Introduction

In the course of developing a generalized factorial function, Bhargava introduced the notion of $p-$orderings of a Dedekind domain [2, 3], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [4] (also KJ) and have also provided a natural framework to extend many classical results in analysis to a $p-$adic setting, such as polynomial approximation and mapping theorems [2, 3, 4].

In this thesis, we examine how a tool based on $p-$orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

Background

Ultrametric basics

We begin by going over definitions and basic results about ultrametric spaces.

Definition. Let (M, ρ) be a metric space, i.e., suppose M is a set and $\rho : M \times M \to \mathbb{R}_{\geq 0}$ is such that:

- $\rho(x, y) = 0$ if and only if $x = y$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any $x, y, z \in M$. If ρ satistifies the ultrametric inequality,

- $\rho(x, z) \leq max(\rho(x, y), \rho(y, z))$

for any $x, y, z \in M$, then (M, ρ) is an **ultrametric space**.

Definition. Let (V, N) be a normed vector space, i.e., suppose V is $\mathbb{F}-$vector space, $\mathbb{F}$ a subfield of $\mathbb{C}$, and $N : V \to \mathbb{R}_{\geq 0}$ is such that:

- $N(x + y) \leq N(x) + N(y)$
- $N(cx) = |c|N(X)$
- $N(x) = 0$ implies $x = 0$

for any $x, y \in V$ and $c \in \mathbb{F}$. We say that N satisfies the **strong trianlge inequality** if

$$N(x+y) \leq max(N(x), N(y))$$

for any $x, y \in V$.

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space, (V, ρ_N), where ρ_N is the metric induced by N, i.e. $\rho_N(x, y) = N(x - y)$, is an ultrametric space.

Definition. The **diameter of** S is $diam(S) = \max\limits_{x,y \in S} \rho(x, y)$.

Proposition. [1] All triangles in an ultrametric space (M, ρ) are either equilateral or isosceles, with at most one short side.

Notation. Let (M, ρ) be a compact ultrametric space and let

$$B_r(a) = \{x \in M \mid \rho(x, a) < r\}$$

denote the open ball of radius r, centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$. Likewise let

$$\overline{B_r(a)} = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the closed ball of radius r, centred at a for some $r \in \mathbb{R}_{\geq 0}$ and $a \in (M, \rho)$.

Proposition. Let $B_r(a)$ be an open ball in an ultrametric space (M, ρ). Then the diameter of B, $d = diam(B) = \sup_{x,y \in B} \rho(x, y)$, is less than or equal to the radius of B.

Proposition. If (M, ρ) is an ultrametric space and $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are balls in (M, ρ), then either $B_{r_1}(x_0) \cap B_{r_2}(y_0) = \emptyset$, $B_{r_1}(x_0) \subseteq B_{r_2}(y_0)$, or $B_{r_2}(x_0) \subseteq B_{r_1}(x_0)$. That is, in an ultrametric space, all balls are either comparable or disjoint.

Proposition. [1] The distance between any two balls in an ultrametric is constant. That is, if $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are two balls in an ultrametric space (M, ρ), then there exists a $c \in \mathbb{R}_{\geq 0}$ such that $\rho(x, y) = c$, $\forall x \in B_{r_1}(x_0)$ and $\forall y \in B_{r_2}(y_0)$

Proof. Write a proof - this is because all triangesl are isosceles. □

Proposition. [1] Every point of a ball in an ultrametric is at its centre. That is, if $B_r(x_0)$ is a ball in an ultrametric space (M, ρ), then $B_r(x) = B_r(x_0)$, $\forall x \in B_r(x_0)$

Proposition. [1] If S is a compact subset of an ultrametric space and Γ_S is the set of all distances occurring between points of S, then Γ_S is a discrete subset of $\mathbb{R}$. In particular if $\mid \Gamma_S \mid = \infty$, then the elements of Γ_S can be indexed by $\mathbb{N}$.

Proof. This is not standard - there is a proof in [1] on page 72 □

It will become useful to write the set of distances occuring in S as a sequence, put in decreasing order. We use the following notation to represent such a sequence

$$\Gamma_S = \{\gamma_0 = d = diam(S), \gamma_1, \gamma_2, \ldots, \gamma_\infty = 0\}$$

where $\gamma_i \in \Gamma_S$ if and only if $\exists x, y \in S$ such that $\rho(x, y) = \gamma_i$ and $\gamma_i < \gamma_j$ if and only if $i > j$.

ρ-orderings, ρ-sequences, and valuative capacity

In what follows let S be a compact subset of an ultrametric space (M, ρ).

Definition. [5] A ρ-**ordering** of S is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ such that $\forall n > 0$, a_n maximizes $\prod_{i=0}^{n-1} \rho(s, a_i)$ over $s \in S$.

Example 1. Suppose S is a finite subset of $(\mathbb{Z}, |\cdot|_2)$, $S = \{0, 2, 8, 3\}$. Then a $\rho-$ordering of S starts (arbitrarily) with $a_0 = 0$, which forces $a_1 = 3$, since $\rho(0, 3) = 1 = diam(S)$. The sequence continues $a_2 = 2$ and $a_3 = 8$, but after this point the sequence becomes arbitrary because $\prod_{i=0}^{n-1} \rho(s, a_i)$ will contain a 0, given by the repeated term. Indeed, for any finite subset S with $|S| = n$, the $\rho-$ordering of S is arbitrary from the n^{th} point on.

