

# Chapter 1

## Introduction

In the course of developing a generalized factorial function, Manjul Bhargava introduced the notion of  $p$ -orderings of a Dedekind domain [?, ?], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [?, ?] and have also provided a natural framework to extend many classical results in analysis to a  $p$ -adic setting, such as polynomial approximation and mapping theorems [?, ?, ?]. In this thesis, we examine how a tool based on  $p$ -orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-archimedean settings.

The historical background to this work comes in two parts. On the one hand, there is the background on logarithmic capacity from potential theory, and secondly, there is the background from Bhargava's  $p$ -orderings. We give a brief summary of each here. A similar treatment, with slightly different perspective, is found in [?]. Jean-Luc Chabert was the first to draw a connection between the two, and many of the known results in this area stem from his work or that of his colleagues. Building on the result in [?], we extend the work by Chabert and colleagues by studying valutive capacity in a more general setting, namely that of an ultrametric space, which may or may not also be a local field. In doing so, we show many properties of capacities are in fact independent of the algebraic structure of a space, although such structure, when it exists, can act as a useful probe.

### 1.1 Logarithmic capacity

The theory of capacity has been developed as a topic in potential theory in a variety of settings. Classically, the notion of capacity was developed over both  $\mathbb{C}$  and  $\mathbb{R}^n$ ,

although the theory has been further developed in a rather general way by Rumely for Berkovich spaces. A significant body of work on the analytic properties of capacity can be found for a number of different contexts. For example, such a treatment of the subject over  $\mathbb{C}$  can be found in [?] and [?], over Berkovich spaces in [?], and over  $\mathbb{Q}_p$  in [?]. We give a brief account of capacity over  $\mathbb{C}$  here, presenting only the most essential definitions and results. One advantage of tracing the historical roots of capacity back to  $\mathbb{C}$  is that the theory in this setting also comes equipped with a physical interpretation. As we are about to see, capacity, in the classical sense, gives a mathematical model for the amount of electrostatic charge a conductor can hold. The exposition below is closely based on Ransford [?].

Even restricting ourself to the definition of capacity of subsets of  $\mathbb{C}$ , we find two paths, one which will give us some physical interpretation and one which will lead more naturally to  $p$ -orderings. We start with the former.

**Definition 1.** [?] Let  $\mu$  be a finite Borel measure on  $\mathbb{C}$  and suppose  $\mu$  has compact support. We associate to  $\mu$  a function

$$p_\mu(x) = \int \log|x - y|d\mu(y)$$

called the **potential function** of  $\mu$ . The **energy** of  $\mu$  is

$$I(\mu) = \int \int \log|x - y|d\mu(y)d\mu(x)$$

**comment from Keith:** is the integral over  $\mathbb{C}$  (resp  $\mathbb{C}^2$ ) or  $\text{supp}(\mu)$ ?

This gives at once the physical interpretation promised above. We interpret the potential function of a measure as giving the potential energy of a point. Viewing the measure as a charge distribution, the double integral gives back the total energy in the system. Now we come upon a physical reality: charged particles in a conductor will naturally distribute themselves as to minimize the energy. This leads to the definition below:

**Definition 2.** [?] Let  $K$  be a compact subset of  $\mathbb{C}$  and let  $\mathcal{P}(K)$  be the set of Borel

probability measures on  $K$ . If  $\nu \in \mathcal{P}(K)$  is such that

$$I(\nu) = \sup_{\mu \in \mathcal{P}(K)} I(\mu)$$

then  $\nu$  is a **equilibrium measure** for  $K$ .

**comment from Keith:** explain why its a sup when you're minimizing the energy

We state the following proposition without proof. A sketch of the proof can be found in [?] and the full details can be found in [?].

**Proposition 1.** [?] *The equilibrium measure exists for every compact set  $K \in \mathbb{C}$ . When finite, the equilibrium measure is unique and isometry-invariant.*

We are now ready to give our first definition of capacity.

**Definition 3.** [?] Let  $K$  be a compact subset of  $\mathbb{C}$ . The logarithmic capacity of  $E$  is

$$C(K) = e^{I(\nu)}$$

where  $\nu$  is the equilibrium measure on  $K$ .

We present below a few results on capacity in  $\mathbb{C}$ , some of which will reappear in the remainder of this work, although the context, and the proofs (omitted here), bear little resemblance to the present case.

**Proposition 2 (Randsford, 5.1.2).** *Let  $K, K_1, K_2$  be compact subsets of  $\mathbb{C}$ .*

1.  $K_1 \subseteq K_2$ , then  $C(K_1) \leq C(K_2)$ .
2.  $C(\alpha K + \beta) = |\alpha|C(K)$  for all  $\alpha, \beta \in \mathbb{C}$ .
3.  $C(K) = C(\delta_e K)$ , where  $\delta_e$  is the exterior boundary of  $K$ .

**Proposition 3 (Randsford, 5.1.4).** *Suppose  $\{B_n\}$  is a sequence of Borel subsets of  $\mathbb{C}$ . Let  $B = \cup_n B_n$  and  $d \geq 0$ .*

1. If  $\text{diam}(B) \leq d$ , then  $C(B) \leq d$  and

$$\frac{1}{\log(\frac{d}{C(B)})} \leq \sum_n \frac{1}{\log(\frac{d}{C(B_n)})}$$

2. If  $\text{dist}(B_j, B_k) \geq d$  whenever  $j \neq k$ , then

$$\frac{1}{\log^+(\frac{d}{C(B)})} \geq \sum_n \frac{1}{\log^+(\frac{d}{C(B_n)})}$$

We now know show an equivalent way of defining of capacity, still over  $\mathbb{C}$ , which starts with the following two definitions due to Fekete [?].

**Definition 4.** Let  $K \subseteq \mathbb{C}$  be a compact subset. Fix  $n \in \mathbb{N}$ , and for  $z = (z_1, \dots, z_n) \in K^n$ , consider

$$\delta_n(z) = \prod_{j < i} |z_i - z_j|^{\frac{2}{(n(n-1))}}$$

An element  $z = (z_1, \dots, z_n) \in K^n$  is called a **Fekete n-tuple** if  $z$  maximizes  $\delta_n$  over all  $n$ -tuples in  $K$ .

Note that since  $K$  is compact by assumption, a Fekete  $n$ -tuple exists for each  $n$ .

**Definition 5.** Let  $K \subseteq \mathbb{C}$  be a compact subset. The **transfinite diameter** of  $K$  is

$$\lim_{n \rightarrow \infty} [\max_z \delta_n(z)]$$

where the maximum is taken over all  $n$ -tuples in  $K$ . That is, the transfinite diameter of  $K$  is  $\lim_{n \rightarrow \infty} \delta_n(z)$ , where  $z$  is a Fekete  $n$ -tuple for each  $n$ .

The following proposition shows the relation to capacity.

**Proposition 4. [Fekete-Szegő Theorem][?]** If  $K$  is a compact subset of  $\mathbb{C}$ , then the transfinite diameter of  $K$  is equal to the logarithmic capacity of  $K$ .

We end this section with an observation about the points  $z_i$  in  $\mathbb{C}$  making up a Fekete  $n$ -tuple. For  $n \geq 2$ , if  $(z_1, \dots, z_{n+1})$  is a Fekete  $(n+1)$ -tuple, then  $(z_1, \dots, z_n)$

is not in general a Fekete  $n$ -tuple. In physical terms, we note that the placement of a new charge in a conductor will almost always change the location of the existing charges in that conductor. Remarkably, this is not the case in ultrametric spaces. Indeed, we are able to build the analogous structure, which we call a  $p$ -ordering or more generally a  $\rho$ -ordering, *recursively*, that is by reusing the points from the previous iteration.

## 1.2 P-orderings

The development of  $p$ -ordering was motivated by the observation that the factorial function had important number-theoretic applications, yet was only defined for the set  $\mathbb{Z}$ . In order to generalize the factorial, Bhargava defined it via an invariant, called the  $p$ -sequence, which could be attached to any subset of a Dedekind domain <sup>1</sup> [?].

We cannot go much further without introducing the following definition.

**Definition 6.** Let  $z \in \mathbb{Z}$  and let  $p$  be any prime. The  $p$ -**adic valuation** of  $z$ , denoted  $v_p(z)$ , is the largest  $n \in \mathbb{N}$  such that  $p^n$  divides  $z \neq 0$  and  $v_p(z) = \infty$  if  $z = 0$ . That is,

$$v_p(z) = \begin{cases} \max\{n \in \mathbb{N}; p^n \mid z\}, & \text{if } z \neq 0 \\ \infty, & \text{otherwise} \end{cases}$$

For  $z_1, z_2 \in \mathbb{Z}$ , we define the  $p$ -**adic metric** by

$$\rho_p(z_1, z_2) = p^{-v_p(z_1 - z_2)}$$

where  $p^\infty$  is taken to be 0.

That the  $p$ -adic metric defined above is truly a metric is easy to see. We will see in the next chapter that it is in fact not just a metric, but also an ultrametric, since it satisfies a strengthened version of the triangle identity. The strong triangle identity is not the only nice property enjoyed by the  $p$ -adic valuation though. Like

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<sup>1</sup>In fact, Bhargava associated  $p$ -sequences to the more general class of Dedekind rings, which are locally principal, Noetherian rings in which all nonzero primes are maximal.

the logarithm, the  $p$ -adic valuation also satisfies:  $v_p(x \cdot y) = v_p(x) + v_p(y)$  for any prime  $p$  and  $x, y$  in  $\mathbb{Z}$ . We are now ready to define  $p$ -orderings, and not long after, to give the connection to Fekete  $n$ -tuples.

**Definition 7.** [?] Let  $S$  be a subset of  $\mathbb{Z}$  and let  $p$  be any prime.<sup>2</sup> A  $p$ -ordering of  $S$  is a sequence,  $\{a_i\}_{i \geq 0}$  in  $S$ , such that  $a_0$  is arbitrary and for  $i > 0$ ,  $a_i$  minimizes

$$v_p\left(\prod_{j < i} (a_i - a_j)\right)$$

over  $z \in S$ .

A  $p$ -ordering in  $S$ , like a Fekete  $n$ -tuple in  $\mathbb{C}$ , is not unique. Indeed, in most of the examples we will explore, there will be infinitely-many choices at each stage of the construction. Nonetheless,  $p$ -orderings give rise to  $p$ -sequences, which are invariants of  $S$ :

**Definition 8.** [?] Let  $S$  be a subset of  $\mathbb{Z}$  and let  $p$  be any prime. Suppose  $\{a_i\}_{i \geq 0}$  is a  $p$ -ordering of  $S$ . The  **$p$ -sequence**, occasionally the **characteristic sequence**, of  $S$  is the sequence defined by  $a_0 = 1$  and for  $i > 0$ ,

$$\delta(i) = \prod_{j=0}^{i-1} v_p(a_i, a_j)$$

It is a fact, not entirely obvious, that the  $p$ -sequence of  $S$  is independent of the  $p$ -ordering used in its construction [?]. To define the generalized factorial, Bhargava considered the product of  $p$ -sequences taken over each prime  $p$  for arbitrary subsets of  $\mathbb{Z}$ . We will go in another direction.

Suppose we were to generalize our definition of Fekete  $n$ -tuple in the obvious way, as below.

**Definition 9.** [?] Let  $(M, \rho)$  be a metric space and suppose  $S \subseteq M$  is a compact

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<sup>2</sup>To apply the definition to a general Dedekind domain, we replace the usual primes with the set of prime ideals in the ring of interest.

subset. Fix  $n \in \mathbb{N}$ , and for  $z = (z_1, \dots, z_n) \in S^n$ , consider

$$\delta_n(z) = \prod_{j < i} \rho(z_i - z_j)^{\frac{2}{(n(n-1))}}$$

An element  $z = (z_1, \dots, z_n) \in S^n$  is called a **generalized Fekete n-tuple** if  $z$  maximizes  $\delta_n$  over all  $n$ -tuples in  $S$ .

What then is the connection to  $p$ -orderings and  $p$ -sequences? Jean-Luc Chabert drew the first connection between these objects when he studied the limit of these sequences not just for the case  $M = \mathbb{Z}$  and  $\rho = \rho_p$ , but in the case that  $M$  is any rank-one valuation domain [?]. We repeat his theorem 4.2 from [?] below,

**Proposition 5.** *Let  $E$  be a subset of  $V$ , a rank-one valuation domain with valuation  $v$ . If  $\{a_i\}_{i \geq 0}$  is  $v$ -ordering<sup>3</sup> of  $E$ , then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0, \dots, x_n \in E} v\left(\prod_{0 \leq j < i \leq n} (x_i - x_j)\right)$$

Chabert called this limit the valuative capacity of  $E$ , and we shall do the same. Of course, if  $v$  is the  $p$ -adic valuation, then minimizing  $v_p(x, a)$  will maximize  $\rho_p(x, a)$ .

- show  $p$ -orderings are generalized FNTs and satisfy the recursive property as promised
- show (state) this is actually true more generally as lead-in to next section
- create better transition or reorganization/flow with above theorem

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<sup>3</sup>A  $v$ -ordering of  $E$  is exactly as expected: a sequence of distinct element  $\{a_i\}_{i \geq 0}$  in  $E$  is  $v$ -ordering of  $E$  if for  $n > 0$ ,

$$v\left(\prod_{k=0}^{n-1} (a_n - a_k)\right) \leq v\left(\prod_{k=0}^{n-1} (x - a_k)\right)$$

for each  $x \in E$ .

## Chapter 2

### Background

The principal context for this thesis is an arbitrary ultrametric space, which is a metric space that also satisfies an additional axiom, sometimes called the ultrametric inequality or (in the case of vector spaces) the strong triangle property. We define ultrametric spaces below and for the rest of this section, we review some of their more important characteristics. The proofs offered in this section are, for the most part, standard and can be found in a number of reference texts, such as [?].

