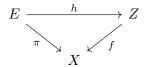
# Exercises

# Section 2

2.1. For a commutative triangle of topological spaces and continuous maps,



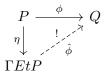
if f and  $\pi$  are étale, so is h (in particular, h is open).

Let  $y \in E$ . Since f is étale,  $\exists U$  an open neighborhood of h(y) such that  $f \mid_U$  is a homeomorphism. Likewise, since  $\pi$  is étale,  $\exists W$  an open neighborhood of y such that  $\pi \mid_W$  is a homeomorphism. Let  $V = h^{-1}(U) \cap W$ . Since h is continuous,  $h^{-1}(U)$  is open, so V is open in E. Since  $V \subseteq W$ ,  $\pi \mid_V$  is a homeomorphism.

Since  $h(V) \subseteq U$ , and  $f|_U$  is a homeomorphism, it only remains to show that h(V) is open, since this will imply that  $f|_{h(v)}$  is a homeomorphism, and in turn that  $h|_V$  is a homeomorphism (since  $\pi|_V$  is a homeomorphism, and the diagram commutes). To see that h(V) is open, note that  $\pi(V)$  is open in X, since  $\pi$  is étale and V is open in E. Further,  $f(h(V)) = \pi(V)$  since the diagram commutes. Then since  $h(V) \subseteq U$  and  $f|_U$  is a homeomorphism, h(V) is open in Z, so  $h|_V$  is a homeomorphism and h is étale.

2.2. (a) Show that Corollary 2.10 determines the associated sheaf up to isomorphism. That is, if R is a sheaf and there exists a map  $\psi: P \to R$  with the same property as  $\eta$  in Corollary 2.10, then  $R \cong \Gamma Et(P)$ .

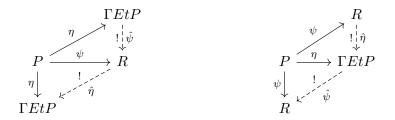
**Corollary.** (2.10) For a map  $\phi: P \to Q$  between presheaves on X, if Q is a sheaf then there exists a unique map of sheaves  $\hat{\phi}$  such that  $\hat{\phi} \circ \eta = \phi$ .



The universal property of  $(\Gamma EtP, \eta)$  implies a unique map  $\hat{\psi}$  such that  $\psi = \hat{\psi} \circ \eta$  and the universal property of  $(R, \psi)$  implies a unique map  $\hat{\eta}$  such that  $\eta = \hat{\eta} \circ \psi$ , i.e. we have the following diagrams:



This implies the following diagrams commute:



In particular,  $\hat{\eta} \circ \hat{\psi}$  is the unique map such that  $\eta = \hat{\eta} \circ \hat{\psi} \circ \eta$  and  $\hat{\psi} \circ \hat{\eta}$  is the unique map such that  $\psi = \hat{\psi} \circ \hat{\eta} \circ \psi$ . But then uniqueness implies  $\hat{\eta} \circ \hat{\psi} = id_{\Gamma EtP}$  and  $\hat{\psi} \circ \hat{\eta} = id_R$ . Then  $R \cong \Gamma EtP$ .

2.2. (b) If P is a subpresheaf of R and R is a sheaf, then let

 $\tilde{P}(U) = \{r \in R(U) \mid \text{ for each } x \in U \text{ there is a neighbourhood } W_x \subseteq U \text{ of } x \text{ such that } (r \mid_{W_x}) \in P(W_x) \}.$ 

Show that  $P \subseteq \tilde{P} \subseteq R$ ,  $\tilde{P}$  is a sheaf and  $P \hookrightarrow \tilde{P}$  has the unique universal property of Corollary 2.10; hence  $\tilde{P}$  is the associated sheaf for P.

## $P \subseteq \tilde{P} \subseteq R$ :

We want to show that for  $U \in X$ , open,  $P(U) \subseteq \tilde{P}(U) \subseteq R(U)$ . So let  $U \subseteq X$  be open. By construction  $\tilde{P}(U) \subseteq R(U)$ , so we need only show  $P(U) \subseteq \tilde{P}(U)$ . Let  $s \in P(U)$ . Since P is a subpresheaf  $s \in R(U)$  and for  $x \in U$ , U itself is a neighborhood of x such that,  $s \mid_{U}$  is in P(U). Then  $s \in \tilde{P}(U)$ .

