Preface

These are the notes of a course on sheaf cohomology, taught by Ieke Moerdijk at Utrecht University in the spring of 1992. The course does not assume familiarity with sheaf theory, and starts with the basics on sheaves and étale spaces. This is followed by an extensive treatise of cohomology, first with arbitrary support and then with compact support. Finally, Verdier duality is discussed. There is an appendix in which the derived category is described in terms of Quillen's theory of closed model structures.

Throughout we have used many definitions and facts from homological algebra with little or no proof. For more details the reader may consult the literature. See, for instance, [2].

1. Sheaves on a Topological Space

1.1. **Sheaves and Presheaves.** Let X be a topological space. A (set-valued) *presheaf* A on X consists of

(1) A set $A(U)$ for each open $U \subseteq X$;
(2) A 'restriction map'

$$\rho_{VU} \colon A(U) \to A(V)$$

for each inclusion of open subsets $V \subseteq U$, such that for $W \subseteq V \subseteq U$,

$$\begin{aligned} \rho_{WV} \circ \rho_{VU} &= \rho_{WU} \\ \rho_{UU} &= 1_{A(U)} \end{aligned}$$

Notation 1.1. For $a \in A(U)$ and $V \subseteq U$, we often write $a|_V$ instead of $\rho_{VU}(a)$.

If each $A(U)$ is an (abelian) group and each ρ_{VU} is a group homomorphism, A is said to be a *presheaf of (abelian) groups*. In a similar way we can define presheaves of R-modules, presheaves of C^*-algebras, etc.

Now let $U = \bigcup_{i \in I} U_i$ be a open cover of the open subset $U \subseteq X$. A family of elements $a_i \in A(U_i)$ is said to be *compatible* if

$$\text{for each pair } i, j \in I, \quad a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}.$$

We call such families also *matching families*. An *amalgamation* for such a family $(a_i)_{i \in I}$ consists of an element $a \in A(U)$ such that $a|_{U_i} = a_i$ for each $i \in I$.

A presheaf is called a *sheaf* (resp. a *separated presheaf*) if each matching family (for any open cover $U = \bigcup_{i \in I} U_i$) has *exactly one* (resp. *at most one*) amalgamation.

1.2. **Morphisms of Presheaves.** A map of (pre-)sheaves of sets on X, $\varphi \colon A \to B$, consists of a family of functions

(1) $$\varphi_U \colon A(U) \to B(U) \qquad \text{for } U \subseteq X \text{ open},$$

such that the following diagram commutes for each inclusion $V \subseteq U$,

$$\begin{tikzcd} A(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho_{VU}"'] & B(U) \arrow[d, "\rho_{VU}"] \\ A(V) \arrow[r, "\varphi_V"'] & B(V) \end{tikzcd} ,$$

i.e., $\varphi_V(a|_V) = \varphi_U(a)|_V$ for all $a \in A(U)$.

This defines the categories of presheaves and sheaves on X (composition and identities are defined componentwise). We denotes these by

$$\mathsf{PSh}\,(X) \quad \text{and} \quad \mathsf{Sh}\,(X).$$

The map φ is an isomorphism if and only if each φ_U is bijective. Note that if $\varphi\colon A \to B$ is an isomorphism of presheaves, then A is a sheaf if and only if B is.

If A and B are sheaves of abelian groups (also called *abelian sheaves*), then a morphism $\varphi\colon A \to B$ is a given by a family φ_U for $U \subseteq X$ open, as in (1), where in addition each φ_U is required to be a group homomorphism. This gives us a category of abelian sheaves on X,

$$\mathsf{Ab}\,(X).$$

This is an abelian category (for the definition of abelian category, see [4, page 194]), as we will see later.

Remark 1.2. Note that each space X gives rise to a category $\mathcal{O}(X)$ with the open subsets of X as objects and a unique arrow $V \to U$ if and only if $V \subseteq U$. Each presheaf on X can also be viewed as a functor $\mathcal{O}(X) \to$ **Sets** and maps between presheaves correspond to natural transformations between the functors in this point of view.

A family of sheaves A^i, $i \in \mathbb{Z}$, with morphisms $d^i\colon A^i \to A^{i+1}$ (which we will also denote by just d), with $d^i \circ d^{i-1} = 0$ for all i (or, $d \circ d = 0$),

$$\cdots \longrightarrow A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \longrightarrow \cdots$$

is called a *cochain complex* of sheaves. However, note that we can also view it as a sheaf of cochain complexes.

A morphism φ of cochain complexes of sheaves is given by a family of morphisms of sheaves,

$$\varphi^i\colon A^i \to B^i,$$

such that the following diagram commutes,

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-2} \arrow[r, "d^{-2}"] \arrow[d, "\varphi^{-2}"'] & A^{-1} \arrow[r, "d^{-1}"] \arrow[d, "\varphi^{-1}"'] & A^0 \arrow[r, "d^0"] \arrow[d, "\varphi^0"'] & A^1 \arrow[r, "d^1"] \arrow[d, "\varphi^1"'] & A^2 \arrow[r] \arrow[d, "\varphi^2"'] & \cdots \\
\cdots \arrow[r] & B^{-2} \arrow[r, "d^{-2}"'] & B^{-1} \arrow[r, "d^{-1}"'] & B^0 \arrow[r, "d^0"'] & B^1 \arrow[r, "d^1"'] & B^2 \arrow[r] & \cdots
\end{tikzcd}$$

This defines the category

$$\mathsf{Ch}\,(X).$$

We will mainly be interested in a subcategory of $\mathsf{Ch}\,(X)$, consisting of those cochain complexes that are bounded below; i.e., $X^i = 0$ for $i \leq i_0$ for some i_0. This category is denoted by

$$\mathsf{Ch}^+\,(X).$$

Examples 1.3. (1) For any open $U \subseteq X$, let $C(U)$ be the set of all continuous real valued functions on U,

$$C(U) = \{f\colon U \to \mathbb{R};\ f \text{ is continuous}\}.$$

This defines a sheaf C on X where ρ_{VU} is given by the usual restriction of functions. Note that this is a sheaf of $\mathbb{R}$-modules.

(2) Let $f\colon E \to X$ be a continuous map. Then we have a *sheaf of sections* denoted by Γ_f (or Γ_c or $\Gamma(-,f)$ or $\Gamma(-,E)$) and defined by

$$\Gamma_f(U) = \{s\colon U \to E;\ s \text{ is continuous}, f \circ s = \mathrm{id}_U\}.$$

In the next section we will see that each sheaf on X is of this form. The reader may check that a map of topological spaces over X,

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & E' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

induces a morphism $\Gamma_\varphi\colon \Gamma_E \to \Gamma_{E'}$ of sheaves, by composition. So Γ may be viewed as a functor from the category of topological spaces over X to $\mathsf{Sh}(X)$. If E and X are C^∞ manifolds and $f\colon E \to X$ is a smooth map there is similarly a sheaf of smooth sections.

(3) For a smooth manifold X, the sheaf Ω^p_X, $p \geq 0$, is defined by taking $\Omega^p_X(U)$ to be the set of smooth p-forms on U. The differential operator gives us a homomorphism,

$$d\colon \Omega^p_X \to \Omega^{p+1}_X,$$

and we obtain a cochain complex of sheaves,

$$\Omega^\bullet_X = \left(\Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \right),$$

which is bounded below; i.e., an object of $\mathsf{Ch}^+(X)$.

(4) If S is a fixed set, we write ΔS for the sheaf defined by taking $\Delta S(U)$ to be the set of locally constant functions $U \to S$. This sheaf is called the *constant sheaf* on X, corresponding to S.

(5) If $f\colon X \to Y$ is a continuous map and A is a sheaf on X, we can define a new sheaf $f_*(A)$ on Y by

$$f_*(A)(V) = A(f^{-1}V)$$

for all open $V \subseteq Y$. (This is a sheaf again, since f^{-1} sends covers to covers.) We will further consider this construction and other related ones in Section 3 below.

2. Stalks, Etale Spaces and Associated Sheaves

2.1. **Definitions.** Let P be a fixed presheaf on a topological space X. For a point $x \in X$, the *stalk* P_x *of* P *at* x is the set of equivalence classes of pairs (U, s), where U is an open neighbourhood of x and $s \in P(U)$. Two such (U, s) and (V, t) are *equivalent*, $(U, s) \sim (V, t)$, if and only if $s|_W = t|_W$ for some neighbourhood $W \subseteq U \cap V$ of x. The equivalence class of (U, s) is denoted by $germ_x(s)$ and is called the *germ of* s *at* x. The stalk P_x is thus the set of germs at x:

$$P_x = \{germ_x(s);\ x \in U,\ s \in P(U)\}.$$

Remark 2.1. For those who have seen more category theory, note that the stalk at x can also be seen as a colimit,

$$P_x = \varinjlim_{U \ni x} P(U).$$

2.2. **Construction of the Etale Space.** From the disjoint union $\coprod_{x\in X} P_x$ of the stalks of P we form a topological space

$$\mathsf{Et}\,(P),$$

called the étale space of P. Its points are germs and its topology is generated by the subsets $B(s)$ of $\mathsf{Et}\,(P)$ defined by

$$B(s) = \{germ_x(s);\, x \in U\},$$

where $U \subset X$ is open and $s \in P(U)$. It is easy to check that these subsets form a basis for a topology on $\mathsf{Et}\,(P)$.

There is an evident projection map

$$\pi_P\colon \mathsf{Et}\,(P) \to X, \qquad \pi_P(germ_x(s)) = x.$$

When P is clear from the context we will also write π for π_P. We will show that this projection map is a local homeomorphism in the sense of the following definition.

Definition 2.2. A function $\pi\colon E \to X$ between topological spaces is called a *local homeomorphism* (or, *étale* map) if for each point $\xi \in E$ there are open neighbourhoods B of ξ and U of $\pi(\xi)$ such that π restricts to a homeomorphism $B \xrightarrow{\sim} U$.

Note that in particular, all étale maps are continuous and open. Some examples are:

(1) All covering projections.
(2) Note that when X is Hausdorff, E not be. For instance, the projections from the double origin non-Hausdorff real line and the branching line (with double branching point) to the real line are both étale.

Lemma 2.3. *The map* $\pi\colon \mathsf{Et}\,(P) \to X$ *is a local homeomorphism.*

Proof. We claim that for each open $U \subseteq X$ and $s \in P(U)$, the restriction of π to the basis open $B(s)$ is a homeomorphism,

$$\pi|_{B(s)}\colon B(s) \xrightarrow{\sim} U. \tag{2}$$

To see this, note first that if $germ_y(t) \in B(s)$, then $germ_y(t) = germ_x(s)$ for some $x \in U$. Hence, $x = y$ and $s|_W = t|_W$ for some sufficiently small neighbourhood W of x. In particular, for $s \in P(U)$ and $t \in P(V)$,

$$B(s) \cap B(t) = B(s|_{W^*}),$$

where $W^* = \bigcup\{W \subseteq U \cap V;\, s|_W = t|_W\}$ is the largest set on which s and t 'agree'. So each open subset of $B(s)$ is of the form $B(s|_W)$ for some open subset $W \subseteq U$.

Now clearly the map $\pi|_{B(s)}$ is bijective, since it has the inverse $x \mapsto germ_x(s)$. Furthermore, $\pi|_{B(s)}$ is continuous since $\pi^{-1}(V) = B(s|_V)$ for any open subset $V \subseteq U$. Also, $\pi|_{B(s)}$ is open, since each open subset of $B(s)$ is of the form $B(s|_W)$ with $W \subseteq U$ open, as we saw before, and $\pi(B(s|_W)) = W \subseteq U$. This concludes our proof of this lemma. □

2.3. **The Associated Sheaf.** We have thus far introduced two operations. For a presheaf P on X we have constructed its étale space over X,

$$\pi\colon \mathsf{Et}\,(P) \to X.$$

And to a topological space over X, $f\colon E \to X$, we have assigned its sheaf of sections Γ_f on X (see Example 1.3(2)).

Definition 2.4. The sheaf $\Gamma \mathsf{Et}\,(P)$ of sections of $\pi\colon \mathsf{Et}\,(P) \to X$ is called the *associated sheaf* of the presheaf P.

There is a canonical map of presheaves

$$\eta_P\colon P \to \Gamma \mathsf{Et}\,(P),$$

defined by

$$\eta_{P,U}(s)(x) = germ_x(s). \tag{3}$$

(When the presheaf P is clear from the context, we will write η instead of η_P.)

Proposition 2.5. *This map η_P is an isomorphism if and only if P is a sheaf.*

Proof. If η is an isomorphism then P must be a sheaf since, $\Gamma \mathsf{Et}\,(P)$ is.
Conversely, suppose that P is a sheaf, then for each open $U \subseteq X$,

(1) η_U *is injective:* let $s, t \in P(U)$ such that $\eta_U(s) = \eta_U(t)$; i.e., $germ_x(s) = germ_x(t)$ for all $x \in U$. Then for each x in U there is an open neighbourhood W_x such that $s|_{W_x} = t|_{W_x}$. Write $t_x \in P(W_x)$ for this element. Note that these t_x with $x \in U$ form a compatible family for the cover $U = \bigcup_{x\in U} W_x$, and both s and t form an amalgamation for this family. Hence, $s = t$ since P is a sheaf.
(2) η_U *is surjective:* Let $\sigma\colon U \to \mathsf{Et}\,(P)$ be a section of π_P. Thus, for each $x \in U$ we have,
$$\sigma(x) = germ_x(s_x),$$
where $s_x \in P(W_x)$ for some neighbourhood W_x of x. The section σ is a continuous map and hence, for each x there exists a neighbourhood $V_x \subseteq W_x$ of x such that $\sigma(y) = germ_y(s_x)$ for all $y \in V_x$. These $s_x|_{V_x}$ form a compatible family as in the previous part of this proof, so they have a unique amalgamation, say $s \in P(U)$. We conclude that $\eta_U(s) = \sigma$ and hence, η is indeed surjective.

□

Next, let $f\colon E \to X$ be any map, let ΓE be its sheaf of sections and $\mathsf{Et}\,(\Gamma E)$ be the corresponding étale space. In this case there is a canonical 'evaluation map',

$$\begin{tikzcd} \mathsf{Et}\,(\Gamma E) \arrow[rr, "\varepsilon_E"] \arrow[dr, "\pi"'] & & E \arrow[dl, "f"] \\ & X & \end{tikzcd},$$

defined by

$$\varepsilon_E(germ_x(s)) = s(x).$$

This map is readily seen to be continuous. Furthermore, it is a map 'over X' in the sense that the above triangle commutes, $f \circ \varepsilon_E = \pi_{\Gamma E}$.

Proposition 2.6. *This evaluation map $\varepsilon_E\colon \mathsf{Et}\,(\Gamma E) \to E$ is a homeomorphism if and only if $f\colon E \to X$ is a local homeomorphism.*

In the proof of this proposition we will use the following lemma.

Lemma 2.7. *For a commutative triangle of topological spaces and continuous maps,*

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

if f and π are étale, so is h (in particular, h is open).

Proof. **Exercise 1** □

Proof of Proposition 2.6. The '$\Rightarrow$' direction is clear. For the '$\Leftarrow$' direction, suppose that f is étale. The map ε_E is continuous and open by Lemma 2.7, so we only need to show that it is bijective.

ε_E *is one-to-one*: Let $germ_x(s)$, $germ_y(t) \in \mathsf{Et}\,(\Gamma E)$ be any two points, represented by sections of f, $s\colon U_x \to E$ and $t\colon U_y \to E$ respectively, on open neighbourhoods U_x of x and U_y of y. Suppose that $\varepsilon_E(germ_x(s)) = \varepsilon_E(germ_y(t))$. By definition of ε_E, $s(x) = t(y)$ and hence, $x = fs(x) = ft(y) = y$. Since f is étale, there is a neighbourhood W of $s(x) = t(y)$ such that f restricts to a homeomorphism $W \xrightarrow{\sim} W_x$ for some neighbourhood W_x of x. Since s and t are continuous, there is a smaller neighbourhood $N_x \subseteq W_x \cap U_x \cap U_y$ of x such that s and t both map N_x into W. But then $s|_{N_x} = t|_{N_x}$ since $fs = \mathrm{id} = ft$ and f is a homeomorphism when restricted to W. Thus, $germ_x(s) = germ_y(t)$.

ε_E *is onto*: Let $e \in E$ be any point. We will construct a neighbourhood U of $x = f(e)$ and a section $s \in \Gamma E(U)$ such that $\epsilon_E(germ_x(s)) = s(x) = e$. Note that e has a neighbourhood V_e such that $f|_{V_e}\colon V_e \to f(V_e)$ is a homeomorphism onto an open subset $f(V_e) \subseteq X$, containing x. Now let $U = f(V_e)$ and $s = (f|_{V_e})^{-1}\colon V_e \to V_e \subseteq E$. Then, $s(x) = e$ as required. □

Proposition 2.8. *For a map $f\colon E \to X$ and a presheaf P on X there is a bijective correspondence between maps of presheaves*

$$\varphi\colon P \to \Gamma E$$

and continuous maps

$$g\colon \mathsf{Et}\,(P) \to E$$

over X.

Proof. Given φ, define g by

$$g(germ_x(s)) = \varphi_U(s)(x), \tag{4}$$

where U is an open neighbourhood of x and $s \in P(U)$. We leave it to the reader to check that g is well-defined and continuous.

Conversely, given the map g, define for an open $U \subseteq X$ and $s \in P(U)$,

$$\varphi_U(s)(x) = g(germ_x(s)).$$

It is straightforward to check that the family φ_U commutes with the restriction maps and hence define a map φ of presheaves. It is also straightforwad to check that the assignments $\varphi \mapsto g$ and $g \mapsto \varphi$ are mutually inverse. □

Corollary 2.9. *Let P and Q be presheaves on X and assume that Q is a sheaf. Then there is a bijective correspondence between presheaf maps*

$$\varphi\colon P \to Q$$

and continuous maps

$$g\colon \mathsf{Et}\,(P) \to \mathsf{Et}\,(Q).$$

Proof. Since $\mathsf{Et}\,(Q) \to X$ is étale by Lemma 2.7, Proposition 2.8 gives us the correspondence between morphisms $P \to \Gamma\mathsf{Et}\,Q$ and $\mathsf{Et}\,P \to \mathsf{Et}\,Q$. However, $\Gamma\mathsf{Et}\,Q \cong Q$ by Proposition 2.6. $\square$

Corollary 2.10. *For a map $\varphi\colon P \to Q$ between presheaves on X, if Q is a sheaf then there exists a unique map of sheaves $\hat{\varphi}$ such that $\hat{\varphi} \circ \eta = \varphi$,*

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\eta"'] & Q \\ \Gamma\mathsf{Et}\,P \arrow[ur, dashed, "!"', "\hat{\varphi}"'] & \end{tikzcd}$$

Proof. By Corollary 2.9, $\varphi\colon P \to Q$ corresponds to a map $g\colon \mathsf{Et}\,P \to \mathsf{Et}\,Q$, and by the descriptions of g in (4) and η in (3), and the fact that $\eta_Q\colon Q \to \Gamma\mathsf{Et}\,Q$ is an isomorphism we conclude that g is the unique map with $\eta_Q^{-1} \circ \Gamma g \circ \eta_P = \varphi$ as in the following square:

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[d, "\eta_P"'] & & Q \arrow[d, "\cong"', "\eta_Q"] \\ \Gamma\mathsf{Et}\,(P) \arrow[rr, dashed, "\Gamma\mathsf{Et}\,(\varphi)=\Gamma g"'] & & \Gamma\mathsf{Et}\,(Q) \end{tikzcd}$$

Define $\hat{\varphi} = \eta_Q^{-1} \circ \Gamma g$. $\square$

2.4. **Summary.** Consider the category $\mathbf{Spaces}/X$ (respectively $\mathbf{EtSpaces}/X$) of topological spaces (resp. étale spaces) over X. Then the previous propositions tell us that we have an adjunction (see [4, p. 80]),

$$\mathbf{Spaces}/X \underset{\mathsf{Et}}{\overset{\Gamma}{\rightleftarrows}} \mathsf{PSh}\,(X), \quad \mathsf{Et}\dashv \Gamma.$$

This adjunction restricts to an equivalence of categories [4, p. 18 and p. 91],

$$\mathbf{EtSpaces}/X \underset{\mathsf{Et}}{\overset{\Gamma}{\rightleftarrows}} \mathsf{Sh}\,(X).$$

Exercise 2

(a) Show that Corollary 2.10 determines the associated sheaf up to isomorphism. That is, if R is a sheaf and there exists a map $\psi\colon P \to R$ with the same property as η in Corollary 2.10, then $R \cong \Gamma\mathsf{Et}\,(P)$.

(b) If P is a subpresheaf of R and R is a sheaf, then let

$$\tilde{P}(U) = \{r \in R(U) \mid \text{for each } x \in U \text{ there is a neighbourhood } W_x \subseteq U \text{ of } x \text{ such that } (r|_{W_x}) \in P(W_x)\}.$$

Show that $P \subseteq \tilde{P} \subseteq R$, $\tilde{P}$ is a sheaf and $P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10; hence, $\tilde{P}$ is the associated sheaf for P.

3. Change of Base

3.1. **Direct and Inverse Image Sheaves.** In this section we study the effect of a continuous map $f\colon X \to Y$ of topological spaces on the categories of sheaves $\mathsf{Sh}\,(X)$ and $\mathsf{Sh}\,(Y)$. In Section 1 we defined already, in Example 1.3(5), the sheaf $f_*(A)$ on Y for each sheaf A on X. This is called the *direct image* of A under f. (Recall that $f_*(A)(U) = A(f^{-1}(U))$.)

In the opposite direction, if B is a sheaf on Y, we can form an *inverse image sheaf* $f^*(B)$ on X, using the following lemma.

Lemma 3.1. *The pullback of an étale map is also étale. That is, given a pullback diagram,*

$$\begin{tikzcd} X \times_Y E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd},$$

then π_1 is étale if p is.

Proof. **Exercise 1** □

To define $f^*(B)$, we first take the étale space $\pi_1\colon \mathsf{Et}\,(B) \to Y$. Then, taking the pullback along f, we obtain the projection map,

$$\pi_1\colon X \times_Y \mathsf{Et}\,(B) \to X.$$

This map is étale by the previous lemma. Next, take its sheaf of sections $\Gamma(X \times_Y \mathsf{Et}\,(B))$. This is the sheaf $f^*(B)$.

Exercise 2 Verify the following equality of stalks:

$$(f^*B)_x \cong B_{f(x)},$$

for each $x \in X$.

We leave it to the reader to verify that a morphism $\varphi\colon A_1 \to A_2$ of sheaves on X induces a morphism $f_*\varphi\colon f_*A_1 \to f_*A_2$ of sheaves on Y and analogously for f^*. So we consider f_* and f^* as functors between the categories of sheaves on X and Y,

$$\begin{tikzcd} \mathsf{Sh}\,(X) \arrow[r, shift right, "f_*"'] & \mathsf{Sh}\,(Y) \arrow[l, shift right, "f^*"'] \end{tikzcd}.$$

The following proposition says that f^* is left adjoint to f_*.

Proposition 3.2. (Adjunction Formula) *Let A be a sheaf on X and B a sheaf on Y. Then there is a bijective correspondence between maps φ of sheaves on Y,*

$$\varphi\colon B \to f_*A$$

and maps ψ of sheaves on X,

$$\psi\colon f^*B \to A,$$

which is natural in A and B.

Proof. We will only indicate the correspondence and leave the details as an exercise. First note that $A \cong \Gamma \mathsf{Et}\, A$ since A is a sheaf. Hence, Proposition 2.8 gives us a bijective correspondence between sheaf maps

$$\psi\colon f^*B \to A \cong \Gamma \mathsf{Et}\,(A)$$

and maps of étale spaces over X,

$$g\colon \mathsf{Et}\,(f^*B) = X \times_Y \mathsf{Et}\,(B) \to \mathsf{Et}\,(A).$$

So it suffices to establish a correspondence between the maps $\varphi\colon B \to f^*A$ in the proposition and these g's.

Given φ, define g by

$$g(y, germ_{\,f(y)}(b)) = germ_{\,y}(\varphi_U(b)),$$

where U is an open neighbourhood of $f(y)$ and $b \in B(U)$. It is easy to check that his map is well-defined on equivalence classes.

For the other direction, given $g\colon X \times_Y \mathsf{Et}\,(B) \to \mathsf{Et}\,(A)$, define $\varphi\colon B \to f_*A$ as follows. Let $U \subseteq Y$ be an open subset and $b \in B(U)$, we want to define $\varphi_U(b) \in f_*A(U) = A(f^{-1}(U))$. By Proposition 2.5 it is sufficient to define a section

$$\bar{b}\colon f^{-1}(U) \to \mathsf{Et}\,(A)$$

of the étale space $\mathsf{Et}\,(A) \to X$ corresponding to the sheaf A. Now, for $y \in f^{-1}(U)$ define

$$\bar{b}(y) = g(y, germ_{\,f(y)}(b)) \in \mathsf{Et}\,(B).$$

□

Remark 3.3. If A is an abelian sheaf, then $\mathsf{Et}\,(A) \to X$ is an étale space with a continuous abelian group structure on the fibers, given by the continuous map

$$\mathsf{Et}\,(A) \times_X \mathsf{Et}\,(A) \xrightarrow{\bullet} \mathsf{Et}\,(A)$$

over X. Conversely, an étale space with this structure gives rise to an abelian sheaf of sections. Thus, if A is an abelian sheaf on X, then f^*A is an abelian sheaf on Y and Proposition 3.2 gives a correspondence of homomorphisms of abelian sheaves,

$$\mathsf{Hom}_{\mathsf{Ab}\,(Y)}(B, f_*A) \cong \mathsf{Hom}_{\mathsf{Ab}\,(X)}(f^*B, A).$$

(We leave it to the reader to verify that in Proposition 3.2, φ is a morphism preserving the group structure if and only if ψ is.)

