

VALUATIVE CAPACITY OF COMPACT SUBSETS OF  
ULTRAMETRIC SPACES

by

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## Abstract

A  $p$ -ordering is a combinatorial concept introduced by Bhargava to generalize the factorial function. K. Johnson noticed in his paper “ *$p$ -orderings, Fekete  $n$ -tuples and capacity in ultrametric spaces*” that  $p$ -orderings also give a construction for Fekete  $n$ -tuples. Fekete  $n$ -tuples, in turn, can be used to compute the capacity of a metric space. In this thesis, we explore some properties of capacity in compact ultrametric spaces.

When our space has algebraic structure, we show how this structure can be exploited to compute capacity. We then develop conditions for computing capacity in spaces that lack algebraic structure by studying the lattice of closed balls in the space. At the end of the thesis, we compute the capacity of  $n$ -fold products of  $(\mathbb{Z}, \rho_{p_i})$ , for a set of  $p$ -adic metrics  $\rho_{p_i}$ . While this is a straightforward process when using a fixed prime, we see that allowing distinct primes on each component produces interesting results even for  $n = 2$ . We conjecture that these spaces have transcendental capacity.

## List of Abbreviations and Symbols Used

$B(x, r)$	the closed ball of radius $r$ centred at $x$
$B^0(x, r)$	the open ball of radius $r$ centred at $x$
$B_i^k$	an element of $S_{\gamma_k}$
$S(x, r)$	the sphere of radius $r$ centred at $x$
$S_{\gamma_k}$	the finite partition of $S$ by closed balls of radius $k$
$T_S$	the lattice of closed balls in $S$
$\Gamma_S$	the decreasing sequence of distances in $S$
$\alpha(n)$	the semi-regular sequence of a set
$\beta(n)$	the structure sequence of a set
$\delta(n)$	the characteristic sequence of a set
$\gamma_k$	the $k^{th}$ element of $\Gamma_S$
$\inf$	the infimum of a set
$ \cdot _p$	the $p$ -adic absolute value
$\mathbb{C}$	the complex numbers
$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\mathbb{Z}$	the integers
$\mu$	a Borel measure on $\mathbb{C}$
$\rho$	an ultrametric
$\rho_N$	an ultrametric, induced by a norm $N$
$\rho_\infty$	the max metric on a product space
$\rho_p$	the $p$ -adic metric
$\sup$	the supremum of a set
$\widehat{\mathbb{Z}}_p$	the $p$ -adic integers
$p$	a fixed prime
$v_p(z)$	the $p$ -adic valuation of $z$

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# Chapter 1

## Introduction

In the course of developing a generalized factorial function, Manjul Bhargava introduced the notion of a  $p$ -ordering of a Dedekind domain [B2, B1], a combinatorial concept which, along with his generalized factorial, provided deep and perhaps unexpected results in number theory. The concepts laid down in these papers have enriched the theory of integer-valued polynomials [B3, J2] and have also provided a natural framework to extend many classical results in analysis to a  $p$ -adic setting, such as polynomial approximation and mapping theorems [B2, B1, B3]. In this thesis, we examine how a tool based on  $p$ -orderings can extend another concept from classical analysis, namely the *valuative capacity* of a set, to non-Archimedean settings.

The historical background to this work comes in two parts. On the one hand, there is the background on logarithmic capacity from potential theory, and secondly, there is the background from Bhargava's  $p$ -orderings. We give a brief summary of each here. A similar treatment, with slightly different perspective, is found in [FP]. Jean-Luc Chabert was the first to draw a connection between the two, and many of the known results in this area stem from his work or that of his colleagues. Building on the result in [J1], we extend the work by Chabert and colleagues by studying valutive capacity in a more general setting, namely that of an ultrametric space, which may or may not also be a local field. In doing so, we show many properties of capacity are in fact independent of the algebraic structure of a space, although such structure, when it exists, can act as a useful probe.

### 1.1 Logarithmic capacity

The theory of capacity has been developed as a topic in potential theory in a variety of settings. Classically, the notion of capacity was developed over both  $\mathbb{C}$  and  $\mathbb{R}^n$ , although the theory has been further developed in a rather general way by Rumely for Berkovich spaces. A significant body of work on the analytic properties of capacity can be found for a number of different contexts. For example, such a treatment of the subject over  $\mathbb{C}$  can be found in [W] and [Ra1], over Berkovich spaces in [BR], and over  $\mathbb{Q}_p$  in [Ca]. We give a brief account of capacity over  $\mathbb{C}$  here, presenting only the most essential definitions and results. One advantage of tracing the historical roots of capacity back to  $\mathbb{C}$  is that the theory in this setting also comes equipped with a physical interpretation. As we are about to see, capacity in the classical sense gives a mathematical model for the amount of electrostatic charge a conductor can hold. The exposition below is closely based on [Ra1] and [Ra2].

Even restricting ourself to the definition of capacity of subsets of  $\mathbb{C}$ , we find two paths, one which will give us some physical interpretation, and one which will lead more naturally to  $p$ -orderings. We start with the former.

**Definition 1.** [Ra2] Let  $\mu$  be a finite Borel measure on  $\mathbb{C}$  and suppose  $\mu$  has compact support. We associate to  $\mu$  a function,  $p_\mu : \mathbb{C} \rightarrow (-\infty, \infty]$ , given by

$$p_\mu(x) = \int \log \frac{1}{|x - y|} d\mu(y)$$

called the **potential function** of  $\mu$ . The **energy** of  $\mu$  is

$$I(\mu) = \int \int \log \frac{1}{|x - y|} d\mu(y) d\mu(x).$$

This gives at once the physical interpretation promised above. We interpret the potential function of a measure as giving the potential energy of a point. Viewing the measure as a charge distribution, the double integral gives back the total energy in the system. Now we come upon a physical reality: charged particles in a conductor will naturally distribute themselves in order to minimize the energy. This leads to

the definition below:

**Definition 2.** [Ra2] Let  $K$  be a compact subset of  $\mathbb{C}$  and let  $\mathcal{P}(K)$  be the set of Borel probability measures on  $K$ . If  $\nu \in \mathcal{P}(K)$  is such that

$$I(\nu) = \inf_{\mu \in \mathcal{P}(K)} I(\mu)$$

then  $\nu$  is a **equilibrium measure** for  $K$ .

We state the following proposition without proof. A sketch of the proof can be found in [FP] and the full details can be found in [Ra1].

**Proposition 1.** [Ra1] *An equilibrium measure exists for every compact set  $K \in \mathbb{C}$ . When finite, the equilibrium measure is unique and isometry-invariant.*

We are now ready to give our first definition of capacity.

**Definition 3.** [Ra2] Let  $K$  be a compact subset of  $\mathbb{C}$ . The logarithmic capacity of  $K$  is

$$C(K) = e^{-I(\nu)}$$

where  $\nu$  is an equilibrium measure on  $K$ . If  $I(\nu) = \infty$ , then we understand that  $C(K) = 0$ .

We present below a few results on capacity in  $\mathbb{C}$ , some of which will reappear in the remainder of this work, although the context, and the proofs (omitted here), bear little resemblance to the present case.

**Proposition 2.** ([Ra1], 5.1.2) *Let  $K, K_1, K_2$  be compact subsets of  $\mathbb{C}$ .*

1. *If  $K_1 \subseteq K_2$ , then  $C(K_1) \leq C(K_2)$ .*
2.  *$C(\alpha K + \beta) = |\alpha|C(K)$  for all  $\alpha, \beta \in \mathbb{C}$ .*



3.  $C(K) = C(\delta_e K)$ , where  $\delta_e$  is the exterior boundary.<sup>1</sup>

**Proposition 3.** ([Ra1], 5.1.4) Suppose  $\{B_n\}$  is a sequence of Borel subsets of  $\mathbb{C}$ . Let  $B = \cup_n B_n$  and  $d \geq 0$ .

1. If  $\text{diam}(B) \leq d$ , then  $C(B) \leq d$  and

$$\frac{1}{\log(\frac{d}{C(B)})} \leq \sum_n \frac{1}{\log(\frac{d}{C(B_n)})}$$

2. If  $\text{dist}(B_j, B_k) \geq d$  whenever  $j \neq k$ , then

$$\frac{1}{\log^+(\frac{d}{C(B)})} \geq \sum_n \frac{1}{\log^+(\frac{d}{C(B_n)})}$$

where  $\log^+(x) = \max(\log(x), 0)$ .

We now show an equivalent way of defining capacity, still over  $\mathbb{C}$ , which starts with the following two definitions due to Fekete [F].

**Definition 4.** [F] Let  $K \subseteq \mathbb{C}$  be a compact subset. Fix  $n \in \mathbb{N}$ , and for  $z = (z_1, \dots, z_n) \in K^n$ , consider

$$\delta_n(z) = \prod_{j < i} |z_i - z_j|^{\frac{2}{n(n-1)}}$$

An element  $z = (z_1, \dots, z_n) \in K^n$  is called a **Fekete  $n$ -tuple** if  $z$  maximizes  $\delta_n$  over all  $n$ -tuples in  $K$ .

Note that since  $K$  is compact by assumption, a Fekete  $n$ -tuple exists for each  $n$ .

**Definition 5.** Let  $K \subseteq \mathbb{C}$  be a compact subset. The **transfinite diameter** of  $K$  is

$$\lim_{n \rightarrow \infty} [\max_z \delta_n(z)]$$

---

<sup>1</sup>The exterior boundary of a compact subset,  $K$ , of  $\mathbb{C}$  is the boundary of the unbounded, connected component of  $U = \mathbb{C} \setminus K$ .

where the maximum is taken over all  $n$ -tuples in  $K$ . That is, the transfinite diameter of  $K$  is  $\lim_{n \rightarrow \infty} \delta_n(z)$ , where  $z$  is a Fekete  $n$ -tuple for each  $n$ .

The following proposition shows the relation to capacity.

**Proposition 4.** (*[F], Fekete-Szegö Theorem*) *If  $K$  is a compact subset of  $\mathbb{C}$ , then the transfinite diameter of  $K$  is equal to the logarithmic capacity of  $K$ .*

We end this section with an observation about the points  $z_i$  in  $\mathbb{C}$  (or some subset thereof) making up a Fekete  $n$ -tuple. For  $n \geq 2$ , if  $(z_1, \dots, z_n)$  is a Fekete  $n$ -tuple, then in general there is no  $z_{n+1}$  available such that  $(z_1, \dots, z_n, z_{n+1})$  is a Fekete  $(n+1)$ -tuple. In physical terms, we note that the placement of a new charge in a conductor will almost always change the location of the existing charges in that conductor. Remarkably, this is not the case in ultrametric spaces. Indeed, we are able to build the analogous structure, which we call a  $p$ -ordering or more generally a  $\rho$ -ordering, *recursively*, that is by reusing the points from the previous iteration.

## 1.2 P-orderings

The development of  $p$ -orderings was motivated by the observation that the factorial function had important number-theoretic applications, yet was only defined for the set  $\mathbb{Z}$ . In order to generalize the factorial, Bhargava defined it via an invariant, called the  $p$ -sequence, which could be attached to any subset of a Dedekind domain<sup>2</sup> [B2].

We cannot go much further without introducing the following definition.

**Definition 6.** Let  $z \in \mathbb{Z}$  and let  $p$  be any prime. The  $p$ -**adic valuation** of  $z$ , denoted  $v_p(z)$ , is the largest  $n \in \mathbb{N}$  such that  $p^n$  divides  $z \neq 0$  and  $v_p(z) = \infty$  if  $z = 0$ . That is,

$$v_p(z) = \begin{cases} \max\{n \in \mathbb{N}; p^n \mid z\}, & \text{if } z \neq 0 \\ \infty, & \text{otherwise} \end{cases}$$

---

<sup>2</sup>In fact, Bhargava associated  $p$ -sequences to the more general class of Dedekind rings, which are locally principal, Noetherian rings in which all nonzero primes are maximal.

For  $z \in \mathbb{Z}$ , we define the  $p$ -**adic absolute value** by

$$|z|_p = p^{-v_p(z)}$$

and the  $p$ -**adic metric** accordingly; that is, for  $z_1, z_2 \in \mathbb{Z}$

$$\rho_p(z_1, z_2) = p^{-v_p(z_1 - z_2)}$$

where  $p^{-\infty}$  is taken to be 0.

It is worth pausing to make a few comments about the above definitions. That the  $p$ -adic metric is truly a metric is easy to see. In fact, we will see in the next chapter that it is not just a metric, but also an ultrametric, since the  $p$ -adic absolute value satisfies a strengthened version of the triangle identity. The strong triangle identity is not the only interesting property at hand, though. Like the logarithm, the  $p$ -adic valuation also satisfies  $v_p(x \cdot y) = v_p(x) + v_p(y)$  for any prime  $p$  and  $x, y$  in  $\mathbb{Z}$ . Moreover, we note that the  $p$ -adic valuation and  $p$ -adic metric have an interesting relationship with each other: two points whose difference has a relatively small valuation will have a relatively large distance between them and vice versa.

We are now ready to define  $p$ -orderings, and not long after, to give the connection to Fekete  $n$ -tuples.

**Definition 7.** [B2] Let  $S$  be a subset of  $\mathbb{Z}$  and let  $p$  be any prime.<sup>3</sup> A  $p$ -**ordering** of  $S$  is a sequence,  $\{a_i\}_{i \geq 0}$  in  $S$ , such that  $a_0$  is arbitrary and for  $i > 0$ ,  $a_i$  minimizes

$$v_p\left(\prod_{j < i} (z - a_j)\right)$$

over  $z \in S$ .

A  $p$ -ordering in  $S$ , like a Fekete  $n$ -tuple in  $\mathbb{C}$ , is not unique. Indeed, in most of the examples we will explore, there will be infinitely-many choices at each stage

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<sup>3</sup>To apply the definition to a general Dedekind domain, we replace the usual primes with the set of prime ideals in the ring of interest.

of the construction. Nonetheless,  $p$ -orderings give rise to  $p$ -sequences, which are invariants of  $S$ :

**Definition 8.** [B2] Let  $S$  be a subset of  $\mathbb{Z}$  and let  $p$  be any prime. Suppose  $\{a_i\}_{i \geq 0}$  is a  $p$ -ordering of  $S$ . The **p-sequence**, occasionally the **characteristic sequence**, of  $S$  is the sequence defined by  $\delta(0) = 1$  and for  $i > 0$ ,

$$\delta(i) = v_p\left(\prod_{j=0}^{i-1} (a_i - a_j)\right)$$

It is a fact, not entirely obvious, that the  $p$ -sequence of  $S$  is independent of the  $p$ -ordering used in its construction [B2]. To define the generalized factorial, Bhargava considered the product of  $p$ -sequences taken over each prime  $p$  for arbitrary subsets of  $\mathbb{Z}$ . We will go in another direction.

Suppose we were to generalize our definition of Fekete  $n$ -tuple in the obvious way, as below.

**Definition 9.** Let  $(M, \rho)$  be a metric space and suppose  $S \subseteq M$  is a compact subset. Fix  $n \in \mathbb{N}$ , and for  $z = (z_1, \dots, z_n) \in S^n$ , consider

$$\delta_n(z) = \prod_{j < i} \rho(z_i - z_j)^{\frac{2}{(n(n-1))}}$$

An element  $z = (z_1, \dots, z_n) \in S^n$  is called a **generalized Fekete n-tuple** if  $z$  maximizes  $\delta_n$  over all  $n$ -tuples in  $S$ .

What then is the connection to  $p$ -orderings and  $p$ -sequences? Suppose  $S$  is a subset of  $\mathbb{Z}$  and that  $\{a_i\}_{i \geq 0}$  is a  $p$ -ordering of  $S$  for some prime  $p$ . Then of course from the definition of  $p$ -orderings, we know that for  $n > 0$ ,

$$v_p\left(\prod_{j < n} (a_n - a_j)\right) \leq v_p\left(\prod_{j < n} (z - a_j)\right)$$

for  $z \in S$ . Something more is true though, namely,

$$v_p\left(\prod_{j < n} (a_n - a_j)\right) \leq v_p\left(\prod_{i=0}^n (x_i - x_j)\right)$$

for  $x_i, x_j \in S$  [B2]. That is, when we pick  $a_n$  to minimize the  $p$ -adic valuation over  $\prod_{j < n} (z - a_j)$ , we actually achieve the minimum over the product of *all pairs of  $n$  differences in  $S$* . Since minimizing  $v_p(x_i - x_j)$  is the same as maximizing  $\rho_p(x_i, x_j)$ , we have the following remarkable fact: if  $\{a_i\}_{i \geq 0}$  is a  $p$ -ordering of  $S$ , then  $\{a_i\}_{i=0}^n$  is a generalized Fekete  $n$ -tuple for  $(S, \rho_p)$  for each  $n$ . In particular,  $p$ -orderings give a recursive construction for generalized Fekete  $n$ -tuples.

