Chapter 1

Introduction

In the course of developing a generalized factorial function, Manjul Bhargava introduced the notion of $p-$orderings of a Dedekind domain [B1, B2], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [B3, J2] and have also provided a natural framework to extend many classical results in analysis to a $p-$adic setting, such as polynomial approximation and mapping theorems [B1, B2, B3]. In this thesis, we examine how a tool based on $p-$orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

The historical background to this work comes in two parts. On the one hand, there is the background on logarithmic capacity from potential theory, and secondly, there is the background from Bhargava's $p-$orderings. We give a brief summary of each here. A similar treatment, with slightly different perspective, is found in [FP]. Jean-Luc Chabert was the first to draw a connection between the two, and many of the known results in this area stem from his work or that of his colleagues. Building on the result in [J1], we extend the work by Chabert and colleagues by studying valuative capacity in a more general setting, namely that of an ultrametric space, which may or may not also be a local field. In doing so, we show many properties of capacity are in fact independent of the algebraic structure of a space, although such structure, when it exists, can act as a useful probe.

1.1 Logarithmic capacity

The theory of capacity has been developed as a topic in potential theory in a variety of settings. Classically, the notion of capacity was developed over both $\mathbb{C}$ and $\mathbb{R}^n$, although the theory has been further developed in a rather general way by Rumely for Berkovich spaces. A signficant body of work on the analytic properties of capacity can be found for a number of different contexts. For example, such a treatment of the subject over $\mathbb{C}$ can be found in [W] and [Ra], over Berkovich spaces in [BR], and over $\mathbb{Q}_p$ in [Ca]. We give a brief account of capacity over $\mathbb{C}$ here, presenting only the most essential definitions and results. One advantage of tracing the historical roots of capacity back to $\mathbb{C}$ is that the theory in this setting also comes equipped with a physical interpretation. As we are about to see, capacity in the classical sense gives a mathematical model for the amount of electrostatic charge a conductor can hold. The exposition below is closely based on Ransford [Ra].

Even restricting ourself to the definition of capacity of subsets of $\mathbb{C}$, we find two paths, one which will give us some physical interpretation, and one which will lead more naturally to $p-$orderings. We start with the former.

Definition 1. [Ra] Let μ be a finite Borel measure on $\mathbb{C}$ and suppose μ has compact support. We associate to μ a function, $p_\mu : \mathbb{C} \to [-\infty, \infty)$, given by

$$p_\mu(x) = \int \log|x - y| d\mu(y)$$

called the **potential function** of μ. The **energy** of μ is

$$I(\mu) = \int \int \log|x - y| d\mu(y) d\mu(x)$$

This gives at once the physical interpretation promised above. We interpret the potential function of a measure as giving the potential energy of a point. Viewing the measure as a charge distribution, the double integral gives back the total energy in the system. Now we come upon a physical reality: charged particles in a conductor will naturally distribute themselves as to minimize the energy. This leads to the definition below:

Definition 2. [Ra] Let K be a compact subset of $\mathbb{C}$ and let $\mathcal{P}(K)$ be the set of Borel probability measures on K. If $\nu \in \mathcal{P}(K)$ is such that

$$I(\nu) = \sup_{\mu \in \mathcal{P}(k)} I(\mu)$$

then ν is a **equilibrium measure** for K.

comment from Keith: explain why its a sup when you're minimizing the energy; I think this is similar as to how minimizing the valuation (often) leads to maximizing the distances - the less potential energy you take up at each iteration the more total charge you can add

We state the following proposition without proof. A sketch of the proof can be found in [FP] and the full details can be found in [Ra].

Proposition 1. *[Ra] The equilibrium measure exists for every compact set $K \in \mathbb{C}$. When finite, the equilibrium measure is unique and isometry-invariant.*

We are now ready to give our first definition of capacity.

Definition 3. [Ra] Let K be a compact subset of $\mathbb{C}$. The logarithmic capacity of E is

$$C(K) = e^{I(\nu)}$$

where ν is an equilibrium measure on K. If $I(\nu) = -\infty$, then we understand that $C(K) = 0$.

We present below a few results on capacity in $\mathbb{C}$, some of which will reappear in the remainder of this work, although the context, and the proofs (omitted here), bear little resemblence to the present case.

Proposition 2. *([Ra], 5.1.2) Let K, K_1, K_2 be compact subsets of $\mathbb{C}$.*

1. $K_1 \subseteq K_2$*, then* $C(K_1) \leq C(K_2)$*.*

2. $C(\alpha K + \beta) = |\alpha| C(K)$ *for all* $\alpha, \beta \in \mathbb{C}$*.*

3. $C(K) = C(\delta_e K)$*, where* δ_e *is the exterior boundary.*[1]

Proposition 3. *([Ra], 5.1.4) Suppose* $\{B_n\}$ *is a sequence of Borel subsets of* $\mathbb{C}$*. Let* $B = \cup_n B_n$ *and* $d \geq 0$*.*

1. If $diam(b) \leq d$*, then* $C(B) \leq d$ *and*

$$\frac{1}{log(\frac{d}{C(B)})} \leq \sum_n \frac{1}{log(\frac{d}{C(B_n)})}$$

2. If $dist(B_j, B_k) \geq d$ *whenever* $j \neq k$*, then*

$$\frac{1}{log^+(\frac{d}{C(B)})} \geq \sum_n \frac{1}{log^+(\frac{d}{C(B_n)})}$$

where $log^+(x) = \max(log(x), 0)$*.*

We now know show an equivalent way of defining of capacity, still over $\mathbb{C}$, which starts with the following two definitions due to Fekete [F].