3.2. **Open Subspaces.** In this and the following sections we will consider the special case where X is a subspace of Y. For this section we will assume that X is an *open* subspace of Y and write $i\colon X \hookrightarrow Y$ for the inclusion map. Then each sheaf B on Y restricts to a sheaf $B|_X$ on X in the evident way: if $U \subseteq X$ is open in X, it is open in Y and we define

$$B|_X(U) = B(U).$$

The reader may check that

$$B|_X \cong i^*(B).$$

(Hint: check that $\mathsf{Et}\,(B|_X) \cong X \times_Y \mathsf{Et}\,(B)$.)

Conversely, if we have a sheaf A on the open subspace X, then $i_*(A)$ is the sheaf on Y which 'extends' A in the sense that

$$i_*(A)|_X = A.$$

(This follows immediately from the definition: $i_*(A)(V) = A(i^{-1}(V)) = A(V \cap X)$.) To compute the stalks of $i_*(A)$, first note that as a general fact, if S is any sheaf on X then for the empty subset $\emptyset \subseteq X$ we have that $S(\emptyset)$ is a singleton (the unique section on the emptyset). We write this as $\{0\}$. Thus, for the stalk of i_*A at any point y we have

$$i_*(A)_y = \begin{cases} A_y & \text{if } y \in X, \\ \{0\} & \text{if } y \notin \overline{X}. \end{cases}$$

(Here we write $\overline{X}$ for the closure of X in Y.) However, the description of $i_*(A)_y$ is more complicated if y lies on the boundary of X.

There is yet another way to 'extend' a sheaf A on X to a sheaf on Y. This construction is easiest described in terms of étale spaces. Note that the sheaf A gives rise to the étale space $\pi_A\colon \mathsf{Et}\,(A) \to X$, but $i \hookrightarrow X$ is also étale. Hence, the composition

$$i \circ \pi_A\colon \mathsf{Et}\,(A) \to Y$$

is étale and corresponds to a sheaf on Y, which we denote by $i_!(A)$. In other words, for $V \subseteq Y$ open,

$$i_!(A)(V) = \begin{cases} A(V) & \text{if } V \subset X \\ \emptyset & \text{otherwise.} \end{cases}$$

We also have an adjunction formula for $i_!$.

Exercise 3 Prove that

$$\mathsf{Hom}_{\mathsf{Sh}\,(Y)}(i_!A, B) \cong \mathsf{Hom}_{\mathsf{Sh}\,(X)}(A, i^*B)$$

where $i\colon X \hookrightarrow Y$ is an open inclusion.

Note that the construction above cannot be used for abelian sheaves. If A is abelian, i_*A is abelian as well, but in general this is not the case for $i_!A$. To define $i_!A$ as an abelian sheaf and retain the adjunction formula, we first define a presheaf $\hat{A}$ on Y:

$$\hat{A}(V) = \begin{cases} A(V) & \text{if } V \subseteq X \\ \{0\} \text{ (the trivial group)} & \text{otherwise.} \end{cases}$$

Then let $i_!A$ be the associated sheaf of $\hat{A}$. To give a more explicit description of $i_!A$ we will make use of the notion of the *support* of an element.

Definition 3.4. If A is an abelian sheaf on a topological space X and $a \in A(U)$ where $U \subseteq X$ is open, then

$$\begin{aligned} \mathsf{supp}(a) &= U - \bigcup\{W \subseteq U;\ W \text{ open, and } a|_W = 0\} \\ &= \{x \in U;\ \mathit{germ}_x(a) \neq \mathit{germ}_x(0)\}. \end{aligned}$$

Note that $\mathsf{supp}(a)$ is a closed subset of U for each $a \in A(U)$. We can now define for $A \in \mathsf{Ab}\,(X)$ and $V \subseteq Y$ open,

(5)

$$\begin{aligned} i_!(A) = \{\hat{a};\ a \in A(V \cap X) &\text{ and there is an open } N \subseteq V \text{ such that } V = X \cup N \\ &\text{and } a|_{N \cap X} = 0\} \\ = \{\hat{a};\ a \in A(V \cap X), &\ \mathsf{supp}(a) \text{ is closed in } V\}. \end{aligned}$$

Note that this definition makes sense since $\mathsf{supp}(a)$ is closed as a subspace of $X \cap V$, but need not be closed in V. The restriction maps are inherited from A: if $U \subseteq V$

and $\hat{a} \in i_!A(V)$, then $\hat{a}|_U$ (in $i_!(A)$) is the element corresponding to $a|_{X\cap U}$ (in A). We remark that if $U \cap X = \emptyset$ then $\hat{a}|_U = 0$ by definition. Further, note that $i_!A$ thus defined is a sheaf. The reader can either take (5) as the definition of $i_!A$ or prove that it is indeed the associated sheaf of the presheaf $\hat{A} \subseteq i_*A$ (use Exercise 2(b) in Section 2.4).

We conclude that we have two functors, both denoted by $i_!$:

$$i_!\colon \mathsf{Sh}\,(X) \to \mathsf{Sh}\,(Y),$$

and

$$i_!\colon \mathsf{Ab}\,(X) \to \mathsf{Ab}\,(Y).$$

The second $i_!$ naturally extends to chain complexes

$$i_!\colon \mathsf{Ch}(X) \to \mathsf{Ch}(Y).$$

If $A \in \mathsf{Ab}\,(X)$ and $a \in A(V \cap X)$ for $V \subset Y$ open, then $0 \in A(V - \mathsf{supp}(a))$ and a are compatible elements for the cover $V = (V - \mathsf{supp}(a)) \cup (X \cap V)$. Their amalgamation gives us the extension $\hat{a} \in i_!A(V)$ of a. This is also called the 'extension by zero' of a. This construction gives for each sheaf $B \in \mathsf{Ab}\,(Y)$ a map

$$\varepsilon_B\colon i_!i^*(B) \to B \tag{6}$$

This maps is defined by: let $b \in i^*(B)(X \cap V) = B(X \cap V)$; then $\mathsf{supp}(b)$ is closed in V, so define

$$\varepsilon_{B,V}(b) = \hat{b}.$$

Note that $\varepsilon_{B,V}$ is a monomorphism. This is the counit of the adjunction given by the natural correspondence between morphisms $\varphi\colon i_!A \to B$ and $\psi\colon A \to i^*B$ for $A \in \mathsf{Ab}\,(X)$ and $B \in \mathsf{Ab}\,(Y)$. This correspondence can be established using extension by zero.

So in both cases (for sheaves of sets and for abelian sheaves) we have the following chain of adjunctions,

$$i_! \dashv i^* \dashv i_*,$$

where $\dashv$ stands for '*is left adjoint to*'.

3.3. **Closed Subspaces.** Let $Z \subseteq Y$ be a closed subspace and write $j\colon Z \hookrightarrow Y$ for the inclusion map. Again, we consider the functors

$$j_*\colon \mathsf{Sh}\,(Z) \leftrightarrows \mathsf{Sh}\,(Y) : j^*$$

and

$$j_*\colon \mathsf{Ab}\,(Z) \leftrightarrows \mathsf{Ab}\,(Y) : j^*$$

in a bit more detail. The morphism j_* is easily described: let A be a sheaf on Z and $U \subseteq Y$ an open subset, then

$$j_*A(U) = A(U \cap Z),$$

and for the stalk at a point $y \in Y$ we have

$$(j_*A)_y = \begin{cases} A_y & \text{if } y \in Z \\ \{0\} & \text{if } y \notin Z. \end{cases}$$

For a sheaf B on Y, we again denote j^*B by $B|_Z$. For a point $z \in Z$ we have

$$(B|_Z)_z = B_z.$$

Thus, for any sheaf A on Z we see that

$$(j_*A)|_Z \cong A,$$

and j_*A may again be regarded as an 'extension' of A. However, this time we are able to extend the chain of adjunctions on the other side by an additional functor,

$$j^! \colon \mathsf{Ab}\,(Y) \to \mathsf{Ab}\,(Z).$$

The adjunction formula becomes

$$\mathsf{Hom}_{\mathsf{Ab}\,(Y)}(j_*A, B) \cong \mathsf{Hom}_{\mathsf{Ab}\,(Z)}(A, j^!B). \tag{7}$$

Thus, there is a chain of adjunctions as follows:

$$j^* \dashv j_* \dashv j^!.$$

We define $j^!B$ as follows. Let $V \subseteq Z$ be open and choose an open $U \subseteq Y$ such that $V = U \cap Z$. (There is a canonical choice, $U = V \cup (Y - Z)$, since Z is closed.) Let

$$\begin{aligned} j^!B(V) &= \{b \in B(U);\, \mathsf{supp}(b) \subseteq Z\} \\ &= \{b \in B(U);\, b|(U - Z) = 0\}. \end{aligned}$$

Hence, $j^!B(V)$ is the set of those sections over $U \supseteq V$ which coincide with the zero section on $U - Z$ (so this set does not depend on the choice of the open set U). One readily checks that $j^!B$ is a sheaf again.

In the exercise below we will see that $j^!B$ is isomorphic to a subsheaf of j^*B.

Definition 3.5. For two sheaves P and Q on a space X, P is said to be a *subsheaf* of Q if $P(U) \subseteq Q(U)$ for each open subset $U \subseteq X$ and if the restriction maps $P(U) \to P(V)$ (for $U \supseteq V$) agree with those of Q.

Exercise 4

(a) Verify the adjunction formula involving j_* and $j^!$ in (7).
(b) Show that $j^!B$ is isomorphic to a subsheaf of j^*B.

3.4. **Extension to Locally Closed Subspaces.** Let $Z \subseteq X$ be locally closed,. This means that $Z = N \cap F$ for some closed subset F and some open subset N of X. Let $h \colon Z \hookrightarrow X$ be the inclusion map. In this section we will construct adjoint functors

$$h_! \colon \mathsf{Ab}\,(Z) \leftrightarrows \mathsf{Ab}\,(X) : h^!,$$

with the corresponding adjunction formula; i.e., for $A \in Ab(X)$ and $B \in \mathsf{Ab}\,(Z)$,

$$\mathsf{Hom}_{\mathsf{Ab}\,(X)}(h_!B, A) \cong \mathsf{Hom}_{\mathsf{Ab}\,(Z)}(B, h^!A).$$

We will use the functors $i_!$ and $j^!$ from the previous sections in our new construction. We will also see that this new construction extends those of the previous sections: when Z is an open subspace, $h_!$ is the same as $i_!$ in Section 3.2 and $h^* = h^!$, and when Z is a closed subspace, $h^!$ is the same as $j^!$ in Section 3.3 whereas $h_! = h_*$ (see Exercise 5 below).

We first factor the inclusion h into an open and a closed one:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "j"', "\text{closed}"] & & X \\ & N \arrow[ur, "i", "\text{open}"'] & \end{tikzcd}$$

By the results from Sections 3.2 and 3.3 we have adjunctions $j^* \dashv j_* \dashv j^!$ and $i_! \dashv i^* \dashv i_*$. By composition we obtain the adjunction,

$$i_! j_* \dashv j^! i^*.$$

This corresponds to the following composition of adjunction formulas for A and B as before:

$$\mathsf{Hom}_{\mathsf{Ab}\,(X)}(i_! j_* B, A) \cong \mathsf{Hom}_{\mathsf{Ab}\,(N)}(j_* B, i^* A) \cong \mathsf{Hom}_{\mathsf{Ab}\,(Z)}(B, j^! i^* A).$$

Thus, we can define $h_! = i_! j_*$ and $h^! = j^! i^*$. Explicitly written out for A and B as above and open subsets $U \subseteq X$ and $V \subseteq Z$, this becomes,

$$h_!(B)(U) = \{b \in B(Z \cap U);\ \mathsf{supp}(b) \text{ is closed in } U\}$$

and

$$h^!(A)(V) = \{a \in A(V \cup (N - Z));\ \mathsf{supp}(a) \subseteq Z\}.$$

Exercise 5 Let $h\colon \hookrightarrow X$ be an inclusion map.

(a) Show that there are natural isomorphisms (see [4, p.16])
- $h_* h^* \cong \mathrm{id} \cong h_! h^*$ if h is open;
- $h_* h^* \cong h_* h^!$ if h is closed.

(b) If $h\colon Z \hookrightarrow X$ is locally closed, prove that the definitions of $h_!$ and $h^!$ do not depend on the choice of the factorization $h = i \circ j$. Also, prove that $h_! h^! \cong \mathrm{id}$.

(c) Conclude that for a locally closed subspace $h\colon Z \hookrightarrow X$,
- $h_! \cong h_!$ and $h^! \cong h^*$ if h is open;
- $h_! \cong h_*$ and $h^! \cong h^!$ if h is closed.

Note that the first and last of these isomorphisms are not entirely tautological: when h is open, $h_!$ has been defined separately for h as an open map and h as a locally closed map, and analogously for $h^!$ in the closed case.

(d) For two composable locally closed inclusions $W \overset{k}{\hookrightarrow} Z \overset{h}{\hookrightarrow} X$, show that $h_! k_! \cong (hk)_!$ and $k^! h^! \cong (hk)^!$.

4. Cohomology

In this section 'sheaf' always means 'abelian sheaf on the space X'.

4.1. **Exactness.**

A sequence of maps of sheaves

$$\cdots \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow \cdots$$

is said to be *exact at B* if for each point $x \in X$ the sequence of induced maps on the stalks

$$\cdots \longrightarrow A_x \xrightarrow{\varphi_x} B_x \xrightarrow{\psi_x} C_x \longrightarrow \cdots$$

is exact at B_x as a sequence of abelian groups; i.e., $\mathsf{ker}(\psi_x) = \mathsf{im}(\psi_x)$. The whole sequence is called exact if it is exact at every sheaf in the sequence.

4.2. **Kernel, Cokernel and Sum.** It is also possible to define exactness of a sequence of sheaves directly in terms of the kernels and cokernels of the sheaf maps, provided they are appropriately defined (within the category of sheaves). For a map $\alpha\colon F \to G$ of sheaves, the *kernel*, $\ker(\alpha) \subseteq F$, can be defined in the 'naive' way,

$$\ker(\alpha)(U) = \{f \in F(U);\ \alpha_U(f) = 0\}$$

for every open U in X. The reader may check that this defines a sheaf with the universal property of a kernel (see [4, p.187]). That is, a sequence $0 \to A \xrightarrow{\varphi} B$ is exact at A if and only if $0 \longrightarrow A(U) \xrightarrow{\varphi_U} B(U)$ is exact for all open $U \subseteq X$ and this is the case if and only if $\ker(\varphi) = 0$. Then φ is called a *monomorphism*.

A similar naive definition for the *image* does not work, because in general it produces a presheaf rather than a sheaf. Instead, we define for a map $\alpha\colon F \to G$,

$$\mathsf{im}(\alpha)(U) = \{g \in G(U);\ \text{each } x \in U \text{ has an open neighbourhood } V_x \text{ such that } g|_{V_x} = \alpha_{V_x}(f) \text{ for some } f \in F(V_x)\}.$$

(This is the associated sheaf for the presheaf given by the naive 'pointwise' definition; see Section 2.4 Exercise 2(b).) So $\mathsf{im}(\alpha)$ consists of elements of G that are *locally* in the image of α. The reader may check that a sequence $B \xrightarrow{\psi} C \to 0$ is exact if and only if $\mathsf{im}(\psi) \cong C$. In that case, ψ is called an *epimorphism*. With this definition we also have that $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is exact at B precisely when $\ker(\psi) = \mathsf{im}(\varphi)$.

We can similarly define the *cokernel* of a sheaf map $\alpha\colon F \to G$, by taking the associated sheaf. First define the presheaf P by

$$P(U) = \mathsf{coker}(F(U) \xrightarrow{\alpha_U} G(U)), \qquad \text{for } U \subseteq X \text{ open}.$$

Then define the sheaf $\mathsf{coker}(\alpha)$ to be the associated sheaf of P. But there is also another way to describe this sheaf. Start with the étale space $\mathsf{Et}\,(G)$ and define the equivalence relation R on this space by $\mathit{germ}_x(g_1) \sim \mathit{germ}_y(g_2)$ if and only if $x = y$ and $g_1|_W - g_2|_W = \alpha_W(f)$ for some neighbourhood W of x and some $f \in F(W)$. Note that $R \subseteq \mathsf{Et}\,(G) \times_X \mathsf{Et}\,(G)$ is an open subspace and the quotient $\mathsf{Et}\,(G)/R$ is again étale over X. Now $\mathsf{coker}(\alpha)$ is the sheaf of sections of $\pi\colon \mathsf{Et}\,(G)/R \to X$.

Exercise 1

(a) Show that with the notation as above, $\Gamma(-,\pi)$ is indeed isomorphic to the sheafification of P.
(b) Show that $\mathsf{coker}(\alpha)$ has the following universal property: given any other sheaf H, a map $\chi\colon G \to H$ factors through $\mathsf{coker}(\alpha)$ if and only if $\chi \circ \alpha = 0$.
(c) Show that $\mathsf{coker}(\alpha)_x = \mathsf{coker}(\alpha_x)$.

For two sheaves F and G, their *direct sum* $F \oplus G$ can be defined in a naive way as

$$(F \oplus G)(U) = F(U) \oplus G(U),$$

since this definition yields a sheaf.

Exercise 2 Show that $\mathsf{Ab}\,(X)$ is an abelian category with kernel, cokernel and sum defined as above (see [4, p.194]).

4.3. **Injectives.** A sheaf I is said to be *injective* if for any injective sheaf map $i\colon A \rightarrowtail B$, any map $\varphi\colon A \to I$ can be extended to a map $\bar{\varphi}\colon B \to I$,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\varphi"'] & B \arrow[dl, dashed, "\bar{\varphi}"] \\ I & . \end{tikzcd}$$

A *resolution* of a sheaf A is an exact sequence,

$$0 \to A \to B^0 \to B^1 \to B^2 \to \cdots$$

It is said to be an *injective resolution* if all B^p for $p \geq 0$ are injective. We usually denote each sheaf map $B^p \to B^{p+1}$ in the resolution by d.

Lemma 4.1. *For each sheaf A on X there exists an injective sheaf map $A \rightarrowtail I$ into an injective sheaf.* ($\mathsf{Ab}\,(X)$ *is said to have* 'enough injectives'.)

Proof. The usual category of abelian groups has enough injectives (see [2, p. 32, Proposition 7.4]). So for each $x \in X$ we can embed the stalk A_x into an injective abelian group I_x by $\sigma_x\colon A_x \hookrightarrow I_x$. Now define the sheaf I by

$$I(U) = \prod_{x \in U} I_x.$$

A can be embedded into I by the map $\sigma\colon A \to I$ defined by

$$(\sigma_U(a))_x = \sigma_x(germ_x(a))$$

for all open $U \subseteq X$ and $a \in A(U)$. Now for any sheaf B a map $\varphi\colon B \to I$ is completely determined by a collection of homomorphisms $\varphi_x\colon B_x \to I_x$ (for $x \in X$). It follows easily that I is injective (just extend the induced stalk maps). For a different proof, see the appendix. $\square$

It follows now by a standard argument that each sheaf A has an injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

Namely, embed A is an injective sheaf by $\sigma\colon A \rightarrowtail I^0$, then embed $\mathsf{coker}(\sigma)$ into an injective sheaf, $\mathsf{coker}(\sigma) \rightarrowtail I^1$. Then we have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\sigma"] & I^0 \arrow[rr, dashed] \arrow[dr] & & I^1 \\ & & & \mathsf{coker}(\sigma) \arrow[ur] & \end{tikzcd}$$

and you can continue by embedding $\mathsf{coker}(I^0 \to I^1)$ into an injective I^2, etc.

4.4. **Cohomology.** Consider the *global sections functor*,

$$\Gamma\colon \mathsf{Ab}\,(X) \to \textbf{Abelian Groups}, \qquad A \mapsto A(X).$$

For each $A \in \mathsf{Ab}\,(X)$, choose an injective resolution,

$$0 \to A \to I^0 \to I^1 \to \cdots$$

Then the n-th cohomology group of X with values in A is defined as

$$\begin{aligned} H^n(X; A) &= H^n(\Gamma I^\bullet) \\ &= \mathsf{ker}(\Gamma I^n \to \Gamma I^{n+1}) / \mathsf{im}(\Gamma I^{n-1} \to \Gamma I^n), \end{aligned}$$

where $I^m = 0$ if $m < 0$. In other words, $H^n(X; A) = R^n\Gamma A$, the *n-th right derived functor* of Γ, evaluated at A (see [2, p.134]). It follows from general theory on right derived functors that $H^n(X; A)$ does not depend on the choice of the resolution $I^\bullet$. (The reader can also find a proof in the appendix on derived categories.) The following properties of the cohomology groups are easily established.

Proposition 4.2. (a) *$H^0(X; A) \cong \Gamma A$;*
(b) *$H^n(X; I) \cong 0$ for $n > 0$ if I is injective.*

Proof. To prove (a), take an injective resolution

$$0 \to A \overset{\sigma}{\rightarrowtail} I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots \tag{8}$$

Then $H^0(X; A) = \mathsf{ker}(\Gamma I^0 \xrightarrow{\Gamma d} \Gamma I^1)$ (since $\Gamma I^{-1} = 0$). Obviously, if $a \in A(X)$ then $\sigma(a) \in \mathsf{ker}(\Gamma d) = H^0(I^\bullet)$, so $\Gamma A \subseteq H^0(X; A)$. Conversely, if $b \in I^0$ and $db = 0$, then by exactness of (8), $b \in \mathsf{im}(\sigma)$. So there is a cover $X = \bigcup_{j \in J} U_j$ such that $b|_{U_j} = \sigma_{U_j}(a_j)$ for some $a_j \in A(U_j)$. These a_j are compatible, since σ is a monomorphism. Hence, they have an amalgamation $a \in A(X)$ and $\sigma_X(a) = b$. (The latter can be seen as follows: $\sigma_X(a)|_{U_j} = \sigma_{U_j}(a_j) = b|_{U_j}$ and I^0 is a sheaf.) We conclude that $\Gamma A = H^0(X; A)$. To prove (b), consider the injective resolution with $I^0 = I$ and $I^n = 0$ for all $n > 0$,

$$0 \to I \to I \to 0.$$

It is clear that $H^n(X; I) = 0$ for $n > 0$. □

4.5. **Functoriality.** The following proposition describes the effect of sheaf maps $\varphi\colon A \to B$ and continuous maps $f\colon Y \to X$ on the cohomology groups. We see that $H^n(-; -)$ is contravariant in the first coordinate and covariant in the second.

Proposition 4.3. (a) *A map $\varphi\colon A \to B$ induces a group homomorphism*

$$\varphi_*\colon H^n(X; A) \to H^n(X; B);$$

(b) *A map $f\colon Y \to X$ induces a group homomorphism*

$$f^*\colon H^n(X; A) \to H^n(Y; f^*A).$$

Proof. Part (a) follows immediately from Exercise 3 below. To prove part (b), apply f_* to an injective resolution of f^*A,

$$0 \to f^*A \overset{\tau}{\rightarrowtail} J^0 \to J^1 \to \cdots \quad .$$

This yields a sequence,

$$0 \to f_*f^*A \overset{f_*\tau}{\rightarrowtail} f_*J^0 \to f_*J^1 \to \cdots$$

which needs not be exact anymore. However, the f_*J^i are still injective as follows from the adjunction formula for f_* and f^*, and the fact that f^* preserves monomorphisms. Now let

$$0 \to A \overset{\sigma}{\rightarrowtail} I^0 \to I^1 \to I^2 \to \cdots$$

be an injective resolution of A. By Exercise 3 below we can extend the unit $\eta_A\colon A \to f_8f^*A$ of the adjunction $f^* \dashv f_*$ to a map of complexes. (Note that η is the map corresponding to $\mathrm{id}_{f^*A}\colon f^*A \to f^*A$ in the adjunction formula.)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "\eta_A"] & I^0 \arrow[r] \arrow[d, dashed] & I^1 \arrow[r] \arrow[d, dashed] & I^2 \arrow[r] \arrow[d, dashed] & \cdots \\
0 \arrow[r] & f_*f^*A \arrow[r] & f_*J^0 \arrow[r] & f_*J^1 \arrow[r] & f_*J^2 \arrow[r] & \cdots
\end{tikzcd}$$

Apply Γ to this diagram and by Exercise 3 and 4 below we obtain a homomorphism,

$$\begin{aligned}
H^n(X;A) \to \mathsf{ker}(\Gamma f_* J^n \to \Gamma f_* J^{n+1})/\mathsf{im}(\Gamma f_* J^{n-1} \to \Gamma f_* J^n) \\
\cong R^n\Gamma f_*(f^*A) \cong R^n\Gamma(f^*A) \cong H^n(Y, f^*A).
\end{aligned}$$

This completes the proof of the proposition. □

Exercise 3

(a) [2, p.126] Let $0 \to A \hookrightarrow I^0 \to I^1 \to \cdots$ and $0 \to B \to B \hookrightarrow J^0 \to J^1 \to J^2 \to \cdots$ be injective resolutions of A and B respectively. Show that a map $\varphi\colon A \to B$ extends to a map of complexes:

(9)
$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "\varphi"] & I^0 \arrow[r] \arrow[d, dashed, "\varphi^0"] & I^1 \arrow[r] \arrow[d, dashed, "\varphi^1"] & I^2 \arrow[r] \arrow[d, dashed, "\varphi^2"] & \cdots \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots
\end{tikzcd}$$

[Hint: use the exactness of the upper row and the injectivity of the J^i.]