The first connection between these objects was made by Jean-Luc Chabert in [Ch] when he studied the limit of these sequences not just for the case  $M = \mathbb{Z}$  and  $\rho = \rho_p$ , but in the case that  $M$  is any rank-one valuation domain [Ch]. We repeat his Theorem 4.2 from [Ch] below,

**Proposition 5.** *Let  $E$  be a subset of  $V$ , a rank-one valuation domain with valuation  $v$ . If  $\{a_i\}_{i \geq 0}$  is a  $v$ -ordering<sup>4</sup> of  $E$ , then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0, \dots, x_n \in E} v\left(\prod_{0 \leq j < i \leq n} (x_i - x_j)\right)$$

Chabert called this limit the valutive capacity of  $E$ , and we shall do the same. The following result by Johnson in [J1], in which  $p$ -ordering has been replaced by  $\rho$ -ordering, provides the foundation for the rest of this work:

**Proposition.** *([J1], Theorem 1) If  $S$  is a compact subset of an ultrametric space  $(M, \rho)$ , then the first  $n$  terms of a  $\rho$ -ordering of  $S$  always give a Fekete  $n$ -tuple of  $S$  and all Fekete  $n$ -tuples of  $S$  arise in this way.*

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<sup>4</sup>A  $v$ -ordering of  $E$  is exactly as expected: a sequence of distinct element  $\{a_i\}_{i \geq 0}$  in  $E$  is  $v$ -ordering of  $E$  if for  $n > 0$ ,

$$v\left(\prod_{k=0}^{n-1} (a_n - a_k)\right) \leq v\left(\prod_{k=0}^{n-1} (x - a_k)\right)$$

for each  $x \in E$ .

One important consequence of this remark is that it gives a way to define capacity in a general ultrametric space. By replacing the notion of  $p$ -ordering (or  $v$ -ordering) with the more general notion of  $\rho$ -ordering, we are able to give a definition of valutive capacity for a general ultrametric space, without appealing to any algebraic (or measure-theoretic) structure. Of course, we have yet to say what a  $\rho$ -ordering is. We take this up, along with the necessary background from ultrametric spaces, in the next chapter.

## Chapter 2

### Capacity and Ultrametric spaces

#### 2.1 Ultrametric basics

The principal context for this thesis is an arbitrary ultrametric space, which is a metric space that also satisfies an additional axiom, sometimes called the ultrametric inequality or (in the case of vector spaces) the strong triangle property. We define ultrametric spaces below and for the rest of this section, we review some of their more important properties. The proofs offered in this section are, for the most part, standard and can be found in a number of reference texts, such as [Ro].

**Definition 10.** Let  $(M, \rho)$  be a metric space; that is, suppose  $M$  is a set and  $\rho$  is a map,  $\rho : M \times M \rightarrow \mathbb{R}_{\geq 0}$  such that:

- (i)  $\rho(x, y) = 0$  if and only if  $x = y$
- (ii)  $\rho(x, y) = \rho(y, x)$
- (iii)  $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

for any  $x, y, z \in M$ . If  $\rho$  also satisfies the ultrametric inequality,

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z))$$

for any  $x, y, z \in M$ , then  $(M, \rho)$  is an **ultrametric space**.

A special case of an ultrametric space, and one where much of the previous work on this topic has been completed, is one where the metric has been derived from a norm on a vector space.

**Definition 11.** Let  $(V, N)$  be a normed vector space; that is, suppose  $V$  is an  $\mathbb{F}$ -vector space, for  $\mathbb{F}$  some subfield of  $\mathbb{C}$ , and  $N : V \rightarrow \mathbb{R}_{\geq 0}$  is such that:

$$(i) \ N(x + y) \leq N(x) + N(y)$$

$$(ii) \ N(cx) = |c| \ N(x)$$

$$(iii) \ N(x) = 0 \text{ implies } x = 0$$

for any  $x, y \in V$  and  $c \in \mathbb{F}$ . We say that  $N$  satisfies the **strong triangle inequality** if

$$N(x + y) \leq \max(N(x), N(y))$$

for any  $x, y \in V$ .

**Proposition 6.** *Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle inequality. Then the metric space  $(V, \rho_N)$ , where  $\rho_N$  is the metric induced by  $\rho_N(x, y) = N(x - y)$ , is an ultrametric space.*

*Proof.* We take for granted that  $(V, \rho_N)$  is a metric space and also note that

$$N(x + z) \leq \max(N(x), N(z))$$

implies

$$\rho_N(x, z) \leq \max(\rho_N(x, 0), \rho_N(z, 0)) \leq \max(\rho_N(x, y), \rho_N(y, z))$$

□

**Notation.** If  $(V, N)$  is a normed vector space, then the metric induced by  $N$  will be denoted  $\rho_N$ .

When ultrametric spaces come from spaces with algebraic structure, such as normed vector spaces, some of this structure carries over into the metric space structure in a rather nice way:

**Proposition 7.** *[Ro] Let  $S$  be a group equipped with a (right) invariant ultrametric,  $\rho$ . If  $B = B(0, r)$  is a (closed) ball centred at the neutral element of  $S$ , that is  $B = \{x \in S; \rho(x, 0) \leq r\}$ , then  $B$  is a subgroup of  $S$ .*

*Proof.* Let  $x, y \in B$ . Then

$$\rho(x - y, 0) = \rho(x, y) \leq \max(\rho(x, 0), \rho(y, 0)) \leq r,$$



so that  $x - y \in B$ . Note that although we have used additive notation here, we do not require that our group  $S$  be Abelian.  $\square$

In the previous chapter, we claimed that the  $p$ -adic metric was an ultrametric on the set  $\mathbb{Z}$ . Indeed,  $(\mathbb{Z}, \rho_p)$  and the closely related space of  $p$ -adic integers, denoted  $\widehat{\mathbb{Z}}_p$ , are the canonical examples of an ultrametric space.

**Example 1.** Let  $p$  be any prime and consider the metric space  $(\mathbb{Z}, \rho_p)$ . To see that  $(\mathbb{Z}, \rho_p)$  is an ultrametric space, we must show that  $\rho_p$  satisfies the ultrametric inequality, or equivalently, that  $p$ -adic absolute value satisfies the strong triangle inequality. Let  $x, y$  be in  $\mathbb{Z}$  and suppose  $v_p(x) = n_x$  and  $v_p(y) = n_y$ . Then if  $n = \min(n_x, n_y)$ ,  $p^n$  divides  $x$  and  $p^n$  divides  $y$ , so  $p^n$  divides  $x + y$ . We see now that  $v_p(x + y) \geq \min(v_p(x), v_p(y))$  and in turn  $|x + y|_p \leq \max(|x|_p, |y|_p)$ .

**Example 2.** Let  $p$  be any prime. If

$$z = \sum_{i \geq 0} b_i p^i$$

is such that  $b_i \in \{0, \dots, p-1\}$  for all  $i$ , then we say that  $z$  is a  **$p$ -adic integer**. If  $z = \sum_{i \geq 0} b_i p^i$ , we note that if only a finite number of the coefficients of  $z$  are non-zero, then  $\sum_{i \geq 0} b_i p^i$  is a representation in base  $p$  of some element of  $\mathbb{Z}$ . We can define the  **$p$ -adic order** of a  $p$ -adic integer, denoted  $\text{ord}_p(z)$ , in a way that agrees with the  $p$ -adic valuation when  $\sum_{i \geq 0} b_i p^i$  is in  $\mathbb{Z}$ . We let  $\text{ord}_p(z)$  be the smallest  $i$  such that  $b_i \neq 0$ . The  $p$ -adic integers are both a ring<sup>1</sup> and an ultrametric space with the metric induced by  $\text{ord}_p(z)$ . For a given prime,  $p$ , we denote the  $p$ -adic integers by  $\widehat{\mathbb{Z}}_p$ .

In what follows, we will often refer to  $p$ -adic spaces and it will not make much of a difference whether the reader prefers to think of being in  $(\mathbb{Z}, \rho_p)$  or  $\widehat{\mathbb{Z}}_p$ . The reason is this: when forming  $\rho_p$ -orderings of subsets of either space we are always able to do so by selecting elements with a finite number of non-zero coefficients, that is, by

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<sup>1</sup>The ring operations carry over on the coefficients of  $p$ -adic integers in the expected way from  $\mathbb{Z}/p\mathbb{Z}$ , as long as special care is taken to keep track of carries.

selecting elements from  $\mathbb{Z}$  itself.

Ultrametric spaces exhibit many properties unlike those of traditional metric spaces, and we review a few of these below. Of particular interest to us is the behavior between (closed) balls in an ultrametric space.

**Notation.** Let  $(M, \rho)$  be a compact ultrametric space and let

$$B(a, r) = \{x \in M \mid \rho(x, a) \leq r\}$$

denote the *closed* ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{>0}$  and  $a \in (M, \rho)$ . Let

$$B^0(a, r) = \{x \in M \mid \rho(x, a) < r\}$$

denote the *open* ball of radius  $r$ , centred at  $a$  for some  $r \in \mathbb{R}_{>0}$  and  $a \in (M, \rho)$ .

In the above notation, we break with convention in that we denote a closed ball without using any decoration. This is because before too long we will work exclusively with closed balls. We are able to do this because for the most part, the notion of open and closed ball in an ultrametric space overlap, although we will need a few more facts before showing this.

**Definition 12.** Let  $S$  be a subset of an ultrametric space. The **diameter of  $S$**  is  $\text{diam}(S) = \sup_{x, y \in S} \rho(x, y)$ . Note that if  $S$  is compact,  $\text{diam}(S) = \max_{x, y \in S} \rho(x, y)$ .

**Proposition 8.** Let  $B = B(a, r)$  be a ball in an ultrametric space  $(M, \rho)$ . Then the diameter of  $B$  is less than or equal to the radius of  $B$ .

*Proof.* Suppose  $d = \text{diam}(B) > r$ . This would imply there exists  $x, y$  in  $B$  such that  $\rho(x, y) > r$ , in particular  $\rho(x, y)$  is strictly greater than  $\max(\rho(x, a), \rho(y, a))$ , which is a contradiction since  $\rho$  is an ultrametric.  $\square$

The following example shows that we can obtain a strict inequality in the above proposition.

**Example 3.** Let  $M$  be a non-empty set. Let  $\rho$  be the ultrametric given by,

$$\rho(x, y) = \begin{cases} 0, & \text{if } x=y \\ 1, & \text{if } x \neq y \end{cases}$$

for  $x, y \in M$ . Then  $B(x, \frac{1}{2})$  has radius  $\frac{1}{2}$  and diameter 0 for any  $x$  in  $M$ .

In the following proposition, we describe the triangles in an ultrametric space, and the result is more or less a restatement, in geometric terms, of the ultrametric inequality.

**Proposition 9.** *All triangles in an ultrametric space  $(M, \rho)$  are either equilateral or isosceles, with at most one short side.*

*Proof.* Let  $x, y$ , and  $z$  be three points in an ultrametric space  $(M, \rho)$ . We show that  $\rho(x, y) \neq \rho(x, z)$  and  $\rho(x, y) \neq \rho(y, z)$  implies  $\rho(x, y) < \rho(x, z) = \rho(y, z)$ .

If  $\rho(x, z) \neq \rho(y, z)$ , then without loss,  $\rho(x, z) > \rho(y, z)$ . At the same time, the ultrametric inequality implies

$$\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$$

and  $\rho(y, z) \leq \max(\rho(x, y), \rho(x, z))$ . The first inequality implies  $\rho(x, y) < \rho(x, z)$ , which means the second inequality implies  $\rho(y, z) < \rho(x, z)$ . This is a contradiction, so we must have  $\rho(x, z) = \rho(y, z)$ .

To see that  $\rho(x, y) < \rho(x, z)$ , simply note that  $\rho(x, y) \leq \max(\rho(x, z), \rho(y, z))$   $\square$

With this result in hand, we are able to quickly demonstrate some of the properties of balls, which are of fundamental importance to us. We see below that the ultrametric inequality, perhaps innocuous on the surface, quickly implies ultrametric balls are markedly different from their Archimedean counterparts.

**Proposition 10.** *Every point of a ball in an ultrametric space is at its centre. That is, if  $B(x_0, r)$  is a ball in an ultrametric space  $(M, \rho)$ , then  $B(x, r) = B(x_0, r)$ ,  $\forall x \in B(x_0, r)$*

*Proof.* Let  $a \in B(x, r)$ . Then  $\rho(a, x) \leq r$  and since

$$\rho(a, x_0) \leq \max(\rho(a, x), \rho(x, x_0)) \leq r$$

we must have  $a \in B(x_0, r)$  and  $B(x, r) \subseteq B(x_0, r)$ . A similar argument shows  $B(x_0, r) \subseteq B(x, r)$ .  $\square$

**Proposition 11.** *If  $(M, \rho)$  is an ultrametric space and  $B(x_0, r_1)$  and  $B(y_0, r_2)$  are balls in  $(M, \rho)$ , then either  $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$ ,  $B(x_0, r_1) \subseteq B(y_0, r_2)$ , or  $B(y_0, r_2) \subseteq B(x_0, r_1)$ . That is, in an ultrametric space, all balls are either comparable or disjoint.*

*Proof.* Suppose  $B(x_0, r_1) \cap B(y_0, r_2) \neq \emptyset$  and let  $z$  be a point in the intersection. We show that if there exists an  $a \in B(y_0, r_2)$  such that  $a \notin B(x_0, r_1)$ , then  $B(x_0, r_1) \subseteq B(y_0, r_2)$ . Let  $x \in B(x_0, r_1)$ . Then we must have  $\rho(x, z) < \rho(x, a)$ , since  $z \in B(x_0, r_1) = B(x, r_1)$  and  $a$  is not. Since the triangle with vertices  $(a, x, z)$  is isosceles with at most one short side, we must have  $\rho(x, a) = \rho(a, z) \leq r_2$ , since  $a \in B(y_0, r_2) = B(z, r_2)$ . Then  $x \in B(y_0, r_2)$ .  $\square$

**Proposition 12.** *The distance between points in two non-overlapping balls in an ultrametric is constant. That is, if  $B(x_0, r_1)$  and  $B(y_0, r_2)$  are two balls in an ultrametric space with  $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$ , then there exists a  $c \in \mathbb{R}_{>0}$  such that  $\rho(x, y) = c$ ,  $\forall x \in B(x_0, r_1)$  and  $\forall y \in B(y_0, r_2)$ .*

*Proof.* Suppose  $\rho(x_0, y_0) = c$  and let  $x \in B(x_0, r_1)$  and  $y \in B(y_0, r_2)$  be arbitrary. Consider the triangle formed by  $(x_0, y_0, y)$ . Since  $\rho(x_0, y_0) = c$  and  $\rho(y, y_0) \leq r_2 < c$ , we must have  $\rho(x_0, y) = c$  because triangles in an ultrametric space have at most one short side. Now consider the triangle formed by  $(x_0, x, y)$ . Since  $\rho(x_0, y) = c$  and  $\rho(x, x_0) \leq r_1 < c$ , we must have  $\rho(x, y) = c$ .  $\square$

We will get a bit closer to showing the relationship between open and closed balls with the following results and will pick up a few other useful facts along the way.

We start with another definition.

**Definition 13.** If  $(M, \rho)$  is an ultrametric space, then for  $x_0 \in M$  and  $r \in \mathbb{R}_{>0}$ ,

$$S(x_0, r) = \{x \in M; \rho(x, x_0) = r\}$$

is the **sphere of radius  $r$  at  $x_0$** .

**Lemma 1.** ([Ro]) *Spheres (of positive radius) in an ultrametric space are both open and closed as sets.*

*Proof.* [Ro] A sphere in any metric space is closed, so we need only show a sphere is also open in an ultrametric space. We show a sphere,  $S = S(x_0, r)$ , is equal to a union of open sets,  $S = \cup_{x \in S} B^0(x, r)$ .

Let  $B = B^0(x, s)$  be an open ball that does not contain some  $x_0$ . Let  $r = \rho(x_0, x)$ . We must have  $r \geq s$ , so then (since all triangles are isosocles) every point in  $B$  lies in  $S(x_0, r)$ , that is  $B \subseteq S(x_0, r)$ . Then for any  $x \in S(x_0, r)$ ,  $B^0(x, r) \subseteq S(x_0, r)$  and

$$\bigcup_{x \in S(x_0, r)} B^0(x, r) \subseteq S(x_0, r)$$

The reverse inequality is clear since the union is taken over points of  $S$ . □

**Proposition 13.** ([Ro]) *The open balls in an ultrametric space are closed sets and the closed balls are open sets.*

*Proof.* The proof follows immediately from the result that spheres are both open and closed: to see that closed balls are open sets, note that for a closed ball,  $B(x_0, r)$ ,

$$B(x_0, r) = B^0(x_0, r) \cup S(x_0, r)$$

Likewise, to see that open balls are closed sets, note that

$$B^0(x_0, r) = B(x_0, r) \setminus S(x_0, r)$$

□

The following proposition is now easy to see, although the result is both unintuitive and important for our purposes.