#### $\tilde{P}$ is a sheaf:

Let U be an open set in X and let  $\cup U_i$ ,  $i \in I$ , be an open cover for U. Suppose  $\{a_i\} \in \tilde{P}(U_i)$ , for  $i \in I$  is a compatible family for the  $U_i$ , i.e.,  $a_i \mid_{U_i \cap U_j} = a_j \mid_{U_i \cap U_j}$ . For each  $i \in I$  and each  $x \in U_i$ , choose  $W_x^i$  such that  $(a_i \mid_{W_x^i}) \in P(W_x^i)$ . Then  $\cup_i \cup_{x \in U_i} W_x^i$  is an open cover for U (since  $\cup_x W_x^i$  is an open cover for each  $U_i$ ). Moreover,  $a_i \mid_{W_x^i \cap W_y^j} = a_j \mid_{W_x^i \cap W_y^j}$  because  $(a_i \mid_{U_i \cap U_j}) \mid_{W_x^i \cap W_y^j} = (a_j \mid_{U_i \cap U_j}) \mid_{W_x^i \cap W_y^j}$ . So  $\{a_i\} \in P(W_x^i)$  is a compatible family in  $\mathbb{R}$ .

Since R is a sheaf, let a e the unique amalgamation. We want to show  $a \in \tilde{P}(U)$ , i.e.,  $\forall x \in U, \exists$  a neighborhood  $V_x$  of x such that  $a \mid_{V_x} \in P(V_x)$ . But since a is the amalgamation of the family  $\{a_i\} \in P(W_x^i)$ , if  $x \in U$ , then  $x \in U_i$ , for some i, so  $a \mid_{W_x^i} = a_i$  and  $W_x$  was chosen so that  $a_i \mid_{W_x^i} \in P(W_x^i)$ . Then take  $V_x$  to be  $W_x^i$  (for appropriate i). Then  $a \in \tilde{P}(U)$ , so  $\tilde{P}$  is a sheaf.

# $P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10:

Let  $i: P \hookrightarrow \tilde{P}$  be the map that injects P(U) into  $\tilde{P}(U)$ . We must show that for a map between presheaves,  $\phi: P \to Q$ , where Q is a sheaf, there is a unique  $\tilde{\phi}$  such that  $\tilde{\phi} \circ i = \phi$ :

So let Q be a sheaf and  $\phi: P \to Q$  be a morphism. For each open set  $U \subseteq X$ , we define  $\tilde{\phi}_U: \tilde{P}(U) \to Q(U)$  as follows: let  $s \in \tilde{P}(U)$  and let  $\cup_{x \in U} W_x$  be a cover of U such that  $(s \mid_{W_x}) \in P(W_x)$  for each  $x \in U$ . Consider the family  $\phi_{W_x}(s \mid_{W_x})$  for  $x \in U$ . This family is compatible because commutativity of the restriction maps implies  $(s \mid_{W_x}) \mid_{W_x \cap W_y} = (s \mid_{W_y}) \mid_{W_x \cap W_y} = s \mid_{W_x \cap W_y}$  and naturality of  $\phi$  implies for  $x, y \in U$ :

$$\phi(s \mid W_x) \mid_{W_x \cap W_y} = \phi((s \mid W_x) \mid_{W_x \cap W_y}) = \phi(s \mid_{W_x \cap W_y})$$
$$= \phi((s \mid_{W_y} \mid_{W_x \cap W_y})) = \phi(s \mid_{W_y}) \mid_{W_x \cap W_y}.$$

Then since Q is a sheaf, let  $t \in Q(U)$  be the unique amalgamation of the  $\phi_{W_x}(s|_{W_x})$ 's and define  $\tilde{\phi}(s) = t$ . In fact, by naturality of  $\tilde{\phi}$ , we must choose the amalgamation to ensure that the following commutes for each  $W_x$ :

$$s \in \tilde{P}(U) \xrightarrow{\tilde{\phi}_{U}} Q(U) \ni t$$

$$\downarrow^{\rho} \qquad \qquad \downarrow^{\rho}$$

$$(s \mid_{W_{x}}) \in \tilde{P}(W_{x}) \xrightarrow{\tilde{\phi}_{W_{x}}} Q(W_{x}) \ni \tilde{\phi}(s \mid_{W_{x}}) = \phi(s \mid_{W_{x}})$$

since the bottom row must be  $\tilde{\phi}(s|_{W_x}) = \phi(s|_{W_x})$  by the requirement that  $\tilde{\phi} \circ i = \phi$ . So then uniqueness of  $\tilde{\phi}$  is implied so long as  $\tilde{\phi}$  is well-defined.

To see that  $\tilde{\phi}$  is well-defined, let  $\cup_{x\in U}V_x$  be another cover of U with the property that  $(s\mid_{V_x})\in P(V_x)$ , so that  $\phi_{V_x}(s\mid_{V_x})$  is again a compatible family. We must show that t is also the amalgamation of this family, i.e.,  $t\mid_{V_x}=\phi_{V_x}(s\mid_{V_x})$  for each  $V_x$ .

Consider the cover of U formed by the refinement of the  $W_x$ 's and  $V_x$ 's,  $U = \bigcup_{x \in U} W_x \bigcup_{x \in U} V_x$ . This contains the compatible family

$$\{\{\phi(s\mid_{W_x})\}_{x\in U}, \{\phi(s\mid_{V_x})\}_{x\in U}\}$$

.

