Exercises

Section 2

2.1. For a commutative triangle of topological spaces and continuous maps,

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

if f and π are étale, so is h (in particular, h is open).

Let $y \in E$. Since f is étale, $\exists U$ an open neighborhood of $h(y)$ such that $f\mid_U$ is a homeomorphism. Likewise, since π is étale, $\exists W$ an open neighborhood of y such that $\pi\mid_W$ is a homeomorphism. Let $V = h^{-1}(U) \cap W$. Since h is continuous, $h^{-1}(U)$ is open, so V is open in E. Since $V \subseteq W$, $\pi\mid_V$ is a homeomorphism.

Since $h(V) \subseteq U$, and $f\mid_U$ is a homeomorphism, it only remains to show that $h(V)$ is open, since this will imply that $f\mid_{h(v)}$ is a homeomorphism, and in turn that $h\mid_V$ is a homeomorphism (since $\pi\mid_V$ is a homeomorphism, and the diagram commutes). To see that $h(V)$ is open, note that $\pi(V)$ is open in X, since π is étale and V is open in E. Further, $f(h(V)) = \pi(V)$ since the diagram commutes. Then since $h(V) \subseteq U$ and$f\mid_U$ is a homeomorphism, $h(V)$ is open in Z, so $h\mid_V$ is a homeomorphism and h is étale.

2.2. (a) Show that Corollary 2.10 determines the associated sheaf up to isomorphism. That is, if R is a sheaf and there exists a map $\psi : P \to R$ with the same property as η in Corollary 2.10, then $R \cong \Gamma Et(P)$.

Corollary. (2.10) For a map $\phi : P \to Q$ between presheaves on X, if Q is a sheaf then there exists a unique map of sheaves $\hat{\phi}$ such that $\hat{\phi} \circ \eta = \phi$.

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\eta"'] & Q \\ \Gamma EtP \arrow[ur, dashed, "!", "\hat{\phi}"'] & \end{tikzcd}$$

The universal property of $(\Gamma EtP, \eta)$ implies a unique map $\hat{\psi}$ such that $\psi = \hat{\psi} \circ \eta$ and the universal property of (R, ψ) implies a unique map $\hat{\eta}$ such that $\eta = \hat{\eta} \circ \psi$, i.e. we have the following diagrams:

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\eta"'] & R \\ \Gamma EtP \arrow[ur, dashed, "!", "\hat{\psi}"'] & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\psi"'] & \Gamma EtP \\ R \arrow[ur, dashed, "!", "\hat{\eta}"'] & \end{tikzcd}$$

This implies the following diagrams commute:

$$\begin{tikzcd} & \Gamma EtP \arrow[d, dashed, "!"', "\hat{\psi}"] \\ P \arrow[ur, "\eta"] \arrow[r, "\psi"] \arrow[d, "\eta"'] & R \arrow[dl, dashed, "!"', "\hat{\eta}"] \\ \Gamma EtP & \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[d, dashed, "!"', "\hat{\eta}"] \\ P \arrow[ur, "\psi"] \arrow[r, "\eta"] \arrow[d, "\psi"'] & \Gamma EtP \arrow[dl, dashed, "!"', "\hat{\psi}"] \\ R & \end{tikzcd}$$

In particular, $\hat{\eta} \circ \hat{\psi}$ is the unique map such that $\eta = \hat{\eta} \circ \hat{\psi} \circ \eta$ and $\hat{\psi} \circ \hat{\eta}$ is the unique map such that $\psi = \hat{\psi} \circ \hat{\eta} \circ \psi$. But then uniqueness implies $\hat{\eta} \circ \hat{\psi} = id_{\Gamma EtP}$ and $\hat{\psi} \circ \hat{\eta} = id_R$. Then $R \cong \Gamma EtP$.