**Proposition 14.** *Suppose  $S$  is a compact subset of an ultrametric space  $(M, \rho)$  and that  $\cup_{i \in I} B(x_i, r_i)$  is a cover of  $S$  by closed balls in  $S$ . Then there exists  $i_1, \dots, i_n$ , a finite subset of  $I$ , such that  $\cup_{j=1}^n B(x_{i_j}, r_{i_j})$  is a partition of  $S$ .*

*Proof.* Since  $S$  is compact and  $\rho$  is an ultrametric,  $\cup_{i \in I} B(x_i, r_i)$  is an open cover and contains a finite subcover of  $S$ . Say this subcover is given by the elements  $i_1, \dots, i_{n'} \in I$ , and suppose this is not a partition. That is, suppose for some  $i_i, i_j$ ,  $B(x_{i_i}, r_{i_i}) \cap B(x_{i_j}, r_{i_j}) \neq \emptyset$ . Then, without loss of generality, we must have  $B(x_{i_i}, r_{i_i}) \subseteq B(x_{i_j}, r_{i_j})$ , so that the removal of  $B(x_{i_i}, r_{i_i})$  is still a cover of  $S$ . We continue this process a finite number of times, since the subcover was finite to begin with, to arrive at a finite partition of  $S$ .  $\square$

In fact, a slightly stronger statement than the above is true:

**Corollary 1.** *Suppose  $S$  is a compact subset of an ultrametric space  $(M, \rho)$  and that  $B(x_0, r)$  is a closed ball in  $S$ . Then, there exists a finite partition of  $S$  having  $B(x_0, r)$  as an element.*

*Proof.* Let  $\mathcal{C}$  be the cover of  $S$  given by  $\cup_{x \in S} B(x, r) \cap S$ . From the proposition, we can select a finite subcover of  $\mathcal{C}$  that is a partition of  $S$ . Suppose  $B(y, r) \cap S$  is the element in this partition containing  $x_0$ . Then since  $B(y, r)$  and  $B(x_0, r)$  are equal in  $M$ ,  $B(y, r) \cap S = B(x_0, r) \cap S = B(x_0, r)$ .  $\square$

We end this section by making a few comments about the set of distances that occur between the points of a compact ultrametric space.

**Proposition 15.** *([Ro]) Let  $S$  be a compact subset of an ultrametric space,  $(M, \rho)$*

- (i) *For  $m \in (M \setminus S)$ , let  $f_m : S \rightarrow \mathbb{R}$ , be the function defined by  $f_m(s) = \rho(m, s)$ . Then  $\text{Im}(f_m)$  is finite for all  $m \in (M \setminus S)$ .*

(ii) For  $a \in S$ , let  $\phi_a : S \setminus \{a\} \rightarrow \mathbb{R}$  be the function defined by  $\phi_a(x) = \rho(x, a)$ . Then  $\text{Im}(\phi_a)$  is a discrete subset of  $\mathbb{R}$  for all  $a \in S$ .

*Proof.* ([Ro])

(i) The fibers of  $f_m$ ,  $f_m^{-1}(s)$ , for  $s \in S$ , form a cover of  $S$ . In fact, they form an open partition. Since  $S$  is compact by assumption, we must have that this partition is finite, and so the image of  $f_m$  was also finite.

(ii) Let  $\epsilon > 0$ . Let  $B^0(a, \epsilon)$  be the open ball,  $B^0(a, \epsilon) = \{x \in S; \rho(x, a) < \epsilon\}$ . Then  $(S \setminus B^0(a, \epsilon))$  is compact, and so from the above we know that  $\phi_a$  restricted to  $(S \setminus B^0(a, \epsilon))$  has finite range (let  $M = S$  and  $S = (S \setminus B^0(a, \epsilon))$  and apply (i)). Then the sets

$$[\epsilon, \infty) \cap \{\rho(s, a); s \in S, s \neq a\}$$

are finite and  $\text{Im}(\phi_a)$  is discrete.

□

This leads to the following definition.

**Definition 14.** If  $(M, \rho)$  is an ultrametric space, we say  $M$  is **discretely-valued** if the set  $\Gamma_M = \{r \in \mathbb{R}; \exists x, y \in M \text{ such that } \rho(x, y) = r\}$  is a discrete subset of  $\mathbb{R}$ .

All of the examples of ultrametric spaces that we see in this work, indeed all of the examples that we know of, are discretely-valued. In fact, if  $M$  is a compact group with a translation-invariant ultrametric, then  $M$  is discretely-valued since the sets  $\phi_a$  are then equal for all  $a$  in  $M$ . Now we have the following question.

**Question 1.** Are there mild conditions under which a compact ultrametric space is discretely-valued? In particular, are there conditions that do not appeal to some algebraic structure in  $M$ ?

When this is the case, it will become useful to write the set of distances occurring in  $S$  as a sequence, put in decreasing order.

**Notation.** If  $S$  is a compact, discretely-valued ultrametric space, then we denote the set of distances between points of  $S$  by

$$\Gamma_S = \{\gamma_0 = d = \text{diam}(S), \gamma_1, \gamma_2, \dots, \gamma_\infty = 0\}$$

where  $\gamma_i \in \Gamma_S$  if and only if  $\exists x, y \in S$  such that  $\rho(x, y) = \gamma_i$  and  $\gamma_i < \gamma_j$  if and only if  $i > j$ .

We end this section with the following corollary.

**Corollary 2.** ([Ro]) *Let  $B(a, r)$  be a closed ball in a compact, discretely-valued ultrametric space. Then there exists  $r' > r \in \mathbb{R}$  such that  $B(a, r) = \{x \in M \mid \rho(x, a) < r'\}$ ; that is, every closed ball is also an open ball with the same centre and slightly larger radius.*

## 2.2 $\rho$ -orderings, $\rho$ -sequences, and valutive capacity

We are now in a position to give a general definition of  $p$ -orderings and in turn,  $p$ -sequences and valutive capacity. The observation that an analogous notion of  $p$ -ordering can be defined for a general ultrametric space, and that these structures coincide with Fekete  $n$ -tuples, is due to [J1]. The exploration of this idea makes up the remainder of this work.

**Definition 15.** [J1] Let  $S$  be a subset of an ultrametric space  $(M, \rho)$ . A  $\rho$ -ordering of  $S$  is a sequence  $\{a_i\}_{i \geq 0}$  in  $S$  such that  $a_0$  is arbitrary and  $\forall n > 0$ ,  $a_n$  maximizes

$$\prod_{i=0}^{n-1} \rho(s, a_i)$$

over  $s \in S$ .

The above generalizes the definition of  $p$ -orderings for  $\mathbb{Z}$ , since maximizing the  $p$ -adic distance between two points in  $\mathbb{Z}$  (or  $\widehat{\mathbb{Z}_p}$ ) is the same as minimizing the  $p$ -adic valuation of the difference of two points. In particular,  $\{a_i\}_{i \geq 0}$  is a  $p$ -ordering of  $S$ ,



a subset of  $\mathbb{Z}$ , if and only if it is a  $\rho_p$ -ordering of  $(S, \rho_p)$ . Let us see an example of the simplest kind, i.e., for a finite set  $S$ .

**Example 4.** Suppose  $S$  is the finite subset of  $(\mathbb{Z}, \rho_2)$ , given by  $S = \{0, 2, 8, 3\}$ . Then a  $\rho_2$ -ordering of  $S$  starts (arbitrarily) with  $a_0 = 0$ , which forces  $a_1 = 3$ , since  $\rho_2(0, 3) = 1 = \text{diam}(S)$ . The sequence continues  $a_2 = 2$  and  $a_3 = 8$ , but after this point the sequence becomes arbitrary because  $\prod_{i=0}^{n-1} \rho(s, a_i)$  will contain a 0, given by the repeated term. Indeed, for any finite subset  $S$  with  $|S| = n$  the  $\rho$ -ordering of  $S$  is arbitrary from the  $n^{\text{th}}$  point on.

We now give the definition of a  $\rho$ -sequence for an ultrametric space, generalizing the notion of a  $p$ -sequence.

**Definition 16.** [J1] Let  $\{a_i\}_{i \geq 0}$  be a  $\rho$ -ordering of  $S$ . The  $\rho$ -sequence of  $S$  is defined by letting  $\delta(0) = 1$  and for  $n > 0$ ,

$$\delta(n) = \prod_{i=0}^{n-1} \rho(a_n, a_i)$$

The two propositions that follow are the critical observations. The first one tells us that we can use the  $\rho$ -sequence of  $S$  as an invariant and the second one motivates the definition of valutive capacity. The proofs of each are given in [J1].

**Proposition 16.** ([J1], Lemma 1) *The  $\rho$ -sequence of  $S$  is well-defined so long as  $S$  is compact and  $\rho$  is an ultrametric. That is, the  $\rho$ -sequence of a compact subset of an ultrametric spaces does not depend on the choice of  $\rho$ -ordering of  $S$ .*

**Proposition 17.** ([J1], Theorem 1) *If  $S$  is a compact subset of an ultrametric space  $(M, \rho)$ , then the first  $n$  terms of a  $\rho$ -ordering of  $S$  give a Fekete  $n$ -tuple of  $S$  and all Fekete  $n$ -tuples of  $S$  arise in this way.*

Armed with the notion of a well-defined  $\rho$ -sequence for an ultrametric space, and the knowledge that it gives a construction for Fekete  $n$ -tuples in that space, we define the valutive capacity of  $S$ , where  $S$  is any compact subset of an ultrametric

space.

**Definition 17.** [J1] Let  $S$  be a compact subset of an ultrametric space  $(M, \rho)$  and let  $\delta(n)$  be the  $\rho$ -sequence of  $S$ . The **valuative capacity** of  $S$  is

$$\omega(S) := \lim_{n \rightarrow \infty} \delta(n)^{1/n}$$

We spend the rest of this chapter establishing some basic results on valutive capacity. These results form the start of our toolkit for calculating the capacities of specific sets. They also show that many of the properties of capacity from  $\mathbb{C}$  carry over to the non-Archimedean case in a natural way.

Let us assume from this point on that  $S$  is always a compact subset of an ultrametric space, unless stated otherwise.

**Proposition 18.**  $\omega(S)$  is finite. If  $S$  itself is finite, then  $\omega(S) = 0$ .

A compact set  $E \subseteq \mathbb{C}$  is said to be polar if the logarithmic capacity of  $E$  is 0 [Ra1]. Polar sets play a central role in potential theory and the theory of logarithmic capacity, which raises the following question:

**Question 2.** Are there ultrametric spaces that have some *infinite* subset  $S$  with  $\omega(S) = 0$ ?

We also have the expected result on monotonicity for valutive capacity:

**Proposition 19.** ([J1], Lemma 4) If  $S$  and  $T$  are compact subsets of an ultrametric space such that  $S \subseteq T$  then  $\omega(S) \leq \omega(T)$ .

We show now some results on the interaction between the algebraic structure of the space and valutive capacity. These results can be powerful tools for calculating capacities, in particular, when they are combined with the subadditivity result that follows.

**Proposition 20.** (*Translation Invariance*) If  $(M, \rho)$  is a compact ultrametric space and also a topological group for which  $\rho$  is (left) invariant under the group operation, then  $\omega$  is also (left)-invariant. That is, if  $\rho(x, y) = \rho(g + x, g + y)$ ,  $\forall g, x, y \in M$ , then  $\omega(g + S) = \omega(S)$ , for  $S \subseteq M$ .

*Proof.* Let  $\{a_i\}_{i \geq 0}$  be a  $\rho$ -ordering for  $S$ . Then  $\{g + a_i\}_{i \geq 0}$  is a  $\rho$ -ordering for  $g + S$ . Then

$$\begin{aligned}
 \omega(g + S) &= \lim_{n \rightarrow \infty} \delta(n)^{1/n} \\
 &= \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(g + a_n, g + a_i) \right]^{1/n} \\
 &= \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} \\
 &= \omega(S)
 \end{aligned}$$

□

**Example 5.** Note that  $\rho_p$  is translation invariant for each  $p$  since for any  $x, y$ , we have  $\rho_p(x, y) = p^{-v_p(x-y)} = p^{-v_p((a+x)-(a+y))} = \rho_p(a+x, a+y)$ . Then  $\omega(a+S) = \omega(S)$  for  $S \subseteq (\mathbb{Z}_p, \rho_p)$ .

**Proposition 21.** (*Scaling*) Let  $(V, N)$  be a normed vector space and suppose  $N$  satisfies the strong triangle identity, so that  $(V, \rho_N)$  is an ultrametric space. Then if  $N$  is multiplicative, so is  $\omega$ . That is, if  $N(gx) = N(g)N(x), \forall g, x \in V$ , then  $\omega(gS) = N(g)\omega(S)$ , for  $g \in V$  and  $S \subseteq M$ .

*Proof.* Let  $\rho_N$  be the metric induced by  $N$ , so that  $\rho_N(x, y) = N(x - y), \forall x, y \in V$ . Let  $\{a_i\}_{i \geq 0}$  be a  $\rho_N$ -ordering for  $S$  and let  $u, v$  be in  $gS$  with  $u = gs_i$  and  $v = gs_j$

for some  $s_i, s_j \in S$ . Then, since  $N$  is multiplicative,

$$\begin{aligned}
& \rho(u, v) \\
&= \rho(g s_i, g s_j) \\
&= N(g s_i - g s_j) \\
&= N(g(s_i - s_j)) \\
&= N(g)N(s_i - s_j) \\
&= N(g)\rho(s_i, s_j),
\end{aligned}$$

so that  $\{g a_i\}_{i \geq 0}$  is a  $\rho_N$ -ordering for  $gS$ . Then,

$$\begin{aligned}
\omega(gS) &= \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(g a_n, g a_i) \right]^{1/n} = \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} N(g) \rho(a_n, a_i) \right]^{1/n} \\
&= \lim_{n \rightarrow \infty} \left[ N(g)^n \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g) \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^{n-1} \rho(a_n, a_i) \right]^{1/n} = N(g) \omega(S)
\end{aligned}$$

□

**Example 6.** Since  $\rho_p$  is multiplicative, we have that  $\omega(mS) = |m|_p \cdot \omega(S)$  for  $m \in \mathbb{Z}$  and  $S \subseteq \mathbb{Z}$ . In particular,  $\omega(p\mathbb{Z}) = \frac{1}{p} \cdot \omega(\mathbb{Z})$ .

The following proposition is from [J1], where it is given for some  $S$  written as the union of two subsets, although it is easily seen to be true for  $S$  equal to any finite union, so long as the other assumptions remain satisfied.

**Proposition 22.** ([J1], Proposition 10) (Decomposition) *If  $\text{diam}(S) = d$  and  $S = \cup_i^n A_i$  for  $A_i$  compact subsets of  $M$  with  $\rho(a, b) = d$ , for all  $a \in A_i$  and for all  $b \in A_j$ , for each  $i$  and  $j$ , then*

$$\frac{1}{\log(\omega(S)/d)} = \sum_{i=1}^n \frac{1}{\log(\omega(A_i)/d)}$$

**Example 7.** We are now in a position to compute the valuative capacity of  $(\mathbb{Z}, \rho_p)$ . For any  $p$ , we note that  $\mathbb{Z}$  can be decomposed into  $p$  closed balls of radius  $\frac{1}{p}$ , which

are equal to the cosets of  $\mathbb{Z}$  modulo  $p$ . Since  $\text{diam}(S) = 1$ , this gives

$$\frac{1}{\log(\omega(\mathbb{Z}))} = \sum_{i=0}^{p-1} \frac{1}{\log(\omega(p\mathbb{Z} + i))} = \frac{p}{\log(\omega(p\mathbb{Z}))} = \frac{p}{\log(\frac{1}{p} \cdot \omega(\mathbb{Z}))}$$

Now we have,

$$\log(\omega(\mathbb{Z})^p) = \log\left(\frac{1}{p} \cdot \omega(\mathbb{Z})\right)$$

Since we have logs on both sides, we are able to use any base we like, so that,

$$\omega(\mathbb{Z})^p - \frac{\omega(\mathbb{Z})}{p} = 0$$

and  $\omega(\mathbb{Z}) = p^{\frac{1}{1-p}} = p^{\frac{-1}{p-1}}$ .

We can apply the same reasoning to any partition of  $S$  made up of sets that all have the same capacity and meeting the requirement that their pairwise distances are all equal to the diameter of  $S$ .

