# Completeness in Generalized Ultrametric Spaces

Nathanael Leedom Ackerman
presented by Anne Johnson

August 29, 2018

*"The goal of this paper is to characterize the relationship between four important notions of completeness for Γ- ultrametric spaces: spherical completeness, Cauchy completeness, strong Cauchy completeness and injectivity"*

Instead, we'll talk about:

- equivalence of categories between the category of flabby, separated presheaves on $\Gamma^{op}$ and the category of $\Gamma-$ultrametric spaces and non-expanding maps.
- Injective hulls

# Some background

### Definition

A partial order $(\Gamma, \leq_\Gamma)$ is a **complete lattice** if every subset of $\Gamma$ has a supremum and infimum. A complete lattice $(\Gamma, \leq_\Gamma)$ is an **op-frame** if it satisfies

$$(\forall x \in \Gamma, A \subseteq \Gamma) x \vee_\Gamma (\wedge_{\alpha \in A}^\Gamma) = \wedge_{\alpha \in A}^\Gamma (x \vee_\Gamma \alpha)$$

# Some background

Notation:

- $(\Gamma, \leq_\Gamma)$ or $\Gamma$ – complete lattice, always an op-frame
- $\bot_\Gamma$ – least element, ie $\wedge^\Gamma \{\gamma : \gamma \in \Gamma\}$
- $\top_\Gamma$ – greatest element, ie $\vee^\Gamma \{\gamma : \gamma \in \Gamma\}$
- $\Gamma^{op}$ – $\Gamma$ but with the order reversed

# Some background

### Definition

A $\Gamma-$**ultrametric space** is a pair $(M, d_M)$, where $M$ is a set and $d_M : M \times M \to \Gamma$ is a function that satisfies:

- $(\forall x, y \in M)\ d_M(x, y) = \bot \iff x = y$
- $(\forall x, y \in M)\ d_M(x, y) = d_M(y, x)$
- $(\forall x, y, z, \in M)\ d_M(x, y) \vee d_M(y, z) \geq d_M(x, z)$

### Definition

A **non-exanding map** between $\Gamma-$ultrametric spaces $(M, d_M)$ and $(N, d_N)$ is a function $f : M \to N$ such that $(\forall a, b \in M)$
$d_N(f(a), f(b)) \leq d_M(a, b)$

# Some background

A few things to note

- $\Gamma-$ultrametric spaces and non-exanding maps form a category, denoted $UMet(\Gamma)$
- Given a $\Gamma-$ ultrametric space $(M, d_M)$, the balls of fixed radius $\gamma$ for $\gamma \in \Gamma$ also form a generalized ultrametric space, with the distance function taking values in
  $\Gamma_\gamma = \{\zeta \in \Gamma : \zeta \geq_\Gamma \gamma\}$

Denote by $(M_\gamma, d_{M_\gamma})$ the generalized ultrametric space having points
$$M_\gamma = \{B^M(x; \gamma) : x \in M\}$$

and distance function

$$d_{M_\gamma}(B^M(x; \gamma), B^M(y; \gamma)) = d_M(x, y) \vee \gamma$$

Lastly if $f : (M, d_M) \to (N, d_N)$ is a map of $\Gamma-$ultrametric spaces, let $f_\gamma : (M_\gamma, d_{M_\gamma}) \to (N_\gamma, d_{N_\gamma})$ be such that

$$f(B^M(x; \gamma)) = B^N(f(x), \gamma)$$

Then $f_\gamma$ is non-expanding and well-defined.

**Lemma 2.7** $(\cdot)_\gamma$ is a functor from $UMet(\Gamma)$ to $UMet(\Gamma_\gamma)$

Let's see an example

# Motivation

*"In order to show that each Γ−ultrametric space has a minimal, strongly Cauchy complete extension, we first show that the category of Γ−ultrametric spaces is equivalent to the category of flabby separated presheaves on $\Gamma^{op}$."*

- ▶ Theorem 2.25: a Γ−ultrametric space is injective if and only if is strongly Cauchy complete.
- ▶ Then an injective hull of a Γ-ultrametric space is a minimal extension that is strongly Cauchy complete
- ▶ Idea: show every object in $Flab(\Gamma^{op})$ has an injective hull and use the equivalence.

More notation:

- $Flab(\Gamma^{op})$ : the category of flabby separated presheaves on $\Gamma^{op}$
- $Sep(\Gamma^{op})$ : the category of separated presheaves on $\Gamma^{op}$
- $i_\Gamma : Flab(\Gamma^{op}) \rightarrow Sep(\Gamma^{op})$ : the inclusion functor

First we show an equivalence of categories between $UMet(\Gamma)$ and $Flab(\Gamma^{op})$ via two functors, $F_\Gamma$ and $G_\Gamma$.