(b) Show that a map of compexes as in (9) induces a homomorphism of cohomology groups

$$H^n(X;A) \to H^n(X;B).$$

Exercise 4

(a) Let $f\colon Y \to X$ be a map of topological spaces. Show that there is natural isomorphism $\Gamma_Y \xrightarrow{\sim} \Gamma_X \circ f_*$,

$$\begin{tikzcd}
\mathsf{Ab}\,(Y) \arrow[rr, "f_*"] \arrow[dr, "\Gamma_Y"'] & & \mathsf{Ab}\,(X) \arrow[dl, "\Gamma_X"] \\
& \textbf{Abelian Groups} &
\end{tikzcd}$$

(b) Show that a natural isomorphism between left exact functors $\tau\colon T_1 \xrightarrow{\sim} T_2$ induces a natural isomorphism $R^nT_1 \xrightarrow{\sim} R^nT_2$ for each n.

4.6. **Facts from Homological Algebra.** We review here a couple of well-known facts from homological algebra that will be used in the remainder of these notes.

(a) If $0 \to A \to B \to C \to 0$ is an exact sequence, there are injective resolutions $0 \to A \to I^\bullet$, $0 \to B \to J^\bullet$ and $0 \to C \to K^\bullet$, which fit into an exact sequence of complexes, $0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0$. Thus,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where vertically we have injective resolutions and the rows are exact. (Take this as an exercise: first choose an injective $C \rightarrowtail K^0$, then take a pullback to obtain P. Now embed A into an injective I^0 and pushout to get Q as in the following diagram,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{po}"] & P \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^0 \arrow[r] & Q \arrow[r] & K^0 \arrow[r] & 0
\end{tikzcd}$$

The last row of this diagram splits, so Q is injective. (Check this!) Write J^0 for Q and continue like this.)

(b) If $0 \to A \to B^0 \to B^1 \to \cdots$ is a resolution (or just any augmented chain complex), it can be completed to an 'injective resolution of this complex':

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B^0 \arrow[r] \arrow[d] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & J^{00} \arrow[r] \arrow[d] & J^{01} \arrow[r] \arrow[d] & J^{02} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & J^{10} \arrow[r] \arrow[d] & J^{11} \arrow[r] \arrow[d] & J^{12} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

So $I^\bullet$ is a resolution of A, $J^{\bullet k}$ is a resolution of B^k for each $k \geq 0$ and each row is exact if $B^\bullet$ is a resolution of A. (See [2, p.301].)

4.7. **The Long Exact Sequence in Cohomology.** In this section we will present some of the basic results on sheaf cohomology.

Proposition 4.4. *An exact sequence of sheaves* $0 \to A \to B \to C \to 0$ *yields a long exact sequence in cohomology,*

$$0 \to H^0(X;A) \to H^0(X;B) \to H^0(X;C) \to H^1(X;A) \to \cdots$$
$$\cdots \to H^n(X;A) \to H^n(X;B) \to H^n(X;C) \to H^{n+1}(X;A) \to \cdots$$

Proof. Take a 'resolution' of the short exact sequence as in Section 4.6 Part (1). By Lemma 4.6 below, $0 \to \Gamma(I^\bullet) \to \Gamma(J^\bullet) \to \Gamma(K^\bullet) \to 0$ is an exact sequence of complexes. Now a standard argument (as in the Algebraic Topology Notes, Appendix A.5) proves the proposition. Do we perhaps want to add a different reference? $\square$

Lemma 4.5. *If A is an injective sheaf, then each restriction map $\rho_{VU}\colon A(U) \to A(V)$ is surjective.*

Proof. For any open $U \subseteq X$, define a sheaf $\mathbb{Z}_U$ by

$$\mathbb{Z}_U(W) = \{f\colon W \to \mathbb{Z};\ f \text{ is locally constant and } f(x) = 0 \text{ if } x \notin U\}$$

for all open $W \subseteq X$. So $\mathbb{Z}_X$ is the constant sheaf $\Delta\mathbb{Z}$, and if $U \supseteq V$ there is an evident inclusion $\mathbb{Z}_V \rightarrowtail \mathbb{Z}_U$. Now sheaf maps $\varphi\colon \mathbb{Z}_U \to A$ correspond bijectively to elements $a_\varphi \in A(U)$. We will often denote this type of correspondences by a horizontal bar,

$$\frac{\varphi\colon \mathbb{Z}_U \to A}{a_\varphi \in A(U)}$$

Indeed, given $\varphi\colon \mathbb{Z}_U \to A$, define $a_\varphi = \varphi_U(1)$, where $1\colon U \to \mathbb{Z}$ is the constant function with value 1. Conversely, given an element $a \in A(U)$, define for each open $W \subseteq X$ and $f \in \mathbb{Z}_U(W)$,

$$\varphi_W(f) = f \cdot a \in A(W).$$

Here $f \cdot a$ is defined in the evident way: if $W = \bigcup_{i\in I} W_i$ with $f|_{W_i} = n_i$, a constant function, then $(f \cdot a)|_{W_i} = (n_i \cdot a)|_{W_i}$, where $n_i \cdot a$ stands for the sum $a+a+a+\cdots+a$ with n_i summands.

Now apply the injectivity of A to

$$\begin{tikzcd}
\mathbb{Z}_V \arrow[r, "\varphi"] \arrow[d] & A \\
\mathbb{Z}_U \arrow[ur, dashed, "\psi"'] &
\end{tikzcd},$$

then the correspondence between $\varphi\colon \mathbb{Z}_U \to A$ and $a_\varphi \in A(U)$ above shows precisely that any $a \in A(U)$ can be extended to an $\tilde{a} \in A(U)$. $\square$

Lemma 4.6. *If in the exact sequence* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *the sheaf A is injective, then* $0 \to \Gamma A \xrightarrow{\Gamma\varphi} \Gamma B \xrightarrow{\Gamma\psi} \Gamma C \to 0$ *is exact.*

Proof. Exactness at ΓA is clear ($\Gamma\varphi\colon \Gamma A \to \Gamma B$ is again a monomorphism), and exactness at ΓB can be shown in the same way as in the proof of Proposition 4.2(1). To prove that $\Gamma\psi\colon \Gamma B \to \Gamma C$ is surjective, choose $c \in \Gamma C = C(X)$. By Zorn's Lemma there exists a maximal open subset $U \subseteq X$ for which there is a

$b \in B(U)$ with $\psi(b) = c|_U$. Suppose that $U \neq X$. Pick $x \notin U$. By exactness of $0 \to A \to B \to C \to 0$ at C there is a neighbourhood W of x and an element $b_x \in B(W)$ such that $\psi_W(b_x) = C|_W$. Then $\psi(b_x|_{W\cap U}) = \psi(b|_{W\cap U})$. Note that $0 \to A(W\cap U) \to B(W\cap U) \to C(W\cap U) \to 0$ is exact at $B(W\cap U)$ (the proof is the same as that for global sections). Hence, $(b|_{W\cap U}) - (b_x|_{W\cap U}) = \varphi(a)$ for some $a \in A(W\cap U)$. By Lemma 4.5, we can extend this element a to a global section $\tilde{a} \in A(X)$ (so, $\tilde{a}|_{W\cap U} = a$). But $b - (\varphi(\tilde{a})|_U)$ agrees with b_x on the overlap $W\cap U$. So these two elements amalgamate to a section b' on $W\cup U$:

$$\begin{aligned} b'|_U &= b - (\varphi(\tilde{a})|_U), \\ b'|_W &= b_x, \\ \psi(b') &= c. \end{aligned}$$

This contradicts the maximality of U and completes the proof. □

4.8. **Acyclic Sheaves.** A sheaf B on X is said to be *acyclic* $H^n(X;B) = 0$ for $n > 0$. A resolution of a sheaf A,

$$0 \to A \to B^0 \to B^1 \to B^2 \to \cdots$$

is acyclic if each B^i is. Acyclic resolutions can be used to compute cohomology:

Proposition 4.7. *Let* $0 \to A \to B^0 \to B^1 \to B^2 \to \cdots$ *be an acyclic resolution of* A. *Then*

$$\begin{aligned} H^n(X;A) &\cong H^n(\Gamma B^\bullet) \\ &= \mathsf{ker}(\Gamma B^n \to \Gamma B^{n+1})/\mathsf{im}(\Gamma B^{n-1} \to \Gamma B^n) \end{aligned}$$

Proof. Consider a resolution of $0 \to A \to B^\bullet$ as in Section 4.6 Part (2) and apply Γ to it. The n-th row of the double complex $\Gamma(J^{\bullet\bullet})$ computes $H^\bullet(X;I^n)$ and is therefore exact by Proposition 4.2(2). The m-th column computes $H^\bullet(X;B^m)$ and is therefore exact by the acyclicity of B^m. Thus, we have a double complex with exact rows and columns. It follows (see Algebraic Topology Notes, Appendix A.7) that

$$H_h^n H_v^0(\Gamma J^{\bullet\bullet}) \cong H_v^n H_h^0(\Gamma J^{\bullet\bullet}).$$

So, by Proposition 4.2(1),

$$H^n(\Gamma I^\bullet) \cong H^n(\Gamma B^\bullet)$$

and the left hand side is $H^n(X;A)$ by definition. This proves the proposition. □

5. Acyclic Sheaves

In Section 4.8 we saw that the sheaf cohomology groups can be calculated using general acyclic sheaves instead of just the injective ones. In this section we introduce several types of acyclic sheaves: flabby, soft and fine sheaves. Flabby sheaves are convenient because it is easy to construct flabby resolutions for abelian sheaves. Soft and fine sheaves are useful when X is paracompact; for instance, when X is a manifold. At the end of this section, in Example 5.22 we will use fine sheaves, a special kind of soft sheaves, to prove De Rham's Theorem for sheaf cohomology on manifolds.

5.1. **Flabby Sheaves.**

Definition 5.1. A sheaf A on a space X is called *flabby* if for each open subset $U \subseteq X$ the restriction $\rho_{UX} \colon A(X) \to A(U)$ is surjective.

Remark 5.2. In French these sheaves are called *flasque*.

Example 5.3. In Lemma 4.5 we proved that each injective sheaf is flabby. Note that if A is flabby, so is the sheaf $A|_U$ for any open $U \subseteq X$ (equivalently, every restriction $\rho_{VU} \colon A(U) \to A(V)$ is surjective).

Proposition 5.4. *Every flabby sheaf is acyclic.*

The proof of this proposition is based on the following two lemmas.

Lemma 5.5. *If* $0 \to A \to B \to C \to 0$ *is exact and* A *is flabby, then* $0 \to \Gamma A \to \Gamma B \to \Gamma C \to 0$ *is again exact.*

Proof. The proof is identical to that of Lemma 4.6 where this fact is proved for the case that A is injective. The only property of A that was actually used in that proof is that A has surjective restriction maps. $\square$

Lemma 5.6. *If* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *is exact and* A *and* B *are flabby, so is* C*.*

Proof. Take $U \subseteq X$ open and an element $c \in C(U)$. Since $A|_U$ is flabby we may apply Lemma 5.5 to the restriction sheaves and we see that

$$0 \to A(U) \xrightarrow{\varphi_U} B(U) \xrightarrow{\psi_U} C(U) \to 0$$

is exact. Thus, there is an element $b \in B(U)$ such that $\psi_U(b) = c$. Since B is flabby, b can be extended to $b' \in B(X) = \Gamma B$. Then $\psi_X(b')$ is the required extension of c in $C(X) = \Gamma C$. $\square$

Proof of Proposition 5.4. Let A be a flabby sheaf and let $0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$ be an injective resolution of A. Write $B^1 = \mathsf{im}(I^0 \to I^1)$ to obtain a short exact sequence,

$$0 \to A \to I^0 \to B^1 \to 0 \tag{10}$$

Applying Proposition 4.4 we get a long exact sequence,

$$0 \to H^0(X;A) \to H^0(X;I^0) \to H^0(X;B^1) \to H^1(X;A) \to \cdots \tag{11}$$

$$\cdots \to H^{n-1}(X;B^1) \to H^n(X;A) \to H^n(X;I^0) \to H^n(X;B^1) \to \cdots \tag{12}$$

Since A is flabby,

$$0 \to \Gamma A \to \Gamma I^0 \to \Gamma B^1 \to 0$$

is exact by (10) and Lemma 5.5. Further, $H^1(X;I^0) = 0$ by Proposition 4.2(2). So (11) gives that

$$0 \to H^0(X;A) \to H^0(X;I^0) \to H^0(X;B^1) \to 0 \to H^1(X;A) \to 0$$

is exact. Hence, $H^1(X;A) = 0$. We conclude that the first cohomology group of X with values in any flabby sheaf is zero. Applying Lemma 5.6 to (10) we find that B^1 is also flabby, so $H^1(X;B) = 0$.

We complete the proof by induction: suppose that $H^k(X;F)=0$ for all $k<n$ and all flabby F. Then in particular, $H^{n-1}(X;B^1)=0$ as in the case $n=2$. But also, $H^n(X;I^0)=0$ by injectivity of I^0, so the second line in (11) becomes

$$\cdots \to 0 \to H^n(X;A) \to 0 \to \cdots$$

and we conclude that $H^n(X;A)=0$. This ends the proof. $\square$

Corollary 5.7. *For a flabby resolution* $0\to A\to B^0\to B^1\to B^2\to\cdots$ *of a sheaf* A*,*

$$H^n(X;A)\cong H^n(\Gamma B^\bullet).$$

Proof. By Proposition 5.4 and Proposition 4.7. $\square$

Example 5.8. *The Godement Resolution* Let A be an abelian sheaf on X. We will construct a flabby resolution of A. Let X_{dis} be the same points as X but with the discrete topology, and let

$$p\colon X_{\text{dis}}\to X$$

be the map defined by $p(x)=x$. If B is any sheaf on X_{dis}, then clearly p_*B is flabby. (We see this as follows: if $V\subseteq U$ and $b\in p_*B(V)$ then $b\in B(V)$ and $0\in B(U-V)$ amalgamate to an element $\tilde{b}\in B(U)=p_*B(U)$ such that $\rho_{VU}(\tilde{b})=b$ in p_*B.) In particular, the evident map induced by the adjunction $p^*\dashv p_*$,

$$\eta_A\colon A\to p_*p^*A$$

is an embedding of A into a flabby sheaf. Iterating this we obtain a flabby resolution

$$0\to A\to B^0\to B^1\to B^2\to\cdots,$$

where $B^0=p_*p^*A$. For B^1 we take $B^1=p_*p^*(\mathsf{coker}(A\xrightarrow{\eta}B^0))$. This is a flabby sheaf containing $\mathsf{coker}(A\xrightarrow{\eta}B^0)$ and hence the sequence is exact at B^0. We continue by taking $B^2=p_*p^*(\mathsf{coker}(B^0\to B^1))$ and so on: $B^{i+1}=p_*p^*(\mathsf{coker}(B^{i-1}\to B^i))$.

Remark 5.9. There is also a slightly different 'cosimplicial' version of this flabby resolution. Define $C^n=(p_*p^*)\dots(p_*p^*)(A)=(p_*p^*)^{n+1}A$, obtained by applying p_*p^* $n+1$ times and

$$d=\sum_{i=0}^{n}(-1)^i(p_*p^*)^i\eta_{(p_*p^*)^{n-i}(A)}\colon C^{n-1}\to C^n.$$

The reader may check that this gives an exact sequence [3, p.17].

5.2. **Soft Sheaves.** From now on we will assume that our space X is *paracompact*. Perhaps add a short review of the relevant properties of paracompact spaces?

In this case there is a larger class of acyclic sheaves with nice properties. Before we can give the definition of this class of sheaves, we will first generalize our notion of *section* of a sheaf.

Definition 5.10. Let A be a sheaf on X and let $S\subseteq X$ be any subset. A *section of A over S* is a section of the map $\pi\colon \mathsf{Et}(A)\to X$ over S. We write $\Gamma(S,A)$ for the set of sections of A over S.

Write for $S \subseteq X$ an open subset, $\Gamma(S, A) = A(S)$ as in Section 2. In general, a section $s \in \Gamma(S, A)$ is represented by germs $s(x) = germ_x(a_x)$ for each $x \in S$, where $a_x \in A(U_x)$ for some neighbourhood U_x of x. Since the map π is étale, we can choose U_x small enough such that $s(y) = germ_y(a_x)$ for all $y \in S \cap U_x$. Note that $a_x(y) = a_{x'}(y)$ for all $y \in U_x \cap U_{x'} \cap S$ but not necessarily for all $y \in U_x \cap U_{x'}$.

Definition 5.11. A sheaf A on X is called *soft* (in French, *mou*) if for each *closed* $S \subseteq X$, the restriction map $\Gamma(X; A) \to \Gamma(S; A)$ is surjective.

Remark 5.12. For a paracompact space X, being soft is a *local property* of the sheaf A. This means: if $\mathcal{U} = \{U_i;\, i \in I\}$ is an open cover of X such that each restriction $A|_{U_i}$ is a soft sheaf on U_i, then A is a soft sheaf on X. We prove this as follows: Let $S \subseteq X$ be a closed subset and $s \in \Gamma(S, A)$. Assume, by paracompactness of X, that the cover $\mathcal{U}$ is locally finite and choose a locally finite refinement $\mathcal{V} = \{V_i;\, i \in I\}$ with $\overline{V}_i \subseteq U_i$ for each $i \in I$. Now consider all pairs (J, t) where $J \subseteq I$ and $t \in \Gamma(\bigcup_{j \in J} \overline{V}_j, A)$ extends $s \in \Gamma((\bigcup_{j \in J} \overline{V}_j) \cap S, A)$. These pairs are ordered by inclusion in the evident way:

$$(J, t) \leq (J', t')$$

if and only if

$$J \subseteq J' \text{ and } t' \text{ extends } t.$$

We apply Zorn's Lemma to obtain a maximal element (J_0, t_0). Suppose that $J_0 \neq I$ and choose an index $i \in I - J_0$. Now $U_i \cap ((\bigcup_{j \in J_0} \overline{V}_j) \cup S)$ is closed in U_i (by local finiteness). Hence, by assumption there exists a section t' which extends $(t_0 \cup s)|_{U_i} \in \Gamma(U_i \cap ((\bigcup_{j \in J_0} \overline{V}_j) \cup S), A)$. (Here, $t_0 \cup s$ denotes the amalgamation of t_0 and s.) But then t_0 and t' are compatible and can be glued to a section on $\bigcup_{j \in J_0 \cup \{i\}} \overline{V}_j$ and this contradicts the maximality of J_0. Hence $J_0 = I$ and this proves the remark.

Proposition 5.13. *A soft sheaf on a paracompact space is acyclic.*

Just as for flabby sheaves, this follows from the following two lemmas.

Lemma 5.14. *If* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *is exact and* A *is soft, then* $0 \to \Gamma A \to \Gamma B \to \Gamma C \to 0$ *is exact.*

Proof. As we saw in the proof of Lemma 4.6, it suffices to show that $\Gamma\psi \colon \Gamma B \to \Gamma C$ is surjective. To this end take $c \in \Gamma C$. Since $B \xrightarrow{\psi} C \to 0$ is exact, there is a cover $\{U_i;\, i \in I\}$ of X with for each $i \in I$ a section $b_i \in B(U_i)$ with $\psi(b_i) = c|_{U_i}$. Let $\{V_i;\, i \in I\}$ be a locally finite cover such that $\overline{V}_i \subseteq U_i$ for each $i \in I$. Consider all pairs (J, b) where $J \subseteq I$ and $b \in \Gamma(\bigcup_{j \in J} \overline{V}_j, B)$ such that $\psi(b) = c|_{(\bigcup_{j \in J} \overline{V}_j)}$. These pairs are ordered by inclusion, as in the previous remark: $(J, b) \leq (J', b')$ if and only if $J \subseteq J'$ and b' extends b. By Zorn's Lemma, there exists a maximal pair (J_0, b_0). Suppose that $J_0 \neq I$ and choose $i \in J_0 - I$. Since both $\psi(b_i) = c|_{U_i}$ and $\psi(b_0) = c|_{(\bigcup_{j \in J_0} V_j)}$ there exists an element $a \in A((\bigcup_{j \in J_0} \overline{V}_j) \cap \overline{V}_i, A)$ such that

$$b_0|_{\overline{V}_i \cap (\bigcup_{j \in J_0} \overline{V}_j)} - b_i|_{(\bigcup_{j \in J_0} \overline{V}_j) \cap \overline{V}_i} = \varphi(a).$$

Since $(V_i)_{i \in I}$ is locally finite, $(\bigcup_{j \in J_0} \overline{V}_j) \cap \overline{V}_i$ is closed in X. Hence, since A is soft there exists an extension $\tilde{a} \in A(X)$ of a. Now replace b_i by $b'_i = b_i + \varphi(\tilde{a}|_{\overline{V}_i}) \in$

$\Gamma(\overline{V}_i, B)$. Then b_0 and b'_i are compatible and their amalgamation is a section

$$b_0 \cup b'_i \in \Gamma(\bigcup_{j \in J_0 \cup \{i\}} \overline{V}_j, B).$$

This contradicts the maximality of (J_0, b_0). We conclude that $J_0 = I$ and the sequence

$$0 \to \Gamma A \to \Gamma B \to \Gamma C \to 0$$

is exact. $\square$

Remark 5.15. In the proof of this lemma it would have been sufficient to find an extension $\tilde{a}$ of a over $\overline{V}_i$ rather than all of X.

Lemma 5.16. *If $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is exact and A and B are soft, then so is C.*

Proof. Let $c \in \Gamma(S, C)$ where $S \subseteq X$ is closed. Then S is itself a paracompact space and $A|_S$ is soft (since closed subsets of S are closed in X as well). By applying Lemma 5.14 with $X = S$ we find that

$$0 \to \Gamma(S, A) \to \Gamma(S, B) \to \Gamma(S, C) \to 0$$

is exact. Thus, there is an element $b \in \Gamma(S, B)$ with $\psi(b) = c$. Since B is soft, we can extend b to $\tilde{b} \in \Gamma B$. And we get that $\psi(\tilde{b}) \in \Gamma C$ is an extension of c as required. $\square$

The relation between the notions of ‘soft’ and ‘flabby’ for sheaves on a paracompact space is based on the following proposition.

Proposition 5.17. *Let A be a sheaf on a paracompact space. Then any section $s \in \Gamma(S, A)$ over a closed subset $S \subseteq X$ can be extended to a neighbourhood $N \supseteq S$.*

More generally, it will become clear from the proof of this proposition that this applies to any subset $S \subseteq X$ with the property that for any neighbourhood $U \supseteq S$ there exists a paracompact subspace T with $S \subseteq T \subseteq U$.

Corollary 5.18. *Any flabby sheaf on a paracompact space is soft.*

Proof of Proposition 5.17. Since S is itself paracompact, we can find a locally finite cover $\mathcal{U} = \{U_i;\, i \in I\}$ of S and sections $s_i \in A(U_i)$ such that $s(y) = \mathit{germ}_y(s_i)$ for all $y \in U_i \cap S$. Let $\mathcal{V} = \{V_i;\, i \in I\}$ be a locally finite refinement of $\mathcal{U}$ with $\overline{V}_i \subseteq U_i$ for all $i \in I$. Furthermore, let $N = \{x \in \bigcup_{i \in I} V_i;\, s_i(x) = s_j(x) \text{ whenever } x \in \overline{V}_i \cap \overline{V}_j\}$. Note that N is open. Also note that the family $s_i|_{\overline{V}_i \cap N}$ with $i \in I$ is compatible; hence, there is an amalgamation $s \in \Gamma(N, A)$. So it suffices to show that N is a neighbourhood of S. Now pick $x \in S$. By local finiteness, there is finite number p such that $x \in \overline{V}_{i_1}, \ldots, \overline{V}_{i_p}$, but no others. Then let N_x be a small neighbourhood of x which meets only $\overline{V}_{i_1}, \ldots, \overline{V}_{i_p}$ and such that $N_x \subseteq U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_p}$. Since $x \in S$, we have that

$$s(x) = \mathit{germ}_x(s_{i_1}) = \cdots = \mathit{germ}_x(s_{i_p}).$$

Hence, there is a neighbourhood $O_x \subseteq N_x$ such that

$$s_{i_1}|_{O_x} = \cdots = s_{i_p}|_{O_x}.$$

Then $O_x \subseteq N$ since O_x meets no other $\overline{V}_i$. This completes the proof of Proposition 5.14. $\square$

5.3. **Fine Sheaves.** The characterization of the following class of acyclic sheaves uses the fact that paracompact spaces allow for partitions of unity subordinate to locally finite covers.

