Sheaf Cohomology Final Exam for Anne Johnson, September 2018

1. Let $\varphi\colon F \to G$ be a morphism of sheaves on a space X.

   (a) Show that the induced map $\mathrm{Et}(\varphi)\colon \mathrm{Et}(F) \to \mathrm{Et}(G)$ between the associated étale spaces over X is surjective if and only if for every open set U of X, and every section $s \in G(U)$, there exists an open cover $(U_i)_{i\in I}$ of U and sections $t_i \in F(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ in $G(U_i)$ for all $i \in I$.

$Et(\phi)$ is surjective

$$\Longleftrightarrow$$

$\phi_x : F_x \to G_x$ is surjective for all $x \in X$

$$\Longleftrightarrow$$

$\forall germ_x(s) \in G_x$, $\exists U_x \ni x$ such that $s\mid_{U_x} = \phi_{U_x}(t_x)$, for some $t_x \in F(U_x)$ and for each $x \in X$; that is, each germ in the stalk of G at x is represented by some s locally in the image of ϕ

$$\Longleftrightarrow$$

for every open set U of X, and every section $s \in G(U)$, there exists an open cover $(U_i)_{i\in I}$ of U and sections $t_i \in F(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ in $G(U_i)$ for all $i \in I$ (given such U_i's covering U, we must have $x \in U_i$ for some i, so take $U_x = U_i$. Conversely, take the U_x's (for all $x \in U$) to be the required cover)

(b) Give an example of a surjective morphism of sheaves $\varphi\colon F \to G$, and an open set U such that $\varphi_U\colon F(U) \to G(U)$ is not surjective.

Let F be the sheaf on S^1 built from the double cover of the circle over itself, $\delta : S^1 \to S^1$ with $\delta(cos(\theta), sin(\theta)) = (cos(2\theta), sin(2\theta))$. Let G be the sheaf on S^1 built from the trivial covering of the circle $\pi : S^1 \to S^1$.Then $\forall U$, $G(U)$ has a single section, given by the identity function. Let us call it $0_{G,U}$. Also note that for all $U \neq X$, $F(U)$ has exactly two sections. Let us call them $0_{F,U}$ and $1_{F,U}$. Define a sheaf map $\phi : F \to G$ by

$$\phi_U(F(U))(0_{F,U}) = \phi_U(F(U))(1_{F,U}) = 0_{G,U}$$

for $U \neq X$. Then ϕ is natural and ϕ gives a surjection on the stalks, i.e $Et(\phi)$ is surjective. Taking $U = X = S^1$, however, we see $\phi(F(U)) = \phi(F(S^1)) = \phi(\emptyset)$ has empty image, so ϕ_{S^1} is not surjective.

2. Let $[0, 1]$ denote the closed interval and $(0, 1)$ denote the open interval and $i : (0, 1) \to [0, 1]$ the inclusion map. Write $\Delta_{(0,1)}\mathbb{Z}$ and $\Delta_{[0,1]}\mathbb{Z}$ for the constant sheaves on $(0, 1)$ and $[0, 1]$ respectively. Let

$$\varphi : i_!\Delta_{(0,1)}\mathbb{Z} \to \Delta_{[0,1]}\mathbb{Z}$$

be the adjunct of the identity map

$$\Delta_{(0,1)}\mathbb{Z} \to \Delta_{(0,1)}\mathbb{Z} = i^*\Delta_{[0,1]}\mathbb{Z}.$$

Calculate the cokernel of φ. Give it both as an étale space and as a sheaf defined as a functor.

Define the presheaf P on X by $P(U) = coker(i_!\Delta_{(0,1)}\mathbb{Z}(U) \to \Delta_{[0,1]}\mathbb{Z}(U)) = \Delta_{[0,1]}\mathbb{Z}(U)/im_{\phi_U}$. Suppose $s \in \Delta_{[0,1]}\mathbb{Z}(U)$. Then $s \in im_{\phi_U}$ if the support of s is closed in U, but this happens just in case 0 and 1 are not in the support of s. In this case $P(U)$ gives back the stalks of $\Delta_{[0,1]}\mathbb{Z}$ at 0 and 1. This gives the following definition of P for U open in X:

$$P(U) = \begin{cases} \mathbb{Z} \times \mathbb{Z}, & \text{if } 0, 1 \in U \\ \mathbb{Z}, & \text{if exactly one of 0 or } 1 \in U \\ 0, & otherwise \end{cases}$$

