Chapter 1

Introduction

In the course of developing a generalized factorial function, Manjul Bhargava introduced the notion of $p-$orderings of a Dedekind domain [B1, B2], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [B3, J2] and have also provided a natural framework to extend many classical results in analysis to a $p-$adic setting, such as polynomial approximation and mapping theorems [B1, B2, B3]. In this thesis, we examine how a tool based on $p-$orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-Archimedean settings.

The historical background to this work comes in two parts. On the one hand, there is the background on logarithmic capacity from potential theory, and secondly, there is the background from Bhargava's $p-$orderings. We give a brief summary of each here. A similar treatment, with slightly different perspective, is found in [FP]. Jean-Luc Chabert was the first to draw a connection between the two, and many of the known results in this area stem from his work or that of his colleagues. Building on the result in [J1], we extend the work by Chabert and colleagues by studying valuative capacity in a more general setting, namely that of an ultrametric space, which may or may not also be a local field. In doing so, we show many properties of capacity are in fact independent of the algebraic structure of a space, although such structure, when it exists, can act as a useful probe.

1.1 Logarithmic capacity

The theory of capacity has been developed as a topic in potential theory in a variety of settings. Classically, the notion of capacity was developed over both $\mathbb{C}$ and $\mathbb{R}^n$, although the theory has been further developed in a rather general way by Rumely for Berkovich spaces. A signficant body of work on the analytic properties of capacity can be found for a number of different contexts. For example, such a treatment of the subject over $\mathbb{C}$ can be found in [W] and [Ra1], over Berkovich spaces in [BR], and over $\mathbb{Q}_p$ in [Ca]. We give a brief account of capacity over $\mathbb{C}$ here, presenting only the most essential definitions and results. One advantage of tracing the historical roots of capacity back to $\mathbb{C}$ is that the theory in this setting also comes equipped with a physical interpretation. As we are about to see, capacity in the classical sense gives a mathematical model for the amount of electrostatic charge a conductor can hold. The exposition below is closely based on [Ra1] and [Ra2].

Even restricting ourself to the definition of capacity of subsets of $\mathbb{C}$, we find two paths, one which will give us some physical interpretation, and one which will lead more naturally to $p-$orderings. We start with the former.

Definition 1. [Ra2] Let μ be a finite Borel measure on $\mathbb{C}$ and suppose μ has compact support. We associate to μ a function, $p_\mu : \mathbb{C} \to (-\infty, \infty]$, given by

$$p_\mu(x) = \int \log \frac{1}{|x-y|} d\mu(y)$$

called the **potential function** of μ. The **energy** of μ is

$$I(\mu) = \int \int \log \frac{1}{|x-y|} d\mu(y) d\mu(x)$$

This gives at once the physical interpretation promised above. We interpret the potential function of a measure as giving the potential energy of a point. Viewing the measure as a charge distribution, the double integral gives back the total energy in the system. Now we come upon a physical reality: charged particles in a conductor will naturally distribute themselves in order to minimize the energy. This leads to the definition below:

Definition 2. [Ra2] Let K be a compact subset of $\mathbb{C}$ and let $\mathcal{P}(K)$ be the set of Borel probability measures on K. If $\nu \in \mathcal{P}(K)$ is such that

$$I(\nu) = \inf_{\mu \in \mathcal{P}(k)} I(\mu)$$

then ν is a **equilibrium measure** for K.

We state the following proposition without proof. A sketch of the proof can be found in [FP] and the full details can be found in [Ra1].

Proposition 1. *[Ra1] An equilibrium measure exists for every compact set $K \in \mathbb{C}$. When finite, the equilibrium measure is unique and isometry-invariant.*

We are now ready to give our first definition of capacity.

Definition 3. [Ra2] Let K be a compact subset of $\mathbb{C}$. The logarithmic capacity of E is

$$C(K) = e^{-I(\nu)}$$

where ν is an equilibrium measure on K. If $I(\nu) = \infty$, then we understand that $C(K) = 0$.

We present below a few results on capacity in $\mathbb{C}$, some of which will reappear in the remainder of this work, although the context, and the proofs (omitted here), bear little resemblence to the present case.

Proposition 2. *([Ra1], 5.1.2) Let K, K_1, K_2 be compact subsets of $\mathbb{C}$.*

1. *$K_1 \subseteq K_2$, then $C(K_1) \leq C(K_2)$.*

2. *$C(\alpha K + \beta) = |\alpha| C(K)$ for all $\alpha, \beta \in \mathbb{C}$.*

3. $C(K) = C(\delta_e K)$, *where* δ_e *is the exterior boundary.*[1]

Proposition 3. *([Ra1], 5.1.4) Suppose* $\{B_n\}$ *is a sequence of Borel subsets of* $\mathbb{C}$. *Let* $B = \cup_n B_n$ *and* $d \geq 0$.

1. *If* $diam(b) \leq d$, *then* $C(B) \leq d$ *and*

$$\frac{1}{log(\frac{d}{C(B)})} \leq \sum_n \frac{1}{log(\frac{d}{C(B_n)})}$$

2. *If* $dist(B_j, B_k) \geq d$ *whenever* $j \neq k$, *then*

$$\frac{1}{log^+(\frac{d}{C(B)})} \geq \sum_n \frac{1}{log^+(\frac{d}{C(B_n)})}$$

where $log^+(x) = \max(log(x), 0)$.

We now know show an equivalent way of defining of capacity, still over $\mathbb{C}$, which starts with the following two definitions due to Fekete [F].

Definition 4. [F] Let $K \subseteq \mathbb{C}$ be a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in K^n$, consider

$$\delta_n(z) = \prod_{j<i} |z_i - z_j|^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in K^n$ is called a **Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in K.

Note that since K is compact by assumption, a Fekete $n-$tuple exists for each n.

Definition 5. Let $K \subseteq \mathbb{C}$ be a compact subset. The **transfinite diameter** of K is

$$\lim_{n\to\infty} [\max_z \ \delta_n(z)]$$

[1]The exterior boundary of a compact subset, K, of $\mathbb{C}$ is the boundary of the unbounded, connected component of $U = \mathbb{C} \setminus K$.

where the maximum is taken over all $n-$tuples in K. That is, the transfinite diameter of K is $\lim_{n\to\infty}\delta_n(z)$, where z is a Fekete $n-$tuple for each n.

The following proposition shows the relation to capacity.

Proposition 4. *([F], **Fekete-Szegö Theorem**) If K is a compact subset of $\mathbb{C}$, then the transfinite diameter of K is equal to the logarithmic capacity of K.*

We end this section with an observation about the points z_i in $\mathbb{C}$ (or some subset thereof) making up a Fekete n-tuple. For $n \geq 2$, if $(z_1, \ldots, z_n)$ is a Fekete $n-$tuple, then in general there is no z_{n+1} available such that $(z_1, \ldots, z_n, z_{n+1})$ a Fekete $(n+1)-$tuple. In physical terms, we note that the placement of a new charge in a conductor will almost always change the location of the existing charges in that conductor. Remarkably, this is not the case in ultrametric spaces. Indeed, we are able to build the analogous structure, which we call a $p-$ordering or more generally a $\rho-$ordering, *recursively*, that is by reusing the points from the previous iteration.

1.2 P-orderings

The development of $p-$ordering was motivated by the observation that the factorial function had important number-theoretic applications, yet was only defined for the set $\mathbb{Z}$. In order to generalize the factorial, Bhargava defined it via an invariant, called the $p-$sequence, which could be attached to any subset of a Dedekind domain [2] [B1].

We cannot go much further without introducing the following definition.

Definition 6. Let $z \in \mathbb{Z}$ and let p be any prime. The $p-$**adic valuation** of z, denoted $v_p(z)$, is the largest $n \in \mathbb{N}$ such that p^n divides $z \neq 0$ and $v_p(z) = \infty$ if $z = 0$. That is,

$$v_p(z) = \begin{cases} \max\{n \in \mathbb{N}; p^n \mid z\}, & \text{if } z \neq 0 \\ \infty, & \text{otherwise} \end{cases}$$

[2] In fact, Bhargava associated $p-$ sequences to the more general class of Dedekind rings, which are locally principal, Noetherian rings in which all nonzero primes are maximal.

For $z \in \mathbb{Z}$, we definite the $p-$**adic absoluste value** by

$$|z|_p = p^{-v_p(z)}$$

and the $p-$**adic metric** accordingly; that is, for $z_1, z_2 \in \mathbb{Z}$

$$\rho_p(z_1, z_2) = p^{-v_p(z_1 - z_2)}$$

where $p^{-\infty}$ is taken to be 0.

It is worth pausing to make a few comments about the above definitions. That the $p-$adic metric is truly a metric is easy to see. In fact, we will see in the next chapter that it is not just a metric, but also an ultrametric, since the $p-$adic absolute value satistfies a strengthen version of the triangle identity. The strong triangle identity is not the only interesting property at hand though. Like the logarithm, the $p-$adic valuation also satisfies: $v_p(x \cdot y) = v_p(x) + v_p(y)$ for any prime p and x, y in $\mathbb{Z}$. Moreover, we note that the $p-$adic valuation and $p-$adic metric have an interesting relationship with each other: two points whose difference has a relatively small valuation will have a relatively large distance between them and vice versa.

We are now ready to define $p-$orderings, and not long after, to give the connection to Fekete $n-$tuples.

Definition 7. [B1] Let S be a subset of $\mathbb{Z}$ and let p be any prime.[3] A p**-ordering** of S is a sequence, $\{a_i\}_{i \geq 0}$ in S, such that a_0 is arbitrary and for $i > 0$, a_i minimizes

$$v_p(\prod_{j<i}(z - a_j))$$

over $z \in S$.

A $p-$ordering in S, like a Fekete $n-$tuple in $\mathbb{C}$, is not unique. Indeed, in most of the examples we will explore, there will be infinitely-many choices at each stage

[3]To apply the definition to a general Dedekind domain, we replace the usual primes with the set of primes ideals in the ring of interest.

of the construction. Nonetheless, $p-$orderings give rise to $p-$sequences, which are invariants of S:

Definition 8. [B1] Let S be a subset of $\mathbb{Z}$ and let p be any prime. Suppose $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S. The **p-sequence**, occasionally the **characteristic sequence**, of S is the sequence defined by $\delta(0) = 1$ and for $i > 0$,

$$\delta(i) = v_p(\prod_{j=0}^{i-1}(a_i - a_j))$$

It is a fact, not entirely obvious, that the $p-$sequence of S is independent of the $p-$ordering used in its construction [B1]. To define the generalized factorial, Bhargava considered the product of $p-$sequences taken over each prime p for arbitrary subsets of $\mathbb{Z}$. We will go in another direction.

Suppose we were to generalize our definition of Fekete $n-$tuple in the obvious way, as below.

Definition 9. Let (M, ρ) be a metric space and suppose $S \subseteq M$ is a compact subset. Fix $n \in \mathbb{N}$, and for $z = (z_1, \ldots, z_n) \in S^n$, consider

$$\delta_n(z) = \prod_{j<i} \rho(z_i - z_j)^{\frac{2}{(n(n-1))}}$$

An element $z = (z_1, \ldots, z_n) \in S^n$ is called a **generalized Fekete n-tuple** if z maximizes δ_n over all $n-$tuples in S.

What then is the connection to $p-$orderings and $p-$sequences? Suppose S is a subset of $\mathbb{Z}$ and that $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S for some prime p. Then of course from the definition of $p-$orderings, we know that for $i > 0$,

$$v_p(\prod_{j<i}(a_i - a_j)) \leq v_p(\prod_{j<i}(z - a_j))$$

for $z \in S$. Something more is true though, namely,

$$v_p(\prod_{j<i}(a_i - a_j)) \leq v_p(\prod(x_i - x_j))$$

for $x_i, x_j \in S$ [B1]. That is, when we pick a_i to minimize the $p-$adic valuation over $\prod_{j<i}(z - a_j)$, we actually achieve the minimum over the product of *all pairs of differences in* S. Since minimizing $v_p(x_i - x_j)$ is the same as maximizing $\rho_p(x_i, x_j)$, we have the following remarkable fact: if $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S, then $\{a_i\}_{i=0}^n$ is a generalized Fekete $n-$tuple for (S, ρ_p) for each n. In particular, $p-$orderings give a recursive construction for generalized Fekete $n-$tuples.

The first connection between these objects was made by Jean-Luc Chabert in [Ch] when he studied the limit of these sequences not just for the case $M = \mathbb{Z}$ and $\rho = \rho_p$, but in the case that M is any rank-one valuation domain [Ch]. We repeat his theorem 4.2 from [Ch] below,

Proposition 5. *Let E be a subset of V, a rank-one valuation domain with valuation v. If $\{a_i\}_{i\geq 0}$ is $v-$ordering* [4] *of E, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0,\ldots,x_n\in E} v(\prod_{0\leq j<i\leq n}(x_i - x_j))$$

Chabert called this limit the valuative capacity of E, and we shall do the same. The following result by Johnson in [J1], in which $p-$ordering has been replaced by $\rho-$ordering, provides the foundation for the rest of this work:

Proposition. *([J1], Theorem 1) If S is a compact subset of an ultrametric space (M, ρ), then the first n terms of a ρ-ordering of S always give a Fekete n-tuple of S and all Fekete n-tuples of S arise in this way.*

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[4] A $v-$ordering of E is exactly as expected: a sequence of distinct element $\{a_i\}_{i\geq 0}$ in E is $v-$ordering of E if for $n > 0$,

$$v(\prod_{k=0}^{n-1}(a_n - a_k)) \leq v(\prod_{k=0}^{n-1}(x - a_k)$$

for each $x \in E$.

One important consequence of this remark is that it gives a way to define capacity in a general ultrametric space. By replacing the notion of $p-$ordering (or $v-$ordering) with the more general notion of $\rho-$ordering, we are able to give a definition of valuative capacity for a general ultrametric space, without appealing to any algebraic (or measure-theoretic) structure. Of course, we have yet to say what a $\rho-$ordering is. We take this up, along with the necessary background from ultrametric spaces, in the next chapter.

