k} \cong T_{B_j^k}$, $\forall k \in \mathbb{N}$ and $i, j \in \beta(k)$, and where the isomorphism is understood as an isomorphism of trees. That is, S is semi-regular if each ball of radius γ_k breaks into the same number of balls of radius γ_{k+1}, for all k. If there exists an $n \in \mathbb{N}$ such that $T_{B_i^N} \cong T_{B_j^N}$ for all $N \geq n$, i.e. each ball of radius

γ_N breaks into the same number of balls of radius γ_{N+1} for $N \geq n$, then we say S is **eventually semi-regular**.

Definition 23. Suppose S is a compact subset of an ultrametric space and S is semi-regular. The α**-sequence** of S is the sequence given by $\alpha(k) = \frac{\beta(k+1)}{\beta(k)}$, which is in $\mathbb{N}$ for each k. That is, if B_i^k is an element of S_{γ_k}, then $\alpha(k)$ is equal to the number of children of B_i^k in T_s. Since S is semi-regular, this number does not depend on i.

Lemma 4. $\lfloor \frac{n}{q} \rfloor$ *counts the numbers strictly less than* n *that are congruent to* n mod q.

Proof. (sketch) Every multiple of q produces exactly one of the numbers from 1 to q and exactly one of those is the residue class of n modulo q. The remainder is the residue class of n itself and since we only want the numbers strictly less than n, we ignore this by taking the floor. ☐

Lemma 5.

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n+kb}{ab} \rfloor$$

for $n, a, b \in \mathbb{N}$. *In particular,*

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{2b} \rfloor = \lfloor \frac{n+b}{2b} \rfloor$$

for $n, b \in \mathbb{N}$.

Proof.

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor = \lfloor a \cdot \frac{n}{ab} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=0}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor - \lfloor \frac{n}{ab} \rfloor \text{ (*)}$$

$$= \sum_{k=1}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n+kb}{ab} \rfloor$$

where the final step in (*) is due to Hermite's identity: $\lfloor nx \rfloor = \sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor$, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. ☐

Lemma 6. *If* S *is semi-regular and* δ *denotes the canonical* ρ*-ordering of* S, *that is, a* $\rho-$*ordering formed by pulling from left to right in* T_s, *then*

$$\rho(\delta(n), \delta(m)) = \gamma_k$$

if, and only if,

$$n = m \mod \beta(k) \text{ and } n \neq m \mod \beta(k+1)$$

Proof. Since S is semi-regular, every sequence of $\beta(k)$ terms in δ will be from each of distinct elements of S_{γ_k} (for any k). Moreover, since δ is a canonical $\rho-$ordering, we always pull from the elements of S_{γ_k} in the same order. Then $\delta(n)$ and $\delta(m)$ are descendents of some B_j^k if, and only if, $n = m \mod \beta(k)$. Then the result follows since $\rho(\delta(n), \delta(m)) = \gamma_k$ if, and only if, B_i^k for some $i \in 1, \ldots, \beta(k)$ is the join of $B_i^n \ni \delta(n)$ and $B_{i'}^m \ni \delta(m)$.

□

Proposition 25. *If S is a semi-regular ultrametric space, σ is the characteristic sequence of S, β is the structure sequence of S, and α is the sequence describing the semi-regularity, then*

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Proof. The exponent of γ_k in the n^{th} term of the characteristic sequence is the number of m strictly less than n such that $\rho(\sigma(n), \sigma(m)) = \gamma_k$. By the lemma above, this the number of $m < n$ such that $m = n \mod \beta(k)$ and $m \neq n \mod \beta(k+1)$, which by the previous lemma is $\lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor$. Then we have:

$$\begin{aligned} & v_{\gamma_k}(\sigma(n)) \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k)\alpha(k)} \rfloor, \text{ because } S \text{ is semi-regular} \\ &= \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor \end{aligned}$$

□

Example 12. Consider the ultrametric space $(\mathbb{Z}, |\cdot|_p)$ for any prime p. Then $\beta(k) =$

p^k and $\alpha(k) = p$ for any k. The above gives

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor$$

and since $\gamma_k = p^{-k}$, $\forall k$, we have

$$\begin{aligned} &v_{\frac{1}{p}}(\sigma(n)) \\ &= \sum_{k=1}^{\infty} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \sum_{k=1}^{\lceil log_p(n) \rceil} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor + 2\lfloor \frac{n}{p^2} \rfloor - 2\lfloor \frac{n}{p^3} \rfloor + \ldots + \lceil log_p(n) \rceil \lfloor \frac{n}{p^{\lceil log_p(n) \rceil}} \rfloor \\ &= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \ldots + \lfloor \frac{n}{p^{\lceil log_p(n) \rceil}} \rfloor \\ &= \sum_{k=1}^{\lceil log_p(n) \rceil} \lfloor \frac{n}{p^k} \rfloor \\ &= \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor \\ &= v_p(n!) \end{aligned}$$

since $\lfloor \frac{n}{p^k} \rfloor = 0$ if $p^k > n \iff log(p^k) > log(n) \iff k > log_p(n)$, so that we are able to recover the well-known Legendre's formula.

