

Chapter 1

Introduction

In the course of developing a generalized factorial function, Manjul Bhargava introduced the notion of p -orderings of a Dedekind domain [B1, B2], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [B3, J2] and have also provided a natural framework to extend many classical results in analysis to a p -adic setting, such as polynomial approximation and mapping theorems [B1, B2, B3]. In this thesis, we examine how a tool based on p -orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-Archimedean settings.

The historical background to this work comes in two parts. On the one hand, there is the background on logarithmic capacity from potential theory, and secondly, there is the background from Bhargava's p -orderings. We give a brief summary of each here. A similar treatment, with slightly different perspective, is found in [FP]. Jean-Luc Chabert was the first to draw a connection between the two, and many of the known results in this area stem from his work or that of his colleagues. Building on the result in [J1], we extend the work by Chabert and colleagues by studying valutive capacity in a more general setting, namely that of an ultrametric space, which may or may not also be a local field. In doing so, we show many properties of capacity are in fact independent of the algebraic structure of a space, although such structure, when it exists, can act as a useful probe.

1.1 Logarithmic capacity

The theory of capacity has been developed as a topic in potential theory in a variety of settings. Classically, the notion of capacity was developed over both \mathbb{C} and \mathbb{R}^n , although the theory has been further developed in a rather general way by Rumely for Berkovich spaces. A significant body of work on the analytic properties of capacity can be found for a number of different contexts. For example, such a treatment of the subject over \mathbb{C} can be found in [W] and [Ra1], over Berkovich spaces in [BR], and over \mathbb{Q}_p in [Ca]. We give a brief account of capacity over \mathbb{C} here, presenting only the most essential definitions and results. One advantage of tracing the historical roots of capacity back to \mathbb{C} is that the theory in this setting also comes equipped with a physical interpretation. As we are about to see, capacity in the classical sense gives a mathematical model for the amount of electrostatic charge a conductor can hold. The exposition below is closely based on [Ra1] and [Ra2].

Even restricting ourself to the definition of capacity of subsets of \mathbb{C} , we find two paths, one which will give us some physical interpretation, and one which will lead more naturally to p -orderings. We start with the former.

Definition 1. [Ra2] Let μ be a finite Borel measure on \mathbb{C} and suppose μ has compact support. We associate to μ a function, $p_\mu : \mathbb{C} \rightarrow (-\infty, \infty]$, given by

$$p_\mu(x) = \int \log \frac{1}{|x - y|} d\mu(y)$$

called the **potential function** of μ . The **energy** of μ is

$$I(\mu) = \int \int \log \frac{1}{|x - y|} d\mu(y) d\mu(x)$$

This gives at once the physical interpretation promised above. We interpret the potential function of a measure as giving the potential energy of a point. Viewing the measure as a charge distribution, the double integral gives back the total energy in the system. Now we come upon a physical reality: charged particles in a conductor will naturally distribute themselves in order to minimize the energy. This leads to

the definition below:

Definition 2. [Ra2] Let K be a compact subset of \mathbb{C} and let $\mathcal{P}(K)$ be the set of Borel probability measures on K . If $\nu \in \mathcal{P}(K)$ is such that

$$I(\nu) = \inf_{\mu \in \mathcal{P}(K)} I(\mu)$$

then ν is a **equilibrium measure** for K .

We state the following proposition without proof. A sketch of the proof can be found in [FP] and the full details can be found in [Ra1].

Proposition 1. [Ra1] *An equilibrium measure exists for every compact set $K \in \mathbb{C}$. When finite, the equilibrium measure is unique and isometry-invariant.*

We are now ready to give our first definition of capacity.

Definition 3. [Ra2] Let K be a compact subset of \mathbb{C} . The logarithmic capacity of E is

$$C(K) = e^{-I(\nu)}$$

where ν is an equilibrium measure on K . If $I(\nu) = \infty$, then we understand that $C(K) = 0$.

We present below a few results on capacity in \mathbb{C} , some of which will reappear in the remainder of this work, although the context, and the proofs (omitted here), bear little resemblance to the present case.

Proposition 2. ([Ra1], 5.1.2) *Let K, K_1, K_2 be compact subsets of \mathbb{C} .*

1. $K_1 \subseteq K_2$, then $C(K_1) \leq C(K_2)$.
2. $C(\alpha K + \beta) = |\alpha|C(K)$ for all $\alpha, \beta \in \mathbb{C}$.

3. $C(K) = C(\delta_e K)$, where δ_e is the exterior boundary.¹

Proposition 3. ([Ra1], 5.1.4) Suppose $\{B_n\}$ is a sequence of Borel subsets of \mathbb{C} . Let $B = \cup_n B_n$ and $d \geq 0$.

1. If $\text{diam}(b) \leq d$, then $C(B) \leq d$ and

$$\frac{1}{\log(\frac{d}{C(B)})} \leq \sum_n \frac{1}{\log(\frac{d}{C(B_n)})}$$

2. If $\text{dist}(B_j, B_k) \geq d$ whenever $j \neq k$, then

$$\frac{1}{\log^+(\frac{d}{C(B)})} \geq \sum_n \frac{1}{\log^+(\frac{d}{C(B_n)})}$$

where $\log^+(x) = \max(\log(x), 0)$.

We now know show an equivalent way of defining of capacity, still over \mathbb{C} , which starts with the following two definitions due to Fekete [F].

Definition 4. [F] Let $K \subseteq \mathbb{C}$ be a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \dots, z_n) \in K^n$, consider

$$\delta_n(z) = \prod_{j < i} |z_i - z_j|^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \dots, z_n) \in K^n$ is called a **Fekete n-tuple** if z maximizes δ_n over all n -tuples in K .

Note that since K is compact by assumption, a Fekete n -tuple exists for each n .

Definition 5. Let $K \subseteq \mathbb{C}$ be a compact subset. The **transfinite diameter** of K is

$$\lim_{n \rightarrow \infty} [\max_z \delta_n(z)]$$

¹The exterior boundary of a compact subset, K , of \mathbb{C} is the boundary of the unbounded, connected component of $U = \mathbb{C} \setminus K$.

where the maximum is taken over all n -tuples in K . That is, the transfinite diameter of K is $\lim_{n \rightarrow \infty} \delta_n(z)$, where z is a Fekete n -tuple for each n .

The following proposition shows the relation to capacity.

Proposition 4. (*[F], Fekete-Szegő Theorem*) *If K is a compact subset of \mathbb{C} , then the transfinite diameter of K is equal to the logarithmic capacity of K .*

We end this section with an observation about the points z_i in \mathbb{C} (or some subset thereof) making up a Fekete n -tuple. For $n \geq 2$, if (z_1, \dots, z_n) is a Fekete n -tuple, then in general there is no z_{n+1} available such that $(z_1, \dots, z_n, z_{n+1})$ a Fekete $(n+1)$ -tuple. In physical terms, we note that the placement of a new charge in a conductor will almost always change the location of the existing charges in that conductor. Remarkably, this is not the case in ultrametric spaces. Indeed, we are able to build the analogous structure, which we call a p -ordering or more generally a ρ -ordering, *recursively*, that is by reusing the points from the previous iteration.

1.2 P-orderings

The development of p -ordering was motivated by the observation that the factorial function had important number-theoretic applications, yet was only defined for the set \mathbb{Z} . In order to generalize the factorial, Bhargava defined it via an invariant, called the p -sequence, which could be attached to any subset of a Dedekind domain ² [B1].

We cannot go much further without introducing the following definition.

Definition 6. Let $z \in \mathbb{Z}$ and let p be any prime. The p -**adic valuation** of z , denoted $v_p(z)$, is the largest $n \in \mathbb{N}$ such that p^n divides $z \neq 0$ and $v_p(z) = \infty$ if $z = 0$. That is,

$$v_p(z) = \begin{cases} \max\{n \in \mathbb{N}; p^n \mid z\}, & \text{if } z \neq 0 \\ \infty, & \text{otherwise} \end{cases}$$

²In fact, Bhargava associated p -sequences to the more general class of Dedekind rings, which are locally principal, Noetherian rings in which all nonzero primes are maximal.

For $z \in \mathbb{Z}$, we define the p -**adic absolute value** by

$$|z|_p = p^{-v_p(z)}$$

and the p -**adic metric** accordingly; that is, for $z_1, z_2 \in \mathbb{Z}$

$$\rho_p(z_1, z_2) = p^{-v_p(z_1 - z_2)}$$

where $p^{-\infty}$ is taken to be 0.

It is worth pausing to make a few comments about the above definitions. That the p -adic metric is truly a metric is easy to see. In fact, we will see in the next chapter that it is not just a metric, but also an ultrametric, since the p -adic absolute value satisfies a strengthened version of the triangle identity. The strong triangle identity is not the only interesting property at hand though. Like the logarithm, the p -adic valuation also satisfies: $v_p(x \cdot y) = v_p(x) + v_p(y)$ for any prime p and x, y in \mathbb{Z} . Moreover, we note that the p -adic valuation and p -adic metric have an interesting relationship with each other: two points whose difference has a relatively small valuation will have a relatively large distance between them and vice versa.

We are now ready to define p -orderings, and not long after, to give the connection to Fekete n -tuples.

Definition 7. [B1] Let S be a subset of \mathbb{Z} and let p be any prime.³ A p -**ordering** of S is a sequence, $\{a_i\}_{i \geq 0}$ in S , such that a_0 is arbitrary and for $i > 0$, a_i minimizes

$$v_p\left(\prod_{j < i} (z - a_j)\right)$$

over $z \in S$.

A p -ordering in S , like a Fekete n -tuple in \mathbb{C} , is not unique. Indeed, in most of the examples we will explore, there will be infinitely-many choices at each stage

³To apply the definition to a general Dedekind domain, we replace the usual primes with the set of prime ideals in the ring of interest.

of the construction. Nonetheless, p -orderings give rise to p -sequences, which are invariants of S :

Definition 8. [B1] Let S be a subset of \mathbb{Z} and let p be any prime. Suppose $\{a_i\}_{i \geq 0}$ is a p -ordering of S . The **p-sequence**, occasionally the **characteristic sequence**, of S is the sequence defined by $\delta(0) = 1$ and for $i > 0$,

$$\delta(i) = v_p\left(\prod_{j=0}^{i-1} (a_i - a_j)\right)$$

It is a fact, not entirely obvious, that the p -sequence of S is independent of the p -ordering used in its construction [B1]. To define the generalized factorial, Bhargava considered the product of p -sequences taken over each prime p for arbitrary subsets of \mathbb{Z} . We will go in another direction.

Suppose we were to generalize our definition of Fekete n -tuple in the obvious way, as below.