Chapter 2

Capacity and Ultrametric spaces

2.1 Ultrametric basics

The principal context for this thesis is an arbitrary ultrametric space, which is a metric space that also satisifies an additional axion, sometimes called the ultrametric inequality or (in the case of vector spaces) the strong triangle propery. We define ultrametric spaces below and for the rest of this section, we review some of their more important properties. The proofs offered in this section are, for the most part, standard and can be found in a number of reference texts, such as [Ro].

Definition 10. Let (M, ρ) be a metric space; that is, suppose M is a set and ρ is a map, $\rho : M \times M \to \mathbb{R}_{\geq 0}$ such that:

(i) $\rho(x, y) = 0$ if and only if $x = y$

(ii) $\rho(x, y) = \rho(y, x)$

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any $x, y, z \in M$. If ρ also satisfies the ultrametric inequality,

$$\rho(x, z) \leq \max\left(\rho(x, y), \rho(y, z)\right)$$

for any $x, y, z \in M$, then (M, ρ) is an **ultrametric space**.

A special case of an ultrametric space, and one where much of the previous work on this topic has been completed, is one where the metric has been derived from a norm on a vector space.

Definition 11. Let (V, N) be a normed vector space; that is, suppose V is an $\mathbb{F}-$vector space, for $\mathbb{F}$ some subfield of $\mathbb{C}$, and $N : V \to \mathbb{R}_{\geq 0}$ is such that:

(i) $N(x+y) \leq N(x) + N(y)$

(ii) $N(cx) = |c|\ N(X)$

(iii) $N(x) = 0$ implies $x = 0$

for any $x, y \in V$ and $c \in \mathbb{F}$. We say that N satisfies the **strong trianlge inequality** if

$$N(x+y) \leq \max(N(x), N(y))$$

for any $x, y \in V$.

Proposition 6. *Let (V, N) be a normed vector space and suppose N satisfies the strong triangle inequality. Then the metric space (V, ρ_N), where ρ_N is the metric induced by $\rho_N(x, y) = N(x - y)$, is an ultrametric space.*

Proof. We take for granted that (V, ρ_N) is a metric space and also note that

$$N(x+z) \leq \max(N(x), N(z))$$

implies

$$\rho_N(x, z) \leq \max(\rho_N(x, 0), \rho_N(z, 0)) \leq \max(\rho_N(x, y), \rho_N(y, z))$$

□

Notation. If (V, N) is a normed vector space, then the metric induced by N will be denoted ρ_N.

When ultrametric spaces come from spaces with algebraic structure, such as normed vector spaces, some of this structure carries over into metric spaces structure in a rather nice way:

Proposition 7. *[Ro] Let S be a group equipped with a (right) invariant ultrametric, ρ. If $B = B(0, r)$ is a (closed) ball centred at the neutral element of S, that is $B = \{x \in S; \rho(x, 0) \leq r\}$, then B is a subgroup of S.*

Proof. Let $x, y \in B$. Then

$$\rho(x - y, 0) = \rho(x, y) \leq \max(\rho(x, 0), \rho(y, 0)) \leq r,$$

so that $x - y \in B$. $\square$

In the previous chapter, we claimed that the $p-$adic metric was an ultrametric on the set $\mathbb{Z}$. Indeed, $(\mathbb{Z}, \rho_p)$ and the closely related space of $p-$adic integers, denoted $\widehat{\mathbb{Z}_p}$, are the canonical examples of an ultrametric space.

Example 1. Let p be any prime and consider the metric space $(\mathbb{Z}, \rho_p)$. To see that $(\mathbb{Z}, \rho_p)$ is an ultrametric space, we must show that ρ_p satisfies the ultrametric inequality, or equivalently, that $p-$adic absolute value satisfies the strong triangle inequality. Let x, y be in $\mathbb{Z}$ and suppose $v_p(x) = n_x$ and $v_p(y) = n_y$. Then if $n = \min(n_x, n_y)$, p^n divides x and p^n divides y, so p^n divides $x + y$. We see now that $v_p(x + y) \geq \min(v_p(x), v_p(y))$ and in turn $|x + y|_p \leq \max(|x|_p, |y|_p)$.

Example 2. Let p be any prime. If

$$z = \sum_{i \geq 0} b_i p^i$$

is such that $b_i \in \{0, \ldots, p - 1\}$ for all i, then we say that z is a $p-$**adic integer**. If $z = \sum_{i \geq 0} b_i p^i$, we note that if only a finite number of the coefficients of z are non-zero, then $\sum_{i \geq 0} b_i p^i$ is a representation in base p of some element of $\mathbb{Z}$. We can define the $p-$**adic order** of a $p-$adic integer, denoted $ord_p(z)$, in a way that agrees with the $p-$adic valuation when $\sum_{i \geq 0} b_i p^i$ is in $\mathbb{Z}$. We let $ord_p(z)$ be the smallest i such that $b_i \neq 0$. The $p-$adic integers are both a ring[1] and an ultrametric space with the metric induced by $ord_p(z)$. For a given prime, p, we denote the $p-$adic integers by $\widehat{\mathbb{Z}_p}$.

In what follows, we will often refer to $p-$adic spaces and it will not make much of a difference whether the reader prefers to think of being in $(\mathbb{Z}, \rho_p)$ or $\widehat{\mathbb{Z}_p}$. The reason is this: when forming ρ_p-orderings of subsets of either space we are always able to do so by selecting elements with finite number of non-zero coefficients, that is, by selecting elements from $\mathbb{Z}$ itself.

[1]The ring operations carry over on the coefficients of $p-$adic integers in the expected way from $\mathbb{Z}/p\mathbb{Z}$, as long as special care is taken to keep track of carries.

Ultrametric spaces exhibit many properties unlike those of traditional metric spaces, and we review of few of these below. Of particular interest to us is the behavior between (closed) balls in an ultrametric space.

Notation. Let (M, ρ) be a compact ultrametric space and let

$$B(a, r) = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the *closed* ball of radius r, centred at a for some $r \in \mathbb{R}_{>0}$ and $a \in (M, \rho)$. Let

$$B^0(a, r) = \{x \in M \mid \rho(x, a) < r\}$$

denote the *open* ball of radius r, centred at a for some $r \in \mathbb{R}_{>0}$ and $a \in (M, \rho)$.

In the above notation, we break with convention in that we denote a closed ball without using any decoration. This is because before too long we will work exclusively with closed balls. We are able to do this because for the most part, the notion of open and closed ball in an ultrametric space overlap, although we will need a few more facts before showing this.

Definition 12. Let S be a subset of an ultrametric space. The **diameter of** S is $diam(S) = \sup\limits_{x,y \in S} \rho(x, y)$. Note that if S is compact, $diam(S) = \max\limits_{x,y \in S} \rho(x, y)$.

Proposition 8. *Let $B = B(a, r)$ be a ball in an ultrametric space (M, ρ). Then the diameter of B is less than or equal to the radius of B.*

Proof. Suppose $d = diam(B) > r$. This would imply there exists x, y in B such that $\rho(x, y) > r$, in particular $\rho(x, y)$ is strictly greater than $\max(\rho(x, a), \rho(y, a))$, which is a contradiction since ρ is an ultrametric. $\square$

The following example shows that we can obtain a strict inequality in the above proposition.

Example 3. Let M be a non-empty set. Let ρ be the ultrametric given by,

$$\rho(x,y) = \begin{cases} 0, & \text{if x=y} \\ 1, & \text{if x } \neq \text{ y} \end{cases}$$

for $x, y \in M$. Then $B(x, \frac{1}{2})$ has radius $\frac{1}{2}$ and diameter 0 for any x in M.

In the following proposition, we describe the triangles in an ultrametric space, and the result is more or less a restatement, in geometric terms, of the ultrametric inequality.

Proposition 9. *All triangles in an ultrametric space (M, ρ) are either equilateral or isosceles, with at most one short side.*

Proof. Let x, y, and z be three points in an ultrametric space (M, ρ). We show that $\rho(x, y) \neq \rho(x, z)$ and $\rho(x, y) \neq \rho(y, z)$ implies $\rho(x, y) < \rho(x, z) = \rho(y, z)$.

If $\rho(x, z) \neq \rho(y, z)$, then without loss, $\rho(x, z) > \rho(y, z)$. At the same time, the ultrametric inequality implies $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ and $\rho(y, z) \leq \max(\rho(x, y), \rho(x, z))$. The first inequality implies $\rho(x, y) < \rho(x, z)$, which means the second inequality implies $\rho(y, z) < \rho(x, z)$. This is a contradiction, so we must have $\rho(x, z) = \rho(y, z)$.

To see that $\rho(x, y) < \rho(x, z)$, simply note that $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$ □

With this result in hand, we are able to quickly demonstrate some of the properties of balls, which are of fundamental importance to us. We see below that the ultrametric inequality, perhaps innocuous on the surface, quickly implies ultrametric balls are markedly different from their Archimedean counterparts.

Proposition 10. *Every point of a ball in an ultrametric is at its centre. That is, if $B(x_0, r)$ is a ball in an ultrametric space (M, ρ), then $B(x, r) = B(x_0, r)$, $\forall x \in B(x_0, r)$*

Proof. Let $a \in B(x, r)$. Then $\rho(a, x) \le r$ and since

$$\rho(a, x_0) \le \max(\rho(a, x), \rho(x, x_0)) \le r$$

we must have $a \in B(x_0, r)$ and $B(x, r) \subseteq B(x_0, r)$ A similar argument shows $B(x_0, r) \subseteq B(x, r)$. □

Proposition 11. *If (M, ρ) is an ultrametric space and $B(x_0, r_1)$ and $B(y_0, r_2)$ are balls in (M, ρ), then either $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$, $B(x_0, r_1) \subseteq B(y_0, r_2)$, or $B(x_0, r_1) \subseteq B(x_0, r_1)$. That is, in an ultrametric space, all balls are either comparable or disjoint.*

Proof. Suppose $B(x_0, r_1) \cap B(y_0, r_2) \neq \emptyset$ and let z be a point in the intersection. We show that if there exists an $a \in B(y_0, r_2)$ such that $a \notin B(x_0, r_1)$, then $B(x_0, r_1) \subseteq B(y_0, r_2)$. Let $x \in B(x_0, r_1)$. Then we must have $\rho(x, z) < \rho(x, a)$, since $z \in B(x_0, r_1) = B(x, r_1)$ and a is not. Since the triangle with vertices (a, x, z) is isocolces with at most one short side, we must have $\rho(x, a) = \rho(a, z) \le r_2$, since $a \in B(y_0, r_1) = B(z, r_2)$. Then $x \in B(y_0, r_1)$. □

Proposition 12. *The distance between points in two non-overlapping balls in an ultrametric is constant. That is, if $B(x_0, r_1)$ and $B(y_0, r_2)$ are two balls in an ultrametric space with $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$, then there exists a $c \in \mathbb{R}_{>0}$ such that $\rho(x, y) = c$, $\forall x \in B(x_0, r_1)$ and $\forall y \in B(y_0, r_2)$.*

Proof. Suppose $\rho(x_0, y_0) = c$ and let $x \in B(x_0, r_1)$ and $y \in B(y_0, r_2)$ be arbitrary. Consider the triangle formed by (x_0, y_0, y). Since $\rho(x_0, y_0) = c$ and $\rho(y, y_0) \le r_2 < c$, we must have $\rho(x_0, y) = c$ because triangles in an ultrametric space have at most one short side. Now consider the triangle formed by (x_0, x, y). Since $\rho(x_0, y) = c$ and $\rho(x, x_0) \le r_1 < c$, we must have $\rho(x, y) = c$. □

We will get a bit closer to showing the relationship between open and closed balls with the following results and will pick up a few other useful facts along the way. We start with another definition.

Definition 13. If (M, ρ) is an ultrametric space, then for $x_0 \in M$ and $r \in \mathbb{R}_{>0}$,

$$S(x_0, r) = \{x \in M; \rho(x, x_0) = r\}$$

is the **sphere of radius** r **at** x_0.

Lemma 1. *([Ro]) Spheres (of positive radius) in an ultrametric space are both open and closed as sets.*

Proof. [Ro] A sphere in any metric space is closed, so we need only show a sphere is also open in an ultrametric space. We show a sphere, $S = S(x_0, r)$, is equal to a union of open sets, $S = \cup_{x \in S} B^0(x, r)$.

Let $B = B^0(x, s)$ be an open ball that does not contain some x_0. Let $r = \rho(x_0, x)$. We must have $r \geq s$, so then (since all triangles are isosocles) every point in B lies in $S(x_0, r)$, that is $B \subseteq S(x_0, r)$. Then for any $x \in S(x_0, r)$, $B^0(x, r) \subseteq S(x_0, r)$ and

$$\bigcup_{x \in S(x_0, r)} B^0(x, r) \subseteq S(x_0, r)$$

The reverse inequality is clear since the union is taken over points of S. $\square$

Proposition 13. *([Ro]) The open balls in an ultrametric space are closed sets and the closed balls are open sets.*

Proof. The proof follows immediately from the result that spheres are both open and closed: to see that closed balls are open sets, note that for a closed ball, $B(x_0, r)$,

$$B(x_0, r) = B^0(x_0, r) \cup S(x_0, r)$$

Likewise, to see that open balls are closed sets, note that

$$B^0(x_0, r) = B(x_0, r) \setminus S(x_0, r)$$

$\square$

The following proposition is now easy to see, although the result is both unintuitive and important for our purposes.

