Computing a ρ-ordering

In the previous section, we defined valuative capacity for a compact subset S of an ultrametric space (M, ρ). We also got a glimpse into the way the valuative capacity of S interacts with its other properties, such as the set of distances occurring in S and the lattice of closed balls in S (or equivalently, if S has enough structure, the lattice of subgroups).

In this section, we offer a more detailed study of the interaction between the valuative capacity of S and the lattice of closed balls in S. In particular, we show how, in all cases (with S compact), the latter can be used to compute the first n terms of a $\rho-$ordering of S (for any $n < \infty$) and how, in some cases, this extends to being able to compute the valuative capacity of S.

We begin by letting S be, as before, a compact subset of an ultrametric space (M, ρ), and by letting $\Gamma_S = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in S. Now fix some $k \in \mathbb{N}$, and consider for a moment the set of closed balls of radius γ_k in S. We could denote these alternatively by $B^M(x, \gamma_k) \cap S$ or by $B^S(x, \gamma_k)$, but when there is no risk of confusion, we will denote them simply by $B(x, \gamma_k)$. Clearly, the set $\{B(x, \gamma_k); x \in S\}$ forms a cover of S and since S is compact, we must have some $x_1, \ldots, x_n$ such that $S = \cup_{i=1}^n B(x_i, \gamma_k)$. In fact, since ρ is an ultrametric, we can pick the x_i's so that $\cup_{i=1}^n B(x_i, \gamma_k)$ will be a disjoint union and therefore a partition of S. Note that both n and the x_i's depend on our fixed k, but that n is independent of the x_i's, since any choice of centres is equivalent. We rephrase this with following definition and lemma:

Definition. For S and Γ_S as above, and $k \in \mathbb{N}$, fixed, define $\sim_k$ to be the relation on S given by

$$x \sim_k y \text{ if and only if } \rho(x, y) \leq \gamma_k$$

i.e., $x \sim_k y$ if and only if $B_{\gamma_k}(x) = B_{\gamma_k}(y)$.

The fact $\sim_k$ is an equivalence relation on S is equivalent to the observation that every point in a ultrametric ball is at its centre:

Lemma. Let S and Γ_S be as above, then $\sim_k$ is an equivalence relation on S.

Proof. $\sim_k$ is clearly reflexive and symmetric, since ρ is a metric. Transitivity results from the ultrametric property of ρ: if $x \sim_k y$ and $y \sim_k z$, then

$$\rho(x, z) \leq \max(\rho(x, y), \rho(z, y)) \leq \gamma_k$$

so $x \sim_k z$. □

We denote the set of equivalence classes of $S/\sim_k$ by S_{γ_k}. We have defined S_{γ_k} to be the set of equivalence classes in S under the relation $\sim_k$, which is equivalent to letting S_{γ_k} be the set of closed balls of fixed radius γ_k in S. We now offer a third perspective on the elements on S_{γ_k}, which is due to [**?**],

Lemma. For each k, the elements of S_{γ_k}, that is, the closed balls of radius γ_k, themselves form an ultrametric space, where the metric is given by:

$$\rho_k(B(x, \gamma_k), B(y, \gamma_k)) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) > \gamma_k \\ 0, & \text{if } \rho(x, y) \leq \gamma_k, \text{ i.e., } B(x, \gamma_k) = B(y, \gamma_k) \end{cases}$$

Proof. ρ_k is reflexive, symmetric and transitive since ρ is. Likewise, ρ_k satisfies the ultrametric property, since ρ does: let $B(x, \gamma_k), B(y, \gamma_k)$ and $B(z, \gamma_k)$ be any three elements of S_{γ_k} and suppose $\rho_k(B(x, \gamma_k), B(y, \gamma_k)) > 0$. Then,

$$\begin{aligned} \gamma_k &< \rho_k(B(x, \gamma_k), B(y, \gamma_k)) \\ &= \rho(x, y) \leq \max(\rho(x, z), \rho(y, z)) \\ &= \max(\rho_k(B(x, \gamma_k), B(z, \gamma_k)), \rho_k(B(y, \gamma_k), B(z, \gamma_k))) \end{aligned}$$

since $\gamma_k < \max(\rho(x,z), \rho(y,z))$ implies that at least one of $\rho_k(B(x,\gamma_k), B(z,\gamma_k))$ or $\rho_k(B(y,\gamma_k), B(z,\gamma_k))$ is greater than 0. $\square$

So now the elements of S_{γ_k} may be viewed as either equivalence classes, closed balls of fixed radius, or points in a new metric space. We make a final definition and introduce some notation before moving on.