**Corollary 3.** *Suppose  $S = \cup_i^n S_i$  with  $\rho(S_i, S_j) = d = \text{diam}(S)$  and also  $\omega(S_i) = \omega(S_j)$ ,  $\forall i, j$ . Let  $r \in \mathbb{R}$  be such that  $\omega(S_i) = r\omega(S)$ ,  $\forall i$ . Then  $\omega(S) = r^{\frac{1}{n-1}}$ .*

Now we note that a partition of  $S$  into closed balls will satisfy the hypotheses if the pairwise distance between elements is equal to the diameter of  $S$ . In particular, if  $B(x_i, r_i)$  is a collection of closed balls such that the pairwise-distance between any  $B(x_i, r_i)$  and  $B(x_j, r_j)$  is constant, then if we know the capacity of each  $B(x_i, r_i)$ , we can compute the capacity of their union. If  $M$  is discretely-valued, then we can say more.

**Corollary 4.** *(Joins of computable sets are computable) Let  $M$  be a compact, discretely-valued ultrametric space. Let  $\Gamma_M = \{\gamma_0, \gamma_1, \dots, \gamma_\infty = 0\}$  be the set of distances in  $M$ . Suppose that  $S = B(x, \gamma_i)$ , for some  $x$  and  $i$ , is the union of  $n \geq 2$  balls of radius  $\gamma_{i+1}$ , that is,  $S = \cup_{j=1}^n B(x_j, \gamma_{i+1})$  is a join in the lattice of closed balls in  $M$ . Then*

$$\frac{1}{\log\left(\frac{\omega(B(x, \gamma_i))}{\gamma_{i+1}}\right)} = \sum_{j=1}^n \frac{1}{\log\left(\frac{\omega(B(x_j, \gamma_{i+1}))}{\gamma_{i+1}}\right)}$$

Of course, if  $M$  is a group, then we know the elements in these partitions are cosets, and if the metric is translation-invariant, then they each have the same capacity. We take up this last corollary in significant detail in the next chapter, obtaining some formulae for valutive capacity with various restrictions on  $\Gamma_M$  or related structures.

## Chapter 3

### $\rho$ -orderings and the structure of $S$

In the previous section, we defined valutive capacity for a compact subset  $S$  of an ultrametric space  $(M, \rho)$ . We also got a glimpse into the way the valutive capacity of  $S$  interacts with its other properties, such as the set of distances occurring in  $S$  and the lattice of closed balls in  $S$  (or equivalently, if  $S$  has enough structure, a lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valutive capacity of  $S$  and the lattice of closed balls in  $S$ . In particular, we show how, if  $S$  is compact and discretely-valued, the lattice of closed balls can be used to compute the first  $n$  terms of a  $\rho$ -ordering of  $S$  (for any  $n < \infty$ ).

Similar results have been found for the special case of ultrametric fields in [CEF]. We extend these results by moving to a more general setting, showing that much can be said about capacity in  $S$  without appealing to any underlying algebraic structure. Significant portions of the theory developed in this chapter and the ones that follow were guided by an empirical investigation into the capacity of product spaces, which we describe in the final chapter. The code that performed these calculations is included in the appendix.

We assume throughout this section that  $S$  is a compact, discretely-valued subset of an ultrametric space  $(M, \rho)$ .

#### Subspaces of $S$

In the section we explore the subspaces of  $S$  formed by considering closed balls of some fixed radius. Recall from the previous section that if  $S$  is compact and

discretely-valued, then the set of distances occurring in  $S$  is a discrete, bounded subset of  $\mathbb{R}$  and so we may represent the set of distances by a sequence in decreasing order. As before, let the decreasing sequence of distances in  $S$  be given by  $\Gamma_S = \{\gamma_0 = \text{diam}(S), \gamma_1, \dots, \gamma_\infty = 0\}$ .

Now fix some  $k \in \mathbb{N}$ , and consider for a moment the set of closed balls of radius  $\gamma_k$  in  $S$ . We could denote these alternatively by  $B^M(x, \gamma_k) \cap S$  or by  $B^S(x, \gamma_k)$ , but when there is no risk of confusion, we will denote them simply by  $B(x, \gamma_k)$ . Clearly, the set  $\{B(x, \gamma_k); x \in S\}$  forms a cover of  $S$ . Although we have built the cover using closed balls, since we are in an ultrametric space, this gives an open cover of  $S$  (in fact, each element in the cover is not only an open set, but also an open ball for some radius slightly bigger than  $\gamma_k$ ). Then since  $S$  is compact, we must have some  $x_1, \dots, x_n$  such that  $S = \cup_{i=1}^n B(x_i, \gamma_k)$ . In fact, since  $\rho$  is an ultrametric, we can pick the  $x_i$ 's so that  $\cup_{i=1}^n B(x_i, \gamma_k)$  will be a disjoint union and therefore a finite partition of  $S$ . Note that both  $n$  and the  $x_i$ 's depend on our fixed  $k$ , but that  $n$  is independent of the  $x_i$ 's, since any choice of centres is equivalent. We rephrase this with the following definition and lemma:

**Definition 18.** For  $S$  and  $\Gamma_S$  as above, and  $k \in \mathbb{N}$ , fixed, define  $\sim_k$  to be the relation on  $S$  given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e.,  $x \sim_k y$  if and only if  $B_{\gamma_k}(x) = B_{\gamma_k}(y)$ .

The fact that  $\sim_k$  is an equivalence relation on  $S$  is equivalent to the observation that every point in a ultrametric ball is at its centre:

**Lemma 2.** *Let  $S$  and  $\Gamma_S$  be as above; then  $\sim_k$  is an equivalence relation on  $S$ .*

*Proof.*  $\sim_k$  is clearly reflexive and symmetric, since  $\rho$  is a metric. Transitivity results from the ultrametric property of  $\rho$ : if  $x \sim_k y$  and  $y \sim_k z$ , then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(y, z)) \leq \gamma_k$$



so  $x \sim_k z$ . □

We denote the set of equivalence classes of  $S / \sim_k$  by  $S_{\gamma_k}$ . We have defined  $S_{\gamma_k}$  to be the set of equivalence classes in  $S$  under the relation  $\sim_k$ , which is equivalent to letting  $S_{\gamma_k}$  be the set of closed balls of fixed radius  $\gamma_k$  in  $S$ . We now offer a third perspective on the elements on  $S_{\gamma_k}$ , which is due to [Ac]:

**Lemma 3.** *For each  $k$ , the elements of  $S_{\gamma_k}$ , that is, the closed balls of radius  $\gamma_k$ , themselves form an ultrametric space, where the metric is given by:*

$$\rho_k(B(x, \gamma_k), B(y, \gamma_k)) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } B(x, \gamma_k) = B(y, \gamma_k) \end{cases}$$

*Proof.*  $\rho_k$  is symmetric since  $\rho$  is. Likewise,  $\rho_k$  satisfies the ultrametric property, since  $\rho$  does: let  $B(x, \gamma_k), B(y, \gamma_k)$  and  $B(z, \gamma_k)$  be any three elements of  $S_{\gamma_k}$  and suppose  $\rho_k(B(x, \gamma_k), B(y, \gamma_k)) > 0$ . Then,

$$\begin{aligned} & \gamma_k \\ & < \rho_k(B(x, \gamma_k), B(y, \gamma_k)) \\ & = \rho(x, y) \leq \max(\rho(x, z), \rho(y, z)) \\ & = \max(\rho_k(B(x, \gamma_k), B(z, \gamma_k)), \rho_k(B(y, \gamma_k), B(z, \gamma_k))) \end{aligned}$$

since  $\gamma_k < \max(\rho(x, z), \rho(y, z))$  implies that at least one of  $\rho_k(B(x, \gamma_k), B(z, \gamma_k))$  or  $\rho_k(B(y, \gamma_k), B(z, \gamma_k))$  is greater than 0. □

So now the elements of  $S_{\gamma_k}$  may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

**Definition 19.** Let  $S$  and  $\Gamma_S$  be as above. Define  $\beta(i)_{i \geq 0}$  to be the sequence given by  $\beta(i) = |S_{\gamma_i}|$ , which is an invariant of  $S$  and which counts the number of connected components of  $S_{\gamma_i}$  (that is, the points of  $S_{\gamma_i}$ ), when viewed as a metric space. When necessary, we use  $\beta^S(i)$  to denote the  $\beta$  sequence for a given, compact ultrametric

space  $S$ . Adopting the terminology in [FP], we call  $\beta^S(i)$  the **structure sequence** of  $S$ .

**Notation 1.** Let  $S_{\gamma_k}$  be as above. We denote the elements of  $S_{\gamma_k}$  by  $B_1^k, \dots, B_{\beta(k)}^k$  or by  $B_1^{S,k}, \dots, B_{\beta(k)}^{S,k}$ , when necessary.

We return to the sequence  $\beta(i)$  at the end of this section. For now, we show how a  $\rho$ -ordering of  $S$  can be built recursively from the spaces  $S_{\gamma_k}$ . This begins by noting that the spaces themselves can be built recursively:

**Observation 1.** Let  $S$ ,  $\Gamma_S$ , and  $S_{\gamma_k}$  be as above. Then  $S_{\gamma_{k+1}}$  can be constructed by partitioning each of the closed balls in  $S_{\gamma_k}$  into closed balls of radius  $\gamma_{k+1}$  and taking their union: Let  $B(x_i, \gamma_k)$  be an element of  $S_{\gamma_k}$ , denoted by  $B_i^k$ . Then, there exists  $x_{i,1}, \dots, x_{i,l_i} \in B_i^k$  such that,

$$B_i^k = \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1})$$

and

$$B(x_{i,j}, \gamma_{k+1}) \cap B(x_{i,j'}, \gamma_{k+1}) = \emptyset, \forall j, j' \in 1 : l_i$$

and so

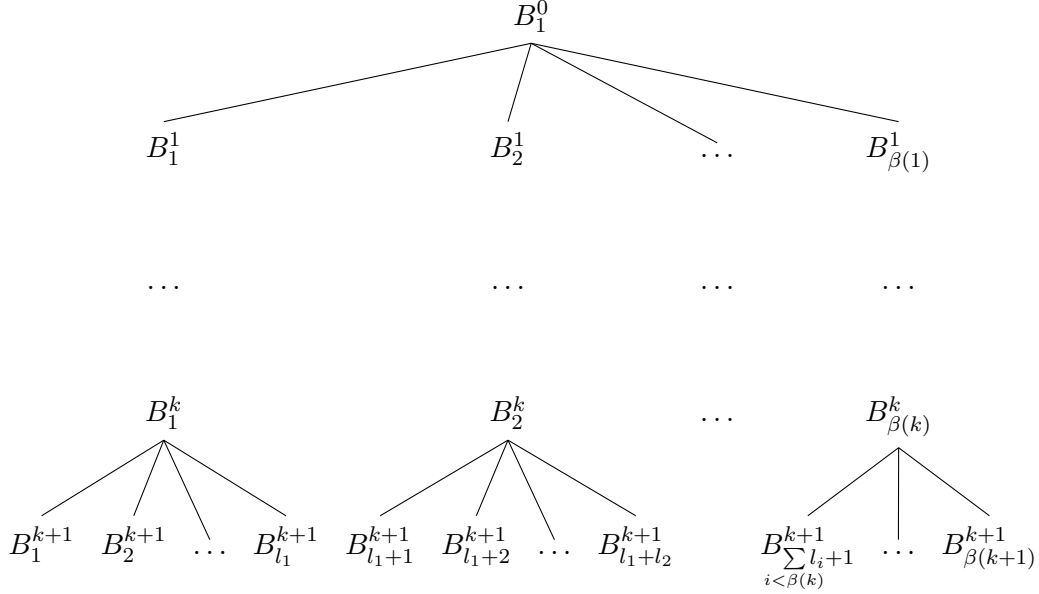
$$S_{\gamma_{k+1}} = \bigcup_{i=1}^{\beta(k)} \bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1}) = \bigcup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where  $\bigcup_{j=1}^{l_i} B(x_{i,j}, \gamma_k) = B(x_i, \gamma_{k+1}) = B_i^k, \forall i$ .

Since  $S$  is compact, hence bounded, if we represent this process schematically we obtain a tree, where the root node is  $B_1^0 = B(x, \gamma_0)$ , for any choice of  $x \in S$ , and the children of any given  $B_n^m$  are such that they form a partition of their join. Since we will often refer to this schematic representation, we define it below.

**Definition 20.** If  $S$  is a compact subset of an ultrametric space, then  $T_s$  is the tree whose vertices are  $B_i^k$ , that is the elements of  $S_{\gamma_k}$ , and whose edge-set,  $E$ , is given by  $(B_k^i, B_l^j) \in E$  if and only if  $j = i + 1$  and  $B_l^j \subseteq B_k^i$  for some choice of representatives

$B(x_k, \gamma_i)$  and  $B(x_l, \gamma_j)$ , as shown below:



Before going on, first note that we have drawn  $T_S$  such that leftmost child of some  $B_i^k$  is  $B_j^{k+1}$  where  $j$  is minimal among the children of  $B_i^k$ , and then continued in increasing order. In general, if we draw  $T_S$  so that the children of a given vertex are depicted in increasing order according to their index, then each choice of indexing for the elements of  $S_{\gamma_k}$  produces a different graphical representation of  $T_S$ . The structures produced by different choices of indices are clearly isomorphic as trees, and as we will see by the end of the section, each choice of indexing will be valid for our purposes as well.

Of central importance to us is the distance between two vertices in  $T_S$ . Since each vertex represents an element of  $S_{\gamma_k}$ , that is a closed ball in an ultrametric space, it is well-defined to let the distance between vertices be equal to the distance between a choice of centres for those balls. Note that if the distance between  $B_i^k$  and  $B_j^l$  is taken to be  $\rho(x_i, x_j)$ , for some choice of  $x_i \in B_i^k$  and  $x_j \in B_j^l$ , say  $\rho(x_i, x_j) = \gamma_n$ , then the join of  $B_i^k$  and  $B_j^l$  is some  $B_x^n$ .

**Lemma 4.** *If  $B_i^k$  and  $B_j^l$  are two vertices in  $T_S$ , then  $\rho(x_i, x_j)$ , for any choice of  $x_i \in B_i^k$  and  $x_j \in B_j^l$ , is equal to the diameter of the join of  $B_i^k$  and  $B_j^l$ .*

*Proof.* Let  $B_i^k$  and  $B_j^l$  be two (distinct) vertices in  $T_S$  and let  $B_x^n$  be their join. The diameter of  $B_x^n$  is  $\gamma_n$  since  $B_x^n = B(x_0, \gamma_n)$  for some  $x_0$ . Since  $\rho$  is an ultrametric, the distance between any  $x_i \in B_i^k$  and  $x_j \in B_j^l$  is constant, and must be equal to the diameter of the smallest ball containing both of them, that is  $\gamma_n$ .  $\square$

In particular, we have that for any  $k$  and any  $i < \beta(k)$ , the distances between the children of  $B_i^k$  will be  $\gamma_k$  and for any  $i \neq j$  the distance between the children of  $B_i^k$  and  $B_j^k$  will be equal to the distance between  $B_i^k$  and  $B_j^k$  (which will be some  $\gamma_m, m < k$ ).

### Recursive $\rho$ -orderings

In this section, we show how the recursive partitioning of  $S$  into the spaces  $S_{\gamma_k}$  gives rise to a  $\rho$ -ordering of  $S$ . We first note that without loss of generality, for any  $k \in \mathbb{N}$ , we can reindex the  $B_i^k$ 's so that they give the first  $\beta(k)$  terms of a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , when the latter is viewed as a (finite) metric space. In the first proposition below, we note that if the  $B_i^k$ 's are so indexed, then finding a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is straightforward: select a  $B_j^{k+1}$  from each of the  $B_i^k$ 's in order and then start over.

**Proposition 23.** *Let  $S$  be a compact, discretely-valued subset of an ultrametric space  $(M, \rho)$  and  $\Gamma_S$ , the set of distances in  $S$ . If  $S_{\gamma_k}$  is the partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the elements are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ , then the first  $\beta(k+1)$  terms in a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  can be found by selecting at each stage  $n$ , a child from  $B_{\bar{n}}^k$ , where  $\bar{n} = n \bmod \beta(k) + r$  and  $r$  is minimal in  $\{0, \dots, \beta(k) - 1\}$  such that  $B_{n \bmod \beta(k) + r}^k$  still has unused children.*

*Proof.* Let  $S$ ,  $S_{\gamma_k}$ , and  $S_{\gamma_{k+1}}$  be as above. In particular, suppose the elements of  $S_{\gamma_k}$  are indexed according to a  $\rho_k$ -ordering. Denote the elements of  $S_{\gamma_{k+1}}$  by  $B_{i,j}^{k+1}$  where the first subscript indicates that the element is a child of  $B_i^k$ . To form a  $\rho_{k+1}$  ordering of  $S_{\gamma_{k+1}}$ , we must maximize the product of distances at each step  $n$ .