2.2. (b) If P is a subpresheaf of R and R is a sheaf, then let

$$\tilde{P}(U) = \{r \in R(U) \mid \text{for each } x \in U \text{ there is a neighbourhood } W_x \subseteq U \text{ of } x \\ \text{such that } (r\mid_{W_x}) \in P(W_x)\}.$$

Show that $P \subseteq \tilde{P} \subseteq R, \tilde{P}$ is a sheaf and $P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10; hence $\tilde{P}$ is the associated sheaf for P.

<u>$P \subseteq \tilde{P} \subseteq R$:</u>
We want to show that for $U \in X$, open, $P(U) \subseteq \tilde{P}(U) \subseteq R(U)$. So let $U \subseteq X$ be open. By construction $\tilde{P}(U) \subseteq R(U)$, so we need only show $P(U) \subseteq \tilde{P}(U)$. Let $s \in P(U)$. Since P is a subpresheaf $s \in R(U)$ and for $x \in U$, U itself is a neighborhood of x such that, $s\mid_U$ is in $P(U)$. Then $s \in \tilde{P}(U)$.

<u>$\tilde{P}$ is a sheaf:</u>
Let U be an open set in X and let $\cup U_i$, $i \in I$, be an open cover for U. Suppose $\{a_i\} \in \tilde{P}(U_i)$, for $i \in I$ is a compatible family for the U_i, i.e., $a_i\mid_{U_i \cap U_j} = a_j\mid_{U_i \cap U_j}$. For each $i \in I$ and each $x \in U_i$, choose W_x^i such that $(a_i\mid_{W_x^i}) \in P(W_x^i)$. Then $\cup_i \cup_{x \in U_i} W_x^i$ is an open cover for U (since $\cup_x W_x^i$ is an open cover for each U_i). Moreover, $a_i\mid_{W_x^i \cap W_y^j} = a_j\mid_{W_x^i \cap W_y^j}$ because $(a_i\mid_{U_i \cap U_j})\mid_{W_x^i \cap W_y^j} = (a_j\mid_{U_i \cap U_j})\mid_{W_x^i \cap W_y^j}$. So $\{a_i\} \in P(W_x^i)$ is a compatible family in R.

Since R is a sheaf, let a e the unique amalgamation. We want to show $a \in \tilde{P}(U)$, i.e., $\forall x \in U, \exists$ a neighborhood V_x of x such that $a\mid_{V_x} \in P(V_x)$. But since a is the amalgamation of the family $\{a_i\} \in P(W_x^i)$, if $x \in U$, then $x \in U_i$, for some i, so $a\mid_{W_X^i} = a_i$ and W_x was chosen so that $a_i\mid_{W_x^i} \in P(W_x^i)$. Then take V_x to be W_x^i (for appropriate i). Then $a \in \tilde{P}(U)$, so $\tilde{P}$ is a sheaf.

$P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10:

Let $i : P \hookrightarrow \tilde{P}$ be the map that injects $P(U)$ into $\tilde{P}(U)$. We must show that for a map between presheaves, $\phi : P \to Q$, where Q is a sheaf, there is a unique $\tilde{\phi}$ such that $\tilde{\phi} \circ i = \phi$:

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, hook, "i"'] & Q \\ \tilde{P} \arrow[ur, dashed, "!"', "\tilde{\phi}"] & \end{tikzcd}$$

So let Q be a sheaf and $\phi : P \to Q$ be a morphism. For each open set $U \subseteq X$, we define $\tilde{\phi}_U : \tilde{P}(U) \to Q(U)$ as follows: let $s \in \tilde{P}(U)$ and let $\cup_{x \in U} W_x$ be a cover of U such that $(s\mid_{W_x}) \in P(W_x)$ for each $x \in U$. Consider the family $\phi_{W_x}(s\mid_{W_x})$ for $x \in U$. This family is compatible because commutativity of the restriction maps implies $(s\mid_{W_x})\mid_{W_x \cap W_y} = (s\mid_{W_y})\mid_{W_x \cap W_y} = s\mid_{W_x \cap W_y}$ and naturality of ϕ implies for $x, y \in U$:

$$\phi(s\mid_{W_x})\mid_{W_x \cap W_y} = \phi((s\mid_{W_x})\mid_{W_x \cap W_y}) = \phi(s\mid_{W_x \cap W_y})$$

$$= \phi((s\mid_{W_y}\mid_{W_x \cap W_y})) = \phi(s\mid_{W_y})\mid_{W_x \cap W_y}\ .$$

