Introduction

In the course of developing a generalized factorial function, Bhargava introduced the notion of $p-$orderings of a Dedekind domain [2, 3], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [4] (also KJ) and have also provided a natural framework to extend many classical results in analysis to a $p-$adic setting, such as polynomial approximation and mapping theorems [2, 3, 4].

In this thesis, we examine how a tool based on $p-$orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

Background

Ultrametric basics

Definition. Let (M, ρ) be a metric space. If ρ satistifies the ultrametric inequality

$$\rho(x, z) \leq max(\rho(x, y), \rho(y, z)), \forall x, y, z \in M$$

then (M, ρ) is an **ultrametric space**.

Definition. Let (V, N) be a normed vector space. Then N satisfies the **strong trianlge inequality** if

$$N(x + y) \leq max(N(x), N(y)), \forall x, y \in V$$

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space, (V, ρ_N), where ρ_N is the metric induced by N, is an ultrametric space.

Proposition. [1] All triangles in an ultrametric space (M, ρ) are either equilateral or isocoles, with at most one short side.

Proposition. [1] If S is a compact subset of an ultrametric space and Γ_S is the set of all distances occurring between points of S, then Γ_S is a discrete subset of $\mathbb{R}$. In particular if $\mid \Gamma_S \mid = \infty$, then the elements of Γ_S can be indexed by $\mathbb{N}$.

Let (M, ρ) be a compact ultrametric space and let

$$B_r(a) = \{x \in M \mid \rho(x, a) < r\}$$

denote the open ball of radius r, centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$. Likewise let

$$\overline{B_r(a)} = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the closed ball of radius r, centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$.

Proposition. Let $B_r(a)$ be a ball in an ultrametric space (M, ρ). Then the diameter of B, $d = diam(B) = \sup_{x,y \in B} \rho(x, y)$, is less than or equal to the radius of B.

Proposition. If (M, ρ) is an ultrametric space and $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are balls in (M, ρ), then either $B_{r_1}(x_0) \cap B_{r_2}(y_0) = \emptyset$, $B_{r_1}(x_0) \subseteq B_{r_2}(y_0)$, or $B_{r_2}(x_0) \subseteq B_{r_1}(x_0)$. That is, in an ultrametric space, all balls are either comparable or disjoint.

Proposition. [1] The distance between any two balls in an ultrametric is constant. That is, if $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are two balls in an ultrametric space (M, ρ), then $\rho(x, y) = c$ for some $c \in \mathbb{R}$ and $\forall x \in B_{r_1}(x_0)$ and $\forall y \in B_{r_2}(y_0)$

Proposition. [1] Every point of a ball in an ultrametric is at its centre. That is, if $B_r(x_0)$ is a ball in an ultrametric space (M, ρ), then $B_r(x) = B_r(x_0)$, $\forall x \in B_r(x_0)$

ρ-orderings, ρ-sequences, and valuative capacity

In what follows let S be a compact subset of an ultrametric space (M, ρ).

Definition. [5] A **ρ-ordering** of S is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ such that $\forall n > 0$, a_n maximizes $\prod_{i=0}^{n-1} \rho(s, a_i)$ over $s \in S$.

Definition. [5] The **ρ-sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is $\prod_{i=o}^{n-1} \rho(a_n, a_i)$.

Proposition. [5] The ρ-sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ-sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ-ordering of S.

Definition. [5] Let $\gamma(n)$ be the ρ-sequence of S. The **valuative capacity** of S is

$$\omega(S) := \lim_{n \to \infty} \gamma(n)^{1/n}$$

Proposition. [5] For S and $\gamma(n)$ as above, $\lim_{n\to\infty} \gamma(n)^{1/n} = r < \infty$.

Proposition. If $S \subseteq M$ is a finite subset of an ultrametric space, then $\omega(S) = 0$.

Proposition. (upper bound) If $diam(S) := \max_{x,y \in S} \rho(x, y) = d$, then $\omega(S) < d$.