Definition 4. [F] Let $K \subseteq \mathbb{C}$ be a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in K^n$, consider

$$\delta_n(z) = \prod_{j<i} |z_i - z_j|^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in K^n$ is called a **Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in K.

Note that since K is compact by assumption, a Fekete $n-$tuple exists for each n.

[1] The exterior boundary of a compact subset, K, of $\mathbb{C}$ is the boundary of the unbounded, connected component of $U = \mathbb{C} \setminus K$.

Definition 5. Let $K \subseteq \mathbb{C}$ be a compact subset. The **transfinite diameter** of K is

$$\lim_{n\to\infty}[\max_{z} \ \delta_n(z)]$$

where the maximum is taken over all $n-$tuples in K. That is, the transfinite diameter of K is $\lim_{n\to\infty}\delta_n(z)$, where z is a Fekete $n-$tuple for each n.

The following proposition shows the relation to capacity.

Proposition 4. *([F], **Fekete-Szegö Theorem**) If K is a compact subset of $\mathbb{C}$, then the transfinite diameter of K is equal to the logarithmic capacity of K.*

We end this section with an observation about the points z_i in $\mathbb{C}$ (or some subset thereof) making up a Fekete n-tuple. For $n \geq 2$, if $(z_1, \ldots, z_n)$ is a Fekete $n-$tuple, then in general there is no z_{n+1} available such that $(z_1, \ldots, z_n, z_{n+1})$ a Fekete $(n+1)-$tuple. In physical terms, we note that the placement of a new charge in a conductor will almost always change the location of the existing charges in that conductor. Remarkably, this is not the case in ultrametric spaces. Indeed, we are able to build the analogous structure, which we call a $p-$ordering or more generally a $\rho-$ordering, *recursively*, that is by reusing the points from the previous iteration.

1.2 P-orderings

The development of $p-$ordering was motivated by the observation that the factorial function had important number-theoretic applications, yet was only defined for the set $\mathbb{Z}$. In order to generalize the factorial, Bhargava defined it via an invariant, called the $p-$sequence, which could be attached to any subset of a Dedekind domain [2] [B1].

We cannot go much further without introducing the following definition.

Definition 6. Let $z \in \mathbb{Z}$ and let p be any prime. The $p-$**adic valuation** of z, denoted $v_p(z)$, is the largest $n \in \mathbb{N}$ such that p^n divides $z \neq 0$ and $v_p(z) = \infty$ if

[2] In fact, Bhargava associated $p-$ sequences to the more general class of Dedekind rings, which are locally principal, Noetherian rings in which all nonzero primes are maximal.

$z = 0$. That is,

$$v_p(z) = \begin{cases} \max\{n \in \mathbb{N}; p^n \mid z\}, & \text{if } z \neq 0 \\ \infty, & \text{otherwise} \end{cases}$$

For $z \in \mathbb{Z}$, we definite the $p-$**adic absoluste value** by

$$|z|_p = p^{-v_p(z)}$$

and the $p-$**adic metric** accordingly; that is, for $z_1, z_2 \in \mathbb{Z}$

$$\rho_p(z_1, z_2) = p^{-v_p(z_1 - z_2)}$$

where $p^{-\infty}$ is taken to be 0.

It is worth pausing to make a few comments about the above definitions. That the $p-$adic metric is truly a metric is easy to see. In fact, we will see in the next chapter that it is not just a metric, but also an ultrametric, since the $p-$adic absolute value satistfies a strengthen version of the triangle identity. The strong triangle identity is not the only interesting property at hand though. Like the logarithm, the $p-$adic valuation also satisfies: $v_p(x \cdot y) = v_p(x) + v_p(y)$ for any prime p and x, y in $\mathbb{Z}$. Moreover, we note that the $p-$adic valuation and $p-$adic metric have an interesting relationship with each other: two points whose difference has a relatively small valuation will have a relatively large distance between them and vice versa.

We are now ready to define $p-$orderings, and not long after, to give the connection to Fekete $n-$tuples.

Definition 7. [B1] Let S be a subset of $\mathbb{Z}$ and let p be any prime.[3] A p**-ordering** of S is a sequence, $\{a_i\}_{i \geq 0}$ in S, such that a_0 is arbitrary and for $i > 0$, a_i minimizes

$$v_p(\prod_{j<i}(z - a_j))$$

[3] To apply the definition to a general Dedekind domain, we replace the usual primes with the set of primes ideals in the ring of interest.

over $z \in S$.

A $p-$ordering in S, like a Fekete $n-$tuple in $\mathbb{C}$, is not unique. Indeed, in most of the examples we will explore, there will be infinitely-many choices at each stage of the construction. Nonetheless, $p-$orderings give rise to $p-$sequences, which are invariants of S:

Definition 8. [B1] Let S be a subset of $\mathbb{Z}$ and let p be any prime. Suppose $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S. The **p-sequence**, occasionally the **characteristic sequence**, of S is the sequence defined by $\delta(0) = 1$ and for $i > 0$,

$$\delta(i) = v_p(\prod_{j=0}^{i-1}(a_i - a_j))$$

It is a fact, not entirely obvious, that the $p-$sequence of S is independent of the $p-$ordering used in its construction [B1]. To define the generalized factorial, Bhargava considered the product of $p-$sequences taken over each prime p for arbitrary subsets of $\mathbb{Z}$. We will go in another direction.

