

# Chapter 1

## Introduction

In the course of developing a generalized factorial function, Bhargava introduced the notion of  $p$ -orderings of a Dedekind domain [?, ?], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [?] (also KJ) and have also provided a natural framework to extend many classical results in analysis to a  $p$ -adic setting, such as polynomial approximation and mapping theorems [?, ?, ?].

In this thesis, we examine how a tool based on  $p$ -orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

## Chapter 2

### Background

The principal context for this thesis is an arbitrary ultrametric space, which is a metric space that also satisfies an additional axiom, sometimes called the ultrametric inequality or (in the case of vector spaces) the strong triangle property. We define ultrametric spaces below and for the rest of this section, we review some of their more important characteristics. The proofs offered in this section are, for the most part, standard and can be found in a number of reference texts, such as [?].

**Definition 1.** Let  $(M, \rho)$  be a metric space; that is, suppose  $M$  is a set and  $\rho : M \times M \rightarrow \mathbb{R}_{\geq 0}$  is such that:

- (i)  $\rho(x, y) = 0$  if and only if  $x = y$
- (ii)  $\rho(x, y) = \rho(y, x)$
- (iii)  $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any  $x, y, z \in M$ . If  $\rho$  satisfies the ultrametric inequality,

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z))$$

for any  $x, y, z \in M$ , then  $(M, \rho)$  is an **ultrametric space**.

A special case of an ultrametric space, and one where much of the previous work on this topic has been completed, is one where the metric has been derived from a norm on a vector space.

**Definition 2.** Let  $(V, N)$  be a normed vector space; that is, suppose  $V$  is  $\mathbb{F}$ -vector space, for  $\mathbb{F}$  some subfield of  $\mathbb{C}$ , and  $N : V \rightarrow \mathbb{R}_{\geq 0}$  is such that:

$$(i) \quad N(x + y) \leq N(x) + N(y)$$

$$(ii) \quad N(cx) = |c| N(x)$$

$$(iii) \quad N(x) = 0 \text{ implies } x = 0$$

for any  $x, y \in V$  and  $c \in \mathbb{F}$ . We say that  $N$  satisfies the **strong triangle inequality** if

$$N(x + y) \leq \max(N(x), N(y))$$

for any  $x, y \in V$ .

**Proposition 1.** *Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle inequality. Then the metric space,  $(V, \rho_N)$ , where  $\rho_N$  is the metric induced by  $N$ , that is,  $\rho_N(x, y) = N(x - y)$ , is an ultrametric space.*

*Proof.* We take for granted that  $(V, \rho_N)$  is a metric space and also note that

$$N(x + z) \leq \max(N(x), N(z))$$

implies

$$\rho_N(x, z) \leq \max(\rho_N(x, 0), \rho_N(z, 0)) \leq \max(\rho_N(x, y), \rho_N(y, z))$$

□

**Notation.** If  $(V, N)$  is a normed vector space, then the metric induced by  $N$  is denoted  $\rho_N$ .

Ultrametric spaces exhibit properties much unlike traditional metric spaces, and we review a few of these below. Of particular interest to us is the behavior between (closed) balls in an ultrametric space.

**Notation.** Let  $(M, \rho)$  be a compact ultrametric space and let

$$B(a, r) = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the *closed* ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{\geq 0}$  and  $a \in (M, \rho)$ .

In the above notation, we break from convention in that we denote a closed ball without using any decoration and by omitting any notation for an open ball. This because, for the most part, the notion of open and closed ball in an ultrametric space overlap, although we will need a few more facts before showing this.

**Definition 3.** Let  $S$  be a subset of an ultrametric space. The **diameter of  $S$**  is  $\text{diam}(S) = \max_{x,y \in S} \rho(x,y)$ .

**Proposition 2.** Let  $B = B(a, r)$  be a (closed) ball in an ultrametric space  $(M, \rho)$ . Then the diameter of  $B$  is less than or equal to the radius of  $B$ .

*Proof.* Suppose  $d = \text{diam}(B) > r$ . This would imply there exists  $x, y$  in  $B$  such that  $\rho(x, y) > r$ , in particular  $\rho(x, y)$  is strictly greater than  $\max(\rho(x, a), \rho(y, a))$ , which is a contradiction since  $\rho$  is an ultrametric.  $\square$

**Proposition 3.** All triangles in an ultrametric space  $(M, \rho)$  are either equilateral or isosceles, with at most one short side.

*Proof.* Let  $x, y$ , and  $z$  be three points in an ultrametric space  $(M, \rho)$ . We show that  $\rho(x, y) \neq \rho(x, z)$  and  $\rho(x, y) \neq \rho(y, z)$  implies  $\rho(x, y) < \rho(x, z) = \rho(y, z)$ .

If  $\rho(x, z) \neq \rho(y, z)$ , then without loss,  $\rho(x, z) > \rho(y, z)$ . At the same time, the ultrametric inequality implies  $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$  and  $\rho(y, z) \leq \max(\rho(x, y), \rho(x, z))$ . The first inequality implies  $\rho(x, y) < \rho(x, z)$ , which means the second inequality implies  $\rho(y, z) < \rho(x, z)$ . This is a contradiction, so we must have  $\rho(x, z) = \rho(y, z)$ .

To see that  $\rho(x, y) < \rho(x, z)$ , simply note that  $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$   $\square$

**Proposition 4.** If  $(M, \rho)$  is an ultrametric space and  $B(x, r_1)$  and  $B(y, r_2)$  are balls in  $(M, \rho)$ , then either  $B(x, r_1) \cap B(y, r_2) = \emptyset$ ,  $B(x, r_1) \subseteq B(y, r_2)$ , or  $B(x, r_1) \subseteq B(y, r_1)$ . That is, in an ultrametric space, all balls are either comparable or disjoint.

*Proof.* Suppose  $B(x, r_1) \cap B(y, r_2) \neq \emptyset$  and let  $z$  be a point in the intersection. If there exists an  $a \in B(y, r_2)$  such that  $\rho(x, a) < r_1$  then  $B(x, r_1) \subseteq B(y, r_2)$ .  $\square$

**Proposition 5.** The distance between any two distinct balls in an ultrametric is constant. That is, if  $B(x_0, r_1)$  and  $B(y_0, r_2)$  are two balls in an ultrametric space

with  $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$ , then there exists a  $c \in \mathbb{R}_{\geq 0}$  such that  $\rho(x, y) = c$ ,  $\forall x \in B(x_0, r_1)$  and  $\forall y \in B(y_0, r_2)$ .

*Proof.* Suppose  $\rho(x_0, y_0) = c$  and let  $x \in B(x_0, r_1)$  and  $y \in B(y_0, r_2)$  be arbitrary. Consider the triangle formed by  $(x_0, y_0, y)$ . Since  $\rho(x_0, y_0) = c$  and  $\rho(y, y_0) \leq r_2 < c$ , we must have  $\rho(x_0, y) = c$  because triangles in an ultrametric space have at most one short side. Now consider the triangle formed by  $(x_0, x, y)$ . Since  $\rho(x_0, y) = c$  and  $\rho(x, x_0) \leq r_1 < c$ , we must have  $\rho(x, y) = c$ .  $\square$

**Proposition 6.** *Every point of a ball in an ultrametric is at its centre. That is, if  $B(x_0, r)$  is a ball in an ultrametric space  $(M, \rho)$ , then  $B(x, r) = B(x_0, r)$ ,  $\forall x \in B(x_0, r)$*

*Proof.* Let  $a \in B(x, r)$ . Then  $\rho(a, x) \leq r$  and since

$$\rho(a, x_0) \leq \max(\rho(a, x), \rho(x, x_0)) \leq r$$

we must have  $a \in B(x_0, r)$  and  $B(x, r) \subseteq B(x_0, r)$ . A similar argument shows  $B(x_0, r) \subseteq B(x, r)$ .  $\square$

**Proposition 7.** *Let  $B(a, r)$  be a closed ball in an ultrametric space. Then there exists  $r' > r \in \mathbb{R}$  such that  $B(a, r) = \{x \in M \mid \rho(x, a) < r'\}$ ; that is, every closed ball is also an open ball with the same centre and slightly larger radius.*

The following proposition is easy to see, although the result is both unintuitive and rather important for our purposes.

**Proposition 8.** *Suppose  $S$  is a compact subset of an ultrametric space  $(M, \rho)$  and that  $\cup_{i \in I} B(x_i, r_i)$  is a cover of  $S$  by (closed) balls in  $S$ . Then there exists  $i_1, \dots, i_n$ , a finite subset of  $I$ , such that  $\cup_{j=1}^n B(x_{i_j}, r_{i_j})$  is a partition of  $S$ .*

*Proof.* Since  $S$  is compact,  $\cup_{i \in I} B(x_i, r_i)$  contains a finite subcover of  $S$ . Say this subcover is given by the elements  $i_1, \dots, i_{n'}$ , and suppose this is not a partition. That is, suppose for some  $i_i, i_j$ ,  $B(x_{i_i}, r_{i_i}) \cap B(x_{i_j}, r_{i_j}) \neq \emptyset$ . Then, without loss of generality, we must have  $B(x_{i_i}, r_{i_i}) \subseteq B(x_{i_j}, r_{i_j})$ , so that the removal of  $B(x_{i_i}, r_{i_i})$  is still a cover of  $S$ . We continue this process a finite number of times, since the subcover was finite to begin with, to arrive at a partition of  $S$ .  $\square$

In fact, a slightly stronger statement than the above is true:

**Corollary 1.** *Suppose  $S$  is a compact subset of an ultrametric space  $(M, \rho)$  and that  $B(x_0, r)$  is a (closed) ball in  $S$ . Then, there exists a finite partition of  $S$  having  $B(x_0, r)$  as an element.*

*Proof.* Let  $\mathcal{C}$  be the cover of  $S$  given by  $\cup_{x \in S} B(x, r) \cap S$ . From the proposition, we can select a finite subcover of  $\mathcal{C}$  that is a partition of  $S$ . Suppose  $B(y, r) \cap S$  is the element in this partition containing  $x_0$ . Then since  $B(y, r)$  and  $B(x_0, r)$  are equal in  $M$ ,  $B(y, r) \cap S = B(x_0, r) \cap S = B(x_0, r)$ .  $\square$

**Proposition 9.** *[?] Let  $S$  be a group equipped with a (right) invariant ultrametric,  $\rho$ . If  $B = B(0, r)$  is a (closed) ball centred at the neutral element of  $S$ , then  $B$  is a subgroup of  $S$ .*

*Proof.* Let  $x, y \in B$ . Then

$$\rho(x - y, 0) = \rho(x, y) \leq \max(\rho(x, 0), \rho(y, 0)) \leq r,$$

so that  $x - y \in B$ .  $\square$

**Proposition 10.** *[?] If  $S$  is a compact subset of an ultrametric space and  $\Gamma_S$  is the set of all distances occurring between points of  $S$ , then  $\{\Gamma_S \setminus 0\}$  is a discrete subset of  $\mathbb{R}^+$ . In particular, if  $S$  is compact, then  $\Gamma_S$  is countable.*

*Proof.* If  $S = \{a\}$ , then  $\{\Gamma_S \setminus 0\} = \emptyset$  and we are done. Suppose then  $S$  has at least two points and let  $\mathcal{U} = \{\Gamma_S \setminus 0\}$ . Let  $p \in S$  be arbitrary and consider the function  $\phi_p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  given by  $\epsilon \mapsto B(p, \epsilon) \cap \mathcal{U}$ . We show  $\phi_p$  is locally constant for all  $p \in \mathcal{U}$ .  $\square$

It will become useful to write the set of distances occurring in  $S$  as a sequence, put in decreasing order. We use the following notation to represent such a sequence

$$\Gamma_S = \{\gamma_0 = d = \text{diam}(S), \gamma_1, \gamma_2, \dots, \gamma_\infty = 0\}$$

where  $\gamma_i \in \Gamma_S$  if and only if  $\exists x, y \in S$  such that  $\rho(x, y) = \gamma_i$  and  $\gamma_i < \gamma_j$  if and only if  $i > j$ .

