

# Valuative capacity of compact subsets of ultrametric spaces

Anne Johnson

August 5, 2019

# Background

## Definition

([F]) Let  $K \subseteq \mathbb{C}$  be a compact subset. Fix  $n \in \mathbb{N}$ , and for  $z = (z_1, \dots, z_n) \in K^n$ , consider

$$\delta_n(z) = \prod_{j < i} |z_i - z_j|^{\frac{2}{(n(n-1))}}$$

An element  $z = (z_1, \dots, z_n) \in K^n$  is called a **Fekete  $n$ -tuple** if  $z$  maximizes  $\delta_n$  over all  $n$ -tuples in  $K$ .

# Background

## Definition

[F] Let  $K$  be a compact subset of a metric space,  $(M, \rho)$ . Fix  $n \in \mathbb{N}$ , and for  $z = (z_1, \dots, z_n) \in K^n$ , consider

$$\delta_n(z) = \prod_{j < i} \rho(z_i, z_j)^{\frac{2}{(n(n-1))}}$$

An element  $z = (z_1, \dots, z_n) \in K^n$  is called a **generalized Fekete  $n$ -tuple** if  $z$  maximizes  $\delta_n$  over all  $n$ -tuples in  $K$ .

# Background

## Definition

([B2]) Let  $S$  be a subset of  $\mathbb{Z}$  and let  $p$  be any prime. A  **$p$ -ordering** of  $S$  is a sequence,  $\{a_i\}_{i \geq 0}$  in  $S$ , such that  $a_0$  is arbitrary and for  $i > 0$ ,  $a_i$  minimizes

$$v_p\left(\prod_{j < i} (z - a_j)\right)$$

over  $z \in S$ .

# Background

## Definition

([B2]) Let  $S$  be a subset of  $\mathbb{Z}$  and let  $p$  be any prime. A  **$p$ -ordering** of  $S$  is a sequence,  $\{a_i\}_{i \geq 0}$  in  $S$ , such that  $a_0$  is arbitrary and for  $i > 0$ ,  $a_i$  minimizes

$$v_p\left(\prod_{j < i} (z - a_j)\right)$$

over  $z \in S$ .

- ▶  $p$ -orderings give a recursive construction for generalized Fekete  $n$ -tuples.

# Background

## Definition

([B2]) Let  $S$  be a subset of  $\mathbb{Z}$  and let  $p$  be any prime. A  **$p$ -ordering** of  $S$  is a sequence,  $\{a_i\}_{i \geq 0}$  in  $S$ , such that  $a_0$  is arbitrary and for  $i > 0$ ,  $a_i$  minimizes

$$v_p\left(\prod_{j < i} (z - a_j)\right)$$

over  $z \in S$ .

- ▶  $p$ -orderings give a recursive construction for generalized Fekete  $n$ -tuples!

# Background

## Definition

([F]) Let  $K \subseteq \mathbb{C}$  be a compact subset. The **transfinite diameter** of  $K$  is

$$\lim_{n \rightarrow \infty} \left[ \max_z \delta_n(z) \right]$$

where the maximum is taken over all  $n$ -tuples in  $K$ .

# Background

## Proposition

([Ch], theorem 4.2) Let  $E$  be a subset of  $V$ , a rank-one valuation domain with valuation  $v$ . If  $\{a_i\}_{i \geq 0}$  is  $v$ -ordering of  $E$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} v(a_n - a_k) = \frac{2}{n(n+1)} \inf_{x_0, \dots, x_n \in E} v\left(\prod_{0 \leq j < i \leq n} (x_i - x_j)\right)$$



# Valuative capacity: set-up

## Definition

([J1]) Let  $S$  be a compact subset of  $(M, \rho)$ . A  $\rho$ -**ordering** of  $S$  is a sequence,  $\{a_i\}_{i \geq 0}$  in  $S$ , such that  $a_0$  is arbitrary and for  $i > 0$ ,  $a_i$  maximizes

$$\prod_{j < i} \rho(z, a_j))$$

over  $z \in S$ .

# Valuative capacity: set-up

## Definition

([J1]) Let  $S$  be a compact subset of  $(M, \rho)$ . A  $\rho$ -**ordering** of  $S$  is a sequence,  $\{a_i\}_{i \geq 0}$  in  $S$ , such that  $a_0$  is arbitrary and for  $i > 0$ ,  $a_i$  maximizes

$$\prod_{j < i} \rho(z, a_j))$$

over  $z \in S$ .

- ▶ The terms  $\prod_{i=0}^n \rho(a_n - a_i)$  do not depend on the choice of  $\rho$ -ordering. We call this the  $\rho$ -**sequence** of  $S$ .

# Valuative capacity: set-up

- ▶  $\rho$ -orderings give a recursive construction for generalized Fekete  $n$ -tuples.

