Introduction

In the course of developing a generalized factorial function, Bhargava introduced the notion of $p-$orderings of a Dedekind domain [2, 3], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [4] (also KJ) and have also provided a natural framework to extend many classical results in analysis to a $p-$adic setting, such as polynomial approximation and mapping theorems [2, 3, 4].

In this thesis, we examine how a tool based on $p-$orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

Background

Ultrametric basics

Definition. Let (M, ρ) be a metric space. If ρ satistifies the ultrametric inequality

$$\rho(x, z) \leq max(\rho(x, y), \rho(y, z)), \forall x, y, z \in M$$

then (M, ρ) is an **ultrametric space**.

Definition. Let (V, N) be a normed vector space. Then N satisfies the **strong trianlge inequality** if

$$N(x + y) \leq max(N(x), N(y)), \forall x, y \in V$$

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space, (V, ρ_N), where ρ_N is the metric induced by N, is an ultrametric space.

Proposition. [1] All triangles in an ultrametric space (M, ρ) are either equilateral or isocoles, with at most one short side.

Proposition. [1] If S is a compact subset of an ultrametric space and Γ_S is the set of all distances occurring between points of S, then Γ_S is a discrete subset of $\mathbb{R}$. In particular if $\mid \Gamma_S \mid = \infty$, then the elements of Γ_S can be indexed by $\mathbb{N}$.

Let (M, ρ) be a compact ultrametric space and let

$$B_r(a) = \{x \in M \mid \rho(x, a) < r\}$$

denote the open ball of radius r, centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$. Likewise let

$$\overline{B_r(a)} = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the closed ball of radius r, centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$.

Proposition. Let $B_r(a)$ be a ball in an ultrametric space (M, ρ). Then the diameter of B, $d = diam(B) = \sup_{x,y \in B} \rho(x, y)$, is less than or equal to the radius of B.

Proposition. If (M, ρ) is an ultrametric space and $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are balls in (M, ρ), then either $B_{r_1}(x_0) \cap B_{r_2}(y_0) = \emptyset$, $B_{r_1}(x_0) \subseteq B_{r_2}(y_0)$, or $B_{r_2}(x_0) \subseteq B_{r_1}(x_0)$. That is, in an ultrametric space, all balls are either comparable or disjoint.

Proposition. [1] The distance between any two balls in an ultrametric is constant. That is, if $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are two balls in an ultrametric space (M, ρ), then $\rho(x, y) = c$ for some $c \in \mathbb{R}$ and $\forall x \in B_{r_1}(x_0)$ and $\forall y \in B_{r_2}(y_0)$

Proposition. [1] Every point of a ball in an ultrametric is at its centre. That is, if $B_r(x_0)$ is a ball in an ultrametric space (M, ρ), then $B_r(x) = B_r(x_0)$, $\forall x \in B_r(x_0)$

ρ-orderings, ρ-sequences, and valuative capacity

In what follows let S be a compact subset of an ultrametric space (M, ρ).

Definition. [5] A **ρ-ordering** of S is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ such that $\forall n > 0$, a_n maximizes $\prod_{i=0}^{n-1} \rho(s, a_i)$ over $s \in S$.

Definition. [5] The **ρ-sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is $\prod_{i=o}^{n-1} \rho(a_n, a_i)$.

Proposition. [5] The ρ-sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ-sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ-ordering of S.

Definition. [5] Let $\gamma(n)$ be the ρ-sequence of S. The **valuative capacity** of S is

$$\omega(S) := \lim_{n\to\infty} \gamma(n)^{1/n}$$

Proposition. [5] For S and $\gamma(n)$ as above, $\lim_{n\to\infty} \gamma(n)^{1/n} = r < \infty$.

Proposition. If $S \subseteq M$ is a finite subset of an ultrametric space, then $\omega(S) = 0$.

Proposition. (upper bound) If $diam(S) := \max_{x,y\in S} \rho(x, y) = d$, then $\omega(S) < d$.

Proof. Since d is the diameter of S, the n^{th} term of the ρ-sequence of S is bounded by d^n and so $\lim_{n\to\infty} \gamma(n)^{1/n} = d$ if and only if $\gamma(n) = d^n$, $\forall n$. This implies $\rho(a_n, a_i) = d$, $\forall n$ and $\forall i < n$, but then $\rho(a_i, a_j) = d$, $\forall i, j$, since the ρ-sequence is maximized at each n. This means $\omega(S) < d$ would imply that the cover of S, $\cup_{a_i} B_d(a_i)$ is in fact an infinite partition, contradicting the compactness of S. Then $\omega(S) = \lim_{n\to\infty} \gamma(n)^{1/n} < d$. $\square$

This doesn't work because $\cup_{a_i} B_d(a_i)$ could fail to be a cover -when does this happen

Proposition. (translation invariance) Let (M, ρ) be a compact ultrametric space and suppose M is also a topological group. If ρ is (left) invariant under the group operation, then so is ω. That is, if $\rho(x,y) = \rho(gx,gy)$, $\forall g,x,y \in M$, then $\omega(gS) = \omega(S)$, for $S \subseteq M$.

Proof. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS. Then

$$\omega(gS) = \lim_{n\to\infty} \gamma(n)^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = \omega(S)$$

$\square$

Example 1. With the notation of the previous section, note that for $x, y \in (\mathbb{Z}_p, |\cdot|_p)$, $\rho_p(x,y) =| x-y |_p = p^{-\nu_p(x-y)} = p^{-\nu_p((a+x)-(a+y))} =| (a+x)-(a+y) |_p = \rho_p(a+x, a+y)$ so that $\omega(a+S) = \omega(S)$ for $S \in (\mathbb{Z}_p, |\cdot|_p)$.

