

Introduction

In the course of developing a generalized factorial function, Bhargava introduced the notion of p -orderings of a Dedekind domain [2, 3], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [4] (also KJ) and have also provided a natural framework to extend many classical results in analysis to a p -adic setting, such as polynomial approximation and mapping theorems [2, 3, 4].

In this thesis, we examine how a tool based on p -orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

Background

Ultrametric basics

Definition. Let (M, ρ) be a metric space. If ρ satisfies the ultrametric inequality

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)), \forall x, y, z \in M$$

then (M, ρ) is an **ultrametric space**.

Definition. Let (V, N) be a normed vector space. Then N satisfies the **strong triangle inequality** if

$$N(x + y) \leq \max(N(x), N(y)), \forall x, y \in V$$

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space, (V, ρ_N) , where ρ_N is the metric induced by N , is an ultrametric space.

Proposition. [1] All triangles in an ultrametric space (M, ρ) are either equilateral or isosceles, with at most one short side.

Proposition. [1] If S is a compact subset of an ultrametric space and Γ_S is the set of all distances occurring between points of S , then Γ_S is a discrete subset of \mathbb{R} . In particular if $|\Gamma_S| = \infty$, then the elements of Γ_S can be indexed by \mathbb{N} .

Let (M, ρ) be a compact ultrametric space and let

$$B_r(a) = \{x \in M \mid \rho(x, a) < r\}$$

denote the open ball of radius r , centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$. Likewise let

$$\overline{B_r(a)} = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the closed ball of radius r , centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$.

Proposition. Let $B_r(a)$ be a ball in an ultrametric space (M, ρ) . Then the diameter of B , $d = \text{diam}(B) = \sup_{x, y \in B} \rho(x, y)$, is less than or equal to the radius of B .

Proposition. If (M, ρ) is an ultrametric space and $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are balls in (M, ρ) , then either $B_{r_1}(x_0) \cap B_{r_2}(y_0) = \emptyset$, $B_{r_1}(x_0) \subseteq B_{r_2}(y_0)$, or $B_{r_2}(y_0) \subseteq B_{r_1}(x_0)$. That is, in an ultrametric space, all balls are either comparable or disjoint.

Proposition. [1] The distance between any two balls in an ultrametric is constant. That is, if $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are two balls in an ultrametric space (M, ρ) , then $\rho(x, y) = c$ for some $c \in \mathbb{R}$ and $\forall x \in B_{r_1}(x_0)$ and $\forall y \in B_{r_2}(y_0)$

Proposition. [1] Every point of a ball in an ultrametric is at its centre. That is, if $B_r(x_0)$ is a ball in an ultrametric space (M, ρ) , then $B_r(x) = B_r(x_0)$, $\forall x \in B_r(x_0)$

ρ -orderings, ρ -sequences, and valuative capacity

In what follows let S be a compact subset of an ultrametric space (M, ρ) .

Definition. [5] A ρ -ordering of S is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ such that $\forall n > 0$, a_n maximizes $\prod_{i=0}^{n-1} \rho(s, a_i)$ over $s \in S$.

Definition. [5] The ρ -sequence of S is the sequence whose 0th-term is 1 and whose n^{th} term, for $n > 0$, is $\prod_{i=0}^{n-1} \rho(a_n, a_i)$.

Proposition. [5] The ρ -sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ -sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ -ordering of S .

Definition. [5] Let $\gamma(n)$ be the ρ -sequence of S . The **valuative capacity** of S is

$$\omega(S) := \lim_{n \rightarrow \infty} \gamma(n)^{1/n}$$

Proposition. [5] For S and $\gamma(n)$ as above, $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = r < \infty$.

Proposition. If $S \subseteq M$ is a finite subset of an ultrametric space, then $\omega(S) = 0$.

Proposition. (upper bound) If $\text{diam}(S) := \max_{x,y \in S} \rho(x, y) = d$, then $\omega(S) < d$.