Proof. Since d is the diameter of S, the n^{th} term of the ρ-sequence of S is bounded by d^n and so $\lim_{n\to\infty} \gamma(n)^{1/n} = d$ if and only if $\gamma(n) = d^n$, $\forall n$. This implies $\rho(a_n, a_i) = d$, $\forall n$ and $\forall i < n$, but then $\rho(a_i, a_j) = d$, $\forall i, j$, since the ρ-sequence is maximized at each n. This means $\omega(S) < d$ would imply that the cover of S, $\cup_{a_i} B_d(a_i)$ is in fact an infinite partition, contradicting the compactness of S. Then $\omega(S) = \lim_{n\to\infty} \gamma(n)^{1/n} < d$. □

Proposition. (translation invariance) Let (M, ρ) be a compact ultrametric space and suppose M is also a topological group. If ρ is (left) invariant under the group operation, then so is ω. That is, if $\rho(x, y) = \rho(gx, gy)$, $\forall g, x, y \in M$, then $\omega(gS) = \omega(S)$, for $S \subseteq M$.

Proof. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS. Then

$$\omega(gS) = \lim_{n\to\infty} \gamma(n)^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = \omega(S)$$

□

Example 1. With the notation of the previous section, note that for $x, y \in (\mathbb{Z}_p, |\cdot|_p)$, $\rho_p(x,y) = | x-y |_p = p^{-\nu_p(x-y)} = p^{-\nu_p((a+x)-(a+y))} = | (a+x)-(a+y) |_p = \rho_p(a+x, a+y)$ so that $\omega(a+S) = \omega(S)$ for $S \in (\mathbb{Z}_p, |\cdot|_p)$.

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity. Then if N is multiplicative, so is ω. That is, if $N(gx) = N(g)N(x), \forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.

Proof. Let ρ be the metric induced by N, so that $\rho(x,y) = N(x-y), \forall x, y \in V$. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then since N is multiplicative, for $u, v \in gS$, $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$,

$$\rho(u,v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS and

$$\omega(gS) = \lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty}[\prod_{i=0}^{n-1}N(g)\rho(a_n, a_i)]^{1/n}$$

$$= \lim_{n\to\infty}[N(g)^n\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\omega(S)$$

□

Example 2. Since $|\cdot|_p$ is multiplicative, $\omega(mS) = | m |_p \, \omega(S)$ for $m \in \mathbb{Z}_p$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) = | p |_p \, \omega(\mathbb{Z}) = \frac{1}{p} * p^{\frac{1}{1-p}} = p^{-p/p-1}$.

Proposition. [5](subadditivity) If $diam(S) := \max_{x,y\in S}\rho(x,y) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n}\frac{1}{log(\omega(A_i)/d)}$$

Corollary. Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = diam(S)$ and also $\omega(S_i) = \omega(S_j)$, $\forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S)$, $\forall i$. Then $\omega(S) = r^{\frac{1}{n-1}} \cdot d$. In particular if $S = \mathbb{Z}$ and

$(M, \rho) = (\mathbb{Z}, |\cdot|_p)$ then $\omega(S) = (\frac{1}{p})^{1/p-1}$ for any prime p.

Corollary. (Joins of computable sets are computable) Let $\Gamma_M = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in M. Suppose that $S = B_{\gamma_i}(x)$, for some x and i, is the union of 2 or more balls of radius γ_{i+1}, i.e., $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$ is a join in the lattice of open sets in M, then

$$\frac{1}{log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^{n} \frac{1}{log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

Computing a ρ-ordering

We describe an algorithm for computing the ρ-ordering of a set recursively and discuss some immediate corollaries.

Let $S \subseteq M$ be a compact subset of an ultrametric space (M, ρ). Let $\Gamma_S = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in S. Note that for each $k \in \mathbb{N}$, the closed balls of radius γ_k partition S, i.e., $S = S_{\gamma_k} := \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$, where both n and the x_i's depend on k. In what follows, fix a $k \in \mathbb{N}$ and let $S_{\gamma_k} = \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$ be such a partition. Note that we can construct $S_{\gamma_{k+1}}$ by partitioning each of the $\overline{B_{\gamma_k}(x_i)}$, i.e.,

$$S = S_{\gamma_{k+1}} = \cup_{i=1}^{n} \cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})}$$

where $1 \leq l_i \leq n$ and $\cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})} = \overline{B_{\gamma_k}(x_i)}$, $\forall i$. We denote by $x_{i,j}$ the centre of a ball of radius γ_{k+1} partitioning the ball $B_{\gamma_k}(x_i)$. Without loss of generality, when $j = 1$, assume $x_{i,j} = x_i$, $\forall i$.