Definition. [5] The ρ-**sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is $\prod_{i=o}^{n-1} \rho(a_n, a_i)$.

Proposition. [5] The ρ-sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ-sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ-ordering of S.

Definition. [5] Let $\gamma(n)$ be the ρ-sequence of S. The **valuative capacity** of S is

$$\omega(S) := \lim_{n\to\infty} \gamma(n)^{1/n}$$

Proposition. [5] For S and $\gamma(n)$ as above, $\lim_{n\to\infty} \gamma(n)^{1/n} = r < \infty$.

Proposition. If $S \subseteq M$ is a finite subset of an ultrametric space, then $\omega(S) = 0$.

Proposition. (upper bound) If $diam(S) = d$, then $\omega(S) < d$.

Proof. Since d is the diameter of S, the n^{th} term of the ρ-sequence of S is bounded by d^n and so $\lim_{n\to\infty} \gamma(n)^{1/n} = d$ if and only if $\gamma(n) = d^n$, $\forall n$. This implies $\rho(a_n, a_i) = d$, $\forall n$ and $\forall i < n$, but then $\rho(a_i, a_j) = d$, $\forall i, j$, since the ρ-sequence is maximized at each

n. This means $\omega(S) < d$ would imply that the cover of S, $\cup_{a_i} B_d(a_i)$ is in fact an infinite partition, contradicting the compactness of S. Then $\omega(S) = \lim_{n\to\infty} \gamma(n)^{1/n} < d$. $\square$

This doesn't work because $\cup_{a_i} B_d(a_i)$ could fail to be a cover -when does this happen

Proposition. (translation invariance) If (M, ρ) be a compact ultrametric space and s also a topological group for which ρ is (left) invariant under the group operation, then ω is also (left)-invariant. That is, if $\rho(x, y) = \rho(gx, gy)$, $\forall g, x, y \in M$, then $\omega(gS) = \omega(S)$, for $S \subseteq M$.

Proof. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS. Then

$$\omega(gS) = \lim_{n\to\infty} \gamma(n)^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = \omega(S)$$

$\square$

Example 2. With the notation of the previous section, note that for $x, y \in (\mathbb{Z}_p, |\cdot|_p)$, $\rho_p(x, y) = \mid x - y \mid_p = p^{-\nu_p(x-y)} = p^{-\nu_p((a+x)-(a+y))} = \mid (a + x) - (a + y) \mid_p = \rho_p(a + x, a + y)$ so that $\omega(a + S) = \omega(S)$ for $S \subseteq (\mathbb{Z}_p, |\cdot|_p)$.

Proposition. Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity. Then if N is multiplicative, so is ω. That is, if $N(gx) = N(g)N(x)$,$\forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.

Proof. Let ρ be the metric induced by N, so that $\rho(x, y) = N(x - y), \forall x, y \in V$. Let $\{a_i\}_{i=0}^{\infty}$ be a ρ-ordering for S. Then since N is multiplicative, for $u, v \in gS$, $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then $\{ga_i\}_{i=0}^{\infty}$ is a ρ-ordering for gS and

$$\omega(gS) = \lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty}[\prod_{i=0}^{n-1}N(g)\rho(a_n, a_i)]^{1/n}$$

$$= \lim_{n\to\infty}[N(g)^n\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\omega(S)$$

$\square$

Example 3. Since $|\cdot|_p$ is multiplicative, $\omega(mS) =| m |_p \omega(S)$ for $m \in \mathbb{Z}_p$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) =| p |_p \omega(\mathbb{Z}) = \frac{1}{p}\cdot p^{\frac{1}{1-p}} = p^{-p/p-1}$.

The following proposition is from [5], where it is given for some S written as the union of two subsets, although it is easily seen to be true for S equal to any finite union, so long as the other assumptions remain satisfied.

Proposition. [5](subadditivity) If $diam(S) := \max_{x,y\in S}\rho(x, y) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n}\frac{1}{log(\omega(A_i)/d)}$$

Example 4.

- Also talk about when cosets and open balls coincide

Corollary. Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = diam(S)$ and also $\omega(S_i) = \omega(S_j)$, $\forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S)$, $\forall i$. Then $\omega(S) = r^{\frac{1}{n-1}}\cdot d$. In particular if $S = \mathbb{Z}$ and $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$ then $\omega(S) = (\frac{1}{p})^{1/p-1}$ for any prime p.

Corollary. (Joins of computable sets are computable) Let $\Gamma_M = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in M. Suppose that $S = B_{\gamma_i}(x)$, for some x and i, is the union of 2 or more balls of radius γ_{i+1}, i.e., $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$ is a join in the lattice of open sets in M, then

$$\frac{1}{log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^{n}\frac{1}{log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

Computing a ρ-ordering

In the previous section, we defined valuative capacity for a compact subset S of an ultrametric space (M, ρ). We also got a glimpse into the way the valuative capacity of S interacts with its other properties, such as the set of distances occurring in S and the lattice of closed balls in S (or equivalently, if S has enough structure, the lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valuative capacity of S and the lattice of closed balls in S. In particular, we show how, in all cases (with S compact), the latter can be used to compute the first n terms of a $\rho-$ordering of S (for any $n < \infty$) and how, in some cases, this extends to being able to compute the valuative capacity of S.