The amalgamation of this family, say  $\bar{t}$ , must have the property that  $\bar{t}\mid_{W_x} = \phi(s\mid_{W_x})$ , and since Q is a sheaf, hence separated, this implies that  $\bar{t} = t$ . Then  $t\mid_{V_x} = \phi_{V_x}(s\mid_{V_x})$  for each  $V_x$ , so  $\tilde{\phi}$  is well-defined. In particular, note that  $\tilde{\phi} \circ i = \phi$ , since  $s \in P(U)$  implies that U is a cover of U for which  $s\mid_{U} \in P(U)$  so  $\tilde{\phi}(s) = \phi(s\mid_{U}) = \phi(s)$ .

Then  $\tilde{\phi}$  is the required unique sheaf map satisfying  $\tilde{\phi} \circ i = \phi$ , so  $\tilde{P}$  is the associated sheaf of P.

## Section 3

3.1 The pullback of an étale map is also étale. That is, given a pullback diagram,

$$\begin{array}{ccc} X \times_Y E & \xrightarrow{\pi_2} & E \\ \downarrow^{\pi_1} & & \downarrow^p \\ X & \xrightarrow{f} & Y \end{array}$$

then  $\pi_1$  is étale if p is.

Let  $(u, e) \in X \times_Y E$ . Since p is étale, there exists a  $U \subseteq E$ , open neighborhood of  $e = \pi_2(u, e)$ , such that  $p : U \to p(U)$  is a homeomorphism. Since  $f : X \to Y$  is continuous (and p is open), the preimage of p(U) under f is open in X. Let  $W \subseteq X$  be the preimage of p(U). Since p(e) = f(u),  $f(u) \in p(U)$  and so  $u \in W$ .

Since  $W \times U$  is open in  $X \times E$  under the product topology,  $V = W \times U \cap X \times_Y E$  is an open neighborhood of (u,e) in  $X \times_Y E$  under the subspace topology. I claim that  $\pi_1 \mid_V$  is a homeomorphism. Since  $\pi_1$  is a projection map,  $\pi_1 \mid_V$  is continuous. We must show there exists a continuous  $g: \pi_1(V) \to X \times_Y E$  such that  $g \circ \pi_1 = id$ . Since  $p \mid_U$  is a homeomorphism, we have continuous  $p^{-1}: f(\pi_1(V)) \to E$  such that  $p^{-1} \circ p = id_E$ . Then  $p^{-1}(f(\pi_1(V))) = p^{-1}(p(\pi_2(V))) = \pi_2(V)$ . Let  $i: EtB \hookrightarrow X \times_Y EtB$  be the inclusion map. Then let  $g: \pi_1(V) \to X \times_Y EtB$  be defined by  $x \mapsto (x, i \circ p^{-1} \circ f(x))$ . Then g is continuous and  $g \circ \pi_1 \mid_V = id_{X \times_Y E}$ , so that  $\pi_1 \mid_V$  is a homeomorphism. Then  $\pi_1$  is étale.

#### 3.2. Verify the following equality of stalks:

$$(f^*B)_x \cong B_{f(x)}$$

for each  $x \in X$ .

First note that since  $f^*(B) = \Gamma(X \times_Y EtB)$  and since  $\pi_1$  is étale,  $Et\Gamma(X \times_Y EtB) \cong X \times_Y EtB$  by the results in section 2. That is,  $Et(f^*(B)) \cong X \times_Y EtB = \{(x, e) \in X \times EtB \mid f(x) = \pi_B(e)\}.$ 

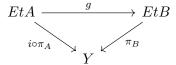
Fix an  $x_0 \in X$ . The stalk of  $f^*(B)$  at  $x_0$  is  $(f^*(B))_{x_0} = \{s \in Et(f^*(B)) \mid \pi_B(s) = x_0\} = \{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\}$ . Then the map  $\pi_2$  gives us a bijection  $\{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\} \rightarrow \{e \in EtB \mid f(x_0) = \pi_B(e)\}$ , which is exactly the stalk of  $B_{f(x_0)}$ .

#### 3.3. Prove that

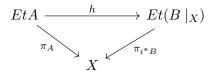
$$Hom_{Sh(Y)}(i_!A, B) \cong Hom_{Sh(X)}(A, i^*B)$$

where  $i: X \hookrightarrow Y$  is an open inclusion.

