

Sheaf Cohomology Final Exam for Anne Johnson, September 2018

1. Let $\varphi: F \rightarrow G$ be a morphism of sheaves on a space X .

- (a) Show that the induced map $\text{Et}(\varphi): \text{Et}(F) \rightarrow \text{Et}(G)$ between the associated étale spaces over X is surjective if and only if for every open set U of X , and every section $s \in G(U)$, there exists an open cover $(U_i)_{i \in I}$ of U and sections $t_i \in F(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ in $G(U_i)$ for all $i \in I$.

(Forwards direction)

Let $\phi: F \rightarrow G$ be a morphism of sheaves on X and suppose the induced map on the associated étale spaces is surjective. Denote points in the étale space of F by $\text{germ}_y^F(a)$ and points in the étale space of G by $\text{germ}_z^G(b)$. Pick some U an open subset of X and $s \in G(U)$. Since $\text{Et}(\phi)$ is surjective, for each $x \in U$, pick $\text{germ}_x^F(t) \in \text{Et}(F)$ such that $\text{Et}(\phi)(\text{germ}_x^F(t)) = \text{germ}_x^G(s)$.

Take representatives (t_x, U_x) for each $x \in U$. Then $\cup_{x \in X} U_x$ covers U and $t_x \in F(U_x)$ is such that $\phi(t_x) = s|_{U_x}$.

(Backwards direction)

Let $\phi: F \rightarrow G$ be a morphism of sheaves on X and suppose for every open set U of X , and every section $s \in G(U)$, there exists an open cover $(U_i)_{i \in I}$ of U and sections $t_i \in F(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ in $G(U_i)$ for all $i \in I$.

Let $\text{germ}_x(s)$ be a element of $\text{Et}(G)$ represented by some (s, U) . Let U_i and t_i be as above. First note that the $t_i \in F(U_i)$ form a compatible family in the U_i since naturality of ϕ implies

$$\phi(t_i|_{U_i \cap U_j}) = \phi(t_j|_{U_i \cap U_j}) = s|_{U_i \cap U_j}$$

Suppose that $t_i|_{U_i \cap U_j}$ and $t_j|_{U_i \cap U_j}$ were distinct elements of $F(U_i \cap U_j)$. **remember youre trying to show the induced mpa is surjective not phi itself** Then we must have $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. Since F is a sheaf, let $t \in F(U)$ be the amalgamation. Then $\text{Et}(\phi)(\text{germ}_x(t)) = \text{germ}_x(s)$ so $\text{Et}(\phi)$ is surjective.

- (b) Give an example of a surjective morphism of sheaves $\varphi: F \rightarrow G$, and an open set U such that $\varphi_U: F(U) \rightarrow G(U)$ is not surjective.

Use two sheaves on the circle and show that the global section is not surjective.

2. Let $[0, 1]$ denote the closed interval and $(0, 1)$ denote the open interval and $i : (0, 1) \rightarrow [0, 1]$ the inclusion map. Write $\Delta_{(0,1)}\mathbb{Z}$ and $\Delta_{[0,1]}\mathbb{Z}$ for the constant sheaves on $(0, 1)$ and $[0, 1]$ respectively. Let

$$\varphi : i_! \Delta_{(0,1)}\mathbb{Z} \rightarrow \Delta_{[0,1]}\mathbb{Z}$$

be the adjunct of the identity map

$$\Delta_{(0,1)}\mathbb{Z} \rightarrow \Delta_{(0,1)}\mathbb{Z} = i^* \Delta_{[0,1]}\mathbb{Z}.$$

Calculate the cokernel of φ . Give it both as an étale space and as a sheaf defined as a functor.

3. Describe the direct sum $F \oplus G$ of two sheaves F and G on a space X . Then show that $H^n(X; F \oplus G) \cong H^n(X; F) \oplus H^n(X; G)$; i.e., sheaf cohomology commutes with taking direct sums.

Let F and G be sheaves on a space X . We define the direct product of F and G to be the sheaf given by $F \oplus G(U) = F(U) \oplus G(U)$ for U open in X . To see that $H^n(X; F \oplus G) \cong H^n(X; F) \oplus H^n(X; G)$, first note that we can build an exact sequence of sheaves by using the injection and projection maps as follows

$$0 \rightarrow F \rightarrow F \oplus G \rightarrow G \rightarrow 0$$

Let I_F^\bullet , I^\bullet , and I_G^\bullet be resolutions of F , $F \oplus G$, and G , respectively that fit into the following map of chain complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \longrightarrow & F \oplus G & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_F^0 & \longrightarrow & I^0 & \longrightarrow & I_G^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_F^1 & \longrightarrow & I^1 & \longrightarrow & I_G^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

where the columns are injective resolutions and the rows are exact. Such resolutions exist by section 4.6. Now apply the global section functor to this complex:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma F & \longrightarrow & \Gamma F \oplus G & \longrightarrow & \Gamma G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma I_F^0 & \longrightarrow & \Gamma I^0 & \longrightarrow & \Gamma I_G^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma I_F^1 & \longrightarrow & \Gamma I^1 & \longrightarrow & \Gamma I_G^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

and note that since the columns in the first diagram were injective, the rows in the second diagram are still exact (Lemma 4.6). Moreover, the first row is split exact since

$$\Gamma F \oplus G = F \oplus G(X) = F(X) \oplus G(X) = \Gamma F \oplus \Gamma G.$$