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity. Then if N is multiplicative, so is ω. That is, if $N(gx) = N(g)N(x)$,$\forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.

Proof. Let ρ be the metric induced by N, so that $\rho(x,y) = N(x-y), \forall x, y \in V$. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then since N is multiplicative, for $u, v \in gS$, $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$,

$$\rho(u,v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS and

$$\omega(gS) = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} N(g)\rho(a_n, a_i)]^{1/n}$$

$$= \lim_{n\to\infty} [N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g) \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g)\omega(S)$$

$\square$

Example 2. Since $|\cdot|_p$ is multiplicative, $\omega(mS) = | m |_p \omega(S)$ for $m \in \mathbb{Z}_p$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) = | p |_p \omega(\mathbb{Z}) = \frac{1}{p} * p^{\frac{1}{1-p}} = p^{-p/p-1}$.

Proposition. [5](subadditivity) If $diam(S) := \max_{x,y \in S} \rho(x,y) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n} \frac{1}{log(\omega(A_i)/d)}$$

- Put in an example here.
- Also talk about how cosets and open balls coincide and when the above is satisfied

Corollary. Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = diam(S)$ and also $\omega(S_i) = \omega(S_j)$, $\forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S)$, $\forall i$. Then $\omega(S) = r^{\frac{1}{n-1}} \cdot d$. In particular if $S = \mathbb{Z}$ and $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$ then $\omega(S) = (\frac{1}{p})^{1/p-1}$ for any prime p.

Corollary. (Joins of computable sets are computable) Let $\Gamma_M = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in M. Suppose that $S = B_{\gamma_i}(x)$, for some x and i, is the union of 2 or more balls of radius γ_{i+1}, i.e., $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$ is a join in the lattice of open sets in M, then

$$\frac{1}{log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^{n} \frac{1}{log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

Computing a ρ-ordering

We describe an algorithm for computing the ρ-ordering of a set recursively and discuss some immediate corollaries.

Let $S \subseteq M$ be a compact subset of an ultrametric space (M, ρ). Let $\Gamma_S = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in S. Note that for each $k \in \mathbb{N}$, the closed balls of radius γ_k partition S, i.e., $S = S_{\gamma_k} := \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$, where both n and the x_i's depend on k. In what follows, fix a $k \in \mathbb{N}$ and let $S_{\gamma_k} = \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$ be such a partition. Note that we can construct $S_{\gamma_{k+1}}$ by partitioning each of the $\overline{B_{\gamma_k}(x_i)}$, i.e.,

$$S = S_{\gamma_{k+1}} = \cup_{i=1}^{n} \cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})}$$

where $1 \leq l_i \leq n$ and $\cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})} = \overline{B_{\gamma_k}(x_i)}$, $\forall i$. We denote by $x_{i,j}$ the centre of a ball of radius γ_{k+1} partitioning the ball $B_{\gamma_k}(x_i)$. Without loss of generality, when $j = 1$, assume $x_{i,j} = x_i$, $\forall i$.

We now make the following observation due to [6],

Lemma. For each k, the elements of S_{γ_k}, that is, the closed balls of radius γ_k, themselves form an ultrametric space, where

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x,y), & \text{if } \rho(x,y) > \gamma_k \\ 0, & \text{if } \rho(x,y) \leq \gamma_k \text{, i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

We note that since S is assumed to be compact, S_{γ_k} is a finite metric space $\forall k < \infty$ and $S_{\gamma_\infty} = \cup_{x \in S} \overline{B_0(x)} = \cup_{x \in S} x = S$ and $\rho_\infty = \rho$. Now view S_{γ_k}, for fixed $k < \infty$ as a finite ultrametric space and represent its $n < \infty$ elements by their centres, the x_i's. Without loss of genearlity, we can reindex the x_i's so that they give the first n terms of a ρ_k-ordering of S_{γ_k}. The following proposition is the main result of this section.

Proposition. Given S a compact subset of an ultrametric space M and Γ_S, the set of distances in S, if S_{γ_k} is a partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the centres of the balls are indexed according to a ρ_k-ordering of S_{γ_k}, then a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ can be found by forming a matrix, A_k, whose $(i,j)^{th}$-entry is $x_{i,j}$, as shown below, and then concatenating the rows (where the columns are padded by * if necessary).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \dots & x_{n,l_n} \end{pmatrix}$$

Proof. Note that the entries in each column are points in the ball $B_{\gamma_k}(x_i)$ so that the pairwise distance between columns is constant and always exceeds the distance between elements within a column. Moreover, the columns are organized such that for any j, $x_{n,j}$ maximizes$\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j})$ since $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j}) = \prod_{i=1}^{n-1} \rho(x_{n,1}, x_{i,1}) = \prod_{i=1}^{n-1} \rho(x_n, x_i)$ and the x_i's are indexed according a ρ_k-ordering of S_{γ_k}.

Then a $\rho_{\gamma_{k+1}}$-ordering of $S_{\gamma_{k+1}}$ is obtained by minimizing the number of elements from any one column and by taking the points $x_{i,j}$ (for fixed j) in sequence. For example, by concatenating the rows. $\square$

Corollary. Interweaving the bottown row of the lattice of closed balls for a set S gives a ρ-ordering of S.

Corollary. Suppose S and T are compact subsets of an ultrametric space M with $\Gamma_S = \Gamma_T$ and $\mid S_{\gamma_k} \mid = \mid T_{\gamma_k} \mid$, $\forall k$. Then $\omega(S) = \omega(T)$.

