

# Introduction

In the course of developing a generalized factorial function, Bhargava introduced the notion of  $p$ -orderings of a Dedekind domain [2, 3], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [4] (also KJ) and have also provided a natural framework to extend many classical results in analysis to a  $p$ -adic setting, such as polynomial approximation and mapping theorems [2, 3, 4].

In this thesis, we examine how a tool based on  $p$ -orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

# Background

## Ultrametric basics

**Definition.** Let  $(M, \rho)$  be a metric space. If  $\rho$  satisfies the ultrametric inequality

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)), \forall x, y, z \in M$$

then  $(M, \rho)$  is an **ultrametric space**.

**Definition.** Let  $(V, N)$  be a normed vector space. Then  $N$  satisfies the **strong triangle inequality** if

$$N(x + y) \leq \max(N(x), N(y)), \forall x, y \in V$$

**Proposition.** Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle inequality. Then the metric space,  $(V, \rho_N)$ , where  $\rho_N$  is the metric induced by  $N$ , is an ultrametric space.

**Proposition.** [1] All triangles in an ultrametric space  $(M, \rho)$  are either equilateral or isosceles, with at most one short side.

**Proposition.** [1] If  $S$  is a compact subset of an ultrametric space and  $\Gamma_S$  is the set of all distances occurring between points of  $S$ , then  $\Gamma_S$  is a discrete subset of  $\mathbb{R}$ . In particular if  $|\Gamma_S| = \infty$ , then the elements of  $\Gamma_S$  can be indexed by  $\mathbb{N}$ .

Let  $(M, \rho)$  be a compact ultrametric space and let

$$B_r(a) = \{x \in M \mid \rho(x, a) < r\}$$

denote the open ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{\geq 0}$  and  $a \in (M, \rho)$ . Likewise let

$$\overline{B_r(a)} = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the closed ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{\geq 0}$  and  $a \in (M, \rho)$ .

**Proposition.** Let  $B_r(a)$  be a ball in an ultrametric space  $(M, \rho)$ . Then the diameter of  $B$ ,  $d = \text{diam}(B) = \sup_{x, y \in B} \rho(x, y)$ , is less than or equal to the radius of  $B$ .

**Proposition.** If  $(M, \rho)$  is an ultrametric space and  $B_{r_1}(x_0)$  and  $B_{r_2}(y_0)$  are balls in  $(M, \rho)$ , then either  $B_{r_1}(x_0) \cap B_{r_2}(y_0) = \emptyset$ ,  $B_{r_1}(x_0) \subseteq B_{r_2}(y_0)$ , or  $B_{r_2}(y_0) \subseteq B_{r_1}(x_0)$ . That is, in an ultrametric space, all balls are either comparable or disjoint.

**Proposition.** [1] The distance between any two balls in an ultrametric is constant. That is, if  $B_{r_1}(x_0)$  and  $B_{r_2}(y_0)$  are two balls in an ultrametric space  $(M, \rho)$ , then  $\rho(x, y) = c$  for some  $c \in \mathbb{R}$  and  $\forall x \in B_{r_1}(x_0)$  and  $\forall y \in B_{r_2}(y_0)$

**Proposition.** [1] Every point of a ball in an ultrametric is at its centre. That is, if  $B_r(x_0)$  is a ball in an ultrametric space  $(M, \rho)$ , then  $B_r(x) = B_r(x_0)$ ,  $\forall x \in B_r(x_0)$

## $\rho$ -orderings, $\rho$ -sequences, and valuative capacity

In what follows let  $S$  be a compact subset of an ultrametric space  $(M, \rho)$ .

**Definition.** [5] A  $\rho$ -ordering of  $S$  is a sequence  $\{a_i\}_{i=0}^{\infty} \subseteq S$  such that  $\forall n > 0$ ,  $a_n$  maximizes  $\prod_{i=0}^{n-1} \rho(s, a_i)$  over  $s \in S$ .

**Definition.** [5] The  $\rho$ -sequence of  $S$  is the sequence whose 0<sup>th</sup>-term is 1 and whose  $n^{\text{th}}$  term, for  $n > 0$ , is  $\prod_{i=0}^{n-1} \rho(a_n, a_i)$ .