Suppose we were to generalize our definition of Fekete $n-$tuple in the obvious way, as below.

Definition 9. Let (M, ρ) be a metric space and suppose $S \subseteq M$ is a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in S^n$, consider

$$\delta_n(z) = \prod_{j<i} \rho(z_i - z_j)^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in S^n$ is called a **generalized Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in S.

What then is the connection to $p-$orderings and $p-$sequences? Suppose S is a subset of $\mathbb{Z}$ and that $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S for some prime p. Then of course from the definition of $p-$orderings, we know that for $i > 0$,

$$v_p(\prod_{j<i}(a_i - a_j)) \leq v_p(\prod_{j<i}(z - a_j))$$

for $z \in S$. Something more is true though, namely,

$$v_p(\prod_{j<i}(a_i - a_j)) \leq v_p(\prod(x_i - x_j))$$

for $x_i, x_j \in S$ [B1]. That is, when we pick a_i to minimize the $p-$adic valuation over $\prod_{j<i}(z - a_j)$, we actually achieve the minimum over the product of *all pairs of differences in* S. Since minimizing $v_p(x_i - x_j)$ is the same as maximizing $\rho_p(x_i, x_j)$, we have the following remarkable fact: if $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S, then $\{a_i\}_{i=0}^n$ is a generalized Fekete $n-$tuple for (S, ρ_p) for each n. In particular, $p-$orderings give a recursive construction for generalized Fekete $n-$tuples.

The first connection between these objects was made by Jean-Luc Chabert in [Ch] when he studied the limit of these sequences not just for the case $M = \mathbb{Z}$ and $\rho = \rho_p$, but in the case that M is any rank-one valuation domain [Ch]. We repeat his theorem 4.2 from [Ch] below,

Proposition 5. *Let E be a subset of V, a rank-one valuation domain with valuation v. If $\{a_i\}_{i\geq 0}$ is $v-$ordering* [4] *of E, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0,\ldots,x_n \in E} v(\prod_{0\leq j<i\leq n}(x_i - x_j))$$

Chabert called this limit the valuative capacity of E, and we shall do the same. By replacing the notion of $p-$ordering (or $v-$ordering) with the more general notion of $\rho-$ordering, we are able to give a definition of valuative capacity for a general ultrametric space, without appealing to any algebraic (or measure-theoretic) structure. The following result by Johnson in [J1] provides the foundation for the rest of this work:

Proposition 6. *([J1], Theorem 1) If S is a compact subset of an ultrametric space*

[4] A $v-$ordering of E is exactly as expected: a sequence of distinct element $\{a_i\}_{i\geq 0}$ in E is $v-$ordering of E if for $n > 0$,

$$v(\prod_{k=0}^{n-1}(a_n - a_k)) \leq v(\prod_{k=0}^{n-1}(x - a_k)$$

for each $x \in E$.

(M, ρ), then the first n terms of a ρ-ordering of S always give a Fekete n-tuple of S and all Fekete n-tuples of S arise in this way.

One important consequence of this remark is that it gives a way to define capacity in a general ultrametric space: we define an analogous structure to a $p-$sequence, which we call a $\rho-$sequence, and consider the limit.

Chapter 2

Capacity and Ultrametric spaces

2.1 Ultrametric basics

The principal context for this thesis is an arbitrary ultrametric space, which is a metric space that also satisifies an additional axion, sometimes called the ultrametric inequality or (in the case of vector spaces) the strong triangle propery. We define ultrametric spaces below and for the rest of this section, we review some of their more important characteristics. The proofs offered in this section are, for the most part, standard and can be found in a number of reference texts, such as [Ro].

Definition 10. Let (M, ρ) be a metric space; that is, suppose M is a set and $\rho : M \times M \to \mathbb{R}_{\geq 0}$ is such that:

(i) $\rho(x, y) = 0$ if and only if $x = y$

(ii) $\rho(x, y) = \rho(y, x)$

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any $x, y, z \in M$. If ρ satistifies the ultrametric inequality,

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z))$$

for any $x, y, z \in M$, then (M, ρ) is an **ultrametric space**.

A special case of an ultrametric space, and one where much of the previous work on this topic has been completed, is one where the metric has been derived from a norm on a vector space.

Definition 11. Let (V, N) be a normed vector space; that is, suppose V is $\mathbb{F}-$vector space, for $\mathbb{F}$ some subfield of $\mathbb{C}$, and $N : V \to \mathbb{R}_{\geq 0}$ is such that:

(i) $N(x+y) \leq N(x) + N(y)$

(ii) $N(cx) = |c|\ N(X)$

(iii) $N(x) = 0$ implies $x = 0$

for any $x, y \in V$ and $c \in \mathbb{F}$. We say that N satisfies the **strong trianlge inequality** if

$$N(x+y) \leq \max(N(x), N(y))$$

for any $x, y \in V$.