Proof. Since d is the diameter of S , the n^{th} term of the ρ -sequence of S is bounded by d^n and so $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = d$ if and only if $\gamma(n) = d^n$, $\forall n$. This implies $\rho(a_n, a_i) = d$, $\forall n$ and $\forall i < n$, but then $\rho(a_i, a_j) = d$, $\forall i, j$, since the ρ -sequence is maximized at each n . This means $\omega(S) < d$ would imply that the cover of S , $\cup_{a_i} B_d(a_i)$ is in fact an infinite partition, contradicting the compactness of S . Then $\omega(S) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} < d$. \square

Proposition. (translation invariance) Let (M, ρ) be a compact ultrametric space and suppose M is also a topological group. If ρ is (left) invariant under the group operation, then so is ω . That is, if $\rho(x, y) = \rho(gx, gy)$, $\forall g, x, y \in M$, then $\omega(gS) = \omega(S)$, for $S \subseteq M$.

Proof. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ -ordering for S . Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ -ordering for gS . Then

$$\omega(gS) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} = \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = \omega(S)$$

□

Example 1. With the notation of the previous section, note that for $x, y \in (\mathbb{Z}_p, |\cdot|_p)$, $\rho_p(x, y) = |x - y|_p = p^{-\nu_p(x-y)} = p^{-\nu_p((a+x)-(a+y))} = |(a+x) - (a+y)|_p = \rho_p(a+x, a+y)$ so that $\omega(a+S) = \omega(S)$ for $S \in (\mathbb{Z}_p, |\cdot|_p)$.

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity. Then if N is multiplicative, so is ω . That is, if $N(gx) = N(g)N(x), \forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.

Proof. Let ρ be the metric induced by N , so that $\rho(x, y) = N(x - y), \forall x, y \in V$. Let $\{a_i\}_{i=0}^\infty$ be a ρ -ordering for S . Then since N is multiplicative, for $u, v \in gS$, $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then $\{ga_i\}_{i=0}^\infty$ is a ρ -ordering for gS and

$$\begin{aligned} \omega(gS) &= \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} N(g)\rho(a_n, a_i) \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} [N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g) \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g)\omega(S) \end{aligned}$$

□

Example 2. Since $|\cdot|_p$ is multiplicative, $\omega(mS) = |m|_p \omega(S)$ for $m \in \mathbb{Z}_p$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) = |p|_p \omega(\mathbb{Z}) = \frac{1}{p} * p^{\frac{1}{1-p}} = p^{-p/p-1}$.

Proposition. [5](subadditivity) If $\text{diam}(S) := \max_{x,y \in S} \rho(x, y) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then

$$\frac{1}{\log(\omega(S)/d)} = \sum_{i=1}^n \frac{1}{\log(\omega(A_i)/d)}$$

Corollary. Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = \text{diam}(S)$ and also $\omega(S_i) = \omega(S_j), \forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S), \forall i$. Then $\omega(S) = r^{\frac{1}{n-1}} \cdot d$. In particular if $S = \mathbb{Z}$ and

$(M, \rho) = (\mathbb{Z}, |\cdot|_p)$ then $\omega(S) = (\frac{1}{p})^{1/p-1}$ for any prime p .

Corollary. (Joins of computable sets are computable) Let $\Gamma_M = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$ be the set of distances in M . Suppose that $S = B_{\gamma_i}(x)$, for some x and i , is the union of 2 or more balls of radius γ_{i+1} , i.e., $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$ is a join in the lattice of open sets in M , then

$$\frac{1}{\log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^n \frac{1}{\log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

Computing a ρ -ordering

We describe an algorithm for computing the ρ -ordering of a set recursively and discuss some immediate corollaries.

Let $S \subseteq M$ be a compact subset of an ultrametric space (M, ρ) . Let $\Gamma_S = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$ be the set of distances in S . Note that for each $k \in \mathbb{N}$, the closed balls of radius γ_k partition S , i.e., $S = S_{\gamma_k} := \cup_{i=1}^n \overline{B_{\gamma_k}(x_i)}$, where both n and the x_i 's depend on k . In what follows, fix a $k \in \mathbb{N}$ and let $S_{\gamma_k} = \cup_{i=1}^n \overline{B_{\gamma_k}(x_i)}$ be such a partition. Note that we can construct $S_{\gamma_{k+1}}$ by partitioning each of the $\overline{B_{\gamma_k}(x_i)}$, i.e.,

$$S = S_{\gamma_{k+1}} = \cup_{i=1}^n \cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})}$$

where $1 \leq l_i \leq n$ and $\cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})} = \overline{B_{\gamma_k}(x_i)}$, $\forall i$. We denote by $x_{i,j}$ the centre of a ball of radius γ_{k+1} partitioning the ball $B_{\gamma_k}(x_i)$. Without loss of generality, when $j = 1$, assume $x_{i,j} = x_i$, $\forall i$.