**Proposition.** [5] The  $\rho$ -sequence of  $S$  is well-defined so long as  $S$  is compact and  $\rho$  is an ultrametric. That is, the  $\rho$ -sequence of a compact subset of an ultrametric spaces does not depend on the choice of  $\rho$ -ordering of  $S$ .

**Definition.** [5] Let  $\gamma(n)$  be the  $\rho$ -sequence of  $S$ . The **valuative capacity** of  $S$  is

$$\omega(S) := \lim_{n \rightarrow \infty} \gamma(n)^{1/n}$$

**Proposition.** [5] For  $S$  and  $\gamma(n)$  as above,  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = r < \infty$ .

**Proposition.** If  $S \subseteq M$  is a finite subset of an ultrametric space, then  $\omega(S) = 0$ .

**Proposition.** (upper bound) If  $\text{diam}(S) := \max_{x,y \in S} \rho(x, y) = d$ , then  $\omega(S) < d$ .

*Proof.* Since  $d$  is the diameter of  $S$ , the  $n^{\text{th}}$  term of the  $\rho$ -sequence of  $S$  is bounded by  $d^n$  and so  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = d$  if and only if  $\gamma(n) = d^n$ ,  $\forall n$ . This implies  $\rho(a_n, a_i) = d$ ,  $\forall n$  and  $\forall i < n$ , but then  $\rho(a_i, a_j) = d$ ,  $\forall i, j$ , since the  $\rho$ -sequence is maximized at each  $n$ . This means  $\omega(S) < d$  would imply that the cover of  $S$ ,  $\cup_{a_i} B_d(a_i)$  is in fact an infinite partition, contradicting the compactness of  $S$ . Then  $\omega(S) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} < d$ .  $\square$

**Proposition.** (translation invariance) Let  $(M, \rho)$  be a compact ultrametric space and suppose  $M$  is also a topological group. If  $\rho$  is (left) invariant under the group operation, then so is  $\omega$ . That is, if  $\rho(x, y) = \rho(gx, gy)$ ,  $\forall g, x, y \in M$ , then  $\omega(gS) = \omega(S)$ , for  $S \subseteq M$ .

*Proof.* Let  $\{a_i\}_{i=0}^{\infty}$  be a  $\rho$ -ordering for  $S$ . Then  $\{ga_i\}_{i=0}^{\infty}$  is a  $\rho$ -ordering for  $gS$ . Then

$$\omega(gS) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = \omega(S)$$

□

**Example 1.** With the notation of the previous section, note that for  $x, y \in (\mathbb{Z}_p, |\cdot|_p)$ ,  $\rho_p(x, y) = |x - y|_p = p^{-\nu_p(x-y)} = p^{-\nu_p((a+x)-(a+y))} = |(a+x) - (a+y)|_p = \rho_p(a+x, a+y)$  so that  $\omega(a+S) = \omega(S)$  for  $S \in (\mathbb{Z}_p, |\cdot|_p)$ .

**Proposition.** Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle identity. Then if  $N$  is multiplicative, so is  $\omega$ . That is, if  $N(gx) = N(g)N(x), \forall g, x \in V$ , then  $\omega(gS) = N(g)\omega(S)$ , for  $g \in V$  and  $S \subseteq M$ .

*Proof.* Let  $\rho$  be the metric induced by  $N$ , so that  $\rho(x, y) = N(x - y), \forall x, y \in V$ . Let  $\{a_i\}_{i=0}^\infty$  be a  $\rho$ -ordering for  $S$ . Then since  $N$  is multiplicative, for  $u, v \in gS$ ,  $u = gs_i$  and  $v = gs_j$  for some  $s_i, s_j \in S$ ,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then  $\{ga_i\}_{i=0}^\infty$  is a  $\rho$ -ordering for  $gS$  and

$$\begin{aligned} \omega(gS) &= \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} N(g)\rho(a_n, a_i) \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} [N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g) \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g)\omega(S) \end{aligned}$$

□

**Example 2.** Since  $|\cdot|_p$  is multiplicative,  $\omega(mS) = |m|_p \omega(S)$  for  $m \in \mathbb{Z}_p$  and  $S \subseteq \mathbb{Z}$ . In particular,  $\omega(p\mathbb{Z}) = |p|_p \omega(\mathbb{Z}) = \frac{1}{p} * p^{\frac{1}{1-p}} = p^{-p/p-1}$ .