Note that the stalks of P are:

$$P_x = \begin{cases} \mathbb{Z}, & \text{if } x = 0 \text{ or } x = 1 \\ 0, & otherwise \end{cases}$$

Since $coker(\phi)$ is the associated sheaf of P, this describes the étale space of $coker(\phi)$. As a functor, $coker(\phi)(U)$ is the set of continuous functions from U into $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{Z}$ or 0, respectively. Note that this also describes the pushforward sheaf, $h_*\Delta\mathbb{Z}$, for $h : \{0, 1\} = [0, 1] \setminus (0, 1) \hookrightarrow [0, 1]$, since $h_*\Delta\mathbb{Z}(U) = \Delta\mathbb{Z}(h^{-1}(U))$. Then we may also describe $coker(\phi)$ as the sheaf $h_*\Delta\mathbb{Z}$.

3. Describe the direct sum $F \oplus G$ of two sheaves F and G on a space X. Then show that $H^n(X; F \oplus G) \cong H^n(X; F) \oplus H^n(X; G)$; i.e., sheaf cohomology commutes with taking direct sums.

Let F and G be sheaves on a space X. We define the direct sum of F and G to be the sheaf given by $F \oplus G(U) = F(U) \oplus G(U)$ for U open in X. To see that $H^n(X; F \oplus G) \cong H^n(X; F) \oplus H^n(X; G)$, let $I_F^\bullet$ and $I_G^\bullet$ be injective resolutions of $F$and G, respectively. Then note that $I_F^\bullet \oplus I_G^\bullet$ is an injective resolution of $F \oplus G$: each $I_F^n \oplus I_G^n$ is an injective sheaf because if ϕ is an injective sheaf map from some A to $I_F^n \oplus I_G^n$ and $i : A \to B$ is also an injective map, as below

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\phi"'] & B \\ & I_F^n \oplus I_G^n & \end{tikzcd}$$

Then ϕ can be lifted to a map $\tilde{\phi} : B \to I_F^n \oplus I_G^n$, since if $\phi(a) = (f_a, g_a)$, define $\phi_F(a) = f_a \in I_F^n$ and use the injective property of I_F^n to lift ϕ_F to $\tilde{\phi_F}$, and likewise for a ϕ_G. Then take $\tilde{\phi}$ to be the map $b \in B \mapsto (\tilde{\phi_F}(b), \tilde{\phi_G}(b))$. Taking the co-boundary maps to be the direct sum of the maps in $I_F^\bullet$ and $I_G^\bullet$, we get immediately that $F \oplus G \to I_F^0 \oplus I_G^0 \to I_F^1 \oplus I_G^1 \to \dots$ is exact.

The we have the following map of chain complexes:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & F \oplus G \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_F^0 \arrow[r] \arrow[d] & I_F^0 \oplus I_G^0 \arrow[r] \arrow[d] & I_G^0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_F^1 \arrow[r] \arrow[d] & I_F^1 \oplus I_G^1 \arrow[r] \arrow[d] & I_G^1 \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

where the columns are injective resolutions and the rows are split exact. Now apply the global section functor to this complex:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma F \arrow[r] \arrow[d] & \Gamma F \oplus G \arrow[r] \arrow[d] & \Gamma G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma I_F^0 \arrow[r] \arrow[d] & \Gamma I_F^0 \oplus I_G^0 \arrow[r] \arrow[d] & \Gamma I_G^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma I_F^1 \arrow[r] \arrow[d] & \Gamma I_F^1 \oplus I_G^1 \arrow[r] \arrow[d] & \Gamma I_G^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Note that the rows are still split exact (because each I_F^n is injective (which implies that the rows are exact) and since $\Gamma I_F^n \oplus I_G^n = I_F^n \oplus I_G^n(X) = I_F^n(X) \oplus I_G^n$), and the restriction of the maps in the rows to the homology groups is again split exact:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma F \arrow[r] \arrow[d] & \Gamma F \oplus G \arrow[r] \arrow[d] & \Gamma G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X;F) \arrow[r] \arrow[d] & H^0(X;F \oplus G) \arrow[r] \arrow[d] & H^0(X;G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(X;F) \arrow[r] \arrow[d] & H^1(X;F \oplus G) \arrow[r] \arrow[d] & H^1(X;G) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Since we have for each n