**Definition 10.** Let  $(M, \rho)$  be a metric space; that is, suppose  $M$  is a set and  $\rho : M \times M \rightarrow \mathbb{R}_{\geq 0}$  is such that:

- (i)  $\rho(x, y) = 0$  if and only if  $x = y$
- (ii)  $\rho(x, y) = \rho(y, x)$
- (iii)  $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any  $x, y, z \in M$ . If  $\rho$  satisfies the ultrametric inequality,

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z))$$

for any  $x, y, z \in M$ , then  $(M, \rho)$  is an **ultrametric space**.

A special case of an ultrametric space, and one where much of the previous work on this topic has been completed, is one where the metric has been derived from a norm on a vector space.

**Definition 11.** Let  $(V, N)$  be a normed vector space; that is, suppose  $V$  is  $\mathbb{F}$ -vector space, for  $\mathbb{F}$  some subfield of  $\mathbb{C}$ , and  $N : V \rightarrow \mathbb{R}_{\geq 0}$  is such that:



$$(i) \quad N(x + y) \leq N(x) + N(y)$$

$$(ii) \quad N(cx) = |c| N(x)$$

$$(iii) \quad N(x) = 0 \text{ implies } x = 0$$

for any  $x, y \in V$  and  $c \in \mathbb{F}$ . We say that  $N$  satisfies the **strong triangle inequality** if

$$N(x + y) \leq \max(N(x), N(y))$$

for any  $x, y \in V$ .

**Proposition 6.** *Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle inequality. Then the metric space,  $(V, \rho_N)$ , where  $\rho_N$  is the metric induced by  $N$ , that is,  $\rho_N(x, y) = N(x - y)$ , is an ultrametric space.*

*Proof.* We take for granted that  $(V, \rho_N)$  is a metric space and also note that

$$N(x + z) \leq \max(N(x), N(z))$$

implies

$$\rho_N(x, z) \leq \max(\rho_N(x, 0), \rho_N(z, 0)) \leq \max(\rho_N(x, y), \rho_N(y, z))$$

□

**Notation.** If  $(V, N)$  is a normed vector space, then the metric induced by  $N$  is denoted  $\rho_N$ .

When ultrametric spaces come from spaces with algebraic structure, such as normed vector spaces, some of this structure carries over into metric spaces structure in a rather nice way:

**Proposition 7.** *[?] Let  $S$  be a group equipped with a (right) invariant ultrametric,  $\rho$ . If  $B = B(0, r)$  is a (closed) ball centred at the neutral element of  $S$ , then  $B$  is a subgroup of  $S$ .*

*Proof.* Let  $x, y \in B$ . Then

$$\rho(x - y, 0) = \rho(x, y) \leq \max(\rho(x, 0), \rho(y, 0)) \leq r,$$

so that  $x - y \in B$ . □

The canonical example of an ultrametric space, and one which is derived from a norm, is  $p$ -adic integers,  $\mathbb{Z}_p$  for any prime  $p$ .

**Definition 12.**  $p$ -adic valuation

It is easy to see that the  $p$ -adic valuation exhibits the following properties, which makes it nice to work with:

- $v_p(ab) = v_p(a) + v_p(b)$
- *anotherone*

**Example 1.**  $\mathbb{Z}_p$

We give a second example below, showing a space that does not readily come equipped with algebraic structure:

**Example 2.** snowflake metric

Ultrametric spaces exhibit properties much unlike traditional metric spaces, and we review a few of these below. Of particular interest to us is the behavior between (closed) balls in an ultrametric space.

**Notation.** Let  $(M, \rho)$  be a compact ultrametric space and let

$$B(a, r) = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the *closed* ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{>0}$  and  $a \in (M, \rho)$ .

In the above notation, we break from convention in that we denote a closed ball without using any decoration and in that we omit any notation for an open ball. This is because, for the most part, the notion of open and closed ball in an ultrametric space overlap, although we will need a few more facts before showing this.

**Definition 13.** Let  $S$  be a subset of an ultrametric space. The **diameter of  $S$**  is  $\text{diam}(S) = \sup_{x,y \in S} \rho(x,y)$ . Note that if  $S$  is compact,  $\text{diam}(S) = \max_{x,y \in S} \rho(x,y)$ .

**Proposition 8.** Let  $B = B(a, r)$  be a (closed) ball in an ultrametric space  $(M, \rho)$ . Then the diameter of  $B$  is less than or equal to the radius of  $B$ .

*Proof.* Suppose  $d = \text{diam}(B) > r$ . This would imply there exists  $x, y$  in  $B$  such that  $\rho(x, y) > r$ , in particular  $\rho(x, y)$  is strictly greater than  $\max(\rho(x, a), \rho(y, a))$ , which is a contradiction since  $\rho$  is an ultrametric.  $\square$

In the following proposition, we describe the triangles in an ultrametric space, and the result is more or less a restatement, in geometric terms, of the ultrametric inequality.

**Proposition 9.** All triangles in an ultrametric space  $(M, \rho)$  are either equilateral or isosceles, with at most one short side.

*Proof.* Let  $x, y$ , and  $z$  be three points in an ultrametric space  $(M, \rho)$ . We show that  $\rho(x, y) \neq \rho(x, z)$  and  $\rho(x, y) \neq \rho(y, z)$  implies  $\rho(x, y) < \rho(x, z) = \rho(y, z)$ .

If  $\rho(x, z) \neq \rho(y, z)$ , then without loss,  $\rho(x, z) > \rho(y, z)$ . At the same time, the ultrametric inequality implies  $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$  and  $\rho(y, z) \leq \max(\rho(x, y), \rho(x, z))$ . The first inequality implies  $\rho(x, y) < \rho(x, z)$ , which means the second inequality implies  $\rho(y, z) < \rho(x, z)$ . This is a contradiction, so we must have  $\rho(x, z) = \rho(y, z)$ .

To see that  $\rho(x, y) < \rho(x, z)$ , simply note that  $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$   $\square$

With this result in hand, we are able to quickly demonstrate some of the properties of (closed) balls, which are of fundamental importance to us. We see below that the ultrametric inequality, perhaps innocuous on the surface, quickly implies ultrametric balls are markedly different from their archimedean counterparts.

**Proposition 10.** Every point of a ball in an ultrametric is at its centre. That is, if  $B(x_0, r)$  is a ball in an ultrametric space  $(M, \rho)$ , then  $B(x, r) = B(x_0, r)$ ,  $\forall x \in B(x_0, r)$

*Proof.* Let  $a \in B(x, r)$ . Then  $\rho(a, x) \leq r$  and since

$$\rho(a, x_0) \leq \max(\rho(a, x), \rho(x, x_0)) \leq r$$

we must have  $a \in B(x_0, r)$  and  $B(x, r) \subseteq B(x_0, r)$ . A similar argument shows  $B(x_0, r) \subseteq B(x, r)$ .  $\square$

**Proposition 11.** *If  $(M, \rho)$  is an ultrametric space and  $B(x_0, r_1)$  and  $B(y_0, r_2)$  are balls in  $(M, \rho)$ , then either  $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$ ,  $B(x_0, r_1) \subseteq B(y_0, r_2)$ , or  $B(x_0, r_1) \subseteq B(x_0, r_1)$ . That is, in an ultrametric space, all balls are either comparable or disjoint.*

*Proof.* Suppose  $B(x_0, r_1) \cap B(y_0, r_2) \neq \emptyset$  and let  $z$  be a point in the intersection. We show that if there exists an  $a \in B(y_0, r_2)$  such that  $a \notin B(x_0, r_1)$ , then  $B(x_0, r_1) \subseteq B(y_0, r_2)$ . Let  $x \in B(x_0, r_1)$ . Then we must have  $\rho(x, z) < \rho(x, a)$ , since  $z \in B(x_0, r_1) = B(x, r_1)$  and  $a$  is not. Since the triangle with vertices  $(a, x, z)$  is isocolces with at most one short side, we must have  $\rho(x, a) = \rho(a, z) \leq r_2$ , since  $a \in B(y_0, r_1) = B(z, r_2)$ . Then  $x \in B(y_0, r_1)$ .  $\square$

**Proposition 12.** *The distance between any two non-overlapping balls in an ultrametric is constant. That is, if  $B(x_0, r_1)$  and  $B(y_0, r_2)$  are two balls in an ultrametric space with  $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$ , then there exists a  $c \in \mathbb{R}_{\geq 0}$  such that  $\rho(x, y) = c$ ,  $\forall x \in B(x_0, r_1)$  and  $\forall y \in B(y_0, r_2)$ .*

*Proof.* Suppose  $\rho(x_0, y_0) = c$  and let  $x \in B(x_0, r_1)$  and  $y \in B(y_0, r_2)$  be arbitrary. Consider the triangle formed by  $(x_0, y_0, y)$ . Since  $\rho(x_0, y_0) = c$  and  $\rho(y, y_0) \leq r_2 < c$ , we must have  $\rho(x_0, y) = c$  because triangles in an ultrametric space have at most one short side. Now consider the triangle formed by  $(x_0, x, y)$ . Since  $\rho(x_0, y) = c$  and  $\rho(x, x_0) \leq r_1 < c$ , we must have  $\rho(x, y) = c$ .  $\square$

Before moving on, we give another example of an ultrametric space, showing again the quick and unusal implications of Proposition 4.

**Example 3.** Let  $S$  be a compact subset of an ultrametric space,  $(M, \rho)$ , and then let  $\mathbf{Triag}_S$  be the space whose points are the set of distinct (up to labeling) triangles

in  $S$ . If  $t$  is an element of  $\mathbf{Triag}_S$ , denote by  $long_t$  the length of the long side of  $t$  and by  $short_t$  the length of the short side of  $t$ . Then define  $\Delta_S(t, t') = |long_t - long_{t'}| + |short_t - short_{t'}|$ , for  $t$  and  $t' \in \mathbf{Triag}_S$ . Now note  $long_t$  and  $short_t$  are in  $\mathbb{R}$  and so  $\Delta_S(t, t')$  is always in  $\mathbb{R}_{\geq 0}$ . It's also clear that  $\Delta_S(t, t') = 0$  if, and only if,  $t$  and  $t'$  have the same length sides, in which case they are equal in  $\mathbf{Triag}_S$ . Now let  $t_a, t_b$  and  $t$  be any three points in  $\mathbf{Triag}_S$ . Without loss of generality, assume  $\max(\Delta_S(t_a, t), \Delta_S(t_b, t)) = \Delta_S(t_b, t)$ , that is,

$$|long_{t_a} - long_t| + |short_{t_a} - short_t| \leq |long_t - long_{t_b}| + |short_t - short_{t_b}|$$

We must show

$$|long_{t_a} - long_{t_b}| + |short_{t_a} - short_{t_b}| \leq |long_t - long_{t_b}| + |short_t - short_{t_b}|$$

You can easily construct a counter example with all equilateral triangles (which abouts to the fact that  $\mathbb{R}$  is not normally an ultrametric space)

Note that an analogous construction would not work on the set of line segments in  $S$ .

The following proposition is easy to see, although the result is both unintuitive and important for our purposes.

**Proposition 13.** *Suppose  $S$  is a compact subset of an ultrametric space  $(M, \rho)$  and that  $\cup_{i \in I} B(x_i, r_i)$  is a cover of  $S$  by (closed) balls in  $S$ . Then there exists  $i_1, \dots, i_n$ , a finite subset of  $I$ , such that  $\cup_{j=1}^n B(x_{i_j}, r_{i_j})$  is a partition of  $S$ .*

*Proof.* Since  $S$  is compact,  $\cup_{i \in I} B(x_i, r_i)$  contains a finite subcover of  $S$ . Say this subcover is given by the elements  $i_1, \dots, i_{n'}$  in  $I$ , and suppose this is not a partition. That is, suppose for some  $i_i, i_j$ ,  $B(x_{i_i}, r_{i_i}) \cap B(x_{i_j}, r_{i_j}) \neq \emptyset$ . Then, without loss of generality, we must have  $B(x_{i_i}, r_{i_i}) \subseteq B(x_{i_j}, r_{i_j})$ , so that the removal of  $B(x_{i_i}, r_{i_i})$  is still a cover of  $S$ . We continue this process a finite number of times, since the subcover was finite to begin with, to arrive at a partition of  $S$ .  $\square$

In fact, a slightly stronger statement than the above is true:

**Corollary 1.** *Suppose  $S$  is a compact subset of an ultrametric space  $(M, \rho)$  and that  $B(x_0, r)$  is a (closed) ball in  $S$ . Then, there exists a finite partition of  $S$  having  $B(x_0, r)$  as an element.*

*Proof.* Let  $\mathcal{C}$  be the cover of  $S$  given by  $\cup_{x \in S} B(x, r) \cap S$ . From the proposition, we can select a finite subcover of  $\mathcal{C}$  that is a partition of  $S$ . Suppose  $B(y, r) \cap S$  is the element in this partition containing  $x_0$ . Then since  $B(y, r)$  and  $B(x_0, r)$  are equal in  $M$ ,  $B(y, r) \cap S = B(x_0, r) \cap S = B(x_0, r)$ .  $\square$

We end this section by making a few comments about the set of distances that occur between the points of a compact ultrametric space.

**Proposition 14.** *[?] Let  $S$  is a compact subset of an ultrametric space. For  $a \in S$ , let  $\phi_a : S \setminus \{a\} \rightarrow \mathbb{R}$  be the function defined by  $\phi_a(x) = \rho(x, a)$ . Then  $\text{Im}(\phi_a)$  is a discrete subset of  $\mathbb{R}$  for all  $a \in S$ .*

*Proof.* [?]  $\square$

**Corollary 2.** *[?] Let  $B(a, r)$  be a closed ball in an ultrametric space. Then there exists  $r' > r \in \mathbb{R}$  such that  $B(a, r) = \{x \in M \mid \rho(x, a) < r'\}$ ; that is, every closed ball is also an open ball with the same centre and slightly larger radius.*

*Proof.*  $\square$

**Corollary 3.** *[?] If  $S$  is a compact subset of an ultrametric space and  $\Gamma_S$  is the set of all distances occurring between points of  $S$ , then  $\Gamma_S$  is countable; that is, there is an injective function from  $\Gamma_S \mapsto \mathbb{N}$ .*

*Proof.*  $\square$

It will become useful to write the set of distances occurring in  $S$  as a sequence, put in decreasing order.