# Valuative capacity: set-up

- ▶  $\rho$ -orderings give a recursive construction for generalized Fekete  $n$ -tuples.
- ▶ The limit

$$\omega(S) := \lim_{n \rightarrow \infty} \left[ \prod_{i=0}^n \rho(a_n - a_i) \right]^{\frac{1}{n}}$$

is called the **valuative capacity** of  $S$ .

# Valuative capacity: facts

Valuative capacity has the following properties:

- ▶ *translation-invariance*, i.e.,  $\omega(a + S) = \omega(S)$   
(under a translation-invariant operation)

# Valuative capacity: facts

Valuative capacity has the following properties:

- ▶ *translation-invariance*, i.e.,  $\omega(a + S) = \omega(S)$   
(under a translation-invariant operation)
- ▶ *scaling*, i.e.,  $\omega(bS) = |b|\omega(S)$   
(under a multiplicative norm)

# Valuative capacity: facts

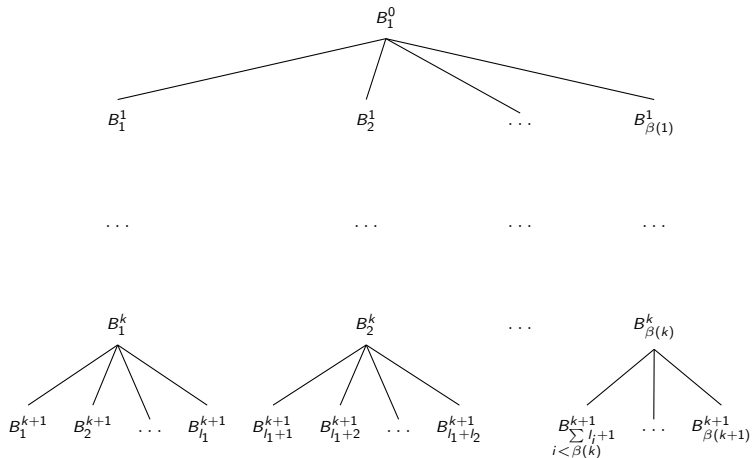
Valuative capacity has the following properties:

- ▶ *translation-invariance*, i.e.,  $\omega(a + S) = \omega(S)$   
(under a translation-invariant operation)
- ▶ *scaling*, i.e.,  $\omega(bS) = |b|\omega(S)$   
(under a multiplicative norm)
- ▶ *decomposition*, i.e.,

$$\frac{1}{\log(\frac{\omega(S)}{d})} = \sum_{i=1}^n \frac{1}{\log(\frac{\omega(A_i)}{d})}$$

for  $d = \text{diam}(S)$  and  $\rho(A_i, A_j) = d, \forall i, j$

# Recursive $\rho$ -orderings





## Recursive $\rho$ -orderings

- ▶  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by selecting elements of each  $B_i^k$  in order as much as possible, and skipping to  $B_{i+1}^k$ , when it is not possible.

## Recursive $\rho$ -orderings

- ▶  $\rho_{k+1}$ -ordering of  $S_{\gamma_{k+1}}$  is found by selecting elements of each  $B_i^k$  in order as much as possible, and skipping to  $B_{i+1}^k$ , when it is not possible.
- ▶ to build a  $\rho$ -ordering of  $S$  from the above, it suffices only to make a choice of centres for each of  $B_i^{k'}$ 's.

# Semi-regularity

## Proposition

If  $S$  is a semi-regular ultrametric space,  $\delta$  is the characteristic sequence of  $S$ ,  $\beta$  is the structure sequence of  $S$ , and  $\alpha$  is the sequence describing the semi-regularity, then

$$v_{\gamma_k}(\delta(n)) = \lfloor \frac{n}{\beta(k)} \rfloor - \lfloor \frac{n}{\beta(k+1)} \rfloor = \sum_{j=1}^{\alpha(k)-1} \lfloor \frac{n + j \cdot \beta(k)}{\alpha(k)\beta(k)} \rfloor$$

# Regularity

## Proposition

Let  $S$  be a regular, tame subset of a compact ultrametric space with  $\gamma_k = q^{c_k}$  for some  $c_k \in \mathbb{Q}$  and for all  $k \in \mathbb{N} \cup 0$ . Then

$$v_q(\delta(n)) = c_0 n + \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \left\lfloor \frac{n}{q^k} \right\rfloor$$

and

$$\log_q(\omega(S)) = \lim_{n \rightarrow \infty} c_0 + \frac{1}{n} \sum_{k=1}^{\infty} (c_k - c_{k-1}) \cdot \left\lfloor \frac{n}{q^k} \right\rfloor$$

- ▶ Semi-regularity implies the presence of a “well-balanced” partition of  $S$ .

- ▶ Semi-regularity implies the presence of a “well-balanced” partition of  $S$ .
- ▶ Regularity shows us that we can recover a notion of scaling, when the conditions are right.

- ▶ Semi-regularity implies the presence of a “well-balanced” partition of  $S$ .
- ▶ Regularity shows us that we can recover a notion of scaling, when the conditions are right.
- ▶ Taken together, they extend the toolkit for computing capacity without appealing to algebraic structure.