Definition 9. Let (M, ρ) be a metric space and suppose $S \subseteq M$ is a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \dots, z_n) \in S^n$, consider

$$\delta_n(z) = \prod_{j < i} \rho(z_i - z_j)^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \dots, z_n) \in S^n$ is called a **generalized Fekete n-tuple** if z maximizes δ_n over all n -tuples in S .

What then is the connection to p -orderings and p -sequences? Suppose S is a subset of \mathbb{Z} and that $\{a_i\}_{i \geq 0}$ is a p -ordering of S for some prime p . Then of course from the definition of p -orderings, we know that for $i > 0$,

$$v_p\left(\prod_{j < i} (a_i - a_j)\right) \leq v_p\left(\prod_{j < i} (z - a_j)\right)$$

for $z \in S$. Something more is true though, namely,

$$v_p\left(\prod_{j<i}(a_i - a_j)\right) \leq v_p\left(\prod(x_i - x_j)\right)$$

for $x_i, x_j \in S$ [B1]. That is, when we pick a_i to minimize the p -adic valuation over $\prod_{j<i}(z - a_j)$, we actually achieve the minimum over the product of *all pairs of differences in S* . Since minimizing $v_p(x_i - x_j)$ is the same as maximizing $\rho_p(x_i, x_j)$, we have the following remarkable fact: if $\{a_i\}_{i \geq 0}$ is a p -ordering of S , then $\{a_i\}_{i=0}^n$ is a generalized Fekete n -tuple for (S, ρ_p) for each n . In particular, p -orderings give a recursive construction for generalized Fekete n -tuples.

The first connection between these objects was made by Jean-Luc Chabert in [Ch] when he studied the limit of these sequences not just for the case $M = \mathbb{Z}$ and $\rho = \rho_p$, but in the case that M is any rank-one valuation domain [Ch]. We repeat his theorem 4.2 from [Ch] below,

Proposition 5. *Let E be a subset of V , a rank-one valuation domain with valuation v . If $\{a_i\}_{i \geq 0}$ is v -ordering⁴ of E , then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0, \dots, x_n \in E} v\left(\prod_{0 \leq j < i \leq n} (x_i - x_j)\right)$$

Chabert called this limit the valutive capacity of E , and we shall do the same. The following result by Johnson in [J1], in which p -ordering has been replaced by ρ -ordering, provides the foundation for the rest of this work:

Proposition 6. *([J1], Theorem 1) If S is a compact subset of an ultrametric space (M, ρ) , then the first n terms of a ρ -ordering of S always give a Fekete n -tuple of S and all Fekete n -tuples of S arise in this way.*

⁴A v -ordering of E is exactly as expected: a sequence of distinct element $\{a_i\}_{i \geq 0}$ in E is v -ordering of E if for $n > 0$,

$$v\left(\prod_{k=0}^{n-1} (a_n - a_k)\right) \leq v\left(\prod_{k=0}^{n-1} (x - a_k)\right)$$

for each $x \in E$.

One important consequence of this remark is that it gives a way to define capacity in a general ultrametric space. By replacing the notion of p -ordering (or v -ordering) with the more general notion of ρ -ordering, we are able to give a definition of valutive capacity for a general ultrametric space, without appealing to any algebraic (or measure-theoretic) structure. Of course, we have yet to say what a ρ -ordering is. We take this up, along with the necessary background from ultrametric spaces, in the next chapter.

Chapter 2

Capacity and Ultrametric spaces

2.1 Ultrametric basics

The principal context for this thesis is an arbitrary ultrametric space, which is a metric space that also satisfies an additional axion, sometimes called the ultrametric inequality or (in the case of vector spaces) the strong triangle property. We define ultrametric spaces below and for the rest of this section, we review some of their more important characteristics. The proofs offered in this section are, for the most part, standard and can be found in a number of reference texts, such as [Ro].

Definition 10. Let (M, ρ) be a metric space; that is, suppose M is a set and $\rho : M \times M \rightarrow \mathbb{R}_{\geq 0}$ is such that:

- (i) $\rho(x, y) = 0$ if and only if $x = y$
- (ii) $\rho(x, y) = \rho(y, x)$
- (iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any $x, y, z \in M$. If ρ satisfies the ultrametric inequality,

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z))$$

for any $x, y, z \in M$, then (M, ρ) is an **ultrametric space**.

A special case of an ultrametric space, and one where much of the previous work on this topic has been completed, is one where the metric has been derived from a norm on a vector space.

Definition 11. Let (V, N) be a normed vector space; that is, suppose V is \mathbb{F} -vector space, for \mathbb{F} some subfield of \mathbb{C} , and $N : V \rightarrow \mathbb{R}_{\geq 0}$ is such that:

$$(i) \ N(x + y) \leq N(x) + N(y)$$

$$(ii) \ N(cx) = |c| \ N(x)$$

$$(iii) \ N(x) = 0 \text{ implies } x = 0$$

for any $x, y \in V$ and $c \in \mathbb{F}$. We say that N satisfies the **strong triangle inequality** if

$$N(x + y) \leq \max(N(x), N(y))$$

for any $x, y \in V$.

Proposition 7. *Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space, (V, ρ_N) , where ρ_N is the metric induced by N , that is, $\rho_N(x, y) = N(x - y)$, is an ultrametric space.*

Proof. We take for granted that (V, ρ_N) is a metric space and also note that

$$N(x + z) \leq \max(N(x), N(z))$$

implies

$$\rho_N(x, z) \leq \max(\rho_N(x, 0), \rho_N(z, 0)) \leq \max(\rho_N(x, y), \rho_N(y, z))$$

□

Notation. If (V, N) is a normed vector space, then the metric induced by N is denoted ρ_N .

When ultrametric spaces come from spaces with algebraic structure, such as normed vector spaces, some of this structure carries over into metric spaces structure in a rather nice way:

Proposition 8. *[Ro] Let S be a group equipped with a (right) invariant ultrametric, ρ . If $B = B(0, r)$ is a (closed) ball centred at the neutral element of S , that is $B = \{x \in S; \rho(x, 0) \leq r\}$, then B is a subgroup of S .*

Proof. Let $x, y \in B$. Then

$$\rho(x - y, 0) = \rho(x, y) \leq \max(\rho(x, 0), \rho(y, 0)) \leq r,$$

so that $x - y \in B$. □

In the previous chapter, we claimed that the p -adic metric was an ultrametric on the set \mathbb{Z} . Indeed, (\mathbb{Z}, ρ_p) and the closely related space of p -adic integers, denoted $\mathbb{Z}_{(p)}$, are the canonical examples of an ultrametric space.

Example 1. Let p be any prime and consider the metric space (\mathbb{Z}, ρ_p) . To see that (\mathbb{Z}, ρ_p) is an ultrametric space, we must show that ρ_p satisfies the ultrametric inequality, or equivalently, that p -adic absolute value satisfies the strong triangle inequality. Let x, y be in \mathbb{Z} and suppose $v_p(x) = n_x$ and $v_p(y) = n_y$. Then if $n = \min(n_x, n_y)$, p^n divides x and p^n divides y , so p^n divides $x + y$. We see now that $v_p(x + y) \geq \min(v_p(x), v_p(y))$ and in turn $|x + y|_p \leq \max(|x|_p, |y|_p)$.

Example 2. Let p be any prime. If

$$z = \sum_{i \geq 0} b_i p^i$$

is such that $b_i \in \{0, \dots, p - 1\}$ for all i , then we say that z is a **p -adic integer**. If $z = \sum_{i \geq 0} b_i p^i$, we note that if only a finite number of the coefficients of z are non-zero, then $\sum_{i \geq 0} b_i p^i$ is a representation in base p of some element of \mathbb{Z} . We can define the **p -adic order** of a p -adic integer, denoted $\text{ord}_p(z)$, in a way that agrees with the p -adic valuation when $\sum_{i \geq 0} b_i p^i$ is in \mathbb{Z} . We let $\text{ord}_p(z)$ be the smallest i such that $b_i \neq 0$. The p -adic integers are both a ring¹ and an ultrametric space with the metric induced by $\text{ord}_p(z)$. For a given prime, p , we denote the p -adic integers by $\mathbb{Z}_{(p)}$.

In what follows, we will often refer to p -adic spaces and it will not make much of a difference whether the reader prefers to think of being in (\mathbb{Z}, ρ_p) or $\mathbb{Z}_{(p)}$. The reason is this: when forming ρ_p -orderings of subsets of either space we are always able to do so by selecting elements with finite number of non-zero coefficients, that is, by selecting elements from \mathbb{Z} itself.

¹The ring operations carry over on the coefficients of p -adic integers in the expected way from $\mathbb{Z}/p\mathbb{Z}$, as long as special care is taken to keep track of carries.

Ultrametric spaces exhibit properties much unlike traditional metric spaces, and we review a few of these below. Of particular interest to us is the behavior between (closed) balls in an ultrametric space.

Notation. Let (M, ρ) be a compact ultrametric space and let

$$B(a, r) = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the *closed* ball of radius r , centred at a for some $r \in \mathbb{R}_{>0}$ and $a \in (M, \rho)$. Let

$$B^0(a, r) = \{x \in M \mid \rho(x, a) < r\}$$

denote the *open* ball of radius r , centred at a for some $r \in \mathbb{R}_{>0}$ and $a \in (M, \rho)$.

In the above notation, we break from convention in that we denote a closed ball without using any decoration. This is because before too long we will work exclusively with closed balls. We are able to do this because for the most part, the notion of open and closed ball in an ultrametric space overlap, although we will need a few more facts before showing this.

Definition 12. Let S be a subset of an ultrametric space. The **diameter of S** is $\text{diam}(S) = \sup_{x, y \in S} \rho(x, y)$. Note that if S is compact, $\text{diam}(S) = \max_{x, y \in S} \rho(x, y)$.

Proposition 9. Let $B = B(a, r)$ be a ball in an ultrametric space (M, ρ) . Then the diameter of B is less than or equal to the radius of B .

Proof. Suppose $d = \text{diam}(B) > r$. This would imply there exists x, y in B such that $\rho(x, y) > r$, in particular $\rho(x, y)$ is strictly greater than $\max(\rho(x, a), \rho(y, a))$, which is a contradiction since ρ is an ultrametric. \square

In the following proposition, we describe the triangles in an ultrametric space, and the result is more or less a restatement, in geometric terms, of the ultrametric inequality.

Proposition 10. *All triangles in an ultrametric space (M, ρ) are either equilateral or isosceles, with at most one short side.*

Proof. Let x, y , and z be three points in an ultrametric space (M, ρ) . We show that $\rho(x, y) \neq \rho(x, z)$ and $\rho(x, y) \neq \rho(y, z)$ implies $\rho(x, y) < \rho(x, z) = \rho(y, z)$.