Proposition 14. *Suppose S is a compact subset of an ultrametric space (M,ρ) and that $\cup_{i\in I}B(x_i,r_i)$ is a cover of S by closed balls in S. Then there exists $i_1,\ldots,i_n$, a finite subset of I, such that $\cup_{j=1}^{j=n}B(x_{i_j},r_{i_j})$ is a partition of S.*

Proof. Since S is compact and ρ is an ultrametric, $\cup_{i\in I}B(x_i,r_i)$ is an open cover and contains a finite subcover of S. Say this subcover is given by the elements $i_1,\ldots,i_{n'}\in I$, and suppose this is not a partition. That is, suppose for some i_i,i_j, $B(x_{i_i},r_{i_i})\cap B(x_{i_j},r_{i_j})\neq\emptyset$. Then, without loss of generality, we must have $B(x_{i_i},r_{i_i})\subseteq B(x_{i_j},r_{i_j})$, so that the removal of $B(x_{i_i},r_{i_i})$ is still a cover of S. We continue this process a finite number of times, since the subcover was finite to begin with, to arrive at a finite partition of S. □

In fact, a slightly stronger statement then the above is true:

Corollary 1. *Suppose S is a compact subset of an ultrametric space (M,ρ) and that $B(x_0,r)$ is a closed ball in S. Then, there exists a finite partition of S having $B(x_0,r)$ as an element.*

Proof. Let $\mathcal{C}$ be the cover of S given by $\cup_{x\in S}B(x,r)\cap S$. From the proposition, we can select a finite subcover of $\mathcal{C}$ that is a partition of S. Suppose $B(y,r)\cap S$ is the element in this partition containing x_0. Then since $B(y,r)$ and $B(x_0,r)$ are equal in M, $B(y,r)\cap S=B(x_0,r)\cap S=B(x_0,r)$. □

We end this section by making a few comments about the set of distances that occur between the points of a compact ultrametric space.

Proposition 15. *([Ro]) Let S be a compact subset of an ultrametric space, (M,ρ)*

(i) For $m\in(M\setminus S)$, let $f_m:S\to\mathbb{R}$, be the function defined by $f_m(s)=\rho(m,s)$. Then $Im(f_m)$ is finite for all $m\in(M\setminus S)$.

(ii) For $a\in S$, let $\phi_a:S\setminus\{a\}\to\mathbb{R}$ be the function defined by $\phi_a(x)=\rho(x,a)$. Then $Im(\phi_a)$ is a discrete subset of $\mathbb{R}$ for all $a\in S$.

Proof. ([Ro])

(i) The fibers of f_m, $f_m^{-1}(s)$, for $s \in S$, form a cover of S. In fact, they form an open partition. Since S is compact by assumption, we must have that this partition is finite, and so the image of f_m was also finite.

(ii) Let $\epsilon > 0$. Let $B^0(a,\epsilon)$ be the open ball, $B^0(a,\epsilon) = \{x \in S; \rho(x,\epsilon) < \epsilon\}$. Then $(S \setminus B^0(a,\epsilon))$ is compact, and so from the above we know that ϕ_a restricted to $(S \setminus B^0(a,\epsilon))$ has finite range (let $M = S$ and $S = (S \setminus B^0(a,\epsilon))$ and apply (i)). Then the sets

$$[\epsilon, \infty) \cap \{\rho(s,a); s \in S, x \neq a\}$$

are finite and $Im(\phi_a)$ is discrete.

□

This leads to the following definition.

Definition 14. If (M,ρ) is an ultrametric space, we say M is **discretely-valued** if the set $\Gamma_S = \{r \in \mathbb{R}; \exists x, y \in M \text{ such that } \rho(x,y) = r\}$ is a discrete subset of $\mathbb{R}$.

If (M,ρ) has a translation-invariant ultrametric then clearly M is discretely-valued since the sets ϕ_a are then equal for all a in M. Now we have the following question.

Question 1. Are there mild conditions under which a compact ultrametric space is discretely-valued? In particular, are there conditions that do not appeal to some algebraic structure in M?

When this is the case, it will become useful to write the set of distances occuring in S as a sequence, put in decreasing order.

Notation. If S is a compact, discretely-valued ultrametric space, then we denote the set of distances between points of S by

$$\Gamma_S = \{\gamma_0 = d = diam(S), \gamma_1, \gamma_2, \ldots, \gamma_\infty = 0\}$$

where $\gamma_i \in \Gamma_S$ if and only if $\exists x, y \in S$ such that $\rho(x,y) = \gamma_i$ and $\gamma_i < \gamma_j$ if and only if $i > j$.

We end this section with the following corollary.

Corollary 2. *([Ro]) Let $B(a,r)$ be a closed ball in an compact, discretely-valued ultrametric space. Then there exists $r' > r \in \mathbb{R}$ such that $B(a,r) = \{x \in M \mid \rho(x,a) < r'\}$; that is, every closed ball is also an open ball with the same centre and slightly larger radius.*

ρ-orderings, ρ-sequences, and valuative capacity

We are now in a position to give a general definition of $p-$orderings and in turn, $p-$sequences and valuative capacity. The observation that an analogous notion of $p-$ordering can be defined for a general ultrametric space, and that these structures coincide with Fekete $n-$tuples, is due to [J1]. The exploration of this idea makes up the remainder of this work.

Definition 15. [J1] Let S be a subset of an ultrametric space (M, ρ). A **ρ-ordering** of S is a sequence $\{a_i\}_{i\geq 0}$ in S such that a_0 is arbitrary and $\forall n > 0$, a_n maximizes

$$\prod_{i=0}^{n-1} \rho(s, a_i)$$

over $s \in S$.

The above generalizes the definition of $p-$orderings for $\mathbb{Z}$, since maximizing the $p-$adic distance between two points in $\mathbb{Z}$ (or $\widehat{\mathbb{Z}_p}$) is the same as minimizing the $p-$adic valuation of the difference of two points. In particular, $\{a_i\}_{i\geq 0}$ is a $p-$ordering of S, a subset of $\mathbb{Z}$, if and only if it is a ρ_p-ordering of (S, ρ_p). Let us see an example of the simplest kind, i.e., for a finite set S.

Example 4. Suppose S is the finite subset of $(\mathbb{Z}, \rho_2)$, given by $S = \{0, 2, 8, 3\}$. Then a ρ_2-ordering of S starts (arbitrarily) with $a_0 = 0$, which forces $a_1 = 3$, since

$\rho_2(0,3) = 1 = diam(S)$. The sequence continues $a_2 = 2$ and $a_3 = 8$, but after this point the sequence becomes arbitrary because $\prod_{i=0}^{n-1} \rho(s, a_i)$ will contain a 0, given by the repeated term. Indeed, for any finite subset S with $|S| = n$ the $\rho-$ordering of S is arbitrary from the n^{th} point on.

We now give the definition of a $\rho-$sequence for an ultrametric space, generalizing the notion of a $p-$sequence.

Definition 16. [J1] Let $\{a_i\}_{i\geq 0}$ be a $\rho-$ordering of S. The **ρ-sequence** of S is defined by letting $\delta(0) = 1$ and for $n > 0$,

$$\delta(n) = \prod_{i=0}^{n-1} \rho(a_n, a_i)$$

The two propositions that follow are the critical observations. The first one tells us that we can use the $\rho-$sequence of S as an invariant and the second one motivates the definition of valuative capacity. The proofs of each are given in [J1].

Proposition 16. *([J1], Lemma 1) The ρ-sequence of S is well-defined so long as S is compact and ρ is an ultrametric. That is, the ρ-sequence of a compact subset of an ultrametric spaces does not depend on the choice of ρ-ordering of S.*

Proposition 17. *([J1], Theorem 1) If S is a compact subset of an ultrametric space (M, ρ), then the first n terms of a ρ-ordering of S give a Fekete n-tuple of S and all Fekete n-tuples of S arise in this way.*

Armed with the notion of a well-defined $\rho-$sequence for an ultrametric space, and the knowledge that it gives a construction for Fekete $n-$tuples in that space, we define the valuative capacity of S, where S is any compact subset of an ultrametric space.

Definition 17. [J1] Let S be a compact subset of an ultrametric space (M, ρ) and let $\delta(n)$ be the ρ-sequence of S. The **valuative capacity** of S is

$$\omega(S) := \lim_{n\to\infty} \delta(n)^{1/n}$$

We spend the rest of this chapter establishing some basic results on valuative capacity. These results form the start of our toolkit for calculating the capacities of specifics sets. They also show that many of the properties of capacity from $\mathbb{C}$ carry over to the non-Archimedean case in a natural way.

Let us assume from this point on that S is always a compact subset of an ultrametric space, unless stated otherwise.

Proposition 18. *([J1]) $\omega(S)$ is finite. If S itself is finite, then $\omega(S) = 0$.*

A compact set $E \subseteq \mathbb{C}$ is said to be polar if the logarithmic capacity of E is 0 [Ra1]. Polar sets play a central role in potential theory and the theory of logarithmic capacity, which raises the following question:

Question 2. Are there ultrametric spaces that have some *infinite* subset S with $\omega(S) = 0$?

We also have the expected result on monotoncity for valuative capacity:

Proposition 19. *([J1], Lemma 4) If S and T are compact subsets of an ultrametric space such that $S \subseteq T$ then $\omega(S) \leq \omega(T)$.*

We show now some results on the interaction between the algebraic structure of the space and valuative capacity. These results can be powerful tools for calculating capacities, in particular, when they are combined with the subadditivity result that follows.

Proposition 20. *(translation invariance) If (M, ρ) is a compact ultrametric space and also a topological group for which ρ is (left) invariant under the group operation, then ω is also (left)-invariant. That is, if $\rho(x, y) = \rho(g + x, g + y)$, $\forall g, x, y \in M$, then $\omega(g + S) = \omega(S)$, for $S \subseteq M$.*

Proof. Let $\{a_i\}_{i\geq 0}$ be a ρ-ordering for S. Then $\{g+a_i\}_{i\geq 0}$ is a ρ-ordering for $g+S$. Then

$$\omega(g+S) = \lim_{n\to\infty} \delta(n)^{1/n}$$

$$= \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(g+a_n, g+a_i)]^{1/n}$$

$$= \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = \omega(S)$$

□

Example 5. Note that ρ_p is translation invariant for each p since for any x, y, we have $\rho_p(x,y) = p^{-v_p(x-y)} = p^{-v_p((a+x)-(a+y))} = \rho_p(a+x, a+y)$. Then $\omega(a+S) = \omega(S)$ for $S \subseteq (\mathbb{Z}_p, \rho_p)$.

Proposition 21. *(scaling) Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity, so that (V, ρ_N) is an ultrametric space. Then if N is multiplicative, so is ω. That is, if $N(gx) = N(g)N(x), \forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.*

Proof. Let ρ_N be the metric induced by N, so that $\rho_N(x,y) = N(x-y), \forall x, y \in V$. Let $\{a_i\}_{i\geq 0}$ be a ρ_N-ordering for S and let u, v be in gS with $u = gs_i$ and $v = gs_j$ for some $s_i, s_j \in S$. Then, since N is multiplicative,

$$\rho(u,v) = \rho(gs_i, gs_j) = N(gs_i - gs_j)$$

$$= N(g(s_i - s_j)) = N(g)N(s_i - s_j) = N(g)\rho(s_i, s_j),$$

so that $\{ga_i\}_{i\geq 0}$ is a ρ_N-ordering for gS. Then,

$$\omega(gS) = \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(ga_n, ga_i)]^{1/n} = \lim_{n\to\infty} [\prod_{i=0}^{n-1} N(g)\rho(a_n, a_i)]^{1/n}$$

$$= \lim_{n\to\infty} [N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g) \lim_{n\to\infty} [\prod_{i=0}^{n-1} \rho(a_n, a_i)]^{1/n} = N(g)\omega(S)$$

□

Example 6. Since ρ_p is multiplicative, we have that $\omega(mS) = |m|_p \cdot \omega(S)$ for $m \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}$. In particular, $\omega(p\mathbb{Z}) = \frac{1}{p} \cdot \omega(\mathbb{Z})$.

The following proposition is from [J1], where it is given for some S written as the union of two subsets, although it is easily seen to be true for S equal to any finite union, so long as the other assumptions remain satisfied.

Proposition 22. *([J1], Proposition 10) (subadditivity) If $diam(S) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(A_i, A_j) = d, \forall i, j$, then*

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n} \frac{1}{log(\omega(A_i)/d)}$$

Example 7. We are now in a position to compute the valuative capacity of $(\mathbb{Z}, \rho_p)$. For any p, we note that $\mathbb{Z}$ can be decomposed into p closed balls of radius $\frac{1}{p}$, which are equal to the cosets of $\mathbb{Z}$ modulo p. Since $diam(S) = 1$, this gives

$$\frac{1}{log(\omega(\mathbb{Z}))} = \sum_{i=0}^{p-1} \frac{1}{log(\omega(p\mathbb{Z} + i))} = \frac{p}{log(\omega(p\mathbb{Z}))} = \frac{p}{log(\frac{1}{p} \cdot \omega(\mathbb{Z}))}$$

Now we have,

$$log(\omega(\mathbb{Z})^p) = log(\frac{1}{p} \cdot \omega(\mathbb{Z}))$$

so that,

$$\omega(\mathbb{Z})^p - \frac{\omega(\mathbb{Z})}{p} = 0$$

and $\omega(\mathbb{Z}) = p^{\frac{1}{1-p}} = p^{\frac{-1}{p-1}}$.

We can apply the same reasoning to any partition of S made up of sets that all have the same capacity and meeting the requirement that their pairwise distances are all equal to the diameter of S.

Corollary 3. *Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = diam(S)$ and also $\omega(S_i) = \omega(S_j), \forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S), \forall i$. Then $\omega(S) = r^{\frac{1}{n-1}}$.*

Now we note that a partition of S into closed balls wil satisfy the hypotheses if the distance between each ball is equal to the diameter of S. In particular, if

$B(x_i, r_i)$ is a collection of closed balls such that the pairwise-distance between any $B(x_i, r_i)$ and $B(x_j, r_j)$ is constant, then if we know the capacity of each $B(x_i, r_i)$, we can compute the capacity of their union. If M is discretely-valued, then we can say more.