We now make the following observation due to [6],

Lemma. For each k, the elements of S_{γ_k}, that is, the closed balls of radius γ_k, themselves form an ultrametric space, where

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x,y), & \text{if } \rho(x,y) > \gamma_k \\ 0, & \text{if } \rho(x,y) \leq \gamma_k, \text{ i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

We note that since S is assumed to be compact, S_{γ_k} is a finite metric space $\forall k < \infty$ and $S_{\gamma_\infty} = \cup_{x \in S} \overline{B_0(x)} = \cup_{x \in S} x = S$ and $\rho_\infty = \rho$. Now view S_{γ_k}, for fixed $k < \infty$ as a finite ultrametric space and represent its $n < \infty$ elements by their centres, the x_i's. Without loss of genearlity, we can reindex the x_i's so that they give the first n terms of a ρ_k-ordering of S_{γ_k}. The following proposition is the main result of this section.

Proposition. Given S a compact subset of an ultrametric space M and Γ_S, the set of distances in S, if S_{γ_k} is a partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the centres of the balls are indexed according to a ρ_k-ordering of S_{γ_k}, then a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ can be found by forming a matrix, A_k, whose $(i,j)^{th}$-entry is $x_{i,j}$, as shown below, and then concatenating the rows (where the columns are padded by * if necessary).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \dots & x_{n,l_n} \end{pmatrix}$$

Proof. Note that the entries in each column are points in the ball $B_{\gamma_k}(x_i)$ so that the pairwise distance between columns is constant and always exceeds the distance between elements within a column. Moreover, the columns are organized such that for any j, $x_{n,j}$ maximizes$\prod_{i=1}^{n-1} \rho_(x_{n,j}, x_{i,j})$ since $\prod_{i=1}^{n-1} \rho_(x_{n,j}, x_{i,j}) = \prod_{i=1}^{n-1} \rho_(x_{n,1}, x_{i,1}) = \prod_{i=1}^{n-1} \rho_(x_n, x_i)$ and the x_i's are indexed according a ρ_k-ordering of S_{γ_k}.

Then a $\rho_{\gamma_{k+1}}$-ordering of $S_{\gamma_{k+1}}$ is obtained by minimizing the number of elements from any one column and by taking the points $x_{i,j}$ (for fixed j) in sequence. For example, by concatenating the rows. □

Corollary. Interweaving the bottown row of the lattice of closed balls for a set S gives a ρ-ordering of S.

Corollary. Suppose S and T are compact subsets of an ultrametric space M with $\Gamma_S = \Gamma_T$ and $\mid S_{\gamma_k} \mid = \mid T_{\gamma_k} \mid$, $\forall k$. Then $\omega(S) = \omega(T)$.

Corollary. (regularity) Suppose that S is such that $\forall k$, any $B_{\gamma_k}(x) = \cup_{j=1}^{l} B_{\gamma_{k+1}}(x_j)$, that is, every ball in S breaks into exactly l smaller balls.

algebraic extensions

Type up the example Keith gave you.

product space

Example 3. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$ and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1)), \rho_p(x_2, y_2))$. Consider it also as a topological group with operation $(g_1, g_2) + (x_1, x_2) = (g_1 + x_1, g_2 + x_2)$. Then $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1)), \rho_p(x_2, y_2) = \max(\rho_p(g_1 + x_1, g_1 + y_1)), \rho_p(g_2 + x_2, g_2 + y_2) = \rho_{p,\infty}(((g_1, g_2) + (x_1, x_2)), ((g_1, g_2) + (y_1, y_2)))$, and $\omega((g_1, g_2) + S) = \omega(S)$ for $S \in (\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$.

Example 4. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, |\cdot|_{p,\infty})$ be the vector space with elements $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$ and norm $\mid (x_1, x_2) \mid_{p,\infty} = \max(\mid x_1 \mid_p, \mid x_2 \mid_p)$. Then $\mid (g, g)(x_1, x_2) \mid_{p,\infty} = \max(\mid gx_1 \mid_p, \mid gx_2 \mid_p) = \max(\mid g \mid_p \mid x_1 \mid_p, \mid g \mid_p \mid x_2 \mid_p = \mid (g, g) \mid_p \mid (x_1, x_2) \mid_p$, so that $\mid \cdot \mid_{p,\infty}$ is multiplicative for $(g, g) \in \mathbb{Z}_p \times \mathbb{Z}_p$. Then $\omega((g, g)S) = \mid (g, g) \mid_p \omega(S)$. In particular, $\omega((p, p)\mathbb{Z} \times \mathbb{Z}) = \omega(p\mathbb{Z} \times p\mathbb{Z}) = \mid (p, p) \mid_p \omega(\mathbb{Z} \times \mathbb{Z}) = p^{-1}\omega(\mathbb{Z} \times \mathbb{Z})$.

projective space

Background from Gerritzen and van der Put

Background results from [7]. Let k be a field that is complete with respect to a non-archimedean valuation and let K be a complete and algebraically closed field containing k.