Proposition 7. *Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space, (V, ρ_N), where ρ_N is the metric induced by N, that is, $\rho_N(x, y) = N(x - y)$, is an ultrametric space.*

Proof. We take for granted that (V, ρ_N) is a metric space and also note that

$$N(x+z) \leq \max(N(x), N(z))$$

implies

$$\rho_N(x, z) \leq \max(\rho_N(x, 0), \rho_N(z, 0)) \leq \max(\rho_N(x, y), \rho_N(y, z))$$

□

Notation. If (V, N) is a normed vector space, then the metric induced by N is denoted ρ_N.

When ultrametric spaces come from spaces with algebraic structure, such as normed vector spaces, some of this structure carries over into metric spaces structure in a rather nice way:

Proposition 8. *[Ro] Let S be a group equipped with a (right) invariant ultrametric, ρ. If $B = B(0, r)$ is a (closed) ball centred at the neutral element of S, that is $B = \{x \in S; \rho(x, 0) \leq r\}$, then B is a subgroup of S.*

Proof. Let $x, y \in B$. Then

$$\rho(x - y, 0) = \rho(x, y) \leq \max(\rho(x, 0), \rho(y, 0)) \leq r,$$

so that $x - y \in B$. $\square$

In the previous chapter, we claimed that the $p-$adic metric was an ultrametric on the set $\mathbb{Z}$. Indeed, $(\mathbb{Z}, \rho_p)$ and the closely related space of $p-$adic integers, denoted $\mathbb{Z}_{(p)}$, are the canonical examples of an ultrametric space.

Example 1. Let p be any prime and consider the metric space $(\mathbb{Z}, \rho_p)$. To see that $(\mathbb{Z}, \rho_p)$ is an ultrametric space, we must show that ρ_p satisfies the ultrametric inequality, or equivalently, that $p-$adic absolute value satisfies the strong triangle inequality. Let x, y be in $\mathbb{Z}$ and suppose $v_p(x) = n_x$ and $v_p(y) = n_y$. Then if $n = \min(n_x, n_y)$, p^n divides x and p^n divides y, so p^n divides $x + y$. We see now that $v_p(x + y) \geq \min(v_p(x), v_p(y))$ and in turn $|x + y|_p \leq \max(|x|_p, |y|_p)$.

Example 2. Let p be any prime. If

$$z = \sum_{i \geq 0} b_i p^i$$

is such that $b_i \in \{0, \ldots, p-1\}$ for all i, then we say that z is a $p-$**adic integer**. If $z = \sum_{i \geq 0} b_i p^i$, we note that if only a finite number of the coefficients of z are non-zero, then $\sum_{i \geq 0} b_i p^i$ is a representation in base p of some element of $\mathbb{Z}$. We can define the $p-$adic order of a $p-$adic integer, denoted $ord_p(z)$, in a way that agrees with the $p-$adic valuation when $\sum_{i \geq 0} b_i p^i$ is in $\mathbb{Z}$, by letting $ord_p(z)$ be the smallest i such that $b_i \neq 0$. The $p-$adic integers are both a ring[1] and an ultrametric space with the metric induced by $ord_p(z)$. For a given prime, p, we denote the $p-$adic integers by $\mathbb{Z}_{(p)}$.

In what follows, we will often refer to $p-$adic spaces and it will not make much of a difference whether the reader prefers to think of being in $(\mathbb{Z}, \rho_p)$ or $\mathbb{Z}_{(p)}$. The reason is this: when forming ρ_p-orderings of subsets of either space we are always able to do so by selecting elements with finite number of non-zero coefficients, that is, by selecting elements from $\mathbb{Z}$ itself.

[1] The ring operations carry over on the coefficients of $p-$adic integers in the expected way from $\mathbb{Z}/p\mathbb{Z}$, as long as special care is taken to keep track of carries.

Ultrametric spaces exhibit properties much unlike traditional metric spaces, and we review of few of these below. Of particular interest to us is the behavior between (closed) balls in an ultrametric space.

Notation. Let (M, ρ) be a compact ultrametric space and let

$$B(a, r) = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the *closed* ball of radius r, centred at a for some $r \in \mathbb{R}_{>0}$ and $a \in (M, \rho)$. Let

$$B^0(a, r) = \{x \in M \mid \rho(x, a) < r\}$$

denote the *open* ball of radius r, centred at a for some $r \in \mathbb{R}_{>0}$ and $a \in (M, \rho)$.

In the above notation, we break from convention in that we denote a closed ball without using any decoration. This is because before too long we will work exclusively with closed balls. We are able to do this because for the most part, the notion of open and closed ball in an ultrametric space overlap, although we will need a few more facts before showing this.

Definition 12. Let S be a subset of an ultrametric space. The **diameter of** S is $diam(S) = \sup\limits_{x,y \in S} \rho(x, y)$. Note that if S is compact, $diam(S) = \max\limits_{x,y \in S} \rho(x, y)$.

Proposition 9. *Let $B = B(a, r)$ be a ball in an ultrametric space (M, ρ). Then the diameter of B is less than or equal to the radius of B.*

Proof. Suppose $d = diam(B) > r$. This would imply there exists x, y in B such that $\rho(x, y) > r$, in particular $\rho(x, y)$ is strictly greater than $\max(\rho(x, a), \rho(y, a))$, which is a contradiction since ρ is an ultrametric. $\square$

In the following proposition, we describe the triangles in an ultrametric space, and the result is more or less a restatement, in geometric terms, of the ultrametric inequality.