Then since Q is a sheaf, let $t \in Q(U)$ be the unique amalgamation of the $\phi_{W_x}(s\mid_{W_x})$'s and define $\tilde{\phi}(s) = t$. In fact, by naturality of $\tilde{\phi}$, we must choose the amalgamation to ensure that the following commutes for each W_x:

$$\begin{tikzcd} s \in \tilde{P}(U) \arrow[r, "\tilde{\phi}_U"] \arrow[d, "\rho"'] & Q(U) \ni t \arrow[d, "\rho"] \\ (s\mid_{W_x}) \in \tilde{P}(W_x) \arrow[r, "\tilde{\phi}_{W_x}"'] & Q(W_x) \ni \tilde{\phi}(s\mid_{W_x}) = \phi(s\mid_{W_x}) \end{tikzcd}$$

since the bottom row must be $\tilde{\phi}(s\mid_{W_x}) = \phi(s\mid_{W_x})$ by the requirement that $\tilde{\phi} \circ i = \phi$. So then uniqueness of $\tilde{\phi}$ is implied so long as $\tilde{\phi}$ is well-defined.

To see that $\tilde{\phi}$ is well-defined, let $\cup_{x \in U} V_x$ be another cover of U with the property that $(s\mid_{V_x}) \in P(V_x)$, so that $\phi_{V_x}(s\mid_{V_x})$ is again a compatible family. We must show that t is also the amalgamation of this family, i.e., $t\mid_{V_x} = \phi_{V_x}(s\mid_{V_x})$ for each V_x.

Consider the cover of U formed by the refinement of the W_x's and V_x's, $U = \cup_{x \in U} W_x \cup_{x \in U} V_x$. This contains the compatible family

$$\{\{\phi(s\mid_{W_x})\}_{x\in U}, \{\phi(s\mid_{v_x})\}_{x\in U}\}$$

.

The amalgamation of this family, say $\bar{t}$, must have the property that $\bar{t}\mid_{W_x} = \phi(s\mid_{W_x})$, and since Q is a sheaf, hence separated, this implies that $\bar{t} = t$. Then $t\mid_{V_x} = \phi_{V_x}(s\mid_{V_x})$ for each V_x. In particular, note that $\tilde{\phi} \circ i = \phi$, since $s \in P(U)$ implies that U is a cover of U for which $s\mid_U \in P(U)$ so $\tilde{\phi}(s) = \phi(s\mid_U) = \phi(s)$.

The $\tilde{\phi}$ is the required unique map satisfying $\tilde{\phi} \circ i = \phi$, so $\tilde{P}$ is the associated sheaf of P.

Section 3

3.1 The pullback of an étale map is also étale. That is, given a pullback diagram,

$$\begin{tikzcd} X \times_Y E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

then π_1 is étale if p is.

Let $(u, e) \in X \times_Y E$. Since p is étale, there exists a $U \subseteq E$, open neighborhood of $e = \pi_2(u, e)$, such that $p : U \to p(U)$ is a homeomorphism. Since $f : X \to Y$ is continuous (and p is open), the preimage of $p(U)$ under f is open in X. Let $W \subseteq X$ be the preimage of $p(U)$. Since $p(e) = f(u)$, $f(u) \in p(U)$ and so $u \in W$.