Some comment about how the above starts to build up a toolkit independent of algebraic structure in S.

Corollary 7. *If G is a compact ultrametric group, then G is semi-regular.*

Proof. Since G is a group, each ball centred at 0 is in fact a subgroup of G. Then each set of elements of S_{γ_k} is a collection of cosets of $G/B(0, \gamma_k)$. Since G is assumed to be compact, $G/B(0, \gamma_k)$ is finite and so Lagrange's theorem implies the result. □

Semi-regularity gives a notion of translation invariance to ultrametric spaces that are not themselves groups. In the previous section, we observed that spaces which admitted both translation invariance and scaling had valuative capacity that could be computed explicity via the subadditivity formula. Is there a way to define this for ultrametric which are not normed vector spaces?

Definition 24. Let S be a semi-regular compact subset of an ultrametric space. If there exists a $q \in \mathbb{N}$ such that $\alpha(n) = q$, for all n, then S is said to be **regular**. If there exists a q and N in $\mathbb{N}$ such that $\alpha(n) = q$, for all $n \geq N$, then S is said to be **eventually regular**.

Alt.:

Definition 25. Let S be a semi-regular compact subset of an ultrametric space. If there exists a $q_1, \ldots, q_m \in \mathbb{N}$ such that $\alpha(n) = q_i$ if, and only if, $n = i \mod m$, that is, α has an infinitely-repeating finite subsequence of length m, then we say S is **periodic**. If $m = 1$, then S is **regular**.

S is regular just in case the $\alpha-$sequence of S is constant. In what follows, we assume that S is regular, but the results carry over to the case where S is peroidic with the necessary adjustments.

If S is regular with $\alpha(k) = q$, for all k:

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

$$v_{q^{c_k}}(\sigma(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

$$v_q(\sigma(n)) = \sum_{k=0}^{\infty} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor)$$

$$v_q(\sigma(n)) = \sum_{k=0}^{\lceil log_q(n) \rceil} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor)$$

$$= c_0 n - c_0 \lfloor \frac{n}{q} \rfloor + c_1 \lfloor \frac{n}{q} \rfloor - c_1 \lfloor \frac{n}{q^2} \rfloor + c_2 \lfloor \frac{n}{q^2} \rfloor - c_2 \lfloor \frac{n}{q^3} \rfloor \ldots + \lceil log_q(n) \rceil \lfloor \frac{n}{q^{\lceil log_q(n) \rceil}} \rfloor - \lceil log_q(n) \rceil \lfloor \frac{n}{q^{\lceil log_q(n) \rceil}} \rfloor$$

$$v_q(\sigma(n)) = \sum_{k=0}^{\lceil log_q(n) \rceil} (c_{k+1} - c_k) \cdot (\lfloor \frac{n}{q^{k+1}} \rfloor)$$

Note that the coefficient $(c_{k+1} - c_k)$ is always positive: since Γ is (strictly) decreasing and $\gamma_k = q^{c_k}$, the sequence c_k is also be strictly descreasing (since $\gamma_k = q^{c_k}$

implies $log_q(\gamma_k) = c_k$ and log is a monotone function). The above implies that:

$$\omega(S) = \lim_{n\to\infty} \sum_{k=0}^{\lceil log_q(n) \rceil} (c_{k+1} - c_k) \cdot (\lfloor \frac{n}{q^{k+1}} \rfloor)$$

Chapter 4

Product spaces of $\mathbb{Z}_p$

As a first point of departure, a natural space to consider is the product space of ultrametric spaces, for example $\mathbb{Z}^n$ (or $\mathbb{Z}_p^n$ or $\mathbb{Q}_p^n$), for some $n > 1$. If we restrict our attention to bounded subsets, then a natural candidate for an ultrametric on the product space is the L_∞ metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = sup_i\{\rho(x_i, y_i)\}$$

where ρ is the metric from the base space. In fact, since we have only defined valuative capacity for compact subsets of an ultrametric spaces, there is no loss of generality by restricting our metric to bounded spaces. We also see that no problems arise in letting both M and ρ vary between components of the space, as long as each M_i remains bounded and each ρ_i is an ultrametric.