**Proposition.** [5](subadditivity) If  $\text{diam}(S) := \max_{x,y \in S} \rho(x, y) = d$  and  $S = \cup_i^n A_i$  for  $A_i$  compact subsets of  $M$  with  $\rho(A_i, A_j) = d, \forall i, j$ , then

$$\frac{1}{\log(\omega(S)/d)} = \sum_{i=1}^n \frac{1}{\log(\omega(A_i)/d)}$$

**Corollary.** Suppose  $S = \cup_i^n S_i$  with  $\rho(S_i, S_j) = d = \text{diam}(S)$  and also  $\omega(S_i) = \omega(S_j), \forall i, j$ . Let  $r \in \mathbb{R}$  be such that  $\omega(S_i) = r\omega(S), \forall i$ . Then  $\omega(S) = r^{\frac{1}{n-1}} \cdot d$ . In particular if  $S = \mathbb{Z}$  and

$(M, \rho) = (\mathbb{Z}, |\cdot|_p)$  then  $\omega(S) = (\frac{1}{p})^{1/p-1}$  for any prime  $p$ .

**Corollary.** (Joins of computable sets are computable) Let  $\Gamma_M = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$  be the set of distances in  $M$ . Suppose that  $S = B_{\gamma_i}(x)$ , for some  $x$  and  $i$ , is the union of 2 or more balls of radius  $\gamma_{i+1}$ , i.e.,  $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$  is a join in the lattice of open sets in  $M$ , then

$$\frac{1}{\log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^n \frac{1}{\log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

## Computing a $\rho$ -ordering

We describe an algorithm for computing the  $\rho$ -ordering of a set recursively and discuss some immediate corollaries.

Let  $S \subseteq M$  be a compact subset of an ultrametric space  $(M, \rho)$ . Let  $\Gamma_S = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$  be the set of distances in  $S$ . Note that for each  $k \in \mathbb{N}$ , the closed balls of radius  $\gamma_k$  partition  $S$ , i.e.,  $S = S_{\gamma_k} := \cup_{i=1}^n \overline{B_{\gamma_k}(x_i)}$ , where both  $n$  and the  $x_i$ 's depend on  $k$ . In what follows, fix a  $k \in \mathbb{N}$  and let  $S_{\gamma_k} = \cup_{i=1}^n \overline{B_{\gamma_k}(x_i)}$  be such a partition. Note that we can construct  $S_{\gamma_{k+1}}$  by partitioning each of the  $\overline{B_{\gamma_k}(x_i)}$ , i.e.,

$$S = S_{\gamma_{k+1}} = \cup_{i=1}^n \cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})}$$

where  $1 \leq l_i \leq n$  and  $\cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})} = \overline{B_{\gamma_k}(x_i)}$ ,  $\forall i$ . We denote by  $x_{i,j}$  the centre of a ball of radius  $\gamma_{k+1}$  partitioning the ball  $B_{\gamma_k}(x_i)$ . Without loss of generality, when  $j = 1$ , assume  $x_{i,j} = x_i$ ,  $\forall i$ .

We now make the following observation due to [6],

**Lemma.** For each  $k$ , the elements of  $S_{\gamma_k}$ , that is, the closed balls of radius  $\gamma_k$ , themselves form an ultrametric space, where

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

We note that since  $S$  is assumed to be compact,  $S_{\gamma_k}$  is a finite metric space  $\forall k < \infty$  and  $S_{\gamma_\infty} = \cup_{x \in S} \overline{B_0(x)} = \cup_{x \in S} x = S$  and  $\rho_\infty = \rho$ . Now view  $S_{\gamma_k}$ , for fixed  $k < \infty$  as a finite ultrametric space and represent its  $n < \infty$  elements by their centres, the  $x_i$ 's. Without loss of generality, we can reindex the  $x_i$ 's so that they give the first  $n$  terms of a  $\rho_k$ -ordering of  $S_{\gamma_k}$ . The following proposition is the main result of this section.