- i think this coincides with translation invariance when there is a group operation

Corollary. (regularity) Suppose that S is such that $\forall k$, any $B_{\gamma_k}(x) = \cup_{j=1}^{l} B_{\gamma_{k+1}}(x_j)$, that is, every ball in S breaks into exactly l smaller balls.

Product space

As a first point of departure, a natural space to consider is the product space of ultrametric spaces, for example $\mathbb{Z}^n$ (or $\mathbb{Z}_p^n$ or $\mathbb{Q}_p^n$), for some $n > 1$. If we restrict our attention to bounded subsets, then a natural candidate for an ultrametric on the product space is the L_∞ metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = sup_i\{\rho(x_i, y_i)\}$$

where ρ is the metric from the base space. In fact, since we have only defined valuative capacity for compact subsets of an ultrametric spaces, there is no loss of generality by restricting our metric to bounded spaces. We also see that no problems arise in letting both M and ρ vary between components of the space, as long as each M_i remains bounded and each ρ_i is an ultrametric.

Proposition. Let (M_i, ρ_i) for i in some finite or countably infinite index set I be a collection of metric spaces and suppose ρ_i is a bounded ultrametric for all i. Then (M, ρ_∞) is an ultrametric space, where $M = M_1 \times M_2 \times M_3 \times \ldots$ and ρ_∞ is the L_∞ metric described above.

Proof. Let (M, ρ_∞) be the product of ultrametric spaces as above and let x and y be two points in the space. Clearly, $\rho_\infty(x, y) \geq 0$ since each $\rho_i(x_i, y_i) \geq 0$, and $\rho_\infty(x, y) = 0 \iff \rho_i(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$. The fact that ρ_∞ is symmetric is also an easy consequence of the fact that each ρ_i is symmetric since $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$

implies $sup_i\{\rho_i(x_i, y_i)\} = sup_i\{\rho_i(y_i, x_i)\}$. To see that ρ_∞ is an ultrametric, note that if $z = z_i$ is any other point of M, then

$$\begin{aligned}
\rho_\infty(x,y) &= sup_i\{\rho_i(x_i, y_i)\} \\
&\leq sup_i\{max(\rho_i(x_i, z_i), \rho_i(y_i, z_i))\} && \text{since each } \rho_i \text{ is an ultrametric} \\
&\leq max(\rho_i(x_i, z_i)\rho_i(y_i, z_i)), \forall i \\
&\leq max(sup_i\{\rho_i(x_i, z_i)\}, sup_i\{\rho_i(y_i, z_i)\} \\
&= max(\rho_\infty(x,z), \rho_\infty(y,z))
\end{aligned}$$

□

We show a few quick results ultrametric spaces formed as product spaces, which allows us to quickly calculate the valuative capacity of a few subsets.

Proposition. Suppose (M, ρ_∞) is the product of ultrametric spaces (M_i, ρ_i) and each M_i is a topological group with operation $+$. Then ρ_∞ is (left) translation invariant if each ρ_i is, in which case valuative capacity is also (left) translation invariant.

Proof. Let (M, ρ_∞) be as above. Suppose also that

$$\rho(x_i, y_i) = \rho(s_i + x_i, s_i + y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each ρ_i is (left) translation invariant. Then,

$$\rho_\infty(s + x, s + y) = sup_i\{\rho(s_i + x_i, s_i + y_i)\} = sup_i\{\rho(x_i, y_i)\} = \rho_\infty(x, y).$$

so that ρ_∞ is translation invariant. Proposition xyz implies valuative capacity is as well. □

Proposition. Suppose (M, ρ_∞) is the product of ultrametric spaces (M_i, ρ_i) and each M_i is in fact a normed vector space (with each ρ_i being the metric derived from the norm on M_i). Then ρ_∞ is multiplicative if the norm producing each ρ_i is multiplicative, in which valuative capacity is as well.

Proof. Similar. $\square$

Example 3. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$ and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1)), \rho_p(x_2, y_2))$, where ρ_p is the p-adic metric for some fixed prime p. Since ρ_p is translation invariant and multiplicative in $\mathbb{Z}_p$, valuative capacity is also translation invariant and multiplicative in $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$.

Example 4. Let $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x \in \mathbb{Z}_{p_1}, y \in \mathbb{Z}_{p_2}\}$ for two distinct primes, $p_1 \neq p_2$, and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_{p_1}(x_1, y_1)), \rho_{p_2}(x_2, y_2))$, where ρ_{p_i} is the p-adic metric. Since each ρ_{p_i} is translation invariant and multiplicative in $\mathbb{Z}_{p_i}$, valuative capacity is also translation invariant and multiplicative in $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$.

n-fold products

What is the valuative capacity of $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ from the example above? Suppose $p = 2$. Using translation invariance, scaling and subaddivity, we can compute the result by first noting that we can write $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always $1 = diam(\mathbb{Z}_2 \times \mathbb{Z}_2)$, subadditivity implies that

$$\frac{1}{log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

$$= \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))}$$

$$= \frac{4}{log(\|2\|_2 * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{log(\frac{1}{2} * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{log(\frac{1}{2}) + log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

Taking logs base 2, we have that

$$\omega(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^{\frac{-1+log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} 2^{\frac{log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} (2^{log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))})^{\frac{1}{4}} = 2^{\frac{-1}{4}} \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\frac{1}{4}}$$

so that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a solution of the equation $x^4 - \frac{x}{2}$, for which there is a single real positive root, given by $2^{-1/3}$.