**Notation.** If  $S$  is a compact (hence bounded) ultrametric space, then we denote the set of distances between points of  $S$  by

$$\Gamma_S = \{\gamma_0 = d = \text{diam}(S), \gamma_1, \gamma_2, \dots, \gamma_\infty = 0\}$$

where  $\gamma_i \in \Gamma_S$  if and only if  $\exists x, y \in S$  such that  $\rho(x, y) = \gamma_i$  and  $\gamma_i < \gamma_j$  if and only if  $i > j$ .

To make use of regularity in  $S$  we should, before continuing, also put some constraints on our sequence of distances  $\Gamma_S$ .

**Definition 14.** Let  $S$  be a compact subset of an ultrametric space and  $\Gamma_S$  is the sequence of decreasing distances in  $S$ . Then we say  $S$  is **well-behaved** (I guess - or tame? reasonable?), if  $S$  is semi-regular and  $\gamma_k = \alpha(k)^{c_k}$  for all  $k \in \mathbb{N}$  and some  $c_k \in \mathbb{Z}$ .

**Example 4.** Any ultrametric space where  $\rho$  is induced from a valuation domain.

### $\rho$ -orderings, $\rho$ -sequences, and valuative capacity

We now come to the central theme of this work. The observation that an analogous notion of  $p$ -ordering can be defined for a general ultrametric space, and that these structures coincide with Fekete  $n$ -tuples, is due to [?]. The exploration of this idea makes up the remainder of this work.

In what follows, we let  $S$  be a compact subset of an ultrametric space  $(M, \rho)$ .

**Definition 15.** [?] A  $\rho$ -ordering of  $S$  is a sequence  $\{a_i\}_{i=0}^{\infty}$  in  $S$  such that  $a_0$  is arbitrary and  $\forall n > 0$ ,  $a_n$  maximizes  $\prod_{i=0}^{n-1} \rho(s, a_i)$  over  $s \in S$ .

**Example 5.** Suppose  $S$  is a finite subset of  $(\mathbb{Z}, \rho_2)$ ,  $S = \{0, 2, 8, 3\}$ . Then a  $\rho$ -ordering of  $S$  starts (arbitrarily) with  $a_0 = 0$ , which forces  $a_1 = 3$ , since  $\rho(0, 3) = 1 = \text{diam}(S)$ . The sequence continues  $a_2 = 2$  and  $a_3 = 8$ , but after this point the sequence becomes arbitrary because  $\prod_{i=0}^{n-1} \rho(s, a_i)$  will contain a 0, given by the repeated term. Indeed, for any finite subset  $S$  with  $|S| = n$ , the  $\rho$ -ordering of  $S$  is arbitrary from the  $n^{\text{th}}$  point on.

**Definition 16.** [?] Let  $\sigma(i)$  be a  $\rho$ -ordering of  $S$ . The  $\rho$ -sequence of  $S$  is defined by letting  $a_0 = 1$  and  $a_n = \prod_{i=0}^{n-1} \rho(\sigma(n), \sigma(i))$ , for  $n > 0$ .

- this gives back the  $n$ -th diameter - the limit is the transfinite diameter and we take it as the definition of capacity
- clean up all the sequence definitions to use the same format
- check the historical background and this section to make sure you've said enough about feketen-tuples and  $p$ -orderings corresponding
- go through the JLC chabert reference
- check that youve used the same notation for closed balls

**Proposition 15.** [?] *The  $\rho$ -sequence of  $S$  is well-defined so long as  $S$  is compact and  $\rho$  is an ultrametric. That is, the  $\rho$ -sequence of a compact subset of an ultrametric spaces does not depend on the choice of  $\rho$ -ordering of  $S$ .*



**Definition 17.** [?] Let  $\gamma(n)$  be the  $\rho$ -sequence of  $S$ . The **valuative capacity** of  $S$  is

$$\omega(S) := \lim_{n \rightarrow \infty} \gamma(n)^{1/n}$$

**Proposition 16.** [?] For  $S$  and  $\gamma(n)$  as above,  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = r < \infty$ .

**Proposition 17.** If  $S \subseteq M$  is a finite subset of an ultrametric space, then  $\omega(S) = 0$ .

**Proposition 18.** (upper bound) If  $\text{diam}(S) = d$ , then  $\omega(S) < d$ .

*Proof.* Since  $d$  is the diameter of  $S$ , the  $n^{\text{th}}$  term of the  $\rho$ -sequence of  $S$  is bounded by  $d^n$  and so  $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = d$  if, and only if,  $\gamma(n) = d^n$ ,  $\forall n$ . This implies  $\rho(a_n, a_i) = d$ ,  $\forall n$  and  $\forall i < n$ , but then  $\rho(a_i, a_j) = d$ ,  $\forall i, j$ , since the  $\rho$ -sequence is maximized at each  $n$ . This means  $\omega(S) < d$  would imply that the cover of  $S$ ,  $\cup_{a_i} B_d(a_i)$  is in fact an infinite partition, contradicting the compactness of  $S$ . Then  $\omega(S) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} < d$ .  $\square$

This doesn't work because  $\cup_{a_i} B_d(a_i)$  could fail to be a cover - the  $\rho$ -ordering will be a dense subset under mild conditions - then being compact should imply a contradiction since we should have  $\bar{S} = S$  for  $S$  compact.

Make a comment about algebraic structure here

**Proposition 19.** (translation invariance) If  $(M, \rho)$  be a compact ultrametric space and  $s$  also a topological group for which  $\rho$  is (left) invariant under the group operation, then  $\omega$  is also (left)-invariant. That is, if  $\rho(x, y) = \rho(gx, gy)$ ,  $\forall g, x, y \in M$ , then  $\omega(gS) = \omega(S)$ , for  $S \subseteq M$ .

*Proof.* Let  $\{a_i\}_{i=0}^\infty$  be a  $\rho$ -ordering for  $S$ . Then  $\{ga_i\}_{i=0}^\infty$  is a  $\rho$ -ordering for  $gS$ . Then

$$\omega(gS) = \lim_{n \rightarrow \infty} \gamma(n)^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = \omega(S)$$

$\square$

**Example 6.** With the notation of the previous section, note that  $\rho_p$  is translation invariant since for  $x, y \in (\mathbb{Z}_p, \rho_p)$ ,  $\rho_p(x, y) = p^{-v_p(x-y)} = p^{-v_p((a+x)-(a+y))} = \rho_p(a+x, a+y)$ . Then  $\omega(a+S) = \omega(S)$  for  $S \subseteq (\mathbb{Z}_p, \rho_p)$ .

**Proposition 20.** (*scaling*) Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle identity, so that  $(V, \rho_N)$  is an ultrametric space. Then if  $N$  is multiplicative, so is  $\omega$ . That is, if  $N(gx) = N(g)N(x), \forall g, x \in V$ , then  $\omega(gS) = N(g)\omega(S)$ , for  $g \in V$  and  $S \subseteq M$ .

*Proof.* Let  $\rho$  be the metric induced by  $N$ , so that  $\rho(x, y) = N(x - y), \forall x, y \in V$ . Let  $\{a_i\}_{i=0}^\infty$  be a  $\rho$ -ordering for  $S$ . Then since  $N$  is multiplicative, for  $u, v \in gS$ ,  $u = gs_i$  and  $v = gs_j$  for some  $s_i, s_j \in S$ ,

$$\rho(u, v) = \rho(gs_i, gs_j) = N(gs_i - gs_j) = N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j).$$

Then  $\{ga_i\}_{i=0}^\infty$  is a  $\rho$ -ordering for  $gS$  and

$$\begin{aligned} \omega(gS) &= \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(ga_n, ga_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} N(g)\rho(a_n, a_i) \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} [N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g) \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g)\omega(S) \end{aligned}$$

□

**Example 7.** Since  $\rho_p$  is multiplicative,  $\omega(mS) = v_p(m) \cdot \omega(S)$  for  $m \in \mathbb{Z}_p$  and  $S \subseteq \mathbb{Z}$ . In particular,  $\omega(p\mathbb{Z}) = \frac{1}{p} \cdot \omega(\mathbb{Z})$ .

This is important and you should say so. The following proposition is from [?], where it is given for some  $S$  written as the union of two subsets, although it is easily seen to be true for  $S$  equal to any finite union, so long as the other assumptions remain satisfied.

**Proposition 21.** [?](*subadditivity*) If  $\text{diam}(S) := \max_{x, y \in S} \rho(x, y) = d$  and  $S = \cup_i^n A_i$  for  $A_i$  compact subsets of  $M$  with  $\rho(A_i, A_j) = d, \forall i, j$ , then

$$\frac{1}{\log(\omega(S)/d)} = \sum_{i=1}^n \frac{1}{\log(\omega(A_i)/d)}$$

**Example 8.**  $\omega(\mathbb{Z})$

**Corollary 4.** Suppose  $S = \cup_i^n S_i$  with  $\rho(S_i, S_j) = d = \text{diam}(S)$  and also  $\omega(S_i) =$

$\omega(S_j), \forall i, j$  . Let  $r \in \mathbb{R}$  be such that  $\omega(S_i) = r\omega(S), \forall i$ . Then  $\omega(S) = r^{\frac{1}{n-1}} \cdot d$ . In particular if  $S = \mathbb{Z}$  and  $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$  then  $\omega(S) = (\frac{1}{p})^{1/p-1}$  for any prime  $p$ .

**Example 9.**

**Corollary 5.** (*Joins of computable sets are computable*) Let  $\Gamma_M = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$  be the set of distances in  $M$ . Suppose that  $S = B_{\gamma_i}(x)$ , for some  $x$  and  $i$ , is the union of 2 or more balls of radius  $\gamma_{i+1}$ , i.e.,  $S = \cup_{j=1}^n B_{\gamma_{i+1}}(x_j)$  is a join in the lattice of open sets in  $M$ , then

$$\frac{1}{\log(\omega(S)/\gamma_{i+1})} = \sum_{j=1}^n \frac{1}{\log(\omega(B_{\gamma_{i+1}}(x_j))/\gamma_{i+1})}$$

## Chapter 3

### $\rho$ -orderings and the structure of $S$

In the previous section, we defined valutive capacity for a compact subset  $S$  of an ultrametric space  $(M, \rho)$ . We also got a glimpse into the way the valutive capacity of  $S$  interacts with its other properties, such as the set of distances occurring in  $S$  and the lattice of closed balls in  $S$  (or equivalently, if  $S$  has enough structure, a lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valutive capacity of  $S$  and the lattice of closed balls in  $S$ . In particular, we show how, in all cases (with  $S$  compact), the latter can be used to compute the first  $n$  terms of a  $\rho$ -ordering of  $S$  (for any  $n < \infty$ ) and how, in some cases, this extends to being able to compute the valutive capacity of  $S$ .

### Subspaces of $S$

In the section we explore the subspaces of  $S$  formed by considering closed balls of some fixed radius. We begin by letting  $S$  be, as before, a compact subset of an ultrametric space  $(M, \rho)$ . Recall from the previous section that if  $S$  is compact, then the set of distances occurring in  $S$  is a discrete, bounded subset of  $\mathbb{R}$  and so we may represent the set of distances by a sequence in decreasing order. As before, let the decreasing sequence of distances in  $S$  be given by  $\Gamma_S = \{\gamma_0 = \text{diam}(S), \gamma_1, \dots, \gamma_\infty = 0\}$ .

Now fix some  $k \in \mathbb{N}$ , and consider for a moment the set of closed balls of radius  $\gamma_k$  in  $S$ . We could denote these alternatively by  $B^M(x, \gamma_k) \cap S$  or by  $B^S(x, \gamma_k)$ , but when there is no risk of confusion, we will denote them simply by  $B(x, \gamma_k)$ . Clearly, the set  $\{B(x, \gamma_k); x \in S\}$  forms a cover of  $S$ . Although we have build the cover using closed balls, since each set in an ultrametric is clopen, this gives an open cover of  $S$

(in fact, each element is not only an open set, but also an open ball for some radius slightly bigger than  $\gamma_k$ ). Then since  $S$  is compact, we must have some  $x_1, \dots, x_n$  such that  $S = \cup_{i=1}^n B(x_i, \gamma_k)$ . In fact, since  $\rho$  is an ultrametric, we can pick the  $x_i$ 's so that  $\cup_{i=1}^n B(x_i, \gamma_k)$  will be a disjoint union and therefore a partition of  $S$ . Note that both  $n$  and the  $x_i$ 's depend on our fixed  $k$ , but that  $n$  is independent of the  $x_i$ 's, since any choice of centres is equivalent. We rephrase this with following definition and lemma:

**Definition 18.** For  $S$  and  $\Gamma_S$  as above, and  $k \in \mathbb{N}$ , fixed, define  $\sim_k$  to be the relation on  $S$  given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e.,  $x \sim_k y$  if and only if  $B_{\gamma_k}(x) = B_{\gamma_k}(y)$ .