Proposition 26. *Let (M_i, ρ_i) for i in some finite or countably infinite index set, I, be a collection of metric spaces and suppose ρ_i is a bounded ultrametric for all i. Then (M, ρ_∞) is an ultrametric space, where $M = M_1 \times M_2 \times M_3 \times \ldots$ and ρ_∞ is the L_∞ metric described above.*

Proof. Let (M, ρ_∞) be the product of ultrametric spaces as above and let x and y be two points in the space. Clearly, $\rho_\infty(x, y) \geq 0$ since each $\rho_i(x_i, y_i) \geq 0$, and $\rho_\infty(x, y) = 0 \iff \rho_i(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$. The fact that ρ_∞ is symmetric is also an easy consequence of the fact that each ρ_i is symmetric since $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$ implies $sup_i\{\rho_i(x_i, y_i)\} = sup_i\{\rho_i(y_i, x_i)\}$. To see that ρ_∞ is an ultrametric, note that if $z = z_i$ is any other point of M, then

$$\begin{aligned}
\rho_\infty(x,y) &= \sup_i\{\rho_i(x_i,y_i)\} \\
&\leq \sup_i\{max(\rho_i(x_i,z_i),\rho_i(y_i,z_i))\} && \text{since each } \rho_i \text{ is an ultrametric} \\
&\leq \max(\sup_i\{\rho_i(x_i,z_i)\},\sup_i\{\rho_i(y_i,z_i)\}) && * \\
&= max(\rho_\infty(x,z),\rho_\infty(y,z))
\end{aligned}$$

* Let $M = max(sup_i(\{a_i\}, sup_j(\{b_j\})))$, then $M \geq a_i, \forall i$ and $M \geq b_i, \forall i$, so $M \geq max(a_i, b_i), \forall i$, hence $M \geq sup_i(max(a_i, b_i)$. □

We first show that translation invariance carries over into product spaces under the expected conditions.

Proposition 27. *Suppose (M, ρ_∞) is the product of ultrametric spaces (M_i, ρ_i) and each M_i is a topological group with operation $+$. Then ρ_∞ is (left) translation invariant if each ρ_i is, in which case valuative capacity is also (left) translation invariant.*

Proof. Let (M, ρ_∞) be as above. Suppose also that

$$\rho_i(x_i, y_i) = \rho_i(s_i + x_i, s_i + y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each ρ_i is (left) translation invariant. Then,

$$\rho_\infty(s + x, s + y) = \sup_i\{\rho_i(s_i + x_i, s_i + y_i)\} = \sup_i\{\rho_i(x_i, y_i)\} = \rho_\infty(x, y).$$

so that ρ_∞ is translation invariant. Proposition **??** implies valuative capacity is as well. □

In the next proposition, we show that scaling carries over to product space as well, although the conditions are now more restrictive. In contrast to the proposition above, here we do not allow the spaces to vary between components.

Proposition 28. *Let (m, ρ_N) be an ultrametric space, where ρ_N is the metric induced by some norm N. Let (M, ρ_∞) be the ultrametric space formed by taking products of m, along with the L_∞ metric defined above. Then if ρ_N is multiplicative on m,*

ρ_∞ is multiplicative on M, in the sense that $\rho_\infty(cx, cy) = |c|_{\rho_N}\ \rho_\infty(x, y)$, for $c = (c, c, c, \ldots), x, y \in M$.

Proof. Let M, ρ, and ρ_∞ be as above. Then,

$$\begin{aligned}
&\rho_\infty(cx, cy) \\
&= \sup_i\{\rho_N(c_i x_i, c_i y_i)\} \\
&= \sup_i\{|c|_{\rho_N}\ \rho_N(x_i, y_i)\} \\
&= |c|_{\rho_N}\ \sup_i\{\rho_N(x_i, y_i)\} \\
&= |c|_{\rho_N}\ \rho_\infty(x_i, y_i)
\end{aligned}$$

□

Corollary 8. *Let S be a subset of (M, ρ_∞), where M is the product of an ultrametric space (m, ρ_N), which is itself a normed vector space with a multiplicative norm inducing ρ_N. If $c = (c, c, c, \ldots)$ is an element of M with constant value on each component, then $\omega(cS) = |c|_{\rho_N}\ \omega(S)$.*

Proof. Note that if $\{a_j\}_{j=0}^{\infty}$ is a ρ_∞ ordering of S, then $\{ca_j\}_{j=0}^{\infty}$ is a ρ_∞ ordering of cS. □

We now introduce two examples, whose explorations take up a large portion of the following section.

Example 13. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$ and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1)), \rho_p(x_2, y_2))$, where ρ_p is the p-adic metric for some fixed prime p. Since ρ_p is translation invariant and multiplicative in $\mathbb{Z}_p$, valuative capacity is also translation invariant and multiplicative in $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$.