To compute the valuative capacity for a 2-fold product for an arbitary prime p, note that we can always decompose $\mathbb{Z}_p \times \mathbb{Z}_p$ into a union of p^2 sets each of the form $\{p\mathbb{Z}_p + s \times p\mathbb{Z}_p + t\}$ for $s, t \in (0, \ldots, p-1)$, and the pairwise distance between these sets will always be $1 = diam(\mathbb{Z}_p \times \mathbb{Z}_p)$ (to see this, either note that we can always find co-prime elements, or note that each set is an closed ball of radius $1/p$ centred at (s,t) and so the distance between them must be greater than $1/p$, and 1 is the only possible distance greater than $1/p$ in $\mathbb{Z}_p \times \mathbb{Z}_p$). Then, we combine our tools as before to obtain the equation,

$$\frac{1}{log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(\|p\|_p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(1/p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, taking logs base p, we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p) = p^{\frac{-1}{p^2}} \omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{\frac{1}{p^2}}$$

So that $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$ is a solution of the equation $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p})$ over $\mathbb{R}$ and since $\mathbb{R}$ is a division ring, this means the positive solutions are given by solving $x^{p^2-1} - \frac{1}{p}$. Solutions of this equation are of the form $p^{\frac{-1}{p^2-1}}$ times a p^2-1 root of unity, and so there is exactly one positive, real solution, namely $p^{\frac{-1}{p^2-1}}$ itself. Then the valulative capacity of the entire product space $\mathbb{Z}_p \times \mathbb{Z}_p$ is $p^{\frac{-1}{p^2-1}}$. In fact, from here it is not hard to see that by taking the n-fold product, we would end up with the same equation except that the exponent of p would become n rather than 2. We arrive at the following result:

Proposition. Let $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$ be the ultrametric space with points equal to the n-fold product of $\mathbb{Z}_p$ (for $n < \infty$) for some fixed prime p. The valuative capacity of M is $(\frac{1}{p})^{\frac{1}{p^n-1}}$.

Proof. Above. □

Taking $n = 1$, we see that this agrees with the valuative capacity of $\mathbb{Z}_p$ computed in the last chapter.

We end this section with two observations on the results above. First, recall that in computing the valuative capacity of these spaces, we were ultimately reduced to finding solutions to polynomials of the form $x^{p^n} - \frac{x}{p}$ for some n and for some p. The first observation is that these polynomials are $\mathbb{Z}$-valued on $p\mathbb{Z}$, that is, they are elements of $Int(p\mathbb{Z}, \mathbb{Z})$. We might ask then, what sort of polynomials would arise in finding the valuative capacity of spaces such as $(\mathbb{Z}_2 \times \mathbb{Z}_3, \rho_\infty)$ or in computing the valuative capacity of infinite product spaces, such as $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots$ for either some fixed prime p or over each prime.

Secondly, we observe the asymptotic behavior of capacity in these spaces. For a fixed prime p, $(\frac{1}{p})^{\frac{1}{p^n-1}}$ is an monotone, increasing sequence in n with $lim_{n\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. For fixed n, the sequence in p is also montone, increasing, again with $lim_{p\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. In both cases, the limiting value is equal to the diameter of space. Indeed, we can observe that the sequence $\{(0, 0, \ldots), (1, 0, \ldots), (0, 1, \ldots), \ldots\}$, in which the first element has only zeros and the n-th element has a single 1 in the $(n-1)$-th component, is a $\rho-$ordering for both $(\mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, \rho_{p,\infty})$ and $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots, \rho_{P,\infty})$, since the distance between elements in this sequence (in either metric space) is always 1. If we could show that these spaces are compact, this would gives a valuative capacity of $lim_{n\to\infty}(1^n)^{(1/n)} = 1$

for both spaces. We explore this more in the following section.

Product topology

In considering the product space of ultrametric spaces, we may wonder whether the chosen metric also gives back the product topology on the space. For products formed by taking some finite number of copies, the answer is positive. We give the necessary background and show this fact, adapting the proof in Munkres (20.3) to the case of ultrametric spaces.

Definition-Proposition. (Munkres) Suppose X_i, for i in some index set I, is a family of topological spaces. Let $\pi_j : \prod_{i \in I} X_i \to X_j$ be the map given by projection onto the j-th component, that is $\pi_j(x) = \pi_j((x_i)_{i \in I}) = x_j$. For each $j \in I$, let $\mathcal{S}_j$ be the collection

$$\mathcal{S}_j = \{\pi_j^{-1}(U_j) \mid U_j \text{ open in } X_j\}$$

Let $\mathcal{S}$ be the union of the $\mathcal{S}_j$ over $j \in I$, $\mathcal{S} = \cup_{j \in I} \mathcal{S}_j$. Then $\mathcal{S}$ is a subbasis that generates a topology on $\prod_{i \in I} X_i$ called the **product topology**.

The basis, $\mathcal{B}$, generated by $\mathcal{S}$ in the definition above is the set of all finite intersections of elements in $\mathcal{S}$. That is $B \in \mathcal{B}$ if there exists $S_1, S_2, \ldots, S_n$ in $\mathcal{S}$ such that $B = S_1 \cap S_2 \cap \ldots S_n$. A useful description of the basis for the product topology also appears in Munkres, as below:

Proposition. (Munkres 19.2) Suppose X_i, for i in some index set I, is a family of topological spaces and denote by $\mathcal{B}_i$ the basis for the topology on X_i. Let

$$\mathcal{B}_P = \prod_{i \in I} B_i, \text{ for } B_i \in \mathcal{B}_i \text{ and } B_i = X_i \text{ for all but finitely-many } i \in I.$$

then $\mathcal{B}_P$ is a basis for the product topology on $\prod_{i \in I} X_i$.