If $\rho(x, z) \neq \rho(y, z)$, then without loss, $\rho(x, z) > \rho(y, z)$. At the same time, the ultrametric inequality implies $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ and $\rho(y, z) \leq \max(\rho(x, y), \rho(x, z))$. The first inequality implies $\rho(x, y) < \rho(x, z)$, which means the second inequality implies $\rho(y, z) < \rho(x, z)$. This is a contradiction, so we must have $\rho(x, z) = \rho(y, z)$.

To see that $\rho(x, y) < \rho(x, z)$, simply note that $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ \square

With this result in hand, we are able to quickly demonstrate some of the properties of balls, which are of fundamental importance to us. We see below that the ultrametric inequality, perhaps innocuous on the surface, quickly implies ultrametric balls are markedly different from their Archimedean counterparts.

Proposition 11. *Every point of a ball in an ultrametric is at its centre. That is, if $B(x_0, r)$ is a ball in an ultrametric space (M, ρ) , then $B(x, r) = B(x_0, r)$, $\forall x \in B(x_0, r)$*

Proof. Let $a \in B(x, r)$. Then $\rho(a, x) \leq r$ and since

$$\rho(a, x_0) \leq \max(\rho(a, x), \rho(x, x_0)) \leq r$$

we must have $a \in B(x_0, r)$ and $B(x, r) \subseteq B(x_0, r)$. A similar argument shows $B(x_0, r) \subseteq B(x, r)$. \square

Proposition 12. *If (M, ρ) is an ultrametric space and $B(x_0, r_1)$ and $B(y_0, r_2)$ are balls in (M, ρ) , then either $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$, $B(x_0, r_1) \subseteq B(y_0, r_2)$, or $B(x_0, r_1) \subseteq B(x_0, r_1)$. That is, in an ultrametric space, all balls are either comparable or disjoint.*

Proof. Suppose $B(x_0, r_1) \cap B(y_0, r_2) \neq \emptyset$ and let z be a point in the intersection. We show that if there exists an $a \in B(y_0, r_2)$ such that $a \notin B(x_0, r_1)$, then $B(x_0, r_1) \subseteq$

$B(y_0, r_2)$. Let $x \in B(x_0, r_1)$. Then we must have $\rho(x, z) < \rho(x, a)$, since $z \in B(x_0, r_1) = B(x, r_1)$ and a is not. Since the triangle with vertices (a, x, z) is isosceles with at most one short side, we must have $\rho(x, a) = \rho(a, z) \leq r_2$, since $a \in B(y_0, r_1) = B(z, r_2)$. Then $x \in B(y_0, r_1)$. \square

Proposition 13. *The distance between any two non-overlapping balls in an ultrametric is constant. That is, if $B(x_0, r_1)$ and $B(y_0, r_2)$ are two balls in an ultrametric space with $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$, then there exists a $c \in \mathbb{R}_{>0}$ such that $\rho(x, y) = c$, $\forall x \in B(x_0, r_1)$ and $\forall y \in B(y_0, r_2)$.*

Proof. Suppose $\rho(x_0, y_0) = c$ and let $x \in B(x_0, r_1)$ and $y \in B(y_0, r_2)$ be arbitrary. Consider the triangle formed by (x_0, y_0, y) . Since $\rho(x_0, y_0) = c$ and $\rho(y, y_0) \leq r_2 < c$, we must have $\rho(x_0, y) = c$ because triangles in an ultrametric space have at most one short side. Now consider the triangle formed by (x_0, x, y) . Since $\rho(x_0, y) = c$ and $\rho(x, x_0) \leq r_1 < c$, we must have $\rho(x, y) = c$. \square

We will get a bit closer to showing the relationship between open and closed balls with the following results and will pick up a few other useful facts along the way. We start with another definition.

Definition 13. If (M, ρ) is an ultrametric space, then for $x_0 \in M$ and $r \in \mathbb{R}_{>0}$,

$$S(x_0, r) = \{x \in M; \rho(x, x_0) = r\}$$

is the **sphere of radius r at x_0** .

Lemma 1. *Spheres (of positive radius) in an ultrametric space are both open and closed as sets.*

Proof. [Ro] A sphere in any metric space is closed, so we need only show a sphere is also open in an ultrametric space. We show a sphere, $S = S(x_0, r)$, is equal to a union of open sets, $S = \cup_{x \in S} B^0(x, r)$.

Let $B = B^0(x, s)$ be an open ball that does not contain some x_0 . Let $r = \rho(x_0, x)$. We must have $r \geq s$, so then (since all triangles are isosceles) every point in B lies

in $S(x_0, r)$, that is $B \subseteq S(x_0, r)$. Then for any $x \in S(x_0, r)$, $B^0(x, r) \subseteq S(x_0, r)$ and

$$\bigcup_{x \in S(x_0, r)} B^0(x, r) \subseteq S(x_0, r)$$

The reverse inequality is clear since the union is taken over points of S . \square

Proposition 14. *The open balls in an ultrametric space are closed sets and the closed balls are open sets.*

Proof. The proof follows immediately from the result that spheres are both open and closed: to see that closed balls are open sets, note that for a closed ball, $B(x_0, r)$,

$$B(x_0, r) = B^0(x_0, r) \cup S(x_0, r)$$

Likewise, to see that open balls are closed sets, note that

$$B^0(x_0, r) = B(x_0, r) \setminus S(x_0, r)$$

\square

The following proposition is now easy to see, although the result is both unintuitive and important for our purposes.

Proposition 15. *Suppose S is a compact subset of an ultrametric space (M, ρ) and that $\cup_{i \in I} B(x_i, r_i)$ is a cover of S by closed balls in S . Then there exists i_1, \dots, i_n , a finite subset of I , such that $\cup_{j=1}^n B(x_{i_j}, r_{i_j})$ is a partition of S .*

Proof. Since S is compact and ρ is an ultrametric, $\cup_{i \in I} B(x_i, r_i)$ is an open cover and contains a finite subcover of S . Say this subcover is given by the elements $i_1, \dots, i_{n'}$ $\in I$, and suppose this is not a partition. That is, suppose for some i_i, i_j , $B(x_{i_i}, r_{i_i}) \cap B(x_{i_j}, r_{i_j}) \neq \emptyset$. Then, without loss of generality, we must have $B(x_{i_i}, r_{i_i}) \subseteq B(x_{i_j}, r_{i_j})$, so that the removal of $B(x_{i_i}, r_{i_i})$ is still a cover of S . We continue this process a finite number of times, since the subcover was finite to begin with, to arrive at a finite partition of S . \square

In fact, a slightly stronger statement than the above is true:

Corollary 1. *Suppose S is a compact subset of an ultrametric space (M, ρ) and that $B(x_0, r)$ is a closed ball in S . Then, there exists a finite partition of S having $B(x_0, r)$ as an element.*

Proof. Let \mathcal{C} be the cover of S given by $\cup_{x \in S} B(x, r) \cap S$. From the proposition, we can select a finite subcover of \mathcal{C} that is a partition of S . Suppose $B(y, r) \cap S$ is the element in this partition containing x_0 . Then since $B(y, r)$ and $B(x_0, r)$ are equal in M , $B(y, r) \cap S = B(x_0, r) \cap S = B(x_0, r)$. \square

We end this section by making a few comments about the set of distances that occur between the points of a compact ultrametric space.

Proposition 16. *[Ro] Let S be a compact subset of an ultrametric space, (M, ρ)*

(i) For $m \in (M \setminus S)$, let $f_m : S \rightarrow \mathbb{R}$, be the function defined by $f_m(s) = \rho(m, s)$. Then $\text{Im}(f_m)$ is finite for all $m \in (M \setminus S)$.

(ii) For $a \in S$, let $\phi_a : S \setminus \{a\} \rightarrow \mathbb{R}$ be the function defined by $\phi_a(x) = \rho(x, a)$. Then $\text{Im}(\phi_a)$ is a discrete subset of \mathbb{R} for all $a \in S$.

Proof. [Ro]

(i) The fibers of f_m , $f_m^{-1}(s)$, for $s \in S$, form a cover of S . In fact, they form an open partition. Since S is compact by assumption, we must have that this partition is finite, and so the image of f_m was also finite.

(ii) Let $\epsilon > 0$. Let $B^0(a, \epsilon)$ be the open ball, $B^0(a, \epsilon) = \{x \in S; \rho(x, a) < \epsilon\}$. Then $(S \setminus B^0(a, \epsilon))$ is compact, and so from the above we know that ϕ_a restricted to $(S \setminus B^0(a, \epsilon))$ has finite range (let $M = S$ and $S = (S \setminus B^0(a, \epsilon))$ and apply (i)). Then the sets

$$[\epsilon, \infty) \cap \{\rho(s, a); s \in S, s \neq a\}$$

are finite and $\text{Im}(\phi_a)$ is discrete.

\square

This leads to the following definition.

Definition 14. If (M, ρ) is an ultrametric space, we say M is **discretely-valued** if the set $\Gamma_S = \{r \in \mathbb{R}; \exists x, y \in M \text{ such that } \rho(x, y) = r\}$ is a discrete subset of \mathbb{R} .

If (M, ρ) has a translation-invariant ultrametric then clearly M is discretely-valued since the sets ϕ_a are then equal for all a in M . Now we have the following question.

Question 1. Are there mild conditions under which a compact ultrametric space is discretely-valued? In particular, are there conditions that do not appeal to some algebraic structure in M ?

When this is the case, it will become useful to write the set of distances occurring in S as a sequence, put in decreasing order.

Notation. If S is a compact, discretely-valued ultrametric space, then we denote the set of distances between points of S by

$$\Gamma_S = \{\gamma_0 = d = \text{diam}(S), \gamma_1, \gamma_2, \dots, \gamma_\infty = 0\}$$

where $\gamma_i \in \Gamma_S$ if and only if $\exists x, y \in S$ such that $\rho(x, y) = \gamma_i$ and $\gamma_i < \gamma_j$ if and only if $i > j$.

We end this section with the following corollary.

Corollary 2. *[Ro] Let $B(a, r)$ be a closed ball in an compact, discretely-valued ultrametric space. Then there exists $r' > r \in \mathbb{R}$ such that $B(a, r) = \{x \in M \mid \rho(x, a) < r'\}$; that is, every closed ball is also an open ball with the same centre and slightly larger radius.*

ρ -orderings, ρ -sequences, and valuative capacity

We are now in a position to give a general definition of p -orderings and in turn, p -sequences and valuative capacity. The observation that an analogous notion of p -ordering can be defined for a general ultrametric space, and that these structures coincide with Fekete n -tuples, is due to [J1]. The exploration of this idea makes up the remainder of this work.

