Valuative Capacity of some compact subsets of $\mathbb{Z}_p$

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Background: p-orderings, p-sequences

A $p-$**ordering** of an infinite set, $S \subseteq \mathbb{Z}_p$, is a sequence in S such that for $\forall n > 0$, a_n minimizes

$$v_p((x - a_{n-1}) \dots (x - a_0))$$

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cf: A ρ-**ordering** of S, a (compact) subset of an ultrametric space (M, ρ), is a sequence in S such that $\forall n > 0$, a_n maximizes

$$\prod_{i=0}^{n-1} \rho(x, a_i)$$

Background: p-orderings, p-sequences

The p-**sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is

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Background: valuative and logarithm capacity

The **valuative capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$L_p(S) := \lim_{n \to \infty} \frac{w_S(n, p)}{n}$$

where $w_S(n, p)$ is the $p-$sequence of S.

nb: this is the Robin's constant and can be found via the equilibrium measure:

$$L_p(S) = \inf_{\mu \in \mathcal{P}(\bar{S})} \int \int v_p(x - y) d\mu(x) d\mu(y)$$

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The **logarithm capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$V_p(E) := p^{-L_p(E)}$$

nb: this is equal to the transfinite diameter and the Chebychev constant.

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Fares and Petite, Lemma 5.1

Let $A = \{0, 1, .., d-1\}$ be a finite alphabet and $A^{\mathbb{N}}$ be the collection of infinite sequenes with values in A.

Let $p \geq d$ be a prime number and let ϕ be the canonical embedding of $A^{\mathbb{N}}$ into $\mathbb{Z}_p$ via the following continuous map:

$$\phi : A^{\mathbb{N}} \to \mathbb{Z}_p \text{ by } (x_n)_{n\geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

Fares and Petite, Lemma 5.1

Lemma

Let $w_1, w_2, \dots, w_s$ be $s \geq 2$ words with the same length l such that all the first letters are distinct. Let $X \subset A^{\mathbb{N}}$ be the set of sequences such that any factor is a factor of a concatenation of the words $w_1, w_2, \dots, w_s$. Then the set $E := \phi(X) \subset \mathbb{Z}_p$ satisfies:

$$E = \cup_{i=1}^{s} x_i + p^l E, \text{ with } x_i = \phi(w_i 0^\infty)$$

It is a regular compact set and its valuative capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

Fares and Petite, Lemma 5.1

An example:

$w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$
Then $\{x_n\}_{n\geq 0} \in X$ if each term in $\{x_n\}_{n\geq 0}$ is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Digression: projective k-space

Let k be a field that is complete with respect to a non-archimedean valuation.

Definition

The **projective line over** k, denoted $\mathbb{P}^1(k)$, is the space whose points are lines l in k^2 that intersect $(0,0)$.

Proposition

Let $\psi : k \to \mathbb{P}^1(k)$ be the map given by $\psi(\lambda_0) = [1, \lambda_0]$, where $[1, \lambda_0]$ is the line in k^2, $\{\lambda(1, \lambda_0); \lambda \in k^*\}$. Then the image of ψ is $\mathbb{P}^1(k) \setminus [0, 1]$ and is isomorphic to k, so that k is identified with projective space minus a distinguished point, $[0, 1]$, which is denoted by ∞.

Digression: projective k-space

Definition

We denote by $GL(2,k)$ the set of invertible 2×2 matrices over k. A **fractional linear automorphism**, ϕ, of $\mathbb{P}^1(k)$ is a map defined by $z \mapsto \frac{az+b}{cz+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,k)$. The set of fractional linear automorphisms of $\mathbb{P}^1(k)$ is denoted $PGL(2,k)$.

Note that $PGL(2,k) = GL(2,k)/\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right); \lambda \in k^*\}$. In homogeneous coordinates, we can represent the action of ϕ by $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$.

