Valuative Capacity of of some compact subsets of $\mathbb{Z}_p$

Anne Johnson

January 31, 2019

Background: p-orderings, p-sequences

A $p-$**ordering** of an infinite set, $S \subseteq \mathbb{Z}_p$, is a sequence in S such that for $\forall n > 0$, a_n minimizes

$$v_p((x - a_{n-1}) \dots (x - a_0))$$

Background: p-orderings, p-sequences

A $p-$**ordering** of an infinite set, $S \subseteq \mathbb{Z}_p$, is a sequence in S such that for $\forall n > 0$, a_n minimizes

$$v_p((x - a_{n-1}) \dots (x - a_0))$$

cf: A ρ-**ordering** of S, a (compact) subset of an ultrametric space (M, ρ), is a sequence in S such that $\forall n > 0$, a_n maximizes

$$\prod_{i=0}^{n-1} \rho(x, a_i)$$

Background: p-orderings, p-sequences

The p-**sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is

$$v_p((a_n - a_{n-1}) \ldots (a_n - a_0))$$

Background: p-orderings, p-sequences

The p-**sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is

$$v_p((a_n - a_{n-1}) \ldots (a_n - a_0))$$

cf: The ρ-**sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is

$$\prod_{i=0}^{n-1} \rho(a_n, a_i)$$

Background: valuative and logarithm capacity

The **valuative capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$L_p(S) := \lim_{n \to \infty} \frac{w_S(n,p)}{n}$$

where $w_S(n,p)$ is the $p-$sequence of S.

nb: this is the Robin's constant and can be found via the equilibrium measure:

$$L_p(S) = \inf_{\mu \in \mathcal{P}(\bar{S})} \int\int v_p(x-y) d\mu(x) d\mu(y)$$

Background: valuative and logarithm capacity

The **logarithm capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$V_p(E) := p^{-L_p(E)}$$

nb: this is equal to the transfinite diameter and the Chebychev constant.

Background: valuative and logarithm capacity

The **logarithm capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$V_p(E) := p^{-L_p(E)}$$

nb: this is equal to the transfinite diameter and the Chebychev constant.

Fare and Petite, Lemma 5.1

Let $A = \{0, 1, .., d-1\}$ be a finite alphabet and $A^{\mathbb{N}}$ be the collection of infinite sequenes with values in A.

Let $p \geq d$ be a prime number and let ϕ be the canonical embedding of $A^{\mathbb{N}}$ into $\mathbb{Z}_p$ via the following continuous map:

$$\phi : A^{\mathbb{N}} \to \mathbb{Z}_p \text{ by } (x_n)_{n\geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k$$

Fare and Petite, Lemma 5.1

Lemma

Let $w_1, w_2, \dots, w_s$ be $s \geq 2$ words with the same length l such that all the first letters are distinct. Let $X \subset A^{\mathbb{N}}$ be the set of sequences such that any factor is a factor of a concatenation of the words $w_1, w_2, \dots, w_s$. Then the set $E := \phi(X) \subset \mathbb{Z}_p$ satisfies:

$$E = \cup_{i=1}^{s} x_i + p^l E, \text{ with } x_i = \phi(w_i 0^\infty)$$

It is a regular compact set and its valuative capacity is

$$L_p(E) = \frac{l}{s-1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.

Fares and Petite, Lemma 5.1

An example:

$w_1 = 0, w_2 = 2, A = \{0, 1, 2\}, p = d = 3$
Then $\{x_n\}_{n \geq 0} \in X$ if each term in $\{x_n\}_{n \geq 0}$ is either 0 or 2. We have

$$E = 0 + 3E \cup 2 + 3E \text{ and}$$

$$L_p(E) = \frac{1}{2-1} = 1$$

Digression: projective k-space

Let k be a field that is complete with respect to a non-archimedean valuation.

Definition

The **projective line over** k, denoted $\mathbb{P}^1(k)$, is the space whose points are lines l in k^2 that intersect $(0, 0)$.