$$0 \to ker(\Gamma I_F^n \to \Gamma I_F^{n+1})/im(\Gamma I_F^{n-1} \to \Gamma I_F^n) \xrightarrow{\Gamma i|_{H^n(X;F)}}$$
$$ker(\Gamma I_F^n \oplus \Gamma I_G^n \to \Gamma I_F^{n+1} \oplus \Gamma I_g^{n+1})/im(\Gamma I_F^{n-1} \oplus \Gamma I_G^{n-1} \to \Gamma I_F^n \oplus \Gamma I_G^n) \xrightarrow{\Gamma i|_{H^n(X;F \oplus G)}}$$
$$ker(\Gamma I_G^n \to \Gamma I_G^{n+1})/im(\Gamma I_G^{n-1} \to \Gamma I_G^n) \to 0$$

so that we can built the splitting maps by noting that kernels and images themselves commute with direct sums.

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4. Given an abelian group G and a topological space X with a point $p \in X$, define a presheaf G_p on X by

$$G_p(U) = \begin{cases} G, & \text{if } p \in U, \\ 0, & \text{if } p \notin U \end{cases} \quad \text{and} \quad \rho_{VU}(a) = \begin{cases} a & \text{if } p \in V \subseteq U, \\ 0, & \text{if } p \notin V \end{cases}$$

for $V \subseteq U \subseteq X$ open sets.

(a) Verify that this defines a sheaf on X.

Let i be the inclusion of p into X and consider $i_*\Delta G$. For U open in X,

$$i_*\Delta G(U) = \Delta G(i^{-1}(U)) = \begin{cases} \Delta G(\{p\}), & \text{if } p \in U \\ \Delta G(\emptyset), & \text{if } p \notin U \end{cases} = \begin{cases} G, & \text{if } p \in U \\ 0, & \text{if } p \notin U \end{cases}$$

The restriction maps for $i_*\Delta G$ are identity maps, and so they agree with the restriction maps of G_p. Then $i_*\Delta G = G_p$ and since $i_*\Delta G$ is a sheaf, so is G_p.

(b) Describe the stalks at each point of the space.

If X is Hausdorff, then for any $q \neq p$, we can find open neighborhoods N_p and N_q of p and q respectively such that $N_p \cap N_q = \emptyset$, which means that for any $q \neq p$, we have only the 0 germ:

$$G_{p_x} = \begin{cases} 0 & x \neq p \\ G & x = p \end{cases}$$

If X is not Hausdorff, then we must consider whether or not a point $q \neq p$ has a neighborhood that does not intersect with some neighborhood of p. Write $q \notin \bar{p}$ if this is the case and $q \in \bar{p}$ if every neighborhood of q intersects some neighborhood of p. Then we have :

$$G_{p_x} = \begin{cases} 0 & x \notin \bar{p} \\ G & x \in \bar{p} \end{cases}$$

(c) Let F be a sheaf on a one-point space $X = \{P\}$. Show that $H^1(X; F) = 0$.

Let F and $X = \{P\}$ be as above. Let

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots$$

be an injective resolution of F. Consider

$$0 \to \Gamma F \to \Gamma I^0 \to \Gamma I^1 \to \Gamma I^2 \to \dots$$

Since X is a one point space $\Gamma I^\bullet \cong I^\bullet_x$. Then since $I^\bullet$ being exact implies $I^\bullet_x$ is exact (by definition), we have that $\Gamma I^\bullet$ is also exact. This means $H^n(X; F) = 0, \forall n > 0$, in particular for $n = 1$.

(d) Let X be an arbitrary space with $p \in X$ and G_p as defined above. Show that $H^1(X; G_p) = 0$.

Let U be any open subset of X. Then either $G_p(U) = 0$ or $G_p(U) = G$. Since we must have $p \in X$, $G_p(X) = G$, and so $\rho_{X,U}$ is always surjective; that is, G_p is flabby. This implies that G_p is acyclic and so $H^n(X; G_p) = 0, \forall n > 0$, in particular for $n = 1$.

5. The torus T is obtained as the quotient space of $[0,1] \times [0,1]$ by the equivalence relation generated by: $(x,0) \sim (x,1)$ for all $x \in [0,1]$ and $(0,y) \sim (1,y)$ for all $y \in [0,1]$.

   The Klein bottle K is obtained as the quotient space of $[0,1] \times [0,1]$ by the equivalence relation generated by: $(x,0) \sim (x,1)$ for all $x \in [0,1]$ and $(0,y) \sim (1,1-y)$ for all $y \in [0,1]$.