Example 14. Let $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x \in \mathbb{Z}_{p_1}, y \in \mathbb{Z}_{p_2}\}$ for two distinct primes, $p_1 \neq p_2$, and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_{p_1}(x_1, y_1)), \rho_{p_2}(x_2, y_2))$, where both ρ_{p_i} are p-adic metrics. Since each ρ_{p_i} is translation invariant in $\mathbb{Z}_{p_i}$, valuative capacity will be translation invariant in $(\mathbb{Z}_{p_1} \times$

$\mathbb{Z}_{p_2}, \rho_{p,\infty})$; however, unlike the case of $p_1 = p_2$, this space does not have a multiplicative property that allows for scaling.

n-fold products

What is the valuative capacity of $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ from the example above? Suppose $p = 2$. Using translation invariance, scaling and subaddivity, we can compute the result by first noting that we can write $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always $1 = diam(\mathbb{Z}_2 \times \mathbb{Z}_2)$, subadditivity implies that

$$\frac{1}{log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

$$= \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))}$$

$$= \frac{4}{log(\|2\|_2 * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{log(\frac{1}{2} * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{log(\frac{1}{2}) + log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

Taking logs base 2, we have that

$$\omega(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^{\frac{-1 + log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} 2^{\frac{log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} (2^{log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))})^{\frac{1}{4}} = 2^{\frac{-1}{4}} \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\frac{1}{4}}$$

so that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a solution of the equation $x^4 - \frac{x}{2}$, for which there is a single real positive root, given by $2^{-1/3}$.

To compute the valuative capacity for a 2-fold product for an arbitary prime p, note that we can always decompose $\mathbb{Z}_p \times \mathbb{Z}_p$ into a union of p^2 sets each of the form $\{p\mathbb{Z}_p + s \times p\mathbb{Z}_p + t\}$ for $s, t \in (0, \ldots, p-1)$, and the pairwise distance between these sets will always be $1 = diam(\mathbb{Z}_p \times \mathbb{Z}_p)$ (to see this, either note that we can always find co-prime elements, or note that each set is an closed ball of radius $1/p$ centred at (s,t) and so the distance between them must be greater than $1/p$, and 1 is the only possible distance greater than $1/p$ in $\mathbb{Z}_p \times \mathbb{Z}_p$). Then, we combine our tools as before to obtain the equation,

$$\frac{1}{log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(\|p\|_p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(1/p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, taking logs base p, we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p) = p^{\frac{-1}{p^2}} \omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{\frac{1}{p^2}}$$

So that $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$ is a solution of the equation $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p})$ over $\mathbb{R}$. Since $\mathbb{R}$ is a division ring, this means the positive solutions are given by solving $x^{p^2-1} - \frac{1}{p}$. Solutions of this equation are of the form $p^{\frac{-1}{p^2-1}}$ times a $p^2 - 1$ root of unity, and so there is exactly one positive, real solution, namely $p^{\frac{-1}{p^2-1}}$ itself. Then the valulative capacity of the entire product space $\mathbb{Z}_p \times \mathbb{Z}_p$ is $p^{\frac{-1}{p^2-1}}$. In fact, from here it is not hard to see that by taking the n-fold product, we would end up with the same equation except that the exponent of p would become n rather than 2. We arrive at the following result:

Proposition 29. *Let $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$ be the ultrametric space with points equal to the n-fold product of $\mathbb{Z}_p$ (for $n < \infty$) for some fixed prime p. The valuative capacity of M is $(\frac{1}{p})^{\frac{1}{p^n-1}}$.*

Proof. Above. □

Taking $n = 1$, we see that this agrees with the valuative capacity of $\mathbb{Z}_p$ computed in the last chapter.

What about $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ for distinct primes? These spaces do not admit a scaling property, so the same toolset is not available. They are however semi-regular, so we know that

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Suppose $p_1 = 2$ and $p_2 = 3$. Recall that the α sequence of $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$ counts the number of closed balls of radius γ_{k+1} partitioning a closed ball of radius γ_k. In this case, Γ_S is the non-positive powers of 2 or 3 sorted into decreasing order, so that Γ_S starts $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}, \ldots\}$ and $\alpha(S)$ starts $\{6, 2, 3, 2, 2, 3, 2, 3, 2, \ldots\}$. The β sequence of S, which counts the number of distinct balls of a fixed radius, then starts $\{6, 12, 36, 72, 144, \ldots\}$.

We know that the capacity of S will be a product of some negative power of 2 and some negative power of 3. From the lemma **??**, we know that when $\alpha(k) = 2$, we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{2 \cdot \beta(k)} \rfloor$$

and when $\alpha(k) = 3$, we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{3 \cdot \beta(k)} \rfloor + \lfloor \frac{n + 2 \cdot \beta(k)}{3 \cdot \beta(k)} \rfloor$$

We also know that if $\alpha(k) = 2$, then γ_k must be a (negative) power of 2, and likewise if $\alpha(k) = 3$, then γ_k is a power of 3.