Now note that  $\Gamma_{S_{\gamma_k}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$  and  $\Gamma_{S_{\gamma_{k+1}}} = \{\gamma_0, \gamma_1, \dots, \gamma_{k-1}, \gamma_k\}$ . That is, the distances in  $S_{\gamma_{k+1}}$  are the same as the distances in  $S_{\gamma_k}$ , although they also

include the smaller distance  $\gamma_k$ . Since we know that the elements  $B_1^k, \dots, B_{\beta(k)}^k$  already maximize the product of distances in  $\{\gamma_0, \gamma_1, \dots, \gamma_{k-1}\}$ , the first  $\beta(k)$  terms of a  $\rho_{k+1}$ -ordering of  $S_{k+1}$  can be found by taking  $B_{1,j_1}^k, \dots, B_{1,j_{\beta(k)}}^k$  for any choice of  $j$ 's. At this point, any choice of next element will produce a copy of  $\gamma_k$  in the  $\rho_{k+1}$ -sequence; however, if we choose another child of  $B_1^k$ , we are able to keep building the ordering in a canonical fashion, since we know that we will then be able to maximize the product at the next step by choosing another child of  $B_2^k$ .

We see then that a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by minimizing the number of times  $\gamma_k$  is introduced into the  $\rho_{k+1}$ -sequence and maximizing the product among the  $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ , and the latter is already known to be achieved by taking the  $B_i^k$  in order. If the  $B_i^k$ 's all have the same number of children, then we can always select a child of  $B_{\bar{n}}^k$ , where  $\bar{n} = n \bmod \beta(k)$  at each stage  $n$ ,  $n < \beta(k+1)$ , since there will always be one available. On the other hand, suppose the  $B_i^k$  have an unequal number of children and  $n$  is the first step at which all the children of  $B_{\bar{n}}^k$  have been exhausted. What element will maximize the  $\rho_{k+1}$ -sequence?

Consider the space  $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$ . Removal of  $B_{\bar{n}}^k$  will not affect the first  $m$  terms of a  $\rho_k$ -ordering of this space, for  $m < \bar{n}$ , since if a sequence of elements maximizes a function over a set  $X$ , they will also maximize that function of a subset of  $X$  (provided they themselves remain in the subset). Then the  $\rho_k$ -sequence of  $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$  begins  $\{B_1^k, \dots, B_{\bar{n}-1}^k\}$ .

Moreover, if  $B_{\bar{n}+1}^k$  maximizes  $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k)$  over  $S_{\gamma_k}$ , then it also maximizes  $\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)$  over  $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$ , since  $\prod_{i=1}^{\bar{n}} \rho_k(x, B_i^k) = (\prod_{i=1}^{\bar{n}-1} \rho_k(x, B_i^k)) \cdot \rho_k(x, B_{\bar{n}}^k)$ .

Then the  $\rho_k$ -sequence of  $(S_{\gamma_k} \setminus B_{\bar{n}}^k)$  is simply  $\{B_1^k, \dots, B_{\bar{n}-1}^k, B_{\bar{n}+1}^k, \dots, B_{\beta(k)}^k\}$ .

Now we see that a  $\rho_{k+1}$ -sequence of  $S_{\gamma_{k+1}}$  is maximized by simply skipping over  $B_{\bar{n}}^k$ , should all its children be exhausted, and selecting a child from  $B_{\bar{n}+1}^k$ . Then a  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by selecting elements of each  $B_i^k$  in order as much as possible, and skipping to  $B_{i+1}^k$ , when it is not possible.

□

Note that in building the  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  we selected, at each step, a child of some  $B_i^k$ , but we did not concern ourselves over which child was selected. This is because the distances between any two children of some  $B_i^k$  is  $\gamma_k$ , and the distance between any one of them and a child of some  $B_j^k$ ,  $i \neq j$ , is the same. We can now see, as claimed above, that any of the isomorphic versions of  $T_S$  are valid for producing  $\rho$ -orderings. Suppose then that we have created  $T_s$  and (arbitrarily) indexed the children of each vertex. Then, there is no loss of generality in assuming that at each stage, we select a child with smallest index among its siblings, that is, that we select the leftmost available child in  $T_s$ . Since, for ease of indexing, we will assume a  $\rho$ -ordering has been built by this convention, we introduce the following definition.

**Definition 21.** The  $\rho$ -ordering of  $S$  formed by pulling elements from left to right in (a choice of)  $T_s$  is called the **canonical  $\rho$ -ordering** of  $S$  (with respect to  $T_s$ ).

The above proposition quickly leads to a recursive contruction for a  $\rho$ -ordering of  $S$ . Indeed, to build a  $\rho$ -ordering of  $S$  from the above, it suffices only to make a choice of centres for each of the  $B_i^k$ 's.

**Proposition 24.** *Let  $S$  be a compact, discretely-valued subset of an ultrametric space  $(M, \rho)$  and let  $\Gamma_S$  be the set of distances in  $S$ . Let  $S_{\gamma_k}$  be the partition of  $S$  as described above for  $\gamma_k \in \Gamma_S$  with  $k < \infty$ , where the elements are indexed according to a  $\rho_k$ -ordering of  $S_{\gamma_k}$ . Suppose each of the element of  $S_{\gamma_k}$  have also been partitioned into closed balls of radius  $\gamma_{k+1}$ ,  $B_i^k = \cup_{j=1}^{l_i} B_{i,j}^{k+1}, \forall i$ .*

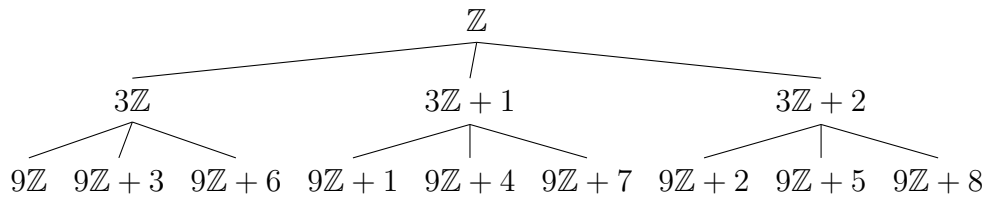
*Let  $x_{i,j}$  denote a choice of centre for the element  $B_{i,j}^{k+1}$ . Then the first  $\beta(k+1)$  elements of a  $\rho$ -ordering of  $S$  can be found by forming a matrix,  $A_k$ , whose  $(i,j)^{th}$  entry is  $x_{i,j}$ , if  $j \leq l_i$  and is otherwise equal to a placeholder,  $*$ , and then concatenating the rows.*

*Proof.* The matrix  $A_k$  is a representation of the  $k^{th}$  and  $(k+1)^{th}$  levels of  $T_S$  where the  $B_i^k$ 's (and  $B_{i,j}^{k+1}$ 's) have been replaced by a choice of centres. Since matrices must

be rectangular, the case where some  $B_i^k$  and  $B_j^k$  have an unequal number children is handled by inserting a placeholder,  $*$ , into  $A_k$ . Moreover, since the  $\rho_{k+1}$  distance between distinct closed balls is just the  $\rho$  distance between a choice of centres of those balls, a choice of centres in a  $\rho_{k+1}$ -ordering gives the beginning of a  $\rho$ -ordering. By the above proposition, we must select elements from each  $B_i^K$  one after the other, which is achieved by selecting one element from each column in order, for example by concatenating the rows (and then deleting  $*$ 's if necessary).  $\square$

We get the most use out of the construction above if, in selecting a choice of centres for the  $B_{i,j}^{k+1}$ 's, we reuse the previous choices as much as possible. Suppose, for example, we have made a choice of centres for the balls of radius  $\gamma_k$  and constructed the matrix  $A_{k-1}$ . At the next iteration, we will need a choice of centres for the balls of radius  $\gamma_{k+1}$ . If  $x_i$  was our choice of representative for  $B_i^k$  and  $x_i \in B_{i,j}^{k+1}$ , we may as well let  $x_i$  be our choice of representative for  $B_{i,j}^{k+1}$ . If we make our choice of centres in this way, then when we concatenate the rows of some  $A_{k-1}$ , we obtain (without loss) the first row of  $A_k$ . We follow this convention in the two examples below.

**Example 8.** Let us use the above to start a  $\rho$ -ordering of  $S = (\mathbb{Z}, \rho_3)$ . We have that  $\Gamma_S = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\}$  and  $T_s$  begins:



We start by finding a  $\rho_0$ -ordering of  $S_{\gamma_0}$ , but this is trivial since  $S_{\gamma_0}$  has only a single element. Let us pick 0 to be our choice of centre for  $B_1^0 = B(0, 1) = \mathbb{Z}$ . As we see from  $T_S$ ,  $S_{\gamma_0}$  is partitioned into 3 closed balls of radius  $\gamma_1 = \frac{1}{3}$ , namely  $3\mathbb{Z}$ ,  $3\mathbb{Z} + 1$ , and  $3\mathbb{Z} + 2$ . A choice of centres is given by 0, 1, and 2, so that  $A_0$  becomes:

$$A_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

To start the  $\rho$ -ordering, concatenate the rows to obtain  $\{0, 1, 2\}$ , and to continue it, make a choice of centres for each of the closed balls of radius  $\gamma_2 = \frac{1}{9}$  partitioning the sets  $3\mathbb{Z} + i$ ,  $i \in 0, 1, 2$ . For example,  $3\mathbb{Z} = 9\mathbb{Z} \cup 9\mathbb{Z} + 3 \cup 9\mathbb{Z} + 6$ , so a choice of centres for  $B_1^1$  is given by  $\{0, 3, 6\}$ . Making choices for the remaining elements, we obtain:

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

To continue the  $\rho$ -ordering we concatenate the rows,  $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ , which also gives the first row of  $A_2$ . The remaining rows are found by partitioning each of the closed balls of radius  $\frac{1}{9}$  and again making a choice of centres:

$$A_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{pmatrix}$$

And so on.

We are able to make two statements following this example. The first is that in starting the  $\rho_3$ -ordering, the fact that  $S_{\gamma_0}$  had only a single element allowed us to get started for free. In fact, all compact ultrametric spaces are bounded, so this is always the case.

The second takeaway is that we found the start of a  $\rho$ -ordering of  $S = (\mathbb{Z}, \rho_3)$  was given by taking the integers starting at 0 in their natural order. If we had continued building the ordering, we would have continued to find this. The fact that the natural ordering on the integers is a  $\rho_p$ -ordering, where  $\rho_p$  is the  $p$ -adic metric for any prime  $p$ , is well known ([B2]), but we give an alternate proof of it here:

**Corollary 5.** *Let  $S$  be the ultrametric space  $(\mathbb{Z}, \rho_p)$ , where  $\rho_p$  is  $p$ -adic metric for any prime  $p$ . The a  $\rho_p$ -ordering of  $S$  can be found by taking the integers, starting at 0, in their natural order.*



*Proof.* We prove the above by induction on  $k$ . First note that for any choice of prime, the elements of  $S_{\gamma_1}$  are the cosets of  $\mathbb{Z}$  modulo  $p$ , so that  $A_1$  has  $p$  columns. Since  $\{0, 1, 2, \dots, p-1\}$  are distributed among each of these cosets, without loss of generality the first row of  $A_1$  is given by  $[0, 1, 2, \dots, p-1]$  in order.

Now suppose that the first row of  $A_k$  is given by  $[0, 1, 2, \dots, n]$  for  $0 < k < k+1$ . We show the first row of  $A_{k+1}$ , and therefore the first  $n'$  elements in a  $\rho_p$ -ordering of  $S$ , where  $n'$  is the column dimension of  $A_{k+1}$ , can be obtained as  $[0, 1, 2, \dots, n, n+1, \dots, n']$ . First note that each closed ball of radius  $p^k = \gamma_k$  is in fact a coset of  $\mathbb{Z}$  modulo  $p^k$ , of which there are  $p$ . Then for any  $k$ ,  $A_k$  is a matrix with  $p^k$  columns and  $p$  rows. In particular,  $n = p^k - 1$ . Let  $i \in \{0, 1, \dots, p^k - 1\}$  be arbitrary. Then  $i$  is in exactly one of the cosets of  $\mathbb{Z}$  modulo  $p^k$  and since the first row of  $A_k$  is  $[0, 1, 2, \dots, p^k - 1]$ , it must have been chosen as our representative of this coset. If we split  $p^k\mathbb{Z} + i$  into balls of radius  $p^{k+1}$ , we have

$$p^k\mathbb{Z} + i = \bigcup_{j=0}^{p-1} p^{k+1}\mathbb{Z} + (p^k j + i)$$

since there will be  $p$  elements in the partition, each of which will be equal to  $i$  modulo  $p^k$  and distinct modulo  $p^{k+1}$ . Then, there is a choice of centres such that the  $i^{th}$  column of  $A_k$  is

$$[i, p^k + i, 2p^k + i, \dots, (p-1)p^k + i]^T$$

filling this in for each  $i$ , we see that  $A_k$  can be obtained as:

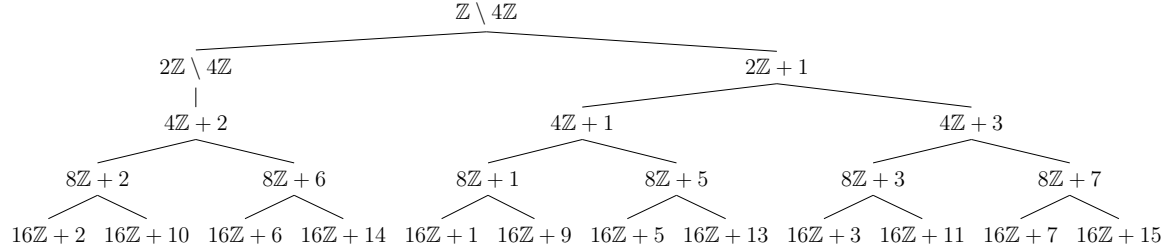
$$A_k = \begin{pmatrix} 0 & 1 & 2 & \dots & p^k - 1 \\ p^k & p^k + 1 & p^k + 2 & \dots & p^k + (p^k - 1) \\ 2p^k & 2p^k + 1 & 2p^k + 2 & \dots & 2p^k + (p^k - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)p^k & (p-1)p^k + 1 & (p-1)p^k + 2 & \dots & (p-1)p^k + (p^k - 1) \end{pmatrix}$$

Concatenating the rows, we see the first row of  $A_{k+1}$  will be

$$[0, 1, 2, \dots, p^k - 1, p^k, \dots, p^{k+1} - 1]$$

as required.  $\square$

**Example 9.** Let us now see an example where there is an uneven number of children between the vertices on a given level. Suppose  $S = \mathbb{Z} \setminus 4\mathbb{Z}$ , a subset of  $(\mathbb{Z}, \rho_2)$ . In this case, we have that  $\Gamma_S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$  and  $T_s$  begins:



Choosing centres for the partition of  $\mathbb{Z}$  into closed balls of radius  $\frac{1}{2}$ , we have:

$$A_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have taken  $S$  to be the complement of  $4\mathbb{Z}$  in  $\mathbb{Z}$ , so  $B(0, \gamma_1)$  has only one child, since  $2\mathbb{Z} \setminus 4\mathbb{Z} = 4\mathbb{Z} + 2$ , while  $B(1, \gamma_1)$  has two. Making a choice of centres, we have:

$$A_1 = \begin{pmatrix} 2 & 1 \\ * & 3 \end{pmatrix}$$

We concatenate the rows, skipping over  $*$ , and again make a choice of centres for the closed balls of radius  $\frac{1}{8}$ :

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \end{pmatrix}$$

One more iteration yields:

$$A_2 = \begin{pmatrix} 2 & 1 & 3 & 6 & 5 & 7 \\ 10 & 9 & 11 & 14 & 13 & 15 \end{pmatrix}$$

So that a  $\rho_2$ -ordering of  $S = \mathbb{Z} \setminus 4\mathbb{Z}$  starts:  $\{2, 1, 3, 6, 5, 7, 10, 9, 11, 14, 13, 15, \dots\}$ .

In the two propositions above, there was notational difficulty that arose when there was an unequal number of children between the vertices on a given level of  $T_s$ .

This difficulty is, in fact, more than a notational inconvenience, and the situation simplifies considerably when it is not the case. We are far from the first to observe this. Amice noted this as far back as her 1964 paper [Am], and it has been observed more recently by Chabert and colleagues, for example in [FP] and [CEF]. In the next chapter, we see that if  $T_S$  has nice enough structure we are able to compute not just  $\rho$ -orderings, but also formulae for  $\rho$ -sequences and, when we are lucky, capacity.