Proposition

Let $\psi : k \to \mathbb{P}^1(k)$ be the map given by $\psi(\lambda_0) = [1, \lambda_0]$, where $[1, \lambda_0]$ is the line in k^2, $\{\lambda(1, \lambda_0); \lambda \in k^*\}$. Then the image of ψ is $\mathbb{P}^1(k) \setminus [0, 1]$ and is isomorphic to k, so that k is identified with projective space minus a distinguished point, $[0, 1]$, which is denoted by ∞.

Digression: projective k-space

Definition

We denote by $GL(2,k)$ the set of invertible 2×2 matrices over k. A **fractional linear automorphism**, ϕ, of $\mathbb{P}^1(k)$ is a map defined by $z \mapsto \frac{az+b}{cz+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,k)$. The set of fractional linear automorphisms of $\mathbb{P}^1(k)$ is denoted $PGL(2,k)$.

Note that $PGL(2,k) = GL(2,k)/\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right); \lambda \in k^*\}$. In homogeneous coordinates, we can represent the action of ϕ by $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$.

Digression: projective k-space

Definition

Suppose Γ is a subgroup of $PGL(2,k)$. A point $p \in \mathbb{P}^1(k)$ is a **limit point of** Γ, if there exists a point q in $\mathbb{P}^1(k)$ and a sequence $\{\gamma_n\}_{n\geq 1}$ in Γ such that $\lim_{n\to\infty} \gamma_n(q) = p$.

Fares and Petite, Lemma 5.1, repharsed (1/2)

Let $x_1, x_2, \ldots, x_s$ be $s \geq 2$ points in $\mathbb{Z}_p$ such that $\mid x_i - x_j \mid_p = 1$, $\forall i, j \in 1, ..., s$. Suppose also that there exists an $l \in \mathbb{N}$ such that $\forall i$,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{l} a_i p^i$$

Fares and Petite, Lemma 5.1, repharsed (2/2)

Let γ_i be the fractional linear automorphism of $\mathbb{P}^1(\mathbb{Q}_p)$ given by $\left(\begin{smallmatrix} p^l & x_i \\ 0 & 1 \end{smallmatrix}\right)$ and let Γ be the subgroup of $PGL(2, \mathbb{Q}_p)$ generated by the γ_i.
If $\mathcal{L}$ is the limit set of Γ, and Z is the subset of $\mathbb{Q}_p$ such that $Z = \psi^{-1}(\mathcal{L})$, (where $\psi : \mathbb{Q}_p \to \mathbb{P}^1(\mathbb{Q}_p)$ is the map given by $\psi(\lambda_0) = [1, \lambda_0]$) then Z is a regular, compact subset of $\mathbb{Z}_p$ satisfying

$$Z = \cup_{i=1}^{s} x_i + p^l Z = \cup_{i=1}^{s} B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s-1}$$

Fares and Petite, Lemma 5.1, repharsed

Sketch of proof:

- We have to show w that the set Z above is equal to $E = \phi(X)$ in the original lemma.
- That that w_i correspond to the x_i is not hard to see.

Fares and Petite, Lemma 5.1, repharsed

What is the limit set of Γ?

- An element of Γ is of the form $\begin{pmatrix} p^{lm} & z \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and z is an element of $\mathbb{Z}_p$ whose coefficient vector is a concatenation of the coefficient vectors of the x_i (for $0 \leq i \leq ml$ and 0 for $i > ml$)
- Let $a = [a_0, a_1] \in \mathbb{P}^1(\mathbb{Q}_p)$ and let $\{\gamma_n\}$ be a sequence in Γ. We have that

$$\lim_{n\to\infty} \gamma_n(a) = \lim_{n\to\infty} [a_0, p^{nl} a_1 + z_n] = [a_0, z],$$

  where the coefficient vector of each z_n is a concatenation of the coefficient vectors of the x_i, for finitely-many terms (and then 0s), and z is an element of $\mathbb{Z}_p$ whose entire coefficient vector is a concatenation of the coefficient vectors of the x_i.

references

Youssef Fares and Samuel Petite, The valuative capacity of subshifts of finite type.

Keith Johnson, P-orderings and Fekete sets