The fact  $\sim_k$  is an equivalence relation on  $S$  is equivalent to the observation that every point in a ultrametric ball is at its centre:

**Lemma 1.** *Let  $S$  and  $\Gamma_S$  be as above, then  $\sim_k$  is an equivalence relation on  $S$ .*

*Proof.*  $\sim_k$  is clearly reflexive and symmetric, since  $\rho$  is a metric. Transitivity results from the ultrametric property of  $\rho$ : if  $x \sim_k y$  and  $y \sim_k z$ , then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)) \leq \gamma_k$$

so  $x \sim_k z$ . □

We denote the set of equivalence classes of  $S / \sim_k$  by  $S_{\gamma_k}$ . We have defined  $S_{\gamma_k}$  to be the set of equivalence classes in  $S$  under the relation  $\sim_k$ , which is equivalent to letting  $S_{\gamma_k}$  be the set of closed balls of fixed radius  $\gamma_k$  in  $S$ . We now offer a third perspective on the elements on  $S_{\gamma_k}$ , which is due to [?],

**Lemma 2.** *For each  $k$ , the elements of  $S_{\gamma_k}$ , that is, the closed balls of radius  $\gamma_k$ , themselves form an ultrametric space, where the metric is given by:*

$$\rho_k(B(x, \gamma_k), B(y, \gamma_k)) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } B(x, \gamma_k) = B(y, \gamma_k) \end{cases}$$

*Proof.*  $\rho_k$  is reflexive, symmetric and transitive since  $\rho$  is. Likewise,  $\rho_k$  satisfies the ultrametric property, since  $\rho$  does: let  $B(x, \gamma_k), B(y, \gamma_k)$  and  $B(z, \gamma_k)$  be any three elements of  $S_{\gamma_k}$  and suppose  $\rho_k(B(x, \gamma_k), B(y, \gamma_k)) > 0$ . Then,

$$\begin{aligned} \gamma_k &< \rho_k(B(x, \gamma_k), B(y, \gamma_k)) \\ &= \rho(x, y) \leq \max(\rho(x, z), \rho(y, z)) \\ &= \max(\rho_k(B(x, \gamma_k), B(z, \gamma_k)), \rho_k(B(y, \gamma_k), B(z, \gamma_k))) \end{aligned}$$

since  $\gamma_k < \max(\rho(x, z), \rho(y, z))$  implies that at least one of  $\rho_k(B(x, \gamma_k), B(z, \gamma_k))$  or  $\rho_k(B(y, \gamma_k), B(z, \gamma_k))$  is greater than 0.  $\square$

So now the elements of  $S_{\gamma_k}$  may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

**Definition 19.** Let  $S$  and  $\Gamma_S$  be as above. Define  $\beta(i)_{i \geq 0}$  to be the sequence given by  $\beta(i) = |S_{\gamma_i}|$ , which is an invariant of  $S$  and which counts the number of connected components of  $S_{\gamma_i}$  (that is, the points of  $S_{\gamma_i}$ ), when viewed as a metric space. When necessary, we use  $\beta^S(i)$  to denote the  $\beta$  sequence for a given, compact ultrametric space  $S$ . Adapting the terminology in [?], we call  $\beta^S(i)$  the **structure sequence** of  $S$ .

**Notation 1.** Let  $S_{\gamma_k}$  be as above. We denote the elements of  $S_{\gamma_k}$  by  $B_1^k, \dots, B_{\beta(k)}^k$  or by  $B_1^{S,k}, \dots, B_{\beta(k)}^{S,k}$ , when necessary.

We return to the sequence  $\beta(i)$  at the end of this section. For now, we show how a  $\rho$ -ordering of  $S$  can be built recursively from the spaces  $S_{\gamma_k}$ . This begins by noting that the spaces themselves can be built recursively:

**Observation 1.** Let  $S$ ,  $\Gamma_S$ , and  $S_{\gamma_k}$  be as above. Then  $S_{\gamma_{k+1}}$  can be constructed by partitioning each of the closed balls in  $S_{\gamma_k}$  into closed balls of radius  $\gamma_{k+1}$  and taking their union: Let  $B(x_i, \gamma_k)$  be an element of  $S_{\gamma_k}$ , denoted by  $B_i^k$ . Then, there exists  $x_{i,1}, \dots, x_{i,l_i} \in B_i^k$  such that,

$$B_i^k = \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1})$$

and

$$B(x_{i,j}, \gamma_{k+1}) \cap B(x_{i,j'}, \gamma_{k+1}) = \emptyset, \forall j, j' \in 1 : l_i$$

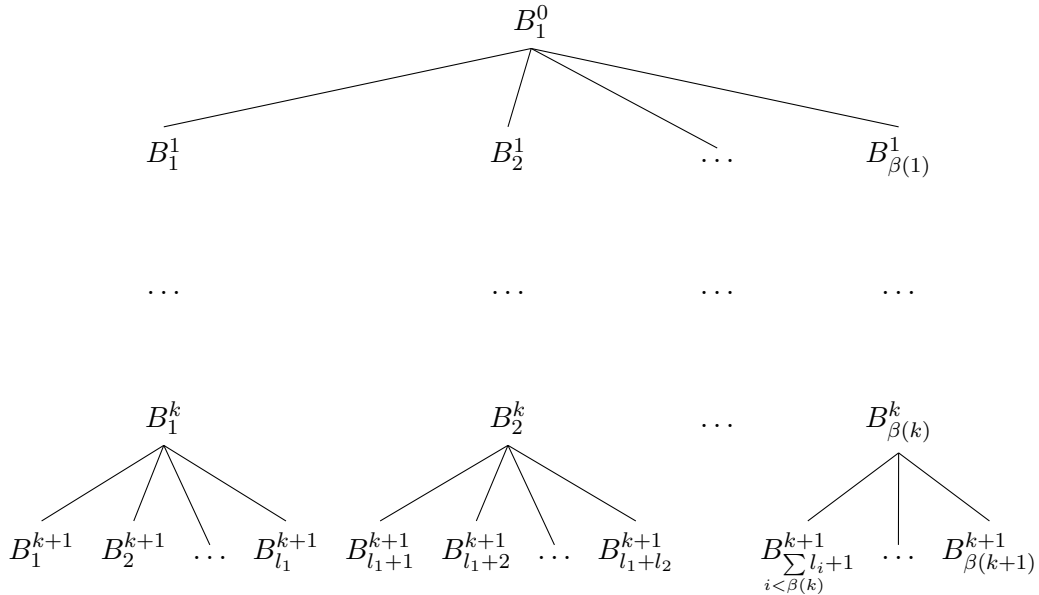
and so

$$S_{\gamma_{k+1}} = \bigcup_{i=1}^{\beta(k)} \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1}) = \bigcup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where  $\bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_k) = B(x_i, \gamma_{k+1}) = B_i^k, \forall i$ .

Since  $S$  is compact, hence bounded, if we represent this process schematically we obtain a tree, where the root node is  $B_1^0 = B(x, \gamma_0)$ , for any choice of  $x \in S$ , and the children of any given  $B_n^m$  are such that they form a partition of their join. Since we will often refer to this schematic representation, we define it below.

**Definition 20.** If  $S$  is a compact subset of an ultrametric space, then  $T_s$  is the tree whose vertices are  $B_i^k$ , that is the elements of  $S_{\gamma_k}$ , and whose edgeset,  $E$ , is given by  $(B_k^i, B_l^j) \in E$  if, and only if,  $j = i + 1$  and  $B_l^j \subseteq B_k^i$  for some choice of representatives  $B(x_k, \gamma_i)$  and  $B(x_l, \gamma_j)$ , as shown below:



Before going on, first note that we have drawn  $T_s$  such that leftmost child of some  $B_i^k$  is  $B_j^{k+1}$  where  $j$  is minimal among the children of  $B_i^k$ , and then continued

in increasing order. In general, if we draw  $T_S$  so that the children of a given vertex are depicted in increasing order according to their index, then each choice of indexing for the elements of  $S_{\gamma_k}$  produces a different graphical representation of  $T_S$ . The structures produced by different choices of indices are clearly isomorphic as trees, and as we will see by the end of the section, each choice of indexing will be valid for our purposes as well.

Of central importance to us is the distance between two vertices in  $T_S$ . Since each vertex represents an element of  $S_{\gamma_k}$ , that is a closed ball in an ultrametric space, it is well-defined to let the distance between vertices be equal to the distance between a choice of centres for those balls. Note that if the distance between  $B_i^k$  and  $B_j^l$  is taken to be  $\rho(x_i, x_j)$ , for some choice of  $x_i \in B_i^k$  and  $x_j \in B_j^l$ , say  $\rho(x_i, x_j) = \gamma_n$ , then the join of  $B_i^k$  and  $B_j^l$  is some  $B_x^n$ .

**Lemma 3.** *If  $B_i^k$  and  $B_j^l$  are two vertices in  $T_S$ , then  $\rho(x_i, x_j)$ , for any choice of  $x_i \in B_i^k$  and  $x_j \in B_j^l$ , is equal to the diameter of the join of  $B_i^k$  and  $B_j^l$ .*

*Proof.* Let  $B_i^k$  and  $B_j^l$  be two (distinct) vertices in  $T_S$  and let  $B_x^n$  be their join. The diameter of  $B_x^n$  is  $\gamma_n$  since  $B_x^n = B(x_0, \gamma_n)$  for some  $x_0$ . Since  $\rho$  is an ultrametric the distance between any  $x_i \in B_i^k$  and  $x_j \in B_j^l$  is constant, and must be equal to the diameter of the smallest ball containing both of them, that is  $\gamma_n$ .  $\square$

In particular, we have that for any  $k$  and any  $i < \beta(k)$ , the distances between the children of  $B_i^k$  will be  $\gamma_k$  and for any  $i \neq j$  the distance between the children of  $B_i^k$  and  $B_j^k$  will be equal to the distance between  $B_i^k$  and  $B_j^k$  (which will be some  $\gamma_m, m < k$ ).

### Recursive $\rho$ -orderings

In this section, we show how the recursive partitioning of  $S$  into the spaces  $S_{\gamma_k}$  gives rise to a  $\rho$ -ordering of  $S$ . We first note that without loss of generality, for any  $k \in \mathbb{N}$ , we can reindex the  $B_i^k$ 's so that they give the first  $\beta(k)$  terms of a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , when the latter is viewed as a (finite) metric space. In the first proposition below, we note that if the  $B_i^k$ 's are so indexed, then finding a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is straightforward: select a  $B_j^{k+1}$  from each of the  $B_i^k$ 's in order and then start over.



**Proposition 22.** *Let be  $S$  a compact subset of an ultrametric space  $(M, \rho)$  and  $\Gamma_S$ , the set of distances in  $S$ . If  $S_{\gamma_k}$  is the partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the elements are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , then the first  $\beta(k+1)$  terms in a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  can be found by selecting at each stage  $n$ , a child from  $B_{\bar{n}}^k$ , where  $\bar{n} = n \bmod \beta(k) + r$  and  $r$  is minimal in  $\{0, \dots, \beta(k) - 1\}$  such that  $B_{n \bmod \beta(k) + r}^k$  still has unused children.*

*Proof.* Let  $S$ ,  $S_{\gamma_K}$ , and  $S_{\gamma_{k+1}}$  be as above. In particular, suppose the elements of  $S_{\gamma_k}$  are indexed according to a  $\rho_k$ -ordering. Denote the elements of  $S_{\gamma_{k+1}}$  by  $B_{i,j}^{k+1}$  where the first subscript indicates that the element is a child of  $B_i^k$ . To form a  $\rho_{k+1}$  ordering of  $S_{\gamma_{k+1}}$ , we must maximize the product of distances at each step  $n$ .

Now note that  $\Gamma_{S_{\gamma_k}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$  and  $\Gamma_{S_{\gamma_{k+1}}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}, \gamma_k\}$ . That is, the distances in  $S_{\gamma_{k+1}}$  are the same as the distances in  $S_{\gamma_k}$ , although they also include the smaller distance  $\gamma_k$ . Since we know that the elements  $B_1^k, \dots, B_{\beta(k)}^k$  already maximizes the product of distances in  $\{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$ , the first  $\beta(k)$  terms of a  $\rho_{k+1}$ -ordering of  $S_{k+1}$  can be found by taking  $B_{1,j_1}^k, \dots, B_{1,j_{\beta(k)}}^k$  for any choice of  $j$ 's. At this point, any choice of next element will produce a copy of  $\gamma_k$  in the  $\rho_{k+1}$ -sequence; however, if we chose another child of  $B_1^k$ , we are able to keep building the ordering in a canonical fashion, since we know that we will then be able to maximize the product at the next step by choosing another child of  $B_2^k$ .

We see then that a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by minimizing the number of times  $\gamma_k$  is introduced into the  $\rho_{k+1}$ -sequence and maximizing the product among the  $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ , and the latter is already known to be achieved by taking the  $B_i^k$  in order. If the  $B_i^k$ 's all have the same number of children, then we can always select a child of  $B_{\bar{n}}^k$ , where  $\bar{n} = n \bmod \beta(k)$  at each stage  $n$ ,  $n < \beta(k+1)$ , since there will always be one available. On the other hand, suppose the  $B_i^k$  have an unequal number of children and  $n$  is the first step at which all the children of  $B_{\bar{n}}^k$  have been exhausted. What element will maximize the  $\rho_{k+1}$ -sequence?

Consider the space  $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$ . Removal of  $B_{\bar{n}}^k$  will not effect the first  $m$  terms of a  $\rho_k$ -ordering of this space, for  $m < \bar{n}$ , since if a sequence of elements maximizes a

function over a set  $X$ , they will also maximize that function of a subset of  $X$  (provided they themselves remain in the subset). Then the  $\rho_k$ -sequence of  $(S_{\gamma_k} \setminus B_n^k)$  begins  $\{B_1^k, \dots, B_{\bar{n}-1}^k\}$ .

Moreover, if  $B_{\bar{n}+1}^k$  maximizes  $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k)$  over  $S_{\gamma_k}$ , then it also maximizes  $\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)$  over  $(S_{\gamma_k} \setminus B_n^k)$ , since  $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k) = (\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)) \cdot \rho_k(x, B_n^k)$ .

Then the  $\rho_k$ -sequence of  $(S_{\gamma_k} \setminus B_n^k)$  is simply  $\{B_1^k, \dots, B_{\bar{n}-1}^k, B_{\bar{n}+1}^k, \dots, B_{\beta(k)}^k\}$ .