Corollary 4. *(Joins of computable sets are computable) Let M be a compact, discretely-valued ultrametric space. Let $\Gamma_M = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in M. Suppose that $S = B(x, \gamma_i)$, for some x and i, is the union of $n \geq 2$ balls of radius γ_{i+1}, that is, $S = \cup_{j=1}^{n} B(x_j, \gamma_{i+1})$ is a join in the lattice of closed balls in M. Then*

$$\frac{1}{log(\frac{\omega(B(x,\gamma_i))}{\gamma_{i+1}})} = \sum_{j=1}^{n} \frac{1}{log(\frac{\omega(B(x_j,\gamma_{i+1}))}{\gamma_{i+1}})}$$

Of course, if M is a group, then we know the elements in these partitions are cosets, and if the metric is translation-invariant, then they each have the same capacity. We take up this last corollary in significant detail in the next chapter, obtaining some formulae for valuative capacity with various restrictions on Γ_M or related structures.

Chapter 3

ρ-orderings and the structure of S

In the previous section, we defined valuative capacity for a compact subset S of an ultrametric space (M, ρ). We also got a glimpse into the way the valuative capacity of S interacts with its other properties, such as the set of distances occurring in S and the lattice of closed balls in S (or equivalently, if S has enough structure, a lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valuative capacity of S and the lattice of closed balls in S. In particular, we show how, if S is compact and discretely-valued, the lattice of closed balls can be used to compute the first n terms of a $\rho-$ordering of S (for any $n < \infty$).

Similar results have been found for the special case of ultrametric fields in [CEF]. We extend these results by moving to a more general setting, showing that much can be said about capacity in S without appealing to any underlying algebraic structure. Significant portions of the theory developed in this chapter and the ones that follow were guided by an empirical investigation into the capacity of product spaces, which we describe in the final chapter. The code that performed these calculations is included in the appendix.

We assume throughout this section that S is a compact, discretely-valued subset of an ultrametric space (M, ρ).

Subspaces of S

In the section we explore the subspaces of S formed by considering closed balls of some fixed radius. Recall from the previous section that if S is compact and discretely-valued, then the set of distances occurring in S is a discrete, bounded subset of $\mathbb{R}$ and so we may represent the set of distances by a sequence in decreasing order. As before, let the decreasing sequence of distances in S be given by $\Gamma_S = \{\gamma_0 = \mathrm{diam}(S), \gamma_1, \ldots, \gamma_\infty = 0\}$.

Now fix some $k \in \mathbb{N}$, and consider for a moment the set of closed balls of radius γ_k in S. We could denote these alternatively by $B^M(x, \gamma_k) \cap S$ or by $B^S(x, \gamma_k)$, but when there is no risk of confusion, we will denote them simply by $B(x, \gamma_k)$. Clearly, the set $\{B(x, \gamma_k); x \in S\}$ forms a cover of S. Although we have built the cover using closed balls, since we are in an ultrametric space, this gives an open cover of S (in fact, each element in the cover is not only an open set, but also an open ball for some radius slightly bigger than γ_k). Then since S is compact, we must have some $x_1, \ldots, x_n$ such that $S = \cup_{i=1}^n B(x_i, \gamma_k)$. In fact, since ρ is an ultrametric, we can pick the x_i's so that $\cup_{i=1}^n B(x_i, \gamma_k)$ will be a disjoint union and therefore a finite partition of S. Note that both n and the x_i's depend on our fixed k, but that n is independent of the x_i's, since any choice of centres is equivalent. We rephrase this with following definition and lemma:

Definition 18. For S and Γ_S as above, and $k \in \mathbb{N}$, fixed, define $\sim_k$ to be the relation on S given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e., $x \sim_k y$ if and only if $B_{\gamma_k}(x) = B_{\gamma_k}(y)$.

The fact $\sim_k$ is an equivalence relation on S is equivalent to the observation that every point in a ultrametric ball is at its centre:

Lemma 2. *Let S and Γ_S be as above, then $\sim_k$ is an equivalence relation on S.*

Proof. $\sim_k$ is clearly reflexive and symmetric, since ρ is a metric. Transitivity results from the ultrametric property of ρ: if $x \sim_k y$ and $y \sim_k z$, then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(z, y)) \leq \gamma_k$$

so $x \sim_k z$. $\square$

We denote the set of equivalence classes of $S/\sim_k$ by S_{γ_k}. We have defined S_{γ_k} to be the set of equivalence classes in S under the relation $\sim_k$, which is equivalent to letting S_{γ_k} be the set of closed balls of fixed radius γ_k in S. We now offer a third perspective on the elements on S_{γ_k}, which is due to [Ac],

Lemma 3. *For each k, the elements of S_{γ_k}, that is, the closed balls of radius γ_k, themselves form an ultrametric space, where the metric is given by:*

$$\rho_k(B(x,\gamma_k), B(y,\gamma_k)) = \begin{cases} \rho(x,y), & \text{if } \rho(x,y) > \gamma_k \\ 0, & \text{if } \rho(x,y) \leq \gamma_k, \text{ i.e., } B(x,\gamma_k) = B(y,\gamma_k) \end{cases}$$

Proof. ρ_k is reflexive, symmetric and transitive since ρ is. Likewise, ρ_k satisfies the ultrametric property, since ρ does: let $B(x,\gamma_k), B(y,\gamma_k)$ and $B(z,\gamma_k)$ be any three elements of S_{γ_k} and suppose $\rho_k(B(x,\gamma_k), B(y,\gamma_k)) > 0$. Then,

$$\begin{aligned} \gamma_k &< \rho_k(B(x,\gamma_k), B(y,\gamma_k)) \\ &= \rho(x,y) \leq \max(\rho(x,z), \rho(y,z)) \\ &= \max(\rho_k(B(x,\gamma_k), B(z,\gamma_k)), \rho_k(B(y,\gamma_k), B(z,\gamma_k))) \end{aligned}$$

since $\gamma_k < \max(\rho(x,z), \rho(y,z))$ implies that at least one of $\rho_k(B(x,\gamma_k), B(z,\gamma_k))$ or $\rho_k(B(y,\gamma_k), B(z,\gamma_k))$ is greater than 0. $\square$

So now the elements of S_{γ_k} may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

Definition 19. Let S and Γ_S be as above. Define $\beta(i)_{i\geq 0}$ to be the sequence given by $\beta(i) = |S_{\gamma_i}|$, which is an invariant of S and which counts the number of connected components of S_{γ_i} (that is, the points of S_{γ_i}), when viewed as a metric space. When necessary, we use $\beta^S(i)$ to denote the β sequence for a given, compact ultrametric space S. Adopting the terminology in [FP], we call $\beta^S(i)$ the **structure sequence** of S.

Notation 1. Let S_{γ_k} be as above. We denote the elements of S_{γ_k} by $B_1^k, \ldots, B_{\beta(k)}^k$ or by $B_1^{S,k}, \ldots, B_{\beta(k)}^{S,k}$, when necessary.

We return to the sequence $\beta(i)$ at the end of this section. For now, we show how a $\rho-$ordering of S can be built recursively from the spaces S_{γ_k}. This begins by noting that the spaces themselves can be built recursively:

Observation 1. Let S, Γ_S, and S_{γ_k} be as above. Then S_{γ_k+1} can be constructed by partitioning each of the closed balls in S_{γ_k} into closed balls of radius γ_{k+1} and taking their union: Let $B(x_i, \gamma_k)$ be an element of S_{γ_k}, denoted by B_i^k. Then, there exists $x_{i,1}, \ldots, x_{i,l_i} \in B_i^k$ such that,

$$B_i^k = \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1})$$

and

$$B(x_{i,j}, \gamma_{k+1}) \cap B(x_{i,j'}, \gamma_{k+1}) = \emptyset,\ \forall j, j' \in 1 : l_i$$

and so

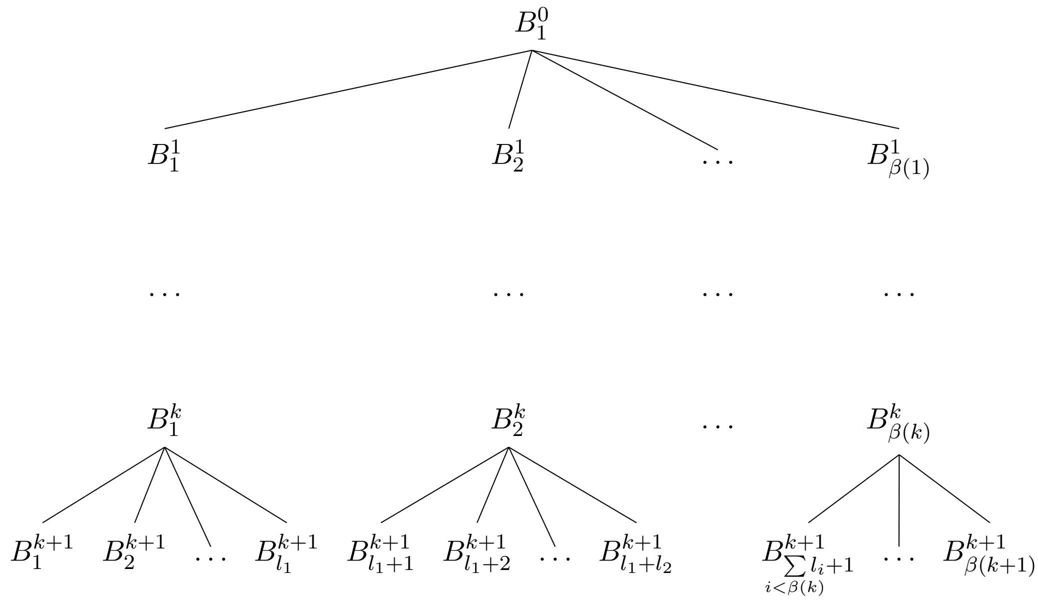
$$S_{\gamma_{k+1}} = \bigcup_{i=1}^{\beta(k)} \cup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1}) = \bigcup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where $\cup_{j=1}^{l_i} B(x_{i,j}, \gamma_k) = B(x_i, \gamma_{k+1}) = B_i^k$, $\forall i$.

Since S is compact, hence bounded, if we represent this process schematically we obtain a tree, where the root node is $B_1^0 = B(x, \gamma_0)$, for any choice of $x \in S$, and the children of any given B_n^m are such that they form a partition of their join. Since we will often refer to this schematic representation, we define it below.

Definition 20. If S is a compact subset of an ultrametric space, then T_s is the tree whose vertices are B_i^k, that is the elements of S_{γ_k}, and whose edgeset, E, is given by $(B_k^i, B_l^j) \in E$ if and only if $j = i + 1$ and $B_l^j \subseteq B_k^i$ for some choice of representatives

$B(x_k, \gamma_i)$ and $B(x_l, \gamma_j)$, as shown below:



Before going on, first note that we have drawn T_S such that leftmost child of some B_i^k is B_j^{k+1} where j is minimal among the children of B_i^k, and then continued in increasing order. In general, if we draw T_S so that the children of a given vertex are depicted in increasing order according to their index, then each choice of indexing for the elements of S_{γ_k} produces a different graphical representation of T_S. The structures produced by different choices of indices are clearly isomorphic as trees, and as we will see by the end of the section, each choice of indexing will be valid for our purposes as well.

Of central importance to us is the distance between two vertices in T_s. Since each vertex represents an element of S_{γ_k}, that is a closed ball in an ultrametric space, it is well-defined to let the distance between vertices be equal to the distance between a choice of centres for those balls. Note that if the distance between B_i^k and B_j^l is taken to be $\rho(x_i, x_j)$, for some choice of $x_i \in B_i^k$ and $x_j \in B_j^l$, say $\rho(x_i, x_j) = \gamma_n$, then the join of B_i^k and B_j^l is some B_x^n.

Lemma 4. *If B_i^k and B_j^l are two vertices in T_S, then $\rho(x_i, x_j)$, for any choice of $x_i \in B_i^k$ and $x_j \in B_j^l$, is equal to the diameter of the join of B_i^k and B_j^l.*

Proof. Let B_i^k and B_j^l be two (distinct) vertices in T_S and let B_x^n be their join. The diameter of B_x^n is γ_n since $B_x^n = B(x_0, \gamma_n)$ for some x_0. Since ρ is an ultrametric the distance between any $x_i \in B_i^k$ and $x_j \in B_j^l$ is constant, and must be equal to the diameter of the smallest ball containing both of them, that is γ_n. □

In particular, we have that for any k and any $i < \beta(k)$, the distances between the children of B_i^k will be γ_k and for any $i \neq j$ the distance between the children of B_i^k and B_j^k will be equal to the distance between B_i^k and B_j^k (which will be some $\gamma_m, m < k$).

Recusive ρ-orderings

In this section, we show how the recursive partioning of S into the spaces S_{γ_k} gives rise to a $\rho-$ordering of S. We first note that without loss of generality, for any $k \in \mathbb{N}$, we can reindex the B_i^k's so that they give the first $\beta(k)$ terms of a ρ_k-ordering of S_{γ_k}, when the latter is viewed as a (finite) metric space. In the first proposition below, we note that if the B_i^k's are so indexed, then finding a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ is straightforward: select a B_j^{k+1} from each of the B_i^k's in order and then start over.

Proposition 23. *Let S be a compact, discretely-valued subset of an ultrametric space (M, ρ) and Γ_S, the set of distances in S. If S_{γ_k} is the partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k-ordering of S_{γ_k}, then the first $\beta(k+1)$ terms in a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ can be found by selecting at each stage n, a child from $B_{\overline{n}}^k$, where $\overline{n} = n \mod \beta(k) + r$ and r is minimal in $\{0, \ldots, \beta(k) - 1\}$ such that $B^k_{n \mod \beta(k)+r}$ still has unused children.*

Proof. Let S, S_{γ_K}, and $S_{\gamma_{k+1}}$ be as above. In particular, suppose the elements of S_{γ_k} are indexed according to a ρ_k-ordering. Denote the elements of $S_{\gamma_{k+1}}$ by $B_{i,j}^{k+1}$ where the first subscript indicates that the elements is a child of B_i^k. To form a ρ_{k+1} ordering of $S_{\gamma_{k+1}}$, we must maximize the product of distances at each step n.