## Chapter 4

### The structure of $T_S$

In the previous chapter, we explored in detail the final Corollary of Chapter 2. This corollary lead us to study the lattice of closed balls in  $S$ , which we called  $T_S$ . In this chapter, we take what we have learned and explore the preceding Corollary, repeated below.

**Corollary.** *Suppose  $S = \cup_i^n S_i$  with  $\rho(S_i, S_j) = d = \text{diam}(S)$  and also  $\omega(S_i) = \omega(S_j)$ ,  $\forall i, j$ . Let  $r \in \mathbb{R}$  be such that  $\omega(S_i) = r\omega(S)$ ,  $\forall i$ . Then  $\omega(S) = r^{\frac{1}{n-1}}$ .*

In particular, we seek answers to the following questions: when does such a partition of  $S$  exist and given such a partition, when are we able to compute the scaling factor  $r$ ? In doing so, we show that the structure of  $T_S$  plays an important role.

#### 4.1 Semi-regularity

In this section, we restrict to the case where in the tree  $T_s$ , for  $S$  a compact, discretely-valued subset of an ultrametric space, every vertex on a given level has the same number of children. In this case, we can attach another sequence to  $S$ , which we call the  $\alpha$ -sequence of  $S$  and which describes, for each level  $k \in \mathbb{N}$ , the size of the partitions on that level. We develop some preliminary lemmas, which we then use to derive formulae for this special case. This situation corresponds to what previous authors ([Am], [CEF], [FP]) have called *regularity*, a term which we reserve for the next section.

In the definitions that follow, we recall that the  $\beta$ -sequence of  $S$  counts the number of elements of  $S_{\gamma_k}$ .

**Definition 22.** Let  $S$  be as before, a compact, discretely-valued subset of an ultrametric space  $(M, \rho)$ . We say that  $S$  is **semi-regular** if  $T_{B_i^k} \cong T_{B_j^k}$ ,  $\forall k \in \mathbb{N}$  and  $i, j \in \beta(k)$ , and where the isomorphism is understood as an isomorphism of trees. That is,  $S$  is semi-regular if each ball of radius  $\gamma_k$  breaks into the same number of balls of radius  $\gamma_{k+1}$ , for all  $k$ . If there exists an  $n \in \mathbb{N}$  such that  $T_{B_i^N} \cong T_{B_j^N}$  for all  $N \geq n$ , that is, each ball of radius  $\gamma_N$  breaks into the same number of balls of radius  $\gamma_{N+1}$  for  $N \geq n$ , then we say  $S$  is **eventually semi-regular**.

**Definition 23.** Suppose  $S$  is a compact, discretely-valued subset of an ultrametric space and  $S$  is semi-regular. The  $\alpha$ -sequence of  $S$  is the sequence given by

$$\alpha(k) = \frac{\beta(k+1)}{\beta(k)}$$

which is in  $\mathbb{N}$  for each  $k$ . That is, if  $B_i^k$  is any element of  $S_{\gamma_k}$ , then  $\alpha(k)$  is equal to the number of children of  $B_i^k$  in  $T_s$ . Since  $S$  is semi-regular, this number does not depend on  $i$ .

**Example 10.** If  $G$  is a compact ultrametric space and also a group, each ball centred at 0 is in fact a subgroup of  $G$ . Then each set of elements of  $S_{\gamma_k}$  is a collection of cosets of  $G/B(0, \gamma_k)$ . Since  $G$  is assumed to be compact,  $G/B(0, \gamma_k)$  is finite and so Lagrange's theorem implies that  $G$  is semi-regular.

We now work towards a formula for the terms in the  $\rho$ -sequence of a semi-regular space  $S$ . We need a few lemmas to get started.

**Lemma 5.** Let  $n$  and  $q$  be in  $\mathbb{N}$ . Then  $\lfloor \frac{n}{q} \rfloor$  counts the numbers in  $\{0, \dots, n-1\}$  that are congruent to  $n \pmod q$ .

*Proof.* By the division algorithm, we know there exists unique  $c, r \in \mathbb{Z}$  such that

$$n = cq + r$$

with  $0 \leq r < q$ . Since  $c$  counts the number of  $q$ -multiples in the set  $\{1, \dots, n\}$ , and each  $q$ -multiple contains exactly one element that is congruent to  $n \pmod q$ , we need

only show  $\lfloor \frac{n}{q} \rfloor = c$ . Simply note that the above implies

$$\frac{n}{q} = c + \frac{r}{q}$$

and we must have  $\frac{r}{q} < 1$ . Then  $c$  is the largest integer such that  $\frac{n}{q} \leq c$ , but this is the definition of  $\lfloor \frac{n}{q} \rfloor$ .  $\square$

**Lemma 6.**

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n+kb}{ab} \rfloor$$

for  $n, a, b \in \mathbb{N}$ . In particular,

$$\lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{2b} \rfloor = \lfloor \frac{n+b}{2b} \rfloor$$

for  $n, b \in \mathbb{N}$ .

*Proof.*

$$\begin{aligned} \lfloor \frac{n}{b} \rfloor - \lfloor \frac{n}{ab} \rfloor &= \lfloor a \cdot \frac{n}{ab} \rfloor - \lfloor \frac{n}{ab} \rfloor = \sum_{k=0}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor - \lfloor \frac{n}{ab} \rfloor \quad (*) \\ &= \sum_{k=1}^{a-1} \lfloor \frac{n}{ab} + \frac{k}{a} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{n+kb}{ab} \rfloor \end{aligned}$$

where the final step in  $(*)$  is due to Hermite's identity:  $\lfloor nx \rfloor = \sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor$ , for  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ .  $\square$

**Lemma 7.** If  $S$  is semi-regular and  $\sigma$  denotes the canonical  $\rho$ -ordering of  $S$ , that is, a  $\rho$ -ordering formed by pulling from left to right in  $T_s$ , then

$$\rho(\sigma(n), \sigma(m)) = \gamma_k$$

if and only if

$$n \equiv m \pmod{\beta(k)} \text{ and } n \not\equiv m \pmod{\beta(k+1)}$$

*Proof.* Since  $S$  is semi-regular, every sequence of  $\beta(k)$  terms in  $\sigma$  will be from each of the distinct elements of  $S_{\gamma_k}$  (for any  $k$ ). Moreover, since  $\sigma$  is a canonical  $\rho$ -ordering, we always pull from the elements of  $S_{\gamma_k}$  in the same order. Then  $\sigma(n)$  and  $\sigma(m)$  are descendants of some  $B_j^k$  if and only if  $n \equiv m \pmod{\beta(k)}$ . Then the result follows

since  $\rho(\sigma(n), \sigma(m)) = \gamma_k$  if and only if  $B_i^k$  for some  $i \in 1, \dots, \beta(k)$  is the join of  $B_i^{k'} \ni \sigma(n)$  and  $B_{i'}^{k'} \ni \sigma(m)$ .

□

We introduce another piece of notation before continuing.

**Notation 2.** Let  $S$  be a compact, discretely-valued subset of an ultrametric space,  $\Gamma_S$  the set of distances in  $S$  and  $\delta(n)$  the characteristic sequence of  $S$ . Suppose  $\gamma_k$  is an element of  $\Gamma_S$ . Then we denote by  $v_{\gamma_k}(\delta(n))$  the exponent of  $\gamma_k$  in the  $n^{\text{th}}$ -term of the characteristic sequence of  $S$ .

**Proposition 25.** *If  $S$  is a semi-regular ultrametric space,  $\delta$  is the characteristic sequence of  $S$ ,  $\beta$  is the structure sequence of  $S$ , and  $\alpha$  is the sequence describing the semi-regularity, then*

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

*Proof.* The exponent of  $\gamma_k$  in the  $n^{\text{th}}$  term of the characteristic sequence is the number of  $m$  strictly less than  $n$  such that  $\rho(\delta(n), \delta(m)) = \gamma_k$ . By Lemma 7, this is the number of  $m < n$  such that  $m = n \pmod{\beta(k)}$  and  $m \neq n \pmod{\beta(k+1)}$ , which by Lemma 5 is  $\lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor$ . Then we have:

$$\begin{aligned} & v_{\gamma_k}(\delta(n)) \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor \\ &= \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k)\alpha(k)} \rfloor, \text{ because } S \text{ is semi-regular} \\ &= \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor \end{aligned}$$

□

**Example 11.** Consider the ultrametric space  $(\mathbb{Z}, \rho_p)$  for any prime  $p$ . Then  $\beta(k) = p^k$  and  $\alpha(k) = p$  for any  $k \in \mathbb{N} \cup 0$ . Proposition 25 gives

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor$$

Now since  $\gamma_k = p^{-k}$ ,  $\forall k$ , we are able to compute the exponent of  $\frac{1}{p}$  in  $\delta(n)$ . We have

$$\begin{aligned} v_{\frac{1}{p}}(\delta(n)) &= \sum_{k=1}^{\infty} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \sum_{k=1}^{\lceil \log_p(n) \rceil} k \cdot (\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor) \\ &= \lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor + 2\lfloor \frac{n}{p^2} \rfloor - 2\lfloor \frac{n}{p^3} \rfloor + \dots + \lceil \log_p(n) \rceil \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\ &= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \dots + \lfloor \frac{n}{p^{\lceil \log_p(n) \rceil}} \rfloor \\ &= \sum_{k=1}^{\lceil \log_p(n) \rceil} \lfloor \frac{n}{p^k} \rfloor \\ &= \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor \end{aligned}$$

We are able to simplify to a finite sum in the above because  $\lfloor \frac{n}{p^k} \rfloor = 0$  if

$$p^k > n \iff \log(p^k) > \log(n) \iff k > \log_p(n)$$

We have already seen that the natural order on the integers gives a  $\rho_p$ -ordering for each  $p$ . So then

$$\sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor = v_{\frac{1}{p}}(\delta(n)) = v_{\frac{1}{p}}\left(\prod_{i=0}^n \frac{1}{p}^{v_p(n-i)}\right) = \sum_{i=0}^{n-1} v_p(n-i) = v_p(n!)$$

so that we are able to recover the well-known Legendre's formula.

We end this section with the following observation.

**Proposition 26.** *Let  $S$  be a semi-regular subset of an ultrametric space  $(M, \rho)$ . Let*



$S_{\gamma_1}$  be the partition of  $S$  described in chapter 3, that is,

$$S_{\gamma_1} = \bigcup_{i=1}^n B(x_i, \gamma_1) = \bigcup_{i=1}^n B_i^1$$

Then  $\rho(B_i^1, B_j^1) = d = \text{diam}(S)$  for any  $i \neq j$  in  $1, \dots, n$  and  $\omega(B_i^1) = \omega(B_j^1)$  for all  $i$  and  $j$ .

*Proof.* The fact that  $\rho(B_i^1, B_j^1) = d = \text{diam}(S)$  for any  $i \neq j$  is clear and does not depend on the fact that  $S$  is semi-regular. In fact, there are plenty of ways to see this, but for example, we simply note  $\rho(B_i^1, B_j^1) \in \Gamma_S$  and  $\gamma_1 < \rho(B_i^1, B_j^1) \leq \gamma_0 = \text{diam}(S)$ .

To see that  $\omega(B_i^1) = \omega(B_j^1)$ , we note that since  $S$  is semi-regular, each  $B_i^1$  is semi-regular as well. Moreover, since  $S$  is semi-regular, the  $\beta$  sequences of  $B_i^1$  and  $B_j^1$  are the same for each  $i$  and  $j$ . Then the result follows: let  $\delta^{B_i^1}(n)$  and  $\delta^{B_j^1}(n)$  be the characteristic sequences of  $B_i^1$  and  $B_j^1$  respectively. We see that for all  $k$ ,

$$v_{\gamma_k}(\delta^{B_i^1}(n)) = \lfloor \frac{n}{\beta^B(k)} \rfloor - \lfloor \frac{n}{\beta^B(k+1)} \rfloor = v_{\gamma_k}(\delta^{B_j^1}(n))$$

where  $\beta^B(k)$  is the  $\beta$  sequence for each  $B_i^1$ .

□

Now we have one answer to our first question: when  $S$  is semi-regular, we can use the elements of  $S_{\gamma_1}$  to build the partition from Corollary 3. The content of that corollary gave a formula for the valutive capacity. Then if  $S$  is semi-regular, the principal obstacle to computing the capacity of  $S$  is the identification of the scaling factor. This leads to our second question: when can we compute  $r$ ?

## 4.2 Regularity

In Example 11, the fact that we were able to reduce to a finite sum was not the only reason we were able to simplify the calculations. It also helped a great deal that the sum was telescoping. What does the fact that we saw a telescoping sum have to do with computing the scaling factor  $r$ ? We explore the inter-relatedness of these situations, and with the definition below, in this section.

**Definition 24.** Let  $S$  be a semi-regular subset of an ultrametric space. If there exists a  $q \in \mathbb{N}$  such that  $\alpha(n) = q$ , for all  $n$ , then  $S$  is said to be **regular**<sup>1</sup>.

So then  $S$  is regular just in case  $S$  is semi-regular and the  $\alpha$ -sequence of  $S$  is constant. We need to make one more definition before we begin calculations.

**Definition 25.** Let  $S$  be a semi-regular subset of an ultrametric space and  $\Gamma_S$  the sequence of decreasing distances in  $S$ . Then we say  $S$  is **tame**, if for  $\gamma_k \in \Gamma_S$ ,

$$\gamma_k = \alpha(k)^{c_k}$$

for some  $c_k \in \mathbb{Q}$  and for all  $k \in \mathbb{N}$ .

Now we see what this situation means for calculations.

**Proposition 27.** *Let  $S$  be a regular, tame subset of a compact ultrametric space with  $\gamma_k = q^{c_k}$  for some  $c_k \in \mathbb{Q}$  and for all  $k \in \mathbb{N} \cup 0$ . Then*

$$v_q(\delta(n)) = c_0 n + \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

and

$$\log_q(\omega(S)) = \lim_{n \rightarrow \infty} c_0 + \frac{1}{n} \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor$$

*Proof.* We know that,

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

and since  $\gamma_k = q^{c_k}$ , we calculate

$$v_{q^{c_k}}(\delta(n)) = \lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor$$

and

$$v_q(\delta(n)) = \sum_{k=0}^{\infty} c_k \cdot (\lfloor \frac{n}{q^k} \rfloor - \lfloor \frac{n}{q^{k+1}} \rfloor)$$

---

<sup>1</sup>This is non-standard: what previous authors ([Am],[CEF],[FP]) have called regular is what we have called semi-regular. Note that  $S$  is regular in the present sense if and only if  $T_S$  is regular in the standard graph theory terminology.

$$\begin{aligned}
&= c_0 n - c_0 \lfloor \frac{n}{q} \rfloor + c_1 \lfloor \frac{n}{q} \rfloor - c_1 \lfloor \frac{n}{q^2} \rfloor + c_2 \lfloor \frac{n}{q^2} \rfloor - c_2 \lfloor \frac{n}{q^3} \rfloor + \\
&\quad \dots - c_{\lceil \log_q(n) \rceil} \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor + \lceil \log_q(n) \rceil \lfloor \frac{n}{q^{\lceil \log_q(n) \rceil}} \rfloor \\
&= c_0 n + \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor
\end{aligned}$$

Then since  $\omega(S) = \lim_{n \rightarrow \infty} \delta(n)^{\frac{1}{n}}$ ,

$$\begin{aligned}
\log_q(\omega(S)) &= \log_q\left(\lim_{n \rightarrow \infty} \delta(n)^{\frac{1}{n}}\right) = \log_q\left(\lim_{n \rightarrow \infty} q^{c_0 n + \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor}\right) \\
&= \lim_{n \rightarrow \infty} \left(c_0 + \frac{1}{n} \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \lfloor \frac{n}{q^k} \rfloor\right)
\end{aligned}$$

□

If, as in the case of  $p$ -adic spaces,  $c_i = -i$  for all  $i$ , then the above simply reduces to  $\lim_{n \rightarrow \infty} \frac{v_q(n!)}{n}$ .