Now we see that  $\rho_{k+1}$ -sequence of  $S_{\gamma_{k+1}}$  is maximized by simply skipping over  $B_n^k$ , should all its children be exhausted, and selecting a child from  $B_{\bar{n}+1}^k$ . Then a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by selecting elements of each  $B_i^k$  in order as much as possible, and skipping to  $B_{i+1}^k$ , when it is not possible.

□

Note that in building the  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  we selected, at each step, a child of some  $B_i^k$ , but we did not concern ourselves over which child was selected. This is because the distances between any two children of some  $B_i^k$  is  $\gamma_k$ , and the distance between any one of them and a child of some  $B_j^k$ ,  $i \neq j$ , is the same. We can now see, as claimed above, that any of the isomorphic versions of  $T_S$  are valid for producing  $\rho$ -orderings. Suppose then that we have created  $T_s$  and (arbitrarily) indexed the children of each vertex. Then, there is no loss of generality in assuming that at each stage, we select a child with smallest index among its siblings, that is, that we select the leftmost available child in  $T_s$ . Since for ease of indexing, we will assume a  $\rho$ -ordering has been built by this convention, we introduce the following definition.

**Definition 21.** The  $\rho$ -ordering of  $S$  formed by pulling elements from left to right in (a choice of)  $T_s$  is call the **canonical  $\rho$ -ordering** of  $S$  (with respect to  $T_s$ ).

The above proposition quickly leads to a recursive construction for a  $\rho$ -ordering of  $S$ . Indeed, to build a  $\rho$ -ordering of  $S$  from the above, it suffices only to make a choice of centres for each of  $B_i^k$ 's.

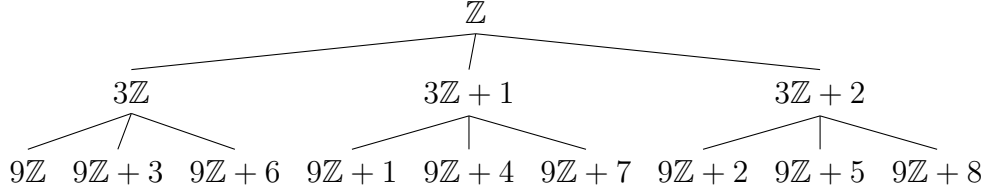
**Proposition 23.** *Let be  $S$  a compact subset of an ultrametric space  $(M, \rho)$  and let  $\Gamma_S$  be the set of distances in  $S$ . Let  $S_{\gamma_k}$  be the partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the elements are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ . Suppose each of the element of  $S_{\gamma_k}$  have also been partitioned into closed balls of radius  $\gamma_{k+1}$ ,  $B_i^k = \cup_{j=1}^{l_i} B_{i,j}^{k+1}, \forall i$ .*

*Let  $x_{i,j}$  denote a choice of centre for the element  $B_{i,j}^{k+1}$ . Then the first  $\beta(k+1)$  elements of a  $\rho$ -ordering of  $S$  can be found by forming a matrix,  $A_k$ , whose  $(i, j)^{th}$ -entry is  $x_{i,j}$ , if  $j \leq l_i$  and \* otherwise, and then concatenating the rows.*

*Proof.* The matrix  $A_k$  is a representation of the  $k^{th}$  and  $(k+1)^{th}$  levels of  $T_S$  where the  $B_i^k$ 's (and  $B_{i,j}^{k+1}$ 's) have been replaced by a choice of centres. Since matrices must be rectangular, the case where some  $B_i^k$  and  $B_j^k$  have an unequal number children is handled by inserting a placeholder, \*, into  $A_k$ . Moreover, since the  $\rho_{k+1}$  distance between distinct closed balls is just the  $\rho$  distance between a choice of centres of those balls, a choice of centres in a  $\rho_{k+1}$ -ordering gives the beginning of a  $\rho$ -ordering. By the above proposition, we must select elements from each  $B_i^K$  one after the other, which is achieved by selecting one element from each column in order, for example by concatenating the rows (and then deleting \*'s if necessary).  $\square$

We get the most use out of the construction above if, in selecting a choice of centres for the  $B_{i,j}^{k+1}$ 's, we reuse the previous the choices as much as possible. Suppose for example we have made a choice of centres for the balls of radius  $\gamma_k$  and constructed the matrix  $A_{k-1}$ . At the next iteration, we will need a choice of centres for the balls of radius  $\gamma_{k+1}$ . If  $x_i$  was our choice of representative for  $B_i^k$  and  $x_i \in B_{i,j}^{k+1}$ , we may as well let  $x_i$  be our choice of representative for  $B_{i,j}^{k+1}$ . If we make our choice of centres in this way, then when we concatenate the rows of some  $A_{k-1}$ , we obtain (without loss) the first row of  $A_k$ . We follow this convention in the two examples below.

**Example 10.** Let us use the above to start a  $\rho$ -ordering of  $S = (\mathbb{Z}, |\cdot|_3)$ . We have that  $\Gamma_S = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\}$  and  $T_s$  begins:



We start by finding a  $\rho_0$ -ordering of  $S_{\gamma_0}$ , but this is trivial since  $S_{\gamma_0}$  has only a single element. Let us pick 0 to be our choice on centre for  $B_1^0 = B(0, 1) = \mathbb{Z}$ . As we see from  $T_S$ ,  $S_{\gamma_0}$  is partitioned into 3 closed balls of radius  $\gamma_1 = \frac{1}{3}$ , namely  $3\mathbb{Z}$ ,  $3\mathbb{Z} + 1$ , and  $3\mathbb{Z} + 2$ . A choice of centres is given by 0, 1, and 2, so that  $A_0$  becomes:

$$A_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

To start the  $\rho$ -ordering, concatenate the rows to obtain  $\{0, 1, 2\}$ , and to continue it, make a choice of centres for each of the closed balls of radius  $\gamma_2 = \frac{1}{9}$  partitioning the sets  $3\mathbb{Z} + i$ ,  $i \in 0, 1, 2$ . For example,  $3\mathbb{Z} = 9\mathbb{Z} \cup 9\mathbb{Z} + 3 \cup 9\mathbb{Z} + 6$ , so a choice of centres for  $B_1^1$  is given by  $\{0, 3, 6\}$ . Making choices for the remaining elements, we obtain:

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

To continue the  $\rho$ -ordering we concatenate the rows,  $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ , which also gives the first row of  $A_2$ . The remaining rows are found by partitioning each of the closed balls of radius  $\frac{1}{9}$  and again making a choice of centres:

$$A_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{pmatrix}$$

And so on.

We are able to make two statements following this example. The first is that in starting the  $\rho$ -ordering, the fact that  $S_{\gamma_0}$  had only a single element allowed us to

get started for free. In fact, all compact ultrametric spaces are bounded, so this is always the case.

The second takeaway is that we found the start of a  $\rho$ -ordering of  $S = (\mathbb{Z}, |\cdot|_3)$  was given by taking the integers starting at 0 in their natural order. If we had continued building the ordering, we would have continued to find this. The fact that the natural ordering on the integers is a  $\rho_p$ -ordering, where  $\rho_p$  is the  $p$ -adic metric for any prime  $p$ , is well known (cf. ...), but we give an alternate proof of it here:

**Corollary 6.** *Let  $S$  be the ultrametric space  $(\mathbb{Z}, \rho_p)$ , where  $\rho_p$  is  $p$ -adic metric for any prime  $p$ . The a  $\rho_p$ -ordering of  $S$  can be found by taking the integers, starting at 0, in their natural order.*

*Proof.* We prove the above by induction on  $k$ . First note that for any choice of prime, the elements of  $S_{\gamma_1}$  are the cosets of  $\mathbb{Z}$  modulo  $p$ , so that  $A_0$  has  $p$  columns. Since  $\{0, 1, 2, \dots, p-1\}$  are distributed among each of these cosets, without loss of generality the first row of  $A_0$  is given by  $[0, 1, 2, \dots, p-1]$  in order.

Now suppose that the first row of  $A_k$  is given by  $[0, 1, 2, \dots, n]$  for  $0 \leq k < k+1$ . We show the first row of  $A_{k+1}$ , and therefore the first  $n'$  elements in a  $\rho_p$ -ordering of  $S$ , where  $n'$  is the column dimension of  $A_{k+1}$ , can be obtained as  $[0, 1, 2, \dots, n, n+1, \dots, n']$ . First note that each closed ball of radius  $p^k = \gamma_k$  is in fact a coset of  $\mathbb{Z}$  modulo  $p^k$ , of which there are  $p$ . Then for any  $k$ ,  $A_k$  is a matrix with  $p^k$  columns and  $p$  rows. In particular,  $n = p^k - 1$ . Let  $i \in \{0, 1, \dots, p^k - 1\}$  be arbitrary. Then  $i$  is in exactly one of the cosets of  $\mathbb{Z}$  modulo  $p^k$  and since the first row of  $A_k$  is  $[0, 1, 2, \dots, p^k - 1]$ , it must have been chosen as our representative of this coset. If we split  $p^k\mathbb{Z} + i$  into balls of radius  $p^{k+1}$ , we have

$$p^k\mathbb{Z} + i = \bigcup_{j=0}^{p-1} p^{k+1}\mathbb{Z} + (p^k j + i)$$

since there will be  $p$  elements in the partition, each of which will be equal to  $i$  modulo  $p^k$  and distinct modulo  $p^{k+1}$ . Then, there is a choice of centres such that the  $i^{th}$  column of  $A_k$  is

$$[i, p^k + i, 2p^k + i, \dots, (p-1)p^k + i]^T$$

filling this in for each  $i$ , we see that  $A_k$  can be obtained as:

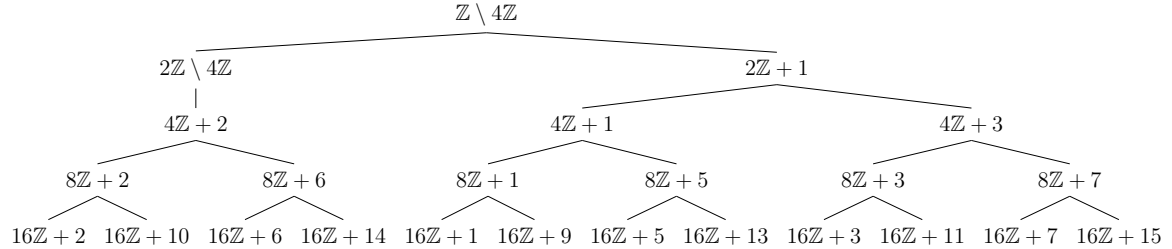
$$A_k = \begin{pmatrix} 0 & 1 & 2 & \dots & p^k - 1 \\ p^k & p^k + 1 & p^k + 2 & \dots & p^k + (p^k - 1) \\ 2p^k & 2p^k + 1 & 2p^k + 2 & \dots & 2p^k + (p^k - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)p^k & (p-1)p^k + 1 & (p-1)p^k + 2 & \dots & (p-1)p^k + (p^k - 1) \end{pmatrix}$$

Concatenating the rows, we see the first row of  $A_{k+1}$  will be

$$[0, 1, 2, \dots, p^k - 1, p^k, \dots, p^{k+1} - 1]$$

as required.  $\square$

**Example 11.** Let us now see an example where there is an uneven number of children between the vertices on a given level. Suppose  $S = \mathbb{Z}_{(2)} \setminus 4\mathbb{Z}$ . In this case, we have that  $\Gamma_S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$  and  $T_s$  begins:



Choosing centres for the partition of  $\mathbb{Z}$  into closed balls of radius  $\frac{1}{2}$ , we have:

$$A_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have taken  $S$  to be the complement of  $4\mathbb{Z}$  in  $\mathbb{Z}_{(2)}$ , so  $B(0, \gamma_1)$  has only one child, since  $2\mathbb{Z} \setminus 4\mathbb{Z} = 4\mathbb{Z} + 2$ , while  $B(1, \gamma_1)$  has two. Making a choice of centres, we have:

$$A_1 = \begin{pmatrix} 2 & 1 \\ * & 3 \end{pmatrix}$$

We concatenate the rows, skipping over  $*$ , and again make a choice of centres for the closed balls of radius  $\frac{1}{8}$ :

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \end{pmatrix}$$

One more iteration yields:

$$A_2 = \begin{pmatrix} 2 & 1 & 3 & 6 & 5 & 7 \\ 10 & 9 & 11 & 14 & 13 & 15 \end{pmatrix}$$

So that a  $\rho_2$ -ordering of  $S = \mathbb{Z}_{(2)} \setminus 4\mathbb{Z}$  starts:  $\{2, 1, 3, 6, 5, 7, 10, 9, 11, 14, 13, 15, \dots\}$ .

In the two propositions above, there was notational difficulty that arose when there was an unequal number of children between the vertices on a given level of  $T_s$ . This difficulty is, in fact, more than a notational inconvenience, and the situation simplifies considerably when it is not the case. We are far from the first to observe this. Amice noted this as far back as her 1964 paper, and it has been observed more recently in several papers by Fares and colleagues. The following section discusses this in more detail, first by supplying some preliminary lemmas and then showing how calculations are simplified in this setting.

## Semi-regularity

In this section, we restrict to the case where in the tree  $T_s$ , for  $S$  some compact subset of an ultrametric space, every vertex on a given level has the same number of children. In this case, we can attach another sequence to  $S$ , which we call the  $\alpha$ -sequence of  $S$  and which describes, for each level  $k \in \mathbb{N}$ , the size of the partitions on that level. We develop some preliminary lemmas, which we then use to derive formulae for this special case.

**Definition 22.** Let  $S$  be as above, a compact subset of an ultrametric space  $(M, \rho)$ . We say that  $S$  is **semi-regular** if  $T_{B_i^k} \cong T_{B_j^k}$ ,  $\forall k \in \mathbb{N}$  and  $i, j \in \beta(k)$ , and where the isomorphism is understood as an isomorphism of trees. That is,  $S$  is semi-regular if each ball of radius  $\gamma_k$  breaks into the same number of balls of radius  $\gamma_{k+1}$ , for all  $k$ . If there exists an  $n \in \mathbb{N}$  such that  $T_{B_i^N} \cong T_{B_j^N}$  for all  $N \geq n$ , i.e. each ball of radius

$\gamma_N$  breaks into the same number of balls of radius  $\gamma_{N+1}$  for  $N \geq n$ , then we say  $S$  is **eventually semi-regular**.