Now note that $\Gamma_{S_{\gamma_k}} = \{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$ and $\Gamma_{S_{\gamma_{k+1}}} = \{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}, \gamma_k\}$. That is, the distances in $S_{\gamma_{k+1}}$ are the same as the distances in S_{γ_k}, although they also include the smaller distance γ_k. Since we know that the elements $B_1^k, \ldots, B_{\beta(k)}^k$ already maximizes the product of distances in $\{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$, the first $\beta(k)$ terms of a ρ_{k+1}-ordering of S_{k+1} can be found by taking $B_{1,j_1}^k, \ldots, B_{1,j_{\beta(k)}}^k$ for any choice of j's. At this point, any choice of next element will produce a copy of γ_k in the ρ_{k+1}-sequence; however, if we chose another child of B_1^k, we are able to keep building the ordering in a canonical fashion, since we know that we will then be able to maximize the product at the next step by chosing another child of B_2^k.

We see then that a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ is found by minimizing the number of times γ_k is introduced into the ρ_{k+1}-sequence and maximizing the product among the $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$, and the latter is already known to be achieved by taking the B_i^k in order. If the B_i^k's all have the same number of children, then we can always select a child of $B_{\overline{n}}^k$, where $\overline{n} = n \mod \beta(k)$ at each stage n, $n < \beta(k+1)$, since there will always be one available. On the other hand, suppose the B_i^k have an unequal number of children and n is the first step at which all the children of $B_{\overline{n}}^k$ have been exhausted. What element will maximize the $\rho_{k+1}-$sequence?

Consider the space $(S_{\gamma_k} \setminus B_{\overline{n}}^k)$. Removal of $B_{\overline{n}}^k$ will not effect the first m terms of a ρ_k-ordering of this space, for $m < \overline{n}$, since if a sequence of elements maximizes a function over a set X, they will also maximize that function of a subset of X (provided they themselves remain in the subset). Then the ρ_k-sequence of $(S_{\gamma_k} \setminus B_{\overline{n}}^k)$ begins $\{B_1^k, \ldots, B_{\overline{n}-1}^k\}$.

Moreover, if $B_{\overline{n}+1}^k$ maximizes $\prod_{i=1}^{\overline{n}} \rho_k(x, B_i^k)$ over S_{γ_k}, then it also maximizes $\prod_{i=1}^{\overline{n}-1} \rho_k(x, B_i^k)$ over $(S_{\gamma_k} \setminus B_{\overline{n}}^k)$, since $\prod_{i=1}^{\overline{n}} \rho_k(x, B_i^k) = (\prod_{i=1}^{\overline{n}-1} \rho_k(x, B_i^k)) \cdot \rho_k(x, B_{\overline{n}}^k)$.

Then the ρ_k-sequence of $(S_{\gamma_k} \setminus B_{\overline{n}}^k)$ is simply $\{B_1^k, \ldots, B_{\overline{n}-1}^k, B_{\overline{n}+1}^k, \ldots, B_{\beta(k)}^k\}$.

Now we see that $\rho_{k+1}-$sequence of $S_{\gamma_{k+1}}$ is maximized by simply skipping over $B_{\overline{n}}^k$, should all its children be exhausted, and selecting a child from $B_{\overline{n}+1}^k$. Then a $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$ is found by selecting elements of each B_i^k in order as much as possible, and skipping to B_{i+1}^k, when it is not possible.

□

Note that in building the $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$ we selected, at each step, a child of some B_i^k, but we did not concern ourselves over which child was selected. This is because the distances between any two children of some B_i^k is γ_k, and the distance between any one of them and a child of some B_j^k, $i \neq j$, is the same. We can now see, as claimed above, that any of the isomorphic versions of T_S are valid for producing $\rho-$orderings. Suppose then that we have created T_s and (arbitrarily) indexed the children of each vertex. Then, there is no loss of genearlity in assuming that at each stage, we select a child with smallest index among its siblings, that is, that we select the leftmost available child in T_s. Since, for ease of indexing, we will assume a $\rho-$ordering has been built by this convention, we introduce the following definition.

Definition 21. The $\rho-$ordering of S formed by pulling elements from left to right in (a choice of) T_s is call the **canonical ρ-ordering** of S (with respect to T_s).

The above proposition quickly leds to a recursive contruction for a $\rho-$ordering of S. Indeed, to build a $\rho-$ordering of S from the above, it suffices only to make a choice of centres for each of B_i^k's.

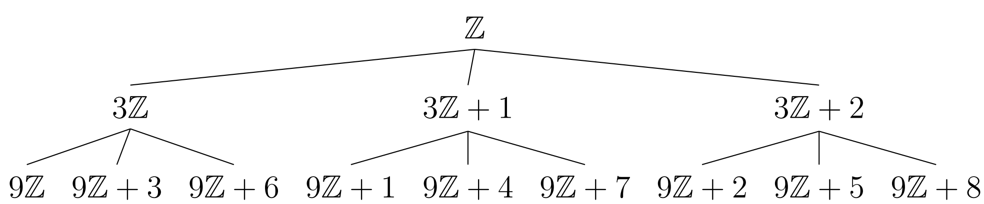
Proposition 24. *Let be S a compact, discretely-valued subset of an ultrametric space (M,ρ) and let Γ_S be the set of distances in S. Let S_{γ_k} be the partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k-ordering of S_{γ_k}. Suppose each of the element of S_{γ_k} have also been partitioned into closed balls of radius γ_{k+1}, $B_i^k = \cup_{j=1}^{l_i} B_{i,j}^{k+1}, \forall i$.*

*Let $x_{i,j}$ denote a choice of centre for the element $B_{i,j}^{k+1}$. Then the first $\beta(k+1)$ elements of a $\rho-$ordering of S can be found by forming a matrix, A_k, whose $(i,j)^{th}$ entry is $x_{i,j}$, if $j \le l_i$ and * otherwise, and then concatenating the rows.*

Proof. The matrix A_k is a representation of the k^{th} and $(k+1)^{th}$ levels of T_S where the B_i^k's (and $B_{i,j}^{k+1}$'s) have been replaced by a choice of centres. Since matrices must be rectangluar, the case where some B_i^k and B_j^k have an unequal number children is handled by inserting a placeholder, *, into A_k. Moreover, since the ρ_{k+1} distance between distinct closed balls is just the ρ distance between a choice of centres of those balls, a choice of centres in a ρ_{k+1}-ordering gives the beginning of a ρ-ordering. By the above proposition, we must select elements from each B_i^K one after the other, which is achieved by selecting one element from each column in order, for example by concatenating the rows (and then deleting *'s if necessary). $\square$

We get the most use out of the construction above if, in selecting a choice of centres for the $B_{i,j}^{k+1}$'s, we reuse the previous choices as much as possible. Suppose for example we have made a choice of centres for the balls of radius γ_k and constructed the matrix A_{k-1}. At the next iteration, we will need a choice of centres for the balls of radius γ_{k+1}. If x_i was our choice of representative for B_i^k and $x_i \in B_{i,j}^{k+1}$, we may as well let x_i be our choice of representative for $B_{i,j}^{k+1}$. If we make our choice of centres in this way, then when we concatenate the rows of some A_{k-1}, we obtain (without loss) the first row of A_k. We follow this convention in the two examples below.

Example 8. Let us use the above to start a ρ-ordering of $S = (\mathbb{Z}, \rho_3)$. We have that $\Gamma_S = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\}$ and T_s begins:



We start by finding a ρ_0-ordering of S_{γ_0}, but this is trival since S_{γ_0} has only a single element. Let us pick 0 to be our choice on centre for $B_1^0 = B(0,1) = \mathbb{Z}$. As we see from T_S, S_{γ_0} is partitioned into 3 closed balls of radius $\gamma_1 = \frac{1}{3}$, namely $3\mathbb{Z}, 3\mathbb{Z}+1$, and $3\mathbb{Z}+2$. A choice of centres is given by $0, 1$, and 2, so that A_0 becomes:

$$A_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

To start the $\rho-$ordering, concatenate the rows to obtain $\{0, 1, 2\}$, and to continue it, make a choice of centres for each of the closed balls of radius $\gamma_2 = \frac{1}{9}$ partitioning the sets $3\mathbb{Z}+i$, $i \in 0,1,2$. For example, $3\mathbb{Z} = 9\mathbb{Z} \cup 9\mathbb{Z}+3 \cup 9\mathbb{Z}+6$, so a choice of centres for B_1^1 is given by $\{0,3,6\}$. Making choices for the remaining elements, we obtain:

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

To continue the $\rho-$ordering we concatenate the rows, $\{0,1,2,3,4,5,6,7,8\}$, which also gives the first row of A_2. The remaining rows are found by partitioning each of the closed balls of radius $\frac{1}{9}$ and again making a choice of centres:

$$A_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{pmatrix}$$

And so on.

We are able to make two statements following this example. The first is that in starting the ρ_3-ordering, the fact that S_{γ_0} had only a single element allowed us to get started for free. In fact, all compact ultrametric spaces are bounded, so this is always the case.

The second takeaway is that we found the start of a $\rho-$ordering of $S = (\mathbb{Z}, \rho_3)$ was given by taking the integers starting at 0 in their natural order. If we had continued building the ordering, we would have continued to find this. The fact that the natural ordering on the integers is a ρ_p-ordering, where ρ_p is the $p-$adic metric for any prime p, is well known ([B1]), but we give an alternate proof of it here:

Corollary 5. *Let S be the ultrametric space $(\mathbb{Z}, \rho_p)$, where ρ_p is $p-$adic metric for any prime p. The a ρ_p-ordering of S can be found by taking the integers, starting at* 0*, in their natural order.*

Proof. We prove the above by induction on k. First note that for any choice of prime, the elements of S_{γ_1} are the cosets of $\mathbb{Z}$ modulo p, so that A_1 has p columns.

Since $\{0, 1, 2 \ldots, p-1\}$ are distributed among each of these cosets, without loss of generality the first row of A_1 is given by $[0, 1, 2, \ldots, p-1]$ in order.

Now suppose that the first row of A_k is given by $[0, 1, 2, \ldots, n]$ for $0 < k < k+1$. We show the first row of A_{k+1}, and therefore the first n' elements in a ρ_p−ordering of S, where n' is the column dimension of A_{k+1}, can be obtained as $[0, 1, 2, \ldots, n, n+1, \ldots, n']$. First note that each closed ball of radius $p^k = \gamma_k$ is in fact a coset of $\mathbb{Z}$ modulo p^k, of which there are p. Then for any k, A_k is a matrix with p^k columns and p rows. In particular, $n = p^k - 1$. Let $i \in \{0, 1, \ldots, p^k - 1\}$ be arbitrary. Then i is in exactly one of the cosets of $\mathbb{Z}$ modulo p^k and since the first row of A_k is $[0, 1, 2, \ldots, p^k - 1]$, it must have been chosen as our representative of this coset. If we split $p^k\mathbb{Z} + i$ into balls of radius p^{k+1}, we have

$$p^k\mathbb{Z} + i = \bigcup_{j=0}^{p-1} p^{k+1}\mathbb{Z} + (p^k j + i)$$

since there will be p elements in the partition, each of which will be equal to i modulo p^k and distinct modulo p^{k+1}. Then, there is a choice of centres such that the i^{th} column of A_k is

$$[i, p^k + i, 2p^k + i, \ldots, (p-1)p^k + i]^T$$

filling this in for each i, we see that A_k can be obtained as:

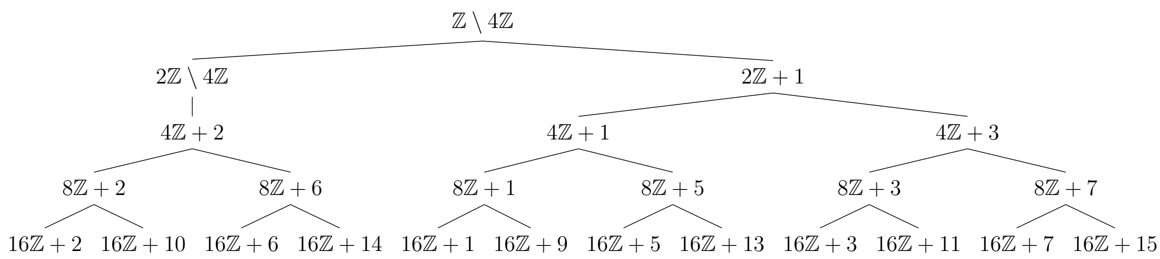
$$A_k = \begin{pmatrix} 0 & 1 & 2 & \ldots & p^k - 1 \\ p^k & p^k + 1 & p^k + 2 & \ldots & p^k + (p^k - 1) \\ 2p^k & 2p^k + 1 & 2p^k + 2 & \ldots & 2p^k + (p^k - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)p^k & (p-1)p^k + 1 & (p-1)p^k + 2 & \ldots & (p-1)p^k + (p^k - 1) \end{pmatrix}$$

Concatenating the rows, we see the first row of A_{k+1} will be

$$[0, 1, 2, \ldots, p^k - 1, p^k, \ldots, p^{k+1} - 1]$$

as required. $\square$

Example 9. Let us now see an example where there is an uneven number of children between the vertices on a given level. Suppose $S = \mathbb{Z} \setminus 4\mathbb{Z}$, a subset of $(\mathbb{Z}, \rho_2)$. In this case, we have that $\Gamma_S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ and T_s begins:



Choosing centres for the partition of $\mathbb{Z}$ into closed balls of radius $\frac{1}{2}$, we have:

$$A_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have taken S to be the complement of $4\mathbb{Z}$ in $\mathbb{Z}$, so $B(0, \gamma_1)$ has only one child, since $2\mathbb{Z} \setminus 4\mathbb{Z} = 4\mathbb{Z} + 2$, while $B(1, \gamma_1)$ has two. Making a choice of centres, we have:

$$A_1 = \begin{pmatrix} 2 & 1 \\ * & 3 \end{pmatrix}$$

We concatenate the rows, skipping over *, and again make a choice of centres for the closed balls of radius $\frac{1}{8}$:

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \end{pmatrix}$$

One more iteration yields:

$$A_2 = \begin{pmatrix} 2 & 1 & 3 & 6 & 5 & 7 \\ 10 & 9 & 11 & 14 & 13 & 15 \end{pmatrix}$$

So that a ρ_2−ordering of $S = \mathbb{Z} \setminus 4\mathbb{Z}$ starts: $\{2, 1, 3, 6, 5, 7, 10, 9, 11, 14, 13, 15, \ldots\}$.