We begin by letting S be, as before, a compact subset of an ultrametric space (M, ρ), and by letting $\Gamma_S = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in S. Now fix some $k \in \mathbb{N}$, and consider for a moment the set of closed balls of radius γ_k in S. We could denote these alternatively by $B^M(x, \gamma_k) \cap S$ or by $B^S(x, \gamma_k)$, but when there is no risk of confusion, we will denote them simply by $B(x, \gamma_k)$. Clearly, the set $\{B(x, \gamma_k); x \in S\}$ forms a cover of S and since S is compact, we must have some $x_1, \ldots, x_n$ such that $S = \cup_{i=1}^n B(x_i, \gamma_k)$. In fact, since ρ is an ultrametric, $\cup_{i=1}^n B(x_i, \gamma_k)$ will be a disjoint union and therefore a partition of S. Note that both n and the x_i's depend on our fixed k, but that n is independent of the x_i's, since any choice of centres is equivalent. We rephrase this with following definition and lemma:

Definition. For S and Γ_S as above, and $k \in \mathbb{N}$, fixed, define $\sim_k$ to be the relation on S given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e., $x \sim_k y$ if and only if $\overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)}$.

The fact $\sim_k$ is an equivalence relation on S is equivalent to the observation that every point in a ultrametric ball is at its centre:

Lemma. Let S and Γ_S be as above, then $\sim_k$ is an equivalence relation on S.

Proof. $\sim_k$ is clearly reflexive and symmetric, since ρ is a metric. Transitivity results from the ultrametric property of ρ: if $x \sim_k y$ and $y \sim_k z$, then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(z, y)) \leq \gamma_k$$

so $x \sim_k z$. □

We denote the set of equivalence classes of $S/\sim_k$ by S_{γ_k}. We have defined S_{γ_k} to be the set of equivalence classes in S under the relation $\sim_k$, which is equivalent to letting S_{γ_k} be the set of closed balls of fixed radius γ_k in S. We now offer a third perspective on the elements on S_{γ_k}, which is due to [6],

Lemma. For each k, the elements of S_{γ_k}, that is, the closed balls of radius γ_k, themselves form an ultrametric space, where the metric is given by:

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

Proof. □

So now the elements of S_{γ_k} may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

Definition. Let S and Γ_S be as above. Define $\beta(i)_{i=0}^{\infty}$ to be the sequence given by $\beta(i) = |S_{\gamma_i}|$, which is an invariant of S and which counts the number of connected components of S_{γ_i} when viewed as a metric space. When necessary, we use $\beta^S(i)$ to denote the β sequence for a given, compact ultrametric space S. Adapting the terminology in [8], we call $\beta^S(i)$ the **structure sequence** of S.

Notation. Let S_{γ_k} be as above. We denote the elements of S_{γ_k} by $B_1^k, \ldots, B_{\beta(k)}^k$ or by $B_1^{S,k}, \ldots, B_{\beta(k)}^{S,k}$, when necessary.

We return to the sequence $\beta(i)$ at the end of this section. For now, we show how a $\rho-$ordering of S can be built recursively from the spaces S_{γ_k}. This begins by noting that the spaces themselves can be built recursively:

Observation. Let S, Γ_S, and S_{γ_k} be as above. Then S_{γ_k+1} can be constructed by partitioning each of the closed balls in S_{γ_k} into closed balls of radius γ_{k+1} and taking their union: Let $\overline{B_{\gamma_k}(x_i)}$ be an element of S_{γ_k}, denoted by B_i^k. Then, there exists $x_{i,1}, \ldots, x_{i,l_i} \in B_i^k$ such that,

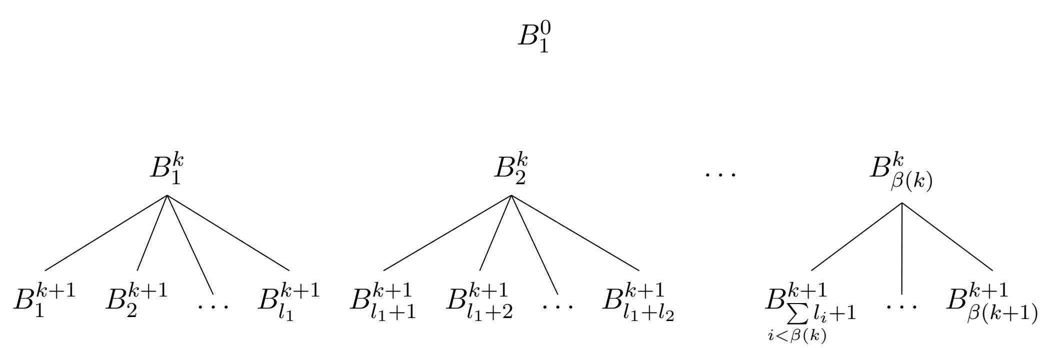
$$B_i^k = \cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})}$$

and so

$$S_{\gamma_{k+1}} = \cup_{i=1}^{\beta(k)} \cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})} = \cup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where $\cup_{j=1}^{l_i} \overline{B_{\gamma_k}(x_{i,j})} = \overline{B_{\gamma_{k+1}}(x_i)} = B_i^k$, $\forall i$.

We can represent this schematically as below:



We denote the above tree by T_s.Since vertices in T_s represent closed balls in an ultrametric space, it is well-defined to take the distance between any two vertices to be the distance between (a choice of) the centres of those closed balls. By construction, the distance between any two vertices will be the diameter of the smallest closed ball that contains both of them. In particular, for any i, the distances between the children of B_i^k will be γ_k and for any $i \neq j$ the distance between the children of B_i^k and B_j^k will be equal to the distance between B_i^k and B_j^k (which will be some $\gamma_m, m < k$).