Definition. [7] The set $\{\lambda \in k; \mid \lambda \mid \leq 1\}$, denoted k^0, is the **valuation ring** of k. It has a unique maximal ideal, denoted k^{00}, given by $\{\lambda \in k; \mid \lambda \mid < 1\}$. The **residue field** of k is $\bar{k} := k^0/k^{00}$.

Definition. [7] The **projective line over** k, denoted $\mathbb{P}^1(k)$, is the space whose points are lines l in k^2 that intersect $(0,0)$ and whose topology and field structure are inherited from k.

We give two equivalent representations for the points of $\mathbb{P}^1(k)$. A point $p \in \mathbb{P}^1(k)$ is an equivalence class of $k^2 \setminus (0,0)$ under the relation $(x,y) \sim (x',y')$ if there exists a $\lambda \in k \setminus 0$ such that $(x,y) = \lambda(x',y')$. Equivalently, suppose that l is a line in k^2 intersecting the origin, that is a point in $\mathbb{P}^1(k)$. We denote l by a representative $[x_0, x_1] \in k^2$ such that $l = \{\lambda(x_0,x_1); \lambda \in k\}$, called homogeneous coordinates of l.

Proposition. [7] Let $\psi : k \to \mathbb{P}^1(k)$ be the map given by $\psi(\lambda_0) = [1, \lambda_0]$, where $[1, \lambda_0]$ is the line in k^2, $\{\lambda(1,\lambda_0); \lambda \in k\}$. Then the image of ψ is $\mathbb{P}^1(k) \setminus [0,1]$ and is isomorphic to k, so that k is identified with projective space minus a distinguished point, $[0,1]$, which is denoted by ∞.

Definition. [7] k is called a **local field** if k is locally compact.

Proposition. [7] The following are equivalent:

1. k is a local field.

2. $\mid k^* \mid \cong \mathbb{Z}$ and $\bar{k}$ is finite, where $k*$ is the set of units in k, ie $k^* = \{\lambda \in k, \lambda \neq 0\}$.

3. k is a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

4. $\mathbb{P}^1(k)$ is compact

Definition. [7] We denote by $GL(2,k)$ the set of invertible 2×2 matrices over k. A **fractional linear automorphism**, ϕ, of $\mathbb{P}^1(k)$ is a map defined by $z \mapsto \frac{az+b}{cz+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,k)$. The set of fractional linear automorphisms of $\mathbb{P}^1(k)$ is denoted $PGL(2,k)$. Note that $PGL(2,k) = GL(2,k)/\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right); \lambda \in k^*\}$. In homogeneous coordinates, we can represent the action of ϕ by $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$.

Definition. [7] Suppose Γ is a subgroup of $PGL(2,k)$. A point $p \in \mathbb{P}^1(k)$ is a **limit point of** Γ, if there exists a point q in $\mathbb{P}^1(k)$ and a sequence $\{\gamma_n\}_{n\geq 1}$ in Γ such that $\lim_{n\to\infty} \gamma_n(q) = p$.

Proposition. [7] If Γ is not a discrete subgroup of $PGL(2,k)$ then every point of $\mathbb{P}^1(k)$ is a limit point of Γ.

Proof. Since Γ is not discrete, the sequence $\{\gamma_n\}_{n\geq 1}$ has a limit γ in Γ. Let p be any point of $\mathbb{P}^1(k)$ and let $q = \gamma^{-1}(p)$. Then $\lim_{n\to\infty} \gamma_n(q) = \lim_{n\to\infty} \gamma_n(\gamma^{-1}(p)) = p$. $\square$

Definition. [7] A subgroup Γ of $PGL(2,k)$ is **discontinuous** if the closure of every orbit of Γ in $\mathbb{P}^1(k)$ is compact and the set of all limit points of Γ is not equal to $\mathbb{P}^1(k)$.

Proposition. [7] If Γ is a discontinuous subgroup of $PGL(2,k)$ and $\mathcal{L}$ is the set of limit points of Γ, then $\mathcal{L}$ is compact, no where dense and if $\mathcal{L}$ contains more than two points, $\mathcal{L}$ is perfect.