Let us first explore the exponent of 2 in $\sigma(n)$. We start by noting that if γ_k is some 2^{-i}, then

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

since there will be a copy of 2 in $\beta(k)$ for every occurence of 2 in $\alpha(0), \ldots, \alpha(k)$, which is also what i counts. So then, the exponent of $\frac{1}{2}$ in the n^{th} characteristic sequence of S is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

What can we say about j, the exponent of 3?

Lemma 7. *Let $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$ and consider the k^{th} element of the β sequence of S, $\beta(k) = 2^i \cdot 3^j$. If k is such that $\gamma_k = 2^{-i}$ for some i, then j counts the numbers $a \in \mathbb{Z}_{\geq 0}$ such that $3^a < 2^i$.*

Proof. Γ_S is strictly monotone decreasing and each γ_k is equal to a non-positive power of 2 or 3. If $\gamma_k = 2^i$, then all non-positive powers of 3 and 2 which are greater than 2^i must be equal to some γ_j, $0 \leq j < k$. That is, 2^i only appears in the Γ_S sequence after all larger powers of 2 and 3 have been exhausted. Since we are only considering the case γ_k is a power of 2, this includes all of the smaller powers of 3. $\square$

Now note that

$$3^a < 2^i \iff log_2(3^a) < log_2(2^i) \iff a \cdot log_2(3) < i$$

So now we are reduced to counting the number of non-negative integers a that satisfy the above for a given i. The number of such a's will simply be the the value of the largest a plus 1 since a satisfying the relation implies all $0 \leq a' \leq a$ solve the relation. Then, we are in fact reduced to finding the largest $a \in \mathbb{Z}$ that satisfies $a < \frac{i}{log_2(3)}$, but this is exactly $\lfloor \frac{i}{log_2(3)} \rfloor$. This in turn gives $j = \lfloor \frac{i}{log_2(3)} \rfloor + 1 = \lceil \frac{i}{log_2(3)} \rceil$, since $\frac{i}{log_2(3)}$ is never an integer. We now revisit our expression for the exponent of $\frac{1}{2}$ and substitute our new found value for j:

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor = \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor - \lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor) \tag{4.1}$$

A symmetric argument shows that exponent of 3 in the i^{th} elements of the $\rho-$sequence of S is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor = \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor - \lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor) \tag{4.2}$$

Conjecture 1. *Products of $\mathbb{Z}_{p_i}$ for distinct primes have transcendental valuative capacity.*

The aperiodicity of the sequence $\lceil \frac{i}{log_2(3)} \rceil$ over i leads us to believe, but not prove, that each of the sums in (4.1) and (4.2) are irrational.

We end this section with an observation on the asymptotic behavior of capacity in these spaces. For a fixed prime p, $(\frac{1}{p})^{\frac{1}{p^n-1}}$ is an monotone, increasing sequence in n with $lim_{n\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. For fixed n, the sequence in p is also montone, increasing, again with $lim_{p\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. In both cases, the limiting value is equal to the diameter of space. Indeed, we can observe that the sequence

$\{(0,0,\ldots),(1,0,\ldots),(0,1,\ldots),\ldots\}$, in which the first element has only zeros and the n-th element has a single 1 in the $(n-1)$-th component, is a ρ−ordering for both $(\mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, \rho_{p,\infty})$ and $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots, \rho_{P,\infty})$, since the distance between elements in this sequence (in either metric space) is always 1. If we could show that these spaces are compact, this would gives a valuative capacity of $lim_{n\to\infty}(1^n)^{(1/n)} = 1$ for both spaces. We explore this more in the following section.

Product topology

In considering the product space of ultrametric spaces, we may wonder whether the chosen metric also gives back the product topology on the space. For products formed by taking some finite number of copies, the answer is positive. We give the necessary background and show this fact, adapting the proof in Munkres (20.3) to the case of ultrametric spaces.