Proposition 10. *All triangles in an ultrametric space* (M, ρ) *are either equilateral or isosceles, with at most one short side.*

Proof. Let x, y, and z be three points in an ultrametric space (M, ρ). We show that $\rho(x, y) \neq \rho(x, z)$ and $\rho(x, y) \neq \rho(y, z)$ implies $\rho(x, y) < \rho(x, z) = \rho(y, z)$.

If $\rho(x, z) \neq \rho(y, z)$, then without loss, $\rho(x, z) > \rho(y, z)$. At the same time, the ultrametric inequality implies $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ and $\rho(y, z) \leq \max(\rho(x, y), \rho(x, z))$. The first inequality implies $\rho(x, y) < \rho(x, z)$, which means the second inequality implies $\rho(y, z) < \rho(x, z)$. This is a contradiction, so we must have $\rho(x, z) = \rho(y, z)$.

To see that $\rho(x, y) < \rho(x, z)$, simply note that $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ $\square$

With this result in hand, we are able to quickly demonstrate some of the properties of balls, which are of fundamental importance to us. We see below that the ultrametric inequality, perhaps innocuous on the surface, quickly implies ultrametric balls are markedly different from their Archimedean counterparts.

Proposition 11. *Every point of a ball in an ultrametric is at its centre. That is, if* $B(x_0, r)$ *is a ball in an ultrametric space* (M, ρ)*, then* $B(x, r) = B(x_0, r)$*,* $\forall x \in B(x_0, r)$

Proof. Let $a \in B(x, r)$. Then $\rho(a, x) \leq r$ and since

$$\rho(a, x_0) \leq \max(\rho(a, x), \rho(x, x_0)) \leq r$$

we must have $a \in B(x_0, r)$ and $B(x, r) \subseteq B(x_0, r)$ A similar argument shows $B(x_0, r) \subseteq B(x, r)$. $\square$

Proposition 12. *If* (M, ρ) *is an ultrametric space and* $B(x_0, r_1)$ *and* $B(y_0, r_2)$ *are balls in* (M, ρ)*, then either* $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$*,* $B(x_0, r_1) \subseteq B(y_0, r_2)$*, or* $B(x_0, r_1) \subseteq B(x_0, r_1)$*. That is, in an ultrametric space, all balls are either comparable or disjoint.*

Proof. Suppose $B(x_0, r_1) \cap B(y_0, r_2) \neq \emptyset$ and let z be a point in the intersection. We show that if there exists an $a \in B(y_0, r_2)$ such that $a \notin B(x_0, r_1)$, then $B(x_0, r_1) \subseteq$

$B(y_0, r_2)$. Let $x \in B(x_0, r_1)$. Then we must have $\rho(x, z) < \rho(x, a)$, since $z \in B(x_0, r_1) = B(x, r_1)$ and a is not. Since the triangle with vertices (a, x, z) is isocolces with at most one short side, we must have $\rho(x, a) = \rho(a, z) \leq r_2$, since $a \in B(y_0, r_1) = B(z, r_2)$. Then $x \in B(y_0, r_1)$. $\square$

Proposition 13. *The distance between any two non-overlapping balls in an ultrametric is constant. That is, if $B(x_0, r_1)$ and $B(y_0, r_2)$ are two balls in an ultrametric space with $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$, then there exists a $c \in \mathbb{R}_{>0}$ such that $\rho(x, y) = c$, $\forall x \in B(x_0, r_1)$ and $\forall y \in B(y_0, r_2)$.*

Proof. Suppose $\rho(x_0, y_0) = c$ and let $x \in B(x_0, r_1)$ and $y \in B(y_0, r_2)$ be arbitrary. Consider the triangle formed by (x_0, y_0, y). Since $\rho(x_0, y_0) = c$ and $\rho(y, y_0) \leq r_2 < c$, we must have $\rho(x_0, y) = c$ because triangles in an ultrametric space have at most one short side. Now consider the triangle formed by (x_0, x, y). Since $\rho(x_0, y) = c$ and $\rho(x, x_0) \leq r_1 < c$, we must have $\rho(x, y) = c$. $\square$

We will get a bit closer to showing the relationship between open and closed balls with the following results and will pick up a few other useful facts along the way. We start with another definition.

Definition 13. If (M, ρ) is an ultrametric space, then

$$S(x_0, r) = \{x \in M; \rho(x, x_0) = r\}$$

is the **sphere of radius r at x_0**.

Lemma 1. *Spheres (of positive radius) in an ultrametric space are both open and closed as sets.*

Proof. A sphere in any metric space is closed, so we need only show a sphere is also open in an ultrametric space. We show a sphere, $S = S(x_0, r)$, is equal to a union of opens, $S = \cup_{x \in S} B^0(x, r)$.