### $\rho$ -orderings, $\rho$ -sequences, and valiative capacity

In what follows let  $S$  be a compact subset of an ultrametric space  $(M, \rho)$ .

**Definition 4.** [?] A  $\rho$ -**ordering** of  $S$  is a sequence  $\{a_i\}_{i=0}^\infty \subseteq S$  such that  $\forall n > 0$ ,  $a_n$  maximizes  $\prod_{i=0}^{n-1} \rho(s, a_i)$  over  $s \in S$ .

**Example 1.** Suppose  $S$  is a finite subset of  $(\mathbb{Z}, |\cdot|_2)$ ,  $S = \{0, 2, 8, 3\}$ . Then a  $\rho$ -ordering of  $S$  starts (arbitrarily) with  $a_0 = 0$ , which forces  $a_1 = 3$ , since  $\rho(0, 3) = 1 = \text{diam}(S)$ . The sequence continues  $a_2 = 2$  and  $a_3 = 8$ , but after this point the sequence becomes arbitrary because  $\prod_{i=0}^{n-1} \rho(s, a_i)$  will contain a 0, given by the repeated term. Indeed, for any finite subset  $S$  with  $|S| = n$ , the  $\rho$ -ordering of  $S$  is arbitrary from the  $n^{\text{th}}$  point on.

**Definition 5.** [?] The  $\rho$ -**sequence** of  $S$  is the sequence whose  $0^{\text{th}}$ -term is 1 and whose  $n^{\text{th}}$  term, for  $n > 0$ , is  $\prod_{i=0}^{n-1} \rho(a_n, a_i)$ .

**Proposition 11.** [?] *The  $\rho$ -sequence of  $S$  is well-defined so long as  $S$  is compact and  $\rho$  is an ultrametric. That is, the  $\rho$ -sequence of a compact subset of an ultrametric spaces does not depend on the choice of  $\rho$ -ordering of  $S$ .*

**Definition 6.** [?] Let  $\gamma(n)$  be the  $\rho$ -sequence of  $S$ . The **valiative capacity** of  $S$  is

$$\omega(S) := \lim_{n \rightarrow \infty} \gamma(n)^{1/n}$$

**Proposition 12.** [?] *For  $S$  and  $\gamma(n)$  as above,  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = r < \infty$ .*

**Proposition 13.** *If  $S \subseteq M$  is a finite subset of an ultrametric space, then  $\omega(S) = 0$ .*

**Proposition 14.** (*upper bound*) *If  $\text{diam}(S) = d$ , then  $\omega(S) < d$ .*

*Proof.* Since  $d$  is the diameter of  $S$ , the  $n^{\text{th}}$  term of the  $\rho$ -sequence of  $S$  is bounded by  $d^n$  and so  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = d$  if and only if  $\gamma(n) = d^n, \forall n$ . This implies  $\rho(a_n, a_i) = d, \forall n$  and  $\forall i < n$ , but then  $\rho(a_i, a_j) = d, \forall i, j$ , since the  $\rho$ -sequence is maximized at each  $n$ . This means  $\omega(S) < d$  would imply that the cover of  $S, \cup_{a_i} B_d(a_i)$  is in fact an infinite partition, contradicting the compactness of  $S$ . Then  $\omega(S) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} < d$ .  $\square$

This doesn't work because  $\cup_{a_i} B_d(a_i)$  could fail to be a cover -when does this happen

**Proposition 15.** (*translation invariance*) If  $(M, \rho)$  be a compact ultrametric space and  $s$  also a topological group for which  $\rho$  is (left) invariant under the group operation, then  $\omega$  is also (left)-invariant. That is, if  $\rho(x, y) = \rho(gx, gy)$ ,  $\forall g, x, y \in M$ , then  $\omega(gS) = \omega(S)$ , for  $S \subseteq M$ .

*Proof.* Let  $\{a_i\}_{i=0}^\infty$  be a  $\rho$ -ordering for  $S$ . Then  $\{ga_i\}_{i=0}^\infty$  is a  $\rho$ -ordering for  $gS$ . Then

$$\omega(gS) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = \omega(S)$$

□

**Example 2.** With the notation of the previous section, note that for  $x, y \in (\mathbb{Z}_p, |\cdot|_p)$ ,  $\rho_p(x, y) = |x - y|_p = p^{-\nu_p(x-y)} = p^{-\nu_p((a+x)-(a+y))} = |(a+x) - (a+y)|_p = \rho_p(a+x, a+y)$  so that  $\omega(a+S) = \omega(S)$  for  $S \subseteq (\mathbb{Z}_p, |\cdot|_p)$ .

**Proposition 16.** Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle identity. Then if  $N$  is multiplicative, so is  $\omega$ . That is, if  $N(gx) = N(g)N(x), \forall g, x \in V$ , then  $\omega(gS) = N(g)\omega(S)$ , for  $g \in V$  and  $S \subseteq M$ .

*Proof.* Let  $\rho$  be the metric induced by  $N$ , so that  $\rho(x, y) = N(x - y), \forall x, y \in V$ . Let  $\{a_i\}_{i=0}^\infty$  be a  $\rho$ -ordering for  $S$ . Then since  $N$  is multiplicative, for  $u, v \in gS$ ,  $u = gs_i$  and  $v = gs_j$  for some  $s_i, s_j \in S$ ,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then  $\{ga_i\}_{i=0}^\infty$  is a  $\rho$ -ordering for  $gS$  and

$$\begin{aligned} \omega(gS) &= \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} N(g)\rho(a_n, a_i) \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} \left[ N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g) \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g)\omega(S) \end{aligned}$$

□



**Example 3.** Since  $|\cdot|_p$  is multiplicative,  $\omega(mS) = |m|_p \omega(S)$  for  $m \in \mathbb{Z}_p$  and  $S \subseteq \mathbb{Z}$ . In particular,  $\omega(p\mathbb{Z}) = |p|_p \omega(\mathbb{Z}) = \frac{1}{p} \cdot p^{\frac{1}{1-p}} = p^{-p/p-1}$ .

The following proposition is from [?], where it is given for some  $S$  written as the union of two subsets, although it is easily seen to be true for  $S$  equal to any finite union, so long as the other assumptions remain satisfied.

**Proposition 17.** [?](subadditivity) *If  $\text{diam}(S) := \max_{x,y \in S} \rho(x,y) = d$  and  $S = \cup_i^n A_i$  for  $A_i$  compact subsets of  $M$  with  $\rho(A_i, A_j) = d, \forall i, j$ , then*

$$\frac{1}{\log(\omega(S)/d)} = \sum_{i=1}^n \frac{1}{\log(\omega(A_i)/d)}$$

**Example 4.**

- Also talk about when cosets and open balls coincide

**Corollary 2.** *Suppose  $S = \cup_i^n S_i$  with  $\rho(S_i, S_j) = d = \text{diam}(S)$  and also  $\omega(S_i) = \omega(S_j), \forall i, j$ . Let  $r \in \mathbb{R}$  be such that  $\omega(S_i) = r\omega(S), \forall i$ . Then  $\omega(S) = r^{\frac{1}{n-1}} \cdot d$ . In particular if  $S = \mathbb{Z}$  and  $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$  then  $\omega(S) = (\frac{1}{p})^{1/p-1}$  for any prime  $p$ .*

**Corollary 3.** (Joins of computable sets are computable) *Let  $\Gamma_M = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$  be the set of distances in  $M$ . Suppose that  $S = B_{\gamma_i}(x)$ , for some  $x$  and  $i$ , is the union of 2 or more balls of radius  $\gamma_{i+1}$ , i.e.,  $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$  is a join in the lattice of open sets in  $M$ , then*

$$\frac{1}{\log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^n \frac{1}{\log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

## Chapter 3

### $\rho$ -orderings and the structure of $S$

In the previous section, we defined valutive capacity for a compact subset  $S$  of an ultrametric space  $(M, \rho)$ . We also got a glimpse into the way the valutive capacity of  $S$  interacts with its other properties, such as the set of distances occurring in  $S$  and the lattice of closed balls in  $S$  (or equivalently, if  $S$  has enough structure, a lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valutive capacity of  $S$  and the lattice of closed balls in  $S$ . In particular, we show how, in all cases (with  $S$  compact), the latter can be used to compute the first  $n$  terms of a  $\rho$ -ordering of  $S$  (for any  $n < \infty$ ) and how, in some cases, this extends to being able to compute the valutive capacity of  $S$ .

### Subspaces of $S$

In the section we explore the subspaces of  $S$  formed by considering closed balls of some fixed radius. We begin by letting  $S$  be, as before, a compact subset of an ultrametric space  $(M, \rho)$ . Recall from the previous section that if  $S$  is compact, then the set of distances occurring in  $S$  is a discrete, bounded subset of  $\mathbb{R}$  and so we may represent the set of distances by a sequence in decreasing order. As before, let the decreasing sequence of distances in  $S$  be given by  $\Gamma_S = \{\gamma_0 = \text{diam}(S), \gamma_1, \dots, \gamma_\infty = 0\}$ .