Definition 5.19. A sheaf A on a paracompact space is *fine* if for each locally finite open cover $\mathcal{U} = \{U_i;\, i \in I\}$ of X there exists an $\mathsf{End}(A)$-valued partition of unity subordinate to $\mathcal{U}$. I.e., there are homomorphisms $\lambda_i \colon A \to A$ such that

(a) $\mathsf{supp}(\lambda_{i,V}(a)) \subseteq U_i$ for all $i \in I$, open $V \subseteq X$ and $a \in A(V)$;
(b) $\sum_{i\in I} \lambda_{i,V}(a) = a$ for all open $V \subseteq X$ and $a \in A(V)$.

Lemma 5.20. *A fine sheaf A on a paracompact space is soft.*

Proof. Let $s \in \Gamma(S, A)$ be a section over a closed subset $S \subseteq X$. By Proposition 5.17, s can be extended to a section $\tilde{s}$ on a neighbourhood N of S. The idea of the proof is now to change $\tilde{s}$ over $N - S$ such that we can extend it by zero outside N. Therefore consider the cover $\{N, X - S\}$ of X. Since A is fine, we have two morphisms, $\lambda, \mu \colon A \to A$ such that

(a) $\mathsf{supp}\,\lambda_V(a) \subseteq N$;
(b) $\mathsf{supp}\,\mu_V(a) \subseteq X - S$;
(c) $\lambda_V(a) + \mu_V(a) = a$.

for all open $V \subseteq X$ and $a \in A(V)$.

By (1), $X = N \cup (X - \mathsf{supp}\,\lambda(\tilde{s}))$ and $\{\lambda(\tilde{s}) \in A(N), 0 \in A(X - \mathsf{supp}\,\lambda(\tilde{s}))\}$ is a compatible family. Let $\bar{s}$ be the amalgamation of this family. We will show that $\bar{s}$ extends s. From (3) we get $\lambda_N(\bar{s}) + \mu_N(\bar{s}) = \bar{s}$. However, $\mu_N(\bar{s})|_S = 0$ by (2), so $\bar{s}|_S = \lambda_N(\tilde{s})|_S = \tilde{s}|_S = s$. □

Corollary 5.21. *Each fine sheaf on a paracompact space is acyclic.*

Example 5.22. (*A version of De Rham's Theorem*) Let X be a smooth manifold and write $\mathbb{R}$ for the constant sheaf $\Delta\mathbb{R}$ of real numbers. By the Poincaré Lemma [1, p.35] the complex of differential forms $\Omega^\bullet$ gives a resolution,

$$0 \to \mathbb{R} \to \Omega^0 \to \Omega^1 \to \cdots$$

of $\mathbb{R}$. The De Rham cohomology of X is by definition the cohomology of the complex

$$0 \to \Gamma\Omega^0 \to \Gamma\Omega^1 \to \cdots$$

By the remark above, each Ω^p is fine, since it is a C_X-module. Hence, there is an isomorphism

$$H^n(X;\mathbb{R}) \xrightarrow{\sim} H^n_{\mathrm{DeRham}}(X).$$

6. Some Exact Sequences for Subspaces

This section will take a closer look at functors between categories of sheaves which are induced by subspaces as in Section 3. We will first derive exactness properties from the fact that they are adjoint functors. Then we will give several short exact sequences involving these functors and study the induced long exact sequences in cohomology. This will lead us to local cohomology, cohomology of a pair (relative cohomology), excision and the Mayer-Vietoris sequences.

6.1. **Exactness Properties of Adjoint Functors.** An additive functor F between abelian categories is called *exact* when it sends a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{13}$$

to a sequence

$$0 \to FA \to FB \to FC \to 0$$

which is again exact. The functor F is called *right exact* (resp. *left exact*) if for the sequence in (13),

$$FA \to FB \to FC \to, 0$$

respectively

$$0 \to FA \to FB \to FC,$$

is exact.

It is clear that a right exact functor preserves epimorphisms and a left exact functor preserves monomorphisms.

If a functor F is left adjoint to another functor, G say,

$$F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

with a natural isomorphism as in the adjunction formula,

$$\mathsf{Hom}_{\mathcal{D}}(FA, E) \cong \mathsf{Hom}_{\mathcal{C}}(A, GE),$$

and F preserves monomorphisms, then G preserves injectives. Indeed, let I be injective in $\mathcal{D}$. To prove that GI is injective, consider the diagrams

$$\begin{tikzcd} M \arrow[r] \arrow[d, tail] & GI \\ N \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} FM \arrow[r] \arrow[d, tail] & I \\ FN \arrow[ur, dashed] & \end{tikzcd} .$$

Since F preserves monomorphisms, $FM \rightarrowtail FN$ is again monic. Furthermore, the morphism $M \to GI$ corresponds to $FM \to I$ via the adjunction formula. Shall we introduce notation for this correspondence? Now a filling $N \dashrightarrow GI$ on the left corresponds via the adjunction formula precisely to one on the right, $FN \dashrightarrow I$, by naturality.

In general, if G has a left adjoint F then G is left exact while F is right exact [4, p.116,197]. For the record, we state explicitly:

Lemma 6.1. *Let $F \colon \mathcal{A} \rightleftarrows \mathcal{B} : G$ be additive functors between abelian categories, where F is left adjoint to G. Then:*

(a) *F is right exact and G is left exact;*
(b) *if in addition F is (left) exact, then G preserves injectives.*

In particular, for a continuous map $f \colon Y \to X$ we have adjoint functors

$$f^* : \mathsf{Ab}\,(X) \rightleftarrows \mathsf{Ab}\,(Y) : f_*$$

and f^* is exact, while f_* preserves injectives.

Recall from Section 3 that an open inclusion $i \colon X \hookrightarrow X$ induces functors

$$i_! \dashv i^* \dashv i_* \colon \mathsf{Ab}\,(X) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \mathsf{Ab}\,(Y),$$

while a closed inclusion $j \colon Z \hookrightarrow Y$ induces

$$j^* \dashv j_* \dashv j^! \colon \mathsf{Ab}\,(Y) \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \mathsf{Ab}\,(Z)$$

Applying the above lemma on adjoint functors we obtain:

(a) i^* and j^* are both exact (in particular, they both preserve monomorphisms);
(b) j_* is also exact and preserves injectives;
(c) $j^!$ preserves injectives;
(d) $i_!$ is exact and (hence) i^* preserves injectives.

All of these results follow from the preceding lemma, except for part (4): $i_!$ is right exact as left adjoint of i^*, but it also preserves monomorphisms as can be seen from the explicit description,

$$i_!(A)(V) = \{a \in A(U \cap V); \mathsf{supp}(a) \text{ is closed in } U\}.$$

So $i_!$ is exact.

6.2. **Short Exact Sequences for Inclusions.** We will now give two short exact sequences for the special case of an open subspace and its (closed) complement. From these sequences and the exactness properties (1) - (4) above we will derive the long exact sequences for local cohomology and cohomology of a pair in the next section.

Lemma 6.2. *Let $Y \subseteq X$ be an open subset with $Z = X - Y$ its closed complement and inclusion maps $i\colon Y \hookrightarrow X$ and $j\colon Z \hookrightarrow X$. For each abelian sheaf A on X there is an exact sequence,*

$$0 \to i_!i^*A \to A \to j_*j^*A \to 0. \tag{14}$$

Furthermore, if A is flabby (for instance, injective) there is an exact sequence,

$$0 \to j_*j^!A \to A \to i_*i^*A \to 0. \tag{15}$$

Proof. In both sequences the maps are the units and counits of the appropriate adjunctions. The proof is a simple matter of spelling out the definitions.

For (14), let $U \subseteq X$ be any open subset. Then,

$$\begin{aligned} i_!i^*(A)(U) &= i_!(A|_Y)(U) \\ &= \{a \in (A|_Y)(Y \cap U); \mathsf{supp}(a) \text{ is closed in } U\} \\ &= \{a \in A(Y \cap U); \mathsf{supp}(a) \text{ is closed in } U\} \end{aligned}$$

and

$$\begin{aligned} j_*j^*(A)(U) &= j^*(A)(U \cap Z) \\ &= \Gamma(U \cap Z, A), \end{aligned}$$

where the notation is as in Definition 5.10. The map $i_!i^*(A) \to A$ is given by 'extension by zero' (see Section 3.2(6)), while $A \to j_*j^*(A)$ is the evident restriction map. Clearly, $i_!i^*A \to A$ is one-to-one, so the sequence is exact at $i_!i^*A$. Furthermore, for a section $a \in A(U)$, the following statements are asily seen to be equivalent:

- $a|_{Z \cap U} = 0$;
- $\mathsf{supp}(a) \subseteq U \cap Y$;
- a is the extension by zero of its restriction to $U \cap Y$.

This gives us exactness at A.

Finally, to prove exactness at j_*j^*A, note that for any $s \in \Gamma(Z \cap U, A)$ and any $x \in U$ there is a neighbourhood V_x of x and a section $a_x \in A(V_x)$ such that $a_x|_{Z \cap V_x} = s|_{Z \cap V_x}$ (if $x \in Z$ take any a_x with $germ_x(a_x) = s(x)$; if $x \notin Z$ take

$V_x \subseteq Y = X - Z$ and a_x any section, for instance zero.) So $A \to j_*j^*A$ is an epimorphism and we conclude that (14) is exact.

To prove (15), let $U \subseteq X$ be an open subset. Then,

$$j_*j^!(A)(U) = j^!(A)(U \cap Z) \tag{16}$$
$$= \{a \in A(U);\, \mathsf{supp}(a) \subseteq Z\} \tag{17}$$

and

$$\begin{aligned} i_*i^*(A)(U) &= i^*A(U \cap Y) \\ &= A(U \cap Y). \end{aligned}$$

The map $j_*j^!A \to A$ is the inclusion, while $A \to i_*i^*A$ is the restriction. Clearly, $j_*j^!A \to A$ is is one-to-one. The sequence (15) is exact at A since for $a \in A(U)$ we have $\mathsf{supp}(a) \subseteq Z$ if and only if $a|_{U\cap Z} = 0$ since Z and Y are complementary subsets of X. Finally, exactness at i_*i^*A is clear from the definition of 'flabby'. $\square$

6.3. **Local Cohomology and Cohomology of a Pair.** We first introduce some notation. For a subset $S \subseteq X$, write

$$\Gamma_S \colon \mathsf{Ab}\,(X) \to \textbf{Abelian Groups}$$

for the functor defined by

$$\Gamma_S(A) = \{a \in \Gamma A;\, \mathsf{supp}(a) \subseteq S\}, \tag{18}$$

and $H^n_S(X; -)$ for the n-th right derived functor of Γ_S. Thus, the groups

$$H^n_S(X; A)$$

are computed as follows. Take an injective resolution

$$0 \to A \to I^\bullet,$$

apply Γ_S to obtain a complex,

$$\Gamma_S(I^0) \to \Gamma_S(I^1) \to \Gamma_S(I^2) \to \cdots,$$

and then take the cohomology. For *closed* subspaces $S \subseteq X$, $H^n_S(X; A)$ is called the *local cohomology* of A with support in S. Note that in this case we can also describe Γ_S as $\Gamma_S = \Gamma j_*j^!$ for $j \colon S \hookrightarrow X$.

For a closed subset $Z \subseteq X$ and the inclusion map $i \colon X - Z \hookrightarrow X$ of its open complement, define the *cohomology of the pair* (X, Z) as

$$H^p(X, Z; A) = H^p(X; i_!i^*A),$$

for any sheaf A on X.

The following proposition gives the connection between the cohomology groups of X, Y and Z and the newly introduced cohomologies.

Proposition 6.3. *For any open subset $Y \subseteq X$ and its closed complement $Z = X - Y$ with inclusion maps as above, there are long exact sequences, for the cohomology of the pair (X, Z) and the local cohomology,*

(a) $\cdots \to H^n(X, Z; A) \to H^n(X; A) \to H^n(Z; A|_Z) \to H^{n+1}(X, Z; A) \to \cdots$
(b) $\cdots \to H^n_Z(X; A) \to H^n(X; A) \to H^n(Y; A|_Y) \to H^{n+1}_Z(X; A) \to \cdots$

Proof. This follows be rewriting the long exact sequences in cohomology induced by the short exact sequences (14) and (15) in Lemma 6.2.

For (1), note that the short exact sequence (14) of Lemma 6.2 yields the long exact sequence (see Proposition 4.4)

$$\cdots \to H^n(X; i_! i^* A) \to H^n(X; A) \to H^n(X; j_* j^* A) \to H^{n+1}(X; i_! i^* A) \to \cdots$$

By definition, $H^n(X; i_! i^* A) = H^n(X, Z; A)$. Furthermore, j_* is exact and preserves injectives, so it sends an injective resolution $0 \to B \to I^\bullet$ of any sheaf B on Z to an injective resolution $0 \to j_* B \to j_* I^\bullet$ of $j_* B$. Since there is a natural isomorphism $\Gamma_X j_* \cong \Gamma_Z$, we conclude that $H^n(X; j_* j^* A) \cong H^n(Z; j^* A)$ (see also, Section 4.5, Exercise 4). As $A|_Z = j^* A$ by definition, we obtain the long exact sequence (a) in this proposition.

To obtain the long exact sequence (b), take an injective resolution $0 \to A \to I^\bullet$; then by (15) in Lemma 6.2 there is an exact sequence of chain complexes,

$$0 \to j_* j^! I^\bullet \to I^\bullet \to i_* i^* I^\bullet \to 0.$$

Since $j_* j^!$ preserves injectives, the sequence

$$0 \to \Gamma j_* j^! I^\bullet \to \Gamma I^\bullet \to \Gamma i_* i^* I^\bullet \to 0$$

is again exact (by Lemma 4.6). This gives us then the following long exact sequence,

$$\cdots \to H^n(\Gamma j_* j^! I^\bullet) \to H^n(\Gamma I^\bullet) \to H^n(\Gamma i_* i^* I^\bullet) \to H^{n+1}(\Gamma j_* j^! I^\bullet) \to \cdots \tag{19}$$

Now, $\Gamma j_* j^! = \Gamma_Z$ by (16) and (18) above, so $H^n(\Gamma j_* j^! I^\bullet)$ gives the local cohomology $H^n_Z(X; A)$. It is clear that $\Gamma I^\bullet$ computes the cohomology of X with values in A and $\Gamma i_* i^* I^\bullet$ computes $H^n(Y; A|_Y)$ since i^* is exact and preserves injectives and $\Gamma i_* \cong \Gamma$. So from (19) we get part (b) of this proposition. □

Remark 6.4. From Proposition 6.3 Part (2) we see that if $A \in \mathsf{Ab}(X)$ is a sheaf such that $A|_U$ is an acyclic sheaf on U for every open $U \subseteq X$ (in particular, for $U = X$), then A is acyclic for local cohomology. (For instance, this holds when A is flabby.)

6.4. **Excision.** If $Y \subseteq X$ is open and $Z \subseteq X$ is closed (but *not* necessarily $Z = X - Y$) then for each injective (or flabby) sheaf we have a short exact sequence,

$$0 \to \Gamma_{Z-Y}(X, I) \to \Gamma_Z(X, I) \to \Gamma_{Z\cap Y}(Y, I) \to 0. \tag{20}$$

Note that exactness at $\Gamma_{Z-Y}(X, I)$ and $\Gamma_Z(X, I)$ follows immediately from the definition of Γ_S. To show exactness at $\Gamma_{Z\cap Y}(Y, I)$, extend $a \in I(Y)$ with $\mathsf{supp}(a) \subseteq Z \cap Y$ first by zero to $\bar{a} \in I(Y \cup (X - Z))$ and then to a global section $\tilde{a} \in I(X)$ by injectivity. It is clear that $\mathsf{supp}(\tilde{a}) \subseteq Z$. Thus, for any $A \in \mathsf{Ab}(X)$ there is a long exact sequence,

$$\cdots \to H^n_{Z-Y}(X; A) \to H^n_Z(X; A) \to H^n_{Z\cap Y}(Y; A) \to \cdots \tag{21}$$

In the special case where $Y \supseteq Z$, the short exact sequence (20) reduces to an isomorphism

$$\Gamma_Z(X, I) \xrightarrow{\sim} \Gamma_Z(Y, I)$$

and the long exact sequence (21) reduces the *excision isomorphism*

$$H^n_Z(X; A) \xrightarrow{\sim} H^n_Z(Y; A).$$

6.5. **Mayer-Vietoris Sequences.** Let $U, V \subseteq X$ be open subsets. Let I be an injective (or just flabby) sheaf on X. Then the following sequence is exact:

$$0 \to I(U \cup V) \to I(U) \oplus I(V) \to I(U \cap V) \to 0, \tag{22}$$

where the maps are given by $s \mapsto (s|_U, s|_V)$ for $s \in I(U \cup V)$ and $(s_1, s_2) \mapsto s_1|_{U\cap V} - s_2|_{U\cap V}$ for $s_1 \in I(U)$ and $s_2 \in I(V)$. Applying this to an injective resolution $0 \to A \to I^\bullet$ of a given sheaf A, we obtain a long exact cohomology sequence,

$$\cdots \to H^n(U\cup V; A) \to H^n(U; A)\oplus H^n(V; A) \to H^n(U\cap V; A) \to H^{n+1}(U\cup V; A) \to \cdots$$

Slightly more subtly, this also works for *closed* subsets $S, T \subseteq X$. Write

$$h\colon S \cap T \hookrightarrow X, \quad i\colon S \hookrightarrow X, \quad j\colon T \hookrightarrow X, \quad k\colon S \cup T \hookrightarrow X$$

for the inclusion maps. As in the proof of Lemma 6.2, we have for any sheaf A on X and any open subset $U \subseteq X$,

$$i_* i^*(A)(U) = \Gamma(S \cap U, A),$$

and similarly for h, j and k. Now consider for each open $U \subseteq X$ the sequence,

$$0 \to \Gamma((S \cup T) \cap U, A) \xrightarrow{\alpha} \Gamma(S \cap U, A) \oplus \Gamma(T \cap U, A) \xrightarrow{\beta} \Gamma(S \cap T \cap U, A) \to 0,$$

similar to (22) above. It is clear that it is exact at $\Gamma((S \cup T) \cap U, A)$ and $\Gamma(S \cap U, A) \oplus \Gamma(T \cap U, A)$. It may not be exact at $\Gamma(S \cap T \cap U, A)$, but since S and T are closed subsets of X, β is at least locally surjective in the sense of Section 4.2. So these sequences, taken for all open $U \subseteq X$, do fit together to give a short exact sequence of sheaves,

$$0 \to k_* k^* A \to i_* i^* A \oplus j_* j^* A \to h_* h^* A \to 0. \tag{23}$$

This yields a long exact cohomology sequence , which as in the proof of Proposition 6.3(1) can be rewritten as

$$\begin{aligned} \cdots \to H^n(S \cup T; A|_{S\cup T}) \to H^n(S; A|_S) \oplus H^n(T; A|_T) \to H^n(S \cap T; A|_{S\cap T}) \to \\ \to H^{n+1}(S \cup T; A|_{S\cup T}) \to \cdots \end{aligned}$$

7. Proper Base Change

In the previous section we saw already that $f_*\colon \mathsf{Ab}(Y) \to \mathsf{Ab}(X)$ is left exact for every continuous map $f\colon Y \to X$. So we can consider the right derived functors $R^q f_*$. We will show that for a proper map f the stalks of these derived functors can be described in terms of the stalks of the cohomology of the fibers:

$$(R^q f_* B)_x \cong H^q(f^{-1}x; B),$$

for any sheaf B on Y. From this result 'proper base change' can be derived.

7.1. **The Functors $R^q f_*$.** Recall from Section 3 that a map $f\colon Y \to X$ of spaces induces a left exact functor

$$f_*\colon \mathsf{Ab}(Y) \to \mathsf{Ab}(X)$$

with

$$f_*(B)(U) \cong B(f^{-1}U)$$

for $B \in \mathsf{Ab}(Y)$ and $U \subseteq X$ open. We saw already in Section 4.5 Exercise 4, that $\Gamma_X \circ f_* \cong \Gamma_Y$, or more explicitly,

$$\Gamma(X, f_* B) \cong \Gamma(Y, B).$$

The q-th right derived functor of f_* is denoted by $R^q f_*$. Recall that for $B \in \mathsf{Ab}\,(Y)$, the sheaf $R^q f_*(B)$ is computed as follows. Take an injective resolution of B,

$$0 \to B \to I^0 \to I^1 \to I^2 \to \cdots ,$$

apply the functor f_*,

$$0 \to f_*B \to f_*I^0 \to f_*I^1 \to f_*I^2 \to \cdots ,$$

and compute the cohomology of $f_*I^\bullet$:

$$R^q f_*(B) = \mathsf{ker}(f_*I^q \to f_*I^{q+1})/\mathsf{im}(f_*I^{q-1} \to f_*I^q).$$

These are the kernel and image in $\mathsf{Ab}\,(X)$, computed by taking the associated sheaf of the 'naive' construction,

$$P\colon U \mapsto \mathsf{ker}(f_*I^q(U) \to f_*I^{q+1}(U))/\mathsf{im}(f_*I^{q-1}(U) \to f_*I^q(U)),$$

which only defines a presheaf as we saw in Section 2. (Note that this construction is completely analogous to that of the usual cohomology groups except that we use $f_*\colon \mathsf{Ab}\,(Y) \to \mathsf{Ab}\,(X)$ instead of $\Gamma\colon \mathsf{Ab}\,(X) \to$ **Abelian Groups**. Thus, the stalk of $R^q f_*(B)$ at a point $x \in X$ is the stalk $P_x = \varinjlim_{x\in U} P(U)$, or:

$$(R^q f_*B)_x = \varinjlim_{x\in U} H^q(f^{-1}U; B|_{f^{-1}U}). \tag{24}$$

For each open neighbourhood U of x in X the inclusion

$$i : f^{-1}x \hookrightarrow f^{-1}U$$

yields a homomorphism

$$i^* : H^q(f^{-1}U; B|_{f^{-1}U}) \to H^q(f^{-1}x; B|_{f^{-1}x}).$$

All these homomorphisms together induce a map

$$(R^q f_*B)_x \to H^q(f^{-1}x; B|_{f^{-1}x}). \tag{25}$$

Thus, the stalk $(R^q f_*B)_x$ is some sort of approximation of the cohomology of the fiber $f^{-1}x$.

7.2. **Proper Base Change.** We will show that the homomorphism (25) is an isomorphism when f is proper with Hausdorff fibers.

Definition 7.1. A map $f\colon Y \to X$ of topological spaces is *proper* if

(a) f is a closed map;
(b) each fiber $f^{-1}x$ is compact.

The reader may check that if a map $f\colon Y \to X$ is proper then for each compact subset $C \subseteq X$, the inverse image $f^{-1}C \subseteq Y$ is again compact.

Now consider for a map $p\colon X' \to X$ the pullback,

$$Y' = \{(x', y) \in X' \times Y;\, px' = fy\},$$

of f along p,

$$\begin{CD} Y' @>{q}>> Y \\ @V{f'}VV @VV{f}V \\ X' @>>{p}> X, \end{CD} \tag{26}$$

with projection maps f' and q. The map f' is said to be obtained from f by *change of base* along p. Note that the fibers of f' are isomorphic to those of f:

$$(f')^{-1}(x') \cong f^{-1}(p(x')),$$

for each $x' \in X'$. So if f is proper, then f' has compact fibers. Furthermore, one can show that f' is closed as well, so it is again proper.

Lemma 7.2. *The pullback of a proper map is a proper map.*

We leave the proof to the reader.

Theorem 7.3. *If $f\colon Y \to X$ is proper and the fibers of f are Hausdorff as subspaces of Y, then the map*

$$(R^q f_* B)_x \to H^q(f^{-1}x; B|_{f^{-1}x})$$

is an isomorphism for any sheaf B on Y.

Corollary 7.4 (Proper Base Change)**.** *For a pullback diagram as in (26) with f and f' proper with Hausdorff fibers, the canonical map*

$$p^*(R^n f_* B) \xrightarrow{\sim} R^n f'_*(q^* B)$$

is an isomorphism for any sheaf B on Y.