Let $B = B^0(x, s)$ be an open ball that does not contain x_0. Let $r = \rho(x_0, x)$. We must have $r \geq s$, so then (since all triangles are isosocles) every point in B lies in

$S(x_0, r)$, that is $B \subseteq S(x_0, r)$. Then for any $x \in S(x_0, r)$, $B^0(x, r) \subseteq S(x_0, r)$ and

$$\bigcup_{x \in S(x_0, r)} B^0(x, r) \subseteq S(x_0, r)$$

The reverse inequality is clear since the union is taken on points of S. $\square$

Proposition 14. *The open balls in an ultrametric space are closed sets and the closed balls are open sets.*

Proof. The proof follows immediately from the result that spheres are clopen: to see that closed balls are open sets, note that for a closed ball, $B(x_0, r)$,

$$B(x_0, r) = B^0(x_0, r) \cup S(x_0, r)$$

Likewise, to see that open balls are closed sets, note that

$$B^0(x_0, r) = B(x_0, r) \setminus S(x_0, r)$$

$\square$

The following proposition is now easy to see, although the result is both unintuitive and important for our purposes.

Proposition 15. *Suppose S is a compact subset of an ultrametric space (M, ρ) and that $\cup_{i \in I} B(x_i, r_i)$ is a cover of S by closed balls in S. Then there exists $i_1, \ldots, i_n$, a finite subset of I, such that $\cup_{j=1}^{j=n} B(x_{i_j}, r_{i_j})$ is a partition of S.*

Proof. Since S is compact and ρ is an ultrametric, $\cup_{i \in I} B(x_i, r_i)$ is an open cover and contains a finite subcover of S. Say this subcover is given by the elements $i_1, \ldots, i_{n'} \in I$, and suppose this is not a partition. That is, suppose for some i_i, i_j, $B(x_{i_i}, r_{i_i}) \cap B(x_{i_j}, r_{i_j}) \neq \emptyset$. Then, without loss of generality, we must have $B(x_{i_i}, r_{i_i}) \subseteq B(x_{i_j}, r_{i_j})$, so that the removal of $B(x_{i_i}, r_{i_i})$ is still a cover of S. We continue this process a finite number of times, since the subcover was finite to begin with, to arrive at a finite partition of S. $\square$

In fact, a slightly stronger statement then the above is true:

Corollary 1. *Suppose S is a compact subset of an ultrametric space (M, ρ) and that $B(x_0, r)$ is a closed ball in S. Then, there exists a finite partition of S having $B(x_0, r)$ as an element.*

Proof. Let $\mathcal{C}$ be the cover of S given by $\cup_{x \in S} B(x, r) \cap S$. From the proposition, we can select a finite subcover of $\mathcal{C}$ that is a partition of S. Suppose $B(y, r) \cap S$ is the element in this partition containing x_0. Then since $B(y, r)$ and $B(x_0, r)$ are equal in M, $B(y, r) \cap S = B(x_0, r) \cap S = B(x_0, r)$. $\square$

We end this section by making a few comments about the set of distances that occur between the points of a compact ultrametric space.

Proposition 16. *[Ro] Let S be a compact subset of an ultrametric space, (M, ρ)*

(i) For $m \in (M \setminus S)$, let $f_m : S \to \mathbb{R}$, be the function defined by $f_m(s) = \rho(m, s)$. Then $Im(f_m)$ is finite for all $m \in (M \setminus S)$.

(ii) For $a \in S$, let $\phi_a : S \setminus \{a\} \to \mathbb{R}$ be the function defined by $\phi_a(x) = \rho(x, a)$. Then $Im(\phi_a)$ is a discrete subset of $\mathbb{R}$ for all $a \in S$.

Proof. [Ro]

(i) The fibers of f_m, $f_m^{-1}(s)$, for $s \in S$, form a cover of S. In fact, they form an open partition. Since S is compact by assumption, we must have that this partition is finite, and so the image of f_m was also finite.

(ii) Let $\epsilon > 0$. Let $B^0(a, \epsilon)$ be the open ball, $B^0(a, \epsilon) = \{x \in S; \rho(x, \epsilon) < \epsilon\}$. Then $(S \setminus B^0(a, \epsilon))$ is compact, and so from the above we know that ϕ_a restricted to $(S \setminus B^0(a, \epsilon))$ has finite range (let $M = S$ and $S = (S \setminus B^0(a, \epsilon))$ and apply (i)). Then the sets

$$[\epsilon, \infty) \cap \{\rho(s, a); s \in S, x \neq a\}$$

are finite and $Im(\phi_a)$ is discrete.

$\square$

This leads to the following definition.

Definition 14. If (M, ρ) is an ultrametric space, we say M is **discretely-valued** if the set $\Gamma_S = \{r \in \mathbb{R}; \exists x, y \in M \text{ such that } \rho(x, y) = r\}$ is a discrete subset of $\mathbb{R}$.

If (M, ρ) has a translation-invariant ultrametric then clearly M is discretely-valued since the sets ϕ_a are then equal for all a in M. Now we have the following question.

Question 1. Are there mild conditions under which a compact ultrametric space is discretely-valued? In particular, are there conditions that do not appeal to the algebraic structure in M?

Now we immediately get the following corollary.

Corollary 2. *[Ro] Let $B(a, r)$ be a closed ball in an compact, discretely-valued ultrametric space. Then there exists $r' > r \in \mathbb{R}$ such that $B(a, r) = \{x \in M \mid \rho(x, a) < r'\}$; that is, every closed ball is also an open ball with the same centre and slightly larger radius.*

It will become useful to write the set of distances occuring in S as a sequence, put in decreasing order.