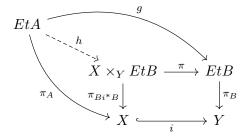
First note that since  $i \circ \pi_A$  is étale,  $Et(i_!A) = Et(\Gamma EtA) = EtA$ . Since B is a sheaf, 2.9 implies  $\phi \in Hom_{Sh(Y)}(i_!A, B)$  corresponds to a map  $g : EtA \to EtB$  such that the following commutes:



By the notes,  $Et(i^*B) = Et(B \mid_X)$  and since  $i^*B$  is a sheaf, 2.9 implies  $\psi \in Hom_{Sh(X)}(A, i^*B)$  corresponds to a map  $h : EtA \to Et(B \mid_X)$  such that the follow commutes:



To see the correspondence between g and h, note that  $Et(B \mid_X) \cong X \times_Y EtB$ , so that given  $g: EtA \to EtB$ , h is the unique map such that the following commutes (and given  $h: EtA \to Et(B \mid_X) \cong X \times_Y EtB$ , g is the map  $\pi \circ h$ ):



3.4 (a) Verify the adjunction formula involving  $j_*$  and  $j^!$  in (7).

We must show that  $Hom_{Ab(Y)}(j_*A, B) \cong Hom_{Ab(Z)}(A, j!B)$ .

Let  $\phi \in Hom_{Ab(Y)}(j_*A, B)$ . Then  $\phi$  is a collection of maps  $\phi_U : j_*A(U) \to B(U)$  for every open set  $U \subseteq Y$ . Note that for  $U \subseteq Y$  such that  $U \cap Z = \emptyset$ ,  $j_*A(U) = A(U \cap Z) = A(\emptyset) = \{0\}$ , so that there is only one choice for  $\phi_U$ . Likewise, if  $U \subset Y$  with  $U \cap \delta Z \neq \emptyset$ , then there exists a  $V \subset U$  such that  $V \cap Z = \emptyset$  and naturality of  $\phi$  implies:

$$j_*A(U) \xrightarrow{\phi_U} B(U)$$

$$\downarrow^{\rho} \qquad \qquad \downarrow^{\rho}$$

$$j_*A(V) = A(V \cap Z) = \{0\} \xrightarrow{\phi_V} B(V)$$

so that  $\phi_U(j_*A(U))$  is the constant 0 map. Finally, suppose U is such that  $U \cap Z = U$ . For each such U, the choices for  $\phi_U$  are the group homomorphisms,  $A(U) \to B(U)$ .

Now consider  $\psi \in Hom_{Ab(Z)}(A, j^!B)$ .  $\psi$  is a collection of maps  $\psi_V$  for each open V in Z. Suppose V is open in Z and  $V \cap \delta Z \neq \emptyset$ . Then  $j^!B(V) = 0$  and so there is only one choice for  $\psi_V$ , the constant 0 map. On the other hand, if  $V \cap \delta Z = \emptyset$ , then the choices for  $\phi_V$  are the group homomorphisms,  $A(V) \to B(V)$ .

Then elements of both  $Hom_{Ab(Z)}(A, j^!B)$  and  $Hom_{Ab(Y)}(j_*A, B)$  correspond to the group homorphisms from  $A(U) \to B(U)$  for every  $U \subseteq Y$  with  $U \cap Z = U$ .

3.4 (b) Show that  $j^!B$  is isomorphic to a subsheaf of  $j^*B$ .

Since  $j^!B$  is a sheaf,we know that  $j^!B \cong \Gamma Etj^!B$ . We show that  $\Gamma Etj^!B$  is a subsheaf of  $j^*B$  and for this is suffices to show that  $\Gamma Etj^!B$  is a subpresheaf; that is, that for V open in Z,  $\Gamma Etj^!B(V) \subseteq j^*B(V)$  and that the restrictions maps of  $\Gamma Etj^!B$  agree with those of  $j^*B$ .

Let V be open in Y, i.e.  $V = Z \cap U$  for some U open in Y. We have

$$j^*B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B) \text{ and } x \in V\}$$

and

$$\Gamma Etj^!B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B), x \in V \text{ and } supp(s) \subseteq Z\}$$

so that  $\Gamma Etj^!B(V)\subseteq j^*B(V)$ . Moreover, the restriction maps agree since they are both restrictions of functions. Then  $\Gamma Etj^!B\cong j^!B$  is a subsheaf of  $j^*B$ .

# Section 4

4.1 (a) Show that with the notation as above,  $\Gamma(-,\pi)$  is indeed isomorphic to the sheaffication of P.

We first fix some notation. Suppose  $U \subseteq X$  and let  $s \in G(U)$  be a section. Let  $\overline{s}$  be the equivalence class of s in  $G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$ , so that points in the étale space of P are denoted  $germ_x(\overline{s})$ . We denote by  $\overline{germ_x(s)}$  the equivalence class of  $germ_x(s)$  under the relation R given above, where  $germ_x(s) \in Et(G)$ .

To show  $\Gamma(-,\pi) \cong P$ , we define a map  $\phi: \Gamma(-,\pi) \to P$  and show it induces an isomorphism on the stalks. For  $U \subseteq X$  and  $\sigma \in \Gamma(-,\pi)(\underline{U})$ , define  $\phi_U: \Gamma(-,\pi)(U) \to P(U)$  by  $\phi_U(\sigma) = \tau$ , where  $\sigma$  is a function  $x \mapsto \overline{germ_x(s)}$  and  $\tau$  is a function  $x \mapsto \overline{germ_x(s)}$  for some  $s \in G(U)$ . Naturality of  $\phi$  is straightforward, since restriction maps are just restrictions of functions.