Now fix some  $k \in \mathbb{N}$ , and consider for a moment the set of closed balls of radius  $\gamma_k$  in  $S$ . We could denote these alternatively by  $B^M(x, \gamma_k) \cap S$  or by  $B^S(x, \gamma_k)$ , but when there is no risk of confusion, we will denote them simply by  $B(x, \gamma_k)$ . Clearly, the set  $\{B(x, \gamma_k); x \in S\}$  forms a cover of  $S$ . Although we have build the cover using closed balls, since each set in an ultrametric is clopen, this gives an open cover of  $S$

(in fact, each element is not only an open set, but also an open ball for some radius slightly bigger than  $\gamma_k$ ). Then since  $S$  is compact, we must have some  $x_1, \dots, x_n$  such that  $S = \cup_{i=1}^n B(x_i, \gamma_k)$ . In fact, since  $\rho$  is an ultrametric, we can pick the  $x_i$ 's so that  $\cup_{i=1}^n B(x_i, \gamma_k)$  will be a disjoint union and therefore a partition of  $S$ . Note that both  $n$  and the  $x_i$ 's depend on our fixed  $k$ , but that  $n$  is independent of the  $x_i$ 's, since any choice of centres is equivalent. We rephrase this with following definition and lemma:

**Definition 7.** For  $S$  and  $\Gamma_S$  as above, and  $k \in \mathbb{N}$ , fixed, define  $\sim_k$  to be the relation on  $S$  given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e.,  $x \sim_k y$  if and only if  $B_{\gamma_k}(x) = B_{\gamma_k}(y)$ .

The fact  $\sim_k$  is an equivalence relation on  $S$  is equivalent to the observation that every point in a ultrametric ball is at its centre:

**Lemma 1.** *Let  $S$  and  $\Gamma_S$  be as above, then  $\sim_k$  is an equivalence relation on  $S$ .*

*Proof.*  $\sim_k$  is clearly reflexive and symmetric, since  $\rho$  is a metric. Transitivity results from the ultrametric property of  $\rho$ : if  $x \sim_k y$  and  $y \sim_k z$ , then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)) \leq \gamma_k$$

so  $x \sim_k z$ . □

We denote the set of equivalence classes of  $S / \sim_k$  by  $S_{\gamma_k}$ . We have defined  $S_{\gamma_k}$  to be the set of equivalence classes in  $S$  under the relation  $\sim_k$ , which is equivalent to letting  $S_{\gamma_k}$  be the set of closed balls of fixed radius  $\gamma_k$  in  $S$ . We now offer a third perspective on the elements on  $S_{\gamma_k}$ , which is due to [?],

**Lemma 2.** *For each  $k$ , the elements of  $S_{\gamma_k}$ , that is, the closed balls of radius  $\gamma_k$ , themselves form an ultrametric space, where the metric is given by:*

$$\rho_k(B(x, \gamma_k), B(y, \gamma_k)) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } B(x, \gamma_k) = B(y, \gamma_k) \end{cases}$$

*Proof.*  $\rho_k$  is reflexive, symmetric and transitive since  $\rho$  is. Likewise,  $\rho_k$  satisfies the ultrametric property, since  $\rho$  does: let  $B(x, \gamma_k), B(y, \gamma_k)$  and  $B(z, \gamma_k)$  be any three elements of  $S_{\gamma_k}$  and suppose  $\rho_k(B(x, \gamma_k), B(y, \gamma_k)) > 0$ . Then,

$$\begin{aligned} \gamma_k &< \rho_k(B(x, \gamma_k), B(y, \gamma_k)) \\ &= \rho(x, y) \leq \max(\rho(x, z), \rho(y, z)) \\ &= \max(\rho_k(B(x, \gamma_k), B(z, \gamma_k)), \rho_k(B(y, \gamma_k), B(z, \gamma_k))) \end{aligned}$$

since  $\gamma_k < \max(\rho(x, z), \rho(y, z))$  implies that at least one of  $\rho_k(B(x, \gamma_k), B(z, \gamma_k))$  or  $\rho_k(B(y, \gamma_k), B(z, \gamma_k))$  is greater than 0.  $\square$

So now the elements of  $S_{\gamma_k}$  may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

**Definition 8.** Let  $S$  and  $\Gamma_S$  be as above. Define  $\beta(i)_{i \geq 0}$  to be the sequence given by  $\beta(i) = |S_{\gamma_i}|$ , which is an invariant of  $S$  and which counts the number of connected components of  $S_{\gamma_i}$  (that is, the points of  $S_{\gamma_i}$ ), when viewed as a metric space. When necessary, we use  $\beta^S(i)$  to denote the  $\beta$  sequence for a given, compact ultrametric space  $S$ . Adapting the terminology in [?], we call  $\beta^S(i)$  the **structure sequence** of  $S$ .

**Notation 1.** Let  $S_{\gamma_k}$  be as above. We denote the elements of  $S_{\gamma_k}$  by  $B_1^k, \dots, B_{\beta(k)}^k$  or by  $B_1^{S,k}, \dots, B_{\beta(k)}^{S,k}$ , when necessary.

We return to the sequence  $\beta(i)$  at the end of this section. For now, we show how a  $\rho$ -ordering of  $S$  can be built recursively from the spaces  $S_{\gamma_k}$ . This begins by noting that the spaces themselves can be built recursively:

**Observation 1.** Let  $S$ ,  $\Gamma_S$ , and  $S_{\gamma_k}$  be as above. Then  $S_{\gamma_{k+1}}$  can be constructed by partitioning each of the closed balls in  $S_{\gamma_k}$  into closed balls of radius  $\gamma_{k+1}$  and taking their union: Let  $B(x_i, \gamma_k)$  be an element of  $S_{\gamma_k}$ , denoted by  $B_i^k$ . Then, there exists  $x_{i,1}, \dots, x_{i,l_i} \in B_i^k$  such that,

$$B_i^k = \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1})$$

and

$$B(x_{i,j}, \gamma_{k+1}) \cap B(x_{i,j'}, \gamma_{k+1}) = \emptyset, \forall j, j' \in 1 : l_i$$

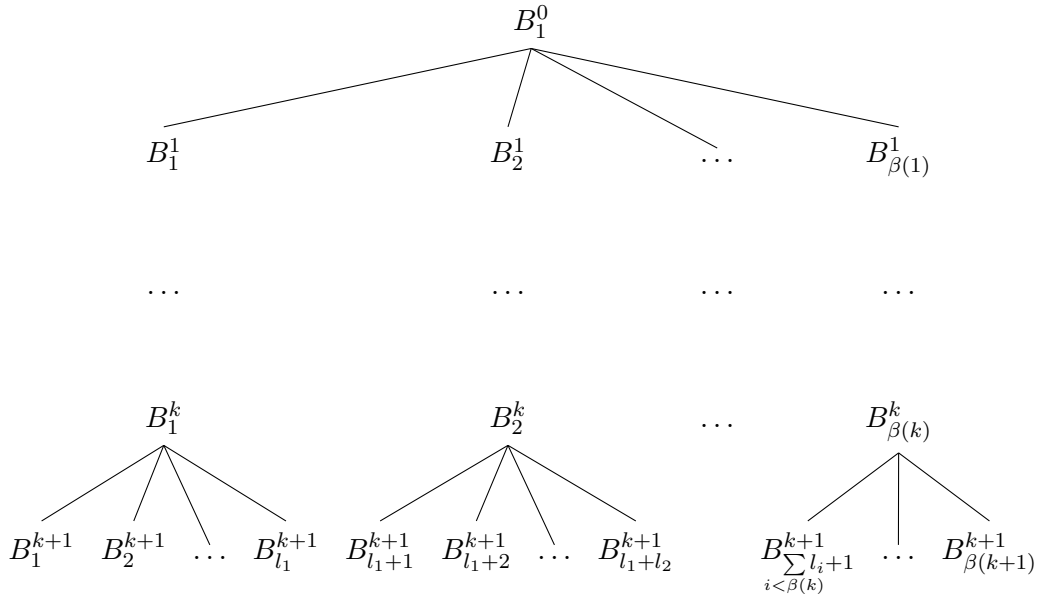
and so

$$S_{\gamma_{k+1}} = \bigcup_{i=1}^{\beta(k)} \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1}) = \bigcup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where  $\bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_k) = B(x_i, \gamma_{k+1}) = B_i^k, \forall i$ .

Since  $S$  is compact, hence bounded, if we represent this process schematically we obtain a tree, where the root node is  $B_1^0 = B(x, \gamma_0)$ , for any choice of  $x \in S$ , and the children of any given  $B_n^m$  are such that they form a partition of their join. Since we will often refer to this schematic representation, we define it below.

**Definition 9.** If  $S$  is a compact subset of an ultrametric space, then  $T_s$  is the tree whose vertices are  $B_i^k$ , that is the elements of  $S_{\gamma_k}$ , and whose edgeset,  $E$ , is given by  $(B_k^i, B_l^j) \in E$  if, and only if,  $j = i + 1$  and  $B_l^j \subseteq B_k^i$  for some choice of representatives  $B(x_k, \gamma_i)$  and  $B(x_l, \gamma_j)$ , as shown below:



Before going on, first note that we have drawn  $T_s$  such that leftmost child of some  $B_i^k$  is  $B_j^{k+1}$  where  $j$  is minimal among the children of  $B_i^k$ , and then continued

in increasing order. In general, if we draw  $T_S$  so that the children of a given vertex are depicted in increasing order according to their index, then each choice of indexing for the elements of  $S_{\gamma_k}$  produces a different graphical representation of  $T_S$ . The structures produced by different choices of indices are clearly isomorphic as trees, and as we will see by the end of the section, each choice of indexing will be valid for our purposes as well.

Of central importance to us is the distance between two vertices in  $T_S$ . Since each vertex represents an element of  $S_{\gamma_k}$ , that is a closed ball in an ultrametric space, it is well-defined to let the distance between vertices be equal to the distance between a choice of centres for those balls. Note that if the distance between  $B_i^k$  and  $B_j^l$  is taken to be  $\rho(x_i, x_j)$ , for some choice of  $x_i \in B_i^k$  and  $x_j \in B_j^l$ , say  $\rho(x_i, x_j) = \gamma_n$ , then the join of  $B_i^k$  and  $B_j^l$  is some  $B_x^n$ .

**Lemma 3.** *If  $B_i^k$  and  $B_j^l$  are two vertices in  $T_S$ , then  $\rho(x_i, x_j)$ , for any choice of  $x_i \in B_i^k$  and  $x_j \in B_j^l$ , is equal to the diameter of the join of  $B_i^k$  and  $B_j^l$ .*

*Proof.* Let  $B_i^k$  and  $B_j^l$  be two (distinct) vertices in  $T_S$  and let  $B_x^n$  be their join. The diameter of  $B_x^n$  is  $\gamma_n$  since  $B_x^n = B(x_0, \gamma_n)$  for some  $x_0$ . Since  $\rho$  is an ultrametric the distance between any  $x_i \in B_i^k$  and  $x_j \in B_j^l$  is constant, and must be equal to the diameter of the smallest ball containing both of them, that is  $\gamma_n$ .  $\square$

In particular, we have that for any  $k$  and any  $i < \beta(k)$ , the distances between the children of  $B_i^k$  will be  $\gamma_k$  and for any  $i \neq j$  the distance between the children of  $B_i^k$  and  $B_j^k$  will be equal to the distance between  $B_i^k$  and  $B_j^k$  (which will be some  $\gamma_m, m < k$ ).