Definition. Let S and Γ_S be as above. Define $\beta(i)_{i=0}^{\infty}$ to be the sequence given by $\beta(i) = |S_{\gamma_i}|$, which is an invariant of S and which counts the number of connected components of S_{γ_i} when viewed as a metric space. When necessary, we use $\beta^S(i)$ to denote the β sequence for a given, compact ultrametric space S. Adapting the terminology in [**?**], we call $\beta^S(i)$ the **structure sequence** of S.

Notation. Let S_{γ_k} be as above. We denote the elements of S_{γ_k} by $B_1^k, \ldots, B_{\beta(k)}^k$ or by $B_1^{S,k}, \ldots, B_{\beta(k)}^{S,k}$, when necessary.

We return to the sequence $\beta(i)$ at the end of this section. For now, we show how a $\rho-$ordering of S can be built recursively from the spaces S_{γ_k}. This begins by noting that the spaces themselves can be built recursively:

Observation. Let S, Γ_S, and S_{γ_k} be as above. Then S_{γ_k+1} can be constructed by partitioning each of the closed balls in S_{γ_k} into closed balls of radius γ_{k+1} and taking their union: Let $B(x_i, \gamma_k)$ be an element of S_{γ_k}, denoted by B_i^k. Then, there exists $x_{i,1}, \ldots, x_{i,l_i} \in B_i^k$ such that,

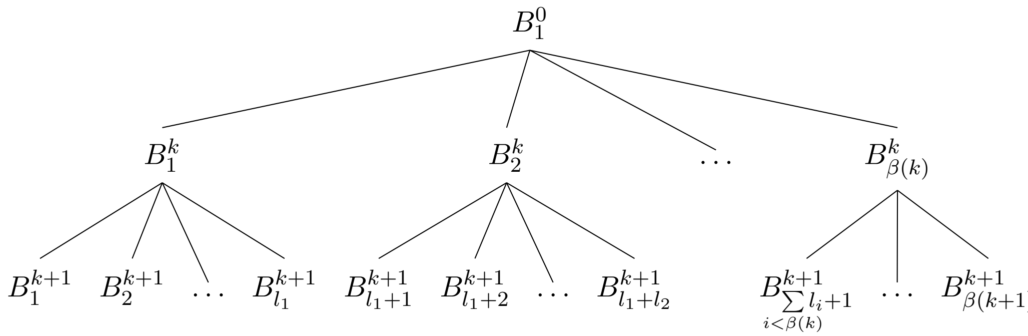
$$B_i^k = \cup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1})$$

and so

$$S_{\gamma_{k+1}} = \cup_{i=1}^{\beta(k)} \cup_{j=1}^{l_i} B(x_{i,j}, \gamma_{k+1}) = \cup_{j=1}^{\beta(k+1)} B_j^{k+1}$$

where $\cup_{j=1}^{l_i} B(x_{i,j}, \gamma_k) = B(x_i, \gamma_{k+1}) = B_i^k$, $\forall i$.

We can represent this schematically as below:



Notation. If S is a compact subset of an ultrametric space, the lattice of closed balls in S, as depicted above, is denoted by T_s.

We make a few observation on the relationships between the vertices in T_s. First we consider the distance between two vertices in T_s. Since each vertex represents a ball in an ultrametric space, it is well-defined to let the distance between vertices be equal to the distance between a choice of centres for those balls. This is equal to the diameter of the smallest closed ball that contains any two vertices, that is, the level of their join. In particular, for any k and any $i < \beta(k)$, the distances between the children of B_i^k will be γ_k and for any $i \neq j$ the distance between the children of B_i^k and B_j^k will be equal to the distance between B_i^k and B_j^k (which will be some $\gamma_m, m < k$).

Finally, note that without loss of genearlity, for any $k \in \mathbb{N}$, we can reindex the B_i^k's so that they give the first $\beta(k)$ terms of a ρ_k-ordering of S_{γ_k}, when the latter is viewed as a (finite) metric space. If the B_i^k's are so indexed, then finding a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ is straightforward: select a B_j^{k+1} from each of the B_i^k's in order and then start over. The following two propositions show how this can be used to recursively build a $\rho-$ordering of S.

Proposition. Let be S a compact subset of an ultrametric space (M, ρ) and Γ_S, the set of distances in S. If S_{γ_k} is a partition of S as described above for $\gamma_k \in \Gamma_S$ with

$k < \infty$, where the elements are indexed according to a ρ_k-ordering of S_{γ_k}, then the first $\beta(k+1)$ terms in a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ can be found by selecting at each stage n, a child from $B^k_{\overline{n}}$, where $\overline{n} = n \mod \beta(k) + r$ and r is minimal in $\{0, \ldots, \beta(k)-1\}$ such that $B^k_{n \mod \beta(k)+r}$ still has unused children.