Since $W \times U$ is open in $X \times E$ under the product topology, $V = W \times U \cap X \times_Y E$ is an open neighborhood of (u, e) in $X \times_Y E$ under the subspace topology. I claim that $\pi_1 |_V$ is a homeomorphism. Since π_1 is a projection map, $\pi_1 |_V$ is continuous. We must show there exists a continuous $g : \pi_1(V) \to X \times_Y E$ such that $g \circ \pi_1 = id$. Since $p |_U$ is a homeomorphism, we have continuous $p^{-1} : f(\pi_1(V)) \to E$ such that $p^{-1} \circ p = id_E$. Then $p^{-1}(f(\pi_1(V))) = p^{-1}(p(\pi_2(V))) = \pi_2(V)$. Let $i : EtB \hookrightarrow X \times_Y EtB$ be the inclusion map. Then let $g : \pi_1(V) \to X \times_Y EtB$ be defined by $x \mapsto (x, i \circ p^{-1} \circ f(x))$. Then g is continuous and $g \circ \pi_1 |_V = id_{X \times_Y E}$, so that $\pi_1 |_V$ is a homeomorphism. Then π_1 is étale.

3.2. Verify the following equality of stalks:

$$(f^*B)_x \cong B_{f(x)}$$

for each $x \in X$.

First note that since $f^*(B) = \Gamma(X \times_Y EtB)$ and since π_1 is étale, $Et\Gamma(X \times_Y EtB) \cong X \times_Y EtB$ by the results in section 2. That is, $Et(f^*(B)) \cong X \times_Y EtB = \{(x, e) \in X \times EtB \mid f(x) = \pi_B(e)\}$.

Fix an $x_0 \in X$. The stalk of $f^*(B)$ at x_0 is $(f^*(B))_{x_0} = \{s \in Et(f^*(B)) \mid \pi_B(s) = x_0\} = \{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\}$. Then the map π_2 gives us a bijection $\{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\} \to \{e \in EtB \mid f(x_0) = \pi_B(e)\}$, which is exactly the stalk of $B_{f(x_0)}$.

3.3. Prove that

$$Hom_{Sh(Y)}(i_!A, B) \cong Hom_{Sh(X)}(A, i^*B)$$

where $i : X \hookrightarrow Y$ is an open inclusion.

First note that since $i \circ \pi_A$ is étale, $Et(i_!A) = Et(\Gamma EtA) = EtA$. Since B is a sheaf, 2.9 implies $\phi \in Hom_{Sh(Y)}(i_!A, B)$ corresponds to a map $g : EtA \to EtB$ such that the following commutes:

$$\begin{tikzcd} EtA \arrow[rr, "g"] \arrow[dr, "i\circ\pi_A"'] & & EtB \arrow[dl, "\pi_B"] \\ & Y & \end{tikzcd}$$

By the notes, $Et(i^*B) = Et(B\mid_X)$ and since i^*B is a sheaf, 2.9 implies $\psi \in Hom_{Sh(X)}(A, i^*B)$ corresponds to a map $h : EtA \to Et(B\mid_X)$ such that the follow commutes:

$$\begin{tikzcd} EtA \arrow[rr, "h"] \arrow[dr, "\pi_A"'] & & Et(B\mid_X) \arrow[dl, "\pi_{i^*B}"] \\ & X & \end{tikzcd}$$

To see the correspondance between g and h, note that $Et(B\mid_X) \cong X \times_Y EtB$, so that given $g : EtA \to EtB$, h is the unique map such that the following commutes (and given $h : EtA \to Et(B\mid_X) \cong X \times_Y EtB$, g is the map $\pi \circ h$):

$$\begin{tikzcd} EtA \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "\pi_A"'] & & \\ & X \times_Y EtB \arrow[r, "\pi"] \arrow[d, "\pi_{Bi^*B}"'] & EtB \arrow[d, "\pi_B"] \\ & X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

3.4 (a) Verify the adjunction formula involving j_* and $j^!$ in (7).

We must show that $Hom_{Ab(Y)}(j_*A, B) \cong Hom_{Ab(Z)}(A, j^!B)$.