Definition-Proposition 1. *(Munkres) Suppose X_i, for i in some index set I, is a family of topological spaces. Let $\pi_j : \prod_{i\in I} X_i \to X_j$ be the map given by projection onto the j-th component, that is $\pi_j(x) = \pi_j((x_i)_{i\in I}) = x_j$. For each $j \in I$, let $\mathcal{S}_j$ be the collection*

$$\mathcal{S}_j = \{\pi_j^{-1}(U_j) \mid U_j \textit{ open in } X_j\}$$

Let $\mathcal{S}$ be the union of the $\mathcal{S}_j$ over $j \in I$, $\mathcal{S} = \cup_{j\in I}\mathcal{S}_j$. Then $\mathcal{S}$ is a subbasis that generates a topology on $\prod_{i\in I} X_i$ called the ***product topology****.*

The basis, $\mathcal{B}$, generated by $\mathcal{S}$ in the definition above is the set of all finite intersections of elements in $\mathcal{S}$. That is $B \in \mathcal{B}$ if there exists $S_1, S_2, \ldots, S_n$ in $\mathcal{S}$ such that $B = S_1 \cap S_2 \cap \ldots S_n$. A useful description of the basis for the product topology also appears in Munkres, as below:

Proposition 30. *(Munkres 19.2) Suppose X_i, for i in some index set I, is a family of topological spaces and denote by $\mathcal{B}_i$ the basis for the topology on X_i. Let*

$$\mathcal{B}_P = \prod_{i \in I} B_i, \textit{ for } B_i \in \mathcal{B}_i \textit{ and } B_i = X_i \textit{ for all but finitely-many } i \in I.$$

then $\mathcal{B}_P$ *is a basis for the product topology on* $\prod_{i \in I} X_i$.

We can now show that the topology induced by the L_∞ metric described above agrees with the product topology for finite products.

Proposition 31. *Let* $M = (M_1 \times M_2 \times \ldots \times M_n, \rho_\infty)$ *be a finite product of bounded, ultrametric spaces and let* ρ_∞ *be the metric described above. Then the topology induced by* ρ_∞ *coincides with the product topology on* $M_1 \times M_2 \times \ldots \times M_n$.

Proof. Let $\mathcal{T}_{\rho_\infty}$ be the topology on $M_1 \times M_2 \times \ldots \times M_n$ induced by ρ_∞ and let $\mathcal{B}_{\rho_\infty}$ be the basis for this topology. Let $\mathcal{T}_P$ be the product topology with basis $\mathcal{B}_P$. We show $\mathcal{T}_P \subset \mathcal{T}_{\rho_\infty}$ and vice versa. For this, it is equivalent (Munkres 13.3) to show that for $z \in M_1 \times M_2 \times \ldots \times M_n$ and $B \in \mathcal{B}_P$ containing z, there is a basis element $B' \in \mathcal{B}_{\rho_\infty}$ such that $z \in B' \subset B$, and vice versa.

So let $z \in M_1 \times M_2 \times \ldots \times M_n$ and suppose $B \in \mathcal{B}_P$ contains z. Since B is in $\mathcal{B}_P$, B is of the form $B_{r_1}(z_1) \times B_{r_2}(z_2) \times \ldots \times B_{r_n}(z_n)$ (since the choice of centres is arbitrary in an ultrametric spaces, we may choose the components of z as the centres without loss of generality). Let $r = \min\{r_i\}$ for $i \in 1, \ldots, n$. Then let B' be the ball $B_r(z)$ in $\mathcal{B}_{\rho_\infty}$. Clearly, $z \in B_r(z)$ and since $r \leq r_i$, $\forall i$, $B_r(z) = B_r(z_1) \times B_r(z_2) \times \ldots \times B_r(z_n) \subset B_{r_1}(z_1) \times B_{r_2}(z_2) \times \ldots \times B_{r_n}(z_n) = B$.

Conversely, suppose $A \in \mathcal{B}_{\rho_\infty}$ and let $y \in A$. To find $A' \in \mathcal{B}_P$ such that $y \in A'$ and $A' \subset A$, simply note that A itself is in $\mathcal{B}_P$.

□

We are now naturally left to ask whether the product topology on *infinite* products of ultrametric spaces coincides with the L_∞ metric. In this case, as in the analogous case of infinite copies of $\mathbb{R}$ and a uniform metric, the answer is negative (at least in general). Forunately, the metric that realizes the product topology on infinite copies of $\mathbb{R}$ can be adapted to the case of ultrametric spaces. We adapt to the proof of Munkres (20.5) to the case of infinite products of ultrametric spaces.

Proposition 32. *Suppose $\boldsymbol{M} = M_1 \times M_2 \times \dots$ is an infinite collection of metric spaces, each with an ultrametric ρ_i which is bounded by 1, that is suppose $\rho_i(x_i, y_i) \leq 1$, for all $x_i, y_i \in M_i$ and for all i. Define a metric d on $\boldsymbol{M}$ as follows:*

$$d(\boldsymbol{x}, \boldsymbol{y}) = sup\{\frac{\rho_i(x_i, y_i)}{i}\}$$

Then d is an ultrametric and induces the product topology on $\boldsymbol{M}$.