- ▶ Semi-regularity implies the presence of a “well-balanced” partition of  $S$ .
- ▶ Regularity shows us that we can recover a notion of scaling, when the conditions are right.
- ▶ Taken together, they extend the toolkit for computing capacity without appealing to algebraic structure!



# Product Space

## Proposition

Let  $(M_i, \rho_i)$ , for  $i$  in some finite index set  $I$ , be a collection of metric spaces and suppose  $\rho_i$  is an ultrametric for each  $i$ . Then  $(M, \rho_\infty)$  is an ultrametric space, where  $M = M_1 \times M_2 \times \dots \times M_n$  and  $\rho_\infty$  is the metric described above.

# Product Space

## Proposition

Let  $(M_i, \rho_i)$ , for  $i$  in some finite index set  $I$ , be a collection of metric spaces and suppose  $\rho_i$  is an ultrametric for each  $i$ . Then  $(M, \rho_\infty)$  is an ultrametric space, where  $M = M_1 \times M_2 \times \dots \times M_n$  and  $\rho_\infty$  is the metric described above.

- ▶ Let  $M = (\mathbb{Z}^n, \rho_{p,\infty})$  be the ultrametric space with points equal to the  $n$ -fold product of  $(\mathbb{Z}, \rho_p)$  (for  $n < \infty$ ) for some fixed prime  $p$ . The valutive capacity of  $M$  is  $(\frac{1}{p})^{\frac{1}{p^n-1}}$ .

# Product Space

## Proposition

Let  $(M_i, \rho_i)$ , for  $i$  in some finite index set  $I$ , be a collection of metric spaces and suppose  $\rho_i$  is an ultrametric for each  $i$ . Then  $(M, \rho_\infty)$  is an ultrametric space, where  $M = M_1 \times M_2 \times \dots \times M_n$  and  $\rho_\infty$  is the metric described above.

- ▶ Let  $M = (\mathbb{Z}^n, \rho_{p,\infty})$  be the ultrametric space with points equal to the  $n$ -fold product of  $(\mathbb{Z}, \rho_p)$  (for  $n < \infty$ ) for some fixed prime  $p$ . The valutive capacity of  $M$  is  $(\frac{1}{p})^{\frac{1}{p^n-1}}$ .
- ▶ What about primes  $p \neq q$ ?

## Product Space: primes $p \neq q$

- ▶ These spaces do not have a scaling property and they are not regular.

## Product Space: primes $p \neq q$

- ▶ These spaces do not have a scaling property and they are not regular.
- ▶ We use the fact that they are semi-regular and study the exponent of each prime in turn.

Product Space:  $(\mathbb{Z}, \rho_2) \times (\mathbb{Z}, \rho_3)$

$$v_{\gamma_{\frac{1}{2}}}(\delta(n)) = \sum_{i=1}^{\infty} i \cdot \left\lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor$$

Product Space:  $(\mathbb{Z}, \rho_2) \times (\mathbb{Z}, \rho_3)$

$$v_{\gamma_{\frac{1}{2}}}(\delta(n)) = \sum_{i=1}^{\infty} i \cdot \left\lfloor \frac{n + 2^i \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}}{2^{i+1} \cdot 3^{\lceil \frac{i}{\log_2(3)} \rceil}} \right\rfloor$$

$$v_{\gamma_{\frac{1}{3}}}(\delta(n)) = \sum_{i=1}^{\infty} i \cdot \left( \left\lfloor \frac{n + 2^{\lceil \frac{i}{\log_3(2)} \rceil} \cdot 3^i}{2^{\lceil \frac{i}{\log_3(2)} \rceil} \cdot 3^{i+1}} \right\rfloor + \left\lfloor \frac{n + 2^{\lceil \frac{i}{\log_3(2)} \rceil + 1} \cdot 3^i}{2^{\lceil \frac{i}{\log_3(2)} \rceil} \cdot 3^{i+1}} \right\rfloor \right)$$






Product Space: primes  $p \neq q$

### Conjecture

Finite products of  $(\mathbb{Z}, \rho_{p_i})$  for distinct primes,  $p_i$ , have transcendental valutive capacity.



# references

-  M. Bhargava,  $P$ -orderings and polynomial functions on arbitrary subsets of Dedekind rings, *J. reine angew. Math.*, **490** (1997), 101-127.
-  M. Bhargava, The factorial function and generalizations, *Am. Math. Monthly* **107** (2000), 783-799.
-  J.-L. Chabert, Generalized factorial ideals, *Commutative algebra, Arab. J. Sci. Eng. Sect. C* **26** (2001), 51-68.
-  M. Fekete, Über die Verteilung der Wurzeln bei gewisser algebraische Gleichungen mit ganzzahligen Koeffizienten, *Math. Zeits.* **17** (1923), 228-249.
-  K. Johnson,  $p$ -orderings, Fekete  $n$ -tuples and capacity in ultrametric spaces (in preparation).