**Proposition.** Given  $S$  a compact subset of an ultrametric space  $M$  and  $\Gamma_S$ , the set of distances in  $S$ , if  $S_{\gamma_k}$  is a partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the centres

of the balls are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , then a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  can be found by forming a matrix,  $A_k$ , whose  $(i, j)^{th}$ -entry is  $x_{i,j}$ , as shown below, and then concatenating the rows (where the columns are padded by  $*$  if necessary).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \dots & x_{n,l_n} \end{pmatrix}$$

*Proof.* Note that the entries in each column are points in the ball  $B_{\gamma_k}(x_i)$  so that the pairwise distance between columns is constant and always exceeds the distance between elements within a column. Moreover, the columns are organized such that for any  $j$ ,  $x_{n,j}$  maximizes  $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j})$  since  $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j}) = \prod_{i=1}^{n-1} \rho(x_{n,1}, x_{i,1}) = \prod_{i=1}^{n-1} \rho(x_n, x_i)$  and the  $x_i$ 's are indexed according a  $\rho_k$ -ordering of  $S_{\gamma_k}$ .

Then a  $\rho_{\gamma_{k+1}}$ -ordering of  $S_{\gamma_{k+1}}$  is obtained by minimizing the number of elements from any one column and by taking the points  $x_{i,j}$  (for fixed  $j$ ) in sequence. For example, by concatenating the rows. □

**Corollary.** Interweaving the bottown row of the lattice of closed balls for a set  $S$  gives a  $\rho$ -ordering of  $S$ .

**Corollary.** Suppose  $S$  and  $T$  are compact subsets of an ultrametric space  $M$  with  $\Gamma_S = \Gamma_T$  and  $|S_{\gamma_k}| = |T_{\gamma_k}|$ ,  $\forall k$ . Then  $\omega(S) = \omega(T)$ .

**Corollary.** (regularity) Suppose that  $S$  is such that  $\forall k$ , any  $B_{\gamma_k}(x) = \cup_{j=1}^l B_{\gamma_{k+1}}(x_j)$ , that is, every ball in  $S$  breaks into exactly  $l$  smaller balls.



# product space

As a first point of departure, a natural space to consider is the product space,  $\mathbb{Z}^n$  (or  $\mathbb{Z}_p^n$  or  $\mathbb{Q}_p^n$ ), for some  $n > 1$ . A natural candidate for an ultrametric on the space is the  $L_\infty$  metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sup_i \{\rho(x_i, y_i)\}$$

where  $\rho$  is the metric from the base space. Then  $\rho_\infty$  is easily seen to be an ultrametric, so long as  $\rho$  is. Indeed, let  $(M, \rho)$  be an ultrametric space and consider two points,  $x$  and  $y$ , in the product space,  $M^n$  for some  $n > 1$ . Clearly,  $\rho_\infty(x, y) \geq 0$  since each  $\rho(x_i, y_i) \geq 0$ , and  $\rho_\infty(x, y) = 0 \iff \rho(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$ . The fact that  $\rho_\infty$  is symmetric is also an easy consequence of the fact that  $\rho$  is symmetric since  $\rho(x_i, y_i) = \rho(y_i, x_i)$  implies  $\sup_i \{\rho(x_i, y_i)\} = \sup_i \{\rho(y_i, x_i)\}$ . To see that  $\rho_\infty$  is an ultrametric, note that since  $\rho$  is an ultrametric  $\rho(x_i, y_i) \leq \max\{\rho(x_i, z_i), \rho(y_i, z_i)\}$ , where  $z_i$  is any other point of  $M$ .

We show a few quick results on the product space.

**Proposition.**  $\rho_\infty$  is translation invariant if  $\rho$  is, in which valutive capacity is translation invariant.

**Proposition.**  $\rho_\infty$  is multiplicative if  $\rho$  is, in which valutive capacity is multiplicative.

Show this an ultrametric, its translation invariant and comes from a multiplicative norm.

**Example 3.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  be the metric space with elements  $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$  and metric  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2))$ . Consider it also as a topological group with operation  $(g_1, g_2) + (x_1, x_2) = (g_1 + x_1, g_2 + x_2)$ . Then  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2)) = \max(\rho_p(g_1 + x_1, g_1 + y_1), \rho_p(g_2 + x_2, g_2 + y_2)) = \rho_{p,\infty}(((g_1, g_2) + (x_1, x_2)), ((g_1, g_2) + (y_1, y_2)))$ , and  $\omega((g_1, g_2) + S) = \omega(S)$  for  $S \in (\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ .

**Example 4.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p, |\cdot|_{p,\infty})$  be the vector space with elements  $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$  and norm  $|(x_1, x_2)|_{p,\infty} = \max(|x_1|_p, |x_2|_p)$ . Then  $|(g, g)(x_1, x_2)|_{p,\infty} = \max(|gx_1|_p, |gx_2|_p) = \max(|g|_p |x_1|_p, |g|_p |x_2|_p) = |g|_p \max(|x_1|_p, |x_2|_p) = |g|_p |(x_1, x_2)|_{p,\infty}$ , so that  $|\cdot|_{p,\infty}$  is multiplicative for  $(g, g) \in \mathbb{Z}_p \times \mathbb{Z}_p$ . Then  $\omega((g, g)S) = |g|_p \omega(S)$ . In particular,  $\omega((p, p)\mathbb{Z} \times \mathbb{Z}) = \omega(p\mathbb{Z} \times p\mathbb{Z}) = |p|_p \omega(\mathbb{Z} \times \mathbb{Z}) = p^{-1} \omega(\mathbb{Z} \times \mathbb{Z})$ .

# projective space

## Background from Gerritzen and van der Put

Background results from [7]. Let  $k$  be a field that is complete with respect to a non-archimedean valuation and let  $K$  be a complete and algebraically closed field containing  $k$ .

**Definition.** [7] The set  $\{\lambda \in k; |\lambda| \leq 1\}$ , denoted  $k^0$ , is the **valuation ring** of  $k$ . It has a unique maximal ideal, denoted  $k^{00}$ , given by  $\{\lambda \in k; |\lambda| < 1\}$ . The **residue field** of  $k$  is  $\bar{k} := k^0/k^{00}$ .

**Definition.** [7] The **projective line over  $k$** , denoted  $\mathbb{P}^1(k)$ , is the space whose points are lines  $l$  in  $k^2$  that intersect  $(0,0)$  and whose topology and field structure are inherited from  $k$ .

We give two equivalent representations for the points of  $\mathbb{P}^1(k)$ . A point  $p \in \mathbb{P}^1(k)$  is an equivalence class of  $k^2 \setminus (0,0)$  under the relation  $(x,y) \sim (x',y')$  if there exists a  $\lambda \in k \setminus 0$  such that  $(x,y) = \lambda(x',y')$ . Equivalently, suppose that  $l$  is a line in  $k^2$  intersecting the origin, that is a point in  $\mathbb{P}^1(k)$ . We denote  $l$  by a representative  $[x_0, x_1] \in k^2$  such that  $l = \{\lambda(x_0, x_1); \lambda \in k\}$ , called homogeneous coordinates of  $l$ .

**Proposition.** [7] Let  $\psi : k \rightarrow \mathbb{P}^1(k)$  be the map given by  $\psi(\lambda_0) = [1, \lambda_0]$ , where  $[1, \lambda_0]$  is the line in  $k^2$ ,  $\{\lambda(1, \lambda_0); \lambda \in k\}$ . Then the image of  $\psi$  is  $\mathbb{P}^1(k) \setminus [0, 1]$  and is isomorphic to  $k$ , so that  $k$  is identified with projective space minus a distinguished point,  $[0, 1]$ , which is denoted by  $\infty$ .

**Definition.** [7]  $k$  is called a **local field** if  $k$  is locally compact.

**Proposition.** [7] The following are equivalent:

1.  $k$  is a local field.
2.  $|k^*| \cong \mathbb{Z}$  and  $\bar{k}$  is finite, where  $k^*$  is the set of units in  $k$ , ie  $k^* = \{\lambda \in k, \lambda \neq 0\}$ .
3.  $k$  is a finite extension of either  $\mathbb{Q}_p$  or  $\mathbb{F}_p((t))$ .
4.  $\mathbb{P}^1(k)$  is compact

**Definition.** [7] We denote by  $GL(2, k)$  the set of invertible  $2 \times 2$  matrices over  $k$ . A **fractional linear automorphism**,  $\phi$ , of  $\mathbb{P}^1(k)$  is a map defined by  $z \mapsto \frac{az+b}{cz+d}$  for some  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, k)$ . The set of fractional linear automorphisms of  $\mathbb{P}^1(k)$  is denoted  $PGL(2, k)$ . Note that  $PGL(2, k) = GL(2, k) / \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \lambda \in k^* \}$ . In homogeneous coordinates, we can represent the action of  $\phi$  by  $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$ .