If  $S$  is semi-regular, we have already seen that the partition of  $S$  given by the elements of  $S_{\gamma_1}$  is such that each element has equal capacity and the pairwise distance between them is equal to the diameter of  $S$ . Now we notice that if  $S$  is regular and tame, then so is  $B(x_i, \gamma_1)$  for each  $i$ . This gives us,

$$\log_q(\omega(B(x_i, \gamma_1))) = \lim_{n \rightarrow \infty} \left[c_1 + \frac{1}{n} \cdot \sum_{k=1}^{\infty} (c_{k+1} - c_k) \lfloor \frac{n}{q^k} \rfloor\right]$$

Putting these together, we can solve for the scaling factor. If

$$\omega(S) = r \cdot \omega(B)$$

then

$$\begin{aligned}
\log_q(r) &= \lim_{n \rightarrow \infty} \left[c_0 + \frac{1}{n} \sum_{k=1}^{\infty} (c_k - c_{k-1}) \lfloor \frac{n}{q^k} \rfloor - c_1 - \frac{1}{n} \cdot \sum_{k=1}^{\infty} (c_{k+1} - c_k) \lfloor \frac{n}{q^k} \rfloor\right] \\
&= \lim_{n \rightarrow \infty} \left[c_0 - c_1 + \frac{1}{n} \sum_{k=1}^{\infty} ((c_k - c_{k-1}) - (c_{k+1} - c_k)) \lfloor \frac{n}{q^k} \rfloor\right]
\end{aligned}$$

When do we know the value of this limit? One case is obvious, namely the case where  $(c_{k+1} - c_k) = (c_k - c_{k-1})$ , which is guaranteed if the distances between each  $c_k$  and  $c_{k+1}$  is constant. In this case, we see right away that the scaling factor  $r$  is equal to  $q^{c_0 - c_1}$ . In particular, this gives an alternate proof for the fact that  $p \cdot \omega(p\mathbb{Z}) = \omega(\mathbb{Z})$  and one which does not rely (directly) on any algebraic structure.

It is now clear that if we want to get the most leverage out of regularity, we need more assumptions on our space than we did for semi-regularity. We have seen something like this before. If  $S$  is a group with translation-invariant metric, we can use translation invariance right away. It implies that the cosets of  $S$  modulo balls centred at 0 all have the same capacity, which allows us to simplify the right-hand side of the subadditivity formula. If  $S$  has a multiplicative norm though, there is one situation in which this property is distinctly more useful. That is, we get the most use out of a multiplicative norm when the subgroups corresponding to the balls centered at 0 are cyclic.

When  $S$  is an ultrametric space with algebraic structure, translation invariance and scaling under a norm can be very effective tools for computing capacity. The results of this chapter give us a sense in which we can generalize this toolkit. Indeed, semi-regularity and regularity respectively provide the analogous notions. Semi-regularity implies the presence of a sort of “well-balanced” partition of  $S$  that we can use in the subadditivity formula. Likewise, regularity shows us that we can recover a notion of scaling, although as with a multiplicative norm, to get the most out of this, the conditions have to be right.

## Chapter 5

### Application: Product spaces of $\mathbb{Z}$

We consider now an application of the last two chapters. A natural space to consider is the product space of ultrametric spaces, for example  $\mathbb{Z}^n$ , for some  $1 < n < \infty$ . A natural candidate for an ultrametric on a finite product space is given by

$$\rho_\infty(x, y) = \rho_\infty((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \max_i \{\rho(x_i, y_i)\}$$

where  $\rho$  is the metric from the base space. We also see that no problems arise in letting both  $M$  and  $\rho$  vary between components of the space, as long as each  $\rho_i$  is an ultrametric.

**Proposition 28.** *Let  $(M_i, \rho_i)$ , for  $i$  in some finite index set  $I$ , be a collection of metric spaces and suppose  $\rho_i$  is an ultrametric for each  $i$ . Then  $(M, \rho_\infty)$  is an ultrametric space, where  $M = M_1 \times M_2 \times M_3 \times \dots \times M_n$  and  $\rho_\infty$  is the metric described above.*

*Proof.* Let  $(M, \rho_\infty)$  be the product of ultrametric spaces as above and let  $x$  and  $y$  be two points in the space. Clearly,  $\rho_\infty(x, y) \geq 0$  since each  $\rho_i(x_i, y_i) \geq 0$ , and  $\rho_\infty(x, y) = 0 \iff \rho_i(x_i, y_i) = 0, \forall i \iff x_i = y_i, \forall i \iff x = y$ . The fact that  $\rho_\infty$  is symmetric is also an easy consequence of the fact that each  $\rho_i$  is symmetric since  $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$  implies  $\max_i \{\rho_i(x_i, y_i)\} = \max_i \{\rho_i(y_i, x_i)\}$ . To see that  $\rho_\infty$  is an ultrametric, note that if  $z = \{z_i\}$  is any other point of  $M$ , then

$$\begin{aligned} & \rho_\infty(x, y) \\ &= \max_i \{\rho_i(x_i, y_i)\} \\ &\leq \max_i \{\max(\rho_i(x_i, z_i), \rho_i(y_i, z_i))\} \quad \text{since each } \rho_i \text{ is an ultrametric} \\ &\leq \max(\max_i \{\rho_i(x_i, z_i)\}, \max_i \{\rho_i(y_i, z_i)\}) \\ &= \max(\rho_\infty(x, z), \rho_\infty(y, z)) \end{aligned}$$

□

If we are going to compute the capacity of product spaces, we must check that compactness is preserved when taking products. By Tychonoff's theorem, it is enough to show that the product metric,  $\rho_\infty$ , gives the product topology on the space. To do this, we adapt the proofs in Munkres ([M]), where they are given for the analogous case of finite products of  $\mathbb{R}$ .

**Definition-Proposition 1.** ([M]) Suppose  $X_i$ , for  $i$  in some index set  $I$ , is a family of topological spaces. Let  $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$  be the map given by projection onto the  $j$ -th component, that is,  $\pi_j(x) = \pi_j((x_i)_{i \in I}) = x_j$ . For each  $j \in I$ , let  $\mathcal{S}_j$  be the collection

$$\mathcal{S}_j = \{\pi_j^{-1}(U_j) \mid U_j \text{ open in } X_j\}$$

Let  $\mathcal{S}$  be the union of the  $\mathcal{S}_j$  over  $j \in I$ ,  $\mathcal{S} = \cup_{j \in I} \mathcal{S}_j$ . Then  $\mathcal{S}$  is a subbasis that generates a topology on  $\prod_{i \in I} X_i$  called the **product topology**.

The basis,  $\mathcal{B}$ , generated by  $\mathcal{S}$  in the definition above is the set of all finite intersections of elements in  $\mathcal{S}$ . That is,  $B \in \mathcal{B}$ , if there exist  $S_1, S_2, \dots, S_n$  in  $\mathcal{S}$  such that  $B = S_1 \cap S_2 \cap \dots \cap S_n$ . A useful description of the basis for the product topology also appears in Munkres, as below:

**Proposition 29.** ([M], Theorem 19.2) Suppose  $X_i$ , for  $i$  in some index set  $I$ , is a family of topological spaces and denote by  $\mathcal{B}_i$  the basis for the topology on  $X_i$ . Let

$$\mathcal{B}_P = \prod_{i \in I} B_i, \text{ for } B_i \in \mathcal{B}_i \text{ and } B_i = X_i \text{ for all but finitely-many } i \in I.$$

then  $\mathcal{B}_P$  is a basis for the product topology on  $\prod_{i \in I} X_i$ .

We can now show that the topology induced by the  $\rho_\infty$  metric described above agrees with the product topology for finite products.

**Proposition 30.** *Let  $M = (M_1 \times M_2 \times \dots \times M_n, \rho_\infty)$  be a finite product of ultrametric spaces and let  $\rho_\infty$  be the metric described above. Then the topology induced by  $\rho_\infty$  coincides with the product topology on  $M_1 \times M_2 \times \dots \times M_n$ .*

*Proof.* (adpated from [M]) Let  $\mathcal{T}_{\rho_\infty}$  be the topology on  $M_1 \times M_2 \times \dots \times M_n$  induced by  $\rho_\infty$  and let  $\mathcal{B}_{\rho_\infty}$  be the basis for this topology. Let  $\mathcal{T}_P$  be the product topology with basis  $\mathcal{B}_P$ . We show  $\mathcal{T}_P \subset \mathcal{T}_{\rho_\infty}$  and vice versa. For this, it is equivalent ([M], Theorem 13.3) to show that for  $z \in M_1 \times M_2 \times \dots \times M_n$  and  $B \in \mathcal{B}_P$  containing  $z$ , there is a basis element  $B' \in \mathcal{B}_{\rho_\infty}$  such that  $z \in B' \subset B$ , and vice versa.

So let  $z \in M_1 \times M_2 \times \dots \times M_n$  and suppose  $B \in \mathcal{B}_P$  contains  $z$ . Since  $B$  is in  $\mathcal{B}_P$ ,  $B$  is of the form  $B(z_1, r_1) \times B(z_2, r_2) \times \dots \times B(z_n, r_n)$  (since the choice of centres is arbitrary in an ultrametric space, we may choose the components of  $z$  as the centres without loss of generality). Let  $r = \min\{r_i\}$  for  $i \in 1, \dots, n$ . Then let  $B'$  be the ball  $B(z, r)$  in  $\mathcal{B}_{\rho_\infty}$ . Clearly,  $z \in B(z, r)$  and since  $r \leq r_i, \forall i$ ,  $B(z, r) = B(z_1, r) \times B(z_2, r) \times \dots \times B(z_n, r) \subset B(z_1, r_1) \times B(z_2, r_2) \times \dots \times B(z_n, r_n) = B$ .

Conversely, suppose  $A \in \mathcal{B}_{\rho_\infty}$  and let  $y \in A$ . To find  $A' \in \mathcal{B}_P$  such that  $y \in A'$  and  $A' \subset A$ , simply note that  $A$  itself is in  $\mathcal{B}_P$ .

□

We are now ready to explore the capacity in these spaces. We first show that translation invariance carries over into product spaces under the expected conditions.

**Proposition 31.** *Suppose  $(M, \rho_\infty)$  is the product of ultrametric spaces  $(M_i, \rho_i)$  and each  $M_i$  is a topological group with operation  $+_i$ . Let  $+$  denote the operation on  $M$  given by  $s + x = (s_1 +_1 x_1, s_2 +_2 x_2, \dots, s_n +_n x_n)$  for  $s = (s_1, \dots, s_n)$  and  $x = (x_1, \dots, x_n)$  in  $(M, \rho_\infty)$ . Then  $\rho_\infty$  is (left) translation invariant under  $+$  if each  $\rho_i$  is (left) translation invariant under  $+_i$ , in which case valiative capacity is also (left) translation invariant.*

*Proof.* Let  $(M, \rho_\infty)$  be as above. Suppose also that

$$\rho_i(x_i, y_i) = \rho_i(s_i +_i x_i, s_i +_i y_i), \forall s_i, x_i, y_i \in M_i, \forall i.$$

that is, suppose each  $\rho_i$  is (left) translation invariant. Then,

$$\rho_\infty(s + x, s + y) = \max_i \{\rho_i(s_i + x_i, s_i + y_i)\} = \max_i \{\rho_i(x_i, y_i)\} = \rho_\infty(x, y).$$

so that  $\rho_\infty$  is translation invariant. Proposition 20 implies that valiative capacity is as well.  $\square$

In the next proposition, we show that scaling carries over to product space as well, although the conditions are now more restrictive. In contrast to the proposition above, here we cannot allow the spaces to vary between components.

**Proposition 32.** *Let  $(m, \rho_N)$  be an ultrametric space, where  $\rho_N$  is the metric induced by some norm  $N$ . Let  $(M, \rho_\infty)$  be the ultrametric space formed by taking products of  $m$ , along with the  $\rho_\infty$  metric defined above. Then if  $\rho_N$  is multiplicative on  $m$ ,  $\rho_\infty$  is multiplicative on  $M$ , in the sense that  $\rho_\infty(cx, cy) = |c|_{\rho_N} \rho_\infty(x, y)$ , for  $c = (c, c, c, \dots)$ ,  $x, y \in M$ .*

*Proof.* Let  $M, \rho$ , and  $\rho_\infty$  be as above. Then,

$$\begin{aligned} \rho_\infty(cx, cy) &= \max_i \{\rho_N(c_i x_i, c_i y_i)\} \\ &= \max_i \{|c|_{\rho_N} \rho_N(x_i, y_i)\} \\ &= |c|_{\rho_N} \max_i \{\rho_N(x_i, y_i)\} \\ &= |c|_{\rho_N} \rho_\infty(x, y) \end{aligned}$$

$\square$

**Corollary 6.** *Let  $S$  be a subset of  $(M, \rho_\infty)$ , where  $M$  is the product of an ultrametric space  $(m, \rho_N)$ , which is itself a normed vector space with a multiplicative norm inducing  $\rho_N$ . If  $c = (c, c, c, \dots)$  is an element of  $M$  with constant value on each component, then  $\omega(cS) = |c|_{\rho_N} \omega(S)$ .*

*Proof.* The result follows by noting that if  $\{a_j\}_{j=0}^\infty$  is a  $\rho_\infty$  ordering of  $S$ , then  $\{ca_j\}_{j=0}^\infty$  is a  $\rho_\infty$  ordering of  $cS$ .  $\square$



We now introduce two examples, the details of which make up the rest of this chapter.

**Example 12.** Let  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  be the metric space with elements  $\{(x, y) \mid x, y \in \mathbb{Z}\}$  and metric  $\rho_{p,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_p(x_1, y_1), \rho_p(x_2, y_2))$ , where  $\rho_p$  is the p-adic metric for some fixed prime  $p$ . Since  $\rho_p$  is translation invariant and multiplicative, valutive capacity is also translation invariant and multiplicative in  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$ .

**Example 13.** Let  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{P,\infty})$  be the metric space with elements  $\{(x, y) \mid x, y \in \mathbb{Z}\}$  and metric  $\rho_{P,\infty}((x_1, x_2), (y_1, y_2)) = \max(\rho_{p_1}(x_1, y_1), \rho_{p_2}(x_2, y_2))$ , for two distinct primes,  $p_1 \neq p_2$ , where both  $\rho_{p_i}$  are p-adic metrics. Since each  $\rho_{p_i}$  is translation invariant in  $\mathbb{Z}$ , valutive capacity will be translation invariant in  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \rho_{P,\infty})$ ; however, unlike the case of  $p_1 = p_2$ , this space does not have a multiplicative property that allows for scaling.

What is the valutive capacity of  $(\mathbb{Z}_p \times \mathbb{Z}_p, \rho_{p,\infty})$  from Example 12? Suppose  $p = 2$ . Using translation invariance, scaling and subadditivity, we can compute the result by first noting that we can write  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as a union, as below,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = (2\mathbb{Z}_2 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1) \cup (2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2) \cup (2\mathbb{Z}_2 + 1, 2\mathbb{Z}_2 + 1).$$

Since the pairwise distances on the right-hand side are always  $1 = \text{diam}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ , the decomposition formula implies that

$$\begin{aligned} & \frac{1}{\log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} \\ &= \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 \times 2\mathbb{Z}_2 + 1))} + \\ & \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2))} + \frac{1}{\log(\omega(2\mathbb{Z}_2 + 1 \times 2\mathbb{Z}_2 + 1))} \\ &= \frac{4}{\log(|2|_2 \cdot \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} = \frac{4}{\log(\frac{1}{2} \cdot \omega(\mathbb{Z}_2 \times \mathbb{Z}_2))} \end{aligned}$$

Then,

$$4 \cdot \log(\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \log\left(\frac{\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)}{2}\right)$$

so that  $\omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$  is a solution of the equation  $x^4 - \frac{x}{2}$ , for which there is a single real positive root, given by  $2^{-1/3}$ .

To compute the valutive capacity for a 2-fold product for an arbitrary prime  $p$ , note that we can always decompose  $\mathbb{Z}_p \times \mathbb{Z}_p$  into a union of  $p^2$  sets, each of the form  $\{(p\mathbb{Z}_p + s) \times (p\mathbb{Z}_p + t)\}$  for  $s, t \in (0, \dots, p-1)$ , and the pairwise distance between these sets will always be  $1 = \text{diam}(\mathbb{Z}_p \times \mathbb{Z}_p)$ . (To see this, either note that we can always find co-prime elements, or note that each set is a closed ball of radius  $1/p$  centred at  $(s, t)$  and so the distance between them must be greater than  $1/p$ , and 1 is the only possible distance greater than  $1/p$  in  $\mathbb{Z}_p \times \mathbb{Z}_p$ ). Then, we combine our tools as before to obtain the equation,

$$\frac{1}{\log(\omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(|p|_p \cdot \omega(\mathbb{Z}_p \times \mathbb{Z}_p))} = \frac{p^2}{\log(\frac{1}{p} \cdot \omega(\mathbb{Z}_p \times \mathbb{Z}_p))}$$

In turn, we have

$$\omega(\mathbb{Z}_p \times \mathbb{Z}_p)^{p^2} = \frac{\omega(\mathbb{Z}_p \times \mathbb{Z}_p)}{p}$$

So that  $\omega(\mathbb{Z}_p \times \mathbb{Z}_p)$  is a solution of the equation  $x^{p^2} - \frac{x}{p} = x(x^{p^2-1} - \frac{1}{p}) = 0$  over  $\mathbb{R}$ . Since  $\mathbb{R}$  is a field, this means the positive solutions are given by solving  $x^{p^2-1} - \frac{1}{p}$ . Solutions of this equation are of the form  $p^{\frac{-1}{p^2-1}}$  times a  $p^2 - 1$  root of unity, and so there is exactly one positive, real solution, namely  $p^{\frac{-1}{p^2-1}}$  itself. Thus the valutive capacity of the entire product space  $\mathbb{Z}_p \times \mathbb{Z}_p$  is  $p^{\frac{-1}{p^2-1}}$ . In fact, from here it is not hard to see that by taking the  $n$ -fold product, we would end up with the same equation except that the exponent of  $p$  would become  $n$  rather than 2. We arrive at the following result:

**Proposition 33.** *Let  $M = (\mathbb{Z}_p^n, \rho_{p,\infty})$  be the ultrametric space with points equal to the  $n$ -fold product of  $(\mathbb{Z}, \rho_p)$  (for  $n < \infty$ ) for some fixed prime  $p$ . The valutive capacity of  $M$  is  $(\frac{1}{p})^{\frac{1}{p^n-1}}$ .*

*Proof.* Above. □

Taking  $n = 1$ , we see that this agrees with the valuative capacity of  $\mathbb{Z}$  computed in the second chapter.

