Example from previous chapter with more detail:

Example 1. Consider the ultrametric space $(\mathbb{Z}, |\cdot|_p)$ for any prime p. Then $\beta(k) = p^k$ and $\alpha(k) = p$ for any k. The above gives

$$v_{\gamma_k}(\sigma(n)) = \lfloor\frac{n}{p^k}\rfloor - \lfloor\frac{n}{p^{k+1}}\rfloor$$

and since $\gamma_k = p^{-k}$, $\forall k$, we have

$$\begin{aligned} & v_{\frac{1}{p}}(\sigma(n)) \\ &= \sum_{k=1}^{\infty} k\cdot(\lfloor\frac{n}{p^k}\rfloor - \lfloor\frac{n}{p^{k+1}}\rfloor) \\ &= \sum_{k=1}^{\lceil log_p(n)\rceil} k\cdot(\lfloor\frac{n}{p^k}\rfloor - \lfloor\frac{n}{p^{k+1}}\rfloor) \\ &= \lfloor\frac{n}{p}\rfloor - \lfloor\frac{n}{p^2}\rfloor + 2\lfloor\frac{n}{p^2}\rfloor - 2\lfloor\frac{n}{p^3}\rfloor + \ldots + \lceil log_p(n)\rceil\lfloor\frac{n}{p^{\lceil log_p(n)\rceil}}\rfloor \\ &= \lfloor\frac{n}{p}\rfloor + \lfloor\frac{n}{p^2}\rfloor + \ldots + \lfloor\frac{n}{p^{\lceil log_p(n)\rceil}}\rfloor \\ &= \sum_{k=1}^{\lceil log_p(n)\rceil}\lfloor\frac{n}{p^k}\rfloor \\ &= \sum_{k=1}^{\infty}\lfloor\frac{n}{p^k}\rfloor \\ &= v_p(n!) \end{aligned}$$

since $\lfloor\frac{n}{p^k}\rfloor = 0$ if $p^k > n \iff log(p^k) > log(n) \iff k > log_p(n)$

The 2-3 case:

What about $(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})$ for distinct primes? These spaces do not admit a scaling property, so the same toolset is not available. They are however semi-regular, so we know that

$$v_{\gamma_k}(\sigma(n)) = \lfloor\frac{n}{\beta(k)}\rfloor - \lfloor\frac{n}{\beta(k+1)}\rfloor = \sum_{j=1}^{\alpha(k)-1}\lfloor\frac{n + j\cdot\beta(k)}{\alpha(k)\beta(k)}\rfloor$$

Suppose $p_1 = 2$ and $p_2 = 3$, so that the α sequence of $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$ is $\alpha = \{6, 2, 3, 2, 2, 3, 2, 3, 2, \ldots\}$ and the β sequence is then $\beta = \{6, 12, 36, 72, 144, \ldots\}$. We know that the capacity of S will be a product of some negative power of 2 and a negative power of 3. From the above, we know that when $\alpha(k) = 2$, we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{2 \cdot \beta(k)} \rfloor$$

and when $\alpha(k) = 3$, we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{3 \cdot \beta(k)} \rfloor + \lfloor \frac{n + 2 \cdot \beta(k)}{3 \cdot \beta(k)} \rfloor$$

We also know that if $\alpha(k) = 2$, then γ_k must be a (negative) power of 2, and likewise if $\alpha(k) = 3$, then γ_k is a power of 3.

Let us first explore the exponent of 2 in $\sigma(n)$. We start by noting that if γ_k is some 2^{-i}, then

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

since there will be a copy of 2 in $\beta(k)$ for every occurence of 2 in $\alpha(0), \ldots, \alpha(k)$, which is also what i counts. So then, the exponent of $\frac{1}{2}$ in the n^{th} characteristic sequence of S is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

What can we say about j, the exponent of 3?

Lemma. Let $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$ and consider the k^{th} element of the β sequence of S, $\beta(k) = 2^i \cdot 3^j$. If k is such that $\gamma_k = 2^{-i}$ for some i, then j counts the numbers a in $\mathbb{Z}_{\geq 0}$ such that $3^a < 2^i$.

Proof. (sketch) 2^i only makes it into the sequence after all smaller powers of 3 and 2 have been used, and since we are only considering the case γ_k is a power of 2, we get all the smaller powers of 3. $\square$

Now note that

$$3^a < 2^i \iff log_2(3^a) < log_2(2^i) \iff a \cdot log_2(3) < i \tag{1}$$

So now we are reduced to counting the number of non-negative integers a that satisfy the above for a given i. Note that the number of such a's will simply be the the value of the largest a plus 1 since a satisfying the relation implies all $0 \leq a' \leq a$ solve the relation. Then, we are in fact reduced to finding the largest $a \in \mathbb{Z}$ that satisfies $a < \frac{i}{log_2(3)}$, but this is exactly $\lfloor \frac{i}{log_2(3)} \rfloor$. This in turn gives $j = \lfloor \frac{i}{log_2(3)} \rfloor + 1 = \lceil \frac{i}{log_2(3)} \rceil$, since $\frac{i}{log_2(3)}$ is never an integer. We now revisit our expression for the exponent of $\frac{1}{2}$ and substitute our new found value for j:

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor$$

$$= \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor - \lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor)$$

Bibliography

[1] Alain M. Robert, A course in p-adic analysis.

[2] Manjul Bhargava, The factorial function and generlizations

[3] P-orderings and polynomial functions on arbitrary subsets of Dedekind rings.

[4] On P-orderings, rings of integer-valued polynomials, and ultrametric analysis

[5] Keith Johnson, P-orderings and Fekete sets

[6] Nate Ackerman, Completeness in Generalized Ultrametric Spaces

[7] Lothar Gerritzen and Marius van der Put, Schottky Groups and Mumford Curves.

[8] Youssef Fares and Samuel Petite. The valuative capcity of subshifts of finite type.

[9] Sabine Evrard and Youssef Fares. p-adic subsets whose factorials satisfy a genearlized Legendre formula.

[10] Jean-Luc Chabert, Sabine Evrard and Youssef Fares. Regular subsets of valued fields and Bhargava's v-orderings