We now make the following observation due to [6],

Lemma. For each k , the elements of S_{γ_k} , that is, the closed balls of radius γ_k , themselves form an ultrametric space, where

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

We note that since S is assumed to be compact, S_{γ_k} is a finite metric space $\forall k < \infty$ and $S_{\gamma_\infty} = \cup_{x \in S} \overline{B_0(x)} = \cup_{x \in S} x = S$ and $\rho_\infty = \rho$. Now view S_{γ_k} , for fixed $k < \infty$ as a finite ultrametric space and represent its $n < \infty$ elements by their centres, the x_i 's. Without loss of generality, we can reindex the x_i 's so that they give the first n terms of a ρ_k -ordering of S_{γ_k} . The following proposition is the main result of this section.

Proposition. Given S a compact subset of an ultrametric space M and Γ_S , the set of distances in S , if S_{γ_k} is a partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the centres

of the balls are indexed according to a ρ_k -ordering of S_{γ_k} , then a ρ_{k+1} -ordering of $S_{\gamma_{k+1}}$ can be found by forming a matrix, A_k , whose $(i, j)^{th}$ -entry is $x_{i,j}$, as shown below, and then concatenating the rows (where the columns are padded by $*$ if necessary).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \dots & x_{n,l_n} \end{pmatrix}$$

Proof. Note that the entries in each column are points in the ball $B_{\gamma_k}(x_i)$ so that the pairwise distance between columns is constant and always exceeds the distance between elements within a column. Moreover, the columns are organized such that for any j , $x_{n,j}$ maximizes $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j})$ since $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j}) = \prod_{i=1}^{n-1} \rho(x_{n,1}, x_{i,1}) = \prod_{i=1}^{n-1} \rho(x_n, x_i)$ and the x_i 's are indexed according a ρ_k -ordering of S_{γ_k} .

Then a $\rho_{\gamma_{k+1}}$ -ordering of $S_{\gamma_{k+1}}$ is obtained by minimizing the number of elements from any one column and by taking the points $x_{i,j}$ (for fixed j) in sequence. For example, by concatenating the rows. □

Corollary. Interweaving the bottown row of the lattice of closed balls for a set S gives a ρ -ordering of S .

Corollary. Suppose S and T are compact subsets of an ultrametric space M with $\Gamma_S = \Gamma_T$ and $|S_{\gamma_k}| = |T_{\gamma_k}|$, $\forall k$. Then $\omega(S) = \omega(T)$.

Corollary. (regularity) Suppose that S is such that $\forall k$, any $B_{\gamma_k}(x) = \cup_{j=1}^l B_{\gamma_{k+1}}(x_j)$, that is, every ball in S breaks into exactly l smaller balls.

product space

As a first point of departure, a natural space to consider is the product space, \mathbb{Z}^n (or \mathbb{Z}_p^n or \mathbb{Q}_p^n), for some $n > 1$. A natural candidate for an ultrametric on the space is the L_∞ metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sup_{i \in 1:n} (\rho(x_i, y_i))$$

where ρ is the metric from the base space. Show this an ultrametric, its translation invariant and comes from a multiplicative norm.

Example 3. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$ and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2))$. Consider it also as a topological group with operation $(g_1, g_2) + (x_1, x_2) = (g_1 + x_1, g_2 + x_2)$. Then $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2)) = \max(\rho_p(g_1 + x_1, g_1 + y_1), \rho_p(g_2 + x_2, g_2 + y_2)) = \rho_{p,\infty}((g_1, g_2) + (x_1, x_2), ((g_1, g_2) + (y_1, y_2)))$, and $\omega((g_1, g_2) + S) = \omega(S)$ for $S \in (\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$.