Proof. We see that d inherits symmetry, injectivity and non-negativity from the requirement that each ρ_i is a metric, just as ρ_∞ did. To see that d satisfies the strong triangle inequality, define a new metric ρ'_i by $\rho'_i(x, y) = \frac{\rho_i(x_i, y_i)}{i}, \forall i$. Then ρ'_i is an ultrametric, since $\rho_i(x, y) \leq \max(\rho_i(x, z), \rho_i(y, z))$ implies $\frac{\rho_i(x,y)}{n} \leq \max(\frac{\rho_i(x,z)}{n}, \frac{\rho_i(y,z)}{n})$ for any $n \in \mathbb{N}$. Then we can view d as the L_∞ metric on the spaces (M_i, ρ'_i), and so d will be an ultrametric as shown in the first proposition of this section.

Now we show d induces the product topology. We first show that metric topology induced by d is finer than the product topology. Let

$$B = B_r^{\mathbf{M}}(\mathbf{z}) = B_r^{M_1}(z_1) \times B_{2r}^{M_2}(z_2) \times B_{3r}^{M_3}(z_3) \times \dots$$

be a basis open in the metric topology. We must find a basis open $B' \ni z$ in the product topology such that $B' \subseteq B$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < r$. Then let B' be the basis open element

$$B' = B_r^{M_1}(z_1) \times B_r^{M_2}(z_2) \times \dots \times B_r^{M_N}(z_N) \times M_{N+1} \times M_{N+2} \times \dots$$

in the product topology. Suppose $\mathbf{y} \in B'$. We must show $\mathbf{y} \in B$, i.e., $d(\mathbf{z}, \mathbf{y}) < r$. Note that for all $i \geq N$,

$$\frac{\rho_i(z_i, y_i)}{i} \leq \frac{1}{N}$$

which means

$$d(\mathbf{z},\mathbf{y}) = \sup\{\frac{\rho_i(z_i,y_i)}{i}\} \leq \max\{\frac{\rho_1(z_1,y_1)}{1}, \frac{\rho_2(z_2,y_2)}{2}, \ldots, \frac{\rho_N(z_N,y_N)}{N}, \frac{1}{N}\}$$

and since N was chosen so that $\frac{1}{N} < r$ and B' was chosen to have balls of radius r in the first N components, we must have $d(\mathbf{z},\mathbf{y}) < r$.

Conversely, ... □

From now on, we refer to the metric d above as the **product metric**. An important consequence of the fact that d achieves the product topology is that Tychnoff's theorem then guarantees that product spaces formed with this metric will be compact, infinite or otherwise. As a result, we can now ask directly about the valuative capacity of some infinite product spaces. We consider two examples.

Example 15. Let $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, d)$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for some fixed prime p and let d be the product metric.

Example 16. Let $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots, \rho_{P,\infty})$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for every prime p and let d be the product metric.

So far we have two methods for computing valuative capacity. Either we can find a useful decomposition that allows us to apply the subadditivity formula, or we can find a $\rho-$ordering and then take the limit of its corresponding $\rho-$sequence.

Conclusion

In this section, we considered the notion of valuative capacity in product spaces, that is, spaces formed by taking copies of ultrametric spaces. In the following sections, we consider vaulative capacity in spaces formed by adding points, that is extension fields, or by both taking copies and adding (a distinguished) point, as in projective spaces. For these purposes, it will be more productive to start working over the field $\mathbb{Q}_p$, instead of $\mathbb{Z}_p$.

Appendix A

Appendix

code/Padic_Product_Ordering.txt

```
with(LinearAlgebra):
with(combinat, cartprod):
with(padic):

ComputePadicProductOrdering := proc (m, components)
# Given a list of primes, p_1,...,p_n, compute the first m terms of a p-infinity ordering of Z_p1 x ... x Z_pn, where p-infinity(x,y)=max(p_j(x_j,y_j)) and p_j is the p_j-adic metric.
# arg m; an integer indicating the number of elements of the ordering to return
# arg components; a list of prime numbers indicating the components of the product space
# return; a matrix where each row is an element in the product space and the i-th row is the i-th element in an ordering

    local numberOfComponents, co_primes, i, n, T, M, v, distances, j, M1, newBlock;

    #will end up with one column per component in the product space
    numberOfComponents := nops(components);

    #the ordering will start with the cartestian product of coprime elts from each component
    #everything up to p-1 is coprime
    co_primes := [[seq(i, i = 0 .. (components[1]-1))]];
    for n from 2 to numberOfComponents do
        co_primes := [op(co_primes), [seq(i, i = 0 .. (components[n]-1))]];
    od;

    #then take the cartestian product to get the first <product of elements in components> elements in the ordering
    T := cartprod(co_primes);
```

```
    M := Matrix([T['nextvalue']()]);
    while not T['finished'] do
        M := <M; T['nextvalue']()>;
    end do;

    #make a list to keep track of the exponent of each prime; start by
       take each prime to the power -1
    v := Vector[row](1 .. numberOfComponents, 1);
    v := convert(v, list);

    #keep adding rows until you have enough points in the ordering
    while RowDimension(M) < m do
        #take each prime to the power of minus the elements in v
        distances := zip(proc (x, y) options operator, arrow; x^(-y) end
            proc, components, v);
        #check each column to see if the max distance was achieved
        for j from 1 to numberOfComponents do
            #if it was then split this column
            if distances[j] = max(distances) then
            #take a snapshot of M before you start - this is what you
               have to add to
            M1 := copy(M, deep);
            #create p-1 new blocks
            for i from 1 to (components[j]-1) do
                    newBlock := copy(M1, deep);
                    newBlock(1..RowDimension(newBlock), j) :=
                       Column(newBlock, [j]) +~ (i*components[j]^v[j]);
                    #add the new block to the master matrix
                    M := Matrix([[M],[newBlock]]);
                od;
                #update the vector of exponents
                v[j] := v[j] + 1;
            end if;
        od;
```

```
      end do;
   return M[1..m,];
end proc;
```