In the two propositions above, there was notational difficulty that arose when there was an unequal number of children between the vertices on a given level of T_s. This difficulty is, in fact, more than a notational inconvenience, and the situation simplifies considerably when it is not the case. We are far from the first to observe this. Amice noted this as far back as her 1964 paper [Am], and it has been observed more recently by Chabert and colleagues, for example in [FP] and [CEF]. In the next chapter, we see that if T_S has nice enough structure we are able to compute not just $\rho-$orderings, but also formulae for $\rho-$sequences and, when we are lucky, capacity.

Chapter 4

The structure of T_S

In the previous chapter, we explored in detail the final Corollary of Chapter 2. This corollary lead us to study the lattice of closed balls in S, which we called T_S. In this chapter, we take what we have learned and explore the preceding Corollary, repeated below.

Corollary. *Suppose $S = \cup_i^n S_i$ with $\rho(S_i, S_j) = d = diam(S)$ and also $\omega(S_i) = \omega(S_j)$, $\forall i, j$. Let $r \in \mathbb{R}$ be such that $\omega(S_i) = r\omega(S)$, $\forall i$. Then $\omega(S) = r^{\frac{1}{n-1}}$.*

In particular, we seek answers to the following questions: when does such a partition of S exist and given such a partition, when are we able to compute the scaling factor r? In doing so, we show that the structure of T_S plays an important role.

4.1 Semi-regularity

In this section, we restrict to the case where in the tree T_s, for S a compact, discretely-valued subset of an ultrametric space, every vertex on a given level has the same number of children. In this case, we can attach another sequence to S, which we call the $\alpha-$sequence of S and which describes, for each level $k \in \mathbb{N}$, the size of the partitions on that level. We develop some preliminary lemmas, which we then use to derive formulae for this special case. This situation corresponds to what previous authors ([Am], [CEF], [FP]) have called *regularity*, a term which we reserve for the next section.

Definition 22. Let S be as before, a compact, discretely-valued subset of an ultrametric space (M, ρ). We say that S is **semi-regular** if $T_{B_i^k} \cong T_{B_j^k}$, $\forall k \in \mathbb{N}$ and $i, j \in \beta(k)$, and where the isomorphism is understood as an isomorphism of trees. That is, S is semi-regular if each ball of radius γ_k breaks into the same number of balls of radius γ_{k+1}, for all k. If there exists an $n \in \mathbb{N}$ such that $T_{B_i^N} \cong T_{B_j^N}$ for all $N \geq n$, that is, each ball of radius γ_N breaks into the same number of balls of radius γ_{N+1} for $N \geq n$, then we say S is **eventually semi-regular**.

Definition 23. Suppose S is a compact, discretely-valued subset of an ultrametric space and S is semi-regular. The α**-sequence** of S is the sequence given by

$$\alpha(k) = \frac{\beta(k+1)}{\beta(k)}$$

which is in $\mathbb{N}$ for each k. That is, if B_i^k is any element of S_{γ_k}, then $\alpha(k)$ is equal to the number of children of B_i^k in T_s. Since S is semi-regular, this number does not depend on i.

Example 10. If G is a compact ultrametric space and also a group, each ball centred at 0 is in fact a subgroup of G. Then each set of elements of S_{γ_k} is a collection of cosets of $G/B(0, \gamma_k)$. Since G is assumed to be compact, $G/B(0, \gamma_k)$ is finite and so Lagrange's theorem implies that G is semi-regular.

We now work towards a formula for the terms in the $\rho-$sequence of a semi-regular space S. We need a few lemmas to get started.

Lemma 5. *Let n and q be in $\mathbb{N}$. Then $\lfloor \frac{n}{q} \rfloor$ counts the numbers strictly less than n that are congruent to $n \mod q$.*

Proof. By the division algorithm, we know there exists unique $c, r \in \mathbb{Z}$ such that

$$n = cq + r$$

with $0 \leq r < q$. Since c counts the number of q-multiples in the set $\{1, \ldots, n-1\}$, and each q-multiple contains exactly one element that is congruent to $n \mod q$, we need only show $\lfloor \frac{n}{q} \rfloor = c$. Simply note that the above implies

$$\frac{n}{q} = c + \frac{r}{q}$$

and we must have $\frac{r}{q} < 1$. Then c is the largest integer such that $\frac{n}{q} \leq c$, but this is the definition of $\lfloor \frac{n}{q} \rfloor$. $\square$

Lemma 6.
$$\lfloor\frac{n}{b}\rfloor - \lfloor\frac{n}{ab}\rfloor = \sum_{k=1}^{a-1}\lfloor\frac{n+kb}{ab}\rfloor$$

for $n, a, b \in \mathbb{N}$. In particular,

$$\lfloor\frac{n}{b}\rfloor - \lfloor\frac{n}{2b}\rfloor = \lfloor\frac{n+b}{2b}\rfloor$$

for $n, b \in \mathbb{N}$.

Proof.
$$\lfloor\frac{n}{b}\rfloor - \lfloor\frac{n}{ab}\rfloor = \lfloor a\cdot\frac{n}{ab}\rfloor - \lfloor\frac{n}{ab}\rfloor = \sum_{k=0}^{a-1}\lfloor\frac{n}{ab}+\frac{k}{a}\rfloor - \lfloor\frac{n}{ab}\rfloor \ (*)$$

$$= \sum_{k=1}^{a-1}\lfloor\frac{n}{ab}+\frac{k}{a}\rfloor = \sum_{k=1}^{a-1}\lfloor\frac{n+kb}{ab}\rfloor$$

where the final step in (*) is due to Hermite's identity: $\lfloor nx\rfloor = \sum_{k=0}^{n-1}\lfloor x+\frac{k}{n}\rfloor$, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. □

Lemma 7. *If S is semi-regular and σ denotes the canonical ρ-ordering of S, that is, a $\rho-$ordering formed by pulling from left to right in T_s, then*

$$\rho(\sigma(n), \sigma(m)) = \gamma_k$$

if and only if

$$n = m \mod \beta(k) \text{ and } n \neq m \mod \beta(k+1)$$

Proof. Since S is semi-regular, every sequence of $\beta(k)$ terms in σ will be from each of distinct elements of S_{γ_k} (for any k). Moreover, since σ is a canonical $\rho-$ordering, we always pull from the elements of S_{γ_k} in the same order. Then $\sigma(n)$ and $\sigma(m)$ are descendents of some B_j^k if and only if $n = m \mod \beta(k)$. Then the result follows since $\rho(\sigma(n), \sigma(m)) = \gamma_k$ if and only if B_i^k for some $i \in 1, \dots, \beta(k)$ is the join of $B_i^n \ni \sigma(n)$ and $B_{i'}^m \ni \sigma(m)$.

□

We introduce another piece of notation before continuing.

Notation 2. Let S be a compact, discretely-valued subset of an ultrametric space, Γ_S the set of distances in S and $\delta(n)$ the characteristic sequence of S. Suppose γ_k is an element of Γ_S. Then we denote by $v_{\gamma_k}(\delta(n))$ the exponent of γ_k in the $n^{th}-$term of the characteristic sequence of S.

Proposition 25. *If S is a semi-regular ultrametric space, δ is the characteristic sequence of S, β is the structure sequence of S, and α is the sequence describing the semi-regularity, then*

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Proof. The exponent of γ_k in the n^{th} term of the characteristic sequence is the number of m strictly less than n such that $\rho(\delta(n), \delta(m)) = \gamma_k$. By the lemma above, this the number of $m < n$ such that $m = n \mod \beta(k)$ and $m \neq n \mod \beta(k+1)$, which by the previous lemma is $\lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor$. Then we have:

$$\begin{aligned} & v_{\gamma_k}(\delta(n)) \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k)\alpha(k)} \rfloor, \text{ because } S \text{ is semi-regular} \\ &= \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor \end{aligned}$$

□

Example 11. Consider the ultrametric space $(\mathbb{Z}, \rho_p)$ for any prime p. Then $\beta(k) = p^k$ and $\alpha(k) = p$ for any $k \in \mathbb{N} \cup 0$. The above gives

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor$$

Now since $\gamma_k = p^{-k}$, $\forall k$, we are able to compute the exponent of $\frac{1}{p}$ in $\delta(n)$. We have

$$v_{\frac{1}{p}}(\delta(n))$$

$$= \sum_{k=1}^{\infty} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor)$$

$$= \sum_{k=1}^{\lceil log_p(n) \rceil} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor)$$

$$= \lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor + 2\lfloor \frac{n}{p^2} \rfloor - 2\lfloor \frac{n}{p^3} \rfloor + \ldots + \lceil log_p(n) \rceil \lfloor \frac{n}{p^{\lceil log_p(n) \rceil}} \rfloor$$

$$= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \ldots + \lfloor \frac{n}{p^{\lceil log_p(n) \rceil}} \rfloor$$

$$= \sum_{k=1}^{\lceil log_p(n) \rceil} \lfloor \frac{n}{p^k} \rfloor$$

$$= \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$$

We are able to simplfy to a finite sum in the above because $\lfloor \frac{n}{p^k} \rfloor = 0$ if

$$p^k > n \iff log(p^k) > log(n) \iff k > log_p(n)$$

We have already seen that the natural order on the integers gives a ρ_p-ordering for each p. So then

$$\sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor = v_{\frac{1}{p}}(\delta(n)) = v_{\frac{1}{p}}(\prod_{i=0}^{n} \frac{1}{p}^{v_p(n-i)}) = \sum_{i=0}^{n-1} v_p(n-i) = v_p(n!)$$

so that we are able to recover the well-known Legendre's formula.

We end this section with the following observation.

Proposition 26. *Let S be a semi-regular subset of an ultrametric space (M, ρ). Let S_{γ_1} be the partition of S described in chapter 3, that is,*

$$S_{\gamma_1} = \bigcup_{i=1}^{b} B(x_i, \gamma_1) = \bigcup_{i=1}^{n} B_i^1$$

Then $\rho(B_i^1, B_j^1) = d = diam(S)$ for any $i \neq j$ in $1, \ldots, n$ and $\omega(B_i^1) = \omega(B_j^1)$ for all i and j.

Proof. The fact that $\rho(B_i^1, B_j^1) = d = diam(S)$ for any $i \neq j$ is clear and does not depend on the fact that S is semi-regular. In fact, there are plently of ways to see this, but for example, we simply note $\rho(B_i^1, B_j^1) \in \Gamma_S$ and $\gamma_1 < \rho(B_i^1, B_j^1) \leq \gamma_0 = diam(S)$.

To see that $\omega(B_i^1) = \omega(B_j^1)$, we note that since S is semi-regular, each B_i^1 is semi-regular as well. Moreover, since S is semi-regular, the β sequences of B_i^1 and B_j^1 are the same for each i and j. Then the result follows: let $\delta^{B_i^1}(n)$ and $\delta^{B_j^1}(n)$ be the characteristic sequences of B_i^1 and B_j^1 respectively. We see that for all k,

$$v_{\gamma_k}(\delta^{B_i^1}(n)) = \lfloor \frac{n}{\beta^B(k)} \rfloor - \lfloor \frac{n}{\beta^B(k+1)} \rfloor = v_{\gamma_k}(\delta^{B_j^1}(n))$$

where $\beta^B(k)$ is the β sequence for each B_i^1.

□

Now we have one answer to our first question: when S is semi-regular, we can use the elements of S_{γ_1} to build the partition from the corollary. The content of that corollary gave a formula for the valuative capacity. Then if S is semi-regular, the principal obstacle to computing the capacity of S is the identification of the scaling factor. This leads to our second question: when can we compute r?

4.2 Regularity

In Example 11, the fact that we were able to reduce to a finite sum was not the only reason we were able to simplfy the calculations. It also helped a great deal that the sum was telescoping. What does the fact that we saw a telescoping sum have to do with computing the scaling factor r? We explore the inter-relatedness of these situations, and with the definition below, in this section.

Definition 24. Let S be a semi-regular subset of an ultrametric space. If there exists a $q \in \mathbb{N}$ such that $\alpha(n) = q$, for all n, then S is said to be **regular**[1].

So then S is regular just in case S is semi-regular and the $\alpha-$sequence of S is constant. We need to make one more definition before we begin calculations.

[1]This is non-standard: what previous authors ([Am],[CEF], [FP]) have called regular is what we have called semi-regular. Note that S is regular in the present sense if and only if T_S is regular in the standard graph theory terminology.

Definition 25. Let S be a semi-regular subset of an ultrametric space and Γ_S is the sequence of decreasing distances in S. Then we say S is **tame**, if for $\gamma_k \in \Gamma_S$,

$$\gamma_k = \alpha(k)^{c_k}$$

for some $c_k \in \mathbb{Q}$ and for all $k \in \mathbb{N}$.

Now we see what this situation means for calculations.