Digression: projective k-space

Definition

Suppose Γ is a subgroup of $PGL(2,k)$. A point $p \in \mathbb{P}^1(k)$ is a **limit point of** Γ, if there exists a point q in $\mathbb{P}^1(k)$ and a sequence $\{\gamma_n\}_{n\geq 1}$ in Γ such that $\lim_{n\to\infty} \gamma_n(q) = p$.

Fares and Petite, Lemma 5.1, rephrased (1/2)

Let $x_1, x_2, \ldots, x_s$ be $s \geq 2$ points in $\mathbb{Z}_p$ such that $\mid x_i - x_j \mid_p = 1$, $\forall i, j \in 1, ..., s$. Suppose also that there exists an $l \in \mathbb{N}$ such that $\forall i$,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{l} a_i p^i$$

Fares and Petite, Lemma 5.1, rephrased (2/2)

Let γ_i be the fractional linear automorphism of $\mathbb{P}^1(\mathbb{Q}_p)$ given by $\left(\begin{smallmatrix} p^l & x_i \\ 0 & 1 \end{smallmatrix}\right)$ and let Γ be the subgroup of $PGL(2, \mathbb{Q}_p)$ generated by the γ_i.

Then Γ has a subgroup H such that the limit set $\mathcal{L}$ of H has the property that $Z = \psi^{-1}(\mathcal{L})$ is equal to $\phi(X)$ in the original lemma. In particular Z is a regular, compact subset of $\mathbb{Z}_p$ satisfying

$$Z = \cup_{i=1}^{s} x_i + p^l Z = \cup_{i=1}^{s} B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

Fares and Petite, Lemma 5.1, rephrased

Sketch of proof:

- We have to show w that the set Z above is equal to $E = \phi(X)$ in the original lemma.
- That that w_i correspond to the x_i is not hard to see.
- What is the limit set of Γ?

Limit set of Γ

Let $\gamma \in \Gamma$.

- If γ is a product of the generators γ_i, then γ is represented by a matrix of the form: $\left(\begin{smallmatrix} p^{lm} & z \\ 0 & 1 \end{smallmatrix}\right)$, where $m \in \mathbb{N}$ and z is an element of $\mathbb{Z}_p$ whose coefficient vector is a concatenation of the coefficient vectors of the x_i (for $0 \leq i \leq ml$ and 0 for $i > ml$).
- For example,

$$\left(\begin{smallmatrix} p^l & x_i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^l & x_j \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^l & x_k \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p^{3l} & p^{2l}x_k + p^l x_j + x_i \\ 0 & 1 \end{smallmatrix}\right)$$

- The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{lm} a_1 + z a_0] \sim [1, p^{lm} \frac{a_1}{a_0} + z]$$

Limit set of Γ

Let $\gamma \in \Gamma$.

- If γ is a product of the inverses of the generators γ_i^{-1}, then γ is represented by a matrix of the form: $\begin{pmatrix} p^{-lm} & -p^{-l}z^{-1} \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and z is as above.
- For example,

$$\begin{pmatrix} p^{-l} & -p^{-l}x_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^{-l} & -p^{-l}x_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^{-l} & -p^{-l}x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{-3l} & -p^{-3l}x_k - p^{-2l}x_j - p^{-l}x_i \\ 0 & 1 \end{pmatrix}$$

- The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{-lm}a_1 - p^{-l}z^{-1}a_0] \sim [1, p^{-l}(p^{-m}\frac{a_1}{a_0} - z^{-1})]$$

Limit set of Γ

Let $\gamma \in \Gamma$.

- If γ is of the form $\gamma_j^{-1}\gamma_i$, for $i \neq j$, then γ is represented by a matrix of the form: $\left(\begin{smallmatrix} 1 & p^{-1}(x_i - x_j) \\ 0 & 1 \end{smallmatrix}\right)$
- The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, a_1 + p^{-l}(x_i - x_j)a_0] \sim [1, \frac{a_1}{a_0} + p^{-l}(x_i - x_j)]$$

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- We quotient the group Γ by the group generated by the translations to obtain H.

references

- Youssef Fares and Samuel Petite, The valuative capacity of subshifts of finite type.
- Keith Johnson, P-orderings and Fekete sets