Example 4. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, |\cdot|_{p,\infty})$ be the vector space with elements $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$ and norm $|(x_1, x_2)|_{p,\infty} = \max(|x_1|_p, |x_2|_p)$. Then $|(g, g)(x_1, x_2)|_{p,\infty} = \max(|gx_1|_p, |gx_2|_p) = \max(|g|_p |x_1|_p, |g|_p |x_2|_p) = |g|_p \max(|x_1|_p, |x_2|_p) = |g|_p |(x_1, x_2)|_{p,\infty}$, so that $|\cdot|_{p,\infty}$ is multiplicative for $(g, g) \in \mathbb{Z}_p \times \mathbb{Z}_p$. Then $\omega((g, g)S) = |g|_p \omega(S)$. In particular, $\omega((p, p)\mathbb{Z} \times \mathbb{Z}) = |p|_p \omega(\mathbb{Z} \times \mathbb{Z}) = p^{-1} \omega(\mathbb{Z} \times \mathbb{Z})$.

projective space

Background from Gerritzen and van der Put

Background results from [7]. Let k be a field that is complete with respect to a non-archimedean valuation and let K be a complete and algebraically closed field containing k .

Definition. [7] The set $\{\lambda \in k; |\lambda| \leq 1\}$, denoted k^0 , is the **valuation ring** of k . It has a unique maximal ideal, denoted k^{00} , given by $\{\lambda \in k; |\lambda| < 1\}$. The **residue field** of k is $\bar{k} := k^0/k^{00}$.

Definition. [7] The **projective line over k** , denoted $\mathbb{P}^1(k)$, is the space whose points are lines l in k^2 that intersect $(0,0)$ and whose topology and field structure are inherited from k .

We give two equivalent representations for the points of $\mathbb{P}^1(k)$. A point $p \in \mathbb{P}^1(k)$ is an equivalence class of $k^2 \setminus (0,0)$ under the relation $(x,y) \sim (x',y')$ if there exists a $\lambda \in k \setminus 0$ such that $(x,y) = \lambda(x',y')$. Equivalently, suppose that l is a line in k^2 intersecting the origin, that is a point in $\mathbb{P}^1(k)$. We denote l by a representative $[x_0, x_1] \in k^2$ such that $l = \{\lambda(x_0, x_1); \lambda \in k\}$, called homogeneous coordinates of l .

Proposition. [7] Let $\psi : k \rightarrow \mathbb{P}^1(k)$ be the map given by $\psi(\lambda_0) = [1, \lambda_0]$, where $[1, \lambda_0]$ is the line in k^2 , $\{\lambda(1, \lambda_0); \lambda \in k\}$. Then the image of ψ is $\mathbb{P}^1(k) \setminus [0, 1]$ and is isomorphic to k , so that k is identified with projective space minus a distinguished point, $[0, 1]$, which is denoted by ∞ .

Definition. [7] k is called a **local field** if k is locally compact.

Proposition. [7] The following are equivalent:

1. k is a local field.
2. $|k^*| \cong \mathbb{Z}$ and \bar{k} is finite, where k^* is the set of units in k , ie $k^* = \{\lambda \in k, \lambda \neq 0\}$.
3. k is a finite extension of either \mathbb{Q}_p or $\mathbb{F}_p((t))$.
4. $\mathbb{P}^1(k)$ is compact

Definition. [7] We denote by $GL(2, k)$ the set of invertible 2×2 matrices over k . A **fractional linear automorphism**, ϕ , of $\mathbb{P}^1(k)$ is a map defined by $z \mapsto \frac{az+b}{cz+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, k)$. The set of fractional linear automorphisms of $\mathbb{P}^1(k)$ is denoted $PGL(2, k)$. Note that $PGL(2, k) = GL(2, k) / \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \lambda \in k^* \}$. In homogeneous coordinates, we can represent the action of ϕ by $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$.

Definition. [7] Suppose Γ is a subgroup of $PGL(2, k)$. A point $p \in \mathbb{P}^1(k)$ is a **limit point of** Γ , if there exists a point q in $\mathbb{P}^1(k)$ and a sequence $\{\gamma_n\}_{n \geq 1}$ in Γ such that $\lim_{n \rightarrow \infty} \gamma_n(q) = p$.

Proposition. [7] If Γ is not a discrete subgroup of $PGL(2, k)$ then every point of $\mathbb{P}^1(k)$ is a limit point of Γ .

Proof. Since Γ is not discrete, the sequence $\{\gamma_n\}_{n \geq 1}$ has a limit γ in Γ . Let p be any point of $\mathbb{P}^1(k)$ and let $q = \gamma^{-1}(p)$. Then $\lim_{n \rightarrow \infty} \gamma_n(q) = \lim_{n \rightarrow \infty} \gamma_n(\gamma^{-1}(p)) = p$. \square

Definition. [7] A subgroup Γ of $PGL(2, k)$ is **discontinuous** if the closure of every orbit of Γ in $\mathbb{P}^1(k)$ is compact and the set of all limit points of Γ is not equal to $\mathbb{P}^1(k)$.