To see that  $\phi$  is well defined, suppose  $germ_x(t)$  is also a representative of  $\overline{germ_x(s)}$ . Then  $\exists W \ni x$  and  $f \in F(W)$  such that  $s \mid_W - t \mid_W = \alpha_W(f)$ . This however gives  $germ_x(\overline{t}) = germ_x(\overline{s})$ , since W is then a neighborhood for which  $\overline{s \mid_W = t \mid_W}$ , so that they are equal as germs.

It remains to show that  $\phi$  induces an isomorphism on the stalks. We show  $\phi_x : \Gamma(-,\pi)_x \to P_x$  is injective and surjective. Fix an  $x \in X$  and suppose  $\phi(\overline{germ_x(a)}) = \overline{germ_x(a)} = \phi(\overline{germ_x(b)})$ . We have to show  $\exists W \ni x$  and  $f \in F(W)$  such that  $a \mid_W - b \mid_W = \alpha_W(f)$ . To see that  $\phi_x$  is surjective.

Let  $germ_x(s)$  be an element of Et(G)/R, and let  $germ_x(s)$  be a representative, so that if  $germ_x(t)$  is another representative of  $germ_x(s)$ , then  $\exists W \ni x$  and  $f \in F(W)$  such that  $s \mid_W - t \mid_W = \alpha_W(f)$ , where s and t are representatives of  $germ_x(s)$  and  $germ_x(t)$  respectively.

Let  $germ_x(a)$  be an element of  $P_x$ , represented by (a,U) for some  $U \ni x$  and  $a \in P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U)) = G(U)/im(\alpha_U)$ . If (b,V) is another representative of  $germ_x(a)$ , then there exists  $W \ni x$  such that  $a \mid_W = b \mid_W$ , i.e.,  $a \mid_W$  and  $b \mid_W$  are in the same equivalence class of  $G(W)/im(\alpha_W)$ , i.e.,  $\exists f \in F(W)$  such that  $s \mid_W -t \mid_W = \alpha_W(f)$ . Then elements of  $G_x/R$  and  $P_x$  are both correspond to elements of  $G_x$  identified via the same relation.

Let  $\overline{germ_x(s)}$  be an element of  $G_x/R$ , and let  $\underline{germ_x(s)}$  be a representative, so that if  $\underline{germ_x(t)}$  is another representative of  $\overline{germ_x(s)}$ , then  $\exists W \ni x$  and  $f \in F(W)$  such that  $s \mid_W -t \mid_W = \alpha_W(f)$ , where s and t are representatives of  $\underline{germ_x(s)}$  and  $\underline{germ_x(t)}$  respectively.

Let  $germ_x(a)$  be an element of  $P_x$ , represented by (a,U) for some  $U\ni x$  and  $a\in P(U)=coker(F(U)\xrightarrow{\alpha_U}G(U))=G(U)/im(\alpha_U)$ . If (b,V) is another representative of  $germ_x(a)$ , then there exists  $W\ni x$  such that  $a\mid_W=b\mid_W$ , i.e.,  $a\mid_W$  and  $b\mid_W$  are in the same equivalence class of  $G(W)/im(\alpha_W)$ , i.e.,  $\exists f\in F(W)$  such that  $s\mid_W-t\mid_W=\alpha_W(f)$ . Then elements of  $G_x/R$  and  $P_x$  are both correspond to elements of  $G_x$  identified via the same relation.

4.1(b) Show that the  $coker(\alpha)$  has the following universal property: given any other sheaf H, a map  $\chi: G \to H$  factors through  $coker(\alpha)$  if and only if  $\chi \circ \alpha = 0$ .

(Forward direction) Let  $\alpha: F \to G$  and  $\chi: G \to H$  and suppose  $\chi \circ \alpha = 0$ . Then for  $U \in X$ ,  $ker(\chi_U) = im(\alpha_U)$ . By the first isomorphism theorem,  $\chi_U(G) \cong G(U)/ker(\chi_U) = G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$ , so that  $\chi = g \circ f$ , where  $f: G \to coker(\alpha)$  is the projection map and  $g: coker(\alpha) \to H$  is an isomorphism.

(Backwards direction) Let  $\alpha: F \to G$  and  $\chi: G \to H$  and suppose  $\chi = g \circ f$  for some  $f: G \to coker(\alpha)$  and  $g: coker(\alpha) \to H$ . Note that  $f \circ \alpha = 0$ , since  $f_U$  has codomain  $coker(F(U) \xrightarrow{\alpha_U} G(U))$  for  $U \in X$ . Then  $\chi \circ \alpha = g \circ f \circ \alpha = g \circ 0 = 0$ .