Definition 15. [J1] Let S be a subset of an ultrametric space (M, ρ) . A ρ -ordering of S is a sequence $\{a_i\}_{i \geq 0}$ in S such that a_0 is arbitrary and $\forall n > 0$, a_n maximizes

$$\prod_{i=0}^{n-1} \rho(s, a_i)$$

over $s \in S$.

The above generalizes the definition of p -orderings for \mathbb{Z} , since maximizing the p -adic distance between two points in \mathbb{Z} (or \mathbb{Z}_p) is the same as minimizing the p -adic valuation of the difference of two points. In particular, $\{a_i\}_{i \geq 0}$ is a p -ordering of S , a subset of \mathbb{Z} , if and only if it is a ρ_p -ordering of (S, ρ_p) . Let us see an example of the simplest kind, that is, for a finite set S .

Example 3. Suppose S is a finite subset of (\mathbb{Z}, ρ_2) , $S = \{0, 2, 8, 3\}$. Then a ρ_2 -ordering of S starts (arbitrarily) with $a_0 = 0$, which forces $a_1 = 3$, since $\rho_2(0, 3) = 1 = \text{diam}(S)$. The sequence continues $a_2 = 2$ and $a_3 = 8$, but after this point the sequence becomes arbitrary because $\prod_{i=0}^{n-1} \rho(s, a_i)$ will contain a 0, given by the repeated term. Indeed, for any finite subset S with $|S| = n$, the ρ -ordering of S is arbitrary from the n^{th} point on.

We now give the definition of a ρ -sequence for an ultrametric space, generalizing the notion of a p -sequence.

Definition 16. [J1] Let $\{a_i\}_{i \geq 0}$ be a ρ -ordering of S . The ρ -sequence of S is

defined by letting $\delta(0) = 1$ and for $n > 0$,

$$\delta(n) = \prod_{i=0}^{n-1} \rho(a_n, a_i)$$

The two propositions that follow are the critical observations. The first one tells us that we can use the ρ -sequence of S as an invariant and the second one motivates the definition of valutive capacity. The proofs of each are given in [J1].

Proposition 17. ([J1], Lemma 1) *The ρ -sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ -sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ -ordering of S .*

Proposition 18. ([J1], Theorem 1) *If S is a compact subset of an ultrametric space (M, ρ) , then the first n terms of a ρ -ordering of S always give a Fekete n -tuple of S and all Fekete n -tuples of S arise in this way.*

Armed with the notion of a well-defined ρ -sequence for an ultrametric space, and the knowledge that it gives a construction for Fekete n -tuples in that space, we define the valutive capacity of S , where S is any compact subset of an ultrametric space.

Definition 17. [J1] Let S be a compact subset of an ultrametric space (M, ρ) and let $\delta(n)$ be the ρ -sequence of S . The **valutive capacity** of S is

$$\omega(S) := \lim_{n \rightarrow \infty} \delta(n)^{1/n}$$

We spend the rest of this chapter showing some basic results on valutive capacity. These results form the start of our toolkit for calculating the capacities of specific sets. They also show that many of the properties of capacity from \mathbb{C} carry over to the non-Archimedean case in a natural way.

Let us assume from this point on that S is always a compact subset of an ultrametric space, unless stated otherwise.

Proposition 19. (*[J1], theorem*) $\omega(S)$ is finite. If S itself is finite, then $\omega(S) = 0$.

A compact set $E \subseteq \mathbb{C}$ is said to be polar if the logarithmic capacity of E is 0 [Ra1]. Polar sets play a central role in potential theory and the theory of logarithmic capacity, which raises the following question:

Question 2. Are there ultrametric spaces that have some *infinite* subset S with $\omega(S) = 0$?

We also have the expected result on monotonicity for vallicative capacity:

Proposition 20. (*[J1], Lemma 4*) If S and T are compact subsets of an ultrametric space such that $S \subseteq T$ then $\omega(S) \leq \omega(T)$.

We show now some results on the interaction between the algebraic structure of the space and vallicative capacity. These results can be powerful tools for calculating capacities, in particular, when they are combined with the subadditivity result that follows.

Proposition 21. (*translation invariance*) If (M, ρ) is a compact ultrametric space and also a topological group for which ρ is (left) invariant under the group operation, then ω is also (left)-invariant. That is, if $\rho(x, y) = \rho(g + x, g + y)$, $\forall g, x, y \in M$, then $\omega(g + S) = \omega(S)$, for $S \subseteq M$.

Proof. Let $\{a_i\}_{i \geq 0}$ be a ρ -ordering for S . Then $\{g + a_i\}_{i \geq 0}$ is a ρ -ordering for $g + S$. Then

$$\begin{aligned} \omega(g + S) &= \lim_{n \rightarrow \infty} \delta(n)^{1/n} \\ &= \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(g + a_n, g + a_i) \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = \omega(S) \end{aligned}$$

□

Example 4. Note that ρ_p is translation invariant for each p since for any x, y , we have $\rho_p(x, y) = p^{-v_p(x-y)} = p^{-v_p((a+x)-(a+y))} = \rho_p(a+x, a+y)$. Then $\omega(a+S) = \omega(S)$ for $S \subseteq (\mathbb{Z}_p, \rho_p)$.

Proposition 22. (*scaling*) Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity, so that (V, ρ_N) is an ultrametric space. Then if N is multiplicative, so is ω . That is, if $N(gx) = N(g)N(x), \forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.

Proof. Let ρ_N be the metric induced by N , so that $\rho_N(x, y) = N(x - y), \forall x, y \in V$. Let $\{a_i\}_{i \geq 0}$ be a ρ_N -ordering for S and let u, v be in gS with $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$. Then, since N is multiplicative,

$$\begin{aligned} \rho(u, v) &= \rho(gs_i, gs_j) = N(gs_i - gs_j) \\ &= N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j), \end{aligned}$$

so that $\{ga_i\}_{i \geq 0}$ is a ρ_N -ordering for gS . Then,

$$\begin{aligned} \omega(gS) &= \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} N(g)\rho(a_n, a_i) \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} \left[N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g) \lim_{n \rightarrow \infty} \left[\prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g)\omega(S) \end{aligned}$$

□

Example 5. Since ρ_p is multiplicative, we have that $\omega(mS) = |m|_p \cdot \omega(S)$ for $m \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) = \frac{1}{p} \cdot \omega(\mathbb{Z})$.

The following proposition is from [J1], where it is given for some S written as the union of two subsets, although it is easily seen to be true for S equal to any finite union, so long as the other assumptions remain satisfied.

Proposition 23. ([J1], Proposition 10) (*subadditivity*) If $\text{diam}(S) = d$ and $S =$

$\cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then

$$\frac{1}{\log(\omega(S)/d)} = \sum_{i=1}^n \frac{1}{\log(\omega(A_i)/d)}$$

Example 6. We are now in a position to compute the valuative capacity of (\mathbb{Z}, ρ_p) . For any p , we note that \mathbb{Z} can be decomposed into p closed balls of radius $\frac{1}{p}$, which are equal to the cosets of \mathbb{Z} modulo p . Since $\text{diam}(S) = 1$, this gives

$$\frac{1}{\log(\omega(\mathbb{Z}))} = \sum_{i=0}^{p-1} \frac{1}{\log(\omega(p\mathbb{Z} + i))} = \frac{p}{\log(\omega(p\mathbb{Z}))} = \frac{p}{\log(\frac{1}{p} \cdot \omega(\mathbb{Z}))}$$

Now we have,

$$\log(\omega(\mathbb{Z})^p) = \log\left(\frac{1}{p} \cdot \omega(\mathbb{Z})\right)$$

so that,

$$\omega(\mathbb{Z})^p - \frac{\omega(\mathbb{Z})}{p} = 0$$

and $\omega(\mathbb{Z}) = p^{\frac{1}{1-p}} = p^{\frac{-1}{p-1}}$.

We can apply the same reasoning to any partition of S made up of sets that all have the same capacity and meeting the requirement that their pairwise distances are all equal to the diameter of S .

Corollary 3. Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = \text{diam}(S)$ and also $\omega(S_i) = \omega(S_j), \forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S), \forall i$. Then $\omega(S) = r^{\frac{1}{n-1}} \cdot d$.

Now we note that a partition of S into closed balls will satisfy the hypotheses if the distance between each ball is equal to the diameter of S . In particular, if $B(x_i, r_i)$ is a collection of closed balls such that the pairwise-distance between any $B(x_i, r_i)$ and $B(x_j, r_j)$ is constant, then if we know the capacity of each $B(x_i, r_i)$, we can compute the capacity of their union. If M is discretely-valued, then we can say more.

Corollary 4. (*Joins of computable sets are computable*) Let M be a compact, discretely-valued ultrametric space. Let $\Gamma_M = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$ be the set of distances in

M . Suppose that $S = B(x, \gamma_i)$, for some x and i , is the union of $n \geq 2$ balls of radius γ_{i+1} , that is, $S = \cup_{j=1}^n B(x_j, \gamma_{i+1})$ is a join in the lattice of closed balls in M . Then

$$\frac{1}{\log(\frac{\omega(B(x, \gamma_i))}{\gamma_{i+1}})} = \sum_{j=1}^n \frac{1}{\log(\frac{\omega(B(x_j, \gamma_{i+1}))}{\gamma_{i+1}})}$$

Of course, if M is a group, then we know the elements in these partitions are cosets, and if the metric is translation-invariant, then they each have the same capacity. We take up this last corollary in significant detail in the next chapter, obtaining some formulae for valutive capacity with various restrictions on Γ_M or related structures.

Chapter 3

ρ -orderings and the structure of S

In the previous section, we defined valutive capacity for a compact subset S of an ultrametric space (M, ρ) . We also got a glimpse into the way the valutive capacity of S interacts with its other properties, such as the set of distances occurring in S and the lattice of closed balls in S (or equivalently, if S has enough structure, a lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valutive capacity of S and the lattice of closed balls in S . In particular, we show how, if S is compact and discretely-valued, the lattice of closed balls can be used to compute the first n terms of a ρ -ordering of S (for any $n < \infty$) and how, in some cases, this extends to being able to derive a formula for the valutive capacity of S .

Similar results have been found for the special case of ultrametric fields in [CEF]. We extend these results by moving to a more general setting, showing that much can be said about capacity in S without appealing to any underlying algebraic structure. Significant portions of the theory developed in this chapter was guided by an empirical investigation into the capacity of product spaces, which we describe in the next chapter. The code that performed these calculations is included in the appendix.

We assume throughout this section that S is a compact, discretely-valued subset of an ultrametric space (M, ρ) .

Subspaces of S

In the section we explore the subspaces of S formed by considering closed balls of some fixed radius. Recall from the previous section that if S is compact and

discretely-valued, then the set of distances occurring in S is a discrete, bounded subset of \mathbb{R} and so we may represent the set of distances by a sequence in decreasing order. As before, let the decreasing sequence of distances in S be given by $\Gamma_S = \{\gamma_0 = \text{diam}(S), \gamma_1, \dots, \gamma_\infty = 0\}$.