**Definition 23.** Suppose  $S$  is a compact subset of an ultrametric space and  $S$  is semi-regular. The  $\alpha$ -**sequence** of  $S$  is the sequence given by  $\alpha(k) = \frac{\beta(k+1)}{\beta(k)}$ , which is in  $\mathbb{N}$  for each  $k$ . That is, if  $B_i^k$  is an element of  $S_{\gamma_k}$ , then  $\alpha(k)$  is equal to the number of children of  $B_i^k$  in  $T_s$ . Since  $S$  is semi-regular, this number does not depend on  $i$ .

**Lemma 4.**  $\lfloor \frac{n}{q} \rfloor$  counts the numbers strictly less than  $n$  that are congruent to  $n \pmod q$ .

*Proof.* (sketch) Every multiple of  $q$  produces exactly one of the numbers from 1 to  $q$  and exactly one of those is the residue class of  $n$  modulo  $q$ . The remainder is the residue class of  $n$  itself and since we only want the numbers strictly less than  $n$ , we ignore this by taking the floor.  $\square$

**Lemma 5.**

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n + kb}{ab} \rfloor$$

for  $n, a, b \in \mathbb{N}$ . In particular,

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{2b} \rfloor = \lfloor \frac{n + b}{2b} \rfloor$$

for  $n, b \in \mathbb{N}$ .

*Proof.*

$$\begin{aligned} \lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor &= \lfloor a \cdot \frac{n}{ab} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=0}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor - \lfloor \frac{n}{ab} \rfloor \quad (*) \\ &= \sum_{k=1}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n + kb}{ab} \rfloor \end{aligned}$$

where the final step in  $(*)$  is due to Hermite's identity:  $\lfloor nx \rfloor = \sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor$ , for  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ .  $\square$

**Lemma 6.** If  $S$  is semi-regular and  $\delta$  denotes the canonical  $\rho$ -ordering of  $S$ , that is, a  $\rho$ -ordering formed by pulling from left to right in  $T_s$ , then

$$\rho(\delta(n), \delta(m)) = \gamma_k$$



if, and only if,

$$n = m \pmod{\beta(k)} \text{ and } n \neq m \pmod{\beta(k+1)}$$

*Proof.* Since  $S$  is semi-regular, every sequence of  $\beta(k)$  terms in  $\delta$  will be from each of distinct elements of  $S_{\gamma_k}$  (for any  $k$ ). Moreover, since  $\delta$  is a canonical  $\rho$ -ordering, we always pull from the elements of  $S_{\gamma_k}$  in the same order. Then  $\delta(n)$  and  $\delta(m)$  are descendents of some  $B_j^k$  if, and only if,  $n = m \pmod{\beta(k)}$ . Then the result follows since  $\rho(\delta(n), \delta(m)) = \gamma_k$  if, and only if,  $B_i^k$  for some  $i \in 1, \dots, \beta(k)$  is the join of  $B_i^n \ni \delta(n)$  and  $B_{i'}^m \ni \delta(m)$ . □

**Proposition 24.** *If  $S$  is a semi-regular ultrametric space,  $\sigma$  is the characteristic sequence of  $S$ ,  $\beta$  is the structure sequence of  $S$ , and  $\alpha$  is the sequence describing the semi-regularity, then*

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

*Proof.* The exponent of  $\gamma_k$  in the  $n^{\text{th}}$  term of the characteristic sequence is the number of  $m$  strictly less than  $n$  such that  $\rho(\sigma(n), \sigma(m)) = \gamma_k$ . By the lemma above, this is the number of  $m < n$  such that  $m = n \pmod{\beta(k)}$  and  $m \neq n \pmod{\beta(k+1)}$ , which by the previous lemma is  $\lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor$ . Then we have:

$$\begin{aligned} & v_{\gamma_k}(\sigma(n)) \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k)\alpha(k)} \rfloor, \text{ because } S \text{ is semi-regular} \\ &= \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor \end{aligned}$$

□

**Example 12.** Consider the ultrametric space  $(\mathbb{Z}, |\cdot|_p)$  for any prime  $p$ . Then  $\beta(k) =$

$p^k$  and  $\alpha(k) = p$  for any  $k$ . The above gives

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor$$

and since  $\gamma_k = p^{-k}$ ,  $\forall k$ , we have

$$\begin{aligned} v_{\frac{1}{p}}(\sigma(n)) &= \sum_{k=1}^{\infty} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \sum_{k=1}^{\lceil \log_p(n) \rceil} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor + 2\lfloor \frac{n}{p^2} \rfloor - 2\lfloor \frac{n}{p^3} \rfloor + \dots + \lceil \log_p(n) \rceil \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\ &= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \dots + \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\ &= \sum_{k=1}^{\lceil \log_p(n) \rceil} \lfloor \frac{n}{p^k} \rfloor \\ &= \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor \\ &= v_p(n!) \end{aligned}$$

since  $\lfloor \frac{n}{p^k} \rfloor = 0$  if  $p^k > n \iff \log(p^k) > \log(n) \iff k > \log_p(n)$ , so that we are able to recover the well-known Legendre's formula.

Some comment about how the above starts to build up a toolkit independent of algebraic structure in  $S$ .

**Corollary 7.** *If  $G$  is a compact ultrametric group, then  $G$  is semi-regular.*

*Proof.* Since  $G$  is a group, each ball centred at 0 is in fact a subgroup of  $G$ . Then each set of elements of  $S_{\gamma_k}$  is a collection of cosets of  $G/B(0, \gamma_k)$ . Since  $G$  is assumed to be compact,  $G/B(0, \gamma_k)$  is finite and so Lagrange's theorem implies the result.  $\square$

Semi-regularity gives a notion of translation invariance to ultrametric spaces that are not themselves groups. In the previous section, we observed that spaces which

admitted both translation invariance and scaling had valutive capacity that could be computed explicitly via the subadditivity formula. Is there a way to define this for ultrametric which are not normed vector spaces?

**Definition 24.** Let  $S$  be a semi-regular compact subset of an ultrametric space. If there exists a  $q \in \mathbb{N}$  such that  $\alpha(n) = q$ , for all  $n$ , then  $S$  is said to be **regular**. If there exists a  $q$  and  $N$  in  $\mathbb{N}$  such that  $\alpha(n) = q$ , for all  $n \geq N$ , then  $S$  is said to be **eventually regular**.

Alt.:

**Definition 25.** Let  $S$  be a semi-regular compact subset of an ultrametric space. If there exists a  $q_1, \dots, q_m \in \mathbb{N}$  such that  $\alpha(n) = q_i$  if, and only if,  $n = i \pmod m$ , that is,  $\alpha$  has an infinitely-repeating finite subsequence of length  $m$ , then we say  $S$  is **periodic**. If  $m = 1$ , then  $S$  is **regular**.

$S$  is regular just in case the  $\alpha$ -sequence of  $S$  is constant. In what follows, we assume that  $S$  is regular, but the results carry over to the case where  $S$  is periodic with the necessary adjustments.

If  $S$  is regular with  $\alpha(k) = q$ , for all  $k$ :

$$\begin{aligned}
v_{\gamma_k}(\sigma(n)) &= \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor \\
v_{q^{c_k}}(\sigma(n)) &= \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor \\
v_q(\sigma(n)) &= \sum_{k=0}^{\infty} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor) \\
v_q(\sigma(n)) &= \sum_{k=0}^{\lceil \log_q(n) \rceil} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor) \\
&= c_0 n - c_0 \lfloor \frac{n}{q} \rfloor + c_1 \lfloor \frac{n}{q} \rfloor - c_1 \lfloor \frac{n}{q^2} \rfloor + c_2 \lfloor \frac{n}{q^2} \rfloor - c_2 \lfloor \frac{n}{q^3} \rfloor \dots + \lceil \log_q(n) \rceil \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor - \lceil \log_q(n) \rceil \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor \\
v_q(\sigma(n)) &= \sum_{k=0}^{\lceil \log_q(n) \rceil} (c_{k+1} - c_k) \cdot (\lfloor \frac{n}{q^{k+1}} \rfloor)
\end{aligned}$$

Note that the coefficient  $(c_{k+1} - c_k)$  is always positive: since  $\Gamma$  is (strictly) decreasing and  $\gamma_k = q^{c_k}$ , the sequence  $c_k$  is also be strictly decreasing (since  $\gamma_k = q^{c_k}$

implies  $\log_q(\gamma_k) = c_k$  and  $\log$  is a monotone function). The above implies that:

$$\omega(S) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\lceil \log_q(n) \rceil} (c_{k+1} - c_k) \cdot (\lfloor \frac{n}{q^{k+1}} \rfloor)$$

## Chapter 4

### Product spaces of $\mathbb{Z}_p$

As a first point of departure, a natural space to consider is the product space of ultrametric spaces, for example  $\mathbb{Z}^n$  (or  $\mathbb{Z}_p^n$  or  $\mathbb{Q}_p^n$ ), for some  $n > 1$ . If we restrict our attention to bounded subsets, then a natural candidate for an ultrametric on the product space is the  $L_\infty$  metric, given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sup_i \{\rho(x_i, y_i)\}$$

where  $\rho$  is the metric from the base space. In fact, since we have only defined valutive capacity for compact subsets of an ultrametric spaces, there is no loss of generality by restricting our metric to bounded spaces. We also see that no problems arise in letting both  $M$  and  $\rho$  vary between components of the space, as long as each  $M_i$  remains bounded and each  $\rho_i$  is an ultrametric.

**Proposition 25.** *Let  $(M_i, \rho_i)$  for  $i$  in some finite or countably infinite index set,  $I$ , be a collection of metric spaces and suppose  $\rho_i$  is a bounded ultrametric for all  $i$ . Then  $(M, \rho_\infty)$  is an ultrametric space, where  $M = M_1 \times M_2 \times M_3 \times \dots$  and  $\rho_\infty$  is the  $L_\infty$  metric described above.*

*Proof.* Let  $(M, \rho_\infty)$  be the product of ultrametric spaces as above and let  $x$  and  $y$  be two points in the space. Clearly,  $\rho_\infty(x, y) \geq 0$  since each  $\rho_i(x_i, y_i) \geq 0$ , and  $\rho_\infty(x, y) = 0 \iff \rho_i(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$ . The fact that  $\rho_\infty$  is symmetric is also an easy consequence of the fact that each  $\rho_i$  is symmetric since  $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$  implies  $\sup_i \{\rho_i(x_i, y_i)\} = \sup_i \{\rho_i(y_i, x_i)\}$ . To see that  $\rho_\infty$  is an

ultrametric, note that if  $z = z_i$  is any other point of  $M$ , then

$$\begin{aligned}
\rho_\infty(x, y) &= \sup_i \{\rho_i(x_i, y_i)\} \\
&\leq \sup_i \{\max(\rho_i(x_i, z_i), \rho_i(y_i, z_i))\} && \text{since each } \rho_i \text{ is an ultrametric} \\
&\leq \max(\sup_i \{\rho_i(x_i, z_i)\}, \sup_i \{\rho_i(y_i, z_i)\}) && * \\
&= \max(\rho_\infty(x, z), \rho_\infty(y, z))
\end{aligned}$$

\* Let  $M = \max(\sup_i(\{a_i\}, \sup_j(\{b_j\})))$ , then  $M \geq a_i, \forall i$  and  $M \geq b_i, \forall i$ , so  $M \geq \max(a_i, b_i), \forall i$ , hence  $M \geq \sup_i(\max(a_i, b_i))$ .  $\square$

We first show that translation invariance carries over into product spaces under the expected conditions.

**Proposition 26.** *Suppose  $(M, \rho_\infty)$  is the product of ultrametric spaces  $(M_i, \rho_i)$  and each  $M_i$  is a topological group with operation  $+$ . Then  $\rho_\infty$  is (left) translation invariant if each  $\rho_i$  is, in which case valutive capacity is also (left) translation invariant.*

*Proof.* Let  $(M, \rho_\infty)$  be as above. Suppose also that

$$\rho_i(x_i, y_i) = \rho_i(s_i + x_i, s_i + y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each  $\rho_i$  is (left) translation invariant. Then,

$$\rho_\infty(s + x, s + y) = \sup_i \{\rho_i(s_i + x_i, s_i + y_i)\} = \sup_i \{\rho_i(x_i, y_i)\} = \rho_\infty(x, y).$$

so that  $\rho_\infty$  is translation invariant. Proposition ?? implies valutive capacity is as well.  $\square$

In the next proposition, we show that scaling carries over to product space as well, although the conditions are now more restrictive. In contrast to the proposition above, here we do not allow the spaces to vary between components.

**Proposition 27.** *Let  $(m, \rho_N)$  be an ultrametric space, where  $\rho_N$  is the metric induced by some norm  $N$ . Let  $(M, \rho_\infty)$  be the ultrametric space formed by taking products of  $m$ , along with the  $L_\infty$  metric defined above. Then if  $\rho_N$  is multiplicative on  $m$ ,*

$\rho_\infty$  is multiplicative on  $M$ , in the sense that  $\rho_\infty(cx, cy) = |c|_{\rho_N} \rho_\infty(x, y)$ , for  $c = (c, c, c, \dots)$ ,  $x, y \in M$ .