Lastly, note that without loss of genearlity, we can reindex the B_i^k's so that they give the first $\beta(k)$ terms of a ρ_k-ordering of S_{γ_k}, when the latter is viewed as a (finite) metric space. If the B_i^k's are so indexed, then finding a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ is straightforward: select a B_j^{k+1} from each of the B_i^k's in order and then start over. The following proposition is the main result of this section.

Proposition. Given S a compact subset of an ultrametric space M and Γ_S, the set of distances in S, if S_{γ_k} is a partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k-ordering of S_{γ_k}, then a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ can be found by forming a matrix, A_k, whose $(i,j)^{th}$-entry is $x_{i,j}$, as shown below, and then concatenating the rows (where the columns are padded by * if necessary).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \dots & x_{n,l_n} \end{pmatrix}$$

Proof. Note that the entries in each column are points in the ball $B_{\gamma_k}(x_i)$ so that the pairwise distance between columns is constant and always exceeds the distance between elements within a column. Moreover, the columns are organized such that for any j, $x_{n,j}$ maximizes$\prod_{i=1}^{n-1} \rho_(x_{n,j}, x_{i,j})$ since $\prod_{i=1}^{n-1} \rho(x_{n,j}, x_{i,j}) = \prod_{i=1}^{n-1} \rho_(x_{n,1}, x_{i,1}) = \prod_{i=1}^{n-1} \rho_(x_n, x_i)$ and the x_i's are indexed according a ρ_k-ordering of S_{γ_k}.

Then a $\rho_{\gamma_{k+1}}$-ordering of $S_{\gamma_{k+1}}$ is obtained by minimizing the number of elements from any one column and by taking the points $x_{i,j}$ (for fixed j) in sequence. For example, by concatenating the rows. $\square$

Example 5. $\mathbb{Z}$ with any prime

Example 6. $\mathbb{Z} \setminus 4\mathbb{Z} \subset (\mathbb{Z}, |\cdot|_2)$

Corollary. Interweaving the bottom row of the lattice of closed balls for a set S gives a ρ-ordering of S.

Semi-regularity

Definition. Let S be as above. We say that S is **semi-regular** if $T_{B_i^k} = T_{B_j^k}$, $\forall k \in \mathbb{N}$ and $i, j \in \beta(k)$. That is, S is semi-regular if each ball of radius γ_k breaks into the same number of balls of radius γ_{k+1}, for all k. If there exists an $n \in \mathbb{N}$ such that $T_{B_i^n} = T_{B_j^n}$, i.e. each ball of radius γ_N breaks into the same number of balls of radius γ_{N+1}, for all $N \geq n$, then we say S is **eventually semi-regular**.

cite Amice or Fares or both here.
Semi-regularity in S reflects horizontal similarity on every level of T_S, and so we expect semi-regularity to simplify the calculation of valuative capacity.

Proposition. Let S be a semi-regular, compact subset of an ultrametric space. Let Γ_S be the set of distances in S and let B be the first element of S_{γ_1}. Let $\sigma^S(i)$ be the characteristic sequence of S and $\sigma^B(i)$ be the characteristic sequence of B. Then,

$$\sigma^S(\beta(0) \cdot n) = \gamma_0^c \cdot \sigma^B(n)$$

where c counts the numbers in 1 to $\beta(0) \cdot n$ that are not divisible by $\beta(0)$.

Proof. (sketch) If $n = 0$, then $c = 0$ and $\beta(0) \cdot n = 0$, and,

$$\sigma^S(0) = 1 = 1 \cdot \sigma^B(0)$$

since the 0^{th}-term of any characteristic sequence is 1 by definition. When $n = 1$, $c = \beta(0) - 1$, so that the right-hand side becomes,

$$\gamma_0^{\beta(0)-1} \cdot \sigma^B(1)$$

Now note that the $\beta(0)^{th}$ term of σ^S will have $\beta(0) - 1$ copies of γ_0 (one from each of the elements of S_{γ_0} not containing the $\beta(0)^{th}$ element) and the remaining term is in the branch B, so it is given by $\sigma^B(1)$, and so,

$$\sigma^S(\beta(0)) = \gamma_0^{\beta(0)-1} \cdot \sigma^B(1)$$

Suppose:

$$\sigma^S(\beta(0) \cdot n) = \gamma_0^{c_n} \cdot \sigma^B(n)$$

for $0 \leq n < n + 1$ and consider $\sigma^S(\beta(0) \cdot (n + 1))$. $\sigma^S(\beta(0) \cdot (n + 1))$ will add $\beta(0)$ more terms to $\sigma^S(\beta(0) \cdot n)$. Moreover, exactly $\beta(0) - 1$ of these additional terms will be in other branches and 1 term will be in B, since any sequence of $\beta(0)$ terms in the $\rho-$ordering of S will be from each of the $\beta(0)$ branches. Each of the $\beta(0) - 1$ terms from other branches will add a copy of γ_0 and remaining term can be found by looking ahead in σ^B, so that

$$
\begin{aligned}
&\sigma^S(\beta(0)\cdot(n+1)) \\
&= \sigma^S(\beta(0)\cdot n)\cdot\gamma_0^{\beta(0)-1}\cdot\frac{\sigma^B(n+1)}{\sigma^B(n)} \\
&= \gamma_0^{c_n}\cdot\sigma^B(n)\cdot\gamma_0^{\beta(0)-1}\cdot\frac{\sigma^B(n+1)}{\sigma^B(n)} \\
&= \gamma_0^{c_n}\cdot\gamma_0^{\beta(0)-1}\cdot\sigma^B(n+1) \\
&= \gamma_0^{c_{n+1}}\cdot\sigma^B(n+1)
\end{aligned}
$$

□

Since semi-regularity requires horizontal similarity at every level of T_S, we can repeat the branch cuts as many times as needed to calculate $\sigma(n)$.

