Completeness in Generalized Ultrametric Spaces

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"The goal of this paper is to characterize the relationship between four important notions of completeness for Γ*- ultrametric spaces: spherical completeness, Cauchy completeness, strong Cauchy completeness and injectivity"*

Instead, we'll talk about:

- equivalence of categories between the category of flabby, separated presheaves on Γ^{op} and the category of $\Gamma-$ultrametric spaces and non-expanding maps.
- Injective hulls

Some background

Definition

A partial order $(\Gamma, \leq_\Gamma)$ is a **complete lattice** if every subset of Γ has a supremum and infimum. A complete lattice $(\Gamma, \leq_\Gamma)$ is an **op-frame** if it satisfies

$$(\forall x \in \Gamma, A \subseteq \Gamma) x \vee_\Gamma (\wedge^\Gamma_{\alpha \in A}) = \wedge^\Gamma_{\alpha \in A}(x \vee_\Gamma \alpha)$$

Some background

Notation:

- $(\Gamma, \leq_\Gamma)$ or Γ – complete lattice, always an op-frame
- $\perp_\Gamma$ – least element, ie $\bigwedge^\Gamma\{\gamma : \gamma \in \Gamma\}$
- $\top_\Gamma$ – greatest element, ie $\bigvee^\Gamma\{\gamma : \gamma \in \Gamma\}$
- Γ^{op} – Γ but with the order reversed

Some background

Definition

A $\Gamma-$**ultrametric space** is a pair (M, d_M), where M is a set and $d_M : M \times M \to \Gamma$ is a function that satisfies:

- $(\forall x, y \in M)\ d_M(x, y) = \perp \iff x = y$
- $(\forall x, y \in M)\ d_M(x, y) = d_M(y, x)$
- $(\forall x, y, z, \in M)\ d_M(x, y) \vee d_M(y, z) \geq d_M(x, z)$

Definition

A **non-exanding map** between $\Gamma-$ultrametric spaces (M, d_M) and (N, d_N) is a function $f : M \to N$ such that $(\forall a, b \in M)$ $d_N(f(a), f(b)) \leq d_M(a, b)$

Some background

A few things to note

- $\Gamma-$ultrametric spaces and non-exanding maps form a category, denoted $UMet(\Gamma)$
- Given a $\Gamma-$ ultrametric space (M, d_M), the balls of fixed radius γ for $\gamma \in \Gamma$ also form a generalized ultrametric space, with the distance function taking values in $\Gamma_\gamma = \{\zeta \in \Gamma : \zeta \geq_\Gamma \gamma\}$

Some background: closed ball functor

Denote by (M_γ, d_{M_γ}) the generalized ultrametric space having points

$$M_\gamma = \{B^M(x;\gamma) : x \in M\}$$

and distance function

$$d_{M_\gamma}(B^M(x;\gamma), B^M(y;\gamma)) = d_M(x,y) \vee \gamma$$

Lastly if $f : (M, d_M) \to (N, d_N)$ is a map of $\Gamma-$ultrametric spaces, let $f_\gamma : (M_\gamma, d_{M_\gamma}) \to (N_\gamma, d_{N_\gamma})$ be such that

$$f(B^M(x;\gamma)) = B^N(f(x), \gamma)$$

Then f_γ is non-expanding and well-defined.

Some background: closed ball functor

Lemma 2.7 $(\cdot)_\gamma$ is a functor from $UMet(\Gamma)$ to $UMet(\Gamma_\gamma)$

Let's see an example

Motivation

"In order to show that each $\Gamma-$ultrametric space has a minimal, strongly Cauchy complete extension, we first show that the category of $\Gamma-$ultrametric spaces is equivalent to the category of flabby separated presheaves on Γ^{op}."

- Theorem 2.25: a $\Gamma-$ultrametric space is injective if and only if is strongly Cauchy complete.
- Then an injective hull of a Γ-ultrametric space is a minimal extension that is strongly Cauchy complete
- Idea: show every object in $Flab(\Gamma^{op})$ has an injective hull and use the equivalence.

More notation:

- $Flab(\Gamma^{op})$: the category of flabby separated presheaves on Γ^{op}
- $Sep(\Gamma^{op})$: the category of separated presheaves on Γ^{op}
- $i_\Gamma : Flab(\Gamma^{op}) \to Sep(\Gamma^{op})$: the inclusion functor

First we show an equivalence of categories between $UMet(\Gamma)$ and $Flab(\Gamma^{op})$ via two functors, F_Γ and G_Γ.

the functor F_Γ

We define $F_\Gamma : UMet(\Gamma) \to Flab(\Gamma^{op})$ as follows:

Definition

Let $(M, d_M) \in UMet(\Gamma)$ and define $F_\Gamma(M, d_M)(\gamma) = (M_\gamma, d_{M_\gamma})$. Let the restriction maps be given by: $B(a, \gamma) \mid_{\gamma^*} = B(a, \gamma^*)$ for $\gamma^* \geq \gamma$.