Proof. **Exercise 1** (Hint: spell out explicitly what this map does on the stalks and apply Theorem 7.3.) □

We now turn to the proof of Theorem 7.3. Let B be a sheaf on $Y$and let $0 \to B \to I^0 \to I^1 \to \cdots$ be an injective resolution of B. Since the set of neighbourhoods $U \subseteq X$ of x is directed [4, p.209], the limit in (24) is directed. So this limit commutes with the operation of taking cohomology (see the Algebraic Topology Notes, Appendix A.11). Hence, (24) can be rewritten as,

(27)
$$(R^q f_* B)_x \cong H^q(\varinjlim_{x \in U} \Gamma(f^{-1}U, I^\bullet)).$$

Now we want to replace this limit over open subsets in X by one over open subsets in Y: since f is a closed map, any open neighbourhood $V \supseteq f^{-1}x$ of the fiber contains a neighbourhood of the form $f^{-1}U$ for some open $U \ni x$ (for instance, $U = X - f(Y - V)$). So

(28)
$$\varinjlim_{x \in U} \Gamma(f^{-1}, ; I^\bullet) \cong \varinjlim_{f^{-1}x \subseteq V} \Gamma(V, I^\bullet),$$

where the second limit is taken over all open subsets $V \subseteq Y$ containing the fibers $f^{-1}(x)$. Next we claim that the evident restriction map

(29)
$$\varinjlim_{f^{-1}x \subseteq V} \Gamma(V, I^q) \to \Gamma(f^{-1}(x), I^q)$$

is an isomorphism. This is a simple consequence of the following lemma.

Lemma 7.5. *Let $K \subseteq Y$ be a compact subset of a space Y and assume that the relative topology on K is Hausdorff. Let A be a sheaf on Y. Then any section $s \in \Gamma(K, A)$ can be extended to an open neighbourhood of K.*

Proof. This proof is a slight improvement of the proof of Proposition 5.17. There we used that Y is paracompact; now we will only use that K is compact Hausdorff.

Let $\{U_1, \ldots, U_n\}$ be a cover of K by open subsets of Y for which there are sections $a_i \in A(U_i)$ with $s(y) = \mathit{germ}_y(a_i)$ for $y \in K \cap U_i$. Since K is compact and Hausdorff, there is a refinement $K \subseteq V_1 \cup \ldots \cup V_n$ with $K \cap \overline{V}_i \subseteq U_i$. Now, as in the proof of Proposition 5.17, let

$$N = \{x \in \bigcup_{i=1}^{n} \overline{V}_i;\ \text{ if } x \in \overline{V}_i \cap \overline{V}_j \text{ then } \mathit{germ}_x(a_i) = \mathit{germ}_x(a_j)\}.$$

Then the $a_i|_{N \cap \overline{V}_i}$ for $i = 1, \ldots, n$ form a compatible family of sections and can hence be amalgamated to a section $a \in \Gamma(N, A)$. So it remains to be shown that N is a neighbourhood of K. Pick $x \in K$ and let $S_x \subseteq \{1, \ldots, n\}$ be the set of indices of the $\overline{V}_i$ which contain x:

$$i \in S_x \text{ if and only if } x \in \overline{V}_i.$$

Find a neighbourhood N_x of x such that $N_x \subseteq U_i$ if $i \in S_x$ and $N_x \subseteq X - \overline{V}_i$ if $i \notin S_x$. For example, $N_x = \left(\bigcap_{i \in S_x} U_i\right) \cap \left(\bigcap_{i \notin S_x} (X - \overline{V}_i)\right)$, as for $i \notin S_x$ it is clear that $x \in X - \overline{V}_i$ and for $i \in S_x$, $x \in \overline{V}_i \cap K \subseteq U_i$.

Since $\mathit{germ}_x(a_i) = \mathit{germ}_x(a_j)$ whenever $i, j \in S_x$ (because $x \in N$), there is a smaller neighbourhood $O_x \subseteq N_x$ such that $a_i|_{O_x} = a_j|_{O_x}$ for all $i, j \in S_x$. Since $O_x \cap \overline{V}_j = \emptyset$ for $j \notin S_x$ it follows that $O_x \subseteq N$. Thus, N is a neighbourhood of K. $\square$

7.3. **The Leray Spectral Sequence.** For a map $f\colon Y \to X$ there are functors

$$\begin{tikzcd} \mathsf{Ab}(Y) \arrow[rr, "f_*"] \arrow[dr, "\Gamma_Y"'] & & \mathsf{Ab}(X) \arrow[dl, "\Gamma_X"] \\ & \textbf{Abelian Groups} & \end{tikzcd} \qquad \Gamma_X \circ f_* = \Gamma_Y.$$

Since f_* preserves injectives (see Section 6.1) there is for each abelian sheaf B on Y a Grothendieck spectral sequence for the composite $\Gamma_Y = \Gamma_X \circ f_*$ [2, p.299],

$$E_2^{pq} = R^p\Gamma_X(R^q f_* B) \Rightarrow R^{p+q}\Gamma_Y B,$$

which we can rewrite as

$$E_2^{p,q} = H^p(X; R^q f_* B) \Rightarrow H^{p+q}(Y; B).$$

This special case of the Grothendieck spectral sequence is called the *Leray spectral sequence* of the map f. (Note that in the case f is proper we already have some information about $R^q f_* B$.) So this gives us the cohomology of Y if we have enough information about $R^q f_* B$ and the cohomology of X.

If $R^q f_* B = 0$ for $q > 0$ then applying f_* to an injective resolution $0 \to B \to I^\bullet$ of B yields an exact sequence,

$$0 \to f_*B \to f_*I^0 \to f_*I^1 \to f_*I^2 \to \cdots$$

and it follows immediately that the map

$$H^p(X; f_*B) \to H^p(Y; B) \tag{30}$$

is an isomorphism (since $\Gamma \circ f_* \cong \Gamma$). Careful inspection of the constructions used shows that this map is the composite

$$H^p(X; f_*B) \to H^p(Y; f^*f_*B) \to H^p(Y; B)$$

of a map of the form of Proposition 4.3(2) and the map induced by the adjunction counit $\varepsilon_B \colon f^*f_*B \to B$. In particular, for a sheaf A on X with $R^q f_*(f^*A) = 0$ for $q > 0$ we obtain an isomorphism

$$H^p(X; f_*f^*A) \to H^p(Y; f^*A),$$

by taking $B = f^*A$ in (30). And in those cases where the unit map $\eta_A \colon A \to f_*f^*A$ is an isomorphism, it thus follows that the map

$$f^* \colon H^p(X; A) \to H^p(Y; f^*A)$$

of Proposition 4.3(2) is an isomorphism. (Use the triangle identity for unit and counit,

$$\begin{tikzcd} f^* \arrow[r, "f^*(\eta)"] \arrow[dr, equal] & f^*f_*f^* \arrow[d, "\varepsilon_{f^*}"] \\ & f^* \end{tikzcd} \quad .)$$

The following proposition tells us that this is the case when $f \colon Y \to X$ is a quotient map with connected fibers.

Proposition 7.6. *When $f \colon Y \to X$ is a quotient map (e.g., f is surjective and open or closed) with connected fibers then the adjunction map $A \to f_*f^*A$ is an isomorphism for each sheaf A on X.*

Proof. For a point $x_0 \in X$, the map between stalks, $A_{x_0} \to (f_*f^*A)_{x_0}$, is the map

$$\varinjlim_{x_0 \in U} A(U) \cong \varinjlim_{x_0 \in U} \Gamma(U, \mathsf{Et}\, A) \to \varinjlim_{x_0 \in U} \Gamma(f^{-1}U, \mathsf{Et}\, f^*A)$$

which is induced by the following pullback

$$\begin{tikzcd} \mathsf{Et}\,(f^*A) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"'] & \mathsf{Et}\, A \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \quad .$$

Now an element $s \in \Gamma(f^{-1}U, f^*A)$ is essentially the same as a continuous map $\bar{s} \colon f^{-1}U \to \mathsf{Et}\, A$ with $\pi \circ \bar{s} = f$. (Given s, let $\bar{s} = \tilde{f} \circ s$ and given $\bar{s}$ let s be the unique map $Y \to \mathsf{Et}\,(f^*A)$ in

$$\begin{tikzcd} Y \arrow[dr, dashed, "s"', "!"] \arrow[drr, bend left, "\bar{s}"] \arrow[ddr, bend right, "\mathrm{id}_Y"'] & & \\ & \mathsf{Et}\,(f^*A) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"'] & \mathsf{Et}\,(A) \arrow[d, "\pi"] \\ & Y \arrow[r, "f"'] & X \end{tikzcd} \quad .)$$

For each $x \in U$, the fiber $\pi^{-1}(x)$ is discrete and $\bar{s}$ sends $f^{-1}(x)$ into $\pi^{-1}(x)$. Thus, by connectivity of $f^{-1}(x)$, the restriction $s|_{f^{-1}(x)}$ is constant. Hence, there is a unique well-defined function

$$t\colon U \to \mathsf{Et}\,(A), \quad t = sf^{-1}.$$

Furthermore, this t is continuous if and only if s is, since f is a quotient map. This shows that $\Gamma(U, A) \to \Gamma(f^{-1}U, f^*A)$ is an isomorphism and proves the proposition. □

Corollary 7.7. *When $f\colon Y \to X$ is a quotient map with connected fibers and $R^q f_*(f^*A) = 0$ for $q > 0$,*

$$f^*\colon H^p(X; A) \to H^p(Y; f^*A)$$

is an isomorphism.

8. Homotopy Invariance

8.1. **Locally Constant Sheaves.** A sheaf A on a space X is said to be *locally constant* if each point $x \in X$ has a neighbourhood U such that $A|_U$ is isomorphic to a constant sheaf ΔS. Then $S = A_x$, the stalk of A at x. This is equivalent to requiring that $\mathsf{Et}\,(A) \to X$ is a covering projection (see Algebraic Topology Notes I.3.1). In particular,

Lemma 8.1. *A locally constant sheaf on a simply connected space is constant.*

For instance, any locally constant sheaf on the unit interval is constant.

Note also that if A is a locally constant sheaf on X, then for any map $f\colon Y \to X$, the sheaf f^*A is locally constant on Y.

Exercise 1. Let X be connected and locally simply connected, and choose a base point x_0. Show that for any locally constant sheaf (of sets) A the group $\pi_1(X, x_0)$ acts on the stalk A_{x_0}. Show that this gives a functor $A \mapsto A_{x_0}$ which is an equivalence of categories between the category of locally constant sheaves on X and the category of sets with a $\pi_1(X, x_0)$-action. (See Algebraic Topology Notes, Exercise ???)

8.2. **Homotopy Invariance.** In this section we prove homotopy invariance for cohomology with values in a locally constant sheaf. (Note that sheaf cohomology is not a homotopy invariant for arbitrary sheaves.)

Theorem 8.2. *Let $f, g\colon Y \to X$ be homotopic maps. Then for any locally constant sheaf A on X, $f^*A \cong g^*A$ and*

$$f^* = g^*\colon H^n(X; A) \to H^n(Y, f^*A),$$

for all $n \geq 0$.

We will prove this theorem using proper base change. In fact everything follows from the following result.

Proposition 8.3. *For any constant sheaf A on the unit interval $[0, 1]$,*

$$H^n([0, 1]; A) = 0 \qquad \textit{for } n > 0.$$

Proof of Theorem 8.2 using Proposition 8.3. Let $H\colon Y\times[0,1]\to X$ be a homotopy from f to g, and consider the diagram

$$\begin{tikzcd} Y \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[dr, "\mathrm{id}"'] & Y\times[0,1] \arrow[r, "H"] \arrow[d, "\pi"] & X \\ & Y, & \end{tikzcd}$$

where $i_0(y)=(y,0)$ and $i_1(y)=(y,1)$; hence, $\pi\circ i_0=\mathrm{id}_Y=\pi\circ i_1$. Since $H\circ i_0=f$ and $H\circ i_1=g$,

$$g^*=i_1^*H^*\colon H^n(X;A)\to H^n(Y;g^*A)$$

and

$$f^*=i_0^*H^*\colon H^n(X;A)\to H^n(Y;f^*A).$$

Thus, it suffices to show that $i_0^*=i_1^*$. But $\mathrm{id}=i_0^*\pi^*=i_1^*\pi^*$, so the result that $i_0^*=i_1^*$ will follow once we show that π^* is an isomorphism, $H^n(Y;\pi_*H^*A)\to H^n(Y\times[0,1];\pi^*\pi_*H^*A)\cong H^n(Y\times[0,1];H^*A)$. However, for any locally constant sheaf B on $Y\times[0,1]$, the restriction $B|_{\pi^{-1}(y)}$ is constant for each $y\in Y$ by Lemma 8.1. Now from Theorem 7.3 and Proposition 8.3 we get

$$(R^q\pi_*B)_y\cong H^q(\{y\}\times[0,1];B)=0 \text{ for } q>0,$$

since $\pi^{-1}(y)\cong[0,1]$ is Hausdorff. Thus, $\pi^*\colon H^q(Y;\pi_*B)\to H^q(Y\times[0,1];B)$ is an isomorphism as required. $\square$

Proof of Proposition 8.3. We first consider the case $A=\Delta\mathbb{Z}$. The exponential exact sequence of abelian groups,

$$0\to\mathbb{Z}\to\mathbb{R}\xrightarrow{\exp}S^1\to 0,$$

where $\exp(x)=e^{2\pi ix}$ yields for any space X an exact sequence of sheaves,

$$0\to\Delta\mathbb{Z}\to C_X(\mathbb{R})\to C_X(S^1)\to 0 \tag{31}$$

where $C_X(\mathbb{R})$ and $C_X(S^1)$ are the sheaves of continuous functions (or, germs of continuous functions) with values in $\mathbb{R}$, respectively S^1. For any paracompact space X, the sheaf $C_X(\mathbb{R})$ is soft (to show this, first use Lemma 5.17 to make a continuous function out of the germs and then use a partition of unity to extend a function on a closed subspace by zero), so $C_X(\mathbb{R})$ is acyclic. So the long exact cohomology sequence induced by (31) yields isomorphisms

$$\delta\colon H^n(X;C_X(S^1))\xrightarrow{\sim}H^{n+1}(X;\Delta\mathbb{Z}) \text{ for } n\geq 1. \tag{32}$$

In the special case that $X=[0,1]$, the sheaf $C_X(S^1)$ is also soft by Lemma 8.4 below and $\Gamma C_X(\mathbb{R})\to\Gamma C_X(S^1)$ is surjective (since any continuous function $[0,1]\to S^1$ can be lifted to a continuous function $[0,1]\to\mathbb{R}$). Thus, the long exact cohomology sequence obtained from (31) gives then $H^n(X;\Delta\mathbb{Z})=0$ for $n>0$.

Next, by induction on p and the exact sequence,

$$0\to\mathbb{Z}\to\mathbb{Z}^p\to\mathbb{Z}^{p-1}\to 0$$

it follows that $H^n([0,1];\mathbb{Z}^p)=0$ for each $p\geq 1$.

Now let A be a finitely generated abelian group. Then A fits into an exact sequence

$$0\to B\to\mathbb{Z}^p\to A\to 0$$

where B is a free group. Thus, in the exact sequence

$$0 \to \Delta B \to \Delta\mathbb{Z}^p \to \Delta A \to 0$$

of sheaves on $[0,1]$, both ΔB and $\Delta\mathbb{Z}^p$ are acyclic. Hence, ΔA is acyclic as well.

Finally, if A is arbitrary, write $A = \varinjlim_{i\in I} A_i$, where A_i ranges over all finitely generated subgroups of A. Let for each $i \in I$,

$$0 \to \Delta A_i \to B_i^0 \to B_i^1 \to B_i^2 \to \cdots$$

be some soft resolution of ΔA_i. Moreover, assume that these are chosen functorially in i; i.e., if $A_i \subseteq A_j$, then $B_i^k \hookrightarrow B_j^k$ and these inclusions are compatible with the d's. Then for each p, $\varinjlim_{i\in I} B_i^p$ is again soft by compactness (see Section **??** (9.??) below). Furthermore, again by compactness of $[0,1]$, for each p,

$$\Gamma \varinjlim_{i\in I} B_i^p \cong \varinjlim_{i\in I} \Gamma B_i^p.$$

Thus,

$$H^n([0,1]; \varinjlim_{i\in I} A_i) \cong \varinjlim_{i\in I} H^n([0,1]; A_i).$$

However, each A_i is finitely generated, so the right hand side vanishes for $n > 0$. Hence, $H^n([0,1]; A) = 0$ for $n > 0$, and this completes the proof of Proposition 8.3. $\square$

Lemma 8.4. *$C_{[0,1]}(S^1)$, the sheaf of (germs of) continuous maps on $[0,1]$ with values in S^1, is soft.*

Proof. Let $K \subseteq [0,1]$ be a closed subset and $f \in \Gamma(K, C_{[0,1]}(S^1))$. First use Lemma 5.17 to extend f to a function $g\colon U \to S^1$ on an open neighbourhood $U \supseteq K$. By compactness, we may assume that U is a union of finitely many disjoint open intervals $U = \bigcup_{i=1}^n (a_i, b_i)$. Now choose smaller closed intervals $[a_i', b_i'] \subset (a_i, b_i)$ such that $K \subseteq \bigcup_{i=1}^n [a_i', b_i'] = L$. Then lift $g|_L$ to a function $\tilde{g}\colon L \to \mathbb{R}$ and extend $\tilde{g}$ to a function $h\colon [0,1] \to \mathbb{R}$, connecting the pieces given by straight lines. Finally, let $\bar{f} = \exp(h)\colon [0,1] \to S^1$. This is the required extension of f. $\square$

Example 8.5. Let S^d be a d-dimensional sphere. For any abelian group A,

$$H^n(S^d; A) = \begin{cases} A & \text{when } n = 0 \text{ or } n = d \text{ provided } d \neq 0; \\ A \oplus A & \text{when } d = n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. **Exercise 2.** (Use induction on d. For $d = 0$ the statement is clear. For $d > 0$ write $S^d = N \cup S$ where N and S are the closed northern and southern hemispheres, so that $N \cap S \cong S^{d-1}$ and use the Mayer-Vietoris sequence.) $\square$

9. Cohomology with Compact Support

In this section all spaces are locally compact and Hausdorff. We will consider cohomology with compact support; i.e., we only consider those sections of a sheaf that have compact support. In general, this is not functorial in X, but when $f\colon Y \to X$ is proper there is a map

$$f^*\colon H_c^n(X; A) \to H_c^n(Y; f^*A), \text{ for all } n \geq 0.$$

Analogous to the case of ordinary cohomology there are long exact sequences for cohomology with compact support. We will also discuss change of base and homotopy invariance. Here again the connection with proper maps appears. We will prove invariance along proper homotopies. However, for base change we need no assumption of propriety.

9.1. **Definitions.** For a sheaf A on X, write

$$\Gamma_c(X, A) = \Gamma_c(A) = \{A \in A(X);\ \mathsf{supp}(a) \text{ is compact}\}.$$

Thus defined, Γ_c is a functor from $\mathsf{Ab}\,(X)$ to the category of abelian groups. Now define *cohomology with compact support* as

$$H^n_c(X; A) = R^n\Gamma_c(A).$$

Thus, $H^n_c(X; A)$ is computed by taking an injective resolution $0 \to A \to I^\bullet$ of A, applying Γ_c to it and taking the cohomology of the resulting complex.

Analogous to the notioin of acyclicity in Section 4.8 we have the notion of Γ_c-acyclicity: a sheaf B on X is said to be Γ_c*-acyclic* if $H^n_c(X; B) = 0$ for all $n \geq 0$. Exactly as in Proposition 4.7, it can be shown that if

$$0 \to A \to B^0 \to B^1 \to B^2 \to \cdots$$

is a resolution of A by Γ_c-acyclic sheaves, then

$$H^n(X; A) = H^n(\Gamma_c B^\bullet).$$

As a special kind of Γ_c-acyclic sheaves we have the c-soft ones. A sheaf B on X is called *c-soft* if for any two compact subsets K and L with $K \subseteq L \subseteq X$, any section of B over K can be extended to a section over L; i.e., the restriction map

$$\Gamma(L, B) \to \Gamma(K, B)$$

is surjective. Note that this is a weakening of the definition of soft given in Section 5.2. Also, note that since X is assumed to be locally compact and Hausdorff, any section $s \in \Gamma(K, B)$ can at least be extended to an open neighbourhood $U \supseteq K$ by Lemma 5.17. If B is flabby, we can then extend the section over U to all of X. So in particular, every flabby sheaf is c-soft.

Proposition 9.1. *A c-soft sheaf on X is Γ_c-acyclic.*

As in the proof of Proposition 5.4 this follows fairly immediately from the following two lemmas.

Lemma 9.2. *If $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is an exact sequence of sheaves on X and A is c-soft, then $0 \to \Gamma_c A \to \Gamma_c B \to \Gamma_c C \to 0$ is exact.*

Proof. As before, it suffices to prove that $\Gamma_c B \to \Gamma_c C$ is surjective. So take a section $s \in C(X)$ with compact support, $K = \mathsf{supp}(s)$. Since $B \to C \to 0$ is exact, and K is compact, there is a finite cover $K \subseteq U_1 \cup U_2 \cup \ldots$
$cupU_n$ for which there are

$$b_i \in B(U_i) \text{ with } \psi(b_i) = s|_{U_i}.$$

Since X is locally compact we may also assume that each $\overline{U}_i$ is compact. Now choose a refinement $K \subseteq V_1 \cup V_2 \cup \ldots \cup V_n$ by open sets V_i with $\overline{V}_i \subseteq U_i$. We will use induction to construct a section

$$t_k \in \Gamma(\overline{V}_1 \cup \ldots \cup \overline{V}_n, B) \text{ with } \psi(t_k) = s, \tag{33}$$

for $k \in \{1, 2, 3 \ldots, n\}$. Here s stands for its restriction $s|_{\overline{V}_1 \cup \ldots \cup \overline{V}_k}$. Take $t_1 = b_1|_{\overline{V}_1}$. Suppose that $t_1, \ldots, t_k$ are defined. Then we have

$$\psi(t_k) = s = \psi(b_{k+1}) \text{ on } (\overline{V}_1 \cup \ldots \cup \overline{V}_k) \cap \overline{V}_{k+1}.$$

We will write M for $(\overline{V}_1 \cup \ldots \cup \overline{V}_k) \cap \overline{V}_{k+1}$ and by exactness at $\Gamma_c(B)$ there is a section $a \in \Gamma_c(M, A)$ with $t_k - b_{k+1} = \varphi(a)$ on M. Since A is c-soft while M and $\overline{V}_{k+1}$ are compact, we extend this a to a section $\tilde{a} \in \Gamma(\overline{V}_{k+1}, A)$.. Then t_k and $b_{k+1} + \varphi(\tilde{a})$ are compatible sections of B. Hence they can be amagamated to a section $t_{k+1} \in \Gamma(\overline{V}_1 \cup \ldots \cup \overline{V}_{k+1}, B)$. This completes the induction and produces the sections in (33). So we have a section

$$t_n \in \Gamma(\overline{V}_1 \cup \ldots \cup \overline{V}_n, B) \text{ with } \psi(t_n) = s.$$

Write $V = V_1 \cup \ldots \cup V_n$ and $U = U_1 \cup \ldots \cup U_n$. Now $\psi(t_n)|_{(\overline{V}-V)} = s|_{\overline{V}-V)} = 0$ since V contains the support K of s. Thus, by exactness at $\Gamma_c B$ there is a section $r \in \Gamma_c(\overline{V} - V, A)$ with $\varphi(r) = t_n|_{(\overline{V}-V)}$. Since A is c-soft we can extend r to a section $\overline{r} \in \Gamma(\overline{V}, A)$. Now let

$$t = t_n - \varphi(\overline{r}) \in \Gamma(\overline{V}, B).$$

Then $\psi(t) = \psi(t_n) = s|_{\overline{V}}$ and $\mathsf{supp}(t) \subseteq V \subseteq \overline{V}$. Hence, t is zero on the boundary of V and can be extended by zero; i.e., there is a section $\tilde{t} \in \Gamma(X, B)$ with $\tilde{t}|_V = t$ and $\tilde{t}|_{X-V} = 0$. Then $\tilde{t} \in \Gamma_c(B)$ and $\psi(\tilde{t}) = s$. □

Lemma 9.3. *If* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *is an exact sequence for which* A *and* B *are c-soft, then so is* C.

Proof. Take a section $s \in \Gamma(K, C)$ where K is a compact subset of X. Consider any bigger compact subset $L \supseteq K$. By Lemma 9.2 applied to K instead of X, there is a section $b \in \Gamma(K, B)$ with $\psi(b) = s$. Since B is c-soft, we can extend b to a section $\tilde{b} \in \Gamma(L, B)$. Then $\psi(\tilde{b})$ is the required extension of s to L. This shows that C is c-soft. □

Remarks 9.4. (a) If R is a c-soft sheaf of rings on X, then any R-module A is again c-soft. (An R-module is a sheaf such that for any open $U \subseteq X$, $A(U)$ is an $R(U)$-module and all restriction maps are R-linear.) Indeed, for a compact set $K \subseteq X$ and a section $a \in \Gamma(K, A)$, we first extend a to a section $\tilde{a} \in \Gamma(U, A)$ for a neighbourhood $U \supseteq K$ (by Lemma 5.17). Now choose a neighbourhood V with $K \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ compact. Since R is c-soft, there exists an $r \in \Gamma(\overline{V}, R)$ with $r|_K = 1$ and $r|_{(\overline{V}-V)} = 0$. Then $b = r \cdot (\tilde{a}|_{\overline{V}}) \in \Gamma(\overline{V}, A)$ agrees with a on K and is zero on the boundary $\overline{V} - V$. So we can extend b to a global section $\tilde{b} \in \Gamma(X, A)$ with $\tilde{b}|_K = a$. (This is even more than is required to show that A is c-soft.)