Corollary. $v_{\gamma_k}(\sigma(n)) = \lceil\frac{n}{\alpha(0)\cdot\ldots\cdot\alpha(k)}\rceil$

Proof. the exponent of γ_k increases every time you land in the appropriate branch, which is every time you hit a multiple of $\alpha(0)\cdot\ldots\cdot\alpha(k)$ □

Example 7. Consider the ultrametric space $(\mathbb{Z}, |\cdot|_p)$ for any prime p. The corollary gives

$$v_1(\sigma(n)) = \lceil\frac{n}{p}\rceil$$

$$v_{\frac{1}{p}}(\sigma(n)) = \lceil\frac{n}{p^2}\rceil$$

$$v_{\frac{1}{p^2}}(\sigma(n)) = \lceil\frac{n}{p^3}\rceil$$

$$v_{\frac{1}{p^3}}(\sigma(n)) = \lceil\frac{n}{p^4}\rceil$$

So that

$$
\begin{aligned}
\sigma(n) &= (\frac{1}{p})^{\lceil\frac{n}{p^2}\rceil+2\cdot\lceil\frac{n}{p^3}\rceil+3\cdot\lceil\frac{n}{p^4}\rceil+\ldots} \\
&= (\frac{1}{p})^{\sum_{i=1}^{\infty} i\cdot\lceil\frac{n}{p^{i+1}}\rceil}
\end{aligned}
$$

Product space

As a first point of departure, a natural space to consider is the product space of ultrametric spaces, for example $\mathbb{Z}^n$ (or $\mathbb{Z}_p^n$ or $\mathbb{Q}_p^n$), for some $n > 1$. If we restrict our attention to bounded subsets, then a natural candidate for an ultrametric on the product space is the L_∞ metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = sup_i\{\rho(x_i, y_i)\}$$

where ρ is the metric from the base space. In fact, since we have only defined valuative capacity for compact subsets of an ultrametric spaces, there is no loss of generality by restricting our metric to bounded spaces. We also see that no problems arise in letting both M and ρ vary between components of the space, as long as each M_i remains bounded and each ρ_i is an ultrametric.

Proposition. Let (M_i, ρ_i) for i in some finite or countably infinite index set I be a collection of metric spaces and suppose ρ_i is a bounded ultrametric for all i. Then (M, ρ_∞) is an ultrametric space, where $M = M_1 \times M_2 \times M_3 \times \ldots$ and ρ_∞ is the L_∞ metric described above.

Proof. Let (M, ρ_∞) be the product of ultrametric spaces as above and let x and y be two points in the space. Clearly, $\rho_\infty(x, y) \geq 0$ since each $\rho_i(x_i, y_i) \geq 0$, and $\rho_\infty(x, y) = 0 \iff \rho_i(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$. The fact that ρ_∞ is symmetric is also an easy consequence of the fact that each ρ_i is symmetric since $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$

implies $sup_i\{\rho_i(x_i, y_i)\} = sup_i\{\rho_i(y_i, x_i)\}$. To see that ρ_∞ is an ultrametric, note that if $z = z_i$ is any other point of M, then

$$\begin{aligned}
\rho_\infty(x,y) &= \sup_i\{\rho_i(x_i, y_i)\} \\
&\leq \sup_i\{max(\rho_i(x_i, z_i), \rho_i(y_i, z_i))\} && \text{since each } \rho_i \text{ is an ultrametric} \\
&\leq \max(\sup_i\{\rho_i(x_i, z_i)\}, \sup_i\{\rho_i(y_i, z_i)\}) && * \\
&= max(\rho_\infty(x,z), \rho_\infty(y,z))
\end{aligned}$$

* Let $M = max(sup_i(\{a_i\}, sup_j(\{b_j\})))$, then $M \geq a_i, \forall i$ and $M \geq b_i, \forall i$, so $M \geq max(a_i, b_i), \forall i$, hence $M \geq sup_i(max(a_i, b_i))$. □

We show a few quick results ultrametric spaces formed as product spaces, which allows us to quickly calculate the valuative capacity of a few subsets.

Proposition. Suppose (M, ρ_∞) is the product of ultrametric spaces (M_i, ρ_i) and each M_i is a topological group with operation $+$. Then ρ_∞ is (left) translation invariant if each ρ_i is, in which case valuative capacity is also (left) translation invariant.