Definition. [7] Let A be an element of $GL(2,k)$ and let a_1 and a_2 be the eigenvalues of A. Then A is called **elliptic** if $a_1 \neq a_2$, but $\mid a_1 \mid = \mid a_2 \mid$. A is called **parabolic** if $a_1 = a_2$, and A is called **hyperbolic** if $\mid a_1 \mid \neq \mid a_2 \mid$.

Example 5. Consider the matrix $T_s = \left(\begin{smallmatrix} p & s \\ 0 & 1 \end{smallmatrix}\right) \in GL(2,\mathbb{Q}_p)$ for some s in $(0,\ldots,p-1)$ (note that $det(T_s) = p$ is invertible in $\mathbb{Q}_p$, so that $T_s \in GL(2,\mathbb{Q}_p)$, although it is not in $GL(2,\mathbb{Z}_p)$). T_s has eigenvalues p and 1 and so T_s is hyperbolic for any choice of s or p. Consider the action of T_s on $\mathbb{Z}_p \subset \mathbb{Q}_p$, where $\mathbb{Z}_p$ is identified with the subspace $\{[1,\lambda]; \lambda \in \mathbb{Z}_p\}$ of $\mathbb{P}^1(\mathbb{Q}_p)$. In homogeneous coordinates, this action is given by $[1,\lambda] \mapsto [1, p\lambda + s]$. Since $\mid (p\lambda + s - s) \mid = \mid p\lambda \mid \leq \frac{1}{p}$, T_s sends λ to $B_{\frac{1}{p}}(s)$. Also note that for $s = 0$, T_s has the effect of shifting the index of λ by 1, that is, if $\lambda = \sum_{i=n}^{\infty} a_i p^i$, where $n = ord(\lambda)$, then $T_0([1,\lambda]) = [1, p\lambda] \rightsquigarrow p\lambda = \sum_{i=n+1}^{\infty} a_{(i-1)} p^i$.

Computation of the capacity of some sets

(F&P, section 5)

Setup

Let $A = \{0, 1, .., d-1\}$ be a finite alphabet and $A^{\mathbb{N}}$ be the collection of infinite sequenes with values in A. Note $A^{\mathbb{N}}$ is a Cantor set, so it is perfect, nowhere dense, and compact.

A basis for the topology is given by the cylinder set: take countably many copies of $\{0, 1, ..., d-1\}$ where each copy has the discrete topology.

Let $p \geq d$ be a prime number and let ϕ be the canonical embedding of $A^{\mathbb{N}}$ into $\mathbb{Z}_p$ via the following continuous (under the above topology) map:

$$\phi : A^{\mathbb{N}} \to \mathbb{Z}_p \text{ by } (x_n)_{n \geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

Lemma. (F&P Lemma 5.1)

Let $w_1, w_2, \dots, w_s$ be $s \geq 2$ words with the same length l such that all the first letters are distinct. Let $X \subset A^{\mathbb{N}}$ be the set of sequences such that any factor is a factor of a concatenation of the words $w_1, w_2, \dots, w_s$. Then the set $E := \phi(X) \subset \mathbb{Z}_p$ satisfies:

$$E = \cup_{i=1}^{s} x_i + p^l E, \text{ with } x_i = \phi(w_i 0^{\infty})$$

It is a regular compact set and its valuative capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

Example 6. $w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$

Then $\{x_n\}_{n\geq 0} \in X$ if each term in $\{x_n\}_{n\geq 0}$ is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Note that we can rephrase the lemma as follows:

Let $x_1, x_2, \ldots, x_s$ be $s \geq 2$ points in $\mathbb{Z}_p$ such that $\mid x_i - x_j \mid_p = 1$, $\forall i, j \in 1, ..., s$. Suppose also that there exists an $l \in \mathbb{N}$ such that $\forall i$,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{l} a_i p^i$$

Let γ_i be the fractional linear automorphism of $\mathbb{P}^1(\mathbb{Q}_p)$ given by $\left(\begin{smallmatrix} p^l & x_i \\ 0 & 1 \end{smallmatrix}\right)$ and let Γ be the subgroup of $PGL(2, \mathbb{Q}_p)$ generated by the γ_i.