We can now show that the topology induced by the L_∞ metric described above agrees with the product topology for finite products.

Proposition. Let $M = (M_1 \times M_2 \times \ldots \times M_n, \rho_\infty)$ be a finite product of bounded, ultrametric spaces and let ρ_∞ be the metric described above. Then the topology induced by ρ_∞ coincides with the product topology on $M_1 \times M_2 \times \ldots \times M_n$.

Proof. Let $\mathcal{T}_{\rho_\infty}$ be the topology on $M_1 \times M_2 \times \ldots \times M_n$ induced by ρ_∞ and let $\mathcal{B}_{\rho_\infty}$ be the basis for this topology. Let $\mathcal{T}_P$ be the product topology with basis $\mathcal{B}_P$. We show $\mathcal{T}_P \subset \mathcal{T}_{\rho_\infty}$ and vice versa. For this, it is equivalent (Munkres 13.3) to show that for $z \in M_1 \times M_2 \times \ldots \times M_n$ and $B \in \mathcal{B}_P$ containing z, there is a basis element $B' \in \mathcal{B}_{\rho_\infty}$ such that $z \in B' \subset B$, and vice versa.

So let $z \in M_1 \times M_2 \times \ldots \times M_n$ and suppose $B \in \mathcal{B}_P$ contains z. Since B is in $\mathcal{B}_P$, B is of the form $B_{r_1}(z_1) \times B_{r_2}(z_2) \times \ldots \times B_{r_n}(z_n)$ (since the choice of centres is arbitrary in an ultrametric spaces, we may choose the components of z as the centres without loss of generality). Let $r = \min\{r_i\}$ for $i \in 1, \ldots, n$. Then let B' be the ball $B_r(z)$ in $\mathcal{B}_{\rho_\infty}$. Clearly, $z \in B_r(z)$ and since $r \leq r_i$, $\forall i$, $B_r(z) = B_r(z_1) \times B_r(z_2) \times \ldots \times B_r(z_n) \subset B_{r_1}(z_1) \times B_{r_2}(z_2) \times \ldots \times B_{r_n}(z_n) = B$.

Conversely, suppose $A \in \mathcal{B}_{\rho_\infty}$ and let $y \in A$. To find $A' \in \mathcal{B}_P$ such that $y \in A'$ and $A' \subset A$, simply note that A itself is in $\mathcal{B}_P$.

$\square$

We are now naturally left to ask whether the product topology on *infinite* products of ultrametric spaces coincides with the L_∞ metric. In this case, as in the analogous case of infinite copies of $\mathbb{R}$ and a uniform metric, the answer is negative (at least in general). Forunately, the metric that realizes the product topology on infinite copies of $\mathbb{R}$ can be adapted to the case of ultrametric spaces. We adapt to the proof of Munkres (20.5) to the case of infinite products of ultrametric spaces.

Proposition. Suppose $\mathbf{M} = M_1 \times M_2 \times \ldots$ is an infinite collection of metric spaces, each with an ultrametric ρ_i which is bounded by 1, that is suppose $\rho_i(x_i, y_i) \leq 1$, for all $x_i, y_i \in M_i$ and for all i. Define a metric d on $\mathbf{M}$ as follows:

$$d(\mathbf{x}, \mathbf{y}) = sup\{\frac{\rho_i(x_i, y_i)}{i}\}$$

Then d is an ultrametric and induces the product topology on $\mathbf{M}$.

Proof. We see that d inherits symmetry, injectivity and non-negativity from the requirement that each ρ_i is a metric, just as ρ_∞ did. To see that d satisfies the strong triangle inequality, define a new metric ρ'_i by $\rho'_i(x, y) = \frac{\rho_i(x_i, y_i)}{i}, \forall i$. Then ρ'_i is an ultrametric, since $\rho_i(x, y) \leq \max(\rho_i(x, z), \rho_i(y, z))$ implies $\frac{\rho_i(x,y)}{n} \leq \max(\frac{\rho_i(x,z)}{n}, \frac{\rho_i(y,z)}{n})$ for any $n \in \mathbb{N}$. Then we can view d as the L_∞ metric on the spaces (M_i, ρ'_i), and so d will be an ultrametric as shown in the first proposition of this section.

Now we show d induces the product topology. We first show that metric topology induced by d is finer than the product topology. Let

$$B = B_r^{\mathbf{M}}(\mathbf{z}) = B_r^{M_1}(z_1) \times B_{2r}^{M_2}(z_2) \times B_{3r}^{M_3}(z_3) \times \ldots$$

be a basis open in the metric topology. We must find a basis open $B' \ni z$ in the product topology such that $B' \subseteq B$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < r$. Then let B' be the basis open element

$$B' = B_r^{M_1}(z_1) \times B_r^{M_2}(z_2) \times \ldots \times B_r^{M_N}(z_N) \times M_{N+1} \times M_{N+2} \times \ldots$$

in the product topology. Suppose $\mathbf{y} \in B'$. We must show $\mathbf{y} \in B$, i.e., $d(\mathbf{z}, \mathbf{y}) < r$. Note that for all $i \geq N$,

$$\frac{\rho_i(z_i, y_i)}{i} \leq \frac{1}{N}$$

which means

$$d(\mathbf{z}, \mathbf{y}) = \sup\{\frac{\rho_i(z_i, y_i)}{i}\} \leq \max\{\frac{\rho_1(z_1, y_1)}{1}, \frac{\rho_2(z_2, y_2)}{2}, \ldots, \frac{\rho_N(z_N, y_N)}{N}, \frac{1}{N}\}$$

and since N was chosen so that $\frac{1}{N} < r$ and B' was chosen to have balls of radius r in the first N components, we must have $d(\mathbf{z}, \mathbf{y}) < r$.