### Recursive $\rho$ -orderings

In this section, we show how the recursive partitioning of  $S$  into the spaces  $S_{\gamma_k}$  gives rise to a  $\rho$ -ordering of  $S$ . We first note that without loss of generality, for any  $k \in \mathbb{N}$ , we can reindex the  $B_i^k$ 's so that they give the first  $\beta(k)$  terms of a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , when the latter is viewed as a (finite) metric space. In the first proposition below, we note that if the  $B_i^k$ 's are so indexed, then finding a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is straightforward: select a  $B_j^{k+1}$  from each of the  $B_i^k$ 's in order and then start over.

**Proposition 18.** *Let be  $S$  a compact subset of an ultrametric space  $(M, \rho)$  and  $\Gamma_S$ , the set of distances in  $S$ . If  $S_{\gamma_k}$  is the partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the elements are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , then the first  $\beta(k+1)$  terms in a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  can be found by selecting at each stage  $n$ , a child from  $B_{\bar{n}}^k$ , where  $\bar{n} = n \bmod \beta(k) + r$  and  $r$  is minimal in  $\{0, \dots, \beta(k) - 1\}$  such that  $B_{n \bmod \beta(k) + r}^k$  still has unused children.*

*Proof.* Let  $S$ ,  $S_{\gamma_K}$ , and  $S_{\gamma_{k+1}}$  be as above. In particular, suppose the elements of  $S_{\gamma_k}$  are indexed according to a  $\rho_k$ -ordering. Denote the elements of  $S_{\gamma_{k+1}}$  by  $B_{i,j}^{k+1}$  where the first subscript indicates that the element is a child of  $B_i^k$ . To form a  $\rho_{k+1}$  ordering of  $S_{\gamma_{k+1}}$ , we must maximize the product of distances at each step  $n$ .

Now note that  $\Gamma_{S_{\gamma_k}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$  and  $\Gamma_{S_{\gamma_{k+1}}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}, \gamma_k\}$ . That is, the distances in  $S_{\gamma_{k+1}}$  are the same as the distances in  $S_{\gamma_k}$ , although they also include the smaller distance  $\gamma_k$ . Since we know that the elements  $B_1^k, \dots, B_{\beta(k)}^k$  already maximizes the product of distances in  $\{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$ , the first  $\beta(k)$  terms of a  $\rho_{k+1}$ -ordering of  $S_{k+1}$  can be found by taking  $B_{1,j_1}^k, \dots, B_{1,j_{\beta(k)}}^k$  for any choice of  $j$ 's. At this point, any choice of next element will produce a copy of  $\gamma_k$  in the  $\rho_{k+1}$ -sequence; however, if we chose another child of  $B_1^k$ , we are able to keep building the ordering in a canonical fashion, since we know that we will then be able to maximize the product at the next step by choosing another child of  $B_2^k$ .

We see then that a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by minimizing the number of times  $\gamma_k$  is introduced into the  $\rho_{k+1}$ -sequence and maximizing the product among the  $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ , and the latter is already known to be achieved by taking the  $B_i^k$  in order. If the  $B_i^k$ 's all have the same number of children, then we can always select a child of  $B_{\bar{n}}^k$ , where  $\bar{n} = n \bmod \beta(k)$  at each stage  $n$ ,  $n < \beta(k+1)$ , since there will always be one available. On the other hand, suppose the  $B_i^k$  have an unequal number of children and  $n$  is the first step at which all the children of  $B_{\bar{n}}^k$  have been exhausted. What element will maximize the  $\rho_{k+1}$ -sequence?

Consider the space  $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$ . Removal of  $B_{\bar{n}}^k$  will not effect the first  $m$  terms of a  $\rho_k$ -ordering of this space, for  $m < \bar{n}$ , since if a sequence of elements maximizes a

function over a set  $X$ , they will also maximize that function of a subset of  $X$  (provided they themselves remain in the subset). Then the  $\rho_k$ -sequence of  $(S_{\gamma_k} \setminus B_n^k)$  begins  $\{B_1^k, \dots, B_{\bar{n}-1}^k\}$ .

Moreover, if  $B_{\bar{n}+1}^k$  maximizes  $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k)$  over  $S_{\gamma_k}$ , then it also maximizes  $\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)$  over  $(S_{\gamma_k} \setminus B_n^k)$ , since  $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k) = (\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)) \cdot \rho_k(x, B_n^k)$ .

Then the  $\rho_k$ -sequence of  $(S_{\gamma_k} \setminus B_n^k)$  is simply  $\{B_1^k, \dots, B_{\bar{n}-1}^k, B_{\bar{n}+1}^k, \dots, B_{\beta(k)}^k\}$ .

Now we see that  $\rho_{k+1}$ -sequence of  $S_{\gamma_{k+1}}$  is maximized by simply skipping over  $B_n^k$ , should all its children be exhausted, and selecting a child from  $B_{\bar{n}+1}^k$ . Then a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by selecting elements of each  $B_i^k$  in order as much as possible, and skipping to  $B_{i+1}^k$ , when it is not possible.

□

Note that in building the  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  we selected, at each step, a child of some  $B_i^k$ , but we did not concern ourselves over which child was selected. This is because the distances between any two children of some  $B_i^k$  is  $\gamma_k$ , and the distance between any one of them and a child of some  $B_j^k$ ,  $i \neq j$ , is the same. We can now see, as claimed above, that any of the isomorphic versions of  $T_S$  are valid for producing  $\rho$ -orderings. Suppose then that we have created  $T_s$  and (arbitrarily) indexed the children of each vertex. Then, there is no loss of generality in assuming that at each stage, we select a child with smallest index among its siblings, that is, that we select the leftmost available child in  $T_s$ . Since for ease of indexing, we will assume a  $\rho$ -ordering has been built by this convention, we introduce the following definition.

**Definition 10.** The  $\rho$ -ordering of  $S$  formed by pulling elements from left to right in (a choice of)  $T_s$  is call the **canonical  $\rho$ -ordering** of  $S$  (with respect to  $T_s$ ).

The above proposition quickly leads to a recursive construction for a  $\rho$ -ordering of  $S$ . Indeed, to build a  $\rho$ -ordering of  $S$  from the above, it suffices only to make a choice of centres for each of  $B_i^k$ 's.



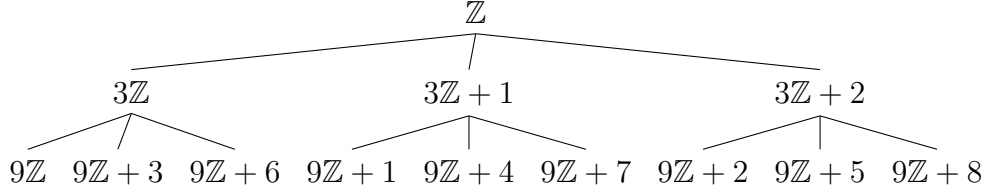
**Proposition 19.** *Let  $S$  be a compact subset of an ultrametric space  $(M, \rho)$  and let  $\Gamma_S$  be the set of distances in  $S$ . Let  $S_{\gamma_k}$  be the partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the elements are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ . Suppose each of the element of  $S_{\gamma_k}$  have also been partitioned into closed balls of radius  $\gamma_{k+1}$ ,  $B_i^k = \cup_{j=1}^{l_i} B_{i,j}^{k+1}, \forall i$ .*

*Let  $x_{i,j}$  denote a choice of centre for the element  $B_{i,j}^{k+1}$ . Then the first  $\beta(k+1)$  elements of a  $\rho$ -ordering of  $S$  can be found by forming a matrix,  $A_k$ , whose  $(i, j)^{th}$ -entry is  $x_{i,j}$ , if  $j \leq l_i$  and  $*$  otherwise, and then concatenating the rows.*

*Proof.* The matrix  $A_k$  is a representation of the  $k^{th}$  and  $(k+1)^{th}$  levels of  $T_S$  where the  $B_i^k$ 's (and  $B_{i,j}^{k+1}$ 's) have been replaced by a choice of centres. Since matrices must be rectangular, the case where some  $B_i^k$  and  $B_j^k$  have an unequal number children is handled by inserting a placeholder,  $*$ , into  $A_k$ . Moreover, since the  $\rho_{k+1}$  distance between distinct closed balls is just the  $\rho$  distance between a choice of centres of those balls, a choice of centres in a  $\rho_{k+1}$ -ordering gives the beginning of a  $\rho$ -ordering. By the above proposition, we must select elements from each  $B_i^K$  one after the other, which is achieved by selecting one element from each column in order, for example by concatenating the rows (and then deleting  $*$ 's if necessary).  $\square$

We get the most use out of the construction above if, in selecting a choice of centres for the  $B_{i,j}^{k+1}$ 's, we reuse the previous the choices as much as possible. Suppose for example we have made a choice of centres for the balls of radius  $\gamma_k$  and constructed the matrix  $A_{k-1}$ . At the next iteration, we will need a choice of centres for the balls of radius  $\gamma_{k+1}$ . If  $x_i$  was our choice of representative for  $B_i^k$  and  $x_i \in B_{i,j}^{k+1}$ , we may as well let  $x_i$  be our choice of representative for  $B_{i,j}^{k+1}$ . If we make our choice of centres in this way, then when we concatenate the rows of some  $A_{k-1}$ , we obtain (without loss) the first row of  $A_k$ . We follow this convention in the two examples below.

**Example 5.** Let us use the above to start a  $\rho$ -ordering of  $S = (\mathbb{Z}, |\cdot|_3)$ . We have that  $\Gamma_S = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\}$  and  $T_s$  begins:



We start by finding a  $\rho_0$ -ordering of  $S_{\gamma_0}$ , but this is trivial since  $S_{\gamma_0}$  has only a single element. Let us pick 0 to be our choice on centre for  $B_1^0 = B(0, 1) = \mathbb{Z}$ . As we see from  $T_S$ ,  $S_{\gamma_0}$  is partitioned into 3 closed balls of radius  $\gamma_1 = \frac{1}{3}$ , namely  $3\mathbb{Z}$ ,  $3\mathbb{Z} + 1$ , and  $3\mathbb{Z} + 2$ . A choice of centres is given by 0, 1, and 2, so that  $A_0$  becomes:

$$A_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

To start the  $\rho$ -ordering, concatenate the rows to obtain  $\{0, 1, 2\}$ , and to continue it, make a choice of centres for each of the closed balls of radius  $\gamma_2 = \frac{1}{9}$  partitioning the sets  $3\mathbb{Z} + i$ ,  $i \in 0, 1, 2$ . For example,  $3\mathbb{Z} = 9\mathbb{Z} \cup 9\mathbb{Z} + 3 \cup 9\mathbb{Z} + 6$ , so a choice of centres for  $B_1^1$  is given by  $\{0, 3, 6\}$ . Making choices for the remaining elements, we obtain:

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

To continue the  $\rho$ -ordering we concatenate the rows,  $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ , which also gives the first row of  $A_2$ . The remaining rows are found by partitioning each of the closed balls of radius  $\frac{1}{9}$  and again making a choice of centres:

$$A_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{pmatrix}$$

And so on.