Proposition 27. *Let S be a regular, tame subset of a compact ultrametric space with $\gamma_k = q^{c_k}$ for some $c_k \in \mathbb{Q}$ and for all $k \in \mathbb{N} \cup 0$. Then*

$$v_q(\delta(n)) = c_0 n + \sum_{k=1}^{\infty}(c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

and

$$log_q(\omega(S)) = \lim_{n\to\infty} c_0 + \frac{1}{n}\sum_{k=1}^{\infty}(c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

Proof. We know that,

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

and since $\gamma_k = q^{c_k}$, we calculate

$$v_{q^{c_k}}(\delta(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

and

$$v_q(\delta(n)) = \sum_{k=0}^{\infty} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor)$$

$$= c_0 n + \sum_{k=1}^{\infty}(c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

Then since $\omega(S) = \lim_{n\to\infty} \delta(n)^{\frac{1}{n}}$,

$$log_q(\omega(S)) = log_q(\lim_{n\to\infty} \delta(n)^{\frac{1}{n}}) = log_q(\lim_{n\to\infty} q^{c_0 n + \sum_{k=1}^{\infty}(c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor^{\frac{1}{n}}})$$

$$= \lim_{n\to\infty} c_0 + \frac{1}{n}\sum_{k=1}^{\infty}(c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

□

Let us make a small digression, motivated by the following observation: if, as in the case of $p-$adic spaces, $c_i = -i$ for all i, then the above simply reduces to $lim_{n\to\infty} \frac{v_q(n!)}{n}$. Moreover, we know that $\{0, 1, 2, \ldots\}$ is simultaneously a $p-$ordering of $\mathbb{Z}$ (with $p-$sequence $v_p(n!)$) for all primes p.

Question 3. Let M be a set and suppose ρ_j is collection of ultrametrics on M such that each (M, ρ_j) is regular and tame with $c_i - c_{i+1} = c$ for some constant c, for all i and for each j. Then if $\{a_i\}_{i\geq 0}$ is a ρ_j-ordering of (M, ρ_j) for some j, is $\{a_i\}_{i\geq 0}$ simultaneously a ρ_j-ordering for all j? In particular, if S is a subset of $\mathbb{Z}$, does S being regular imply that S has simultaneous $p-$ordering for each prime p?

We return now to the question at hand. If S is semi-regular, we have already seen that the partition of S given by the elements of S_{γ_1} is such that each element has equal capacity and the pairwise distance between them is equal to the diameter of S. Now we notice that if S is regular and tame, then so is $B(x_i, \gamma_1)$ for each i. This gives us,

$$log_q(\omega(B(x_i, \gamma_1))) = \lim_{n\to\infty} c_1 + \frac{1}{n} \cdot \sum_{k=1}^{\infty}(c_{k+1} - c_k)\lfloor \frac{n}{q^k} \rfloor$$

Putting these together, we can solve for the scaling factor. If

$$\omega(S) = r \cdot \omega(B)$$

then

$$log_q(r) = \lim_{n\to\infty} c_0 + \frac{1}{n}\sum_{k=1}^{\infty}(c_k - c_{k-1})\lfloor \frac{n}{q^k} \rfloor - c_1 - \frac{1}{n} \cdot \sum_{k=1}^{\infty}(c_{k+1} - c_k)\lfloor \frac{n}{q^k} \rfloor$$

$$= \lim_{n\to\infty} c_0 - c_1 + \frac{1}{n}\sum_{k=1}^{\infty}(c_k - c_{k-1}) - (c_{k+1} - c_k)\lfloor \frac{n}{q^k} \rfloor$$

When do we know the value of this limit? One case is obvious, namely the case where $(c_{k+1} - c_k) = (c_k - c_{k-1})$, which is guarenteed if the distances between each c_k

and c_{k+1} is constant. In this case, we see right away that the scaling factor r is equal to $q^{c_0-c_1}$. In particular, this gives an alternate proof for the fact that $p\cdot\omega(p\mathbb{Z}) = \omega(\mathbb{Z})$ and one which does not rely (directly) on any algebraic structure.

It is now clear that if we want to get the most leverage out of regularity, we need more assumptions on our space than we did for semi-regularity. We have seen something like this before. If S is a group with translation invariant metric, we can use translation invariance right away. It implies the cosets of S modulo balls centred at 0 all have the same capacity, which allows us to simplify the right-hand side of the subadditivity formula. If S has a multiplicative norm though, there is one situation in which this property is distinctly more useful. That is, we get the most use out of a multiplicative norm when the subgroups corresponding to the balls centered at 0 are cyclic.

When S is an ultrametric space with algebraic structure, translation invariance and scaling under a norm can be very effective tools for computing capacity. The results of this chapter give us a sense in which we can generalize this toolkit. Indeed, semi-regularity and regularity respectively provide the analogous notions. Semi-regularity implies the presence of a sort of "well-balanced" partition of S that we can use in the subadditivity formula. Likewise, regularity shows us that we can recover a notion of scaling, although as with a multiplicative norm, to get the most out of this, the conditions have to be right.

Chapter 5

Application: Product spaces of $\mathbb{Z}$

We consider now an application on the above. A natural space to consider is the product space of ultrametric spaces, for example $\mathbb{Z}^n$, for some $1 < n < \infty$. If we restrict our attention to bounded subsets, then a natural candidate for an ultrametric on a finite product space is given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max_i\{\rho(x_i, y_i)\}$$

where ρ is the metric from the base space. In fact, since we have only defined valuative capacity for compact subsets of an ultrametric spaces, there is no loss of generality by restricting our metric to bounded spaces. We also see that no problems arise in letting both M and ρ vary between components of the space, as long as each M_i remains bounded and each ρ_i is an ultrametric.

Proposition 28. *Let (M_i, ρ_i) for i in some finite index set, I, be a collection of metric spaces and suppose ρ_i is a bounded ultrametric for all i. Then (M, ρ_∞) is an ultrametric space, where $M = M_1 \times M_2 \times M_3 \times \ldots \times M_n$ and ρ_∞ is the metric described above.*

Proof. Let (M, ρ_∞) be the product of ultrametric spaces as above and let x and y be two points in the space. Clearly, $\rho_\infty(x, y) \geq 0$ since each $\rho_i(x_i, y_i) \geq 0$, and $\rho_\infty(x, y) = 0 \iff \rho_i(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$. The fact that ρ_∞ is symmetric is also an easy consequence of the fact that each ρ_i is symmetric since $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$ implies $\max_i\{\rho_i(x_i, y_i)\} = \max_i\{\rho_i(y_i, x_i)\}$. To see that ρ_∞ is an ultrametric, note that if $z = z_i$ is any other point of M, then

$$\begin{aligned}
\rho_\infty(x,y) &= \max_i\{\rho_i(x_i,y_i)\} \\
&\le \max_i\{\max(\rho_i(x_i,z_i),\rho_i(y_i,z_i))\} && \text{since each } \rho_i \text{ is an ultrametric} \\
&\le \max(\max_i\{\rho_i(x_i,z_i)\},\max_i\{\rho_i(y_i,z_i)\}) \\
&= \max(\rho_\infty(x,z),\rho_\infty(y,z))
\end{aligned}$$

□

We first show that translation invariance carries over into product spaces under the expected conditions.

Proposition 29. *Suppose* (M,ρ_∞) *is the product of ultrametric spaces* (M_i,ρ_i) *and each* M_i *is a topological group with operation* $+_i$*. Let* $+$ *denote the operation on* M *given by* $s + x = (s_1 +_1 x_1, s_2 +_2 x_2, \ldots, s_n +_n x_n)$ *for* $s = (s_1,\ldots,s_n)$*and* $x = (x_1,\ldots,x_n)$ *in* (M,ρ_∞)*. Then* ρ_∞ *is (left) translation invariant under* $+$ *if each* ρ_i *is (left) translation invariant under* $+_i$*, in which case valuative capacity is also (left) translation invariant.*

Proof. Let (M,ρ_∞) be as above. Suppose also that

$$\rho_i(x_i,y_i) = \rho_i(s_i +_i x_i, s_i +_i y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each ρ_i is (left) translation invariant. Then,

$$\rho_\infty(s+x,s+y) = \max_i\{\rho_i(s_i +_i x_i, s_i +_i y_i)\} = \max_i\{\rho_i(x_i,y_i)\} = \rho_\infty(x,y).$$

so that ρ_∞ is translation invariant. Proposition 20 implies valuative capacity is as well. □

In the next proposition, we show that scaling carries over to product space as well, although the conditions are now more restrictive. In contrast to the proposition above, here we cannot allow the spaces to vary between components.

Proposition 30. *Let (m, ρ_N) be an ultrametric space, where ρ_N is the metric induced by some norm N. Let (M, ρ_∞) be the ultrametric space formed by taking products of m, along with the ρ_∞ metric defined above. Then if ρ_N is multiplicative on m, ρ_∞ is multiplicative on M, in the sense that $\rho_\infty(cx, cy) = |c|_{\rho_N}\ \rho_\infty(x, y)$, for $c = (c, c, c, \ldots), x, y \in M$.*

Proof. Let M, ρ, and ρ_∞ be as above. Then,

$$\begin{aligned}
&\rho_\infty(cx, cy) \\
&= \max_i\{\rho_N(c_i x_i, c_i y_i)\} \\
&= \max_i\{|c|_{\rho_N}\ \rho_N(x_i, y_i)\} \\
&= |c|_{\rho_N}\ \max_i\{\rho_N(x_i, y_i)\} \\
&= |c|_{\rho_N}\ \rho_\infty(x_i, y_i)
\end{aligned}$$

□

Corollary 6. *Let S be a subset of (M, ρ_∞), where M is the product of an ultrametric space (m, ρ_N), which is itself a normed vector space with a multiplicative norm inducing ρ_N. If $c = (c, c, c, \ldots)$ is an element of M with constant value on each component, then $\omega(cS) = |c|_{\rho_N}\ \omega(S)$.*

Proof. The result follows by noting that if $\{a_j\}_{j=0}^\infty$ is a ρ_∞ ordering of S, then $\{ca_j\}_{j=0}^\infty$ is a ρ_∞ ordering of cS. □

We now introduce two examples, the details of which make up the rest of this chapter.

Example 12. Let $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ be the metric space with elements $\{(x, y) \mid x, y \in \mathbb{Z}\}$ and metric $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1)), \rho_p(x_2, y_2))$, where ρ_p is the p-adic metric for some fixed prime p. Since ρ_p is translation invariant and multiplicative, valuative capacity is also translation invariant and multiplicative in $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$.

Example 13. Let $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_\infty)$ be the metric space with elements $\{(x, y) \mid x, y \in \mathbb{Z}\}$ and metric $\rho_\infty((x_1, x_2), (y_1, y_2)) = \max(\rho_{p_1}(x_1, y_1)), \rho_{p_2}(x_2, y_2))$, for two distinct primes, $p_1 \neq p_2$, where both ρ_{p_i} are p-adic metrics. Since each ρ_{p_i} is translation invariant in $\mathbb{Z}_{p_i}$, valuative capacity will be translation invariant in $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{p,\infty})$; however, unlike the case of $p_1 = p_2$, this space does not have a multiplicative property that allows for scaling.

What is the valuative capacity of $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ from the example above? Suppose $p = 2$. Using translation invariance, scaling and subaddivity, we can compute the result by first noting that we can write $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always $1 = diam(\mathbb{Z}_2 \times \mathbb{Z}_2)$, subadditivity implies that

$$\frac{1}{log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

$$= \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))}$$

$$= \frac{4}{log(|2|_2 \cdot \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{log(\frac{1}{2} \cdot \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))}$$

Then,

$$4 \cdot log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)) = log(\frac{\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)}{2})$$

so that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a solution of the equation $x^4 - \frac{x}{2}$, for which there is a single real positive root, given by $2^{-1/3}$.

To compute the valuative capacity for a 2-fold product for an arbitary prime p, note that we can always decompose $\mathbb{Z}_p \times \mathbb{Z}_p$ into a union of p^2 sets each of the form $\{p\mathbb{Z}_p + s \times p\mathbb{Z}_p + t\}$ for $s, t \in (0, \ldots, p-1)$, and the pairwise distance between these sets will always be $1 = diam(\mathbb{Z}_p \times \mathbb{Z}_p)$ (to see this, either note that we can always find co-prime elements, or note that each set is an closed ball of radius $1/p$ centred at (s,t) and so the distance between them must be greater than $1/p$, and 1 is the only possible distance greater than $1/p$ in $\mathbb{Z}_p \times \mathbb{Z}_p$). Then, we combine our tools as before to obtain the equation,

$$\frac{1}{log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(|p|_p \cdot \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{log(\frac{1}{p} \cdot \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{p^2} = \frac{\omega(\mathbb{Z}_p \times \mathbb{Z}_p)}{p}$$

So that $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$ is a solution of the equation $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p})$ over $\mathbb{R}$. Since $\mathbb{R}$ is a division ring, this means the positive solutions are given by solving $x^{p^2-1} - \frac{1}{p}$. Solutions of this equation are of the form $p^{\frac{-1}{p^2-1}}$ times a $p^2 - 1$ root of unity, and so there is exactly one positive, real solution, namely $p^{\frac{-1}{p^2-1}}$ itself. Then the valulative capacity of the entire product space $\mathbb{Z}_p \times \mathbb{Z}_p$ is $p^{\frac{-1}{p^2-1}}$. In fact, from here it is not hard to see that by taking the n-fold product, we would end up with the same equation except that the exponent of p would become n rather than 2. We arrive at the following result:

Proposition 31. *Let $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$ be the ultrametric space with points equal to the n-fold product of $\mathbb{Z}_p$ (for $n < \infty$) for some fixed prime p. The valuative capacity of M is $(\frac{1}{p})^{\frac{1}{p^n-1}}$.*

Proof. Above. □

Taking $n = 1$, we see that this agrees with the valuative capacity of $\mathbb{Z}$ computed in the second chapter.

What about $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ for distinct primes? These spaces do not admit a scaling property, so the same toolset is not available. They are however semi-regular, so we know that

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Suppose $p_1 = 2$ and $p_2 = 3$. Recall that the α sequence of $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$ counts the number of closed balls of radius γ_{k+1} partitioning a closed ball of radius γ_k. In this case, Γ_S is the non-positive powers of 2 or 3 sorted into decreasing order, so that Γ_S starts $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}, \ldots\}$ and $\alpha(S)$ starts $\{6, 2, 3, 2, 2, 3, 2, 3, 2, \ldots\}$. The β sequence of S, which counts the number of distinct balls of a fixed radius, then starts $\{6, 12, 36, 72, 144, \ldots\}$.