Check that:

- The restriction maps are well-defined
- $F_\Gamma(M, d_M)$ is flabby
- $F_\Gamma(M, d_M)$ is separated
- If $f : (M, d_M) \to (N, d_N)$ is a non-expanding map, then $F_\Gamma(f) : F_\Gamma(M, d_M) \to F_\Gamma(N, d_N)$ is a map of flabby separated presheaves.

the functor G_Γ

We define $G_\Gamma : Flab(\Gamma^{op}) \to UMet(\Gamma)$ as follows:

Definition

Let $A \in Flab(\Gamma^{op})$ and define $G_\Gamma(A) = (A(\perp), d_{A_\perp})$, where $d_\perp(a, b) = \wedge\{\gamma : a\,|_\gamma = b\,|_\gamma\}$.

Check that:

- The distance function satisfies the strong triangle identity
- $(\forall a, b \in A(\perp))\; d_\perp(a, b) = \perp \iff a = b$
- If $A, C \in Flab(\Gamma^{op})$ and $f : A \to C$ is a map of presheaves then $G_\Gamma(f) : G_\Gamma(A) \to G_\Gamma(C)$ is a non-expanding map.

Claim 3.19
There is a natural isomorphism $\eta : F_\Gamma \circ G_\Gamma \to id_{Flab(\Gamma^{op})}$ such that $(\forall x \in A(\perp))\ \eta_A(\{x\}) = x$ when $A \in Flab(\Gamma^{op})$.

Proof.
Use the map $\eta_A(B^{G_\Gamma(A)}(a, \gamma)) = a\mid_\gamma$ for $a \in A(\perp)$. Check that:

- η is well-defined and injective (since A being flabby will imply that is surjective)
- η is natural

$\square$

equivalence of categories

Claim 3.20

There is a natural isomorphism $\epsilon : id_{UMet(\Gamma)} \to G_\Gamma \circ F_\Gamma$.

- Suppose $(M', d_{M'}) = G_\gamma(F_\gamma(M, d_M))$.
- Then $M' = \{\{a\} : a \in M\}$ and $d'_M(\{a\}, \{b\}) = \wedge\{\gamma : B^M(a, \gamma) = B^M(b, \gamma)\} = d_M(a, b)$

Theorem 3.12

There is an equivalence of categories between $Flab(\Gamma^{op})$ and $UMet(\Gamma)$

injective hull

Definition

Suppose C is a category and $A \in C$. An **injective hull** for A is a monic $e : A \to I$ where

1. I is injective
2. If $k : A \rightarrowtail I'$ is a monomorphism with I' injective, there is a monomorphism $k' : I \rightarrowtail I'$ such that $k' \circ e = k$.

We show every object of $Flab(\Gamma^{op})$ has an injective hull.

injective hull

Claim 3.23 For every object $\mathcal{F}$ of $Flab(\Gamma^{op})$, the following are equivalent:

1. $\mathcal{F}$ is injective in $Flab(\Gamma^{op})$.
2. $i_\Gamma(\mathcal{F})$ is injective in $Sep(\Gamma^{op})$

In fact, this is enough, since we also have [2],
Claim 3.24 The injective objects of $Sep(\Gamma^{op})$ are exactly the flabby sheaves and every object of $Sep(\Gamma^{op})$ has an injective hull.

injective hull

$\mathcal{F} \in Flab(\Gamma^{op})$ is injective $\Rightarrow i_\Gamma(\mathcal{F})$ is injective in $Sep(\Gamma^{op})$

Proof.

Idea: We have to show that $\mathcal{F}$ is injective when viewed as an object of $Sep(\Gamma^{op})$. Use the fact that $\mathcal{F}$ is injective $\in Flab(\Gamma^{op})$ to build the required maps in $Flab(\Gamma^{op})$ and use the fact that $\mathcal{F}$ is flabby to find appropriate extensions.

$\square$

references

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J. Coulon and J-L.Coulon, "Injectivity in the quasi-topos of separated presheaves on a complete Heyting algebra," Portugal. Math. **47**(3),227-241 (1990).