**Definition.** [7] Suppose  $\Gamma$  is a subgroup of  $PGL(2, k)$ . A point  $p \in \mathbb{P}^1(k)$  is a **limit point of**  $\Gamma$ , if there exists a point  $q$  in  $\mathbb{P}^1(k)$  and a sequence  $\{\gamma_n\}_{n \geq 1}$  in  $\Gamma$  such that  $\lim_{n \rightarrow \infty} \gamma_n(q) = p$ .

**Proposition.** [7] If  $\Gamma$  is not a discrete subgroup of  $PGL(2, k)$  then every point of  $\mathbb{P}^1(k)$  is a limit point of  $\Gamma$ .

*Proof.* Since  $\Gamma$  is not discrete, the sequence  $\{\gamma_n\}_{n \geq 1}$  has a limit  $\gamma$  in  $\Gamma$ . Let  $p$  be any point of  $\mathbb{P}^1(k)$  and let  $q = \gamma^{-1}(p)$ . Then  $\lim_{n \rightarrow \infty} \gamma_n(q) = \lim_{n \rightarrow \infty} \gamma_n(\gamma^{-1}(p)) = p$ .  $\square$

**Definition.** [7] A subgroup  $\Gamma$  of  $PGL(2, k)$  is **discontinuous** if the closure of every orbit of  $\Gamma$  in  $\mathbb{P}^1(k)$  is compact and the set of all limit points of  $\Gamma$  is not equal to  $\mathbb{P}^1(k)$ .

**Proposition.** [7] If  $\Gamma$  is a discontinuous subgroup of  $PGL(2, k)$  and  $\mathcal{L}$  is the set of limit points of  $\Gamma$ , then  $\mathcal{L}$  is compact, nowhere dense and if  $\mathcal{L}$  contains more than two points,  $\mathcal{L}$  is perfect.

**Definition.** [7] Let  $A$  be an element of  $GL(2, k)$  and let  $a_1$  and  $a_2$  be the eigenvalues of  $A$ . Then  $A$  is called **elliptic** if  $a_1 \neq a_2$ , but  $|a_1| = |a_2|$ .  $A$  is called **parabolic** if  $a_1 = a_2$ , and  $A$  is called **hyperbolic** if  $|a_1| \neq |a_2|$ .

**Example 5.** Consider the matrix  $T_s = \begin{pmatrix} p & s \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q}_p)$  for some  $s$  in  $(0, \dots, p-1)$  (note that  $\det(T_s) = p$  is invertible in  $\mathbb{Q}_p$ , so that  $T_s \in GL(2, \mathbb{Q}_p)$ , although it is not in  $GL(2, \mathbb{Z}_p)$ ).  $T_s$  has eigenvalues  $p$  and  $1$  and so  $T_s$  is hyperbolic for any choice of  $s$  or  $p$ . Consider the action of  $T_s$  on  $\mathbb{Z}_p \subset \mathbb{Q}_p$ , where  $\mathbb{Z}_p$  is identified with the subspace  $\{[1, \lambda]; \lambda \in \mathbb{Z}_p\}$  of  $\mathbb{P}^1(\mathbb{Q}_p)$ . In homogeneous

coordinates, this action is given by  $[1, \lambda] \mapsto [1, p\lambda + s]$ . Since  $| (p\lambda + s - s) | = | p\lambda | \leq \frac{1}{p}$ ,  $T_s$  sends  $\lambda$  to  $B_{\frac{1}{p}}(s)$ . Also note that for  $s = 0$ ,  $T_s$  has the effect of shifting the index of  $\lambda$  by 1, that is, if  $\lambda = \sum_{i=n}^{\infty} a_i p^i$ , where  $n = \text{ord}(\lambda)$ , then  $T_0([1, \lambda]) = [1, p\lambda] \rightsquigarrow p\lambda = \sum_{i=n+1}^{\infty} a_{(i-1)} p^i$ .