Now fix some $k \in \mathbb{N}$, and consider for a moment the set of closed balls of radius γ_k in S . We could denote these alternatively by $B^M(x, \gamma_k) \cap S$ or by $B^S(x, \gamma_k)$, but when there is no risk of confusion, we will denote them simply by $B(x, \gamma_k)$. Clearly, the set $\{B(x, \gamma_k); x \in S\}$ forms a cover of S . Although we have build the cover using closed balls, since we are in an ultrametric space, this gives an open cover of S (in fact, each element in the cover is not only an open set, but also an open ball for some radius slightly bigger than γ_k). Then since S is compact, we must have some x_1, \dots, x_n such that $S = \cup_{i=1}^n B(x_i, \gamma_k)$. In fact, since ρ is an ultrametric, we can pick the x_i 's so that $\cup_{i=1}^n B(x_i, \gamma_k)$ will be a disjoint union and therefore a finite partition of S . Note that both n and the x_i 's depend on our fixed k , but that n is independent of the x_i 's, since any choice of centres is equivalent. We rephrase this with following definition and lemma:

Definition 18. For S and Γ_S as above, and $k \in \mathbb{N}$, fixed, define \sim_k to be the relation on S given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e., $x \sim_k y$ if and only if $B_{\gamma_k}(x) = B_{\gamma_k}(y)$.

The fact \sim_k is an equivalence relation on S is equivalent to the observation that every point in a ultrametric ball is at its centre:

Lemma 2. *Let S and Γ_S be as above, then \sim_k is an equivalence relation on S .*

Proof. \sim_k is clearly reflexive and symmetric, since ρ is a metric. Transitivity results from the ultrametric property of ρ : if $x \sim_k y$ and $y \sim_k z$, then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(z, y)) \leq \gamma_k$$

so $x \sim_k z$. □

We denote the set of equivalence classes of S / \sim_k by S_{γ_k} . We have defined S_{γ_k} to be the set of equivalence classes in S under the relation \sim_k , which is equivalent to letting S_{γ_k} be the set of closed balls of fixed radius γ_k in S . We now offer a third perspective on the elements on S_{γ_k} , which is due to [Ac],

Lemma 3. *For each k , the elements of S_{γ_k} , that is, the closed balls of radius γ_k , themselves form an ultrametric space, where the metric is given by:*

$$\rho_k(B(x, \gamma_k), B(y, \gamma_k)) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } B(x, \gamma_k) = B(y, \gamma_k) \end{cases}$$

Proof. ρ_k is reflexive, symmetric and transitive since ρ is. Likewise, ρ_k satisfies the ultrametric property, since ρ does: let $B(x, \gamma_k), B(y, \gamma_k)$ and $B(z, \gamma_k)$ be any three elements of S_{γ_k} and suppose $\rho_k(B(x, \gamma_k), B(y, \gamma_k)) > 0$. Then,

$$\begin{aligned} \gamma_k &< \rho_k(B(x, \gamma_k), B(y, \gamma_k)) \\ &= \rho(x, y) \leq \max(\rho(x, z), \rho(y, z)) \\ &= \max(\rho_k(B(x, \gamma_k), B(z, \gamma_k)), \rho_k(B(y, \gamma_k), B(z, \gamma_k))) \end{aligned}$$

since $\gamma_k < \max(\rho(x, z), \rho(y, z))$ implies that at least one of $\rho_k(B(x, \gamma_k), B(z, \gamma_k))$ or $\rho_k(B(y, \gamma_k), B(z, \gamma_k))$ is greater than 0. □

So now the elements of S_{γ_k} may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

Definition 19. Let S and Γ_S be as above. Define $\beta(i)_{i \geq 0}$ to be the sequence given by $\beta(i) = |S_{\gamma_i}|$, which is an invariant of S and which counts the number of connected components of S_{γ_i} (that is, the points of S_{γ_i}), when viewed as a metric space. When necessary, we use $\beta^S(i)$ to denote the β sequence for a given, compact ultrametric

space S . Adapting the terminology in [FP], we call $\beta^S(i)$ the **structure sequence** of S .

Notation 1. Let S_{γ_k} be as above. We denote the elements of S_{γ_k} by $B_1^k, \dots, B_{\beta(k)}^k$ or by $B_1^{S,k}, \dots, B_{\beta(k)}^{S,k}$, when necessary.

We return to the sequence $\beta(i)$ at the end of this section. For now, we show how a ρ -ordering of S can be built recursively from the spaces S_{γ_k} . This begins by noting that the spaces themselves can be built recursively:

Observation 1. Let S , Γ_S , and S_{γ_k} be as above. Then $S_{\gamma_{k+1}}$ can be constructed by partitioning each of the closed balls in S_{γ_k} into closed balls of radius γ_{k+1} and taking their union: Let $B(x_i, \gamma_k)$ be an element of S_{γ_k} , denoted by B_i^k . Then, there exists $x_{i,1}, \dots, x_{i,l_i} \in B_i^k$ such that,

$$B_i^k = \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1})$$

and

$$B(x_{i,j}, \gamma_{k+1}) \cap B(x_{i,j'}, \gamma_{k+1}) = \emptyset, \forall j, j' \in 1 : l_i$$

and so

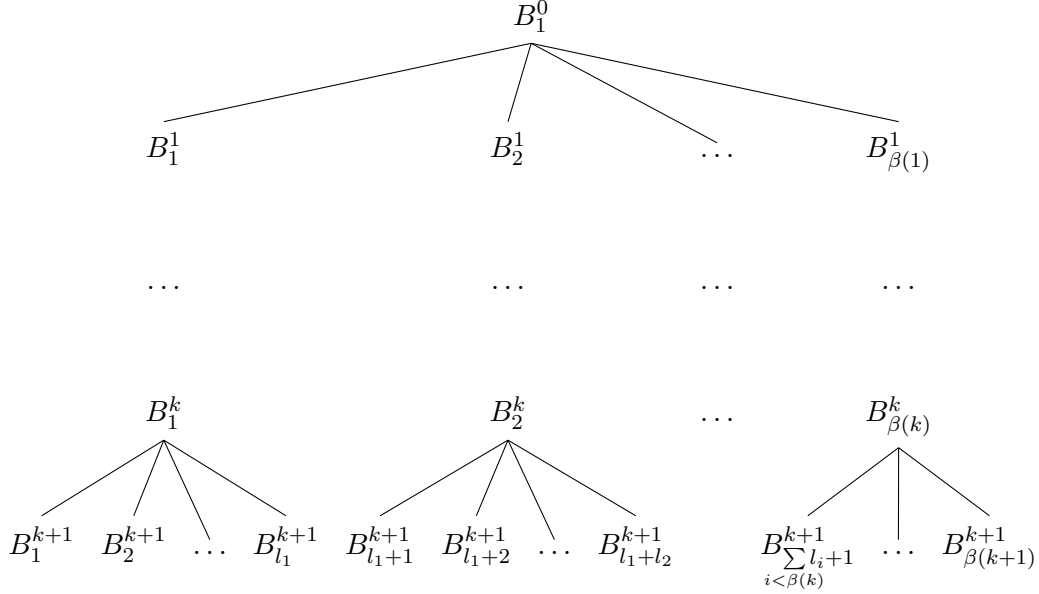
$$S_{\gamma_{k+1}} = \bigcup_{i=1}^{\beta(k)} \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1}) = \bigcup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where $\bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_k) = B(x_i, \gamma_{k+1}) = B_i^k, \forall i$.

Since S is compact, hence bounded, if we represent this process schematically we obtain a tree, where the root node is $B_1^0 = B(x, \gamma_0)$, for any choice of $x \in S$, and the children of any given B_n^m are such that they form a partition of their join. Since we will often refer to this schematic representation, we define it below.

Definition 20. If S is a compact subset of an ultrametric space, then T_s is the tree whose vertices are B_i^k , that is the elements of S_{γ_k} , and whose edgeset, E , is given by $(B_k^i, B_l^j) \in E$ if and only if $j = i + 1$ and $B_l^j \subseteq B_k^i$ for some choice of representatives

$B(x_k, \gamma_i)$ and $B(x_l, \gamma_j)$, as shown below:



Before going on, first note that we have drawn T_S such that leftmost child of some B_i^k is B_j^{k+1} where j is minimal among the children of B_i^k , and then continued in increasing order. In general, if we draw T_S so that the children of a given vertex are depicted in increasing order according to their index, then each choice of indexing for the elements of S_{γ_k} produces a different graphical representation of T_S . The structures produced by different choices of indices are clearly isomorphic as trees, and as we will see by the end of the section, each choice of indexing will be valid for our purposes as well.

Of central importance to us is the distance between two vertices in T_S . Since each vertex represents an element of S_{γ_k} , that is a closed ball in an ultrametric space, it is well-defined to let the distance between vertices be equal to the distance between a choice of centres for those balls. Note that if the distance between B_i^k and B_j^l is taken to be $\rho(x_i, x_j)$, for some choice of $x_i \in B_i^k$ and $x_j \in B_j^l$, say $\rho(x_i, x_j) = \gamma_n$, then the join of B_i^k and B_j^l is some B_x^n .

Lemma 4. *If B_i^k and B_j^l are two vertices in T_S , then $\rho(x_i, x_j)$, for any choice of $x_i \in B_i^k$ and $x_j \in B_j^l$, is equal to the diameter of the join of B_i^k and B_j^l .*

Proof. Let B_i^k and B_j^l be two (distinct) vertices in T_S and let B_x^n be their join. The diameter of B_x^n is γ_n since $B_x^n = B(x_0, \gamma_n)$ for some x_0 . Since ρ is an ultrametric the distance between any $x_i \in B_i^k$ and $x_j \in B_j^l$ is constant, and must be equal to the diameter of the smallest ball containing both of them, that is γ_n . \square

In particular, we have that for any k and any $i < \beta(k)$, the distances between the children of B_i^k will be γ_k and for any $i \neq j$ the distance between the children of B_i^k and B_j^k will be equal to the distance between B_i^k and B_j^k (which will be some $\gamma_m, m < k$).

Recursive ρ -orderings

In this section, we show how the recursive partitioning of S into the spaces S_{γ_k} gives rise to a ρ -ordering of S . We first note that without loss of generality, for any $k \in \mathbb{N}$, we can reindex the B_i^k 's so that they give the first $\beta(k)$ terms of a ρ_k -ordering of S_{γ_k} , when the latter is viewed as a (finite) metric space. In the first proposition below, we note that if the B_i^k 's are so indexed, then finding a ρ_{k+1} -ordering of $S_{\gamma_{k+1}}$ is straightforward: select a B_j^{k+1} from each of the B_i^k 's in order and then start over.