Proof. Let S, S_{γ_K}, and $S_{\gamma_{k+1}}$ be as above. In particular, suppose the elements of S_{γ_k} are indexed according to a ρ_k-ordering. Denote the elements of $S_{\gamma_{k+1}}$ by $B^{k+1}_{i,j}$ where the first subscript indicates that the elements is a child of B^k_i. To form a ρ_{k+1} ordering of $S_{\gamma_{k+1}}$, we must maximize the product of distances at each step n.

Now note that $\Gamma_{S_{\gamma_k}} = \{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$ and $\Gamma_{S_{\gamma_{k+1}}} = \{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}, \gamma_k\}$. That is, the distances in $S_{\gamma_{k+1}}$ are the same as the distances in S_{γ_k}, although they also include the smaller distance γ_k. Since we know that the elements $B^k_1, \ldots, B^k_{\beta(k)}$ already maximizes the product of distances in $\{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$, the first $\beta(k)$ terms of a ρ_{k+1}-ordering of S_{k+1} can be found by taking $B^k_{1,j_1}, \ldots, B^k_{1,j_{\beta(k)}}$ for any choice of j's. At this point, any choice of next element will produce a copy of γ_k in the ρ_{k+1}-sequence; however, if we chose another child of B^k_1, we are able to keep building the ordering in a canonical fashion, since we know that we will then be able to maximize the product at the next step by chosing another child of B^k_2.

We see then that a ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ is found by minimizing the number of times γ_k is introduced into the ρ_{k+1}-sequence and maximizing the product among the $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$, and the latter is already known to be achieved by taking the B^k_i in order. If the B^k_i's all have the same number of children, then we can always select a child of $B^k_{\overline{n}}$, where $\overline{n} = n \mod \beta(K)$ at each stage n, $n < \beta(k+1)$, since there will always be one available. On the other hand, suppose the B^k_i have an unequal number of children and n is the first step at which all the children of $B^k_{\overline{n}}$ have been exhausted. What element will maximize the $\rho_{k+1}-$sequence?

Consider the space $(S_{\gamma_k} \setminus B^k_{\overline{n}})$. Removal of $B^k_{\overline{n}}$ will not effect the first m terms of a ρ_k-ordering of this space, for $m < \overline{n}$, since if a sequence of elements maximizes a function over a set X, they will also maximize that function of a subset of X (provided they themselves remain in the subset). Then the ρ_k-sequence of $(S_{\gamma_k} \setminus B^k_{\overline{n}})$ begins $\{B^k_1, \ldots, B^k_{\overline{n}-1}\}$.

Moreover, if $B^k_{\overline{n}+1}$ maximizes $\prod_{i=1}^{\overline{n}} \rho_k(x, B^k_i)$ over S_{γ_k}, then it also maximizes $\prod_{i=1}^{\overline{n}-1} \rho_k(x, B^k_i)$ over $(S_{\gamma_k} \setminus B^k_{\overline{n}})$, since $\prod_{i=1}^{\overline{n}} \rho_k(x, B^k_i) = (\prod_{i=1}^{\overline{n}-1} \rho_k(x, B^k_i)) \cdot \rho_k(x, B^k_{\overline{n}})$.

Then the ρ_k-sequence of $(S_{\gamma_k} \setminus B^k_{\overline{n}})$ is simply $\{B^k_1, \ldots, B^k_{\overline{n}-1}, B^k_{\overline{n}+1}, \ldots, B^k_{\beta(k)}\}$.

Now we see that $\rho_{k+1}-$sequence of $S_{\gamma_{k+1}}$ is maximized by simply skipping over $B^k_{\overline{n}}$, should all its children be exhausted, and selecting a child from $B^k_{\overline{n}+1}$. Then a $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$ is found by selecting elements of each B^k_i in order as much as possible, and skipping to B^k_{i+1}, when it is not possible.

$\square$

The above results quickly leds to a recursive contruction for a $\rho-$ordering of S.