What about  $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$  for distinct primes? These spaces do not admit a scaling property, so the same toolset is not available. They are, however, semi-regular, so we know that

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

Suppose  $p_1 = 2$  and  $p_2 = 3$ . Recall that the  $\alpha$  sequence of  $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$  counts the number of closed balls of radius  $\gamma_{k+1}$  partitioning a closed ball of radius  $\gamma_k$ . In this case,  $\Gamma_S$  is the non-positive powers of 2 and 3 sorted into decreasing order, so that  $\Gamma_S$  starts  $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}, \dots\}$  and  $\alpha(S)$  starts  $\{6, 2, 3, 2, 2, 3, 2, 3, 2, \dots\}$ . The  $\beta$  sequence of  $S$ , which counts the number of distinct balls of a fixed radius, then starts  $\{6, 12, 36, 72, 144, \dots\}$ .

We know that the capacity of  $S$  will be a product of some negative power of 2 and some negative power of 3. From Lemma 6, we know that when  $\alpha(k) = 2$ , we have

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n + \beta(k)}{2 \cdot \beta(k)} \rfloor$$

and when  $\alpha(k) = 3$ , we have

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n + \beta(k)}{3 \cdot \beta(k)} \rfloor + \lfloor \frac{n + 2 \cdot \beta(k)}{3 \cdot \beta(k)} \rfloor$$

We also know that if  $\alpha(k) = 2$ , then  $\gamma_k$  must be a (negative) power of 2, and likewise if  $\alpha(k) = 3$ , then  $\gamma_k$  is a power of 3.

Let us first explore the exponent of 2 in  $\delta(n)$ . We start by noting that if  $\gamma_k$  is some  $2^{-i}$ , then

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

since there will be a copy of 2 in  $\beta(k)$  for every occurrence of 2 in  $\alpha(0), \dots, \alpha(k)$ , which is also what  $i$  counts. So then, the exponent of  $\frac{1}{2}$  in the  $n^{\text{th}}$  characteristic

sequence of  $S$  is

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^j}{2^{i+1} \cdot 3^j} \rfloor$$

What can we say about  $j$ , the exponent of 3?

**Lemma 8.** *Let  $S = (\mathbb{Z}_2 \times \mathbb{Z}_3)$  and consider the  $k^{th}$  element of the  $\beta$  sequence of  $S$ ,  $\beta(k) = 2^i \cdot 3^j$ . If  $k$  is such that  $\gamma_k = 2^{-i}$  for some  $i$ , then  $j$  counts the numbers  $a \in \mathbb{Z}_{\geq 0}$  such that  $3^a < 2^i$ .*

*Proof.*  $\Gamma_S$  is strictly monotone decreasing and each  $\gamma_k$  is equal to a non-positive power of 2 or 3. If  $\gamma_k = 2^i$ , then all non-positive powers of 3 and 2 which are greater than  $2^i$  must be equal to some  $\gamma_j$ ,  $0 \leq j < k$ . That is,  $2^i$  only appears in the  $\Gamma_S$  sequence after all larger powers of 2 and 3 have been exhausted. Since we are only considering the case  $\gamma_k$  is a power of 2, this includes all of the smaller powers of 3.  $\square$

Now note that

$$3^a < 2^i \iff \log_2(3^a) < \log_2(2^i) \iff a \cdot \log_2(3) < i$$

So now we are reduced to counting the number of non-negative integers  $a$  that satisfy this last inequality for a given  $i$ . The number of such  $a$ 's will simply be the value of the largest  $a$  plus 1 since  $a$  satisfying the relation implies all  $0 \leq a' \leq a$  solve the relation. Then, we are in fact reduced to finding the largest  $a \in \mathbb{Z}$  that satisfies  $a < \frac{i}{\log_2(3)}$ , but this is exactly  $\lfloor \frac{i}{\log_2(3)} \rfloor$ . This in turn gives  $j = \lfloor \frac{i}{\log_2(3)} \rfloor + 1 = \lceil \frac{i}{\log_2(3)} \rceil$ , since  $\frac{i}{\log_2(3)}$  is never an integer. We now revisit our expression for the exponent of  $\frac{1}{2}$  and substitute our new found value for  $j$ :

$$\sum_{i=1}^{\infty} i \cdot \lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \rfloor = \sum_{i=1}^{\infty} i \cdot (\lfloor \frac{n}{2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \rfloor - \lfloor \frac{n}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \rfloor) \quad (5.1)$$

A symmetric argument shows that exponent of  $\frac{1}{3}$  in the  $n^{th}$  element of the  $\rho_{\infty}$ -sequence of  $S$  is

$$\sum_{i=1}^{\infty} i \cdot \left( \left\lfloor \frac{n + 2^{\lceil \frac{i}{\log_3(2)} \rceil} \cdot 3^i}{2^{\lceil \frac{i}{\log_3(2)} \rceil} \cdot 3^{i+1}} \right\rfloor + \left\lfloor \frac{n + 2^{\lceil \frac{i}{\log_3(2)} \rceil + 1} \cdot 3^i}{2^{\lceil \frac{i}{\log_3(2)} \rceil} \cdot 3^{i+1}} \right\rfloor \right) \quad (5.2)$$

The sums that appear in (5.1) and (5.2) are real numbers that we, at present, know little about. However, the aperiodicity of the sequences  $\lceil \frac{i}{\log_2(3)} \rceil$  and  $\lceil \frac{i}{\log_3(2)} \rceil$  over  $i$  leads us to believe, but not prove, that each of the sums are irrational. We have the following conjecture.

**Conjecture 1.** *Finite products of  $(\mathbb{Z}, \rho_{p_i})$  for distinct primes,  $p_i$ , have transcendental valutive capacity.*

What can we say about the *infinite* product of either  $\mathbb{Z}_p$ , for some fixed prime  $p$  or  $\mathbb{Z}_{p_i}$  for each prime? For a fixed prime  $p$ , we can observe that  $(\frac{1}{p})^{\frac{1}{p^n-1}}$  is a monotone, increasing sequence in  $n$  with  $\lim_{n \rightarrow \infty} (\frac{1}{p})^{\frac{1}{p^n-1}} = 1$  and in fact, the sequence  $\{(0, 0, \dots), (1, 0, \dots), (0, 1, \dots), \dots\}$ , in which the first element has only zeros and the  $n$ -th element has a single 1 in the  $(n-1)$ -th component, is a  $\rho$ -ordering. This might lead us to wonder if valutive capacity can obtain the diameter of the space. We are immediately met with a problem however.

Compactness has played no small role in this work. Indeed, we have a definition of valutive capacity only for compact subsets. We are now naturally left to ask whether the product topology on *infinite* products of ultrametric spaces coincides with the  $\rho_\infty$  metric. In this case, as in the analogous case of infinite copies of  $\mathbb{R}$  and a uniform metric, the answer is negative (at least in general). Although, we can find a metric that realizes the product topology on infinite copies of ultrametric spaces (adapting the analogous case from copies of  $\mathbb{R}$ ), there is nothing canonical about this construction. Although this is, on the one hand, a disappointment, on the other hand, it suggests a new avenue to explore: can we define valutive capacity for locally compact spaces, and if so, is capacity computable for any of these spaces?

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## Appendix A

### Maple Code

In this appendix, we include for reference the Maple code that was used to investigate the capacity of product spaces. The result of this investigation also influenced the development of Chapters 3 and 4. There are three procedures listed here.

1. The first procedure, **ComputePadicProductOrdering**, takes as input a list indicating a finite set of primes,  $p_1, \dots, p_n$ , and an integer  $m$  and returns the first  $m$  terms of a  $\rho_\infty$ -ordering of  $(\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n})$ . This is done explicitly by following the algorithm described in Chapter 3.
2. The next procedure, **ComputePartialRhoSeq**, computes the resulting  $\rho_\infty$ -sequence for a product space by multiplying out the distances given by a  $\rho_\infty$ -ordering. It takes as input a metric and a matrix representing a  $\rho_\infty$ -ordering for a product space and returns a number indicating the value of the  $n^{th}$  characteristic sequence, where  $n$  is the row dimension of the input matrix. These two procedures therefore result in the naive computation of the partial characteristic sequence of a product space, found by explicitly calculating a  $\rho_\infty$ -ordering and the  $\rho_\infty$ -sequence in turn.
3. The final procedure, **FastPartialRhoSeq**, exploits the fact that the terms occurring in the characteristic sequence of a product space are all powers of the primes specifying the space. It takes as input the (beginning of) the sequence of decreasing distances and a list of primes, specifying the space. If  $k$  distances are specified in  $A$ , it returns the  $\beta(k)^{th}$  term in the characteristic sequence, where  $\beta$  is the structure sequence.



```

1 with(LinearAlgebra):
2 with(combinat, cartprod):
3 with(padic):
4
5 ComputePadicProductOrdering := proc (m, components)
6 # Given a list of primes, p_1,...,p_n, compute the first m terms of a
   p-infinity ordering of  $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n}$ , where
    $p\text{-infinity}(x,y)=\max(p\text{-j}(x\text{-j},y\text{-j}))$  and  $p\text{-j}$  is the  $p\text{-j}$ -adic metric.
7 # arg m; an integer indicating the number of elements of the ordering to
   return
8 # arg components; a list of prime numbers indicating the components of
   the product space
9 # return; a matrix where each row is an element in the product space and
   the i-th row is the i-th element in an ordering
10
11 local numberOfComponents, co_primes, i, n, T, M, v, distances, j,
   M1, newBlock;
12
13 #will end up with one column per component in the product space
14 numberOfComponents := nops(components);
15
16 #the ordering will start with the cartesian product of coprime elts
   from each component
17 #everything up to p-1 is coprime
18 co_primes := [[seq(i, i = 0 .. (components[1]-1))]];
19 for n from 2 to numberOfComponents do
20     co_primes := [op(co_primes), [seq(i, i = 0 ..
       (components[n]-1))]];
21 od;
22
23 #then take the cartesian product to get the first <product of
   elements in components> elements in the ordering
24 T := cartprod(co_primes);
25 M := Matrix([T['nextvalue']()]);

```

```

26  while not T['finished'] do
27      M := <M; T['nextvalue'](>);
28  end do;
29
30  #make a list to keep track of the exponent of each prime; start by
    take each prime to the power -1
31  v := Vector[row](1 .. numberOfComponents, 1);
32  v := convert(v, list);
33
34  #keep adding rows until you have enough points in the ordering
35  while RowDimension(M) < m do
36      #take each prime to the power of minus the elements in v
37      distances := zip(proc (x, y) options operator, arrow; x^(-y) end
        proc, components, v);
38      #check each column to see if the max distance was achieved
39      for j from 1 to numberOfComponents do
40          #if it was then split this column
41          if distances[j] = max(distances) then
42              #take a snapshot of M before you start - this is what you
                have to add to
43              M1 := copy(M, deep);
44              #create p-1 new blocks
45              for i from 1 to (components[j]-1) do
46                  newBlock := copy(M1, deep);
47                  newBlock(1..RowDimension(newBlock), j) :=
                    Column(newBlock, [j]) +~ (i*components[j]^v[j]);
48                  #add the new block to the master matrix
49                  M := Matrix([M], [newBlock]);
50              od;
51              #update the vector of exponents
52              v[j] := v[j] + 1;
53          end if;
54      od;
55  end do;

```

```
56   return M[1..m,];  
57 end proc;
```

```

1 with(LinearAlgebra):
2 with(combinat, cartprod):
3 with(padic):
4
5 ComputePartialRhoSeq := proc (S, rho)
6 # Given an m by n matrix S whose columns represent points of an
   n-component product space and that has as its i-th row the i-th term
   in a rho-ordering of that space, compute the (m-1)-th partial sum of
   the rho-sequence
7 # note that S and rho must be compatible and no checking is done to
   ensure this
8 # arg S; an n by m matrix representing a rho-ordering, for example as
   created by ComputePadicProductOrdering
9 # arg rho; a compatible metric on the points (rows) in S
10 # return; a real number, corresponding to the (m-1)-th term of the
    partial rho-sequence
11
12 local lastTerm, f, distances, nthTerm;
13
14 #find the last element in the ordering
15 lastTerm := S[RowDimension(S),];
16
17 #make a function that calculates the distance from the i-th row of S
   to the last term in the ordering
18 f := proc (i) options operator, arrow; rho(op(convert(lastTerm,
   list)), op(convert(S[i,], list))) end proc;
19
20 #run over each row to get the set of all m-1 distances
21 distances := map(f, [seq(i, i = 1 .. (RowDimension(S)-1))]);
22
23 #multiply them to get the (m-1)-th term of the rho-ordering
24 partialSum := mul(distances);
25
26 return partialSum

```

27 `end proc;`

---

```

1 FastPartialRhoSeq := proc (A, p:=[2,3])
2 # Given a set of distances A for a product spaces specified in p,
   compute the exponent of each prime in the partial characteristic
   sequence
3 # arg A; a vector indicating the sequence of decreasing distances in
   Z_p_i
4 # arg p; a list of prime numbers indicating the components of the
   product space
5 # return; the beta(k)^th term in the characteristic sequence, where beta
   is the structure sequence and k is the length of A
6
7 local g, h, computePowers, n, shortA, primeExponents, i, thisPrime,
   thisPrimeIndex, B, G, powers, powersOfG, thisPrimeSum;
8
9 === Some helper functions ===
10 #Return the index of every instance of p-multiples in a list
11 #Use to find the index of a given prime in A
12 h := proc(i, L, p) if L[i] mod p = 0 then return i else return NULL fi;
   end proc;
13
14 #Count the number of times the mth element has appeared as a factor
   for the 1..m first elements in a list
15 #Use to compute the (decreasing) sequence of distances in A or a
   subset of A
16 g := proc(m, L)
17   local basePrime;
18
19   basePrime := L[m];
20
21   #since we just want the exponent not actual distance just compute
   what power this would be
22   return ordp(mul(L[1..m]), basePrime);
23
24 end proc;

```

```

25
26 #Compute the appropriate power of an element of G
27 computePowers := proc(m, L)
28     local power;
29
30     if m=nops(L) then
31         power:= L[-1]-1;
32     else
33         power:= mul(L[(m+1)..nops(L)]) * (L[m]-1);
34     end if;
35     return power
36 end proc;
37
38 #compute n, then create a copy of A with the first element deleted to
    ease the indexing
39 n := mul(A);
40 shortA := A[2..nops(A)];
41
42 #compute the terms corresponding to each prime given
43 primeExponents := Vector();
44 for i from 1 to nops(p) do
45     #pull out the prime
46     thisPrime := p[i];
47
48     #first get the index in A of this prime
49     thisPrimeIndex := map(h, [seq(i, i=1..nops(shortA))], shortA,
        thisPrime);
50     B:= shortA[thisPrimeIndex];
51
52     #then find the (exponents for the) distances occuring with this prime
53     G:= map(g, [seq(i, i=1..nops(B))], B);
54
55     #Figure out what power each distance should be raised to
56     powers := map(computePowers, thisPrimeIndex, shortA);

```

```

57
58   #raise each element in G to the given powers
59   powersOfG:= zip(proc (x, y) options operator, arrow; x*(y) end proc,
60                 G, powers);
61
62   #the exponent of this prime in the nth partial will be  $-([$ the sum of
63     the elements in powers $]/n)$ , where n is the product of elements in
64     A (including the first element)
65   thisPrimeSum := add(powersOfG);
66   primeExponents(i) := thisPrimeSum/n;
67
68
69 end proc;

```