Let $\phi \in Hom_{Ab(Y)}(j_*A, B)$. Then ϕ is a collection of maps $\phi_U : j_*A(U) \to B(U)$ for every open set $U \subseteq Y$. Note that for $U \subseteq Y$ such that $U \cap Z = \emptyset$, $j_*A(U) = A(U \cap Z) = A(\emptyset) = \{0\}$, so that there is only one choice for ϕ_U. Likewise, if $U \subset Y$ with $U \cap \delta Z \neq \emptyset$, then there exists a $V \subset U$ such that $V \cap Z = \emptyset$ and naturality of ϕ implies:

$$\begin{array}{ccc} j_*A(U) & \xrightarrow{\phi_U} & B(U) \\ \rho\downarrow & & \downarrow\rho \\ j_*A(V) = A(V \cap Z) = \{0\} & \xrightarrow[\phi_V]{} & B(V) \end{array}$$

so that $\phi_U(j_*A(U))$ is the constant 0 map. Finally, suppose U is such that $U \cap Z = U$. For each such U, the choices for ϕ_U are the group homomorphisms, $A(U) \to B(U)$.

Now consider $\psi \in Hom_{Ab(Z)}(A, j^!B)$. ψ is a collection of maps ψ_V for each open V in Z. Suppose V is open in Z and $V \cap \delta Z \neq \emptyset$. Then $j^!B(V) = 0$ and so there is only one choice for ψ_V, the constant 0 map. On the other hand, if $V \cap \delta Z = \emptyset$, then the choices for ϕ_V are the group homomorphisms, $A(V) \to B(V)$.

Then elements of both $Hom_{Ab(Z)}(A, j^!B)$ and $Hom_{Ab(Y)}(j_*A, B)$ correspond to the group homorphisms from $A(U) \to B(U)$ for every $U \subseteq Y$ with $U \cap Z = U$.

3.4 (b) Show that $j^! B$ is isomorphic to a subsheaf of $j^* B$.

Since $j^! B$ is a sheaf,we know that $j^! B \cong \Gamma Et j^! B$. We show that $\Gamma Et j^! B$ is a subsheaf of $j^* B$ and for this is suffices to show that $\Gamma Et j^! B$ is a sub-presheaf; that is, that for V open in Z, $\Gamma Et j^! B(V) \subseteq j^* B(V)$ and that the restrictions maps of $\Gamma Et j^! B$ agree with those of $j^* B$.

Let V be open in Y, i.e. $V = Z \cap U$ for some U open in Y. We have

$$j^* B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B) \text{ and } x \in V\}$$

and

$$\Gamma Et j^! B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B),\ x \in V \text{ and } supp(s) \subseteq Z\}$$

so that $\Gamma Et j^! B(V) \subseteq j^* B(V)$. Moreover, the restriction maps agree since they are both restrictions of functions. Then $\Gamma Et j^! B \cong j^! B$ is a subsheaf of $j^* B$.

Section 4

4.1 (a) Show that with the notation as above, $\Gamma(-,\pi)$ is indeed isomorphic to the sheaffication of P.

Since $\Gamma(-,\pi)$ is the sheaf of sections of $\pi : Et(G)/R \to X$ and the associated sheaf of P is the sheaf of sections of $\pi_P : Et(P) \to X$, it suffices to show that $Et(G)/R \cong Et(P)$ and for this it suffices to show the stalks are isomorphic, i.e., $G_x/R \cong P_x$.

Define a map $\phi : G_x \to P_x$ by $\phi(germ_x(s)) = germ_x(\overline{s})$, where $\overline{s}$ is the equivalence class of s in $G(U)/im(\alpha_U)$, where (U, x) is a representative of $germ_x(s)$.
Show ϕ is well-defined, i.e., if (V, t) is also a representative of $germ_x(s)$ then $germ_x(\overline{s}) = germ_x(\overline{t})$. Then show that $\phi(germ_x(s)) = \phi(germ_x(t)))$ iff there exists a neighborhood W of x such that $s\mid_W = t\mid_W$. Then the kernel of ϕ is R, so the result follows.