code/Partial_Rho_Sequence.txt

```
with(LinearAlgebra):
with(combinat, cartprod):
with(padic):

ComputePartialRhoSeq := proc (S, rho)
# Given an m by n matrix S whose columns represent points of an n-component product space and that has as its i-th row the i-th term in a rho-ordering of that space, compute the (m-1)-th partial sum of the rho-sequence
# note that S and rho must be compatible and no checking is done to ensure this
# arg S; an n by m matrix representing a rho-ordering, for example as created by ComputePadicProductOrdering
# arg rho; a compatible metric on the points (rows) in S
# return; a real number, correpsonding to the (m-1)-th term of the partial rho-sequence

    local lastTerm, f, distances, nthTerm;

    #find the last element in the ordering
    lastTerm := S[RowDimension(S),];

    #make a function that calculates the distance from the i-th row of S to the last term in the ordering
    f := proc (i) options operator, arrow; rho(op(convert(lastTerm, list)), op(convert(S[i,], list))) end proc;

    #run over each row to get the set of all m-1 distances
    distances := map(f, [seq(i, i = 1 .. (RowDimension(S)-1))]);

    #multiply them to get the (m-1)-th term of the rho-ordering
    partialSum := mul(distances);

```

```
    return partialSum
end proc;
```

code/Fast_Partial_Rho_Sequence.txt

```
FastPartialRhoSeq := proc (A, p:=[2,3])
  local g, h,computePowers,n, shortA, primeExponents,i, thisPrime,
      thisPrimeIndex,B, G, powers, powersOfG, thisPrimeSum;

  #=== Some helper functions ===#
  #Return the index of every instance of p-multiples in a list
  #Use to find the index of a given prime in A
  h := proc(i,L,p) if L[i] mod p = 0 then return i else return NULL fi;
      end proc;

  #Count the number of times the mth element has appeared as a factor
      for the 1..m first elements in a list
  #Use to compute the (decreasing) sequence of distances in A or a
      subset of A
  g := proc(m, L)
    local basePrime;

    basePrime := L[m];

    #since we just want the exponent not actual distance just compute
        what power this would be
    return ordp(mul(L[1..m]), basePrime);

  end proc;

  #Compute the appropriate power of an element of G
  computePowers := proc(m, L)
    local power;

    if m=nops(L) then
      power:= L[-1]-1;
    else
      power:= mul(L[(m+1)..nops(L)])* (L[m]-1);
```

```
    end if;
    return power
  end proc;

  #compute n, then create a copy of A with the first element deleted to
      ease the indexing
  n := mul(A);
  shortA := A[2..nops(A)];

  #compute the terms corrsponding to each prime given
  primeExponents := Vector();
  for i from 1 to nops(p) do
    #pull out the prime
    thisPrime := p[i];

    #first get the index in A of this prime
    thisPrimeIndex := map(h, [seq(i, i=1..nops(shortA))], shortA,
        thisPrime);
    B:= shortA[thisPrimeIndex];

    #then find the (exponents for the) distances occuring with this prime
    G:= map(g,[seq(i, i=1..nops(B))],B);

    #Figure out what power each distance should be raised to
    powers := map(computePowers,thisPrimeIndex, shortA);

    #raise each element in G to the given powers
    powersOfG:= zip(proc (x, y) options operator, arrow; x*(y) end proc,
        G, powers);

    #the exponent of this prime in the nth partial will be -([the sum of
        the elements in powers]/n), where n is the product of elements in
        A (including the first element)
    thisPrimeSum := add(powersOfG);
```

```
    primeExponents(i) := thisPrimeSum/n;

  end do;

  return primeExponents;

end proc;
```

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