Proposition 24. *Let be S a compact, discretely-valued subset of an ultrametric space (M, ρ) and Γ_S , the set of distances in S . If S_{γ_k} is the partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k -ordering of S_{γ_k} , then the first $\beta(k+1)$ terms in a ρ_{k+1} -ordering of $S_{\gamma_{k+1}}$ can be found by selecting at each stage n , a child from $B_{\bar{n}}^k$, where $\bar{n} = n \bmod \beta(k) + r$ and r is minimal in $\{0, \dots, \beta(k) - 1\}$ such that $B_{n \bmod \beta(k) + r}^k$ still has unused children.*

Proof. Let S , S_{γ_k} , and $S_{\gamma_{k+1}}$ be as above. In particular, suppose the elements of S_{γ_k} are indexed according to a ρ_k -ordering. Denote the elements of $S_{\gamma_{k+1}}$ by $B_{i,j}^{k+1}$ where the first subscript indicates that the element is a child of B_i^k . To form a ρ_{k+1} ordering of $S_{\gamma_{k+1}}$, we must maximize the product of distances at each step n .

Now note that $\Gamma_{S_{\gamma_k}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$ and $\Gamma_{S_{\gamma_{k+1}}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}, \gamma_k\}$. That is, the distances in $S_{\gamma_{k+1}}$ are the same as the distances in S_{γ_k} , although they also

include the smaller distance γ_k . Since we know that the elements $B_1^k, \dots, B_{\beta(k)}^k$ already maximizes the product of distances in $\{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$, the first $\beta(k)$ terms of a ρ_{k+1} -ordering of S_{k+1} can be found by taking $B_{1,j_1}^k, \dots, B_{1,j_{\beta(k)}}^k$ for any choice of j 's. At this point, any choice of next element will produce a copy of γ_k in the ρ_{k+1} -sequence; however, if we chose another child of B_1^k , we are able to keep building the ordering in a canonical fashion, since we know that we will then be able to maximize the product at the next step by choosing another child of B_2^k .

We see then that a ρ_{k+1} -ordering of $S_{\gamma_{k+1}}$ is found by minimizing the number of times γ_k is introduced into the ρ_{k+1} -sequence and maximizing the product among the $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$, and the latter is already known to be achieved by taking the B_i^k in order. If the B_i^k 's all have the same number of children, then we can always select a child of $B_{\bar{n}}^k$, where $\bar{n} = n \bmod \beta(k)$ at each stage n , $n < \beta(k+1)$, since there will always be one available. On the other hand, suppose the B_i^k have an unequal number of children and n is the first step at which all the children of $B_{\bar{n}}^k$ have been exhausted. What element will maximize the ρ_{k+1} -sequence?

Consider the space $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$. Removal of $B_{\bar{n}}^k$ will not effect the first m terms of a ρ_k -ordering of this space, for $m < \bar{n}$, since if a sequence of elements maximizes a function over a set X , they will also maximize that function of a subset of X (provided they themselves remain in the subset). Then the ρ_k -sequence of $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$ begins $\{B_1^k, \dots, B_{\bar{n}-1}^k\}$.

Moreover, if $B_{\bar{n}+1}^k$ maximizes $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k)$ over S_{γ_k} , then it also maximizes $\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)$ over $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$, since $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k) = (\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)) \cdot \rho_k(x, B_{\bar{n}}^k)$.

Then the ρ_k -sequence of $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$ is simply $\{B_1^k, \dots, B_{\bar{n}-1}^k, B_{\bar{n}+1}^k, \dots, B_{\beta(k)}^k\}$.

Now we see that ρ_{k+1} -sequence of $S_{\gamma_{k+1}}$ is maximized by simply skipping over $B_{\bar{n}}^k$, should all its children be exhausted, and selecting a child from $B_{\bar{n}+1}^k$. Then a ρ_{k+1} -ordering of $S_{\gamma_{k+1}}$ is found by selecting elements of each B_i^k in order as much as possible, and skipping to B_{i+1}^k , when it is not possible.

□

Note that in building the ρ_{k+1} -ordering of $S_{\gamma_{k+1}}$ we selected, at each step, a child of some B_i^k , but we did not concern ourselves over which child was selected. This is because the distances between any two children of some B_i^k is γ_k , and the distance between any one of them and a child of some B_j^k , $i \neq j$, is the same. We can now see, as claimed above, that any of the isomorphic versions of T_S are valid for producing ρ -orderings. Suppose then that we have created T_s and (arbitrarily) indexed the children of each vertex. Then, there is no loss of generality in assuming that at each stage, we select a child with smallest index among its siblings, that is, that we select the leftmost available child in T_s . Since, for ease of indexing, we will assume a ρ -ordering has been built by this convention, we introduce the following definition.

Definition 21. The ρ -ordering of S formed by pulling elements from left to right in (a choice of) T_s is call the **canonical ρ -ordering** of S (with respect to T_s).

The above proposition quickly leads to a recursive construction for a ρ -ordering of S . Indeed, to build a ρ -ordering of S from the above, it suffices only to make a choice of centres for each of B_i^k 's.

Proposition 25. *Let be S a compact subset of an ultrametric space (M, ρ) and let Γ_S be the set of distances in S . Let S_{γ_k} be the partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k -ordering of S_{γ_k} . Suppose each of the element of S_{γ_k} have also been partitioned into closed balls of radius γ_{k+1} , $B_i^k = \cup_{j=1}^{l_i} B_{i,j}^{k+1}, \forall i$.*

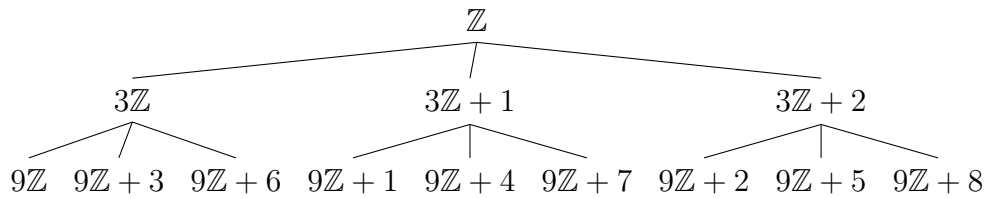
Let $x_{i,j}$ denote a choice of centre for the element $B_{i,j}^{k+1}$. Then the first $\beta(k+1)$ elements of a ρ -ordering of S can be found by forming a matrix, A_k , whose $(i, j)^{th}$ entry is $x_{i,j}$, if $j \leq l_i$ and $$ otherwise, and then concatenating the rows.*

Proof. The matrix A_k is a representation of the k^{th} and $(k+1)^{th}$ levels of T_S where the B_i^k 's (and $B_{i,j}^{k+1}$'s) have been replaced by a choice of centres. Since matrices must be rectangular, the case where some B_i^k and B_j^k have an unequal number children

is handled by inserting a placeholder, $*$, into A_k . Moreover, since the ρ_{k+1} distance between distinct closed balls is just the ρ distance between a choice of centres of those balls, a choice of centres in a ρ_{k+1} -ordering gives the beginning of a ρ -ordering. By the above proposition, we must select elements from each B_i^K one after the other, which is achieved by selecting one element from each column in order, for example by concatenating the rows (and then deleting $*$'s if necessary). \square

We get the most use out of the construction above if, in selecting a choice of centres for the $B_{i,j}^{k+1}$'s, we reuse the previous the choices as much as possible. Suppose for example we have made a choice of centres for the balls of radius γ_k and constructed the matrix A_{k-1} . At the next iteration, we will need a choice of centres for the balls of radius γ_{k+1} . If x_i was our choice of representative for B_i^k and $x_i \in B_{i,j}^{k+1}$, we may as well let x_i be our choice of representative for $B_{i,j}^{k+1}$. If we make our choice of centres in this way, then when we concatenate the rows of some A_{k-1} , we obtain (without loss) the first row of A_k . We follow this convention in the two examples below.

Example 7. Let us use the above to start a ρ -ordering of $S = (\mathbb{Z}, |\cdot|_3)$. We have that $\Gamma_S = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\}$ and T_s begins:



We start by finding a ρ_0 -ordering of S_{γ_0} , but this is trivial since S_{γ_0} has only a single element. Let us pick 0 to be our choice on centre for $B_1^0 = B(0, 1) = \mathbb{Z}$. As we see from T_S , S_{γ_0} is partitioned into 3 closed balls of radius $\gamma_1 = \frac{1}{3}$, namely $3\mathbb{Z}$, $3\mathbb{Z} + 1$, and $3\mathbb{Z} + 2$. A choice of centres is given by 0, 1, and 2, so that A_0 becomes:

$$A_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

To start the ρ -ordering, concatenate the rows to obtain $\{0, 1, 2\}$, and to continue it, make a choice of centres for each of the closed balls of radius $\gamma_2 = \frac{1}{9}$ partitioning the sets $3\mathbb{Z} + i$, $i \in 0, 1, 2$. For example, $3\mathbb{Z} = 9\mathbb{Z} \cup 9\mathbb{Z} + 3 \cup 9\mathbb{Z} + 6$, so a choice of centres for B_1^1 is given by $\{0, 3, 6\}$. Making choices for the remaining elements, we obtain:

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

To continue the ρ -ordering we concatenate the rows, $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, which also gives the first row of A_2 . The remaining rows are found by partitioning each of the closed balls of radius $\frac{1}{9}$ and again making a choice of centres:

$$A_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{pmatrix}$$

And so on.

We are able to make two statements following this example. The first is that in starting the ρ -ordering, the fact that S_{γ_0} had only a single element allowed us to get started for free. In fact, all compact ultrametric spaces are bounded, so this is always the case.

The second takeaway is that we found the start of a ρ -ordering of $S = (\mathbb{Z}, |\cdot|_3)$ was given by taking the integers starting at 0 in their natural order. If we had continued building the ordering, we would have continued to find this. The fact that the natural ordering on the integers is a ρ_p -ordering, where ρ_p is the p -adic metric for any prime p , is well known [B1], but we give an alternate proof of it here:

Corollary 5. *Let S be the ultrametric space (\mathbb{Z}, ρ_p) , where ρ_p is p -adic metric for any prime p . The a ρ_p -ordering of S can be found by taking the integers, starting at 0, in their natural order.*

Proof. We prove the above by induction on k . First note that for any choice of prime, the elements of S_{γ_1} are the cosets of \mathbb{Z} modulo p , so that A_0 has p columns. Since $\{0, 1, 2, \dots, p-1\}$ are distributed among each of these cosets, without loss of generality the first row of A_0 is given by $[0, 1, 2, \dots, p-1]$ in order.