We are able to make two statements following this example. The first is that in starting the  $\rho$ -ordering, the fact that  $S_{\gamma_0}$  had only a single element allowed us to

get started for free. In fact, all compact ultrametric spaces are bounded, so this is always the case.

The second takeaway is that we found the start of a  $\rho$ -ordering of  $S = (\mathbb{Z}, |\cdot|_3)$  was given by taking the integers starting at 0 in their natural order. If we had continued building the ordering, we would have continued to find this. The fact that the natural ordering on the integers is a  $\rho_p$ -ordering, where  $\rho_p$  is the  $p$ -adic metric for any prime  $p$ , is well known (cf. ...), but we give an alternate proof of it here:

**Corollary 4.** *Let  $S$  be the ultrametric space  $(\mathbb{Z}, \rho_p)$ , where  $\rho_p$  is  $p$ -adic metric for any prime  $p$ . The a  $\rho_p$ -ordering of  $S$  can be found by taking the integers, starting at 0, in their natural order.*

*Proof.* We prove the above by induction on  $k$ . First note that for any choice of prime, the elements of  $S_{\gamma_1}$  are the cosets of  $\mathbb{Z}$  modulo  $p$ , so that  $A_0$  has  $p$  columns. Since  $\{0, 1, 2, \dots, p-1\}$  are distributed among each of these cosets, without loss of generality the first row of  $A_0$  is given by  $[0, 1, 2, \dots, p-1]$  in order.

Now suppose that the first row of  $A_k$  is given by  $[0, 1, 2, \dots, n]$  for  $0 \leq k < k+1$ . We show the first row of  $A_{k+1}$ , and therefore the first  $n'$  elements in a  $\rho_p$ -ordering of  $S$ , where  $n'$  is the column dimension of  $A_{k+1}$ , can be obtained as  $[0, 1, 2, \dots, n, n+1, \dots, n']$ . First note that each closed ball of radius  $p^k = \gamma_k$  is in fact a coset of  $\mathbb{Z}$  modulo  $p^k$ , of which there are  $p$ . Then for any  $k$ ,  $A_k$  is a matrix with  $p^k$  columns and  $p$  rows. In particular,  $n = p^k - 1$ . Let  $i \in \{0, 1, \dots, p^k - 1\}$  be arbitrary. Then  $i$  is in exactly one of the cosets of  $\mathbb{Z}$  modulo  $p^k$  and since the first row of  $A_k$  is  $[0, 1, 2, \dots, p^k - 1]$ , it must have been chosen as our representative of this coset. If we split  $p^k\mathbb{Z} + i$  into balls of radius  $p^{k+1}$ , we have

$$p^k\mathbb{Z} + i = \bigcup_{j=0}^{p-1} p^{k+1}\mathbb{Z} + (p^k j + i)$$

since there will be  $p$  elements in the partition, each of which will be equal to  $i$  modulo  $p^k$  and distinct modulo  $p^{k+1}$ . Then, there is a choice of centres such that the  $i^{th}$  column of  $A_k$  is

$$[i, p^k + i, 2p^k + i, \dots, (p-1)p^k + i]^T$$

filling this in for each  $i$ , we see that  $A_k$  can be obtained as:

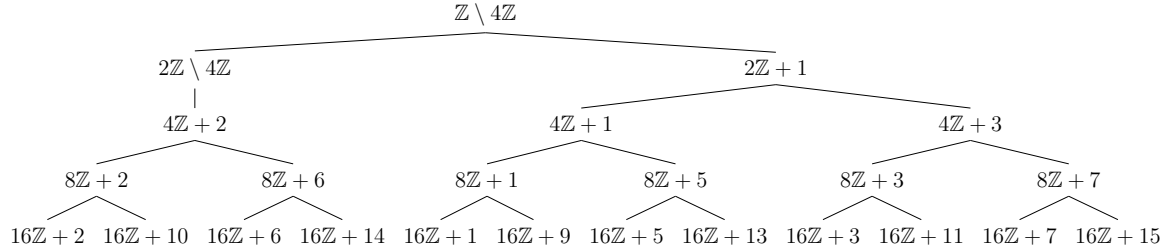
$$A_k = \begin{pmatrix} 0 & 1 & 2 & \dots & p^k - 1 \\ p^k & p^k + 1 & p^k + 2 & \dots & p^k + (p^k - 1) \\ 2p^k & 2p^k + 1 & 2p^k + 2 & \dots & 2p^k + (p^k - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)p^k & (p-1)p^k + 1 & (p-1)p^k + 2 & \dots & (p-1)p^k + (p^k - 1) \end{pmatrix}$$

Concatenating the rows, we see the first row of  $A_{k+1}$  will be

$$[0, 1, 2, \dots, p^k - 1, p^k, \dots, p^{k+1} - 1]$$

as required.  $\square$

**Example 6.** Let us now see an example where there is an uneven number of children between the vertices on a given level. Suppose  $S = \mathbb{Z}_{(2)} \setminus 4\mathbb{Z}$ . In this case, we have that  $\Gamma_S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$  and  $T_s$  begins:



Choosing centres for the partition of  $\mathbb{Z}$  into closed balls of radius  $\frac{1}{2}$ , we have:

$$A_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have taken  $S$  to be the complement of  $4\mathbb{Z}$  in  $\mathbb{Z}_{(2)}$ , so  $B(0, \gamma_1)$  has only one child, since  $2\mathbb{Z} \setminus 4\mathbb{Z} = 4\mathbb{Z} + 2$ , while  $B(1, \gamma_1)$  has two. Making a choice of centres, we have:

$$A_1 = \begin{pmatrix} 2 & 1 \\ * & 3 \end{pmatrix}$$

We concatenate the rows, skipping over  $*$ , and again make a choice of centres for the closed balls of radius  $\frac{1}{8}$ :

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \end{pmatrix}$$

One more iteration yields:

$$A_2 = \begin{pmatrix} 2 & 1 & 3 & 6 & 5 & 7 \\ 10 & 9 & 11 & 14 & 13 & 15 \end{pmatrix}$$

So that a  $\rho_2$ -ordering of  $S = \mathbb{Z}_{(2)} \setminus 4\mathbb{Z}$  starts:  $\{2, 1, 3, 6, 5, 7, 10, 9, 11, 14, 13, 15, \dots\}$ .

In the two propositions above, there was notational difficulty that arose when there was an unequal number of children between the vertices on a given level of  $T_s$ . This difficulty is, in fact, more than a notational inconvenience, and the situation simplifies considerably when it is not the case. We are far from the first to observe this. Amice noted this as far back as her 1964 paper, and it has been observed more recently in several papers by Fares and colleagues. The following section discusses this in more detail, first by supplying some preliminary lemmas and then showing how calculations are simplified in this setting.

### Semi-regularity

In this section, we restrict to the case where in the tree  $T_s$ , for  $S$  some compact subset of an ultrametric space, every vertex on a given level has the same number of children. In this case, we can attach another sequence to  $S$ , which we call the  $\alpha$ -sequence of  $S$  and which describes, for each level  $k \in \mathbb{N}$ , the size of the partitions on that level. We develop some preliminary lemmas, which we then use to derive formulae for this special case.

**Definition 11.** Let  $S$  be as above, a compact subset of an ultrametric space  $(M, \rho)$ . We say that  $S$  is **semi-regular** if  $T_{B_i^k} \cong T_{B_j^k}$ ,  $\forall k \in \mathbb{N}$  and  $i, j \in \beta(k)$ , and where the isomorphism is understood as an isomorphism of trees. That is,  $S$  is semi-regular if each ball of radius  $\gamma_k$  breaks into the same number of balls of radius  $\gamma_{k+1}$ , for all  $k$ . If there exists an  $n \in \mathbb{N}$  such that  $T_{B_i^N} \cong T_{B_j^N}$  for all  $N \geq n$ , i.e. each ball of radius

$\gamma_N$  breaks into the same number of balls of radius  $\gamma_{N+1}$  for  $N \geq n$ , then we say  $S$  is **eventually semi-regular**.

**Definition 12.** Suppose  $S$  is a compact subset of an ultrametric space and  $S$  is semi-regular. The  $\alpha$ -**sequence** of  $S$  is the sequence given by  $\alpha(k) = \frac{\beta(k+1)}{\beta(k)}$ , which is in  $\mathbb{N}$  for each  $k$ . That is, if  $B_i^k$  is an element of  $S_{\gamma_k}$ , then  $\alpha(k)$  is equal to the number of children of  $B_i^k$  in  $T_s$ . Since  $S$  is semi-regular, this number does not depend on  $i$ .

**Lemma 4.**  $\lfloor \frac{n}{q} \rfloor$  counts the numbers strictly less than  $n$  that are congruent to  $n \pmod q$ .

*Proof.* (sketch) Every multiple of  $q$  produces exactly one of the numbers from 1 to  $q$  and exactly one of those is the residue class of  $n$  modulo  $q$ . The remainder is the residue class of  $n$  itself and since we only want the numbers strictly less than  $n$ , we ignore this by taking the floor.  $\square$

**Lemma 5.**

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n + kb}{ab} \rfloor$$

for  $n, a, b \in \mathbb{N}$ . In particular,

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{2b} \rfloor = \lfloor \frac{n + b}{2b} \rfloor$$

for  $n, b \in \mathbb{N}$ .