(b) The property of c-softness is a local property: if A is a sheaf on X such that for some open cover $\mathcal{U} = \{U_i;\, i \in I\}$ of X each $A|_{U_i}$ is a c-soft sheaf on U_i, then A itself is c-soft on X. The proof is analogous to (but simpler than) the proof of Remark 5.12. (It can also be deduced as a consequence of that remark.)

9.2. **Functoriality.** As for ordinary cohomology, a homomorphism of sheaves $A \to B$ induces a homomorphism in cohomology with compact support $H^\bullet_c(X;A) \to H^\bullet_c(X;B)$, and a short exact sequence of sheaves,

$$0 \to A \to B \to C \to 0$$

induces a long exact sequence in cohomology with compact support.

However, contrary to the situation in ordinary cohomology, a map $f\colon Y \to X$ of topological spaces does not generally induce a map $f^*\colon H^n_c(X;A) \to H^n_c(Y;f^*A)$ for $A \in \mathsf{Ab}(X)$. We will now consider the construction of the map f^* a bit more closely. Let A be an abelian sheaf on X and construct injective resolutions

$$0 \to A \to I^\bullet$$

of sheaves on X, and

$$0 \to f^*A \to J^\bullet$$

of sheaves on Y. Then extend the unit of the adjunction,

$$\eta_A\colon A \to f_*f^*A$$

to a map of sheaves of cochain complexes,

$$I^\bullet \to f_*J^\bullet,$$

and then to a map of sections with compact support,

$$\Gamma_c(X, I^\bullet) \to \Gamma_c(X, f_*J^\bullet).$$

Thus far we have been able to do everything completely analogous to what we did for ordinary cohomology. However, at this point we need that

$$\Gamma_c(X, f_*J^\bullet) \cong \Gamma_c(Y, J^\bullet) \tag{34}$$

to obtain a map between cohomology groups, since the right hand side computes the cohomology of f^*A on Y. However, (34) only holds when f is proper as we will prove in Lemma 9.5 below. So in that case we obtain a map

$$f^*\colon H^n_c(X;A) \to H^n_c(Y;f^*A) \text{ for all } n \geq 0,$$

and we conclude that $H^n_c(X;A)$ is *contravariant in X along proper maps.*

Lemma 9.5. *For a proper map $f\colon Y \to X$ and any sheaf B on Y,*

$$\Gamma_c(X, f_*B) \cong \Gamma_c(Y, B).$$

Proof. Since $\Gamma(X, f_*B) = \Gamma(Y, B)$, a section $b \in f_*B(X)$ is the same as a section $b \in B(Y)$. So we only need to show that $\mathsf{supp}_Y(b)$ is compact if and only if $\mathsf{supp}_X(b)$ is compact. Now recall that the support of an element of a sheaf is always closed. So if $\mathsf{supp}_Y(b)$ is compact then $\mathsf{supp}_X(b) \subseteq f(\mathsf{supp}_Y(b))$ is also compact as a closed subset of a compact subset in a Hausdorff space.

For the converse direction, if $\mathsf{supp}_X(b)$ is compact, then $f^{-1}(\mathsf{supp}_X(b))$ is compact because f is proper. Hence, $\mathsf{supp}_Y(b) \subseteq f^{-1}(\mathsf{supp}_X(b))$ is again compact as a closed subset of a compact set. $\square$

9.3. **The Functor $f_!$.** For an arbitrary map $f\colon Y \to X$, we can obtain an isomorphism as in the lemma above by replacing the functor f_* by

$$f_!\colon \mathsf{Ab}\,(Y) \to \mathsf{Ab}\,(X)$$

defined for a sheaf A on Y and an open subset $U \subseteq Y$ by

$$f_!(A)(U) = \{a \in A(f^{-1}U);\ f \text{ restricts to a proper map } \mathsf{supp}(a) \to U\}.$$

(However, note that this does not help us to maker $H^n_c(X; A)$ covariant in X along arbitrary maps, because $f_!$ need not be right adjoint to f^*.) To define the restriction maps of $f_!(A)$ remark that for $V \subseteq U$ and $a \in A(f^{-1}U)$, the square,

$$\begin{tikzcd} \mathsf{supp}(a|_{f^{-1}V}) \arrow[r] \arrow[d] & \mathsf{supp}(a) \arrow[d, "f"] \\ V \arrow[r] & U \end{tikzcd}$$

is a pullback. Hence $\mathsf{supp}(a|_{f^{-1}V}) \to V$ is proper whenever $\mathsf{supp}(a) \to U$ is. Thus, one can define the restriction $f_!(A)(U) \to f_!(A)(V)$ in the evident way, just as for f_*A. With these defined, one readily checks that $f_!(A)$ is again a sheaf. This is clear from the fact that for a map $g\colon Z \to X$ and an open cover $X = \bigcup_{i\in I} U_i$, the map g is proper if and only if all its restrictions $g^{-1}U_i \to U_i$ are (propriety is a local property).

We now list some properties of $f_!$:

Remarks 9.6. (a) For an inclusion $f\colon Y \hookrightarrow X$ of an open or locally closed subspace, the functor $f_!$ defined here is the same as the one constructed in Section 3. (Indeed, an inclusion of a subspace $S \hookrightarrow X$ is proper if and only if the subspace is closed.)

(b) For a proper map $f\colon Y \to X$ the functor $f_!$ is identical to f_*.

(c) For any map $f\colon Y \to X$ and any sheaf B on Y, we have

$$\Gamma_c(X, f_!B) \cong \Gamma_c(Y, B),$$

or succinctly, $\Gamma_c \circ f_! \cong \Gamma_c$. The proof is similar to that of Lemma 9.5 and left as an exercise.

(d) *The functor $f_!$ sends c-soft sheaves to c-soft sheaves.* To prove this, take a c-soft B on Y and compact sets $K \subseteq L$ in X. Suppose $s \in \Gamma(K, f_!A)$. By Lemma 7.5 we can extend s to a section $b \in \Gamma(U, f_!B)$ for a neighbourhood U of K. So $b \in B(f^{-1}U)$ and $f\colon \mathsf{supp}(b) \to U$ is proper. Choose an open subset V with $K \subseteq V \subseteq \overline{V} \subseteq U$ such that $\overline{V}$ is compact. (Remember that all spaces are assumed to be locally compact.) Then $S := \mathsf{supp}(a|_{f^{-1}V}) \to \overline{V}$ is proper and hence S is compact. Now let $W \supseteq S$ be a relatively compact neighbourhood of S in Y, and consider the section $\tilde{b}$ on $(\overline{W} \cap f^{-1}(K)) \cup (\overline{W} - W)$ which agrees with b on $\overline{W} \cap f^{-1}(K)$ and is zero on $\overline{W} - W$. Since B is c-soft, we can extend $\tilde{b}$ to a section $\bar{b}$ on $\overline{W}$. Finally, extend $\bar{b}$ by zero to a section on all of Y. Then $\bar{b} \in \Gamma_c(Y, B)$ since the support of $\bar{b}$ is contained in the compact set $\overline{W}$. Thus, $\bar{b}$ restricts to a section of $f_!(B)$ over the compact set L; i.e., $\bar{b}|_{f^{-1}L} \in \Gamma(L, f_!B)$. Furthermore, $\bar{b}$ extends the given section $s \in \Gamma(K, f_!A)$. For if $y \in f^{-1}K$, then either $y \in \overline{W}$ in which case $b(y) = a = s(y)$, or $y \in f^{-1}K - W$ in which case both b and s vanish at y since W contains the support of s.

(e) It follows that $f_!$ sends c-soft sheaves to Γ_c-acyclic sheaves. Thus, there is a Grothendieck spectral sequence,

$$E_2^{p,q} = H_c^p(X; R^q f_! A) \Rightarrow H_c^{p+q}(Y; A),$$

for any sheaf A on Y.

9.4. **Exact Sequences.** In this and the next sections we will briefly discuss exact sequences for cohomology with compact support (analogous to Section 6) as well as base-change and homotopy invariance (as in Sections 7 and 8).

Proposition 9.7. *For an open subspace $Y \subseteq X$ and its closed complement $Z = X - Y$, there is an exact sequence,*

$$\cdots \to H_c^n(Y; A|_Y) \to H_c^n(X; A) \to H_c^n(Z; A|_Z) \to H_c^{n+1}(Y; A|_Y) \to \cdots$$

for any sheaf A on X.

Proof. As in Section 6 we write $i\colon Y \hookrightarrow X$ and $j\colon Z \hookrightarrow X$ for the inclusion maps. Recall from Lemma 6.2 that there is a short exact sequence,

$$0 \to i_! i^* B \to B \to j_* j^* B \to 0,$$

for any sheaf B on X. Applying this to an injective resolution $I^\bullet$ of the given sheaf A yields an exact sequence of chain complexes,

$$0 \to i_! i^* I^\bullet \to I^\bullet \to j_* j^* I^\bullet \to 0.$$

Here, $i^*(I^n)$ is injective since i^* preserves injectives and $i_! i^*(I^n)$ is c-soft by Remark 9.6(4). So by Lemma 9.2 the sequence

$$0 \to \Gamma_c i_! i^* I^\bullet \to \Gamma_c I^\bullet \to \Gamma_c j_* j^* I^\bullet \to 0 \tag{35}$$

is again exact. Since $\Gamma_c \circ i_! = \Gamma_c$ by Remark 9.6(3) and $\Gamma_c \circ j_* = \Gamma_c \circ j_! = \Gamma_c$ because j is proper, the long exact sequence obtained from (35) is exactly the one in the statement of this proposition. □

Corollary 9.8. *For a sheaf A on X, A is c-soft if and only if $H_c^1(U; A) = 0$ for any open $U \subseteq X$ with $X - U$ compact.*

Proof. For the '⇒' direction, if A is c-soft, so is $A|_U$. Hence, $A|_U$ is Γ_c-acyclic.

For the '⇐' direction, let $K \subseteq X$ be compact. Then the long exact sequence of Proposition 9.7 reduces to

$$0 \to \Gamma_c(X - K, A) \to \Gamma_c(X, A) \to \Gamma_c(K, A) \to 0.$$

So clearly, any section of A over K can be extended to a section over X. □

9.5. **Mayer-Vietoris Sequences.** We have again two versions of the Mayer-Vietoris sequence: one for closed, and one for open subspaces.

Recall from (23) in Section 6 that *closed* subspaces $S, T \subseteq X$ with inclusion maps $h\colon S \cap T \hookrightarrow X$, $i\colon S \hookrightarrow X$, $j\colon T \hookrightarrow X$, $k\colon S \cup T \hookrightarrow X$ yield an exact sequence of sheaves

$$0 \to k_! k^* A \to i_! i^* A \oplus j_! j^* A \to h_! h^* A \to 0 \tag{36}$$

for any sheaf A on X. Since k is a closed inclusion, it is proper. Hence $k_* = k_!$ is exact and preserves c-softness and the same applies to h, i, and j. So we obtain the following Mayer-Vietoris sequence for closed sets:

$$\cdots \to H_c^n(S \cup T; A) \to H_c^n(S; A) \oplus H_c^n(T; A) \to H_c^n(S \cap T; A) \to \cdots \tag{37}$$

For *open* subsets $S = U$ and $T = V$ in X, the sequence (36) is again exact and we obtain

$$(38) \qquad \cdots \to H_c^n(U \cup V; A) \to H_c^n(U; A) \oplus H_c^n(V; A) \to H_c^n(U \cap V; A) \to \cdots$$

9.6. **Base Change.** Analogous to the results in Sections 7 and 8, we obtain:

Proposition 9.9. *For any map $f: Y \to X$ and any sheaf B on Y,*

$$(R^n f_! B)_x \cong H_c^n(f^{-1}x, B).$$

Corollary 9.10. *For a pullback square of locally compact Hausdorff spaces,*

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd} ,$$

the canonical map

$$p^*(R^n f_!(B)) \to R^n f'_!(q^* B)$$

is an isomorphism for any sheaf B on Y.

Corollary 9.11. *Let $f, g: Y \rightrightarrows X$ be proper maps which are homotopic via a proper homotopy $h: Y \times [0,1] \to X$. Then for any locally constant sheaf A on X,*

$$f^* = g^*: H_c^\bullet(X; A) \to H_c^\bullet(Y; f^* A).$$

Proof. The proofs of Corollaries 9.10 and 9.11 given Proposition 9.9 are completely analogous to those given in Sections 7 and 8. The proof of Proposition 9.9 is analogous to that of Theorem 7.3. There the argument was based on the isomorphism,

$$\varinjlim_{x \in U} f_* B(U) \xrightarrow{\sim} \Gamma(f^{-1}(x), B),$$

for a proper map f. Instead, for Proposition 9.9 we prove

$$(39) \qquad \theta_B: \varinjlim_{x \in U} f_! B(U) \xrightarrow{\sim} \Gamma_c(f^{-1}x, B).$$

We will prove (39) for c-soft sheaves B and the deduce the general case from this one using a c-soft resolution. So let B be a c-soft sheaf. To see that θ_B is *surjective*, let $s \in \Gamma(f^{-1}x, B)$ be a section with compact support $K \subseteq f^{-1}x$. Let $W \supseteq K$ be a relatively compact neighbourhood of K, and let $\tilde{s} \in \Gamma((\overline{W} \cap f^{-1}(x)) \cup (\overline{W} - W), B)$ be the section which agrees with s on $\overline{W} \cap f^{-1}(x)$ and is zero on $\overline{W} - W$. Since B is assumed to be c-soft, we can extend $\tilde{s}$ to a section $\bar{s}$ on $\overline{W}$ and then by zero to a section $s' \in \Gamma_c(Y, B) = \Gamma_c(X, f_! B)$. Thus, θ_B is *surjective*; in fact, the morphism $f_! B(X) \to \Gamma_c(f^{-1}(x), B)$ is.

To see that θ_B is *injective*, take $a \in f_! B(U)$ where U Is an open neighbourhood of x, and suppose that $a|_{f^{-1}(x)} = 0$. Thus, $x \notin f(\mathsf{supp}(a))$. The map $f\colon \sup(a) \to U$ is proper and hence closed, so this implies that there is a neighbourhood $V \subseteq U$ of x such that $V \cap f(\mathsf{supp}(a)) = \emptyset$. Thus, $a|_{f^{-1}(V)} = 0$. This shows that (39) is one to one. Note that this argument works for any sheaf B, not just a c-soft one.

To prove surjectivity of θ_B for arbitrary B, choose a c-soft (for instance, injective) resolution,

$$0 \to B \to I^0 \to I^1 \to \cdots$$

Since the θ_{I^n} for $n \geq 0$ are isomorphisms and the rows in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{x\in U} f_! B(U) \arrow[r] \arrow[d, "\theta_B"'] & \varinjlim_{x\in U} f_! I^0(U) \arrow[r] \arrow[d, "\theta_{I^0}"'] & \varinjlim_{x\in U} f_! I^1(U) \arrow[d, "\theta_{I^1}"'] \\
0 \arrow[r] & \Gamma_c(f^{-1}x, B) \arrow[r] & \Gamma_c(f^{-1}x, I^0) \arrow[r] & \Gamma_c(f^{-1}x, I^1)
\end{tikzcd}$$

are exact (the reader may check this), it follows that θ_B is an isomorphism. $\square$

9.7. **Directed Colimits.** Cohomology with compact support commutes with directed colimits.

Proposition 9.12. *For a directed system $\{A_i;\, i \in I\}$ of abelian sheaves on X, the canonical map*

$$\varinjlim_{i\in I} H_c^n(X; A_i) \to H_c^n(X; \varinjlim_{i\in I} A_i) \tag{40}$$

is an isomorphism.

Proof. The sheaf $\varinjlim_{i\in I} A_i$ is computed by first taking, naively, the directed colimit

$$P(U) = \varinjlim_{i\in I} A_i(U)$$

for each open set $U \subseteq X$ and $\varinjlim_{i\in I} A_i$ is the associated sheaf of P. A standard compactness argument shows that for a compact set $K \subseteq X$,

$$\Gamma(K, \varinjlim_{i\in I} A_i) \cong \varinjlim_{i\in I} \Gamma(K, A_i).$$

(Use that $(\varinjlim_{i\in I} A_i)_x \cong \varinjlim_{i\in I} (A_i)_x$ for all $x \in X$.) In particular,

$$\Gamma_c(X, \varinjlim_{i\in I} A_i) \cong \varinjlim_{i\in I} (\Gamma_c(X, A_i)),$$

and $\varinjlim_{i\in I} A_i$ is c-soft if and only if each A_i is. The isomorphism (40) follows by taking a functorial family of c-soft resolutions

$$0 \to A_i \to B_i^0 \to B_i^1 \to B_i^2 \to \cdots \quad .$$

$\square$

9.8. **Example.** Cohomology with compact support is not invariant under arbitrary homotopy (Corollary 9.11 gives only invariance under proper homotopies), but a nice property is that it distinguishes between $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$. More generally, it can be used to define the cohomological dimension of a locally compact Hausdorff space as we will see in the next section.

Proposition 9.13. *For any abelian group A,*

$$H_c^n(\mathbb{R}^d; A) = \begin{cases} A & \text{if } n = d \\ 0 & \text{if } n \neq d. \end{cases}$$

(On the left hand side, A denotes the constant sheaf.)

Proof. Pick a point $p \in S^d$, so that $S^d - \{p\} \cong \mathbb{R}^d$, and consider the long exact sequence of Proposition 9.7,

$$\cdots \to H_c^n(S^d - \{p\}; A) \to H_c^n(S^d; A) \to H_c^n(\{p\}; A) \to H_c^{n+1}(S^d - \{p\}; A) \to \cdots \quad .$$

Now, $H^n_c(\{p\}; A) = A$ if $n = 0$ and is zero otherwise. Furthermore, $H^n_c(S^d; A) = H^n(S^d; A)$ and this group was computed in Example 8.5. The result follows immediately. $\square$

For an open disk $D \subseteq \mathbb{R}^d$, there is an exact sequence

$$\cdots \to H^n_c(D; A) \to H^n_c(\mathbb{R}^d; A) \to H^n_c(\mathbb{R}^d - D; A) \to \cdots \quad .$$

We claim that $H^n_c(\mathbb{R}^d - D; A) = 0$ for all $n \geq 0$, so that

$$H^n_c(D; A) \xrightarrow{\sim} H^n_c(\mathbb{R}^d; A) \tag{41}$$

becomes an isomorphism. Indeed, if we identify $\mathbb{R}^d$ with $S^d - \{p\}$, then $\mathbb{R}^d - D \cong S^d - (D \cup \{p\})$ where D is an open disk on S^d which does not contain the point p. Now consider the first long exact sequence of Proposition 9.7 for $X = S^d - D$, $Y = S^d - (D \cup \{p\})$ and $Z = \{p\}$. Since X and $\{p\}$ are both compact and contractible, the result follows.

10. Cohomological Dimension

10.1. General Properties.

Definition 10.1. For a locally compact Hausdorff space X, its *cohomological dimension* $\mathsf{dim}(X)$ is the smallest $d \in \mathbb{N} - \{\infty\}$ with the property that

$$H^n_c(X; A) = 0 \text{ for every sheaf } A \text{ on } X \text{ and every } n > d.$$

Remark 10.2. Let $U \subseteq X$ be an *open* subset with inclusion map $i\colon U \hookrightarrow X$. Then for any sheaf A on U,

$$H^n_c(U; A) = H^n_c(X; i_!A),$$

since $i_!$ is exact and preserves c-softness. Thus, $\mathsf{dim}(U) \leq \mathsf{dim}(X)$. Similarly, if $j\colon Z \hookrightarrow X$ is the inclusion of a *closed* subspace, then for every sheaf A on Z,

$$H^n_c(Z; A) = H^n_c(X; j_*A).$$

So again, $\mathsf{dim}(Z) \leq \mathsf{dim}(X)$. Combining these two cases as in Section 3.4, it follows that for a *locally closed* $Z \subseteq X$ we have $\mathsf{dim}(Z) \leq \mathsf{dim}(X)$.

Proposition 10.3. *If $\{U_i;\, i \in I\}$ is an open cover of X and $\mathsf{dim}(U) \leq d$ for each index $i \in I$, then $\mathsf{dim}(X) \leq d$.*

Proof. We first show that for open subsets V and W of X, if $\mathsf{dim}(V) \leq d$ and $\mathsf{dim}(W) \leq d$ then also $\mathsf{dim}(V \cup W) \leq d$. Indeed, let A be a sheaf on $V \cup W$ and consider the long exact sequence of Proposition 9.7,

$$\cdots \to H^n_c(V; A|_V) \to H^n_c(V \cup W; A) \to H^n_c(W - V; A|_{W-V}) \to H^{n+1}_c(V; A|_V) \to \cdots$$

By assumption, $H^n_c(V; A) = 0$ for $n > d$ and by Remark 10.2, $H^n_c(W - V; A|_{W-V}) = 0$ for $n > d$ as well. It follows then from exactness that $H^n_c(V \cup W; A) = 0$ for $n > d$. For the cover $\mathcal{U} = \{U_i;\, i \in I\}$ we conclude by induction that for any finite subset $\{i_0, \ldots, i_m\} \subseteq I$ of indices,

$$\mathsf{dim}(U_{i_0} \cup \ldots \cup U_{i_m}) \leq d.$$

Now suppose that $n > d$ and $[\gamma] \in H^n_c(X; A)$ is a cohomology class, represented by a cycle $\gamma \in \Gamma_c(B^n)$ for some c-soft resolution $0 \to A \to B^\bullet$ of A. Let $K = \mathsf{supp}(\gamma)$.

Since K is compact, $K \subseteq U_{i_0} \cup \ldots \cup U_{i_m}$ for some finite set of indices $\{i_0, \ldots, i_m\}$. Write $U = U_{i_0} \cup \ldots \cup U_{i_m}$. Then the long exact sequence of Proposition 9.7 contains

$$\begin{array}{ccccccc} \cdots \longrightarrow H^n(U;A) \longrightarrow & H_c^n(X;A) & \longrightarrow & H_c^n(X-U;A) \longrightarrow \cdots \\ & [\gamma] & \longmapsto & 0 \end{array}$$

since the support of γ is contained in U. For $n > d$, $H^n(U;A) = 0$ by the argument above, and hence $H_c^n(X;A) \longrightarrow H_c^n(X-U;A)$ is injective. Thus, $[\gamma] = 0$ and we conclude that $H_c^n(X;A) = 0$ for $n > d$. □

Proposition 10.4. *Let X be a locally compact Hausdorff space. Then $\mathsf{dim}(X) \leq d$ if and only if every sheaf A on X has a c-soft resolution of length less than or equal to d; i.e., a resolution $0 \to A \to B^0 \to B^1 \to \cdots$ where $B^n = 0$ for $n > d$.*

Proof. The right to left implication is clear.

For the left to right implication, suppose that $\mathsf{dim}(X) \leq d$. Let

$$0 \to A \to B^0 \to B^1 \to \cdots \tag{42}$$

be a possibly infinite c-soft resolution. If $d = \infty$ there is nothing to prove. If $d < \infty$, consider

$$C^{d-1} = \mathsf{im}(B^{d-1} \to B^d).$$

It is enough to show that C^{d-1} is c-soft, because then we can replace the resolution (42) by

$$\cdots \to A \to B^0 \to B^1 \to \cdots \to B^{d-1} \to C^{d-1} \to 0$$

which is of length d.

By Corollary 9.8 we can show that C^{d-1} is c-soft by showing that $H_c^1(U, C^{d-1}) = 0$ for any open $U \subseteq X$. Since (42) is exact, we have that

$$C^n = \mathsf{im}(B^n \to B^{n+1}) = \mathsf{ker}(B^{n+1} \to B^{n+2})$$

for any $n \geq 0$. Hence, there are short exact sequences,

$$0 \to A \to B^0 \to C^0 \to 0 \quad \text{and} \quad 0 \to C^n \to B^{n+1} \to C^{n+1} \to 0.$$

Now use the corresponding long exact sequences in cohomology and the fact that the B^i are c-soft to obtain isomorphisms

$$H_c^i(U;C^0) \xrightarrow{\sim} H_c^{i+1}(U;A) \quad \text{and} \quad H_c^i(U;C^{n+1}) \xrightarrow{\sim} H_c^{i+1}(U;C^n).$$

These can be composed to obtain

$$H_c^1(U;C^{d-1}) \cong H^2(U;C^{d-2}) \cong \cdots \cong H^{d+1}(U;A),$$

and this last group is zero since $\mathsf{dim}(U) \leq \mathsf{dim}(X) \leq d$. □

10.2. **Dimension of a Manifold.** In this section we will show that a d-dimensional manifold M in the sense that M is locally homeomorphic to $\mathbb{R}^d$ has cohomological dimension d in the sense that $\mathsf{dim}(M) = d$.

We start with zero-dimensional manifolds; i.e., discrete spaces. Since any sheaf on a discrete space is soft, we have:

Lemma 10.5. *For a discrete space X, $\mathsf{dim}(X) = 0$.*

We now proceed by induction using the following lemma.