Proposition. [7] If Γ is a discontinuous subgroup of $PGL(2, k)$ and \mathcal{L} is the set of limit points of Γ , then \mathcal{L} is compact, nowhere dense and if \mathcal{L} contains more than two points, \mathcal{L} is perfect.

Definition. [7] Let A be an element of $GL(2, k)$ and let a_1 and a_2 be the eigenvalues of A . Then A is called **elliptic** if $a_1 \neq a_2$, but $|a_1| = |a_2|$. A is called **parabolic** if $a_1 = a_2$, and A is called **hyperbolic** if $|a_1| \neq |a_2|$.

Example 5. Consider the matrix $T_s = \begin{pmatrix} p & s \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q}_p)$ for some s in $(0, \dots, p-1)$ (note that $\det(T_s) = p$ is invertible in \mathbb{Q}_p , so that $T_s \in GL(2, \mathbb{Q}_p)$, although it is not in $GL(2, \mathbb{Z}_p)$). T_s has eigenvalues p and 1 and so T_s is hyperbolic for any choice of s or p . Consider the action of T_s on $\mathbb{Z}_p \subset \mathbb{Q}_p$, where \mathbb{Z}_p is identified with the subspace $\{[1, \lambda]; \lambda \in \mathbb{Z}_p\}$ of $\mathbb{P}^1(\mathbb{Q}_p)$. In homogeneous

coordinates, this action is given by $[1, \lambda] \mapsto [1, p\lambda + s]$. Since $| (p\lambda + s - s) | = | p\lambda | \leq \frac{1}{p}$, T_s sends λ to $B_{\frac{1}{p}}(s)$. Also note that for $s = 0$, T_s has the effect of shifting the index of λ by 1, that is, if $\lambda = \sum_{i=n}^{\infty} a_i p^i$, where $n = \text{ord}(\lambda)$, then $T_0([1, \lambda]) = [1, p\lambda] \rightsquigarrow p\lambda = \sum_{i=n+1}^{\infty} a_{(i-1)} p^i$.

Computation of the capacity of some sets

(F&P, section 5)

Setup

Let $A = \{0, 1, \dots, d-1\}$ be a finite alphabet and $A^{\mathbb{N}}$ be the collection of infinite sequences with values in A . Note $A^{\mathbb{N}}$ is a Cantor set, so it is perfect, nowhere dense, and compact.

A basis for the topology is given by the cylinder set: take countably many copies of $\{0, 1, \dots, d-1\}$ where each copy has the discrete topology.

Let $p \geq d$ be a prime number and let ϕ be the canonical embedding of $A^{\mathbb{N}}$ into \mathbb{Z}_p via the following continuous (under the above topology) map:

$$\phi : A^{\mathbb{N}} \rightarrow \mathbb{Z}_p \text{ by } (x_n)_{n \geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

Lemma. (F&P Lemma 5.1)

Let w_1, w_2, \dots, w_s be $s \geq 2$ words with the same length l such that all the first letters are distinct. Let $X \subset A^{\mathbb{N}}$ be the set of sequences such that any factor is a factor of a concatenation of the words w_1, w_2, \dots, w_s . Then the set $E := \phi(X) \subset \mathbb{Z}_p$ satisfies:

$$E = \cup_{i=1}^s x_i + p^l E, \text{ with } x_i = \phi(w_i 0^\infty)$$

It is a regular compact set and its valutive capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

Example 6. $w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$

Then $\{x_n\}_{n \geq 0} \in X$ if each term in $\{x_n\}_{n \geq 0}$ is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Note that we can rephrase the lemma as follows:

Let x_1, x_2, \dots, x_s be $s \geq 2$ points in \mathbb{Z}_p such that $|x_i - x_j|_p = 1, \forall i, j \in 1, \dots, s$. Suppose also that there exists an $l \in \mathbb{N}$ such that $\forall i$,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^l a_i p^i$$

Let γ_i be the fractional linear automorphism of $\mathbb{P}^1(\mathbb{Q}_p)$ given by $\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix}$ and let Γ be the subgroup of $PGL(2, \mathbb{Q}_p)$ generated by the γ_i .