*Proof.*

$$\begin{aligned} \lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor &= \lfloor a \cdot \frac{n}{ab} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=0}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor - \lfloor \frac{n}{ab} \rfloor \quad (*) \\ &= \sum_{k=1}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n + kb}{ab} \rfloor \end{aligned}$$

where the final step in  $(*)$  is due to Hermite's identity:  $\lfloor nx \rfloor = \sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor$ , for  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ .  $\square$

**Lemma 6.** If  $S$  is semi-regular and  $\delta$  denotes the canonical  $\rho$ -ordering of  $S$ , that is, a  $\rho$ -ordering formed by pulling from left to right in  $T_s$ , then

$$\rho(\delta(n), \delta(m)) = \gamma_k$$

if, and only if,

$$n = m \pmod{\beta(k)} \text{ and } n \neq m \pmod{\beta(k+1)}$$

*Proof.* Since  $S$  is semi-regular, every sequence of  $\beta(k)$  terms in  $\delta$  will be from each of distinct elements of  $S_{\gamma_k}$  (for any  $k$ ). Moreover, since  $\delta$  is a canonical  $\rho$ -ordering, we always pull from the elements of  $S_{\gamma_k}$  in the same order. Then  $\delta(n)$  and  $\delta(m)$  are descendents of some  $B_j^k$  if, and only if,  $n = m \pmod{\beta(k)}$ . Then the result follows since  $\rho(\delta(n), \delta(m)) = \gamma_k$  if, and only if,  $B_i^k$  for some  $i \in 1, \dots, \beta(k)$  is the join of  $B_i^n \ni \delta(n)$  and  $B_{i'}^m \ni \delta(m)$ . □

**Proposition 20.** *If  $S$  is a semi-regular ultrametric space,  $\sigma$  is the characteristic sequence of  $S$ ,  $\beta$  is the structure sequence of  $S$ , and  $\alpha$  is the sequence describing the semi-regularity, then*

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

*Proof.* The exponent of  $\gamma_k$  in the  $n^{\text{th}}$  term of the characteristic sequence is the number of  $m$  strictly less than  $n$  such that  $\rho(\sigma(n), \sigma(m)) = \gamma_k$ . By the lemma above, this is the number of  $m < n$  such that  $m = n \pmod{\beta(k)}$  and  $m \neq n \pmod{\beta(k+1)}$ , which by the previous lemma is  $\lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor$ . Then we have:

$$\begin{aligned} & v_{\gamma_k}(\sigma(n)) \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k)\alpha(k)} \rfloor, \text{ because } S \text{ is semi-regular} \\ &= \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor \end{aligned}$$

□

**Example 7.** Consider the ultrametric space  $(\mathbb{Z}, |\cdot|_p)$  for any prime  $p$ . Then  $\beta(k) = p^k$

and  $\alpha(k) = p$  for any  $k$ . The above gives

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor$$

and since  $\gamma_k = p^{-k}$ ,  $\forall k$ , we have

$$\begin{aligned} v_{\frac{1}{p}}(\sigma(n)) &= \sum_{k=1}^{\infty} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \sum_{k=1}^{\lceil \log_p(n) \rceil} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor + 2\lfloor \frac{n}{p^2} \rfloor - 2\lfloor \frac{n}{p^3} \rfloor + \dots + \lceil \log_p(n) \rceil \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\ &= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \dots + \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\ &= \sum_{k=1}^{\lceil \log_p(n) \rceil} \lfloor \frac{n}{p^k} \rfloor \\ &= \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor \\ &= v_p(n!) \end{aligned}$$

since  $\lfloor \frac{n}{p^k} \rfloor = 0$  if  $p^k > n \iff \log(p^k) > \log(n) \iff k > \log_p(n)$ , so that we are able to recover the well-known Legendre's formula.

Some comment about how the above starts to build up a toolkit independent of algebraic structure in  $S$ .

**Corollary 5.** *If  $G$  is a compact ultrametric group, then  $G$  is semi-regular.*

*Proof.* Since  $G$  is a group, each ball centred at 0 is in fact a subgroup of  $G$ . Then each set of elements of  $S_{\gamma_k}$  is a collection of cosets of  $G/B(0, \gamma_k)$ . Since  $G$  is assumed to be compact,  $G/B(0, \gamma_k)$  is finite and so Lagrange's theorem implies the result.  $\square$

Semi-regularity gives a notion of translation invariance to ultrametric spaces that are not themselves groups. In the previous section, we observed that spaces which



admitted both translation invariance and scaling had valutive capacity that could be computed explicitly via the subadditivity formula. Is there a way to define this for ultrametric which are not normed vector spaces?

**Definition 13.** Let  $S$  be a semi-regular compact subset of an ultrametric space. If there exists a  $q \in \mathbb{N}$  such that  $\alpha(n) = q$ , for all  $n$ , then  $S$  is said to be **regular**. If there exists a  $q$  and  $N$  in  $\mathbb{N}$  such that  $\alpha(n) = q$ , for all  $n \geq N$ , then  $S$  is said to be **eventually regular**.

Alt.:

**Definition 14.** Let  $S$  be a semi-regular compact subset of an ultrametric space. If there exists a  $q_1, \dots, q_m \in \mathbb{N}$  such that  $\alpha(n) = q_i$  if, and only if,  $n = i \pmod m$ , that is,  $\alpha$  has an infinitely-repeating finite subsequence of length  $m$ , then we say  $S$  is **periodic**. If  $m = 1$ , then  $S$  is **regular**.

$S$  is regular just in case the  $\alpha$ -sequence of  $S$  is constant. In what follows, we assume that  $S$  is regular, but the results carry over to the case where  $S$  is periodic with the obvious adjustments. To make use of regularity in  $S$  we should, before continuing, also put some constraints on our sequence of distances  $\Gamma_S$ .

**Definition 15.** Let  $S$  be a compact subset of an ultrametric space and  $\Gamma_S$  is the sequence of decreasing distances in  $S$ . Then we say  $S$  is **well-behaved** (I guess - or tame? reasonable?), if  $S$  is semi-regular and  $\gamma_k = \alpha(k)^{c_k}$  for all  $k \in \mathbb{N}$  and some  $c_k \in \mathbb{Z}$ .

**Example 8.** Any ultrametric space where  $\rho$  is induced from a valuation domain.

If  $S$  is regular with  $\alpha(k) = q$ , for all  $k$ :

$$\begin{aligned} v_{\gamma_k}(\sigma(n)) &= \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor \\ v_{q^{c_k}}(\sigma(n)) &= \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor \\ v_q(\sigma(n)) &= \sum_{k=0}^{\infty} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor) \end{aligned}$$

$$\begin{aligned}
v_q(\sigma(n)) &= \sum_{k=0}^{\lceil \log_q(n) \rceil} c_k \cdot \left( \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor \right) \\
&= c_0 n - c_0 \lfloor \frac{n}{q} \rfloor + c_1 \lfloor \frac{n}{q} \rfloor - c_1 \lfloor \frac{n}{q^2} \rfloor + c_2 \lfloor \frac{n}{q^2} \rfloor - c_2 \lfloor \frac{n}{q^3} \rfloor \dots + \lceil \log_q(n) \rceil \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor - \lceil \log_q(n) \rceil \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor \\
v_q(\sigma(n)) &= \sum_{k=0}^{\lceil \log_q(n) \rceil} (c_{k+1} - c_k) \cdot \left( \lfloor \frac{n}{q^{k+1}} \rfloor \right)
\end{aligned}$$

Note that the coefficient  $(c_{k+1} - c_k)$  is always positive: since  $\Gamma$  is (strictly) decreasing and  $\gamma_k = q^{c_k}$ , the sequence  $c_k$  is also be strictly decreasing (since  $\gamma_k = q^{c_k}$  implies  $\log_q(\gamma_k) = c_k$  and  $\log$  is a monotone function). The above implies that:

$$\omega(S) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\lceil \log_q(n) \rceil} (c_{k+1} - c_k) \cdot \left( \lfloor \frac{n}{q^{k+1}} \rfloor \right)$$

## Chapter 4

### Product space

As a first point of departure, a natural space to consider is the product space of ultrametric spaces, for example  $\mathbb{Z}^n$  (or  $\mathbb{Z}_p^n$  or  $\mathbb{Q}_p^n$ ), for some  $n > 1$ . If we restrict our attention to bounded subsets, then a natural candidate for an ultrametric on the product space is the  $L_\infty$  metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sup_i \{\rho(x_i, y_i)\}$$

where  $\rho$  is the metric from the base space. In fact, since we have only defined valutive capacity for compact subsets of an ultrametric spaces, there is no loss of generality by restricting our metric to bounded spaces. We also see that no problems arise in letting both  $M$  and  $\rho$  vary between components of the space, as long as each  $M_i$  remains bounded and each  $\rho_i$  is an ultrametric.

**Proposition 21.** *Let  $(M_i, \rho_i)$  for  $i$  in some finite or countably infinite index set  $I$  be a collection of metric spaces and suppose  $\rho_i$  is a bounded ultrametric for all  $i$ . Then  $(M, \rho_\infty)$  is an ultrametric space, where  $M = M_1 \times M_2 \times M_3 \times \dots$  and  $\rho_\infty$  is the  $L_\infty$  metric described above.*

*Proof.* Let  $(M, \rho_\infty)$  be the product of ultrametric spaces as above and let  $x$  and  $y$  be two points in the space. Clearly,  $\rho_\infty(x, y) \geq 0$  since each  $\rho_i(x_i, y_i) \geq 0$ , and  $\rho_\infty(x, y) = 0 \iff \rho_i(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$ . The fact that  $\rho_\infty$  is symmetric is also an easy consequence of the fact that each  $\rho_i$  is symmetric since  $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$  implies  $\sup_i \{\rho_i(x_i, y_i)\} = \sup_i \{\rho_i(y_i, x_i)\}$ . To see that  $\rho_\infty$  is an

ultrametric, note that if  $z = z_i$  is any other point of  $M$ , then

$$\begin{aligned}
\rho_\infty(x, y) &= \sup_i \{\rho_i(x_i, y_i)\} \\
&\leq \sup_i \{\max(\rho_i(x_i, z_i), \rho_i(y_i, z_i))\} && \text{since each } \rho_i \text{ is an ultrametric} \\
&\leq \max(\sup_i \{\rho_i(x_i, z_i)\}, \sup_i \{\rho_i(y_i, z_i)\}) && * \\
&= \max(\rho_\infty(x, z), \rho_\infty(y, z))
\end{aligned}$$

\* Let  $M = \max(\sup_i(\{a_i\}, \sup_j(\{b_j\})))$ , then  $M \geq a_i, \forall i$  and  $M \geq b_i, \forall i$ , so  $M \geq \max(a_i, b_i), \forall i$ , hence  $M \geq \sup_i(\max(a_i, b_i))$ .  $\square$

We show a few quick results ultrametric spaces formed as product spaces, which allows us to quickly calculate the valutive capacity of a few subsets.