Conversely, ... $\square$

From now on, we refer to the metric d above as the **product metric**. An important consequence of the fact that d achieves the product topology is that Tychnoff's theorem then guarantees that product spaces formed with this metric will be compact, infinite or otherwise. As a result, we can now ask directly about the valuative capacity of some infinite product spaces. We consider two examples.

Example 5. Let $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, d)$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for some fixed prime p and let d be the product metric.

Example 6. Let $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots, \rho_{P,\infty})$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for every prime p and let d be the product metric.

So far we have two methods for computing valuative capacity. Either we can find a useful decomposition that allows us to apply the subadditivity formula, or we can find a $\rho-$ordering and then take the limit of its corresponding $\rho-$sequence.

Inclusions

From Munkres, Ch. 2 section 18.

Definition. Let X and Y be topological spaces and suppose $f : X \to Y$ is a bijection with inverse function $f^{-1} : Y \to X$. Then f is a **homeomorphism** so long as f and f^{-1} are continuous.

Definition. Let X and Y be topological spaces and suppose $f : X \to Y$ is injective and continuous. Let $Z = f(X) \subseteq Y$ be the image of f in Y. If $f' : X \to Z$ is a homeomorphism, then we say that f is an **topological imbedding** of X in Y.

Proposition. (Munkres 18.4) Let A, X and Y be topological spaces and suppose $f : A \to X \times Y$ is given by $f(a) = (f_1(a), f_2(a))$, where $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if both f_1 and f_2 are.

Conclusion

In this section, we considered the notion of valuative capacity in product spaces, that is, spaces formed by taking copies of ultrametric spaces. In the following sections, we consider vaulative capacity in spaces formed by adding points, that is extension fields, or by both taking copies and adding (a distinguished) point, as in projective spaces. For these purposes, it will be more productive to start working over the field $\mathbb{Q}_p$, instead of $\mathbb{Z}_p$.

projective space

Background from Gerritzen and van der Put

Background results from [7]. Let k be a field that is complete with respect to a non-archimedean valuation and let K be a complete and algebraically closed field containing k.

Definition. [7] The set $\{\lambda \in k; \mid \lambda \mid \leq 1\}$, denoted k^0, is the **valuation ring** of k. It has a unique maximal ideal, denoted k^{00}, given by $\{\lambda \in k; \mid \lambda \mid < 1\}$. The **residue field** of k is $\bar{k} := k^0/k^{00}$.

Definition. [7] The **projective line over** k, denoted $\mathbb{P}^1(k)$, is the space whose points are lines l in k^2 that intersect $(0,0)$ and whose topology and field structure are inherited from k.

We give two equivalent representations for the points of $\mathbb{P}^1(k)$. A point $p \in \mathbb{P}^1(k)$ is an equivalence class of $k^2 \setminus (0,0)$ under the relation $(x,y) \sim (x',y')$ if there exists a $\lambda \in k \setminus 0$ such that $(x,y) = \lambda(x',y')$. Equivalently, suppose that l is a line in k^2 intersecting the origin, that is a point in $\mathbb{P}^1(k)$. We denote l by a representative $[x_0,x_1] \in k^2$ such that $l = \{\lambda(x_0,x_1); \lambda \in k\}$, called homogeneous coordinates of l.

Proposition. [7] Let $\psi : k \to \mathbb{P}^1(k)$ be the map given by $\psi(\lambda_0) = [1,\lambda_0]$, where $[1,\lambda_0]$ is the line in k^2, $\{\lambda(1,\lambda_0); \lambda \in k\}$. Then the image of ψ is $\mathbb{P}^1(k) \setminus [0,1]$ and is isomorphic to k, so that k is identified with projective space minus a distinguished point, $[0,1]$, which is denoted by ∞.

Definition. [7] k is called a **local field** if k is locally compact.

Proposition. [7] The following are equivalent:

1. k is a local field.

2. $\mid k^* \mid \cong \mathbb{Z}$ and $\bar{k}$ is finite, where $k*$ is the set of units in k, ie $k^* = \{\lambda \in k, \lambda \neq 0\}$.

3. k is a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

4. $\mathbb{P}^1(k)$ is compact

Definition. [7] We denote by $GL(2,k)$ the set of invertible 2×2 matrices over k. A **fractional linear automorphism**, ϕ, of $\mathbb{P}^1(k)$ is a map defined by $z \mapsto \frac{az+b}{cz+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,k)$. The set of fractional linear automorphisms of $\mathbb{P}^1(k)$ is denoted $PGL(2,k)$. Note that $PGL(2,k) = GL(2,k)/\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right); \lambda \in k^*\}$. In homogeneous coordinates, we can represent the action of ϕ by $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$.

Definition. [7] Suppose Γ is a subgroup of $PGL(2,k)$. A point $p \in \mathbb{P}^1(k)$ is a **limit point of** Γ, if there exists a point q in $\mathbb{P}^1(k)$ and a sequence $\{\gamma_n\}_{n\geq 1}$ in Γ such that $\lim_{n\to\infty} \gamma_n(q) = p$.

Proposition. [7] If Γ is not a discrete subgroup of $PGL(2,k)$ then every point of $\mathbb{P}^1(k)$ is a limit point of Γ.