   (a) Calculate the cohomology groups of the torus T with values in the constant sheaf $\Delta_T\mathbb{Z}$. (Hint: use Corollary 7.3.2.)

   Note that we can describe the torus as the union of two cylinders $C_1 = [0,1] \times S^1$ and $C_2 = [0,1] \times S^1$ glued together at the ends. We have $T = C_1 \cup C_2$ and $C_1 \cap C_2 = S^1 \oplus S^1$. Then the Mayer-Vietoris sequence gives us:

$$\begin{gathered} 0 \to H^0(T, \Delta_T\mathbb{Z}) \to H^0(C_1; \Delta_T\mathbb{Z}) \oplus H^0(C_2; \Delta_T\mathbb{Z}) \to H^0(S^1 \oplus S^1; \Delta_T\mathbb{Z}) \to \\ H^1(T, \Delta_T\mathbb{Z}) \to H^1(C_1; \Delta_T\mathbb{Z}) \oplus H^1(C_2; \Delta_T\mathbb{Z}) \to H^1(S^1 \oplus S^1; \Delta_T\mathbb{Z}) \to \\ H^2(T, \Delta_T\mathbb{Z}) \to H^2(C_1; \Delta_T\mathbb{Z}) \oplus H^2(C_2; \Delta_T\mathbb{Z}) \to H^2(S^1 \oplus S^1; \Delta_T\mathbb{Z}) \to \end{gathered}$$

   By example 8.5 in the notes and exercise 3 above, this above is equal to

$$\begin{gathered} 0 \to H^0(T, \Delta_T\mathbb{Z}) \to H^0(C_1; \Delta_T\mathbb{Z}) \oplus H^0(C_2; \Delta_T\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \to \\ H^1(T, \Delta_T\mathbb{Z}) \to H^1(C_1; \Delta_T\mathbb{Z}) \oplus H^1(C_2; \Delta_T\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \to \\ H^2(T, \Delta_T\mathbb{Z}) \to H^2(C_1; \Delta_T\mathbb{Z}) \oplus H^2(C_2; \Delta_T\mathbb{Z}) \to 0 \to \end{gathered}$$

   Since a cyclinder is homotopic to S^1 and since $H^0(T, \Delta_T\mathbb{Z}) = \Gamma\Delta_T\mathbb{Z}(T) = \mathbb{Z}$ since T is connected, we have:

$$\begin{gathered} 0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \\ H^1(T, \Delta_T\mathbb{Z}) \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z} \to \\ H^2(T, \Delta_T\mathbb{Z}) \to 0 \to 0 \to \end{gathered}$$

   And since the above is exact, this implies $H^2(T, \Delta_T\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/img$. By section 6.5 of the notes, the map g is given by $(z_1, z_2) \mapsto (z_1 \mid_{C_1 \cap C_2} - z_2 \mid_{C_1 \cap C_2}) = (z_1 - z_2) \oplus (z_2 - z_1) \cong \mathbb{Z}$, so that $H^2(T, \Delta_T\mathbb{Z}) \cong \mathbb{Z}$. In summary:

$$H^n(T; \Delta_T\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \text{ or } n = 2; \\ 0, & \text{if } n > 2 \end{cases}$$

   (b) Use a Mayer-Vietoris sequence to calculate the cohomology groups of the Klein bottle with values in the constant sheaf $\Delta_K\mathbb{Z}$.

   Note that we can describe the space K as the union of two Möbius strips $X = [0,1] \times [0,1]/(x,0) \sim (1-x,1)$ and $Y = [0,1] \times [0,1]/(y,0) \sim (1,1-y)$. Further note that X, Y and $X \cap Y$ are all homotopic to S^1. Then Mayer-Vietoris sequence gives :

$$0 \to H^0(K, \Delta_K\mathbb{Z}) \to H^0(X; \Delta_K\mathbb{Z}) \oplus H^0(Y; \Delta_K\mathbb{Z}) \to H^0(X \cap Y; \Delta_K\mathbb{Z}) \to$$

$$H^1(K,\Delta_K\mathbb{Z}) \to H^1(X;\Delta_K\mathbb{Z}) \oplus H^1(Y;\Delta_K\mathbb{Z}) \to H^1(X\cap Y;\Delta_K\mathbb{Z}) \to$$
$$H^2(K,\Delta_K\mathbb{Z}) \to H^2(X;\Delta_K\mathbb{Z}) \oplus H^2(Y;\Delta_K\mathbb{Z}) \to H^2(X\cap Y;\Delta_K\mathbb{Z}) \to$$

which by example 8.2, for X, Y and $X\cap Y$ homotopic to S^1 gives (and K connected):

$$0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to$$

$$H^1(K,\Delta_K\mathbb{Z}) \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} \mathbb{Z} \to$$
$$H^2(K,\Delta_K\mathbb{Z}) \to 0 \to 0 \to$$

