Valuative capacity of compact subsets of ultrametric spaces

Anne Johnson

August 9, 2019

Background

Definition

([F]) Let $K \subseteq \mathbb{C}$ be a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in K^n$, consider

$$\delta_n(z) = \prod_{j<i} |z_i - z_j|^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in K^n$ is called a **Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in K.

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Let K be a compact subset of a metric space, (M, ρ). Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in K^n$, consider

$$\delta_n(z) = \prod_{j<i} \rho(z_i, z_j)^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in K^n$ is called a **generalized Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in K.

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([B2]) Let S be a subset of $\mathbb{Z}$ and let p be any prime. A p-**ordering** of S is a sequence, $\{a_i\}_{i\geq 0}$ in S, such that a_0 is arbitrary and for $i > 0$, a_i minimizes

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([F])Let $K \subseteq \mathbb{C}$ be a compact subset. The **transfinite diameter** of K is

$$\lim_{n\to\infty}[\max_{z}\ \delta_n(z)]$$

where the maximum is taken over all $n-$tuples in K.

Background

Proposition

([Ch], theorem 4.2) Let E be a subset of V, a rank-one valuation domain with valuation v. If $\{a_i\}_{i\geq 0}$ is $v-$ordering of E, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0,\ldots,x_n\in E} v(\prod_{0\leq j<i\leq n}(x_i - x_j))$$

Valuative capacity: set-up from [J1]

Definition

Let S be a compact subset of (M, ρ). A ρ-**ordering** of S is a sequence, $\{a_i\}_{i \geq 0}$ in S, such that a_0 is arbitrary and for $i > 0$, a_i maximizes

$$\prod_{j<i} \rho(z, a_j))$$

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over $z \in S$.

- If ρ is an ultrametric, the terms $\prod_{i=0}^{n} \rho(a_n - a_j)$ do not depend on the choice of $\rho-$ordering. We call this the $\rho-$**sequence** of S.

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- $\rho-$orderings give a recursive construction for generalized Fekete $n-$tuples.
- The limit

$$\omega(S) := \lim_{n\to\infty}[\prod_{i=0}^{n} \rho(a_n - a_j)]^{\frac{1}{n}}$$

is called the **valuative capacity** of S.

Valuative capacity: facts

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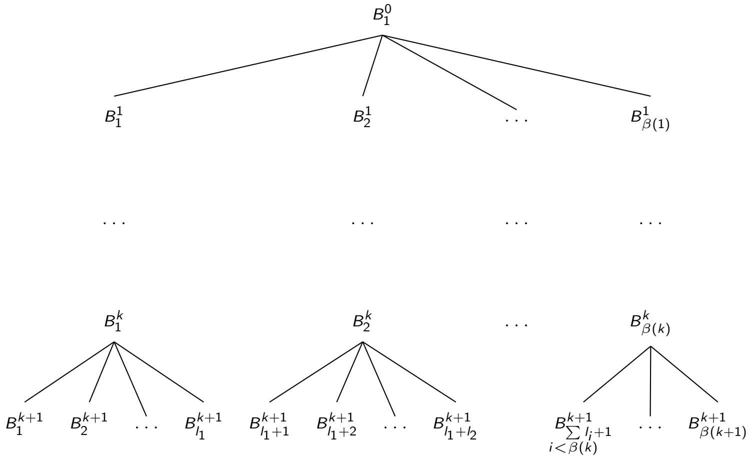
Valuative capacity has the following properties:

- *translation-invariance*, i.e., $\omega(a+S)=\omega(S)$ (under a translation-invariant operation)
- *scaling*, i.e., $\omega(bS)=|b|\omega(S)$ (under a multiplicative norm)
- *decomposition*, i.e.,

$$\frac{1}{log(\frac{\omega(S)}{d})}=\sum_{i=1}^{n}\frac{1}{log(\frac{\omega(A_i)}{d})}$$

for $d=diam(S)$ and $\rho(A_i,A_j)=d,\forall i,j$

Recursive ρ−orderings



Recursive ρ-orderings

- $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$ is found by selecting elements of each B_i^k in order as much as possible, and skipping to B_{i+1}^k, when it is not possible.