**Proposition 22.** *Suppose  $(M, \rho_\infty)$  is the product of ultrametric spaces  $(M_i, \rho_i)$  and each  $M_i$  is a topological group with operation  $+$ . Then  $\rho_\infty$  is (left) translation invariant if each  $\rho_i$  is, in which case valutive capacity is also (left) translation invariant.*

*Proof.* Let  $(M, \rho_\infty)$  be as above. Suppose also that

$$\rho_i(x_i, y_i) = \rho_i(s_i + x_i, s_i + y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each  $\rho_i$  is (left) translation invariant. Then,

$$\rho_\infty(s + x, s + y) = \sup_i \{\rho_i(s_i + x_i, s_i + y_i)\} = \sup_i \{\rho_i(x_i, y_i)\} = \rho_\infty(x, y).$$

so that  $\rho_\infty$  is translation invariant. Proposition *xyz* implies valutive capacity is as well.  $\square$

**Proposition 23.** *Let  $(M, \|\cdot\|)$  be a normed vector space and suppose the norm on  $M$  induces an ultrametric  $\rho$ . Let  $(\mathbf{M}, \rho_\infty)$  be the ultrametric space formed by taking products of  $M$ , along with the  $L_\infty$  metric defined above. Then if  $\rho$  is multiplicative on  $M$ ,  $\rho_\infty$  is multiplicative on  $\mathbf{M}$ , in the sense that  $\rho_\infty(\mathbf{c}\mathbf{x}, \mathbf{c}\mathbf{y}) = |c|_\rho \rho_\infty(\mathbf{x}, \mathbf{y})$ , for  $\mathbf{c} = (c, c, c, \dots)$ ,  $\mathbf{x}, \mathbf{y} \in \mathbf{M}$ .*

*Proof.* Let  $M, \rho$ , and  $\rho_\infty$  be as above. Then,

$$\rho_\infty(\mathbf{c}\mathbf{x}, \mathbf{c}\mathbf{y}) = \sup_i \{\rho(c_i x_i, c_i y_i)\} = \sup_i \{|c|_\rho \rho(x_i, y_i)\} = |c|_\rho \sup_i \{\rho(x_i, y_i)\} = |c|_\rho \rho_\infty(x_i, y_i)$$

□

**Corollary 6.** Let  $\mathbf{S}$  be a subset of  $(\mathbf{M}, \rho_\infty)$ , where  $\mathbf{M}$  is the product of an ultrametric space  $(M, \rho)$ , that is itself a normed vector space with a multiplicative norm inducing  $\rho$ . If  $\mathbf{c} = (c, c, c, \dots)$  is an element of  $\mathbf{M}$  with constant value on each component, then  $\omega(\mathbf{c}\mathbf{S}) = |c|_\rho \omega(\mathbf{S})$ .

*Proof.* If  $\{\mathbf{a}_j\}_{j=0}^\infty$  is a  $\rho_\infty$  ordering of  $\mathbf{S}$ , then  $\{\mathbf{c}\mathbf{a}_j\}_{j=0}^\infty$  is a  $\rho_\infty$  ordering of  $\mathbf{c}\mathbf{S}$ . □

**Example 9.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  be the metric space with elements  $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$  and metric  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2))$ , where  $\rho_p$  is the p-adic metric for some fixed prime  $p$ . Since  $\rho_p$  is translation invariant and multiplicative in  $\mathbb{Z}_p$ , valutive capacity is also translation invariant and multiplicative in  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ .

**Example 10.** Let  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$  be the metric space with elements  $\{(x, y) \mid x \in \mathbb{Z}_{p_1}, y \in \mathbb{Z}_{p_2}\}$  for two distinct primes,  $p_1 \neq p_2$ , and metric  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_{p_1}(x_1, y_1), \rho_{p_2}(x_2, y_2))$ , where  $\rho_{p_i}$  is the p-adic metric. Since each  $\rho_{p_i}$  is translation invariant in  $\mathbb{Z}_{p_i}$ , valutive capacity will be translation invariant in  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$ ; however, unlike the case of  $p_1 = p_2$ , this space does not have a multiplicative property that allows for scaling.

### **$n$ -fold products**

What is the valutive capacity of  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  from the example above? Suppose  $p = 2$ . Using translation invariance, scaling and subadditivity, we can compute the result by first noting that we can write  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always  $1 = \text{diam}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ , subadditivity implies that

$$\begin{aligned}
& \frac{1}{\log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} \\
&= \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))} \\
&= \frac{4}{\log(\|2\|_2 * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{\log(\frac{1}{2} * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{\log(\frac{1}{2}) + \log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}
\end{aligned}$$

Taking logs base 2, we have that

$$\omega(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^{\frac{-1 + \log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} 2^{\frac{\log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} (2^{\log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))})^{\frac{1}{4}} = 2^{\frac{-1}{4}} \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\frac{1}{4}}$$

so that  $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$  is a solution of the equation  $x^4 - \frac{x}{2}$ , for which there is a single real positive root, given by  $2^{-1/3}$ .

To compute the valutive capacity for a 2-fold product for an arbitrary prime  $p$ , note that we can always decompose  $\mathbb{Z}_p \times \mathbb{Z}_p$  into a union of  $p^2$  sets each of the form  $\{p\mathbb{Z}_p + s \times p\mathbb{Z}_p + t\}$  for  $s, t \in (0, \dots, p-1)$ , and the pairwise distance between these sets will always be  $1 = \text{diam}(\mathbb{Z}_p \times \mathbb{Z}_p)$  (to see this, either note that we can always find co-prime elements, or note that each set is an closed ball of radius  $1/p$  centred at  $(s, t)$  and so the distance between them must be greater than  $1/p$ , and 1 is the only possible distance greater than  $1/p$  in  $\mathbb{Z}_p \times \mathbb{Z}_p$ ). Then, we combine our tools as before to obtain the equation,

$$\frac{1}{\log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(\|p\|_p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(1/p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, taking logs base  $p$ , we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p) = p^{\frac{-1}{p^2}} \omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{\frac{1}{p^2}}$$

So that  $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$  is a solution of the equation  $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p})$  over  $\mathbb{R}$  and since  $\mathbb{R}$  is a division ring, this means the positive solutions are given by solving  $x^{p^2-1} - \frac{1}{p}$ . Solutions of this equation are of the form  $p^{\frac{-1}{p^2-1}}$  times a  $p^2 - 1$  root of unity, and so there is exactly one positive, real solution, namely  $p^{\frac{-1}{p^2-1}}$  itself. Then the valutive capacity of the entire product space  $\mathbb{Z}_p \times \mathbb{Z}_p$  is  $p^{\frac{-1}{p^2-1}}$ . In fact, from

here it is not hard to see that by taking the  $n$ -fold product, we would end up with the same equation except that the exponent of  $p$  would become  $n$  rather than 2. We arrive at the following result:

**Proposition 24.** *Let  $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$  be the ultrametric space with points equal to the  $n$ -fold product of  $\mathbb{Z}_p$  (for  $n < \infty$ ) for some fixed prime  $p$ . The valutive capacity of  $M$  is  $(\frac{1}{p})^{\frac{1}{p^n-1}}$ .*

*Proof.* Above. □

Taking  $n = 1$ , we see that this agrees with the valutive capacity of  $\mathbb{Z}_p$  computed in the last chapter.

What about  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$  for distinct primes? These spaces do not admit a scaling property, so the same toolset is not available. They are however semi-regular, so we know that

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Suppose  $p_1 = 2$  and  $p_2 = 3$ , so that the  $\alpha$  sequence of  $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$  is  $\alpha = \{6, 2, 3, 2, 2, 3, 2, 3, 2, \dots\}$  and the  $\beta$  sequence is then  $\beta = \{6, 12, 36, 72, 144, \dots\}$ . We know that the capacity of  $S$  will be a product of some negative power of 2 and a negative power of 3. From the above, we know that when  $\alpha(k) = 2$ , we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{2 \cdot \beta(k)} \rfloor$$

and when  $\alpha(k) = 3$ , we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{3 \cdot \beta(k)} \rfloor + \lfloor \frac{n + 2 \cdot \beta(k)}{3 \cdot \beta(k)} \rfloor$$

We also know that if  $\alpha(k) = 2$ , then  $\gamma_k$  must be a (negative) power of 2, and likewise if  $\alpha(k) = 3$ , then  $\gamma_k$  is a power of 3.

Let us first explore the exponent of 2 in  $\sigma(n)$ . We start by noting that if  $\gamma_k$  is

some  $2^{-i}$ , then

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

since there will be a copy of 2 in  $\beta(k)$  for every occurrence of 2 in  $\alpha(0), \dots, \alpha(k)$ , which is also what  $i$  counts. So then, the exponent of  $\frac{1}{2}$  in the  $n^{\text{th}}$  characteristic sequence of  $S$  is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

What can we say about  $j$ , the exponent of 3?

**Lemma 7.** *Let  $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$  and consider the  $k^{\text{th}}$  element of the  $\beta$  sequence of  $S$ ,  $\beta(k) = 2^i \cdot 3^j$ . If  $k$  is such that  $\gamma_k = 2^{-i}$  for some  $i$ , then  $j$  counts the numbers  $a$  in  $\mathbb{Z}_{\geq 0}$  such that  $3^a < 2^i$ .*

*Proof.* (sketch)  $2^i$  only makes it into the sequence after all smaller powers of 3 and 2 have been used, and since we are only considering the case  $\gamma_k$  is a power of 2, we get all the smaller powers of 3.  $\square$

Now note that

$$3^a < 2^i \iff \log_2(3^a) < \log_2(2^i) \iff a \cdot \log_2(3) < i \quad (4.1)$$

So now we are reduced to counting the number of non-negative integers  $a$  that satisfy the above for a given  $i$ . Note that the number of such  $a$ 's will simply be the value of the largest  $a$  plus 1 since  $a$  satisfying the relation implies all  $0 \leq a' \leq a$  solve the relation. Then, we are in fact reduced to finding the largest  $a \in \mathbb{Z}$  that satisfies  $a < \frac{i}{\log_2(3)}$ , but this is exactly  $\lfloor \frac{i}{\log_2(3)} \rfloor$ . This in turn gives  $j = \lfloor \frac{i}{\log_2(3)} \rfloor + 1 = \lceil \frac{i}{\log_2(3)} \rceil$ , since  $\frac{i}{\log_2(3)}$  is never an integer. We now revisit our expression for the exponent of  $\frac{1}{2}$  and substitute our new found value for  $j$ :

$$\begin{aligned} & \sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \rfloor \\ &= \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \rfloor - \lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \rfloor) \end{aligned}$$

**Claim 1.** *the above is irrational.*



We end this section with an observation on the asymptotic behavior of capacity in these spaces. For a fixed prime  $p$ ,  $(\frac{1}{p})^{\frac{1}{p^n-1}}$  is a monotone, increasing sequence in  $n$  with  $\lim_{n \rightarrow \infty} (\frac{1}{p})^{\frac{1}{p^n-1}} = 1$ . For fixed  $n$ , the sequence in  $p$  is also monotone, increasing, again with  $\lim_{p \rightarrow \infty} (\frac{1}{p})^{\frac{1}{p^n-1}} = 1$ . In both cases, the limiting value is equal to the diameter of space. Indeed, we can observe that the sequence  $\{(0, 0, \dots), (1, 0, \dots), (0, 1, \dots), \dots\}$ , in which the first element has only zeros and the  $n$ -th element has a single 1 in the  $(n-1)$ -th component, is a  $\rho$ -ordering for both  $(\mathbb{Z}_p \times \mathbb{Z}_p \times \dots, \rho_{p,\infty})$  and  $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \dots, \rho_{P,\infty})$ , since the distance between elements in this sequence (in either metric space) is always 1. If we could show that these spaces are compact, this would give a valuative capacity of  $\lim_{n \rightarrow \infty} (1^n)^{(1/n)} = 1$  for both spaces. We explore this more in the following section.