Proposition. Let be S a compact subset of an ultrametric space (M, ρ) and let Γ_S be the set of distances in S. Let S_{γ_k} is a partition of S as described above for $\gamma_k \in \Gamma_S$ with $k < \infty$, where the elements are indexed according to a ρ_k-ordering of S_{γ_k}. Let $x_{i,j}$ denote a choice of centre for the element $B^{k+1}_{i,j}$. Then the first $\beta(k+1)$ elements of a $\rho-$ordering of S can be found by forming a matrix, A_k, whose $(i,j)^{th}$-entry is $x_{i,j}$, as shown below, and then concatenating the rows (where the columns are padded by * if necessary).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \dots & x_{n,l_n} \end{pmatrix}$$

Proof. Note that the entries in each column are points in the ball $B_{\gamma_k}(x_i)$ so that the pairwise distance between columns is constant and always exceeds the distance between elements within a column. Moreover, the columns are organized such that for any j, $x_{n,j}$ maximizes$\prod_{i=1}^{n-1} \rho_(x_{n,j}, x_{i,j})$ since $\prod_{i=1}^{n-1} \rho_(x_{n,j}, x_{i,j}) = \prod_{i=1}^{n-1} \rho_(x_{n,1}, x_{i,1}) = \prod_{i=1}^{n-1} \rho_(x_n, x_i)$ and the x_i's are indexed according a ρ_k-ordering of S_{γ_k}.
Then a $\rho_{\gamma_{k+1}}$-ordering of $S_{\gamma_{k+1}}$ is obtained by minimizing the number of elements from any one column and by taking the points $x_{i,j}$ (for fixed j) in sequence. For example, by concatenating the rows. □

Note that in building the $\rho_{k+1}-$ordering of $S_{\gamma_{k+1}}$, we imposed an ordering on the elements of S_{γ_k}, but not on the individual children of a given B_i^k. Any choice of indexing on these children is equivalent since the distances between any two of them is γ_k and the distance between any one of them and a child of some B_j^k, $i \neq j$, is the same, since each B_i^k is a ball in an ultrametric space. For this reason, it also did not depend which child was chosen at a given stage. Suppose then that we have created T_s and (arbitrarily) indexed the children of each vertex. Then, there is no loss of genearlity in assuming that at each stage, we select a child with smallest index among its siblings, that is, that we select the leftmost available child in T_s. Since for ease of indexing, we will often assume a $\rho-$ordering has been built by this convention, we introduce the following definition.

Definition. The $\rho-$ordering of S formed by pulling elements from left to right in (a choice of) T_s is call the **canonical ρ-ordering** of S (with respect to T_s).

Example 1. $\mathbb{Z}$ with any prime

Example 2. $\mathbb{Z} \setminus 4\mathbb{Z} \subset (\mathbb{Z}, |\cdot|_2)$

Corollary. Interweaving the bottom row of the lattice of closed balls for a set S gives a ρ-ordering of S.

Lemma. If δ denotes the ρ-ordering of S formed by pulling from left to right in T_s, then

$$\rho(\delta(n), \delta(m)) = \gamma_k$$

if and only if

$$n = m \mod \beta(k) \text{ and } n \neq m \mod \beta(k+1)$$

Lemma. $\lfloor \frac{n}{q} \rfloor$ counts the numbers strictly less than n that are congruent to $n \mod q$.

Proof. Every multiple of q produces exactly one of the numbers from 1 to q and exactly one of those is the residue class of n modulo q. The remainder is the residue class of n itself and since we only want the numbers strictly less than n, we ignore this by taking the floor. $\square$

Proposition.

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor$$

Definition. Let S be as above. We say that S is **semi-regular** if $T_{B_i^k} = T_{B_j^k}$, $\forall k \in \mathbb{N}$ and $i, j \in \beta(k)$. That is, S is semi-regular if each ball of radius γ_k breaks into the same number of balls of radius γ_{k+1}, for all k. If there exists an $n \in \mathbb{N}$ such that $T_{B_i^n} = T_{B_j^n}$, i.e. each ball of radius γ_N breaks into the same number of balls of radius γ_{N+1}, for all $N \geq n$, then we say S is **eventually semi-regular**.

cite Amice or Fares or both here.