# the functor $F_\Gamma$

We define $F_\Gamma : UMet(\Gamma) \to Flab(\Gamma^{op})$ as follows:

### Definition

Let $(M, d_M) \in UMet(\Gamma)$ and define $F_\Gamma(M, d_M)(\gamma) = (M_\gamma, d_{M_\gamma})$.
Let the restriction maps be given by: $B(a, \gamma) \mid_{\gamma^*} = B(a, \gamma^*)$ for
$\gamma^* \geq \gamma$.

Check that:

- The restriction maps are well-defined
- $F_\Gamma(M, d_M)$ is flabby, i.e. if $B(a, \gamma) \in F_\Gamma(M, d_M)(\gamma)$, then
  there exists $\tilde{\gamma} \leq \gamma \in \Gamma$ such that $B(a, \gamma) \mid_{\tilde{\gamma}} = B(a, \gamma)$
- $F_\Gamma(M, d_M)$ is separated
- If $f : (M, d_M) \to (N, d_N)$ is a non-expanding map, then
  $F_\Gamma(f) : F_\Gamma(M, d_M) \to F_\Gamma(N, d_N)$ is a map of flabby separated
  presheaves.

# the functor $G_\Gamma$

We define $G_\Gamma : Flab(\Gamma^{op}) \to UMet(\Gamma)$ as follows:

### Definition
Let $A \in Flab(\Gamma^{op})$ and define $G_\Gamma(A) = (A(\bot), d_{A_\bot})$, where $d_\bot(a, b) = \wedge\{\gamma : a\mid_\gamma = b\mid_\gamma\}$.

Check that:

- The distance function satisfies the strong triangle identity
- $(\forall a, b \in A(\bot))\ d_\bot(a, b) = \bot \iff a = b$
- If $A, C \in Flab(\Gamma^{op})$ and $f : A \to C$ is a map of presheaves then $G_\Gamma(f) : G_\Gamma(A) \to G_\Gamma(C)$ is a non-expanding map.

**Claim 3.19**

There is a natural isomorphism $\eta : F_\Gamma \circ G_\Gamma \to id_{Flab(\Gamma^{op})}$ such that $(\forall x \in A(\bot))$ $\eta_A(\{x\}) = x$ when $A \in Flab(\Gamma^{op})$.

Check that:

- $\eta$ is well-defined and injective (since $A$ being flabby will imply that is surjective)
- $\eta$ is natural

# equivalence of categories

**Claim 3.20**

There is a natural isomorphism $\epsilon : id_{UMet(\Gamma)} \to G_\Gamma \circ F_\Gamma$.

- Suppose $(M', d_{M'}) = G_\gamma(F_\gamma(M, d_M))$, so that $M' = \{\{a\} : a \in M\}$.

- Check directly that the map given by $\epsilon_{M,d_m}(a) = \{a\}$ is an isomorphism. Naturality is immediate.

**Theorem 3.12**

There is an equivalence of categories between $Flab(\Gamma^{op})$ and $UMet(\Gamma)$

# injective hull

### Definition

Suppose $C$ is a category and $A \in C$. An **injective hull** for $A$ is a monic $e : A \to I$ where

1. $I$ is injective
2. If $k : A \rightarrowtail I'$ is a monomorphism with $I'$ injective, there is a monomorphism $k' : I \rightarrowtail I'$ such that $k' \circ e = k$.

We show every object of $Flab(\Gamma^{op})$ has an injective hull.

# injective hull

**Claim 3.23** For every object $\mathcal{F}$ of $Flab(\Gamma^{op})$, the following are equivalent:

1. $\mathcal{F}$ is injective in $Flab(\Gamma^{op})$.
2. $i_\Gamma(\mathcal{F})$ is injective in $Sep(\Gamma^{op})$

In fact, this is enough, since we also have [2],
**Claim 3.24** The injective objects of $Sep(\Gamma^{op})$ are exactly the flabby sheaves and every object of $Sep(\Gamma^{op})$ has an injective hull.

# injective hull

$\mathcal{F} \in Flab(\Gamma^{op})$ is injective $\Rightarrow i_{\Gamma}(\mathcal{F})$ is injective in $Sep(\Gamma^{op})$

Proof.

Let $A$ be a separated presheaf on $\Gamma$. Define $A^*$ as follows:

- $A^*(V) = \cup_{U \leq V} A(U) \times \{U\} \times \{V\}$
- If $U \leq V$ and $< x, W, V > \in A^*(V)$ then
  $< x, W, V >|_U = < x |_{U \wedge W}, U \wedge W, U > \in A^*(U)$.

Define $m, m^*, e_A$ □

# references

N.L. Ackerman, "Completeness in Generalized Ultrametric Spaces," p-Adic Numbers, Ultrametric Analysis and Applications, **5**(2),89-105 (2013).

J. Coulon and J-L.Coulon, "Injectivity in the quasi-topos of separated presheaves on a complete Heyting algebra," Portugal. Math. **47**(3),227-241 (1990).