4.1(b) Show that the $coker(\alpha)$ has the following universal property: given any other sheaf H, a map $\chi : G \to H$ factors through $coker(\alpha)$ if and only if $\chi \circ \alpha = 0$.

(Forward direction) Let $\alpha : F \to G$ and $\chi : G \to H$ and suppose $\chi \circ \alpha = 0$. Then for $U \in X$, $ker(\chi_U) = im(\alpha_U)$. By the first isomorphism theorem, $\chi_U(G) \cong G(U)/ker(\chi_U) = G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$, so that $\chi = g \circ f$, where $f : G \to coker(\alpha)$ is the projection map and $g : coker(\alpha) \to H$ is an isomorphism.

(Backwards direction) Let $\alpha : F \to G$ and $\chi : G \to H$ and suppose $\chi = g \circ f$ for some $f : G \to coker(\alpha)$ and $g : coker(\alpha) \to H$. Note that $f \circ \alpha = 0$, since f_U has codomain $coker(F(U) \xrightarrow{\alpha_U} G(U)$ for $U \in X$. Then $\chi \circ \alpha = g \circ f \circ \alpha = g \circ 0 = 0$.

4.1(c) Show that $coker(\alpha)_x = coker(\alpha_x)$.

4.2 Show that $\mathbf{Ab(X)}$ is an Abelian category with kernel, cokernel and sum defined as above.

We first show that $\mathbf{Ab(X)}$ has a 0 object. Let $\Delta\{*\}$ be the constant sheaf on a singleton set. Then for $U \subseteq X$, $\Delta\{*\}(U)$ is the trivial group and every restriction map is the identity, so $\Delta\{*\} \in \mathbf{Ab(X)}$. Moreover, for $A \in \mathbf{Ab(X)}$, there is exactly one map from $A(U) \to \Delta\{*\}(U)$ for every open $U \subseteq X$, namely the constant 0-map, and there is also exactly one map from $\Delta\{*\}(U) \to A(U)$ since group homomorphism must send identity to identity. Then $\Delta\{*\}$ is both initial and terminal, and so serves as the 0 element in $\mathbf{Ab(X)}$. We can now define the 0 morphism in $\mathbf{Ab(X)}$ as the map defined for $A, B \in \mathbf{Ab(X)}$ and $U \in X$, open, by $a \in A(U) \mapsto 0_{B(U)}$.

Let us define the sum of two Abelian sheaves A and B by $A(U) \oplus B(U)$ for U open in X, as in the notes. Then this gives the biproduct of A and B in $\mathbf{Ab(X)}$. We know from the notes that $A \oplus B$ is a sheaf and we observe that for $U \subseteq X$, $A(U) \oplus B(U)$ has an Abelian group structure: let (a, b) and (a', b') be elements of $A(U) \oplus B(U)$ for some U. Then $(a, b) + (a', b') = (a + a', b + b') = (a' + a, b' + b) = (a', b') + (a, b) \in A(U) \oplus B(U)$, since $A(U)$ and $B(U)$ are Abelian groups. This also implies $(a, b) + (0_{A(U)}, 0_{B(U)}) = (a, b) = (0_{A(U)}, 0_{B(U)}) + (a, b)$. To see that this is a biproduct, let $p_j : A_1 \oplus A_2 \to A_j$ be the map defined on $U \subseteq X$ as the projection map $p_{U,j} : A_1(U) \oplus A_2(U) \to A_j(U)$ and i_j be the map defined on $U \subseteq X$ as the injection map $i_j(U) : A_j(U) \to A_1(U) \oplus A_2(U)$ for $j \in \{1, 2\}$ and $A_j \in \mathbf{Ab(X)}$. Then since $p_{U,j}$ and $i_{U,j}$ give the product and coproduct on each $A_1(U) \oplus A_2(U)$, viewed as elements in the category of Abelian groups, their collection gives the product and coproduct on $A_1 \oplus A_2$ (given a map into or out of the biproduct, find a new map that factors through p_j or i_j respectively, by taking the collection of maps implied by the universal property of each $p_{U,j}$ or $i_{U,j}$).