Lemma 10.6. *For a locally compact space X,*

$$\mathsf{dim}(X \times \mathbb{R}) \leq \mathsf{dim}(X) + 1.$$

Proof. If $\mathsf{dim}(X) = \infty$ there is nothing to prove. So assume $\mathsf{dim}(X) = d < \infty$. Let A be a sheaf on $X \times \mathbb{R}$, and let $0 \neq [\gamma] \in H^n_c(X \times \mathbb{R}; A)$ where $n > d+1$. Since the support of γ is compact, there is a closed interval $[a,b]$ such that (writing γ for the evident restriction),

$$0 \neq [\gamma] \in H^n_c(X \times [a,b]; A).$$

Let $c = \frac{1}{2}(a+b)$. Then the Mayer-Vietoris sequence (37) in Section 9.5 for $X \times [a,b] = (X \times [a,c]) \cup (X \times [c,b])$ contains

$$\begin{array}{c} H^{n-1}_c(X \times \{c\}; A) \longrightarrow H^n_c(X \times [a,b]; A) \longrightarrow H^n_c(X \times [a,c]; A) \oplus H^n_c(X \times [c,b]; A) \\ \shortparallel \\ 0 \end{array}$$

where $H^{n-1}_c(X \times \{c\}; A) = H^{n-1}_c(X; A) = 0$ since $n-1 > d = \mathsf{dim}(X)$. Hence the morphism $H^n_c(X \times [a,b]; A) \longrightarrow H^n_c(X \times [a,c]; A) \oplus H^n_c(X \times [c,b]; A)$ is injective and therefore, $[\gamma] \neq 0$ in $H^n_c(X \times [a,c]; A)$ or $[\gamma] \neq 0$ in $H^n_c(X \times [c,b]; A)$. Proceeding in this way, we find a sequence of intervals $[a,b] = [a_0,b_0] \supset [a_1,b_1] \supset [a_2,b_2] \supset \cdots$ of strictly decreasing length such that

$$[\gamma] \neq 0 \text{ in } H^n_c(X \times [a_i,b_i]; A) \quad \text{for } i = 1,2,\ldots \quad .$$

By Lemma 10.7 below we conclude that $[\gamma] \neq 0$ in $H^n_c(X \times \{p\}; A) \cong H^n_c(X; A)$ where $\{p\} = \bigcap_{i \geq 1}[a_i,b_i]$. This contradicts the assumption that $\mathsf{dim}(X) = d < n$. □

Lemma 10.7. *Let $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$ be a decreasing sequence of closed subsets of X. Then for any sheaf A on X,*

$$\varinjlim_k H^n_c(Z_k; A) \cong H^n_c(\bigcap_{k \geq 0} Z_k; A).$$

Proof. Write $j_k \colon Z_k \hookrightarrow X$ and $j \colon \bigcap_{k \geq 0} Z_k \hookrightarrow X$ for the inclusion maps. Then

$$\begin{array}{rcll} H^n_c(Z_k; A) & = & H^n_c(Z_k; j_k^* A) & \text{(by definition)} \\ & = & H^n_c(X; (j_k)_*(j_k)^* A) & \text{(by Lemma 9.5).} \end{array}$$

There is a canonical map

$$(j_k)_*(j_k)^*(A) \to (j_{k+1})_*(j_{k+1})^*(A)$$

for each k, since $j_{k+1} = j_k \circ h_k$ where $h_k \colon Z_{k+1} \hookrightarrow Z_k$. By the explicit description of the functors involved we find that $\varinjlim_k (j_k)_*(j_k)^*(A) \cong j_* j^*(A)$. The lemma follows by Proposition 9.12. □

Proposition 10.8. *A manifold M of ordinary dimension d has cohomological dimension d.*

Proof. By Lemmas 10.5 and 10.6 and Proposition 10.3 we derive that $\mathsf{dim}(M) \leq d$ (where dim still denotes the cohomological dimension).

On the other hand, $H^d_c(\mathbb{R}^d; \Delta\mathbb{Z}) \neq 0$, so for an open subset U of M that is diffeomorphic to $\mathbb{R}^d$, $H^d_c(U; \Delta\mathbb{Z}) \neq 0$. Also, $H^d_c(U; \Delta\mathbb{Z}) \cong H^d_c(M; i_!(\Delta\mathbb{Z}))$ where $i \colon U \hookrightarrow M$ is the inclusion map. Thus, $\mathsf{dim}(M) \geq d$.

We conclude that $\mathsf{dim}(M) = d$. □

11. Dual Sheaves

As before, all spaces are assumed to be locally compact and Hausdorff. From now on, we shall work with sheaves of k-modules for a fixed *field* k, and the associated categories of complexes, denoted by $\mathsf{Ch}_k(X)$ and $\mathsf{Ch}_k^+(X)$. (With some additional work these results can also be proved for the case where k is a Noetherian commutative ring.)

11.1. **The Dual Sheaf.**

Definition 11.1. For a k-sheaf A on X, define a presheaf $\check{A}$ by setting for each open $U \subseteq X$,

$$\begin{aligned}\check{A}(U) &= \mathsf{Hom}_k(\Gamma_c(U,A),k)\\ &= \Gamma_c(U,A)\check{},\end{aligned}$$

the dual of the k-module $\Gamma_c(U,A)$. For $V \subseteq U$, extension by zero defines a map $\Gamma_c(V,A) \to \Gamma_c(U,A)$; the dual of this map is a map

$$\rho_{VU} \colon \check{A}(U) \to \check{A}(V).$$

These maps ρ_{VU}, for all inclusions $V \subseteq U$ of open subsets, give $\check{A}$ the structure of a presheaf.

Lemma 11.2. *If A is c-soft, then $\check{A}$ is a sheaf (called the* dual sheaf*).*

Proof. It is sufficient to show that A has the sheaf property with respect to (i) directed covers and (ii) finite covers. (Recall that for an arbitrary cover $\mathcal{U}$ of X, $\mathcal{V} = \{V;\, V = U_1 \cup U_2 \cup \ldots \cup U_n \text{ for some } n \in \mathbb{N},\, U_i \in \mathcal{U}\}$ is a directed cover of X.) For (i), if $U = \bigcup_i U_i$ is a directed union, then clearly (by Proposition 9.12),

$$\Gamma_c(U,A) \cong \varinjlim_i \Gamma_c(U_i,A).$$

Hence, by taking duals,

$$\begin{aligned}\check{A}(U) &\cong (\varinjlim_i \Gamma_c(U_i,A))\check{}\\ &\cong \varprojlim_i \Gamma_c(U_i,A)\check{}\\ &= \varprojlim_i \check{A}(U_i).\end{aligned}$$

This expresses that $\check{A}$ has the sheaf property with respect to the cover $U = \bigcup_i U_i$, as desired.

For (ii), it suffices to consider a cover $U = V \cup W$ by two open sets (the case for n open sets is trivial for $n = 0$ and $n = 1$ and the rest follows by induction). Since A is assumed c-soft, we have $H_c^1(V \cap W; A) = 0$ by Corollary 9.8. Hence, the exact Mayer-Vietoris sequence (38) starts out as

$$0 \to \Gamma_c(V \cap W, A) \to \Gamma_c(V,A) \oplus \Gamma_c(W,A) \to \Gamma_c(U,A) \to 0 \quad .$$

Dualizing, one obtains an exact sequence

$$0 \leftarrow \check{A}(V \cap W) \leftarrow \check{A}(V) \oplus \check{A}(W) \leftarrow \check{A}(U) \leftarrow 0 \quad .$$

This expresses that $\check{A}$ has the sheaf property with respect to $U = V \cup W$. $\square$

11.2. **The Sheaves k_U.** Let $k_U = i_!(\Delta k)$ where $i\colon U \hookrightarrow X$ is the inclusion of an open subset $U \subseteq X$. (Explicitly, $k_U(W) = \{f\colon W \to k;\ f \text{ is locally constant, } f(x) = 0 \text{ when } x \notin U\}$, defined analogously to the definition of $\mathbb{Z}_U$ in the proof of Lemma 4.5. Then for each k-sheaf C there is a natural 1-1 correspondence

$$\frac{\alpha \in C(U)}{\hat{\alpha}\colon k_U \to C} \tag{43}$$

given by $\hat{\alpha}(1) = \alpha$. In particular, every sheaf C is covered by a sum of copies of k_U for various opens U. For instance, there is an epimorphism,

$$\sum_{\substack{U \subseteq X \text{open} \\ \alpha \in U}} k_U \xrightarrow{\sum \hat{\alpha}} C. \tag{44}$$

Note also that for any k-sheaf C, there is an isomorphism

$$k_U \otimes C \cong i_! i^*(C). \tag{45}$$

Indeed, for any other k-sheaf D, there are 1-1 correspondences,

$$\begin{array}{cl}
k_U \otimes_k C \to D & \\
\hline
& \text{(cf. Appendix ??)} \\
k_U \to \mathsf{Hom}(C, D) & \\
\hline
& \text{by (43)} \\
\alpha \in \mathsf{Hom}(C, D)(U) & \\
\hline
& \text{(by the definition of } \mathsf{Hom}) \\
\alpha\colon C|_U \to D|_U & \\
\hline
& \text{(for } i\colon U \hookrightarrow X \text{ the inclusion)} \\
\alpha\colon i^*C \to i^*D & \\
\hline
& \text{(by the adjunction } i_! \dashv i^*) \\
i_! i^*(C) \to D &
\end{array}$$

Since these correspondences hold for all k-sheaves D, the isomorphism (45) follows by Yoneda. As a special case, for $C = k_V$, we have

$$k_U \otimes k_V \cong k_{U \cap V}. \tag{46}$$

This last isomorphism is also easy to check directly. Alternatively, it is a pretty formal consequence of (45). Indeed, for any pullback square as in Corollary 9.10, the isomorphism given there reads for $n = 0$ as, $p^* f_! \cong f_! q^*$. In particular, for inclusions $i\colon U \hookrightarrow X$ and $j\colon V \hookrightarrow X$ we obtain,

$$\begin{tikzcd}
U \cap V \arrow[r, "k"] \arrow[d, "h"'] & V \arrow[d, "j"] \\
U \arrow[r, "i"'] & X
\end{tikzcd} \qquad \begin{array}{rcl} i^* j_! & \cong & h_! k^*, \\ j^* i_! & \cong & k_! h^*. \end{array} \tag{47}$$

Thus,

$$\begin{aligned} k_U \otimes_k k_V &\cong i_! i^*(k_V) \\ &\cong i_! i^* j_! j^*(\Delta k) \\ &\cong i_! h_! k^* j^*(\Delta k) \qquad \text{(cf. Section 3.4 Exercise 9)} \\ &\cong (ih)_!(ik)^*(\Delta k) \\ &= k_{U\cap V}. \end{aligned}$$

From this, we get for any sheaf B on U and any $V \subseteq X$,

$$\Gamma_c(V, i_! B) \cong \Gamma_c(U \cap V, B). \tag{48}$$

Indeed, using that Γ_c commutes with functors of the form $f_!$ (see Section 9.3 and in particular, Remark 9.6 (3)),

$$\begin{aligned} \Gamma_c(V, i_! B) &= \Gamma_c(V, j^* i_! B) \\ &= \Gamma_c(X, j_! j^* i_! B) \\ &= \Gamma_c(X, j_! k_! h^* B) \\ &= \Gamma_c(U \cap V, h^* B) \\ &= \Gamma_c(U \cap V, B). \end{aligned}$$

11.3. **The Duality Formula.** We will now apply the above results to prove the duality formula and show that the dual sheaf of a c-soft sheaf is injective.

Lemma 11.3. *For k-sheaves A and B on a finite dimensional space X, if A is c-soft so is $B \otimes_k A$.*

Proof. Since (44) gives for any sheaf a covering by sheaves of the form k_U, there exists a resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to B \to 0, \tag{49}$$

where each P_n is of the form $\sum_\alpha k_{U_{n,\alpha}}$ for open subsets $U_{n,\alpha} \subseteq X$. If we write $i_{n,\alpha} \colon U_{n,\alpha} \hookrightarrow X$ for the inclusion, we get

$$P_n \otimes A \cong \sum_\alpha k_{U_n,\alpha} \otimes A \cong \sum_\alpha (i_{n,\alpha})_! (i_{n,\alpha})^*(A),$$

by (45) above. Thus, $P_n \otimes A$ is a sum of c-soft sheaves (see Remark 9.6(4)), hence is itself c-soft. Now let $d = \mathsf{dim}(X)$ and write $K_d := \mathsf{ker}(P_{d-1} \to P_{d-2})$. Since we are working over a field, the sequence

$$0 \to K_d \otimes A \to P_{d-2} \otimes A \to \cdots \to P_0 \otimes A \to B \otimes A \to 0 \tag{50}$$

is exact. Hence, just as in the proof of Proposition 10.4, one has for each open U an isomorphism $H_c^{d+1}(U; K_d \otimes A) \cong H_c^1(U; B \otimes A)$. Since $d + 1 > \mathsf{dim}(X)$, also $H_c^1(U; B \otimes A) = 0$ for each U. By Corollary 9.8, the sheaf $B \otimes A$ is c-soft. □

Theorem 11.4 (Duality Formula)**.** *For k-sheaves A and V on a finite dimensional space X, if A is c-soft there is a canonical isomorphism of sheaves*

$$(B \otimes_k A)\check{} \xrightarrow{\sim} \mathbf{Hom}_k(B, \check{A}). \tag{51}$$

Remark 11.5. The presheaf $(B \otimes_k A)\check{}$ is a sheaf by Lemmas 11.3 and 11.2. Furthermore, $\mathbf{Hom}_k(B, \check{A})$ is the sheaf of k-linear homomorphisms, described explicitly as $\mathbf{Hom}_k(B, \check{A})(U) = \mathsf{Hom}_k(B|_U, \check{A}|_U))$ for any open $U \subseteq X$. Notice that by taking global sections, the isomorphism (51) specializes to an isomorphism of vector spaces,

$$\Gamma_c(X, B \otimes_k A)\check{} \cong \mathbf{Hom}_k(B, \check{A}). \tag{52}$$

Proof of Theorem 11.4. First we describe a natural map of k-sheaves,

$$R = R_B \colon (B \otimes_k A)\check{} \longrightarrow \mathbf{Hom}_k(B, \check{A}),$$

as follows. For $U \subseteq X$ and $\psi \in \Gamma_c(U, B \otimes_k A)\check{}$, define $\bar{\psi} \colon B|_U \to \check{A}|_U$ by setting for any $V \subseteq U$ and any $b \in B(V)$ with $a \in \Gamma_c(V, A)$,

$$\bar{\psi}_V(b)(a) = (\psi|_V)(b \otimes a).$$

Here, $\psi|_V \in \Gamma_c(V, B \otimes_k A)\check{}$ is the restriction of ψ in the (pre)sheaf $B \otimes_k A)\check{}$ and $b \otimes a$ denotes the evident element in $(B \otimes_k A)(V)$. Then set

$$(R_B)_U(\psi) = \bar{\psi}.$$

It is readily verified that R_B is a map of sheaves and is natural in B.

Now consider a resolution of the sheaf B as in (49) above, and take its tensor product with A,

$$\cdots \to P_1 \otimes A \to P_0 \otimes A \to B \otimes A \to 0.$$

Write $L_0 = B \otimes A$, $P_{-1} = B$, and $L_i = \mathsf{ker}(P_{i-1} \otimes A \to P_{i-2} \otimes A)$ for $i \geq 1$, so that for each $i \geq 0$ there is an exact sequence,

$$0 \to L_{i+1} \to P_i \otimes A \to L_i \to 0. \tag{53}$$

By c-softness of $P_i \otimes A$ (for $i \geq -1$) this gives for each $n \geq 1$ an isomorphism $H^n_c(U; L_i) \cong H^{n+1}_c(U, L_{i+1})$. Since X is assumed to be of finite dimension d, it follows that $H^1_c(U; L_i) \cong H^{d+1}_c(U; L_{d+i}) = 0$. Hence, each L_i is again c-soft. In particular, the long exact cohomology sequence associated to the restriction of (53) to any open $U \subseteq X$ gives a short exact sequence,

$$0 \to \Gamma_c(U, L_{i+1}) \to \Gamma_c(U, P_i \otimes A) \to \Gamma_c(U, L_i) \to 0.$$

These short exact sequences fit together into a long exact sequence

$$\cdots \to \Gamma_c(U, P_1 \otimes A) \to \Gamma_c(U, P_0 \otimes A) \to \Gamma_c(U, B \otimes A) \to 0.$$

The dual of this sequence is again exact for any open $U \subseteq X$. Hence, so is the sequence,

$$0 \to (B \otimes A)\check{} \to (P_0 \otimes A)\check{} \to (P_1 \otimes A)\check{} \to \cdots$$

Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & (B \otimes A)\check{} \arrow[r] \arrow[d, "R_B"] & (P_0 \otimes A)\check{} \arrow[r] \arrow[d, "R_{P_0}"] & (P_1 \otimes A)\check{} \arrow[r] \arrow[d, "R_{P_1}"] & \cdots \\
0 \arrow[r] & \mathbf{Hom}_k(B, \check{A}) \arrow[r] & \mathbf{Hom}_k(P_0, \check{A}) \arrow[r] & \mathbf{Hom}_k(P_1, \check{A}) \arrow[r] & \cdots
\end{tikzcd}.$$

The top row has just been shown to be exact, while the bottom row is exact by standard exactness properties of $\mathbf{Hom}_k(-, \check{A})$. Thus, to show that R_B is an isomorphism, it suffices to show that both R_{P_0} and R_{P_1} are. By the construction of the P_i and the naturality of R this means that we have reduced the problem to showing that for each open set U, the map

$$R_{k_U}\colon (k_U \otimes A)\check{} \to \mathbf{Hom}_k(k_U, A) \tag{54}$$

is an isomorphism. Now,

$$\begin{aligned}
\mathbf{Hom}_k(k_U, \check{A})(V) &\cong \mathsf{Hom}_k(k_V, \mathbf{Hom}_k(k_U, \check{A}) \\
&\cong \mathsf{Hom}_k(k_V \otimes k_U, \check{A}) \\
&\cong \mathsf{Hom}_k(k_{U\cap V}, \check{A}) \\
&\cong \check{A}(U \cap V) \\
&\cong \Gamma_c(U \cap V, A)\check{} \\
&\cong \Gamma_c(V, k_U \otimes A)\check{} \\
&\cong (k_U \otimes A)\check{}(V)
\end{aligned}$$

Spelling out the definitions, one finds that the composition of this chain of isomorphisms is exactly the map (54) above (as defined at the beginning of the proof), evaluated at an open subset $V \subseteq X$. We conclude that (54) is an isomorphism and this concludes the proof of the theorem. □

Corollary 11.6. *For a c-soft k-sheaf A on a finite dimensional space X, the dual sheaf $\check{A}$ is injective.*

Proof. By the duality formula, the functor $\mathsf{Hom}_k(-, \check{A}) = \Gamma\mathbf{Hom}_k(-, \check{A})$ is isomorphic to the functor $\Gamma_c(X, - \otimes_k A)\check{}$; this functor is exact since a short exact sequence of sheaves, $0 \to B \to C \to D \to 0$, gives a short exact sequence of c-soft sheaves $0 \to B \otimes A \to C \otimes A \to D \otimes A \to 0$ by Lemma 11.3. □

12. Verdier Duality, Poincaré Duality and Alexander Duality

As in the previous section, we assume X to be a locally compact Hausdorff space of (cohomological) dimension $d < \infty$ and all sheaves to be over a base field k.

12.1. **The Dualizing Complex.** By Proposition 10.4 the constant sheaf $\Delta(k)$ has a c-soft resolution of length d, say

$$0 \to \Delta(k) \to A^0 \to A^1 \to A^2 \to \cdots \to A^d \to 0 \quad . \tag{55}$$

Taking the dual of this complex (with signs as in Appendix **??**) yields a complex $\mathcal{D}^\bullet$ with

$$\mathcal{D}^p = (A^{-p})\check{},$$

and with differential $\mathcal{D}^p \to \mathcal{D}^{p+1}$ the dual of the map $(-1)^{p+1}d\colon A^{-(p+1)} \to A^{-p}$. This complex $\mathcal{D}^\bullet$ is called the *dualizing complex*. Note that by Corollary 11.6 each $\mathcal{D}^p$ is injective and $\mathcal{D}^p = 0$ unless $-d \le p \le 0$.

Theorem 12.1 (The Verdier Duality Formula)**.** *For each complex $I^\bullet$ in $\mathsf{Ch}_k^+(X)$ consisting of injective k-sheaves, there is a natural isomorphism*

$$[I^\bullet, \mathcal{D}^\bullet] \xrightarrow{\sim} [\Gamma_c(X, I^\bullet), k].$$

Remark 12.2. Since restriction to an open $U \subseteq X$ preserves the construction of $\mathcal{D}^\bullet$ as well as injectives, this formula for the subspace U reads,

$$[I^\bullet|_U, \mathcal{D}^\bullet|_U] \cong [\Gamma_c(U, I^\bullet), k].$$

Now apply this to an injective resolution $I^\bullet$ of the constant sheaf $\Delta(k)$, shifted by $p \geq 0$. This gives

$$\begin{aligned} H^{-p}(\mathcal{D}^\bullet(U)) &\cong [\Delta(k)[-p]|_U, \mathcal{D}^\bullet|_U] \quad \text{(appendix\textbf{??})} \\ &\cong [I^\bullet[-p]|_U, \mathcal{D}^\bullet|_U] \\ &\cong [\Gamma_c(U, I^\bullet[-p]), k] \quad \text{(Theorem 12.1 with } U \text{ for } X) \\ &\cong H_c^p(U,k)\check{}. \end{aligned}$$

In particular, the cohomology sheaf $H^{-p}(\mathcal{D}^\bullet)$ of $\mathcal{D}^\bullet$, which is the sheaf associated to the presheaf $U \mapsto H^{-p}(\mathcal{D}^\bullet(U))$, is isomorphic to the sheaf associated to the presheaf $U \mapsto H_c^p(U;k)\check{}$. For $p = d$ (the dimension of X), this last presheaf is already a sheaf. Indeed, the cohomology sheaf $H^{-d}(\mathcal{D}^\bullet)$ is the kernel of $\mathcal{D}^{-d} \to \mathcal{D}^{-d+1}$ since $\mathcal{D}^{-d-1} = 0$. Furthermore, kernels of sheaf maps can be described naively as for presheaves (see Section 4.2). This sheaf,

$$U \mapsto H_c^d(U;k)\check{},$$

is called the *orientation sheaf* and denoted $\mathcal{O}(k)$; i.e.,

$$\mathcal{O}(k) = H_c^d(U;k)\check{}.$$

We shall again consider it in Section 12.3 below.

Proof of Theorem 12.1. We will use the notation from Section 12.1. We will also use that the functor $\Gamma_c\colon \mathsf{Ch}_k^+(X) \to \mathsf{Ch}_k^+$ preserves weak equivalences between complexes of c-soft sheaves. This will be proved in Lemma 12.3 below. Furthermore, for Hom and tensor we use the sign conventions from Appendix Section **??**. So suppose that $I^\bullet$ is a given complex of injectives. Since $\Delta(k)[0] \to A^\bullet$ (with $A^\bullet$ as in (55)) is a weak equivalence by construction, and since we are working over a field,

$$I^\bullet \cong I^\bullet \otimes \Delta(k)[0] \to I^\bullet \otimes A^\bullet$$

is again a weak equivalence. Hence by Lemma 12.3 so is

$$\Gamma_c(X, I^\bullet) \to \Gamma_c(X, I^\bullet \otimes A^\bullet).$$

Taking the dual gives a weak equivalence,

$$\Gamma_c(X, I^\bullet \otimes A^\bullet)\check{} \to \Gamma_c(X, I^\bullet)\check{}.$$

By the duality formula (51), or rather its globalization (52),

$$\Gamma_c(X, I^\bullet \otimes A^\bullet)\check{} \cong \mathsf{Hom}_k(I^\bullet, (A^\bullet)\check{}) = \mathsf{Hom}_k(I^\bullet, \mathcal{D}^\bullet),$$

and this is in fact an isomorphism of complexes (up to a sign). Thus, we have a weak equivalence,

$$\mathsf{Hom}_k(I^\bullet, \mathcal{D}^\bullet) \to \Gamma_c(X, I^\bullet)\check{},$$

and by taking H^0 of these complexes we obtain,

$$[I^\bullet, \mathcal{D}^\bullet] \cong [\Gamma_c(X, I^\bullet), k].$$

$\square$

12.2. **Mapping Cones.** (The I and A in this section have nothing to do with the ones in the previous subsection.) Recall the cylinder $I \otimes A$ of any complex A as constructed in Appendix Section **??**:

$$(I \otimes A)^n = A^{n+1} \oplus A^n \oplus A^n,$$

with coboundary map given by

$$d(a, b, c) = (-da, db - a, dc + a).$$

The cone CA is of course the complex $C(A)$ obtained by factoring out one end of the cylinder, just as in topology:

$$\begin{tikzcd} A \arrow[r, "\partial_0"] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & I \otimes A \arrow[d, "\pi"] & A \arrow[l, "\partial_1"'] \arrow[dl] \\ 0 \arrow[r] & CA & \end{tikzcd},$$

or an exact sequence,

$$0 \longrightarrow A \xrightarrow{\partial_0} I \otimes A \longrightarrow CA \longrightarrow 0 \quad . \tag{56}$$

The mapping cone C_f of map $f\colon A \to B$ of complexes is obtained by gluing B to the bottom of the cone. Again, as in topology,

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & C(A) \arrow[d] \\ B \arrow[r] & C_f \end{tikzcd},$$

$$0 \to A \to B \oplus CA \to C_f \to 0. \tag{57}$$

Thus,

$$C_f^n = A^{n+1} \oplus B^n, \tag{58}$$

with differential

$$d(a, b) = (-da, db - f(a)). \tag{59}$$

Since $\partial_0\colon A \to I \otimes A$ is a weak equivalence, the long exact sequence of (56) gives $H^\bullet(C(A)) = 0$, and hence the long exact sequence of (57) reduces to

$$\cdots \to H^n(A) \to H^n(B) \to H^n(C_f) \to H^{n+1}(A) \to \cdots \tag{60}$$

again as in topology (the Puppe Sequence, as in Algebraic Topology Notes, Page 77). In particular, $f\colon A \to B$ is aweak equivalence if and only if C_f is 'contractible' in the sense that $0 \to C_f$ is a weak equivalence.