Notation. If S is a compact, discretely-valued ultrametric space, then we denote the set of distances between points of S by

$$\Gamma_S = \{\gamma_0 = d = diam(S), \gamma_1, \gamma_2, \ldots, \gamma_\infty = 0\}$$

where $\gamma_i \in \Gamma_S$ if and only if $\exists x, y \in S$ such that $\rho(x, y) = \gamma_i$ and $\gamma_i < \gamma_j$ if and only if $i > j$.

ρ-orderings, ρ-sequences, and valuative capacity

We are now in a position to give a general definition of $p-$orderings and in turn, $p-$sequences and valuative capacity. The observation that an analogous notion of $p-$ordering can be defined for a general ultrametric space, and that these structures coincide with Fekete $n-$tuples, is due to [J1]. The exploration of this idea makes up the remainder of this work.

Definition 15. [J1] Let S be a subset of an ultrametric space (M, ρ). A **ρ-ordering** of S is a sequence $\{a_i\}_{i\geq 0}$ in S such that a_0 is arbitrary and $\forall n > 0$, a_n maximizes

$$\prod_{i=0}^{n-1} \rho(s, a_i)$$

over $s \in S$.

The above generalizes the definition of $p-$orderings for $\mathbb{Z}$, since maximizing the $p-$adic distance between two points in $\mathbb{Z}$ (or $\mathbb{Z}_p$) is the same as minimizing the $p-$adic valuation of the difference of two points. In particular, $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S, a subset of $\mathbb{Z}$, if and only if it is a ρ_p-ordering of (S, ρ_p). Let us see an example of the simplest kind, that is, for a finite set S.

Example 3. Suppose S is a finite subset of $(\mathbb{Z}, \rho_2)$, $S = \{0, 2, 8, 3\}$. Then a ρ_2-ordering of S starts (arbitrarily) with $a_0 = 0$, which forces $a_1 = 3$, since $\rho_2(0, 3) = 1 = diam(S)$. The sequence continues $a_2 = 2$ and $a_3 = 8$, but after this point the sequence becomes arbitrary because $\prod_{i=0}^{n-1} \rho(s, a_i)$ will contain a 0, given by the repeated term. Indeed, for any finite subset S with $|S| = n$, the $\rho-$ordering of S is arbitrary from the n^{th} point on.

We now give the definition of a $\rho-$sequence for an ultrametric space, generalizing the notion of a $p-$sequence.

Definition 16. [J1] Let $\{a_i\}_{i\geq 0}$ be a $\rho-$ordering of S. The **ρ-sequence** of S is defined by letting $\delta(0) = 1$ and for $n > 0$,

$$\delta(n) = \prod_{i=0}^{n-1} \rho(a_n, a_i)$$

The two propositions that follow are the critical observations. The first one tells us that we can use the $\rho-$sequence of S as an invariant and the second one motivates the definition of valuative capacity. The proofs of each are given in [J1].

Proposition 17. *([J1], Lemma 1) The ρ-sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ-sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ-ordering of S.*

Proposition 18. *([J1], Theorem 1) If S is a compact subset of an ultrametric space (M, ρ), then the first n terms of a ρ-ordering of S always give a Fekete n-tuple of S and all Fekete n-tuples of S arise in this way.*

Armed with the notion of a well-defined $\rho-$sequence for an ultrametric space, and the knowledge that it gives a construction for Fekete $n-$tuples in that space, we define the valuative capacity of S, where S is any compact subset of an ultrametric space.

Definition 17. [J1] Let S be a compact subset of an ultrametric space (M, ρ) and let $\delta(n)$ be the ρ-sequence of S. The **valuative capacity** of S is

$$\omega(S) := \lim_{n\to\infty} \delta(n)^{1/n}$$

We spend the rest of this chapter showing some basic results on valuative capacity. These results form the start of our toolkit for calculating the capacities of specifics sets. They also show that many of the properties of capacity from $\mathbb{C}$ carry over to the non-Archimedean case in a natural way.

Let us assume from this point on that S is always a compact subset of an ultrametric space, unless stated otherwisse.

Proposition 19. *([J1], theorem) $\omega(S)$ is finite. If S itself is finite, then $\omega(S) = 0$.*

A compact set $E \subseteq \mathbb{C}$ is said to be polar if the logarithmic capacity of E is 0 [Ra]. Polar sets play a central role in potential theory and the theory of logarithmic capacity, which raises the following question:

Question 2. Are there ultrametric spaces that have some *infinite* subset S with $\omega(S) = 0$?

We also have the expected result on monotoncity for valuative capacity:

Proposition 20. *([J1], Lemma 4) If S and T are compact subsets of an ultrametric space such that $S \subseteq T$ then $\omega(S) \leq \omega(T)$.*

We show now some results on the interaction between the algebraic structure of the space and valuative capacity. These results can be powerful tools for calculating capacities, in particular, when they are combined with the subadditivity result that follows.