*Proof.* Let  $M, \rho$ , and  $\rho_\infty$  be as above. Then,

$$\begin{aligned}
 & \rho_\infty(cx, cy) \\
 &= \sup_i \{ \rho_N(c_i x_i, c_i y_i) \} \\
 &= \sup_i \{ |c|_{\rho_N} \rho_N(x_i, y_i) \} \\
 &= |c|_{\rho_N} \sup_i \{ \rho_N(x_i, y_i) \} \\
 &= |c|_{\rho_N} \rho_\infty(x, y)
 \end{aligned}$$

□

**Corollary 8.** Let  $S$  be a subset of  $(M, \rho_\infty)$ , where  $M$  is the product of an ultrametric space  $(m, \rho_N)$ , which is itself a normed vector space with a multiplicative norm inducing  $\rho_N$ . If  $c = (c, c, c, \dots)$  is an element of  $M$  with constant value on each component, then  $\omega(cS) = |c|_{\rho_N} \omega(S)$ .

*Proof.* Note that if  $\{a_j\}_{j=0}^\infty$  is a  $\rho_\infty$  ordering of  $S$ , then  $\{ca_j\}_{j=0}^\infty$  is a  $\rho_\infty$  ordering of  $cS$ . □

We now introduce two examples, whose explorations take up a large portion of the following section.

**Example 13.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  be the metric space with elements  $\{(x, y) \mid x, y \in \mathbb{Z}_p\}$  and metric  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2))$ , where  $\rho_p$  is the p-adic metric for some fixed prime  $p$ . Since  $\rho_p$  is translation invariant and multiplicative in  $\mathbb{Z}_p$ , valutive capacity is also translation invariant and multiplicative in  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ .

**Example 14.** Let  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$  be the metric space with elements  $\{(x, y) \mid x \in \mathbb{Z}_{p_1}, y \in \mathbb{Z}_{p_2}\}$  for two distinct primes,  $p_1 \neq p_2$ , and metric  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_{p_1}(x_1, y_1), \rho_{p_2}(x_2, y_2))$ , where both  $\rho_{p_i}$  are p-adic metrics. Since each  $\rho_{p_i}$  is translation invariant in  $\mathbb{Z}_{p_i}$ , valutive capacity will be translation invariant in  $(\mathbb{Z}_{p_1} \times$

$\mathbb{Z}_{p_2}, \rho_{p,\infty}$ ); however, unlike the case of  $p_1 = p_2$ , this space does not have a multiplicative property that allows for scaling.

### **$n$ -fold products**

What is the valutive capacity of  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  from the example above? Suppose  $p = 2$ . Using translation invariance, scaling and subadditivity, we can compute the result by first noting that we can write  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always  $1 = \text{diam}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ , subadditivity implies that

$$\begin{aligned} & \frac{1}{\log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} \\ = & \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))} \\ = & \frac{4}{\log(\|2\|_2 * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{\log(\frac{1}{2} * \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{\log(\frac{1}{2}) + \log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} \end{aligned}$$

Taking logs base 2, we have that

$$\omega(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^{\frac{-1 + \log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} 2^{\frac{\log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}{4}} = 2^{\frac{-1}{4}} (2^{\log_2(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))})^{\frac{1}{4}} = 2^{\frac{-1}{4}} \omega(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\frac{1}{4}}$$

so that  $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$  is a solution of the equation  $x^4 - \frac{x}{2}$ , for which there is a single real positive root, given by  $2^{-1/3}$ .

To compute the valutive capacity for a 2-fold product for an arbitrary prime  $p$ , note that we can always decompose  $\mathbb{Z}_p \times \mathbb{Z}_p$  into a union of  $p^2$  sets each of the form  $\{p\mathbb{Z}_p + s \times p\mathbb{Z}_p + t\}$  for  $s, t \in (0, \dots, p-1)$ , and the pairwise distance between these sets will always be  $1 = \text{diam}(\mathbb{Z}_p \times \mathbb{Z}_p)$  (to see this, either note that we can always find co-prime elements, or note that each set is an closed ball of radius  $1/p$  centred at  $(s,t)$  and so the distance between them must be greater than  $1/p$ , and 1 is the only possible distance greater than  $1/p$  in  $\mathbb{Z}_p \times \mathbb{Z}_p$ ). Then, we combine our tools as



before to obtain the equation,

$$\frac{1}{\log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(\|p\|_p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(1/p * \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, taking logs base  $p$ , we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p) = p^{\frac{-1}{p^2}} \omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{\frac{1}{p^2}}$$

So that  $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$  is a solution of the equation  $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p})$  over  $\mathbb{R}$ . Since  $\mathbb{R}$  is a division ring, this means the positive solutions are given by solving  $x^{p^2-1} - \frac{1}{p}$ . Solutions of this equation are of the form  $p^{\frac{-1}{p^2-1}}$  times a  $p^2 - 1$  root of unity, and so there is exactly one positive, real solution, namely  $p^{\frac{-1}{p^2-1}}$  itself. Then the valulative capacity of the entire product space  $\mathbb{Z}_p \times \mathbb{Z}_p$  is  $p^{\frac{-1}{p^2-1}}$ . In fact, from here it is not hard to see that by taking the  $n$ -fold product, we would end up with the same equation except that the exponent of  $p$  would become  $n$  rather than 2. We arrive at the following result:

**Proposition 28.** *Let  $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$  be the ultrametric space with points equal to the  $n$ -fold product of  $\mathbb{Z}_p$  (for  $n < \infty$ ) for some fixed prime  $p$ . The valulative capacity of  $M$  is  $(\frac{1}{p})^{\frac{1}{p^n-1}}$ .*

*Proof.* Above. □

Taking  $n = 1$ , we see that this agrees with the valulative capacity of  $\mathbb{Z}_p$  computed in the last chapter.

What about  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$  for distinct primes? These spaces do not admit a scaling property, so the same toolset is not available. They are however semi-regular, so we know that

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Suppose  $p_1 = 2$  and  $p_2 = 3$ . Recall that the  $\alpha$  sequence of  $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$  counts the number of closed balls of radius  $\gamma_{k+1}$  partitioning a closed ball of radius  $\gamma_k$ . In this case,  $\Gamma_S$  is the non-positive powers of 2 or 3 sorted into decreasing order, so

that  $\Gamma_S$  starts  $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}, \dots\}$  and  $\alpha(S)$  starts  $\{6, 2, 3, 2, 2, 3, 2, 3, 2, \dots\}$ . The  $\beta$  sequence of  $S$ , which counts the number of distinct balls of a fixed radius, then starts  $\{6, 12, 36, 72, 144, \dots\}$ .

We know that the capacity of  $S$  will be a product of some negative power of 2 and some negative power of 3. From the lemma ??, we know that when  $\alpha(k) = 2$ , we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{2 \cdot \beta(k)} \rfloor$$

and when  $\alpha(k) = 3$ , we have

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{3 \cdot \beta(k)} \rfloor + \lfloor \frac{n + 2 \cdot \beta(k)}{3 \cdot \beta(k)} \rfloor$$

We also know that if  $\alpha(k) = 2$ , then  $\gamma_k$  must be a (negative) power of 2, and likewise if  $\alpha(k) = 3$ , then  $\gamma_k$  is a power of 3.

Let us first explore the exponent of 2 in  $\sigma(n)$ . We start by noting that if  $\gamma_k$  is some  $2^{-i}$ , then

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

since there will be a copy of 2 in  $\beta(k)$  for every occurrence of 2 in  $\alpha(0), \dots, \alpha(k)$ , which is also what  $i$  counts. So then, the exponent of  $\frac{1}{2}$  in the  $n^{th}$  characteristic sequence of  $S$  is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

What can we say about  $j$ , the exponent of 3?

**Lemma 7.** *Let  $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$  and consider the  $k^{th}$  element of the  $\beta$  sequence of  $S$ ,  $\beta(k) = 2^i \cdot 3^j$ . If  $k$  is such that  $\gamma_k = 2^{-i}$  for some  $i$ , then  $j$  counts the numbers  $a \in \mathbb{Z}_{\geq 0}$  such that  $3^a < 2^i$ .*

*Proof.*  $\Gamma_S$  is strictly monotone decreasing and each  $\gamma_k$  is equal to a non-positive power of 2 or 3. If  $\gamma_k = 2^i$ , then all non-positive powers of 3 and 2 which are greater than  $2^i$  must be equal to some  $\gamma_j$ ,  $0 \leq j < k$ . That is,  $2^i$  only appears in the  $\Gamma_S$  sequence after all larger powers of 2 and 3 have been exhausted. Since we are only

considering the case  $\gamma_k$  is a power of 2, this includes all of the smaller powers of 3.  $\square$

Now note that

$$3^a < 2^i \iff \log_2(3^a) < \log_2(2^i) \iff a \cdot \log_2(3) < i$$

So now we are reduced to counting the number of non-negative integers  $a$  that satisfy the above for a given  $i$ . The number of such  $a$ 's will simply be the value of the largest  $a$  plus 1 since  $a$  satisfying the relation implies all  $0 \leq a' \leq a$  solve the relation. Then, we are in fact reduced to finding the largest  $a \in \mathbb{Z}$  that satisfies  $a < \frac{i}{\log_2(3)}$ , but this is exactly  $\lfloor \frac{i}{\log_2(3)} \rfloor$ . This in turn gives  $j = \lfloor \frac{i}{\log_2(3)} \rfloor + 1 = \lceil \frac{i}{\log_2(3)} \rceil$ , since  $\frac{i}{\log_2(3)}$  is never an integer. We now revisit our expression for the exponent of  $\frac{1}{2}$  and substitute our new found value for  $j$ :

$$\sum_{i=1}^{\infty} i \cdot \left\lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor = \sum_{i=1}^{\infty} i \cdot \left( \left\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor - \left\lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor \right) \quad (4.1)$$

A symmetric argument shows that exponent of 3 in the  $i^{\text{th}}$  elements of the  $\rho$ -sequence of  $S$  is

$$\sum_{i=1}^{\infty} i \cdot \left\lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor = \sum_{i=1}^{\infty} i \cdot \left( \left\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor - \left\lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor \right) \quad (4.2)$$

**Conjecture 1.** *Products of  $\mathbb{Z}_{p_i}$  for distinct primes have transcendental valiative capacity.*

The aperiodicity of the sequence  $\lceil \frac{i}{\log_2(3)} \rceil$  over  $i$  leads us to believe, but not prove, that each of the sums in (4.1) and (4.2) are irrational.

We end this section with an observation on the asymptotic behavior of capacity in these spaces. For a fixed prime  $p$ ,  $(\frac{1}{p})^{\frac{1}{p^n-1}}$  is an monotone, increasing sequence in  $n$  with  $\lim_{n \rightarrow \infty} (\frac{1}{p})^{\frac{1}{p^n-1}} = 1$ . For fixed  $n$ , the sequence in  $p$  is also montone, increasing, again with  $\lim_{p \rightarrow \infty} (\frac{1}{p})^{\frac{1}{p^n-1}} = 1$ . In both cases, the limiting value is equal to the diameter of space. Indeed, we can observe that the sequence

$\{(0, 0, \dots), (1, 0, \dots), (0, 1, \dots), \dots\}$ , in which the first element has only zeros and the  $n$ -th element has a single 1 in the  $(n - 1)$ -th component, is a  $\rho$ -ordering for both  $(\mathbb{Z}_p \times \mathbb{Z}_p \times \dots, \rho_{p,\infty})$  and  $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \dots, \rho_{P,\infty})$ , since the distance between elements in this sequence (in either metric space) is always 1. If we could show that these spaces are compact, this would give a valuative capacity of  $\lim_{n \rightarrow \infty} (1^n)^{(1/n)} = 1$  for both spaces. We explore this more in the following section.

## Product topology

In considering the product space of ultrametric spaces, we may wonder whether the chosen metric also gives back the product topology on the space. For products formed by taking some finite number of copies, the answer is positive. We give the necessary background and show this fact, adapting the proof in Munkres (20.3) to the case of ultrametric spaces.

**Definition-Proposition 1.** (*Munkres*) Suppose  $X_i$ , for  $i$  in some index set  $I$ , is a family of topological spaces. Let  $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$  be the map given by projection onto the  $j$ -th component, that is  $\pi_j(x) = \pi_j((x_i)_{i \in I}) = x_j$ . For each  $j \in I$ , let  $\mathcal{S}_j$  be the collection

$$\mathcal{S}_j = \{\pi_j^{-1}(U_j) \mid U_j \text{ open in } X_j\}$$

Let  $\mathcal{S}$  be the union of the  $\mathcal{S}_j$  over  $j \in I$ ,  $\mathcal{S} = \cup_{j \in I} \mathcal{S}_j$ . Then  $\mathcal{S}$  is a subbasis that generates a topology on  $\prod_{i \in I} X_i$  called the **product topology**.

The basis,  $\mathcal{B}$ , generated by  $\mathcal{S}$  in the definition above is the set of all finite intersections of elements in  $\mathcal{S}$ . That is  $B \in \mathcal{B}$  if there exists  $S_1, S_2, \dots, S_n$  in  $\mathcal{S}$  such that  $B = S_1 \cap S_2 \cap \dots \cap S_n$ . A useful description of the basis for the product topology also appears in Munkres, as below:

**Proposition 29.** (*Munkres 19.2*) Suppose  $X_i$ , for  $i$  in some index set  $I$ , is a family of topological spaces and denote by  $\mathcal{B}_i$  the basis for the topology on  $X_i$ . Let

$$\mathcal{B}_P = \prod_{i \in I} B_i, \text{ for } B_i \in \mathcal{B}_i \text{ and } B_i = X_i \text{ for all but finitely-many } i \in I.$$

then  $\mathcal{B}_P$  is a basis for the product topology on  $\prod_{i \in I} X_i$ .

We can now show that the topology induced by the  $L_\infty$  metric described above agrees with the product topology for finite products.