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- $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$ is found by selecting elements of each B_i^k in order as much as possible, and skipping to B_{i+1}^k, when it is not possible.
- to build a $\rho-$ordering of S from the above, it suffices only to make a choice of centres for each of B_i^k's.

Semi-regularity

Proposition

If S is a semi-regular ultrametric space, δ is the $\rho-$sequence of S, β is the structure sequence of S, and α is the sequence describing the semi-regularity, then

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Regularity

Proposition

Let S be a regular, tame subset of a compact ultrametric space with $\gamma_k = q^{c_k}$ for some $c_k \in \mathbb{Q}$ and for all $k \in \mathbb{N} \cup 0$. Then

$$v_q(\delta(n)) = c_0 n + \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

and

$$log_q(\omega(S)) = \lim_{n \to \infty} c_0 + \frac{1}{n} \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

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- Taken together, they extend the toolkit for computing capacity without appealing to algebraic structure.

Product Space

Proposition

Let (M_i, ρ_i), for i in some finite index set I, be a collection of metric spaces and suppose ρ_i is an ultrametric for each i. Then (M, ρ_∞) is an ultrametric space, where $M = M_1 \times M_2 \times \ldots \times M_n$ and ρ_∞ is the metric given by,

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n))$$

$$= \max_i \{\rho(x_i, y_i)\}$$

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- Let $M = (\mathbb{Z}^n, \rho_{p,\infty})$ be the ultrametric space with points equal to the n-fold product of $(\mathbb{Z}, \rho_p)$ (for $n < \infty$) for some fixed prime p. The valuative capacity of M is $(\frac{1}{p})^{\frac{1}{p^n - 1}}$.

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- What about primes $p \neq q$?

Product Space: primes $p \neq q$

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- We use the fact that they are semi-regular and study the exponent of each prime in turn.

Product Space: $(\mathbb{Z}, \rho_2) \times (\mathbb{Z}, \rho_3)$

Lemma

Let $S = (\mathbb{Z}, \rho_2) \times (\mathbb{Z}, \rho_3)$ *and consider the* k^{th} *element of the* β *sequence of* S, $\beta(k) = 2^i \cdot 3^j$. *If* k *is such that* $\gamma_k = 2^{-i}$ *for some* i, *then* j *counts the numbers* $a \in \mathbb{Z}_{\geq 0}$ *such that* $3^a < 2^i$.

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Proof.

- Γ_S is strictly monotone decreasing and each γ_k is equal to a non-positive power of 2 or 3.

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- Γ_S is strictly monotone decreasing and each γ_k is equal to a non-positive power of 2 or 3.
- If $\gamma_k = 2^i$, then all non-positive powers of 3 and 2 which are greater than 2^i must be equal to some γ_j, $0 \leq j < k$.

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- Γ_S is strictly monotone decreasing and each γ_k is equal to a non-positive power of 2 or 3.
- If $\gamma_k = 2^i$, then all non-positive powers of 3 and 2 which are greater than 2^i must be equal to some γ_j, $0 \leq j < k$.
- Since we are only considering the case γ_k is a power of 2, this includes all of the smaller powers of 3.

$\square$

Product Space: $(\mathbb{Z}, \rho_2) \times (\mathbb{Z}, \rho_3)$

$$v_{\gamma_{\frac{1}{2}}}(\delta(n)) = \sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor$$

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$$v_{\gamma_{\frac{1}{3}}}(\delta(n)) = \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n + 2^{\lceil \frac{i}{log_3(2)} \rceil} \cdot 3^i}{2^{\lceil \frac{i}{log_3(2)} \rceil} \cdot 3^{i+1}} \rfloor + \lfloor \frac{n + 2^{\lceil \frac{i}{log_3(2)} \rceil + 1} \cdot 3^i}{2^{\lceil \frac{i}{log_3(2)} \rceil} \cdot 3^{i+1}} \rfloor)$$

Product Space: primes $p \neq q$

Conjecture

Finite products of $(\mathbb{Z}, \rho_{p_i})$ for distinct primes, p_i, have transcendental valuative capacity.

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