Corollary. If S is a semi-regular ultrametric space, Γ_S is the sequence of distances in S, and β is the structure sequence of S, then

$$v_{\gamma_k}(\sigma(n)) = \lfloor \frac{n + \beta(k)}{\beta(k+1)} \rfloor$$

Example 3. Consider the ultrametric space $(\mathbb{Z}, |\cdot|_p)$ for any prime p. The corollary gives

$$v_1(\sigma(n)) = \lceil \frac{n}{p} \rceil$$

$$v_{\frac{1}{p}}(\sigma(n)) = \lceil \frac{n+p}{p^2} \rceil$$

$$v_{\frac{1}{p^2}}(\sigma(n)) = \lceil \frac{n+p^2}{p^3} \rceil$$

$$v_{\frac{1}{p^3}}(\sigma(n)) = \lceil \frac{n+p^3}{p^4} \rceil$$

So that

$$\sigma(n) = (\frac{1}{p})^{\lceil \frac{n+p}{p^2} \rceil + 2\cdot\lceil \frac{n+p^2}{p^3} \rceil + 3\cdot\lceil \frac{n+p^3}{p^4} \rceil + \ldots}$$

$$= (\frac{1}{p})^{\sum_{i=1}^{\infty} i\cdot\lceil \frac{n+p^i}{p^{i+1}} \rceil}$$

$$= (\frac{1}{p})^{\sum_{i=1}^{\infty} i\cdot\lceil \frac{pn}{p^{i+1}} \rceil - \lceil \frac{n}{p^{i+1}} \rceil}$$

$$= (\frac{1}{p})^{\sum_{i=1}^{\infty} i\cdot\lceil \frac{n}{p^i} \rceil - \lceil \frac{n}{p^{i+1}} \rceil}$$

$$= (\frac{1}{p})^{\sum_{i=1}^{\infty} \lceil \frac{n}{p^i} \rceil}$$

Corollary. If S is a (eventually) semi-regular ultrametric space and the α sequence of S is (eventually) periodic, then the valuative capacity of S is algebraic.

Semi-regularity in S reflects horizontal similarity on every level of T_S, and so we expect semi-regularity to simplify the calculation of valuative capacity.

Proposition. Let S be a semi-regular, compact subset of an ultrametric space. Let Γ_S be the set of distances in S and let B be the first element of S_{γ_1}. Let $\sigma^S(i)$ be the characteristic sequence of S and $\sigma^B(i)$ be the characteristic sequence of B. Then,

$$\sigma^S(\beta(0)\cdot n) = \gamma_0^c \cdot \sigma^B(n)$$

where c counts the numbers in 1 to $\beta(0) \cdot n$ that are not divisible by $\beta(0)$.

Proof. (sketch) If $n = 0$, then $c = 0$ and $\beta(0) \cdot n = 0$, and,

$$\sigma^S(0) = 1 = 1 \cdot \sigma^B(0)$$

since the 0^{th}-term of any characteristic sequence is 1 by definition. When $n = 1$, $c = \beta(0) - 1$, so that the right-hand side becomes,

$$\gamma_0^{\beta(0)-1} \cdot \sigma^B(1)$$

Now note that the $\beta(0)^{th}$ term of σ^S will have $\beta(0) - 1$ copies of γ_0 (one from each of the elements of S_{γ_0} not containing the $\beta(0)^{th}$ element) and the remaining term is in the branch B, so it is given by $\sigma^B(1)$, and so,

$$\sigma^S(\beta(0)) = \gamma_0^{\beta(0)-1} \cdot \sigma^B(1)$$

Suppose:

$$\sigma^S(\beta(0) \cdot n) = \gamma_0^{c_n} \cdot \sigma^B(n)$$

for $0 \leq n < n + 1$ and consider $\sigma^S(\beta(0) \cdot (n + 1))$. $\sigma^S(\beta(0) \cdot (n + 1))$ will add $\beta(0)$ more terms to $\sigma^S(\beta(0) \cdot n)$. Moreover, exactly $\beta(0) - 1$ of these additional terms will be in other branches and 1 term will be in B, since any sequence of $\beta(0)$ terms in the $\rho-$ordering of S will be from each of the $\beta(0)$ branches. Each of the $\beta(0) - 1$ terms from other branches will add a copy of γ_0 and remaining term can be found by looking ahead in σ^B, so that

$$\begin{aligned}
&\sigma^S(\beta(0) \cdot (n+1)) \\
&= \sigma^S(\beta(0) \cdot n) \cdot \gamma_0^{\beta(0)-1} \cdot \frac{\sigma^B(n+1)}{\sigma^B(n)} \\
&= \gamma_0^{c_n} \cdot \sigma^B(n) \cdot \gamma_0^{\beta(0)-1} \cdot \frac{\sigma^B(n+1)}{\sigma^B(n)} \\
&= \gamma_0^{c_n} \cdot \gamma_0^{\beta(0)-1} \cdot \sigma^B(n+1) \\
&= \gamma_0^{c_{n+1}} \cdot \sigma^B(n+1)
\end{aligned}$$

□

Since semi-regularity requires horizontal similarity at every level of T_S, we can repeat the branch cuts as many times as needed to calculate $\sigma(n)$.

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