## Computation of the capacity of some sets

### (F&P, section 5)

#### Setup

Let  $A = \{0, 1, \dots, d-1\}$  be a finite alphabet and  $A^{\mathbb{N}}$  be the collection of infinite sequences with values in  $A$ . Note  $A^{\mathbb{N}}$  is a Cantor set, so it is perfect, nowhere dense, and compact.

A basis for the topology is given by the cylinder set: take countably many copies of  $\{0, 1, \dots, d-1\}$  where each copy has the discrete topology.

Let  $p \geq d$  be a prime number and let  $\phi$  be the canonical embedding of  $A^{\mathbb{N}}$  into  $\mathbb{Z}_p$  via the following continuous (under the above topology) map:

$$\phi : A^{\mathbb{N}} \rightarrow \mathbb{Z}_p \text{ by } (x_n)_{n \geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

**Lemma.** (F&P Lemma 5.1)

Let  $w_1, w_2, \dots, w_s$  be  $s \geq 2$  words with the same length  $l$  such that all the first letters are distinct. Let  $X \subset A^{\mathbb{N}}$  be the set of sequences such that any factor is a factor of a concatenation of the words  $w_1, w_2, \dots, w_s$ . Then the set  $E := \phi(X) \subset \mathbb{Z}_p$  satisfies:

$$E = \cup_{i=1}^s x_i + p^l E, \text{ with } x_i = \phi(w_i 0^\infty)$$

It is a regular compact set and its valutive capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

**Example 6.**  $w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$

Then  $\{x_n\}_{n \geq 0} \in X$  if each term in  $\{x_n\}_{n \geq 0}$  is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Note that we can rephrase the lemma as follows:

Let  $x_1, x_2, \dots, x_s$  be  $s \geq 2$  points in  $\mathbb{Z}_p$  such that  $|x_i - x_j|_p = 1, \forall i, j \in 1, \dots, s$ . Suppose also that there exists an  $l \in \mathbb{N}$  such that  $\forall i$ ,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^l a_i p^i$$

Let  $\gamma_i$  be the fractional linear automorphism of  $\mathbb{P}^1(\mathbb{Q}_p)$  given by  $\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix}$  and let  $\Gamma$  be the subgroup of  $PGL(2, \mathbb{Q}_p)$  generated by the  $\gamma_i$ .

Then  $\Gamma$  has a subgroup  $H$  such that the limit set  $\mathcal{L}$  of  $H$  has the property that  $Z = \psi^{-1}(\mathcal{L})$  is equal to  $\phi(X)$  in the original lemma. In particular  $Z$  is a regular, compact subset of  $\mathbb{Z}_p$  satisfying

$$Z = \cup_{i=1}^s x_i + p^l Z = \cup_{i=1}^s B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

*Proof.* We must show that the set  $Z$  above is equal to  $E = \phi(X)$  in the original lemma. First note that if  $w_1, w_2, \dots, w_s$  are words in  $A^{\mathbb{N}}$ , then the first letter of each  $w_i$  is distinct if and only if  $| \phi(w_i) - \phi(w_j) |_p = 1, \forall i, j$  (since the pairwise distance is 1 if and only if  $ord(\phi(w_i) - \phi(w_j)) = 0$  for any  $i$  and  $j$ , if and only if the coefficient of  $p^0$  (i.e., the first letter each  $w_i$ ) is different  $\forall i, j$ ). So then the  $x_i$  are just the  $\phi(w_i)$ .

Now consider the limit set of  $\Gamma$ . Let  $\gamma \in \Gamma$ . If  $\gamma$  is a product of the generators  $\gamma_i$ , then  $\gamma$  is represented by a matrix of the form:  $\begin{pmatrix} p^{lm} & z_m \\ 0 & 1 \end{pmatrix}$ , where  $m \in \mathbb{N}$  and  $z_m$  is an element of  $\mathbb{Z}_p$  whose coefficient vector is a concatenation of the coefficient vectors of the  $x_i$  (for  $0 \leq i \leq ml$  and 0 for  $i > ml$ ). For example,  $\gamma_i \gamma_j \gamma_k =$

$$\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^l & x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^l & x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{3l} & p^{2l}x_k + p^l x_j + x_i \\ 0 & 1 \end{pmatrix}$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{lm}a_1 + za_0] \sim [1, p^{lm}\frac{a_1}{a_0} + z_m]$$

As  $m$  tends to infinity, this point tends to  $[1, 0 + z]$ , where  $z$  is an element of  $\mathbb{Z}_p$  whose entire coefficient vector is a concatenation of the  $x_i$ 's. The set  $\psi^{-1}([1, z])$  for all such  $z$  is exactly the set  $E = \phi(X)$ .