Proof. Since Γ is not discrete, the sequence $\{\gamma_n\}_{n\geq 1}$ has a limit γ in Γ. Let p be any point of $\mathbb{P}^1(k)$ and let $q = \gamma^{-1}(p)$. Then $\lim_{n\to\infty} \gamma_n(q) = \lim_{n\to\infty} \gamma_n(\gamma^{-1}(p)) = p$. $\square$

Definition. [7] A subgroup Γ of $PGL(2,k)$ is **discontinuous** if the closure of every orbit of Γ in $\mathbb{P}^1(k)$ is compact and the set of all limit points of Γ is not equal to $\mathbb{P}^1(k)$.

Proposition. [7] If Γ is a discontinuous subgroup of $PGL(2,k)$ and $\mathcal{L}$ is the set of limit points of Γ, then $\mathcal{L}$ is compact, no where dense and if $\mathcal{L}$ contains more than two points, $\mathcal{L}$ is perfect.

Definition. [7] Let A be an element of $GL(2,k)$ and let a_1 and a_2 be the eigenvalues of A. Then A is called **elliptic** if $a_1 \neq a_2$, but $\mid a_1 \mid = \mid a_2 \mid$. A is called **parabolic** if $a_1 = a_2$, and A is called **hyperbolic** if $\mid a_1 \mid \neq \mid a_2 \mid$.

Example 7. Consider the matrix $T_s = \left(\begin{smallmatrix} p & s \\ 0 & 1 \end{smallmatrix}\right) \in GL(2,\mathbb{Q}_p)$ for some s in $(0,\ldots,p-1)$ (note that $det(T_s) = p$ is invertible in $\mathbb{Q}_p$, so that $T_s \in GL(2,\mathbb{Q}_p)$, although it is not in $GL(2,\mathbb{Z}_p)$). T_s has eigenvalues p and 1 and so T_s is hyperbolic for any choice of s or p. Consider the action of T_s on $\mathbb{Z}_p \subset \mathbb{Q}_p$, where $\mathbb{Z}_p$ is identified with the subspace $\{[1,\lambda];\lambda \in \mathbb{Z}_p\}$ of $\mathbb{P}^1(\mathbb{Q}_p)$. In homogeneous coordinates, this action is given by $[1,\lambda] \mapsto [1,p\lambda+s]$. Since $\mid (p\lambda+s-s) \mid = \mid p\lambda \mid \leq \frac{1}{p}$, T_s sends λ to $B_{\frac{1}{p}}(s)$. Also note that for $s=0$, T_s has the effect of shifting the index of λ by 1, that is, if $\lambda = \sum_{i=n}^{\infty} a_i p^i$, where $n = ord(\lambda)$, then $T_0([1,\lambda]) = [1,p\lambda] \rightsquigarrow p\lambda = \sum_{i=n+1}^{\infty} a_{(i-1)} p^i$.

Computation of the capacity of some sets (F&P, section 5)

Setup

Let $A = \{0,1,..,d-1\}$ be a finite alphabet and $A^{\mathbb{N}}$ be the collection of infinite sequenes with values in A. Note $A^{\mathbb{N}}$ is a Cantor set, so it is perfect, nowhere dense, and compact.

A basis for the topology is given by the cylinder set: take countably many copies of $\{0,1,...,d-1\}$ where each copy has the discrete topology.

Let $p \geq d$ be a prime number and let ϕ be the canonical embedding of $A^{\mathbb{N}}$ into $\mathbb{Z}_p$ via the following continuous (under the above topology) map:

$$\phi : A^{\mathbb{N}} \to \mathbb{Z}_p \text{ by } (x_n)_{n\geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

Lemma. (F&P Lemma 5.1)

Let $w_1, w_2, \ldots, w_s$ be $s \geq 2$ words with the same length l such that all the first letters are distinct. Let $X \subset A^{\mathbb{N}}$ be the set of sequences such that any factor is a factor of a concatenation of the words $w_1, w_2, \ldots, w_s$. Then the set $E := \phi(X) \subset \mathbb{Z}_p$ satisfies:

$$E = \cup_{i=1}^{s} x_i + p^l E, \text{ with } x_i = \phi(w_i 0^\infty)$$

It is a regular compact set and its valuative capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

Example 8. $w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$

Then $\{x_n\}_{n\geq 0} \in X$ if each term in $\{x_n\}_{n\geq 0}$ is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Note that we can rephrase the lemma as follows:

Let $x_1, x_2, \ldots, x_s$ be $s \geq 2$ points in $\mathbb{Z}_p$ such that $\mid x_i - x_j \mid_p = 1$, $\forall i, j \in 1, ..., s$. Suppose also that there exists an $l \in \mathbb{N}$ such that $\forall i$,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{l} a_i p^i$$

Let γ_i be the fractional linear automorphism of $\mathbb{P}^1(\mathbb{Q}_p)$ given by $\left(\begin{smallmatrix} p^l & x_i \\ 0 & 1 \end{smallmatrix}\right)$ and let Γ be the subgroup of $PGL(2, \mathbb{Q}_p)$ generated by the γ_i.

Then Γ has a subgroup H such that the limit set $\mathcal{L}$ of H has the property that $Z = \psi^{-1}(\mathcal{L})$ is equal to $\phi(X)$ in the original lemma. In particular Z is a regular, compact subset of $\mathbb{Z}_p$ satisfying

$$Z = \cup_{i=1}^{s} x_i + p^l Z = \cup_{i=1}^{s} B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

Proof. We must show that the set Z above is equal to $E = \phi(X)$ in the original lemma. First note that if $w_1, w_2, ..., w_s$ are words in $A^{\mathbb{N}}$, then the first letter of each w_i is distinct if and only if $\mid \phi(w_i) - \phi(w_j) \mid_p = 1, \forall i, j$ (since the pairwise distance is 1 if and only if $ord(\phi(w_i) - \phi(w_j)) = 0$ for any i and j, if and only if the coefficient of p^0 (i.e., the first letter each w_i) is different $\forall i, j$). So then the x_i are just the $\phi(w_i)$.