Now suppose that the first row of A_k is given by $[0, 1, 2, \dots, n]$ for $0 \leq k < k+1$. We show the first row of A_{k+1} , and therefore the first n' elements in a ρ_p -ordering of S , where n' is the column dimension of A_{k+1} , can be obtained as $[0, 1, 2, \dots, n, n+1, \dots, n']$. First note that each closed ball of radius $p^k = \gamma_k$ is in fact a coset of \mathbb{Z} modulo p^k , of which there are p . Then for any k , A_k is a matrix with p^k columns and p rows. In particular, $n = p^k - 1$. Let $i \in \{0, 1, \dots, p^k - 1\}$ be arbitrary. Then i is in exactly one of the cosets of \mathbb{Z} modulo p^k and since the first row of A_k is $[0, 1, 2, \dots, p^k - 1]$, it must have been chosen as our representative of this coset. If we split $p^k\mathbb{Z} + i$ into balls of radius p^{k+1} , we have

$$p^k\mathbb{Z} + i = \bigcup_{j=0}^{p-1} p^{k+1}\mathbb{Z} + (p^k j + i)$$

since there will be p elements in the partition, each of which will be equal to i modulo p^k and distinct modulo p^{k+1} . Then, there is a choice of centres such that the i^{th} column of A_k is

$$[i, p^k + i, 2p^k + i, \dots, (p-1)p^k + i]^T$$

filling this in for each i , we see that A_k can be obtained as:

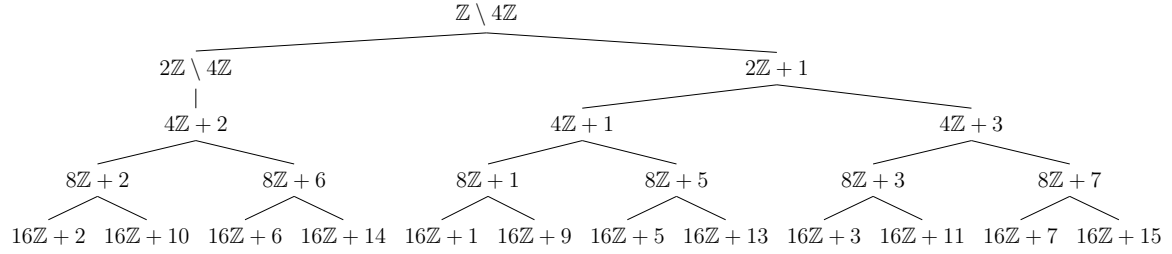
$$A_k = \begin{pmatrix} 0 & 1 & 2 & \dots & p^k - 1 \\ p^k & p^k + 1 & p^k + 2 & \dots & p^k + (p^k - 1) \\ 2p^k & 2p^k + 1 & 2p^k + 2 & \dots & 2p^k + (p^k - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)p^k & (p-1)p^k + 1 & (p-1)p^k + 2 & \dots & (p-1)p^k + (p^k - 1) \end{pmatrix}$$

Concatenating the rows, we see the first row of A_{k+1} will be

$$[0, 1, 2, \dots, p^k - 1, p^k, \dots, p^{k+1} - 1]$$

as required. \square

Example 8. Let us now see an example where there is an uneven number of children between the vertices on a given level. Suppose $S = \mathbb{Z}_{(2)} \setminus 4\mathbb{Z}$. In this case, we have that $\Gamma_S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ and T_s begins:



Choosing centres for the partition of \mathbb{Z} into closed balls of radius $\frac{1}{2}$, we have:

$$A_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have taken S to be the complement of $4\mathbb{Z}$ in $\mathbb{Z}_{(2)}$, so $B(0, \gamma_1)$ has only one child, since $2\mathbb{Z} \setminus 4\mathbb{Z} = 4\mathbb{Z} + 2$, while $B(1, \gamma_1)$ has two. Making a choice of centres, we have:

$$A_1 = \begin{pmatrix} 2 & 1 \\ * & 3 \end{pmatrix}$$

We concatenate the rows, skipping over $*$, and again make a choice of centres for the closed balls of radius $\frac{1}{8}$:

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \end{pmatrix}$$

One more iteration yields:

$$A_2 = \begin{pmatrix} 2 & 1 & 3 & 6 & 5 & 7 \\ 10 & 9 & 11 & 14 & 13 & 15 \end{pmatrix}$$

So that a ρ_2 -ordering of $S = \mathbb{Z}_{(2)} \setminus 4\mathbb{Z}$ starts: $\{2, 1, 3, 6, 5, 7, 10, 9, 11, 14, 13, 15, \dots\}$.

In the two propositions above, there was notational difficulty that arose when there was an unequal number of children between the vertices on a given level of T_s . This difficulty is, in fact, more than a notational inconvenience, and the situation simplifies considerably when it is not the case. We are far from the first to observe this. Amice noted this as far back as her 1964 paper [Am], and it has been observed more recently by Chabert and colleagues, for example in [FP] and [CEF]. The following section discusses this in more detail, first by supplying some preliminary lemmas and then showing how calculations are simplified in this setting.

Semi-regularity

In this section, we restrict to the case where in the tree T_s , for S some compact subset of an ultrametric space, every vertex on a given level has the same number of children. In this case, we can attach another sequence to S , which we call the α -sequence of S and which describes, for each level $k \in \mathbb{N}$, the size of the partitions on that level. We develop some preliminary lemmas, which we then use to derive formulae for this special case.

Definition 22. Let S be as above, a compact, discretely-valued subset of an ultrametric space (M, ρ) . We say that S is **semi-regular** if $T_{B_i^k} \cong T_{B_j^k}$, $\forall k \in \mathbb{N}$ and $i, j \in \beta(k)$, and where the isomorphism is understood as an isomorphism of trees. That is, S is semi-regular if each ball of radius γ_k breaks into the same number of balls of radius γ_{k+1} , for all k . If there exists an $n \in \mathbb{N}$ such that $T_{B_i^N} \cong T_{B_j^N}$ for all $N \geq n$, that is, each ball of radius γ_N breaks into the same number of balls of radius γ_{N+1} for $N \geq n$, then we say S is **eventually semi-regular**.

Definition 23. Suppose S is a compact, discretely-valued subset of an ultrametric space and S is semi-regular. The α -**sequence** of S is the sequence given by

$$\alpha(k) = \frac{\beta(k+1)}{\beta(k)}$$

which is in \mathbb{N} for each k . That is, if B_i^k is an element of S_{γ_k} , then $\alpha(k)$ is equal to the number of children of B_i^k in T_s . Since S is semi-regular, this number does not depend on i .

Example 9. If G is a compact ultrametric space and also a group, each ball centred at 0 is in fact a subgroup of G . Then each set of elements of S_{γ_k} is a collection of cosets of $G/B(0, \gamma_k)$. Since G is assumed to be compact, $G/B(0, \gamma_k)$ is finite and so Lagrange's theorem implies that G is semi-regular.

We now work towards a formula for the terms in the ρ -sequence of a semi-regular space S . We need a few lemmas to get started.

Lemma 5. *Let n and q be in \mathbb{N} . Then $\lfloor \frac{n}{q} \rfloor$ counts the numbers strictly less than n that are congruent to $n \pmod q$.*

Proof. By the division algorithm, we know there exists unique $c, r \in \mathbb{Z}$ such that

$$n = cq + r$$

with $0 \leq r < q$. Since c counts the number of q -multiples in the set $\{1, \dots, n\}$, and each q -multiple contains exactly one element that is congruent to $n \pmod q$, we need only show $\lfloor \frac{n}{q} \rfloor = c$. Simply note the above implies

$$\frac{n}{q} = c + \frac{r}{q}$$

and we must have $\frac{r}{q} < 1$. Then c is the largest integer such that $\frac{n}{q} \leq c$, but this is the definition of $\lfloor \frac{n}{q} \rfloor$. \square

Lemma 6.

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n + kb}{ab} \rfloor$$

for $n, a, b \in \mathbb{N}$. In particular,

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{2b} \rfloor = \lfloor \frac{n + b}{2b} \rfloor$$

for $n, b \in \mathbb{N}$.

Proof.

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor = \lfloor a \cdot \frac{n}{ab} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=0}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor - \lfloor \frac{n}{ab} \rfloor \quad (*)$$

$$= \sum_{k=1}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n + kb}{ab} \rfloor$$

where the final step in (*) is due to Hermite's identity: $\lfloor nx \rfloor = \sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor$, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. \square

Lemma 7. *If S is semi-regular and σ denotes the canonical ρ -ordering of S , that is, a ρ -ordering formed by pulling from left to right in T_s , then*

$$\rho(\sigma(n), \sigma(m)) = \gamma_k$$

if and only if

$$n = m \pmod{\beta(k)} \text{ and } n \neq m \pmod{\beta(k+1)}$$

Proof. Since S is semi-regular, every sequence of $\beta(k)$ terms in σ will be from each of distinct elements of S_{γ_k} (for any k). Moreover, since σ is a canonical ρ -ordering, we always pull from the elements of S_{γ_k} in the same order. Then $\sigma(n)$ and $\sigma(m)$ are descendents of some B_j^k if, and only if, $n = m \pmod{\beta(k)}$. Then the result follows since $\rho(\sigma(n), \sigma(m)) = \gamma_k$ if and only if B_i^k for some $i \in 1, \dots, \beta(k)$ is the join of $B_i^n \ni \sigma(n)$ and $B_i^m \ni \sigma(m)$. \square

We introduce another piece of notation before continuing.

Notation 2. Let S be a compact, discretely-valued subset of an ultrametric space, Γ_S the set of distances in S and $\delta(n)$ the characteristic sequence of S . Suppose γ_k is an element of Γ_S . Then we denote by $v_{\gamma_k}(\delta(n))$ the exponent of γ_k in the n^{th} -term of the characteristic sequence of S .