If $\mathcal{L}$ is the limit set of Γ, and Z is the subset of $\mathbb{Q}_p$ such that $Z = \psi^{-1}(\mathcal{L})$, (where $\psi : \mathbb{Q}_p \to \mathbb{P}^1(\mathbb{Q}_p)$ is the map given by $\psi(\lambda_0) = [1, \lambda_0]$) then Z is a regular, compact subset of $\mathbb{Z}_p$ satisfying

$$Z = \cup_{i=1}^{s} x_i + p^l Z = \cup_{i=1}^{s} B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

Proof. It suffices to show that the set Z above is equal to $E = \phi(X)$ in the original lemma. First note that if $w_1, w_2, ..., w_s$ are words in $A^{\mathbb{N}}$, then the first letter of each w_i is distinct if and only if $\mid \phi(w_i) - \phi(w_j) \mid_p = 1, \forall i, j$ (since the pairwise distance is 1 if and only if $ord(\phi(w_i) - \phi(w_j)) = 0$ for any i and j, if and only if the coefficient of p^0 (i.e., the first letter each w_i) is different $\forall i, j$). So then the x_i are just the $\phi(w_i)$.

We now consider the limit set of Γ. First consider an arbitrary element $\gamma \neq id_\Gamma \in \Gamma$. If $\gamma = \prod_{j\in J}\gamma_j$ for some finite index set J, then we can write γ as

$$\left(\begin{smallmatrix} p^l & a_0+a_1p+\ldots+a_lp^l \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} p^l & b_0+b_1p+\ldots+b_lp^l \\ 0 & 1\end{smallmatrix}\right)\ldots\left(\begin{smallmatrix} p^l & c_0+c_1p+\ldots+c_lp^l \\ 0 & 1\end{smallmatrix}\right)$$

$$=\left(\begin{smallmatrix} p^{l^n} & a_0+a_1p+\ldots+a_lp^l+b_0p^{l+1}+b_1p^{l+2}+\ldots+b_lp^{2l}+\ldots+c_0p^{(n-1)l+1}+c_1p^{(n-1)l+2}\ldots+c_lp^{nl} \\ 0 & 1\end{smallmatrix}\right)$$

where n is the cardinality of J. Then an element of Γ is of the form $\left(\begin{smallmatrix} p^{lm} & z \\ 0 & 1\end{smallmatrix}\right)$, where $m \in \mathbb{N}$ and z is an element of $\mathbb{Z}_p$ whose coefficient vector is a concatenation of the coefficient vectors of the x_i (for $0 \leq i \leq ml$ and 0 for $i > ml$).

Let $a = [a_0, a_1] \in \mathbb{P}^1(\mathbb{Q}_p)$ and let $\{\gamma_n\}$ be a sequence in Γ. We have that

$$\lim_{n\to\infty}\gamma_n(a) = \lim_{n\to\infty}[a_0, p^{nl}a_1 + z_n] = [a_0, z],$$

where the coefficient vector of each z_n is a concatenation of the coefficient vectors of the x_i, for finitely-many terms (and then 0s), and z is an element of $\mathbb{Z}_p$ whose entire coefficient vector is a concatenation of the coefficient vectors of the x_i. Then the limit set of Γ is the set

$$\mathcal{L} = \{[\lambda, x]; \lambda \in \mathbb{Q}_p^*, x \in S\},$$

where S is the set of $x \in \mathbb{Z}_p$, such that the entire coefficient vector of x is a concatenation of the coefficient vectors of the x_i; that is S is the set $S = E = \phi(X)$ in the original lemma. Now we observe that $\psi^{-1}(\{[1, x]; x \in S\}) = S = E = \phi(X)$ and $\psi^{-1}(\{[\lambda, x]; x \in S\}) = \emptyset$ for any other $\lambda \neq 1 \in \mathbb{Q}_p^*$, so that $Z = \psi^{-1}(\mathcal{L}) = S \cup \emptyset = S = E = \phi(X)$, as required.

□

More background:

Definition. [7] A **Schottky group** is a finitely-generated, free and discontinuous subgroup of $PGL(2, k)$

Observation: suppose $z \in \mathbb{Z}_p$ is such that $z = \sum_{i=0}^{\infty} a_ip^i = \sum_{i=0}^{n} a_ip^i$ for some $n \in \mathbb{N}$. Then

$B_{\frac{1}{p^n}}(z)$ is the set of all $y \in \mathbb{Z}_p$ of the form $y = \sum_{i=0}^{n} a_i p^i + \sum_{i=n+1}^{\infty} b_i p^i$.

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