Proposition 21. *(translation invariance) If (M, ρ) is a compact ultrametric space and also a topological group for which ρ is (left) invariant under the group operation, then ω is also (left)-invariant. That is, if $\rho(x, y) = \rho(g + x, g + y)$, $\forall g, x, y \in M$, then $\omega(g + S) = \omega(S)$, for $S \subseteq M$.*

Proof. Let $\{a_i\}_{i \geq 0}$ be a ρ-ordering for S. Then $\{g + a_i\}_{i \geq 0}$ is a ρ-ordering for $g + S$. Then

$$\omega(g + S) = \lim_{n \to \infty} \delta(n)^{1/n}$$

$$= \lim_{n \to \infty} [\prod_{i=0}^{n-1} \rho(g + a_n, g + a_i)]^{1/n}$$

$$= \lim_{n \to \infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = \omega(S)$$

□

Example 4. Note that ρ_p is translation invariant for each p since for any x, y, we have $\rho_p(x, y) = p^{-v_p(x-y)} = p^{-v_p((a+x)-(a+y))} = \rho_p(a+x, a+y)$. Then $\omega(a+S) = \omega(S)$ for $S \subseteq (\mathbb{Z}_p, \rho_p)$.

Proposition 22. *(scaling) Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity, so that (V, ρ_N) is an ultrametric space. Then if N is multiplicative, so is ω. That is, if $N(gx) = N(g)N(x)$,$\forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.*

Proof. Let ρ_N be the metric induced by N, so that $\rho_N(x, y) = N(x - y), \forall x, y \in V$. Let $\{a_i\}_{i \geq 0}$ be a ρ_N-ordering for S and let u, v be in gS with $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$. Then, since N is multiplicative,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j)$$

$$= N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j),$$

so that $\{ga_i\}_{i\geq 0}$ is a ρ_N-ordering for gS. Then,

$$\omega(gS) = \lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty}[\prod_{i=0}^{n-1}N(g)\rho(a_n, a_i)]^{1/n}$$

$$= \lim_{n\to\infty}[N(g)^n\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\lim_{n\to\infty}[\prod_{i=0}^{n-1}\rho(a_n, a_i)]^{1/n} = N(g)\omega(S)$$

□

Example 5. Since ρ_p is multiplicative, we have that $\omega(mS) = |m|_p \cdot \omega(S)$ for $m \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) = \frac{1}{p}\cdot\omega(\mathbb{Z})$.

The following proposition is from [J1], where it is given for some S written as the union of two subsets, although it is easily seen to be true for S equal to any finite union, so long as the other assumptions remain satisfied.

Proposition 23. *([J1], Proposition 10) (subadditivity) If $diam(S) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then*

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n}\frac{1}{log(\omega(A_i)/d)}$$

Example 6. We are now in a position to compute the valuative capacity of $(\mathbb{Z}, \rho_p)$. For any p, we note that the $\mathbb{Z}$ can be decomposed into p closed balls of radius $\frac{1}{p}$, which are equal to the cosets of $\mathbb{Z}$ modulo p. Since $diam(S) = 1$, this gives

$$\frac{1}{log(\omega(\mathbb{Z}))} = \sum_{i=0}^{p-1}\frac{1}{log(\omega(p\mathbb{Z}+i))} = \frac{p}{log(\omega(p\mathbb{Z}))} = \frac{p}{log(\frac{1}{p}\cdot\omega(\mathbb{Z}))}$$

Now we have,

$$log(\omega(\mathbb{Z})^p) = log(\frac{1}{p}\cdot\omega(\mathbb{Z}))$$

so that,

$$\omega(\mathbb{Z})^p - \frac{\omega(\mathbb{Z})}{p} = 0$$

and $\omega(\mathbb{Z}) = p^{\frac{1}{1-p}} = p^{\frac{-1}{p-1}}$.

We can apply the same reasoning to any partition of S made up of sets that all have the same capacity to obtain the following:

Corollary 3. *Suppose* $S = \cup_i^n S_i$ *with* $\rho(S_i, S_j) = d = diam(S)$ *and also* $\omega(S_i) = \omega(S_j)$, $\forall i, j$ *. Let* $r \in \mathbb{R}$ *be such that* $\omega(S_i) = r\omega(S)$, $\forall i$*. Then* $\omega(S) = r^{\frac{1}{n-1}} \cdot d$*.*

Now we note that a partition of S into closed balls satsifies the hypotheses if the distance between each ball is equal to the diameter of S. In particular, if $B(x_i, r_i)$ is a collection of closed balls such that the pairwise-distance between any $B(x_i, r_i)$ and $B(x_j, r_j)$ is constant, then if we know the capacity of each $B(x_i, r_i)$, we can compute the capacity of their union. If M is discretely-valued, then we can say more.

Corollary 4. *(Joins of computable sets are computable) Let* M *be a compact, discretely-valued ultrametric space. Let* $\Gamma_M = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ *be the set of distances in* M*. Suppose that* $S = B(x, \gamma_i)$*, for some* x *and* i*, is the union of* $n \geq 2$ *or more balls of radius* γ_{i+1}*, that is,* $S = \cup_{j=1}^n B(x_j, \gamma_{i+1})$ *is a join in the lattice of closed balls in* M*, then*

$$\frac{1}{log(\frac{\omega(B(x,\gamma_i))}{\gamma_{i+1}})} = \sum_{j=1}^{n} \frac{1}{log(\frac{\omega(B(x_j,\gamma_{i+1}))}{\gamma_{i+1}})}$$

Of course, if M is a group, then we know the elements in these partitions are cosets, and if the metric is translation-invariant, then they each have the same capacity. We take up this last corollary in significant detail in the next chapter, obtaining some formulae for valuative capacity with various restrictions on Γ_M or related structures.

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