But then each row is exact: let d be the map from $I^0 \rightarrow I_G^0$ and take $x \in \Gamma I_G^0$.*** and so by induction, each row is split exact. Then $\Gamma I^n \cong \Gamma I_F^n \oplus \Gamma I_G^n$ which implies $H^n(X; F \oplus G) \cong H^n(X; F) \oplus H^n(X; G)$

4. Given an abelian group G and a topological space X with a point $p \in X$, define a presheaf G_p on X by

$$G_p(U) = \begin{cases} G, & \text{if } p \in U, \\ 0, & \text{if } p \notin U \end{cases} \quad \text{and} \quad \rho_{VU}(a) = \begin{cases} a, & \text{if } p \in V \subseteq U, \\ 0, & \text{if } p \notin V \end{cases}$$

for $V \subseteq U \subseteq X$ open sets.

- (a) Verify that this defines a sheaf on X .

Let i be the inclusion of p into X and consider $i_*\Delta G$. For U open in X ,

$$i_*\Delta G(U) = \Delta G(i^{-1}(U)) = \begin{cases} \Delta G(\{p\}), & \text{if } p \in U \\ \Delta G(\emptyset), & \text{if } p \notin U \end{cases} = \begin{cases} G, & \text{if } p \in U \\ 0, & \text{if } p \notin U \end{cases}$$

The restriction maps for $i_*\Delta G$ are Then $i_*\Delta G = G_p$ and since $i_*\Delta G$ is a sheaf, so is G_p .

- (b) Describe the stalks at each point of the space.

$$G_{p_x} = \begin{cases} 0 & x \notin \bar{p} \\ G & x \in \bar{p} \end{cases}$$

since $x \in \bar{p}$ implies that every open set containing x intersects p , so that every restriction of open sets containing x still has sections $\forall g \in G$.

- (c) Let F be a sheaf on a one-point space $X = \{P\}$. Show that $H^1(X; F) = 0$.
Let F and $X = \{P\}$ be as above. Let

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$$

be an injective resolution of F . Consider

$$0 \rightarrow \Gamma F \rightarrow \Gamma I^0 \rightarrow \Gamma I^1 \rightarrow \Gamma I^2 \rightarrow \dots$$

Since X is a one point space $\Gamma I^\bullet \cong I_x^\bullet$. Then since I^\bullet being exact implies I_x^\bullet is exact (by definition), we have that ΓI^\bullet is also exact. This means $H^n(X; F) = 0, \forall n > 0$, in particular for $n = 1$.

- (d) Let X be an arbitrary space with $p \in X$ and G_p as defined above. Show that $H^1(X; G_p) = 0$.

Note that G_p is flabby for any space X . This implies that G_p is acyclic and so $H^n(X; G_p) = 0, \forall n > 0$, in particular for $n = 1$.

5. The torus T is obtained as the quotient space of $[0, 1] \times [0, 1]$ by the equivalence relation generated by: $(x, 0) \sim (x, 1)$ for all $x \in [0, 1]$ and $(0, y) \sim (1, y)$ for all $y \in [0, 1]$.

The Klein bottle K is obtained as the quotient space of $[0, 1] \times [0, 1]$ by the equivalence relation generated by: $(x, 0) \sim (x, 1)$ for all $x \in [0, 1]$ and $(0, y) \sim (1, 1 - y)$ for all $y \in [0, 1]$.

- (a) Calculate the cohomology groups of the torus T with values in the constant sheaf $\Delta_T \mathbb{Z}$. (Hint: use Corollary 7.3.2.)
- (b) Use a Mayer-Vietoris sequence to calculate the cohomology groups of the Klein bottle with values in the constant sheaf $\Delta_K \mathbb{Z}$.
- (c) Let S be the middle circle of the torus, given by the image of the segment $\{(x, 0) | x \in [0, 1]\}$ in the quotient space. Give the long exact sequences for the pair (T, S) and for the relative cohomology with coefficients in $\Delta_T \mathbb{Z}$ and calculate as many of the groups as you can.
- (d) Let $S' \subset K$ be the image of $\{(x, 0) | x \in [0, 1]\}$ in the quotient space K . Use the long exact sequence from Proposition 9.4.1 and fill in as many groups as you can to find the cohomology with compact support for the open annulus and the open Möbius strip.