4.1(c) Show that  $coker(\alpha)_x = coker(\alpha_x)$ .

Let  $\alpha: F \to G$  for F and G Abelian sheaves on X and let P be the presheaf given by  $P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$ . Since  $\pi_P: Et(P) \to X$  is étale,  $Et\Gamma EtP \cong EtP$ , so the left-hand side is isomorphic to  $P_x$ . On the right-hand side, note that  $coker(\alpha_x) = coker(F_x \xrightarrow{\alpha_x} G_x) = G_x/Im(\alpha_x)$ , so that two points are equivalent in  $coker(\alpha_x)$  iff they are both points in  $G_x$ , say  $germ_x(s)$  and  $germ_y(t)$ , with  $germ_x(s) - germ_y(t) \in Im(\alpha_X)$ , ie  $\exists W \ni x$  and  $f \in F(W)$  such that  $s \mid_W - t \mid_W = \alpha_W(f)$ . This is exactly the relation given by R in the notes, so that the right-hand side is isomorphic to  $G_x/R$ . Then exercise 4.1(a) implies  $coker(\alpha)_x \cong coker(\alpha_x)$ .

4.2 Show that  $\mathbf{Ab}(\mathbf{X})$  is an Abelian category with kernel, cokernel and sum defined as above.

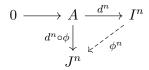
We first show that  $\mathbf{Ab}(\mathbf{X})$  has a 0 object. Let  $\Delta\{*\}$  be the constant sheaf on a singleton set. Then for  $U \subseteq X$ ,  $\Delta\{*\}(U)$  is the trivial group and every restriction map is the identity, so  $\Delta\{*\} \in \mathbf{Ab}(\mathbf{X})$ . Moreover, for  $A \in \mathbf{Ab}(\mathbf{X})$ , there is exactly one map from  $A(U) \to \Delta\{*\}(U)$  for every open  $U \subseteq X$ , namely the constant 0-map, and there is also exactly one map from  $\Delta\{*\}(U) \to A(U)$  since group homomorphism must send identity to identity. Then  $\Delta\{*\}$  is both initial and terminal, and so serves as the 0 element in  $\mathbf{Ab}(\mathbf{X})$ . We can now define the 0 morphism in  $\mathbf{Ab}(\mathbf{X})$  as the map defined for  $A, B \in \mathbf{Ab}(\mathbf{X})$  and  $U \in X$ , open, by  $a \in A(U) \mapsto 0_{B(U)}$ .

Let us define the sum of two Abelian sheaves A and B by  $A(U) \oplus B(U)$ for U open in X, as in the notes. Then this gives the biproduct of Aand B in Ab(X). We know from the notes that  $A \oplus B$  is a sheaf and we observe that for  $U \subseteq X$ ,  $A(U) \oplus B(U)$  has an Abelian group structure: let (a,b) and (a',b') be elements of  $A(U) \oplus B(U)$  for some U. Then  $(a,b) + (a',b') = (a+a',b+b') = (a'+a,b'+b) = (a',b') + (a,b) \in$  $A(U) \oplus B(U)$ , since A(U) and B(U) are Abelian groups. This also implies  $(a,b) + (0_{A(U)}, 0_{B(U)}) = (a,b) = (0_{A(U)}, 0_{B(U)}) + (a,b)$ . To see that this is a biproduct, let  $p_j: A_1 \oplus A_2 \to A_j$  be the map defined on  $U \subseteq X$  as the projection map  $p_{U,i}: A_1(U) \oplus A_2(U) \to A_i(U)$  and  $i_i$  be the map defined on  $U \subseteq X$  as the injection map  $i_j(U): A_j(U) \to A_1(U) \oplus A_2(U)$  for  $j \in \{1, 2\}$ and  $A_i \in \mathbf{Ab}(\mathbf{X})$ . Then since  $p_{U,i}$  and  $i_{U,i}$  give the product and coproduct on each  $A_1(U) \oplus A_2(U)$ , viewed as elements in the category of Abelian groups, their collection gives the product and coproduct on  $A_1 \oplus A_2$  (given a map into or out of the biproduct, find a new map that factors through  $p_i$ or  $i_i$  respectively, by taking the collection of maps implied by the universal property of each  $p_{U,j}$  or  $i_{U,j}$ ).

To see that  $\mathbf{Ab}(\mathbf{X})$  has kernels, note that if  $A, B \in \mathbf{Ab}(\mathbf{X})$  and  $\phi : A \to B$ , then  $\phi$  has a kernel when everything is viewed in the category  $\mathbf{Sh}(\mathbf{X})$ . It remains only to check that  $\ker(\phi)(U) = \{f \in A(U) \mid \phi_U(f) = 0_{B(U)}\}$  is an Abelian group for all U, open in X. But  $\ker(\phi)(U) = \ker(\phi_U)$  is the kernel of the group homomorphism  $\phi_U$ , and so must itself be a subgroup of A(U) for all  $U \in X$ . Then  $\ker(\phi) \in \mathbf{Ab}(\mathbf{X})$ . Likewise, the cokernel of  $\phi$  exists in  $\mathbf{Sh}(\mathbf{X})$ , and so we need only show  $\operatorname{coker}(\phi)(U)$  is an Abelian group,  $\forall U$ .