Now consider the limit set of Γ. Let $\gamma \in \Gamma$. If γ is a product of the generators γ_i, then γ is represented by a matrix of the form: $\left(\begin{smallmatrix} p^{lm} & z_m \\ 0 & 1 \end{smallmatrix}\right)$, where $m \in \mathbb{N}$ and z_m is an element of $\mathbb{Z}_p$ whose coefficient vector is a concatenation of the coefficient vectors of the x_i (for $0 \leq i \leq ml$ and 0 for $i > ml$). For example, $\gamma_i \gamma_j \gamma_k =$

$$\left(\begin{smallmatrix} p^l & x_i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^l & x_j \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^l & x_k \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p^{3l} & p^{2l}x_k + p^l x_j + x_i \\ 0 & 1 \end{smallmatrix}\right)$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{lm} a_1 + z a_0] \sim [1, p^{lm} \frac{a_1}{a_0} + z_m]$$

As m tends to infinity, this point tends to $[1, 0 + z]$, where z is an element of $\mathbb{Z}_p$ whose entire coefficient vector is a concatenation of the x_i's. The set $\psi^{-1}([1, z])$ for all such z is exactly the set $E = \phi(X)$.

Now suppose γ is a product of the inverses of the generators γ_i^{-1}, then γ is represented by a matrix of the form: $\left(\begin{smallmatrix} p^{-lm} & -p^{-l}z^{-1} \\ 0 & 1 \end{smallmatrix}\right)$, where $m \in \mathbb{N}$ and z is as above. For example,

$$\left(\begin{smallmatrix} p^{-l} & -p^{-l}x_i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^{-l} & -p^{-l}x_j \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^{-l} & -p^{-l}x_k \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p^{-3l} & -p^{-3l}x_k - p^{-2l}x_j - p^{-l}x_i \\ 0 & 1 \end{smallmatrix}\right)$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{-lm}a_1 - p^{-lm}za_0] \sim [1, p^{-lm}(\frac{a_1}{a_0} - z)]$$

This time as m grows, the image approaches infinity and so has empty preimage under ψ.

Lastly, we consider elements of Γ made up of both generators and the inverse of generators. These elements will produce translations, either of the form $\left(\begin{smallmatrix} 1 & p^{-l}(x_i - x_j) \\ 0 & 1 \end{smallmatrix}\right)$ if $\gamma = \gamma_j^{-1}\gamma_i$ or $\left(\begin{smallmatrix} 1 & x_j - x_i \\ 0 & 1 \end{smallmatrix}\right)$ if $\gamma = \gamma_j\gamma_i^{-1}$. These elements commute with each other and so the subgroup which they generate is normal. We can quotient by the entire translation subgroup, ie the subgroup generated by $\{\gamma_i\gamma_j^{-1}, \gamma_i^{-1}\gamma_j; \forall i, j \in 1, \ldots, s\}$ to obtain H. Then $\mathcal{L} = \infty \cup \{[1, z] \mid z \in \mathbb{Z}_p$ and the coeffecient vector of z is a concatenation of the coeffiecient vectors of the x_i's$\}$ and $\psi^{-1}(\mathcal{L}) = E = \phi(X)$, as required.

$\square$

H is not a subgroup of $PGL(2, k)$ - the torsion means its not closed, although it is a group (in general) and also a subset of $PGL(2, k)$.

Γ is not discrete in $PGL(2, k)$ - is H or $\{\gamma_i\}$ or $\{\gamma_i^{-1}\}$ a maximal discrete subset?

Does H have a maximal (fg) subgroup and is it interesting?

More background from [7].

Definition. [7] A **Schottky group** is a finitely-generated, free and discontinuous subgroup of $PGL(2, k)$

Notation: Let X be a subset of $\mathbb{P}^1(k)$ and denote by $X^{(3)}$ the set $X \times X \times X \setminus \Delta$, where Δ is the fat diagnol, that is the set $\{x_i = x_j;$ for some $i \neq j,\ i, j \in 1, 2, 3\}$

Notation: Let $a = (a_0, a_1, a_\infty)$ be an element of $X^{(3)}$ for some $X \subseteq \mathbb{P}^1(k)$. We denote by γ_a the element of $PGL(2, k)$ such that $\gamma_a(a_0) = 0$, $\gamma_a(a_1) = 1$, $\gamma_a(a_\infty) = \infty$.

Note that given such a point a, we can calculate the map γ_a explicitly, as follows: Suppose $a = (a_0, a_1, a_\infty)$ and $a_i \neq \infty = [0, 1], \forall i$. Then $\gamma_a(a_0) = 0 = [1, 0]$, $\gamma_a(a_1) = 1 = [1, 1]$, and $\gamma_a(a_\infty) = \infty = [0, 1]$, so that if γ_a is represented by some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have:

Definition. Let x_0, x_1 be two elements of k such that $max(\mid x_0 \mid, \mid x_1 \mid)$, ie both x_0 and x_1 are in the valuation ring of k and at least one is in its residue field. The map R that sends $[x_0, x_1] \mapsto [\bar{x_0}, \bar{x_1}]$, that is that sends each component to its residue class in $\bar{k}$ is called the **standard reduction**.

algebraic extensions

Type up the example Keith gave you.

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