Lemma 12.3. $\Gamma_c\colon \mathsf{Ch}_k^+(X) \to \mathsf{Ch}_k^+$ *preserves weak equivalences between complexes consisting of c-soft sheaves (or, more generally, Γ_c-acyclic sheaves).*

Proof. The proof is a formal consequence of the fact that Γ_c preserves mapping cones; i.e., for $f\colon A^\bullet \to B^\bullet$ one has $C^\bullet_{\Gamma_c(f)} = \Gamma_c(C_f)$. (The first cone here is formed in Ch_k^+, the second one in $\mathsf{Ch}_k^+(X)$.) So, to prove the lemma, suppose that $f\colon A^\bullet \to B^\bullet$ is a weak equivalence as in the lemma. By construction, C_f is also a complex of c-soft sheaves (the sum of c-soft sheaves in (58) is again c-soft). However, $0 \to C_f$ is a weak equivalence, since f is. Hence, $\Gamma_c(C_f)$ computes the cohomology $H_c^\bullet(X;0)$, which is evidently zero. Thus, $0 \to \Gamma_c(C_f)$ is again a weak equivalence, and hence since $\Gamma_c(C_f) = C^\bullet_{\Gamma_c(C_f)}$, the map

$$\Gamma_c(f)\colon \Gamma_c(A^\bullet) \to \Gamma_c(B^\bullet)$$

is a weak equivalence. $\square$

12.3. **Orientability.** For a space X of dimension $d < \infty$, we defined in Remark 12.2 its orientation sheaf $\mathcal{O}(k)$ by

$$\mathcal{O}(k)(U) = H_c^d(U;k)\check{}.$$

If X is a manifold, then for coordinate neighbourhoods $V \subseteq U \subseteq X$ (i.e., both V and U are homeomorphic to $\mathbb{R}^d$) we have by Proposition 9.13 that $H_c^d(U;k) \cong k$ and that extension by zero induces a k-linear isomorphism

$$H_k^d(V;k) \xrightarrow{\sim} H_c^d(U;k).$$

It follows that $\mathcal{O}(k)$ is a *locally constant sheaf* (as in Section 8.1) with stalk k. An *orientation* of X with respect to k is a non-zero global section of $\mathcal{O}(k)$; the space X is said to be *orientable* (with respect to k) if such a section exists. Note that a non-zero section $s \in \Gamma(X, \mathcal{O}(k))$ corresponds to a non-zero k-linear map $\Delta(k) \to \mathcal{O}(k)$, which must then be an isomorphism since $\mathcal{O}(k)$ is locally constant with stalk k. Thus, X is orientable with respect to k if and only if $\mathcal{O}(k)$ is a constant sheaf.

Exercise

(a) Show that X is aways orientable with respect to $k = \mathbb{Z}/2\mathbb{Z}$.
(b) What is the relation between $\mathcal{O}(k)$ and the orientation cover of X (the orientation cover is a double cover, i.e. a covering projection with two sheets, over X)?

12.4. **Poincaré Duality.** From Verdier duality, Poincaré duality and Alexander duality can easily be derived as we will see in this and the next section.

Theorem 12.4. *For a manifold X of dimension $d < \infty$,*

$$H^n(X;\mathcal{O}(k)) \cong H_c^{d-n}(X;k)\check{}.$$

(*In particular, if X is orientable with respect to k then $H^n(X,k) \cong H_c^{d-n}(X;k)\check{}$.*)

Proof. If U is a coordinate neighbourhood in X, then by Proposition 9.13

$$H^n(U;k) = \begin{cases} k & \text{if } n = d; \\ 0 & \text{otherwise.} \end{cases}$$

Since for $0 \le n \le d$ the sheaf $H^{-n}(\mathcal{D}^\bullet)$ is the associated sheaf of the presheaf $U \mapsto H_c^n(U;k)\check{}$ it follows (using the definition of $\mathcal{O}(k)$) that

$$0 \to \mathcal{O}(k) \to \mathcal{D}^{-d} \to \cdots \to \mathcal{D}^{-1} \to \mathcal{D}^0 \to 0$$

is exact. In other words, $\mathcal{D}^\bullet[-d]$ is an injective resolution of $\mathcal{O}(k)$ (see also Section 12.1). Using this resolution to compute $H^\bullet(X;\mathcal{O}(k))$, we find that for any injective resolution $I^\bullet$ of $\Delta(k)$,

$$\begin{aligned} H^{d-n}(X;\mathcal{O}(k)) &\cong [\Delta(k)[n-d],\mathcal{D}^\bullet[-d]] \quad \text{(see } \textit{App.}\textbf{??}) \\ &\cong [\Delta(k)[n],\mathcal{D}^\bullet] \\ &\cong [I^\bullet[n],\mathcal{D}^\bullet] \\ &\cong [\Gamma_c(X,I^\bullet[n]),k] \quad \text{(by Theorem 12.1)} \\ &\cong H^n_c(X;k)\check{}. \end{aligned}$$

Here, the last isomorphism is because taking homotopy classes '$[-,-]$' commutes with taking the k-linear dual. □

12.5. **Alexander Duality.**

Theorem 12.5. *For a manifold X of dimension d and a closed subspace $Z \subseteq X$ there is a natural isomorphism*

$$H^n_Z(X;\mathcal{O}(k)) \cong H^{d-n}_c(Z;k)\check{}.$$

(Again, note that $H^n_Z(X;\mathcal{O}(k)) \cong H^n_Z(X;k)$ if X is orientable.)

Proof. As in Section 12.4 we use the injective resolution $\mathcal{D}^\bullet[-d]$ of $\mathcal{O}(k)$. Let $j\colon Z \hookrightarrow X$ be the inclusion map, and recall from Section 3.3 that there are adjoint functors $j^* \dashv j_* \dashv j^!$. Also, recall that for any sheaf A on X,

$$\begin{aligned} \Gamma_Z(A) &\cong \{a \in \Gamma A;\, \mathsf{supp}(a) \subseteq Z\} \quad \text{(by (}\textbf{??}\text{))} \\ &\cong \Gamma j_* j^! A \quad \text{(by (}\textbf{??}\text{))} \\ &\cong \mathsf{Hom}_X(\Delta(k), j_* j^! A) \\ &\cong \mathsf{Hom}_Z(j^*\Delta(k), j^! A) \\ &\cong \mathsf{Hom}_X(j_* j^* \Delta(k), A). \end{aligned}$$

let $I^\bullet$ be an injective resolution of $j_*j^*\Delta(k)$ (for instance, $I^\bullet$ can be the j_*-image of such a resolution of $j^*\Delta(k)$, since j_* is exact and preserves injectives.) Then,

$$\begin{aligned} H^n_Z(X;\mathcal{O}(k)) &\cong [\Delta(k)[-n],\Gamma_Z(\mathcal{D}^\bullet[-d])] \quad \text{(by defn)} \\ &\cong [(j_*j^*\Delta(k))[-n],\mathcal{D}^\bullet[-d]] \\ &\cong [I^\bullet[-n],\mathcal{D}^\bullet[-d]] \\ &\cong [I^\bullet[d-n],\mathcal{D}^\bullet] \\ &\cong [\Gamma_c(X,I^\bullet[d-n]),k] \quad \text{(Verdier)} \\ &\cong H^{d-n}_c(X;k)\check{}. \end{aligned}$$

□

13. Verdier Duality for a Map

As before, all spaces are assumed to be locally compact Hausdorff and all sheaves are assumed to be sheaves of k-modules for a fixed field. In this section we will prove a 'duality theorem' for a map $f\colon Y \to X$.

13.1. **The Derived Functor $f_!$.** The map $f_!\colon \mathsf{Ch}_k^+(Y) \to \mathsf{Ch}_k^+(X)$ (as defined in Section 9.3) preserves weak equivalences between complexes of c-soft (for instance, injective) sheaves. This is proved exactly as in Lemma 12.3 ($f_!$ preserves mapping cones). It follows that there is a well-defined functor

$$Rf_!\colon \mathsf{Der}_k^+(Y) \to \mathsf{Der}_k^+(X), \tag{61}$$

called the right-derived functor of $f_!$ and constructed as follows. For any $A \in \mathsf{Ch}_k^+(Y)$, choose an injective (or c-soft) resolution $A \hookrightarrow A'$; i.e., $A \to A'$ is a trivial cofibration into a fibrant object as in Appendix **??**. Now define $Rf_!(A)$ to be the object in $\mathsf{Der}_k^+(X)$ represented by $f_!(A')$. In other words, $Rf_!(A)$ is represented by some injective resolution $f_!(A')'$ of $f_!(A')$:

$$\begin{tikzcd} A \arrow[d] & f_!(A) \arrow[d] & \\ A' & f_!(A') \arrow[r] & f_!(A')' = Rf_!(A). \end{tikzcd}$$

Up to isomorphism in the derived category this construction does not depend on the choices of the resolutions A' and $f_!(A')'$. (I.e., for different choices, we will have different functors, but they are all isomorphic.)

The main result of this section can now be stated very briefly as follows.

Theorem 13.1 (Verdier)**.** *The functor $Rf_!$ has a right adjoint.*

If we identify the derived categories of complexes of injective k-sheaves and homotopy classes of maps, then this right adjoint is a functor

$$f^!\colon \mathsf{Der}_k^+(X) \to \mathsf{Der}_k^+(Y)$$

with the property that for any complexes I and J of injective k-sheaves on X, resp. Y, there is an isomorphism

$$[Rf_!(I), J] \cong [I, f^!(J)].$$

Now $Rf_!(I)$ is by definition an injective resolution of $f_!(I)$, so there is a weak equivalence $f_!(I) \to Rf_!(I)$ which induces an isomorphism $[Rf_!(I), J] \xrightarrow{\sim} [f_!(I), J]$ for any complex of injectives J (see Theorem **??** of the appendix). Hence,

$$[f_!(I), J] \cong [I, f^!(J)]. \tag{62}$$

Remark 13.2. For the special case where Y is a point, every k-sheaf on Y is injective. Consider the complex $k[0]$ on $Y = \{\mathrm{pt}\}$ and *define* $\mathcal{D}^\bullet = f^!(k[0])$; then (62) reduces to

$$[f_!(I), k[0]] \cong [I, \mathcal{D}],$$

or, since $f_! = \Gamma_c(X, -)$,

$$[\Gamma_c(X, I), k[0]] \cong [I, \mathcal{D}].$$

This is precisely Theorem 12.1. Conversely, Theorem 13.1 follows from Theorem 12.1 in case Y is a point and k is a field. Since every complex consists of injective sheaves and every exact sequence splits, every complex of injectives over X is isomorphic in the derived category to a direct sum of complexes of the form $k[p]$ with $p \in \mathbb{Z}$. Thus, Theorem 13.1, or rather its explicit formulation (62), is a direct generalization of Theorem 12.1. This observation will motivate the proof, which will be analogous to that of Theorem 12.1. First, we generalize the construction of the dual sheaf and derive the appropriate duality formula.

13.2. **The Construction of $f^!$.** Let A be any k-sheaf on Y, and C be one on X. Define a presheaf $f^!(A,C)$ on Y by setting, for any open $V \subseteq Y$,

$$f^!(A,C)(V) = \mathsf{Hom}(f_!(k_V \otimes A), C). \tag{63}$$

Recall that if we write $j\colon V \hookrightarrow Y$ for the inclusion, then $f^!(k_V \otimes A) \cong f_! j_! j^*(A)$. So, for an open $U \subseteq X$,

$$f_!(k_V \otimes A)(U) = \{a \in A(f^{-1}(U) \cap V);\ f\colon \mathsf{supp}(a) \to U \text{ is proper}\}.$$

For an inclusion of open sets $V \subseteq W$ in Y, the evident map $k_V \hookrightarrow k_W$ induces a map $f_!(k_V \otimes A) \to f_!(k_W \otimes A)$, and hence a 'restriction map'

$$f^!(A,C)(W) \to f^!(A,C)(V),$$

thus giving $f^!$ the structure of a presheaf.

Note that for the special case where X is a point and $C = k$, we have $f^!(A,k) = \check{A}$ as defined in Section 11. This was sufficient in that case, but here we have to consider all k-sheaves C on X and the role of $\check{A}$ is played by the presheaf $f^!(A,C)$. We will use the following analogue of Lemma 11.2.

Lemma 13.3. *If A is c-soft, then $f^!(A,C)$ is a sheaf.*

Proof. As in the proof of Lemma 11.2, we check that $f^!(A,C)$ has the sheaf property with respect to directed covers and finite covers. For directed covers, observe that if $V = \bigcup_{i \in I} V_i$ where $\{V_i; \in I\}$ is directed, then

$$f_!(k_V \otimes A) \cong \varinjlim_i f_!(k_{V_i} \otimes A). \tag{64}$$

(Indeed, $k_V = \varinjlim k_{V_i}$, so $k_V \otimes A = \varinjlim_i (k_{V_i} \otimes A)$; this can be checked at stalks and has nothng to do with directness. Moreover, $f_!$ commutes with directed colimits since Γ_c does and $(f_! B)_x = \Gamma_c(f^{-1}(x), B)$ for any k-sheaf B.) From (64) we obtain immediately,

$$f^!(A,C)(V) \cong \varprojlim_i f^!(A,C)(V_i),$$

which expresses that $f^!(A,C)$ has the sheaf property with respect to the cover $V = \bigcup_{i \in I} V_i$.

For the finite covers, it suffices to consider covers $U = V \cup W$, just as in the proof of Lemma 11.2. Then

$$0 \to k_{V \cap W} \to k_V \oplus k_W \to k_U \to 0$$

is exact, and hence so is

$$0 \to k_{V \cap W} \otimes A \to (k_V \otimes A) \oplus (k_W \otimes A) \to k_U \otimes A \to 0 \quad .$$

Note that all the sheaves occuring in this sequence are c-soft, so the sequence

$$0 \to f_!(k_{V \cap W} \otimes A) \to f_!(k_V \otimes A) \oplus f_!(k_W \otimes A) \to f_!(k_U \otimes A) \to 0$$

is also exact. By homming into C we thus obtain an exact sequence,

$$f^!(A,C)(V \cap W) \leftarrow f^!(A,C)(V) \oplus f^!(A,C)(W) \leftarrow f^!(A,C)(U) \leftarrow 0,$$

which expresses that $f^!(A,C)$ has the sheaf property with respect to the cover $U = V \cup W$. $\square$

13.3. **Duality Formula.** Recall from Section 11 that if A is c-soft then so is $B \otimes_k A$ for any k-sheaf B.

Proposition 13.4 (Duality Formula)**.** *For any k-sheaves A and B on Y with A c-soft and any k-sheaf C on X, there is a natural isomorphism,*

$$R\colon \mathsf{Hom}(f_!(B \otimes A), C) \xrightarrow{\sim} \mathsf{Hom}(B, f^!(A, C)).$$

Remark 13.5. In case $X = \{\mathrm{pt}\}$ and $C = k$ this is precisely the duality formula of Theorem 11.4,

$$\Gamma_c(Y, (A \otimes_k B))\check{} \cong \mathsf{Hom}(B, \check{A}).$$

Proof of Proposition 13.4. We first define the map $R = R_B$ explicitly for an open set $V \subseteq Y$, $b \in B(V)$ and $\psi\colon f_!(B \otimes A) \to C$, define

$$R(\psi)_V(b) \in f_!(A, C)(V) = \mathsf{Hom}(f_!(k_V \otimes A), C)$$

to be the map of sheaves $f_!(k_V \otimes A) \to C$ defined for an open set $U \subseteq X$ and an $a \in f_!(k_V \otimes A)(U) \subseteq A(f^{-1}(U) \cap V)$ by

$$(R(\psi)_V(b))_U(a) = \psi_U(b \otimes a) \in C(U). \tag{65}$$

Here, $b \otimes a$ is the element of $f_!(B \otimes A)(U) \subseteq (B \otimes A)(f^{-1}(U))$ obtained as follows. First, a is given as an element $a \in A(f^{-1}(U) \cap V)$ with the property that $f\colon \mathsf{supp}(a) \to U$ is proper. In particular, if $K \subseteq U$ is any compact set, then $f^{-1}(K) \cap \mathsf{supp}(a)$ is also compact, hence closed in $f^{-1}(U)$. Since U is locally compact, it follows that $\mathsf{supp}(a)$ is closed in $f^{-1}(U)$. Now consider the tensor element $b \otimes a$ (or more explicitly, $b|_{f^{-1}(U)\cap V} \otimes a$ in $(B \otimes A)(f^{-1}(U) \cap V)$). Its support $\mathsf{supp}(b \otimes a)$ is contained in $\mathsf{supp}(a)$, hence $\mathsf{supp}(b \otimes a)$ is also closed in $f^{-1}(U)$. So we can extend $b \otimes a$ by zero to an element (again denoted by) $b \otimes a \in (B \otimes A)(f^{-1}(U))$. This is the $b \otimes a$ occurring in (65). (It indeed lies in the smaller set $f_!(B \otimes A)(U)$, since $f\colon \mathsf{supp}(a) \to U$ is proper.) We leave it to the reader to check that the map R described above is indeed a well-defined map of sheaves and is natural in B.

Next, for $B = k_V$, there is an evident isomorphism,

$$\mathsf{Hom}(f_!(k_V \otimes A), C) \cong \mathsf{Hom}(k_V, f^!(A, C)), \tag{66}$$

by the definition of $f^!(A, C)$ and the characteristic property (43) of the sheaves of the form k_V. Inspecting all the definitions, one sees that this isomorphism is induced by the map $R = R_{k_V}$ just described. Thus, R_B is an isomorphism when $B = k_V$.

For an arbitrary sheaf B, consider the resolution, $0 \leftarrow B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ as in (49) in the proof of Lemma 11.3, where each P_n is a sum of copies of k_V (for various open subsets $V \subseteq Y$). Since R_{k_V} is an isomorphism, it readily follows that R_{P_n} is one as well, for each n. To conclude that R_B is an isomorphism, consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}(f_!(B \otimes A), C) \arrow[r] \arrow[d, "R_B"] & \mathsf{Hom}(f_!(P_0 \otimes A), C) \arrow[r] \arrow[d, "R_{P_0}"] & \mathsf{Hom}(f_!(P_1 \otimes A), C) \arrow[d, "R_{P_1}"] \\
0 \arrow[r] & \mathsf{Hom}(B, f^!(A, C)) \arrow[r] & \mathsf{Hom}(P_0, f^!(A, C)) \arrow[r] & \mathsf{Hom}(P_1, f^!(A, C))
\end{tikzcd}$$

and observe that the rows are exact. (For the opper row, use that $0 \leftarrow B \otimes A \leftarrow P_0 \otimes A \leftarrow P_1 \otimes A \leftarrow \cdots$ is an exact sequence of c-soft sheaves, so applying $f_!$ to it yields another exact sequence.) This completes the proof. $\square$

Corollary 13.6. *If C is injective then so is $f^!(A,C)$.*

Proof. This follows immediately from Proposition 13.4. □

13.4. **The Duality Formula for Complexes.** For complexes of k-sheaves A, B, C, one obtains a similar duality formula

$$\mathsf{Hom}(f_!(B\otimes A),C)\cong \mathsf{Hom}(B,f^!(A,C)) \tag{67}$$

of complexes of k-vector spaces. This follows from Proposition 13.4, using the standard constructions of Hom and tensor for complexes, with sign conventions as in Appendix **??**. (The sign conventions also enter into the definition of $f^!(A,C)$ as a complex.)

13.5. **The Definition of $f^!$ on Derived Categories.** From now on, we fix a c-soft resolution,

$$0\to\Delta(k)\to A^0\to A^1\to A^2\to\cdots$$

of k-sheaves over Y. Define for any complex $C\in\mathsf{Ch}_k^+(Y)$,

$$f^!(C)=f^!(A,C), \tag{68}$$

for this particular resolution A. It follows from Corollary 13.6 that $f^!(A,C)$ is a complex of injectives whenever C is. Thus, $f^!$ as defined in (68) induces a functor

$$f^!\colon \mathsf{Der}_k^+(X)\to\mathsf{Der}_k^+(Y).$$

This functor $f^!$ is the required adjoint for Theorem 13.1.

Proof of Theorem 13.1. Consider the c-soft resolution $0\to\Delta(k)\to A$ fixed above. For any $I\in\mathsf{Ch}_k^+(X)$, this gives a weak equivalence,

$$I=I\otimes\Delta(k)[0]\to I\otimes A.$$

Now, the functor $f_!\colon \mathsf{Ch}_k^+((Y)\to\mathsf{Ch}_k^+(X)$ preserves weak equivalences between complexes of c-soft sheaves (indeed, by Proposition 9.9, $f_!(-)_x\cong\Gamma_c(f^{-1}(x),-)$ and Γ_c preserves weak equivalences by Lemma 12.3). Hence, we obtain a weak equivalence,

$$f_!(I)\to f_!(I\otimes A).$$

Therefore, for any complex $J\in\mathsf{Ch}_k^+(Y)$ consisting of injectives,

$$[f_!(I\otimes A),J]\xrightarrow{\sim}[f_!(I),J].$$

However, by taking H^0 of the isomorphism (67) and the definition (68) of $f^!$, we have also

$$[f_!(I\otimes A),J]\cong[I,f^!J].$$

Thus,

$$[f_!(I),J]\cong[I,f^!(J)],$$

as required in (62). □

Appendix A. The Derived Category

A.1. **Closed Model Categories.** Let $\mathcal{C}$ be a category. A *closed model structure* (in the sense of Quillen) on $\mathcal{C}$ is given by three classes of maps in $\mathcal{C}$: the *fibrations*, the *cofibrations*, and the *weak equivalences*. Together they need to satisfy the following axioms:

CM1 $\mathcal{C}$ has finite limits and colimits.

CM2 For any two composable maps $f\colon C \to D$ and $g\colon B \to C$, if two out of f, g and $f \circ g$ are weak equivalences, so is the third.

CM3 If f is a retract of g; i.e., there are maps i, j, s and r as in

$$\begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "f"'] & \cdot \arrow[r, "r"] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "j"'] & \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

with $sj = 1$ and $ri = 1$ and g is a fibration (or a cofibration, or a weka equivalence) then so is f.

CM4 (Lifting Axiom) In a solid commutative square

(69)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "i"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

where i is a cofibration and p is a fibration, there exists a dashed diagonal which makes the whole diagram commute whenever i or p is also a weak equivalence.

CM5 (Factorization Axiom) Any map f can be factored in two ways: as $f = p \circ i$ where p is a fibration and i is a cofibration as well as a weak equivalence; and as $f = q \circ j$ where j is a cofibration and q is a fibration as well as a weak equivalence.

When $\mathcal{C}$ is equipped wirh such a closed model structure, one also says hat $\mathcal{C}$ is a (Quillen) *closed model category.*

A fibration which is also a weak equivalence is also called a *trivial fibration*; similarly for cofibrations.

One says that a map p has the *right lifting property* (RLP) with respect to a map i if for any commutative square of the form (69) above a diagonal filling exists making both triangles commute. In this case one also says that i has the *left lifting property* (LLP) with respect to p. A map has the RLP (or, LLP) with respect to a class of arrows if it has this property with respect to each arrow in that class. Thus, CM4 says that the fibrations have the RLP with respect to the trivial cofibrations and the cofibrations have the LLP with respect to the trivial fibrations. In this way, the fibrations and cofibrations determine each other.

Proposition A.1.

Proof. □

Appendix B. Appendix 2: Cochain Complexes

Appendix C. Appendix 3: Cochain Complexes of Abelian Sheaves

References

[1] Raoul Bott, Loring W. Tu, *Differential Forms in Algebraic Topology*, Springer Verlag, 1982.

[2] P.J. Hilton, U. Stammbach, *A Course in Homological Algebra*, Springer Verlag, 1971.

[3] L. Illusie, *Complexe Cotangent et Deformations* I, Springer Lecture Notes **239**, Springer Verlag, 1971.

[4] Saunders Mac Lane, *Categories for the Working Mathematician*, second edition, Springer Verlag, 1998.