4.3 Let  $0 \to A \hookrightarrow I^0 \to I^1 \to \dots$  and  $0 \to B \to B \hookrightarrow J^0 \to J^1 \to J^2 \to \dots$  be injective resolutions of A and B respectively. Show that a map  $\phi: A \to B$  extends to a map of complexes:

Since  $0 \to A \hookrightarrow I^0 \to I^1 \to \dots$  is a resolution, in particular exact, for each n, we have an injective map from  $A \to I^n$  given by  $d^n$ . Since  $\phi : A \to B$ , we also have for each n, a map from  $A \to J^n$  given by  $d^n \circ \phi$ . Then injectivity of  $J^n$  implies the required map  $\phi^n : I^n \to J^n$ :



4.3 (b) Show that a map of complexes as above induces a homomorphism of cohomology groups

$$H^n(X;A) \to H^n(X,B)$$

Note that the above map of chain complexes implies a map for  $U \in X$ :

which for U = X gives the map:

$$0 \longrightarrow \Gamma A \longrightarrow \Gamma I^0 \longrightarrow \Gamma I^1 \longrightarrow \Gamma I^2 \longrightarrow \dots$$

$$\downarrow \Gamma \phi \qquad \downarrow \Gamma \phi^0 \qquad \downarrow \Gamma \phi^1 \qquad \downarrow \Gamma \phi^2$$

$$0 \longrightarrow \Gamma B \longrightarrow \Gamma J^0 \longrightarrow \Gamma J^1 \longrightarrow \Gamma J^2 \longrightarrow \dots$$

Since  $H^n(X;A) = \ker(\Gamma I^n \to \Gamma I^{n+1}))/im(\Gamma I^{n-1} \to \Gamma I^n) \subseteq \Gamma I^n$  and likewise  $H^n(X;B) = \ker(\Gamma J^n \to \Gamma J^{n+1}))/im(\Gamma J^{n-1} \to \Gamma J^n) \subseteq \Gamma J^n$ , the maps  $\Gamma \phi^n : \Gamma I^n \to \Gamma J^n$  implies a homomorphism  $\Gamma \phi^n \mid_{H^n(X,A)}: H^n(X;A) \to \Gamma J^n$ . It remains only to check that  $im(\Gamma \phi^n \mid_{H^n(X,A)}) \subseteq H^n(X,B)$ . Let  $\alpha \in H^n(X;A)$ . Then  $\alpha \in \ker(d^n : \Gamma I^n \to \Gamma I^{n+1})$ , so  $\Gamma \phi^{n+1} \circ d^n(\alpha) = 0$ . Since the above diagram commutes, this implies  $d^n \circ \Gamma \phi^n(\alpha) = 0$ , so that  $\Gamma \phi^n(\alpha) \in \ker(d^n : \Gamma J^n \to \Gamma J^{n+1})$ , which implies  $\Gamma \phi^n(\alpha) \in H^n(X;B)$ .

4.4(a) Let  $f: Y \to X$  be a map of topological spaces. Show that there is a natural isomorphism  $\Gamma_Y \to \Gamma_X \circ f_*$ ,

$$Ab(Y) \xrightarrow{\Gamma_X} Ab(X)$$
Abelian Groups

We must find  $\eta_A$  and  $\eta_B$ , isomorphisms, such that the following commutes for all  $A, B \in Ab(Y)$  and  $\phi : A \to B$ :

$$\Gamma_{Y}(A) \xrightarrow{\Gamma_{Y}\phi} \Gamma_{Y}(B)$$

$$\eta_{A} \downarrow \qquad \qquad \downarrow \eta_{B}$$

$$\Gamma_{X} \circ f_{*}(A) \xrightarrow{\Gamma_{x} \circ f_{*}\phi} \Gamma_{X} \circ f_{*}(B)$$

By the definition of  $\Gamma$  and  $f_*$ , this is the same as finding isomorphisms  $\eta_A$  and  $\eta_B$  such that the following commutes:

$$A(Y) \xrightarrow{\phi_Y} B(Y)$$

$$\eta_A \downarrow \qquad \qquad \downarrow \eta_B$$

$$A(f^{-1}(X)) \xrightarrow{\phi_{f^{-1}(X)}} B(f^{-1}(X))$$

But  $f^{-1}(X) = Y$ , so taking  $\eta_A = \rho_{Y,Y} = id_{A(Y)}$  (likewise for  $\eta_B$ ) implies the result, since  $\phi$  is a map  $A \to B$  and so must be natural wrt the restriction maps.