Then Γ has a subgroup H such that the limit set \mathcal{L} of H has the property that $Z = \psi^{-1}(\mathcal{L})$ is equal to $\phi(X)$ in the original lemma. In particular Z is a regular, compact subset of \mathbb{Z}_p satisfying

$$Z = \cup_{i=1}^s x_i + p^l Z = \cup_{i=1}^s B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

Proof. We must show that the set Z above is equal to $E = \phi(X)$ in the original lemma. First note that if w_1, w_2, \dots, w_s are words in $A^{\mathbb{N}}$, then the first letter of each w_i is distinct if and only if $|\phi(w_i) - \phi(w_j)|_p = 1, \forall i, j$ (since the pairwise distance is 1 if and only if $\text{ord}(\phi(w_i) - \phi(w_j)) = 0$ for any i and j , if and only if the coefficient of p^0 (i.e., the first letter each w_i) is different $\forall i, j$). So then the x_i are just the $\phi(w_i)$.

Now consider the limit set of Γ . Let $\gamma \in \Gamma$. If γ is a product of the generators γ_i , then γ is represented by a matrix of the form: $\begin{pmatrix} p^{lm} & z_m \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and z_m is an element of \mathbb{Z}_p whose coefficient vector is a concatenation of the coefficient vectors of the x_i (for $0 \leq i \leq ml$ and 0 for $i > ml$). For example, $\gamma_i \gamma_j \gamma_k =$

$$\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^l & x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^l & x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{3l} & p^{2l}x_k + p^l x_j + x_i \\ 0 & 1 \end{pmatrix}$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{lm}a_1 + za_0] \sim [1, p^{lm}\frac{a_1}{a_0} + z_m]$$

As m tends to infinity, this point tends to $[1, 0 + z]$, where z is an element of \mathbb{Z}_p whose entire coefficient vector is a concatenation of the x_i 's. The set $\psi^{-1}([1, z])$ for all such z is exactly the set $E = \phi(X)$.

Now suppose γ is a product of the inverses of the generators γ_i^{-1} , then γ is represented by a matrix of the form: $\begin{pmatrix} p^{-lm} & -p^{-l}z^{-1} \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and z is as above. For example,

$$\begin{pmatrix} p^{-l} & -p^{-l}x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{-3l} & -p^{-3l}x_k - p^{-2l}x_j - p^{-l}x_i \\ 0 & 1 \end{pmatrix}$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{-lm}a_1 - p^{-lm}za_0] \sim [1, p^{-lm}(\frac{a_1}{a_0} - z)]$$

This time as m grows, the image approaches infinity and so has empty preimage under ψ .

Lastly, we consider elements of Γ made up of both generators and the inverse of generators. These elements will produce translations, either of the form $\begin{pmatrix} 1 & p^{-l}(x_i - x_j) \\ 0 & 1 \end{pmatrix}$ if $\gamma = \gamma_j^{-1}\gamma_i$ or $\begin{pmatrix} 1 & x_j - x_i \\ 0 & 1 \end{pmatrix}$ if $\gamma = \gamma_j\gamma_i^{-1}$. These elements commute with each other and so the subgroup which they generate is normal. We can quotient by the entire translation subgroup, ie the sub-

group generated by $\{\gamma_i\gamma_j^{-1}, \gamma_i^{-1}\gamma_j; \forall i, j \in 1, \dots, s\}$ to obtain H . Then $\mathcal{L} = \infty \cup \{[1, z] \mid z \in \mathbb{Z}_p\}$ and the coefficient vector of z is a concatenation of the coefficient vectors of the x_i 's and $\psi^{-1}(\mathcal{L}) = E = \phi(X)$, as required.

□

More background:

Definition. [7] A **Schottky group** is a finitely-generated, free and discontinuous subgroup of $PGL(2, k)$

algebraic extensions

Type up the example Keith gave you.

Bibliography

- [1] Alain M. Robert, A course in p-adic analysis.
- [2] Manjul Bhargava, The factorial function and generalizations
- [3] P -orderings and polynomial functions on arbitrary subsets of Dedekind rings.
- [4] On P -orderings, rings of integer-valued polynomials, and ultrametric analysis
- [5] Keith Johnson, P -orderings and Fekete sets
- [6] Nate Ackerman, Completeness in Generalized Ultrametric Spaces
- [7] Lothar Gerritzen and Marius van der Put, Schottky Groups and Mumford Curves.