Exactness gives $H^2(K,\Delta_K\mathbb{Z}) \cong \mathbb{Z}/img$ and since the map g is $(z_1,z_2) \mapsto (z_1 - z_2)$, $img \cong \mathbb{Z}$, and $H^2(K,\Delta_K\mathbb{Z}) = \mathbb{Z}/\mathbb{Z} = 0$. In summary:

$$H^n(K;\Delta_T\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n > 1 \end{cases}$$

(c) Let S be the middle circle of the torus, given by the image of the segment $\{(x,0)|\, x \in [0,1]\}$ in the quotient space. Give the long exact sequences for the pair (T,S) and for the relative cohomology with coefficients in $\Delta_T\mathbb{Z}$ and calculate as many of the groups as you can.

The long exact sequence of the cohomology of the pair (S,T) is given by:

$$\ldots \to H^n(T,S;\Delta_T\mathbb{Z}) \to H^n(T;\Delta_T\mathbb{Z}) \to H^n(S;\Delta_T\mathbb{Z}\mid_S)$$

$$H^{n+1}(T,S;\Delta_T\mathbb{Z}) \to \ldots$$

From the results above and the fact that S is homotopic to S^1, we can fill in:

$$H^0(T,S;\Delta_T\mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z} \to$$
$$H^1(T,S;\Delta_T\mathbb{Z}) \to H^1(T;\Delta_T\mathbb{Z}) \to \mathbb{Z} \to$$
$$H^2(T,S;\Delta_T\mathbb{Z}) \to \mathbb{Z} \to 0$$
$$H^3(T,S;\Delta_T\mathbb{Z}) \to 0 \to 0 \to \ldots$$

so that $H^3(T,S;\Delta_T\mathbb{Z}) = 0$ for $n > 2$. We also have $H^0(T,X;\Delta_T\mathbb{Z}) = H^0(T;i_!i^*\Delta_T\mathbb{Z})$, where $i : T\setminus S \hookrightarrow T$, so that $H^0(T,X;\Delta_T\mathbb{Z}) = \Gamma i_!i^*\Delta_T\mathbb{Z} = i_!i^*\Delta_T\mathbb{Z}(T) = \mathbb{Z}$.

For the local cohomology, note that $S = T\setminus Y$, where $Y = T\setminus S$. The long exact sequence for the local cohomology is:

$$\ldots \to H^n_S(T;\Delta_T\mathbb{Z}) \to H^n(T;\Delta_T\mathbb{Z}) \to H^n(Y;\Delta_T\mathbb{Z}\mid_Y) \to$$

$$H^{n+1}_S(T;\Delta_T\mathbb{Z}) \to \ldots$$

From the results above and the fact that Y is homotopic to T, we can fill in:

$$H^0_S(T; \Delta_T\mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z} \to$$

$$H^1_S(T; \Delta_T\mathbb{Z}) \to H^1(T; \Delta_T\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \to$$

$$H^2_S(T; \Delta_T\mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z} \to$$

$$H^3_S(T; \Delta_T\mathbb{Z}) \to 0 \to 0 \to \dots$$

so that the higher homology groups are all zero. For $H^0_S(T; \Delta_T\mathbb{Z})$, note that $H^0_S(T; \Delta_T\mathbb{Z}) = \Gamma_S(\Delta_T\mathbb{Z}) = \{z \in \Gamma(\Delta_T\mathbb{Z}); supp(z) \subseteq S\} = 0$, since the support of 0 is $\emptyset \subseteq S$.

(d) Let $S' \subset K$ be the image of $\{(x, 0) |\, x \in [0, 1]\}$ in the quotient space K. Use the long exact sequence from Proposition 9.4.1 and fill in as many groups as you can to find the cohomology with compact support for the open annulus and the open Möbius strip.