4.4 (b) Show that a natural isomorphism between left exact functors  $\tau: T_1 \to T_2$  induces a natural isomorphism  $R^n T_1 \to R^n T_2$  for each n.

Let  $T_1$  and  $T_2$  be left exact functors from  $\mathcal{C} \to \mathcal{D}$  and let  $\tau: T_1 \to T_2$  be a natural isomorphism. Let  $0 \to C \to I^0 \to I^1 \to ...$  be an injective resolution of C for some  $C \in \mathcal{C}$ . Naturality of  $\tau$  implies that the following commutes:

$$0 \longrightarrow T_1(C) \longrightarrow T_1(I^0) \longrightarrow T_1(I^1) \longrightarrow T_1(I^2) \longrightarrow \dots$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow T_2(C) \longrightarrow T_2(I^0) \longrightarrow T_2(I^1) \longrightarrow T_2(I^2) \longrightarrow \dots$$

Moreover, since  $T_1$  and  $T_2$  are both left exact, each row in the above diagram is exact. Then, by analogous argument to 4.3(b),  $\tau$  restricts to a natural isomorphism  $R^nT_1 \to R^nT_2$  for each n.

## Section 7

7.1 Prove Corollary 7.4 (Proper Base Change): For a pullback diagram as in (26) with f and f' proper with Hausdorff fibers, the canonical map

$$p^*(R^n f_* B) \to R^n f'_*(q^* B)$$

is an isomorphism for any sheaf B on Y.

We show  $p^*(R^n f_* B)_{x'} \cong (R^n f'_*(q^* B))_{x'}$  for  $x' \in X'$ . Let  $x' \in X'$ , B be a sheaf on Y and let the following be a pullback diagram:

$$(f')^{-1}(x') \in Y' \xrightarrow{q} Y \ni f^{-1}(p(x'))$$

$$f' \downarrow \qquad \qquad \downarrow f$$

$$x' \in X' \xrightarrow{p} X \ni p(x')$$

Note that since the above is a pullback diagram, q gives an isomorphism  $q:(f')^{-1}(x')\xrightarrow{\sim} f^{-1}(p(x'))$ , which by Prop (4.3) induces an isomorphism  $H^n(f^{-1}(p(x')); B\mid_{f^{-1}p(x')})\xrightarrow{\sim} H^n((f')^{-1}(x'); q^*B\mid_{(f')^{-1}(x')})$ . Then we have:

$$p^{*}(R^{n}f_{*}B)_{x'} \cong (R^{n}f_{*}B)_{p(x')} \qquad \text{exercise } 3.2$$

$$\cong H^{n}(f^{-1}(p(x')); B \mid_{f^{-1}(p(x'))}) \qquad (7.3)$$

$$\cong H^{n}((f')^{-1}(x'); q^{*}B \mid_{(f')^{-1}(x')}) \qquad \text{pullback and } (4.3)$$

$$\cong (R^{n}f'_{*}(q^{*}B))_{x'} \qquad (7.3)$$

### Section 8

8.1 Let X be connected and locally simply connected, and choose a base point  $x_0$ . Show that for any locally constant sheaf (of sets) A, the group  $\pi_1(X, x_0)$  acts on the stalk  $A_{x_0}$ . Show that this gives a functor  $A \mapsto A_{x_0}$  which is an equivalence of categories between the category of locally constant sheaves on X and the category of sets with a  $\pi_1(X, x_0)$ -action.

Let X be connected and locally simply connected and fix  $x_0$  in X. Let  $p:[0,1] \to X$  be a loop in X based at  $x_0$ . Then since A is a locally constant sheaf, p lifts uniquely to a loop  $\tilde{p}$  in EtA for each  $germ_{x_0}(s)$  in the stalk of A at  $x_0$ , that is, in  $\pi^{-1}(x_0)$ , the fiber of  $x_0$ . Since homotopies also lift to homotopies, two homotopic paths  $p_1$  and  $p_2$  lift to homotopic maps  $\tilde{p}_1$  and  $\tilde{p}_2$  in EtA, so that  $\tilde{p}_1(1) = germ_{x_0}(s) = \tilde{p}_2(1)$ . Then we can define an action of  $\pi_1(X,x_0)$  on  $A_{x_0} = \pi^{-1}(x_0)$  by  $p \cdot s = \tilde{p}_s(1)$ , where p is a loop based at  $x_0$ , s is in the fiber of  $x_0$  (ie s in the stalk of  $A_{x_0}$ ) and  $\tilde{p}_s$  is the unique lift of p for s implied by the fact that  $\pi$  is a covering map. Note that if e is the constant loop, then  $\tilde{p}_s(1) = s$ ,  $\forall s$  so that  $e \cdot s = \tilde{p}_s(1) = s$