To see that $\mathbf{Ab(X)}$ has kernels, note that if $A, B \in \mathbf{Ab(X)}$ and $\phi : A \to B$, then ϕ has a kernel when everything is viewed in the category $\mathbf{Sh(X)}$. It remains only to check that $ker(\phi)(U) = \{f \in A(U) \mid \phi_U(f) = 0_{B(U)}\}$ is an Abelian group for all U, open in X. But $ker(\phi)(U) = ker(\phi_U)$ is the kernel of the group homomorphism ϕ_U, and so must itself be a subgroup of $A(U)$ for all $U \in X$. Then $ker(\phi) \in \mathbf{Ab(X)}$. Likewise, the cokernel of ϕ exists in $\mathbf{Sh(X)}$, and so we need only show $coker(\phi)(U)$ is an Abelian group, $\forall U$.

4.3 Let $0 \to A \hookrightarrow I^0 \to I^1 \to ...$ and $0 \to B \to B \hookrightarrow J^0 \to J^1 \to J^2 \to ...$ be injective resolutions of A and B respectively. Show that a map $\phi : A \to B$ extends to a map of complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "\phi"] & I^0 \arrow[r] \arrow[d, dashed, "\phi^0"] & I^1 \arrow[r] \arrow[d, dashed, "\phi^1"] & I^2 \arrow[r] \arrow[d, dashed, "\phi^2"] & ... \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & ...
\end{tikzcd}$$

Since $0 \to A \hookrightarrow I^0 \to I^1 \to ...$ is a resolution, in particular exact, for each n, we have an injective map from $A \to I^n$ given by d^n. Since $\phi : A \to B$, we also have for each n, a map from $A \to J^n$ given by $d^n \circ \phi$. Then injectivity of J^n implies the required map $\phi^n : I^n \to J^n$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "d^n"] \arrow[d, "d^n \circ \phi"'] & I^n \arrow[dl, dashed, "\phi^n"] \\
& J^n &
\end{tikzcd}$$

4.3 (b) Show that a map of complexes as above induces a homomorphism of cohomology groups

$$H^n(X; A) \to H^n(X, B)$$

Note that the above map of chain complexes implies a map for $U \in X$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A(U) \arrow[r] \arrow[d, "\phi_U"] & I^0(U) \arrow[r] \arrow[d, dashed, "\phi_U^0"] & I^1(U) \arrow[r] \arrow[d, dashed, "\phi_U^1"] & I^2(U) \arrow[r] \arrow[d, dashed, "\phi_U^2"] & ... \\
0 \arrow[r] & B(U) \arrow[r] & J^0(U) \arrow[r] & J^1(U) \arrow[r] & J^2(U) \arrow[r] & ...
\end{tikzcd}$$

which for $U = X$ gives the map:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \Gamma A \arrow[r] \arrow[d, "\Gamma\phi"] & \Gamma I^0 \arrow[r] \arrow[d, dashed, "\Gamma\phi^0"] & \Gamma I^1 \arrow[r] \arrow[d, dashed, "\Gamma\phi^1"] & \Gamma I^2 \arrow[r] \arrow[d, dashed, "\Gamma\phi^2"] & ... \\
0 \arrow[r] & \Gamma B \arrow[r] & \Gamma J^0 \arrow[r] & \Gamma J^1 \arrow[r] & \Gamma J^2 \arrow[r] & ...
\end{tikzcd}$$

Since $H^n(X; A) = ker(\Gamma I^n \to \Gamma I^{n+1}))/im(\Gamma I^{n-1} \to \Gamma I^n) \subseteq \Gamma I^n$ and likewise $H^n(X; B) = ker(\Gamma J^n \to \Gamma J^{n+1}))/im(\Gamma J^{n-1} \to \Gamma J^n) \subseteq \Gamma J^n$, the maps $\Gamma\phi^n : \Gamma I^n \to \Gamma J^n$ implies a homomorphism $\Gamma\phi^n \mid_{H^n(X,A)}: H^n(X; A) \to \Gamma J^n$. It remains only to check that $im(\Gamma\phi^n \mid_{H^n(X,A)}) \subseteq H^n(X, B)$. Get this by diagram chasing.