Proof. Let (M, ρ_∞) be as above. Suppose also that

$$\rho_i(x_i, y_i) = \rho_i(s_i + x_i, s_i + y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each ρ_i is (left) translation invariant. Then,

$$\rho_\infty(s + x, s + y) = \sup_i\{\rho_i(s_i + x_i, s_i + y_i)\} = \sup_i\{\rho_i(x_i, y_i)\} = \rho_\infty(x, y).$$

so that ρ_∞ is translation invariant. Proposition xyz implies valuative capacity is as well. □

Proposition. Let $(M, \|\cdot\|)$ be a normed vector space and suppose the norm on M

induces an ultrametric ρ. Let $(\mathbf{M}, \rho_\infty)$ be the ultrametric space formed by taking products of M, along with the L_∞ metric defined above. Then if ρ is multiplicative on M, ρ_∞ is multiplicative on $\mathbf{M}$, in the sense that $\rho_\infty(\mathbf{cx}, \mathbf{cy}) = |c|_\rho \, \rho_\infty(\mathbf{x}, \mathbf{y})$, for $\mathbf{c} = (c, c, c, \ldots), \mathbf{x}, \mathbf{y} \in \mathbf{M}$.

Proof. Let M, ρ, and ρ_∞ be as above. Then,

$$\rho_\infty(\mathbf{cx}, \mathbf{cy}) = \sup_i \{\rho(c_i x_i, c_i y_i)\} = \sup_i \{|c|_\rho \, \rho(x_i, y_i)\} = |c|_\rho \, \sup_i \{\rho(x_i, y_i)\} = |c|_\rho \, \rho_\infty(x_i, y_i)$$

$\square$

Corollary. Let $\mathbf{S}$ be a subset of $(\mathbf{M}, \rho_\infty)$, where $\mathbf{M}$ is the product of an ultrametric space (M, ρ), that is itself a normed vector space with a multiplicative norm inducing ρ. If $\mathbf{c} = (c, c, c, \ldots)$ is an element of $\mathbf{M}$ with constant value on each component, then $\omega(\mathbf{cS}) = |c|_\rho \, \omega(\mathbf{S})$.

Proof. If $\{\mathbf{a}_j\}_{j=0}^\infty$ is a ρ_∞ ordering of $\mathbf{S}$, then $\{\mathbf{ca}_j\}_{j=0}^\infty$ is a ρ_∞ ordering of $\mathbf{cS}$. $\square$

Example 8. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$ and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1)), \rho_p(x_2, y_2))$, where ρ_p is the p-adic metric for some fixed prime p. Since ρ_p is translation invariant and multiplicative in $\mathbb{Z}_p$, valuative capacity is also translation invariant and multiplicative in $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$.

Example 9. Let $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x \in \mathbb{Z}_{p_1}, y \in \mathbb{Z}_{p_2}\}$ for two distinct primes, $p_1 \neq p_2$, and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_{p_1}(x_1, y_1)), \rho_{p_2}(x_2, y_2))$, where ρ_{p_i} is the p-adic metric. Since each ρ_{p_i} is translation invariant in $\mathbb{Z}_{p_i}$, valuative capacity will be translation invariant in $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$; however, unlike the case of $p_1 = p_2$, this space does not have a multiplicative property that allows for scaling.

n-fold products

What is the valuative capacity of $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ from the example above? Suppose $p = 2$. Using translation invariance, scaling and subaddivity, we can compute the result by first noting that we can write $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always $1 = diam(\mathbb{Z}_2 \times \mathbb{Z}_2)$, subadditivity implies that

$$\frac{1}{log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

$$= \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))}$$

$$= \frac{4}{log(\|2\|_2 * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{log(\frac{1}{2} * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{log(\frac{1}{2}) + log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

Taking logs base 2, we have that

$$\omega(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^{\frac{-1 + log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} 2^{\frac{log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} (2^{log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))})^{\frac{1}{4}} = 2^{\frac{-1}{4}} \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\frac{1}{4}}$$

so that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a solution of the equation $x^4 - \frac{x}{2}$, for which there is a single real positive root, given by $2^{-1/3}$.

To compute the valuative capacity for a 2-fold product for an arbitary prime p, note that we can always decompose $\mathbb{Z}_p \times \mathbb{Z}_p$ into a union of p^2 sets each of the form $\{p\mathbb{Z}_p + s \times p\mathbb{Z}_p + t\}$ for $s, t \in (0, \ldots, p-1)$, and the pairwise distance between these sets will always be $1 = diam(\mathbb{Z}_p \times \mathbb{Z}_p)$ (to see this, either note that we can always find co-prime elements, or note that each set is an closed ball of radius $1/p$ centred at (s,t) and so the distance between them must be greater than $1/p$, and 1 is the only possible distance greater than

$1/p$ in $\mathbb{Z}_p \times \mathbb{Z}_p$). Then, we combine our tools as before to obtain the equation,

$$\frac{1}{log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(\|p\|_p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(1/p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, taking logs base p, we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p) = p^{\frac{-1}{p^2}} \omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{\frac{1}{p^2}}$$

So that $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$ is a solution of the equation $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p})$ over $\mathbb{R}$ and since $\mathbb{R}$ is a division ring, this means the positive solutions are given by solving $x^{p^2-1} - \frac{1}{p}$. Solutions of this equation are of the form $p^{\frac{-1}{p^2-1}}$ times a $p^2 - 1$ root of unity, and so there is exactly one positive, real solution, namely $p^{\frac{-1}{p^2-1}}$ itself. Then the valulative capacity of the entire product space $\mathbb{Z}_p \times \mathbb{Z}_p$ is $p^{\frac{-1}{p^2-1}}$. In fact, from here it is not hard to see that by taking the n-fold product, we would end up with the same equation except that the exponent of p would become n rather than 2. We arrive at the following result:

Proposition. Let $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$ be the ultrametric space with points equal to the n-fold product of $\mathbb{Z}_p$ (for $n < \infty$) for some fixed prime p. The valuative capacity of M is $(\frac{1}{p})^{\frac{1}{p^n-1}}$.