## Product topology

In considering the product space of ultrametric spaces, we may wonder whether the chosen metric also gives back the product topology on the space. For products formed by taking some finite number of copies, the answer is positive. We give the necessary background and show this fact, adapting the proof in Munkres (20.3) to the case of ultrametric spaces.

**Definition-Proposition 1.** (*Munkres*) Suppose  $X_i$ , for  $i$  in some index set  $I$ , is a family of topological spaces. Let  $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$  be the map given by projection onto the  $j$ -th component, that is  $\pi_j(x) = \pi_j((x_i)_{i \in I}) = x_j$ . For each  $j \in I$ , let  $\mathcal{S}_j$  be the collection

$$\mathcal{S}_j = \{\pi_j^{-1}(U_j) \mid U_j \text{ open in } X_j\}$$

Let  $\mathcal{S}$  be the union of the  $\mathcal{S}_j$  over  $j \in I$ ,  $\mathcal{S} = \cup_{j \in I} \mathcal{S}_j$ . Then  $\mathcal{S}$  is a subbasis that generates a topology on  $\prod_{i \in I} X_i$  called the **product topology**.

The basis,  $\mathcal{B}$ , generated by  $\mathcal{S}$  in the definition above is the set of all finite intersections of elements in  $\mathcal{S}$ . That is  $B \in \mathcal{B}$  if there exists  $S_1, S_2, \dots, S_n$  in  $\mathcal{S}$  such that  $B = S_1 \cap S_2 \cap \dots \cap S_n$ . A useful description of the basis for the product topology also appears in Munkres, as below:

**Proposition 25.** (*Munkres 19.2*) Suppose  $X_i$ , for  $i$  in some index set  $I$ , is a family of topological spaces and denote by  $\mathcal{B}_i$  the basis for the topology on  $X_i$ . Let

$$\mathcal{B}_P = \prod_{i \in I} B_i, \text{ for } B_i \in \mathcal{B}_i \text{ and } B_i = X_i \text{ for all but finitely-many } i \in I.$$

then  $\mathcal{B}_P$  is a basis for the product topology on  $\prod_{i \in I} X_i$ .

We can now show that the topology induced by the  $L_\infty$  metric described above agrees with the product topology for finite products.

**Proposition 26.** Let  $M = (M_1 \times M_2 \times \dots \times M_n, \rho_\infty)$  be a finite product of bounded, ultrametric spaces and let  $\rho_\infty$  be the metric described above. Then the topology induced by  $\rho_\infty$  coincides with the product topology on  $M_1 \times M_2 \times \dots \times M_n$ .

*Proof.* Let  $\mathcal{T}_{\rho_\infty}$  be the topology on  $M_1 \times M_2 \times \dots \times M_n$  induced by  $\rho_\infty$  and let  $\mathcal{B}_{\rho_\infty}$  be the basis for this topology. Let  $\mathcal{T}_P$  be the product topology with basis  $\mathcal{B}_P$ . We show  $\mathcal{T}_P \subset \mathcal{T}_{\rho_\infty}$  and vice versa. For this, it is equivalent (Munkres 13.3) to show that for  $z \in M_1 \times M_2 \times \dots \times M_n$  and  $B \in \mathcal{B}_P$  containing  $z$ , there is a basis element  $B' \in \mathcal{B}_{\rho_\infty}$  such that  $z \in B' \subset B$ , and vice versa.

So let  $z \in M_1 \times M_2 \times \dots \times M_n$  and suppose  $B \in \mathcal{B}_P$  contains  $z$ . Since  $B$  is in  $\mathcal{B}_P$ ,  $B$  is of the form  $B_{r_1}(z_1) \times B_{r_2}(z_2) \times \dots \times B_{r_n}(z_n)$  (since the choice of centres is arbitrary in an ultrametric spaces, we may choose the components of  $z$  as the centres without loss of generality). Let  $r = \min\{r_i\}$  for  $i \in 1, \dots, n$ . Then let  $B'$  be the ball  $B_r(z)$  in  $\mathcal{B}_{\rho_\infty}$ . Clearly,  $z \in B_r(z)$  and since  $r \leq r_i, \forall i$ ,  $B_r(z) = B_r(z_1) \times B_r(z_2) \times \dots \times B_r(z_n) \subset B_{r_1}(z_1) \times B_{r_2}(z_2) \times \dots \times B_{r_n}(z_n) = B$ .

Conversely, suppose  $A \in \mathcal{B}_{\rho_\infty}$  and let  $y \in A$ . To find  $A' \in \mathcal{B}_P$  such that  $y \in A'$  and  $A' \subset A$ , simply note that  $A$  itself is in  $\mathcal{B}_P$ .

□

We are now naturally left to ask whether the product topology on *infinite* products of ultrametric spaces coincides with the  $L_\infty$  metric. In this case, as in the

analogous case of infinite copies of  $\mathbb{R}$  and a uniform metric, the answer is negative (at least in general). Fortunately, the metric that realizes the product topology on infinite copies of  $\mathbb{R}$  can be adapted to the case of ultrametric spaces. We adapt to the proof of Munkres (20.5) to the case of infinite products of ultrametric spaces.

**Proposition 27.** *Suppose  $\mathbf{M} = M_1 \times M_2 \times \dots$  is an infinite collection of metric spaces, each with an ultrametric  $\rho_i$  which is bounded by 1, that is suppose  $\rho_i(x_i, y_i) \leq 1$ , for all  $x_i, y_i \in M_i$  and for all  $i$ . Define a metric  $d$  on  $\mathbf{M}$  as follows:*

$$d(\mathbf{x}, \mathbf{y}) = \sup_i \left\{ \frac{\rho_i(x_i, y_i)}{i} \right\}$$

*Then  $d$  is an ultrametric and induces the product topology on  $\mathbf{M}$ .*

*Proof.* We see that  $d$  inherits symmetry, injectivity and non-negativity from the requirement that each  $\rho_i$  is a metric, just as  $\rho_\infty$  did. To see that  $d$  satisfies the strong triangle inequality, define a new metric  $\rho'_i$  by  $\rho'_i(x, y) = \frac{\rho_i(x_i, y_i)}{i}, \forall i$ . Then  $\rho'_i$  is an ultrametric, since  $\rho_i(x, y) \leq \max(\rho_i(x, z), \rho_i(y, z))$  implies  $\frac{\rho_i(x, y)}{n} \leq \max(\frac{\rho_i(x, z)}{n}, \frac{\rho_i(y, z)}{n})$  for any  $n \in \mathbb{N}$ . Then we can view  $d$  as the  $L_\infty$  metric on the spaces  $(M_i, \rho'_i)$ , and so  $d$  will be an ultrametric as shown in the first proposition of this section.

Now we show  $d$  induces the product topology. We first show that metric topology induced by  $d$  is finer than the product topology. Let

$$B = B_r^{\mathbf{M}}(\mathbf{z}) = B_r^{M_1}(z_1) \times B_{2r}^{M_2}(z_2) \times B_{3r}^{M_3}(z_3) \times \dots$$

be a basis open in the metric topology. We must find a basis open  $B' \ni z$  in the product topology such that  $B' \subseteq B$ . Let  $N \in \mathbb{N}$  be such that  $\frac{1}{N} < r$ . Then let  $B'$  be the basis open element

$$B' = B_r^{M_1}(z_1) \times B_r^{M_2}(z_2) \times \dots \times B_r^{M_N}(z_N) \times M_{N+1} \times M_{N+2} \times \dots$$

in the product topology. Suppose  $\mathbf{y} \in B'$ . We must show  $\mathbf{y} \in B$ , i.e.,  $d(\mathbf{z}, \mathbf{y}) < r$ . Note that for all  $i \geq N$ ,

$$\frac{\rho_i(z_i, y_i)}{i} \leq \frac{1}{N}$$

which means

$$d(\mathbf{z}, \mathbf{y}) = \sup_i \left\{ \frac{\rho_i(z_i, y_i)}{i} \right\} \leq \max \left\{ \frac{\rho_1(z_1, y_1)}{1}, \frac{\rho_2(z_2, y_2)}{2}, \dots, \frac{\rho_N(z_N, y_N)}{N}, \frac{1}{N} \right\}$$

and since  $N$  was chosen so that  $\frac{1}{N} < r$  and  $B'$  was chosen to have balls of radius  $r$  in the first  $N$  components, we must have  $d(\mathbf{z}, \mathbf{y}) < r$ .

Conversely, ...

□

From now on, we refer to the metric  $d$  above as the **product metric**. An important consequence of the fact that  $d$  achieves the product topology is that Tychonoff's theorem then guarantees that product spaces formed with this metric will be compact, infinite or otherwise. As a result, we can now ask directly about the valuative capacity of some infinite product spaces. We consider two examples.

**Example 11.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \dots, d)$  be the metric space formed by taking the product of  $(\mathbb{Z}_p, \rho_p)$  for some fixed prime  $p$  and let  $d$  be the product metric.

**Example 12.** Let  $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \dots, \rho_{P,\infty})$  be the metric space formed by taking the product of  $(\mathbb{Z}_p, \rho_p)$  for every prime  $p$  and let  $d$  be the product metric.

So far we have two methods for computing valuative capacity. Either we can find a useful decomposition that allows us to apply the subadditivity formula, or we can find a  $\rho$ -ordering and then take the limit of its corresponding  $\rho$ -sequence.

## Conclusion

In this section, we considered the notion of valuative capacity in product spaces, that is, spaces formed by taking copies of ultrametric spaces. In the following sections, we consider valuative capacity in spaces formed by adding points, that is extension fields, or by both taking copies and adding (a distinguished) point, as in projective spaces. For these purposes, it will be more productive to start working over the field  $\mathbb{Q}_p$ , instead of  $\mathbb{Z}_p$ .

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