Now suppose  $\gamma$  is a product of the inverses of the generators  $\gamma_i^{-1}$ , then  $\gamma$  is represented by a matrix of the form:  $\begin{pmatrix} p^{-lm} & -p^{-l}z^{-1} \\ 0 & 1 \end{pmatrix}$ , where  $m \in \mathbb{N}$  and  $z$  is as above. For example,

$$\begin{pmatrix} p^{-l} & -p^{-l}x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{-3l} & -p^{-3l}x_k - p^{-2l}x_j - p^{-l}x_i \\ 0 & 1 \end{pmatrix}$$

The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{-lm}a_1 - p^{-lm}za_0] \sim [1, p^{-lm}(\frac{a_1}{a_0} - z)]$$

This time as  $m$  grows, the image approaches infinity and so has empty preimage under  $\psi$ .

Lastly, we consider elements of  $\Gamma$  made up of both generators and the inverse of generators. These elements will produce translations, either of the form  $\begin{pmatrix} 1 & p^{-l}(x_i - x_j) \\ 0 & 1 \end{pmatrix}$  if  $\gamma = \gamma_j^{-1}\gamma_i$  or  $\begin{pmatrix} 1 & x_j - x_i \\ 0 & 1 \end{pmatrix}$  if  $\gamma = \gamma_j\gamma_i^{-1}$ . These elements commute with each other and so the subgroup which they generate is normal. We can quotient by the entire translation subgroup, ie the sub-

group generated by  $\{\gamma_i \gamma_j^{-1}, \gamma_i^{-1} \gamma_j; \forall i, j \in 1, \dots, s\}$  to obtain  $H$ . Then  $\mathcal{L} = \infty \cup \{[1, z] \mid z \in \mathbb{Z}_p \text{ and the coefficient vector of } z \text{ is a concatenation of the coefficient vectors of the } x_i\text{'s}\}$  and  $\psi^{-1}(\mathcal{L}) = E = \phi(X)$ , as required.

□

More background from [7].

**Definition.** [7] A **Schottky group** is a finitely-generated, free and discontinuous subgroup of  $PGL(2, k)$

Notation: Let  $X$  be a subset of  $\mathbb{P}^1(k)$  and denote by  $X^{(3)}$  the set  $X \times X \times X \setminus \Delta$ , where  $\Delta$  is the fat diagonal, that is the set  $\{x_i = x_j; \text{ for some } i \neq j, i, j \in 1, 2, 3\}$

Notation: Let  $a = (a_0, a_1, a_\infty)$  be an element of  $X^{(3)}$  for some  $X \subseteq \mathbb{P}^1(k)$ . We denote by  $\gamma_a$  the element of  $PGL(2, k)$  such that  $\gamma_a(a_0) = 0$ ,  $\gamma_a(a_1) = 1$ ,  $\gamma_a(a_\infty) = \infty$ .

Note that given such a point  $a$ , we can calculate the map  $\gamma_a$  explicitly, as follows: Suppose  $a = (a_0, a_1, a_\infty)$  and  $a_i \neq \infty = [0, 1], \forall i$ . Then  $\gamma_a(a_0) = 0 = [1, 0]$ ,  $\gamma_a(a_1) = 1 = [1, 1]$ , and  $\gamma_a(a_\infty) = \infty = [0, 1]$ , so that if  $\gamma_a$  is represented by some  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , we have:

**Definition.** Let  $x_0, x_1$  be two elements of  $k$  such that  $\max(|x_0|, |x_1|)$ , ie both  $x_0$  and  $x_1$  are in the valuation ring of  $k$  and at least one is in its residue field. The map  $R$  that sends  $[x_0, x_1] \mapsto [\bar{x}_0, \bar{x}_1]$ , that is that sends each component to its residue class in  $\bar{k}$  is called the **standard reduction**.



# algebraic extensions

Type up the example Keith gave you.

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