Proof. Above. □

Taking $n = 1$, we see that this agrees with the valuative capacity of $\mathbb{Z}_p$ computed in the last chapter.

- What about $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ for distinct primes?
- These spaces do not admit a scaling property
- They are semi-regular, so their partial sums will be a function of Γ and the structure sequence

We end this section with two observations on the results above. First, recall that in computing the valuative capacity of these spaces, we were ultimately reduced to finding solutions to polynomials of the form $x^{p^n} - \frac{x}{p}$ for some n and for some p. The first observation is that these polynomials are $\mathbb{Z}$-valued on $p\mathbb{Z}$, that is, they are elements of $Int(p\mathbb{Z}, \mathbb{Z})$. We might ask then, what sort of polynomials would arise in finding the valuative capacity of spaces such as $(\mathbb{Z}_2 \times \mathbb{Z}_3, \rho_\infty)$ or in computing the valuative capacity of infinite product spaces, such as $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots$ for either some fixed prime p or over each prime.

Secondly, we observe the asymptotic behavior of capacity in these spaces. For a fixed prime p, $(\frac{1}{p})^{\frac{1}{p^n-1}}$ is an monotone, increasing sequence in n with $lim_{n\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. For fixed n, the sequence in p is also montone, increasing, again with $lim_{p\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. In both cases, the limiting value is equal to the diameter of space. Indeed, we can observe that the sequence $\{(0,0,\ldots),(1,0,\ldots),(0,1,\ldots),\ldots\}$, in which the first element has only zeros and the n-th element has a single 1 in the $(n-1)$-th component, is a $\rho-$ordering for both $(\mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, \rho_{p,\infty})$ and $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots, \rho_{P,\infty})$, since the distance between elements in this sequence (in either metric space) is always 1. If we could show that these spaces are compact, this would gives a valuative capacity of $lim_{n\to\infty}(1^n)^{(1/n)} = 1$ for both spaces. We explore this more in the following section.

Product topology

In considering the product space of ultrametric spaces, we may wonder whether the chosen metric also gives back the product topology on the space. For products formed by taking some finite number of copies, the answer is positive. We give the necessary background and show this fact, adapting the proof in Munkres (20.3) to the case of ultrametric spaces.

Definition-Proposition. (Munkres) Suppose X_i, for i in some index set I, is a family of topological spaces. Let $\pi_j : \prod_{i\in I} X_i \to X_j$ be the map given by projection onto the j-th component, that is $\pi_j(x) = \pi_j((x_i)_{i\in I}) = x_j$. For each $j \in I$, let $\mathcal{S}_j$ be the collection

$$\mathcal{S}_j = \{\pi_j^{-1}(U_j) \mid U_j \text{ open in } X_j\}$$

Let $\mathcal{S}$ be the union of the $\mathcal{S}_j$ over $j \in I$, $\mathcal{S} = \cup_{j\in I}\mathcal{S}_j$. Then $\mathcal{S}$ is a subbasis that generates a topology on $\prod_{i\in I} X_i$ called the **product topology**.

The basis, $\mathcal{B}$, generated by $\mathcal{S}$ in the definition above is the set of all finite intersections of elements in $\mathcal{S}$. That is $B \in \mathcal{B}$ if there exists $S_1, S_2, \ldots, S_n$ in $\mathcal{S}$ such that $B = S_1 \cap S_2 \cap \ldots S_n$. A useful description of the basis for the product topology also appears in Munkres, as below:

Proposition. (Munkres 19.2) Suppose X_i, for i in some index set I, is a family of topological spaces and denote by $\mathcal{B}_i$ the basis for the topology on X_i. Let

$$\mathcal{B}_P = \prod_{i\in I} B_i, \text{ for } B_i \in \mathcal{B}_i \text{ and } B_i = X_i \text{ for all but finitely-many } i \in I.$$

then $\mathcal{B}_P$ is a basis for the product topology on $\prod_{i\in I} X_i$.

We can now show that the topology induced by the L_∞ metric described above agrees with the product topology for finite products.

Proposition. Let $M = (M_1 \times M_2 \times \ldots \times M_n, \rho_\infty)$ be a finite product of bounded, ultrametric spaces and let ρ_∞ be the metric described above. Then the topology induced by ρ_∞ coincides with the product topology on $M_1 \times M_2 \times \ldots \times M_n$.

Proof. Let $\mathcal{T}_{\rho_\infty}$ be the topology on $M_1 \times M_2 \times \ldots \times M_n$ induced by ρ_∞ and let $\mathcal{B}_{\rho_\infty}$ be the basis for this topology. Let $\mathcal{T}_P$ be the product topology with basis $\mathcal{B}_P$. We show $\mathcal{T}_P \subset \mathcal{T}_{\rho_\infty}$ and vice versa. For this, it is equivalent (Munkres 13.3) to show that for $z \in M_1 \times M_2 \times \ldots \times M_n$ and $B \in \mathcal{B}_P$ containing z, there is a basis element $B' \in \mathcal{B}_{\rho_\infty}$ such that $z \in B' \subset B$, and vice versa.