**Proposition 30.** *Let  $M = (M_1 \times M_2 \times \dots \times M_n, \rho_\infty)$  be a finite product of bounded, ultrametric spaces and let  $\rho_\infty$  be the metric described above. Then the topology induced by  $\rho_\infty$  coincides with the product topology on  $M_1 \times M_2 \times \dots \times M_n$ .*

*Proof.* Let  $\mathcal{T}_{\rho_\infty}$  be the topology on  $M_1 \times M_2 \times \dots \times M_n$  induced by  $\rho_\infty$  and let  $\mathcal{B}_{\rho_\infty}$  be the basis for this topology. Let  $\mathcal{T}_P$  be the product topology with basis  $\mathcal{B}_P$ . We show  $\mathcal{T}_P \subset \mathcal{T}_{\rho_\infty}$  and vice versa. For this, it is equivalent (Munkres 13.3) to show that for  $z \in M_1 \times M_2 \times \dots \times M_n$  and  $B \in \mathcal{B}_P$  containing  $z$ , there is a basis element  $B' \in \mathcal{B}_{\rho_\infty}$  such that  $z \in B' \subset B$ , and vice versa.

So let  $z \in M_1 \times M_2 \times \dots \times M_n$  and suppose  $B \in \mathcal{B}_P$  contains  $z$ . Since  $B$  is in  $\mathcal{B}_P$ ,  $B$  is of the form  $B_{r_1}(z_1) \times B_{r_2}(z_2) \times \dots \times B_{r_n}(z_n)$  (since the choice of centres is arbitrary in an ultrametric spaces, we may choose the components of  $z$  as the centres without loss of generality). Let  $r = \min\{r_i\}$  for  $i \in 1, \dots, n$ . Then let  $B'$  be the ball  $B_r(z)$  in  $\mathcal{B}_{\rho_\infty}$ . Clearly,  $z \in B_r(z)$  and since  $r \leq r_i, \forall i$ ,  $B_r(z) = B_r(z_1) \times B_r(z_2) \times \dots \times B_r(z_n) \subset B_{r_1}(z_1) \times B_{r_2}(z_2) \times \dots \times B_{r_n}(z_n) = B$ .

Conversely, suppose  $A \in \mathcal{B}_{\rho_\infty}$  and let  $y \in A$ . To find  $A' \in \mathcal{B}_P$  such that  $y \in A'$  and  $A' \subset A$ , simply note that  $A$  itself is in  $\mathcal{B}_P$ .

□

We are now naturally left to ask whether the product topology on *infinite* products of ultrametric spaces coincides with the  $L_\infty$  metric. In this case, as in the analogous case of infinite copies of  $\mathbb{R}$  and a uniform metric, the answer is negative (at least in general). Fortunately, the metric that realizes the product topology on

infinite copies of  $\mathbb{R}$  can be adapted to the case of ultrametric spaces. We adapt to the proof of Munkres (20.5) to the case of infinite products of ultrametric spaces.

**Proposition 31.** *Suppose  $\mathbf{M} = M_1 \times M_2 \times \dots$  is an infinite collection of metric spaces, each with an ultrametric  $\rho_i$  which is bounded by 1, that is suppose  $\rho_i(x_i, y_i) \leq 1$ , for all  $x_i, y_i \in M_i$  and for all  $i$ . Define a metric  $d$  on  $\mathbf{M}$  as follows:*

$$d(\mathbf{x}, \mathbf{y}) = \sup_i \left\{ \frac{\rho_i(x_i, y_i)}{i} \right\}$$

*Then  $d$  is an ultrametric and induces the product topology on  $\mathbf{M}$ .*

*Proof.* We see that  $d$  inherits symmetry, injectivity and non-negativity from the requirement that each  $\rho_i$  is a metric, just as  $\rho_\infty$  did. To see that  $d$  satisfies the strong triangle inequality, define a new metric  $\rho'_i$  by  $\rho'_i(x, y) = \frac{\rho_i(x_i, y_i)}{i}, \forall i$ . Then  $\rho'_i$  is an ultrametric, since  $\rho_i(x, y) \leq \max(\rho_i(x, z), \rho_i(y, z))$  implies  $\frac{\rho_i(x, y)}{i} \leq \max(\frac{\rho_i(x, z)}{i}, \frac{\rho_i(y, z)}{i})$  for any  $i \in \mathbb{N}$ . Then we can view  $d$  as the  $L_\infty$  metric on the spaces  $(M_i, \rho'_i)$ , and so  $d$  will be an ultrametric as shown in the first proposition of this section.

Now we show  $d$  induces the product topology. We first show that metric topology induced by  $d$  is finer than the product topology. Let

$$B = B_r^{\mathbf{M}}(\mathbf{z}) = B_r^{M_1}(z_1) \times B_r^{M_2}(z_2) \times B_r^{M_3}(z_3) \times \dots$$

be a basis open in the metric topology. We must find a basis open  $B' \ni z$  in the product topology such that  $B' \subseteq B$ . Let  $N \in \mathbb{N}$  be such that  $\frac{1}{N} < r$ . Then let  $B'$  be the basis open element

$$B' = B_r^{M_1}(z_1) \times B_r^{M_2}(z_2) \times \dots \times B_r^{M_N}(z_N) \times M_{N+1} \times M_{N+2} \times \dots$$

in the product topology. Suppose  $\mathbf{y} \in B'$ . We must show  $\mathbf{y} \in B$ , i.e.,  $d(\mathbf{z}, \mathbf{y}) < r$ . Note that for all  $i \geq N$ ,

$$\frac{\rho_i(z_i, y_i)}{i} \leq \frac{1}{N}$$

which means

$$d(\mathbf{z}, \mathbf{y}) = \sup_i \left\{ \frac{\rho_i(z_i, y_i)}{i} \right\} \leq \max \left\{ \frac{\rho_1(z_1, y_1)}{1}, \frac{\rho_2(z_2, y_2)}{2}, \dots, \frac{\rho_N(z_N, y_N)}{N}, \frac{1}{N} \right\}$$

and since  $N$  was chosen so that  $\frac{1}{N} < r$  and  $B'$  was chosen to have balls of radius  $r$  in the first  $N$  components, we must have  $d(\mathbf{z}, \mathbf{y}) < r$ .

Conversely, ... □

From now on, we refer to the metric  $d$  above as the **product metric**. An important consequence of the fact that  $d$  achieves the product topology is that Tychonoff's theorem then guarantees that product spaces formed with this metric will be compact, infinite or otherwise. As a result, we can now ask directly about the valutive capacity of some infinite product spaces. We consider two examples.

**Example 15.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \dots, d)$  be the metric space formed by taking the product of  $(\mathbb{Z}_p, \rho_p)$  for some fixed prime  $p$  and let  $d$  be the product metric.

**Example 16.** Let  $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \dots, \rho_{P,\infty})$  be the metric space formed by taking the product of  $(\mathbb{Z}_p, \rho_p)$  for every prime  $p$  and let  $d$  be the product metric.

So far we have two methods for computing valutive capacity. Either we can find a useful decomposition that allows us to apply the subadditivity formula, or we can find a  $\rho$ -ordering and then take the limit of its corresponding  $\rho$ -sequence.

## Conclusion

In this section, we considered the notion of valutive capacity in product spaces, that is, spaces formed by taking copies of ultrametric spaces. In the following sections, we consider valutive capacity in spaces formed by adding points, that is extension fields, or by both taking copies and adding (a distinguished) point, as in projective spaces. For these purposes, it will be more productive to start working over the field  $\mathbb{Q}_p$ , instead of  $\mathbb{Z}_p$ .

## Appendix A

## Appendix



code/Padic\_Product\_Ordering.txt

```

1 with(LinearAlgebra):
2 with(combinat, cartprod):
3 with(padic):
4
5 ComputePadicProductOrdering := proc (m, components)
6 # Given a list of primes, p_1,...,p_n, compute the first m terms of a
   p-infinity ordering of  $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n}$ , where
    $p\text{-infinity}(x,y) = \max(p\text{-j}(x\text{-j}, y\text{-j}))$  and  $p\text{-j}$  is the  $p\text{-j}$ -adic metric.
7 # arg m; an integer indicating the number of elements of the ordering to
   return
8 # arg components; a list of prime numbers indicating the components of
   the product space
9 # return; a matrix where each row is an element in the product space and
   the i-th row is the i-th element in an ordering
10
11 local numberOfComponents, co_primes, i, n, T, M, v, distances, j,
   M1, newBlock;
12
13 #will end up with one column per component in the product space
14 numberOfComponents := nops(components);
15
16 #the ordering will start with the cartesian product of coprime elts
   from each component
17 #everything up to p-1 is coprime
18 co_primes := [[seq(i, i = 0 .. (components[1]-1))]];
19 for n from 2 to numberOfComponents do
20     co_primes := [op(co_primes), [seq(i, i = 0 ..
       (components[n]-1))]];
21 od;
22
23 #then take the cartesian product to get the first <product of
   elements in components> elements in the ordering
24 T := cartprod(co_primes);

```

```

25 M := Matrix([T['nextvalue']()]);
26 while not T['finished'] do
27     M := <M; T['nextvalue'](>);
28 end do;
29
30 #make a list to keep track of the exponent of each prime; start by
    take each prime to the power -1
31 v := Vector[row](1 .. numberOfComponents, 1);
32 v := convert(v, list);
33
34 #keep adding rows until you have enough points in the ordering
35 while RowDimension(M) < m do
36     #take each prime to the power of minus the elements in v
    distances := zip(proc(x, y) options operator, arrow; x^(-y) end
        proc, components, v);
38 #check each column to see if the max distance was achieved
39 for j from 1 to numberOfComponents do
40     #if it was then split this column
    if distances[j] = max(distances) then
42     #take a snapshot of M before you start - this is what you
        have to add to
43     M1 := copy(M, deep);
44     #create p-1 new blocks
    for i from 1 to (components[j]-1) do
46         newBlock := copy(M1, deep);
47         newBlock(1..RowDimension(newBlock), j) :=
            Column(newBlock, [j]) +~ (i*components[j]^v[j]);
48     #add the new block to the master matrix
    M := Matrix([M], [newBlock]);
50     od;
51     #update the vector of exponents
    v[j] := v[j] + 1;
52 end if;
53 od;
54

```

```
55     end do;  
56     return M[1..m,];  
57 end proc;
```

code/Partial\_Rho\_Sequence.txt

```

1 with(LinearAlgebra):
2 with(combinat, cartprod):
3 with(padic):
4
5 ComputePartialRhoSeq := proc (S, rho)
6 # Given an m by n matrix S whose columns represent points of an
   n-component product space and that has as its i-th row the i-th term
   in a rho-ordering of that space, compute the (m-1)-th partial sum of
   the rho-sequence
7 # note that S and rho must be compatible and no checking is done to
   ensure this
8 # arg S; an n by m matrix representing a rho-ordering, for example as
   created by ComputePadicProductOrdering
9 # arg rho; a compatible metric on the points (rows) in S
10 # return; a real number, corresponding to the (m-1)-th term of the
    partial rho-sequence
11
12 local lastTerm, f, distances, nthTerm;
13
14 #find the last element in the ordering
15 lastTerm := S[RowDimension(S),];
16
17 #make a function that calculates the distance from the i-th row of S
    to the last term in the ordering
18 f := proc (i) options operator, arrow; rho(op(convert(lastTerm,
    list)), op(convert(S[i,], list))) end proc;
19
20 #run over each row to get the set of all m-1 distances
21 distances := map(f, [seq(i, i = 1 .. (RowDimension(S)-1))]);
22
23 #multiply them to get the (m-1)-th term of the rho-ordering
24 partialSum := mul(distances);
25

```

```
26     return partialSum  
27 end proc;
```

code/Fast\_Partial\_Rho\_Sequence.txt

```

1 FastPartialRhoSeq := proc (A, p:=[2,3])
2   local g, h, computePowers, n, shortA, primeExponents, i, thisPrime,
      thisPrimeIndex, B, G, powers, powersOfG, thisPrimeSum;
3
4   == Some helper functions ==
5   #Return the index of every instance of p-multiples in a list
6   #Use to find the index of a given prime in A
7   h := proc(i, L, p) if L[i] mod p = 0 then return i else return NULL fi;
      end proc;
8
9   #Count the number of times the mth element has appeared as a factor
      for the 1..m first elements in a list
10  #Use to compute the (decreasing) sequence of distances in A or a
      subset of A
11  g := proc(m, L)
12    local basePrime;
13
14    basePrime := L[m];
15
16    #since we just want the exponent not actual distance just compute
      what power this would be
17    return ordp(mul(L[1..m]), basePrime);
18
19  end proc;
20
21  #Compute the appropriate power of an element of G
22  computePowers := proc(m, L)
23    local power;
24
25    if m=nops(L) then
26      power:= L[-1]-1;
27    else
28      power:= mul(L[(m+1)..nops(L)]) * (L[m]-1);

```

```

29     end if;
30     return power
31 end proc;
32
33 #compute n, then create a copy of A with the first element deleted to
    ease the indexing
34 n := mul(A);
35 shortA := A[2..nops(A)];
36
37 #compute the terms corresponding to each prime given
38 primeExponents := Vector();
39 for i from 1 to nops(p) do
40     #pull out the prime
41     thisPrime := p[i];
42
43     #first get the index in A of this prime
44     thisPrimeIndex := map(h, [seq(i, i=1..nops(shortA))], shortA,
        thisPrime);
45     B:= shortA[thisPrimeIndex];
46
47     #then find the (exponents for the) distances occuring with this prime
48     G:= map(g, [seq(i, i=1..nops(B))], B);
49
50     #Figure out what power each distance should be raised to
51     powers := map(computePowers, thisPrimeIndex, shortA);
52
53     #raise each element in G to the given powers
54     powersOfG:= zip(proc (x, y) options operator, arrow; x*(y) end proc,
        G, powers);
55
56     #the exponent of this prime in the nth partial will be -([the sum of
        the elements in powers]/n), where n is the product of elements in
        A (including the first element)
57     thisPrimeSum := add(powersOfG);

```

```
58     primeExponents(i) := thisPrimeSum/n;  
59  
60     end do;  
61  
62     return primeExponents;  
63  
64 end proc;
```



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