We know that the capacity of S will be a product of some negative power of 2 and some negative power of 3. From Lemma 6, we know that when $\alpha(k) = 2$, we have

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n + \beta(k)}{2 \cdot \beta(k)} \rfloor$$

and when $\alpha(k) = 3$, we have

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n + \beta(k)}{3 \cdot \beta(k)} \rfloor + \lfloor \frac{n + 2 \cdot \beta(k)}{3 \cdot \beta(k)} \rfloor$$

We also know that if $\alpha(k) = 2$, then γ_k must be a (negative) power of 2, and likewise if $\alpha(k) = 3$, then γ_k is a power of 3.

Let us first explore the exponent of 2 in $\delta(n)$. We start by noting that if γ_k is some 2^{-i}, then

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

since there will be a copy of 2 in $\beta(k)$ for every occurence of 2 in $\alpha(0), \ldots, \alpha(k)$, which is also what i counts. So then, the exponent of $\frac{1}{2}$ in the n^{th} characteristic sequence of S is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

What can we say about j, the exponent of 3?

Lemma 8. *Let $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$ and consider the k^{th} element of the β sequence of S, $\beta(k) = 2^i \cdot 3^j$. If k is such that $\gamma_k = 2^{-i}$ for some i, then j counts the numbers $a \in \mathbb{Z}_{\geq 0}$ such that $3^a < 2^i$.*

Proof. Γ_S is strictly monotone decreasing and each γ_k is equal to a non-positive power of 2 or 3. If $\gamma_k = 2^i$, then all non-positive powers of 3 and 2 which are greater than 2^i must be equal to some γ_j, $0 \leq j < k$. That is, 2^i only appears in the Γ_S sequence after all larger powers of 2 and 3 have been exhausted. Since we are only considering the case γ_k is a power of 2, this includes all of the smaller powers of 3. $\square$

Now note that

$$3^a < 2^i \iff log_2(3^a) < log_2(2^i) \iff a \cdot log_2(3) < i$$

So now we are reduced to counting the number of non-negative integers a that satisfy the above for a given i. The number of such a's will simply be the the value of the largest a plus 1 since a satisfying the relation implies all $0 \leq a' \leq a$ solve the relation. Then, we are in fact reduced to finding the largest $a \in \mathbb{Z}$ that satisfies $a < \frac{i}{log_2(3)}$, but this is exactly $\lfloor \frac{i}{log_2(3)} \rfloor$. This in turn gives $j = \lfloor \frac{i}{log_2(3)} \rfloor + 1 = \lceil \frac{i}{log_2(3)} \rceil$, since $\frac{i}{log_2(3)}$ is never an integer. We now revisit our expression for the exponent of $\frac{1}{2}$ and substitute our new found value for j:

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor = \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor - \lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{log_2(3)} \rceil}} \rfloor) \tag{5.1}$$

A symmetric argument shows that exponent of $\frac{1}{3}$ in the i^{th} element of the ρ_∞−sequence of S is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^{\lceil \frac{i}{log_3(2)} \rceil} \cdot 3^i}{2^{\lceil \frac{i}{log_3(2)} \rceil} \cdot 3^{i+1}} \rfloor = \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n}{2^{\lceil \frac{i}{log_3(2)} \rceil} \cdot 3^i} \rfloor - \lfloor \frac{n}{2^{\lceil \frac{i}{log_3(2)} \rceil} \cdot 3^{i+1}} \rfloor) \tag{5.2}$$

The sums that appear in (5.1) and (5.2) are real numbers that we, at present, know little about. However, the aperiodicity of the sequences $\lceil \frac{i}{log_2(3)} \rceil$ and $\lceil \frac{i}{log_3(2)} \rceil$ over i leads us to believe, but not prove, that each of the sums are irrational. We have the following conjecture.

Conjecture 1. *Finite products of* $(\mathbb{Z}, \rho_{p_i})$ *for distinct primes,* p_i*, have transcendental valuative capacity.*

We end this section with an observation on the asymptotic behavior of capacity in these spaces. For a fixed prime p, $(\frac{1}{p})^{\frac{1}{p^n-1}}$ is an monotone, increasing sequence in n with $lim_{n\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. For fixed n, the sequence in p is also montone, increasing, again with $lim_{p\to\infty}(\frac{1}{p})^{\frac{1}{p^n-1}} = 1$. In both cases, the limiting value is equal to the diameter of space. Indeed, we can observe that the sequence $\{(0, 0, \ldots), (1, 0, \ldots), (0, 1, \ldots), \ldots\}$, in which the first element has only zeros and the n-th element has a single 1 in the $(n-1)$-th component, is a $\rho-$ordering for both $(\mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, \rho_{p,\infty})$ and $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots, \rho_{P,\infty})$, since the distance between elements in this sequence (in either metric space) is always 1.

In considering the product space of ultrametric spaces, we may wonder whether the chosen metric also gives back the product topology on the space. For products formed by taking some finite number of copies, the answer is positive. We give the necessary background and show this fact, adapting the proof in Munkres (20.3) to the case of ultrametric spaces.

Under the ρ_∞ metric these spaces are not compact. This situation is analogous to products of $\mathbb{R}$ and we can find an compact infinite product space but there is nothing canonical about it.

We are now naturally left to ask whether the product topology on *infinite* products of ultrametric spaces coincides with the L_∞ metric. In this case, as in the analogous case of infinite copies of $\mathbb{R}$ and a uniform metric, the answer is negative (at least in general). Forunately, the metric that realizes the product topology on infinite copies of $\mathbb{R}$ can be adapted to the case of ultrametric spaces. We adapt to the proof of Munkres (20.5) to the case of infinite products of ultrametric spaces.

An important consequence of the fact that d achieves the product topology is that Tychnoff's theorem then guarantees that product spaces formed with this metric will be compact, infinite or otherwise. We consider two examples.

Example 14. Let $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots, d)$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for some fixed prime p and let d be the product metric.

Example 15. Let $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots, \rho_{P,\infty})$ be the metric space formed by taking the product of $(\mathbb{Z}_p, \rho_p)$ for every prime p and let d be the product metric.

Appendix

Maple Code

In this appendix, we include for reference the Maple code that was used to investigate the capacity of product spaces. The result of this investigation also influenced the development of Chapters 3 and 4. There are three procedures listed here.

1. The first procedure, `ComputePadicProductOrdering`, takes as input a list indicating a finite set of primes, $p_1, \ldots, p_n$, and an integer m and returns the first m terms of a ρ_∞−ordering of $(\mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_1})$. This is done explicitly by following the algorithm described in Chapter 3.

2. The next procedure, `ComputePartialRhoSeq`, computes the resulting ρ_∞−sequence for a product space by multiplying out the distances given by a ρ_∞−ordering. It takes as input a metric and a matrix representing a ρ_∞−ordering for a product space and returns a number indicating the value of the n^{th} characteristic sequence, where n is the row dimension of the input matrix. These two procedures therefore result in the naive computation of the partial characteristic sequence of a product space, found by explicitly calculating a ρ_∞−ordering and the ρ_∞−sequence in turn.

3. The final procedure, `FastPartialRhoSeq`, exploits the fact that the terms occurring in the characteristic sequence of a product space are all powers of the primes specifying the space. It takes as input the (beginning of) the sequence of decreasing distances and a list of primes, specifying the space. If k distances are specified in A, it returns the $\beta(k)^{th}$ term in the characteristic sequence, where β is the structure sequence.

```
with(LinearAlgebra):
with(combinat, cartprod):
with(padic):

ComputePadicProductOrdering := proc (m, components)
# Given a list of primes, p_1,...,p_n, compute the first m terms of a p-infinity ordering of Z_p1 x ... x Z_pn, where p-infinity(x,y)=max(p_j(x_j,y_j)) and p_j is the p_j-adic metric.
# arg m; an integer indicating the number of elements of the ordering to return
# arg components; a list of prime numbers indicating the components of the product space
# return; a matrix where each row is an element in the product space and the i-th row is the i-th element in an ordering

    local numberOfComponents, co_primes, i, n, T, M, v, distances, j, M1, newBlock;

    #will end up with one column per component in the product space
    numberOfComponents := nops(components);

    #the ordering will start with the cartestian product of coprime elts from each component
    #everything up to p-1 is coprime
    co_primes := [[seq(i, i = 0 .. (components[1]-1))]];
    for n from 2 to numberOfComponents do
        co_primes := [op(co_primes), [seq(i, i = 0 .. (components[n]-1))]];
    od;

    #then take the cartestian product to get the first <product of elements in components> elements in the ordering
    T := cartprod(co_primes);
    M := Matrix([T['nextvalue']()]);
```

```
    while not T['finished'] do
        M := <M; T['nextvalue']()>;
    end do;

    #make a list to keep track of the exponent of each prime; start by
        take each prime to the power -1
    v := Vector[row](1 .. numberOfComponents, 1);
    v := convert(v, list);

    #keep adding rows until you have enough points in the ordering
    while RowDimension(M) < m do
        #take each prime to the power of minus the elements in v
        distances := zip(proc (x, y) options operator, arrow; x^(-y) end
            proc, components, v);
        #check each column to see if the max distance was achieved
        for j from 1 to numberOfComponents do
            #if it was then split this column
            if distances[j] = max(distances) then
            #take a snapshot of M before you start - this is what you
                have to add to
            M1 := copy(M, deep);
            #create p-1 new blocks
            for i from 1 to (components[j]-1) do
                    newBlock := copy(M1, deep);
                    newBlock(1..RowDimension(newBlock), j) :=
                        Column(newBlock, [j]) +~ (i*components[j]^v[j]);
                    #add the new block to the master matrix
                    M := Matrix([[M],[newBlock]]);
                od;
                #update the vector of exponents
                v[j] := v[j] + 1;
            end if;
        od;
    end do;
```

```
    return M[1..m,];
end proc;
```

```
with(LinearAlgebra):
with(combinat, cartprod):
with(padic):

ComputePartialRhoSeq := proc (S, rho)
# Given an m by n matrix S whose columns represent points of an n-component product space and that has as its i-th row the i-th term in a rho-ordering of that space, compute the (m-1)-th partial sum of the rho-sequence
# note that S and rho must be compatible and no checking is done to ensure this
# arg S; an n by m matrix representing a rho-ordering, for example as created by ComputePadicProductOrdering
# arg rho; a compatible metric on the points (rows) in S
# return; a real number, correpsonding to the (m-1)-th term of the partial rho-sequence

    local lastTerm, f, distances, nthTerm;

    #find the last element in the ordering
    lastTerm := S[RowDimension(S),];

    #make a function that calculates the distance from the i-th row of S to the last term in the ordering
    f := proc (i) options operator, arrow; rho(op(convert(lastTerm, list)), op(convert(S[i,], list))) end proc;

    #run over each row to get the set of all m-1 distances
    distances := map(f, [seq(i, i = 1 .. (RowDimension(S)-1))]);

    #multiply them to get the (m-1)-th term of the rho-ordering
    partialSum := mul(distances);

    return partialSum
```

```
end proc;
```

```
FastPartialRhoSeq := proc (A, p:=[2,3])
# Given a set of distances A for a product spaces specified in p,
    compute the exponent of each prime in the partial characteristic
    sequence
# arg A; a vector indicating the sequence of decreasing distances in
    Z_p_i
# arg p; a list of prime numbers indicating the components of the
    product space
# return; the beta(k)^th term in the characteristic sequence, where beta
    is the structure sequence and k is the length of A

  local g, h,computePowers,n, shortA, primeExponents,i, thisPrime,
      thisPrimeIndex,B, G, powers, powersOfG, thisPrimeSum;

  #=== Some helper functions ===#
  #Return the index of every instance of p-multiples in a list
  #Use to find the index of a given prime in A
  h := proc(i,L,p) if L[i] mod p = 0 then return i else return NULL fi;
      end proc;

  #Count the number of times the mth element has appeared as a factor
      for the 1..m first elements in a list
  #Use to compute the (decreasing) sequence of distances in A or a
      subset of A
  g := proc(m, L)
    local basePrime;

    basePrime := L[m];

    #since we just want the exponent not actual distance just compute
        what power this would be
    return ordp(mul(L[1..m]), basePrime);

  end proc;
```

```
#Compute the appropriate power of an element of G
computePowers := proc(m, L)
  local power;

  if m=nops(L) then
    power:= L[-1]-1;
  else
    power:= mul(L[(m+1)..nops(L)])* (L[m]-1);
  end if;
  return power
end proc;

#compute n, then create a copy of A with the first element deleted to
    ease the indexing
n := mul(A);
shortA := A[2..nops(A)];

#compute the terms corrsponding to each prime given
primeExponents := Vector();
for i from 1 to nops(p) do
  #pull out the prime
  thisPrime := p[i];

  #first get the index in A of this prime
  thisPrimeIndex := map(h, [seq(i, i=1..nops(shortA))], shortA,
      thisPrime);
  B:= shortA[thisPrimeIndex];

  #then find the (exponents for the) distances occuring with this prime
  G:= map(g,[seq(i, i=1..nops(B))],B);

  #Figure out what power each distance should be raised to
  powers := map(computePowers,thisPrimeIndex, shortA);
```

```
    #raise each element in G to the given powers
    powersOfG:= zip(proc (x, y) options operator, arrow; x*(y) end proc,
       G, powers);

    #the exponent of this prime in the nth partial will be -([the sum of
       the elements in powers]/n), where n is the product of elements in
       A (including the first element)
    thisPrimeSum := add(powersOfG);
    primeExponents(i) := thisPrimeSum/n;

  end do;

  return primeExponents;

end proc;
```

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