So let $z \in M_1 \times M_2 \times \ldots \times M_n$ and suppose $B \in \mathcal{B}_P$ contains z. Since B is in $\mathcal{B}_P$, B is of the form $B_{r_1}(z_1) \times B_{r_2}(z_2) \times \ldots \times B_{r_n}(z_n)$ (since the choice of centres is arbitrary in an ultrametric spaces, we may choose the components of z as the centres without loss of generality). Let $r = \min\{r_i\}$ for $i \in 1, \ldots, n$. Then let B' be the ball $B_r(z)$ in $\mathcal{B}_{\rho_\infty}$. Clearly, $z \in B_r(z)$ and since $r \leq r_i$, $\forall i$, $B_r(z) = B_r(z_1) \times B_r(z_2) \times \ldots \times B_r(z_n) \subset B_{r_1}(z_1) \times B_{r_2}(z_2) \times \ldots \times B_{r_n}(z_n) = B$.

Conversely, suppose $A \in \mathcal{B}_{\rho_\infty}$ and let $y \in A$. To find $A' \in \mathcal{B}_P$ such that $y \in A'$ and $A' \subset A$, simply note that A itself is in $\mathcal{B}_P$.

$\square$

We are now naturally left to ask whether the product topology on *infinite* products of ultrametric spaces coincides with the L_∞ metric. In this case, as in the analogous case of infinite copies of $\mathbb{R}$ and a uniform metric, the answer is negative (at least in general). Forunately, the metric that realizes the product topology on infinite copies of $\mathbb{R}$ can be adapted to the case of ultrametric spaces. We adapt to the proof of Munkres (20.5) to the case of infinite products of ultrametric spaces.

Proposition. Suppose $\mathbf{M} = M_1 \times M_2 \times \ldots$ is an infinite collection of metric spaces, each with an ultrametric ρ_i which is bounded by 1, that is suppose $\rho_i(x_i, y_i) \leq 1$, for all $x_i, y_i \in M_i$ and for all i. Define a metric d on $\mathbf{M}$ as follows:

$$d(\mathbf{x}, \mathbf{y}) = sup\{\frac{\rho_i(x_i, y_i)}{i}\}$$

Then d is an ultrametric and induces the product topology on $\mathbf{M}$.

Proof. We see that d inherits symmetry, injectivity and non-negativity from the requirement that each ρ_i is a metric, just as ρ_∞ did. To see that d satisfies the strong triangle inequality, define a new metric ρ_i' by $\rho_i'(x,y) = \frac{\rho_i(x_i,y_i)}{i}, \forall i$. Then ρ_i' is an ultrametric, since $\rho_i(x,y) \le \max(\rho_i(x,z), \rho_i(y,z))$ implies $\frac{\rho_i(x,y)}{n} \le \max(\frac{\rho_i(x,z)}{n}, \frac{\rho_i(y,z)}{n})$ for any $n \in \mathbb{N}$. Then we can view d as the L_∞ metric on the spaces (M_i, ρ_i'), and so d will be an ultrametric as shown in the first proposition of this section.

Now we show d induces the product topology. We first show that metric topology induced by d is finer than the product topology. Let

$$B = B_r^{\mathbf{M}}(\mathbf{z}) = B_r^{M_1}(z_1) \times B_{2r}^{M_2}(z_2) \times B_{3r}^{M_3}(z_3) \times \dots$$

be a basis open in the metric topology. We must find a basis open $B' \ni z$ in the product topology such that $B' \subseteq B$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < r$. Then let B' be the basis open element

$$B' = B_r^{M_1}(z_1) \times B_r^{M_2}(z_2) \times \dots \times B_r^{M_N}(z_N) \times M_{N+1} \times M_{N+2} \times \dots$$

in the product topology. Suppose $\mathbf{y} \in B'$. We must show $\mathbf{y} \in B$, i.e., $d(\mathbf{z}, \mathbf{y}) < r$. Note that for all $i \ge N$,

$$\frac{\rho_i(z_i, y_i)}{i} \le \frac{1}{N}$$

which means

$$d(\mathbf{z}, \mathbf{y}) = \sup\{\frac{\rho_i(z_i, y_i)}{i}\} \le \max\{\frac{\rho_1(z_1, y_1)}{1}, \frac{\rho_2(z_2, y_2)}{2}, \dots, \frac{\rho_N(z_N, y_N)}{N}, \frac{1}{N}\}$$

and since N was chosen so that $\frac{1}{N} < r$ and B' was chosen to have balls of radius r in the first N components, we must have $d(\mathbf{z}, \mathbf{y}) < r$.

Conversely, ... □

From now on, we refer to the metric d above as the **product metric**. An important consequence of the fact that d achieves the product topology is that Tychnoff's theorem then guarantees that product spaces formed with this metric will be compact, infinite or otherwise. As a result, we can now ask directly about the valuative capacity of some infinite product spaces. We consider two examples.

Example 10. Let $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, d)$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for some fixed prime p and let d be the product metric.

Example 11. Let $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots, \rho_{P,\infty})$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for every prime p and let d be the product metric.

So far we have two methods for computing valuative capacity. Either we can find a useful decomposition that allows us to apply the subadditivity formula, or we can find a $\rho-$ordering and then take the limit of its corresponding $\rho-$sequence.

Conclusion

In this section, we considered the notion of valuative capacity in product spaces, that is, spaces formed by taking copies of ultrametric spaces. In the following sections, we consider vaulative capacity in spaces formed by adding points, that is extension fields, or by both taking copies and adding (a distinguished) point, as in projective spaces. For these purposes, it will be more productive to start working over the field $\mathbb{Q}_p$, instead of $\mathbb{Z}_p$.

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