Proposition 26. *If S is a semi-regular ultrametric space, δ is the characteristic sequence of S , β is the structure sequence of S , and α is the sequence describing the semi-regularity, then*

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Proof. The exponent of γ_k in the n^{th} term of the characteristic sequence is the number of m strictly less than n such that $\rho(\delta(n), \delta(m)) = \gamma_k$. By the lemma above, this the

number of $m < n$ such that $m \equiv n \pmod{\beta(k)}$ and $m \not\equiv n \pmod{\beta(k+1)}$, which by the previous lemma is $\lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor$. Then we have:

$$\begin{aligned}
& v_{\gamma_k}(\delta(n)) \\
&= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor \\
&= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k)\alpha(k)} \rfloor, \text{ because } S \text{ is semi-regular} \\
&= \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor
\end{aligned}$$

□

Example 10. Consider the ultrametric space (\mathbb{Z}, ρ_p) for any prime p . Then $\beta(k) = p^k$ and $\alpha(k) = p$ for any $k \in \mathbb{N} \cup 0$. The above gives

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor$$

Now since $\gamma_k = p^{-k}$, $\forall k$, we are able to compute the exponent of $\frac{1}{p}$ in $\delta(n)$. We have

$$\begin{aligned}
& v_{\frac{1}{p}}(\sigma(n)) \\
&= \sum_{k=1}^{\infty} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\
&= \sum_{k=1}^{\lceil \log_p(n) \rceil} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\
&= \lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor + 2\lfloor \frac{n}{p^2} \rfloor - 2\lfloor \frac{n}{p^3} \rfloor + \dots + \lceil \log_p(n) \rceil \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\
&= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \dots + \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\
&= \sum_{k=1}^{\lceil \log_p(n) \rceil} \lfloor \frac{n}{p^k} \rfloor \\
&= \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor
\end{aligned}$$

We are able to simplify to a finite sum in the above because $\lfloor \frac{n}{p^k} \rfloor = 0$ if

$$p^k > n \iff \log(p^k) > \log(n) \iff k > \log_p(n)$$

We have already seen that the natural order on the integers gives a ρ_p -ordering for each p . So then

$$v_{\frac{1}{p}}(\delta(n)) = v_{\frac{1}{p}}\left(\prod_{i=0}^n \frac{1}{p}^{v_p(n-i)}\right) = \sum_{i=0}^{n-1} v_p(n-i) = v_p(n!)$$

so that we are able to recover the well-known Legendre's formula.

Definition 24. Let S be a semi-regular compact subset of an ultrametric space. If there exists a $q \in \mathbb{N}$ such that $\alpha(n) = q$, for all n , then S is said to be **regular**. If there exists a q and N in \mathbb{N} such that $\alpha(n) = q$, for all $n \geq N$, then S is said to be **eventually regular**.

Alt.:

Definition 25. Let S be a semi-regular compact subset of an ultrametric space. If there exists a $q_1, \dots, q_m \in \mathbb{N}$ such that $\alpha(n) = q_i$ if, and only if, $n = i \pmod m$, that is, α has an infinitely-repeating finite subsequence of length m , then we say S is **periodic**. If $m = 1$, then S is **regular**.

S is regular just in case the α -sequence of S is constant. In what follows, we assume that S is regular, but the results carry over to the case where S is periodic with the necessary adjustments.

If S is regular with $\alpha(k) = q$, for all k :

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

$$v_{q^{c_k}}(\sigma(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

$$v_q(\sigma(n)) = \sum_{k=0}^{\infty} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor)$$

$$v_q(\sigma(n)) = \sum_{k=0}^{\lceil \log_q(n) \rceil} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor)$$

$$\begin{aligned}
&= c_0 n - c_0 \lfloor \frac{n}{q} \rfloor + c_1 \lfloor \frac{n}{q} \rfloor - c_1 \lfloor \frac{n}{q^2} \rfloor + c_2 \lfloor \frac{n}{q^2} \rfloor - c_2 \lfloor \frac{n}{q^3} \rfloor \dots + \lceil \log_q(n) \rceil \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor - \lceil \log_q(n) \rceil \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor \\
v_q(\sigma(n)) &= \sum_{k=0}^{\lceil \log_q(n) \rceil} (c_{k+1} - c_k) \cdot (\lfloor \frac{n}{q^{k+1}} \rfloor)
\end{aligned}$$

Note that the coefficient $(c_{k+1} - c_k)$ is always positive: since Γ is (strictly) decreasing and $\gamma_k = q^{c_k}$, the sequence c_k is also be strictly decreasing (since $\gamma_k = q^{c_k}$ implies $\log_q(\gamma_k) = c_k$ and \log is a monotone function). The above implies that:

$$\omega(S) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\lceil \log_q(n) \rceil} (c_{k+1} - c_k) \cdot (\lfloor \frac{n}{q^{k+1}} \rfloor)$$

Describe the functions.

code/Padic_Product_Ordering.txt

```

1 with(LinearAlgebra):
2 with(combinat, cartprod):
3 with(padic):
4
5 ComputePadicProductOrdering := proc (m, components)
6 # Given a list of primes, p_1,...,p_n, compute the first m terms of a
   p-infinity ordering of  $Z_{p_1} \times \dots \times Z_{p_n}$ , where
    $p\text{-infinity}(x,y) = \max(p\text{-j}(x\text{-j}, y\text{-j}))$  and  $p\text{-j}$  is the  $p\text{-j}$ -adic metric.
7 # arg m; an integer indicating the number of elements of the ordering to
   return
8 # arg components; a list of prime numbers indicating the components of
   the product space
9 # return; a matrix where each row is an element in the product space and
   the i-th row is the i-th element in an ordering
10
11 local numberOfComponents, co_primes, i, n, T, M, v, distances, j,
   M1, newBlock;
12
13 #will end up with one column per component in the product space
14 numberOfComponents := nops(components);
15
16 #the ordering will start with the cartesian product of coprime elts
   from each component
17 #everything up to p-1 is coprime
18 co_primes := [[seq(i, i = 0 .. (components[1]-1))]];
19 for n from 2 to numberOfComponents do
20     co_primes := [op(co_primes), [seq(i, i = 0 ..
       (components[n]-1))]];
21 od;
22
23 #then take the cartesian product to get the first <product of
   elements in components> elements in the ordering
24 T := cartprod(co_primes);

```

```

25 M := Matrix([T['nextvalue']()]);
26 while not T['finished'] do
27     M := <M; T['nextvalue'](>);
28 end do;
29
30 #make a list to keep track of the exponent of each prime; start by
    take each prime to the power -1
31 v := Vector[row](1 .. numberOfComponents, 1);
32 v := convert(v, list);
33
34 #keep adding rows until you have enough points in the ordering
35 while RowDimension(M) < m do
36     #take each prime to the power of minus the elements in v
    distances := zip(proc (x, y) options operator, arrow; x^(-y) end
        proc, components, v);
38     #check each column to see if the max distance was achieved
39     for j from 1 to numberOfComponents do
40         #if it was then split this column
41         if distances[j] = max(distances) then
42             #take a snapshot of M before you start - this is what you
                have to add to
43             M1 := copy(M, deep);
44             #create p-1 new blocks
45             for i from 1 to (components[j]-1) do
46                 newBlock := copy(M1, deep);
47                 newBlock(1..RowDimension(newBlock), j) :=
                    Column(newBlock, [j]) +~ (i*components[j]^v[j]);
48                 #add the new block to the master matrix
49                 M := Matrix([M], [newBlock]);
50             od;
51             #update the vector of exponents
52             v[j] := v[j] + 1;
53         end if;
54     od;

```

```
55     end do;  
56     return M[1..m,];  
57 end proc;
```

code/Partial_Rho_Sequence.txt

```

1 with(LinearAlgebra):
2 with(combinat, cartprod):
3 with(padic):
4
5 ComputePartialRhoSeq := proc (S, rho)
6 # Given an m by n matrix S whose columns represent points of an
   n-component product space and that has as its i-th row the i-th term
   in a rho-ordering of that space, compute the (m-1)-th partial sum of
   the rho-sequence
7 # note that S and rho must be compatible and no checking is done to
   ensure this
8 # arg S; an n by m matrix representing a rho-ordering, for example as
   created by ComputePadicProductOrdering
9 # arg rho; a compatible metric on the points (rows) in S
10 # return; a real number, corresponding to the (m-1)-th term of the
    partial rho-sequence
11
12 local lastTerm, f, distances, nthTerm;
13
14 #find the last element in the ordering
15 lastTerm := S[RowDimension(S),];
16
17 #make a function that calculates the distance from the i-th row of S
    to the last term in the ordering
18 f := proc (i) options operator, arrow; rho(op(convert(lastTerm,
    list)), op(convert(S[i,], list))) end proc;
19
20 #run over each row to get the set of all m-1 distances
21 distances := map(f, [seq(i, i = 1 .. (RowDimension(S)-1))]);
22
23 #multiply them to get the (m-1)-th term of the rho-ordering
24 partialSum := mul(distances);
25

```

```
26     return partialSum
27 end proc;
```


code/Fast_Partial_Rho_Sequence.txt

```

1 FastPartialRhoSeq := proc (A, p:=[2,3])
2   local g, h, computePowers, n, shortA, primeExponents, i, thisPrime,
      thisPrimeIndex, B, G, powers, powersOfG, thisPrimeSum;
3
4   == Some helper functions ==#
5   #Return the index of every instance of p-multiples in a list
6   #Use to find the index of a given prime in A
7   h := proc(i, L, p) if L[i] mod p = 0 then return i else return NULL fi;
      end proc;
8
9   #Count the number of times the mth element has appeared as a factor
      for the 1..m first elements in a list
10  #Use to compute the (decreasing) sequence of distances in A or a
      subset of A
11  g := proc(m, L)
12    local basePrime;
13
14    basePrime := L[m];
15
16    #since we just want the exponent not actual distance just compute
      what power this would be
17    return ordp(mul(L[1..m]), basePrime);
18
19  end proc;
20
21  #Compute the appropriate power of an element of G
22  computePowers := proc(m, L)
23    local power;
24
25    if m=nops(L) then
26      power:= L[-1]-1;
27    else
28      power:= mul(L[(m+1)..nops(L)]) * (L[m]-1);

```

```

29     end if;
30     return power
31 end proc;
32
33 #compute n, then create a copy of A with the first element deleted to
    ease the indexing
34 n := mul(A);
35 shortA := A[2..nops(A)];
36
37 #compute the terms corresponding to each prime given
38 primeExponents := Vector();
39 for i from 1 to nops(p) do
40     #pull out the prime
41     thisPrime := p[i];
42
43     #first get the index in A of this prime
44     thisPrimeIndex := map(h, [seq(i, i=1..nops(shortA))], shortA,
        thisPrime);
45     B:= shortA[thisPrimeIndex];
46
47     #then find the (exponents for the) distances occuring with this prime
48     G:= map(g, [seq(i, i=1..nops(B))], B);
49
50     #Figure out what power each distance should be raised to
51     powers := map(computePowers, thisPrimeIndex, shortA);
52
53     #raise each element in G to the given powers
54     powersOfG:= zip(proc (x, y) options operator, arrow; x*(y) end proc,
        G, powers);
55
56     #the exponent of this prime in the nth partial will be -([the sum of
        the elements in powers]/n), where n is the product of elements in
        A (including the first element)
57     thisPrimeSum := add(powersOfG);

```

```
58     primeExponents(i) := thisPrimeSum/n;  
59  
60   end do;  
61  
62   return primeExponents;  
63  
64 end proc;
```

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