4.4(a) Let $f : Y \to X$ be a map of topological spaces. Show that there is a natural isomorphism $\Gamma_Y \to \Gamma_X \circ f_*$,

$$\begin{tikzcd} Ab(Y) \arrow[rr, "f_*"] \arrow[dr, "\Gamma_Y"'] & & Ab(X) \arrow[dl, "\Gamma_X"] \\ & \textbf{Abelian Groups} & \end{tikzcd}$$

We must find η_A and η_B, isomorphisms, such that the following commutes for all $A, B \in Ab(Y)$ and $\phi : A \to B$:

$$\begin{tikzcd} \Gamma_Y(A) \arrow[r, "\Gamma_Y \phi"] \arrow[d, "\eta_A"'] & \Gamma_Y(B) \arrow[d, "\eta_B"] \\ \Gamma_X \circ f_*(A) \arrow[r, "\Gamma_x \circ f_* \phi"'] & \Gamma_X \circ f_*(B) \end{tikzcd}$$

By the definition of $\Gamma_\_$ and f_*, this is the same as finding isomorphisms η_A and η_B such that the following commutes:

$$\begin{tikzcd} A(Y) \arrow[r, "\phi_Y"] \arrow[d, "\eta_A"'] & B(Y) \arrow[d, "\eta_B"] \\ A(f^{-1}(X)) \arrow[r, "\phi_{f^{-1}(X)}"'] & B(f^{-1}(X)) \end{tikzcd}$$

But $f^{-1}(X) = Y$, so taking $\eta_A = \rho_{Y,Y} = id_{A(Y)}$ (likewise for η_B) implies the result, since ϕ is a map $A \to B$ and so must be natural wrt the restriction maps.

4.4 (b) Show that a natural isomorphism between left exact functors $\tau : T_1 \to T_2$ induces a natural isomorphism $R^n T_1 \to R^n T_2$ for each n.

Let T_1 and T_2 be left exact functors from $\mathcal{C} \to \mathcal{D}$ and let $\tau : T_1 \to T_2$ be a natural isomorphism. Let $0 \to C \to I^0 \to I^1 \to ...$ be an injective resolution of C for some $C \in \mathcal{C}$. Naturality of τ implies that the following commutes:

$$\begin{tikzcd}
0 \arrow[r] & T_1(C) \arrow[r] \arrow[d, "\wr"'] & T_1(I^0) \arrow[r] \arrow[d, "\wr"'] & T_1(I^1) \arrow[r] \arrow[d, "\wr"'] & T_1(I^2) \arrow[r] \arrow[d, "\wr"'] & ... \\
0 \arrow[r] & T_2(C) \arrow[r] & T_2(I^0) \arrow[r] & T_2(I^1) \arrow[r] & T_2(I^2) \arrow[r] & ...
\end{tikzcd}$$

Moreover, since T_1 and T_2 are both left exact, each row in the above diagram is exact.

Section 7

7.1 Prove Corollary 7.4 (Proper Base Change): For a pullback diagram as in (26) with f and f' proper with Hausdorff fibers, the canonical map

$$p^*(R^n f_* B) \to R^n f'_*(q^* B)$$

is an isomorphism for any sheaf B on Y.

Section 8

8.1 Let X be connected and locally simply connected, and choose a base point x_0. Show that for any locally constant sheaf (of sets) A, the group $\pi_1(X, x_0)$ acts on the stalk A_{x_0}. Show that this gives a functor $A \mapsto A_{x_0}$ which is an equivalence of categories between the category of locally constant sheaves on X, and the category of sets with a $\pi_1(X, x_0) - action$.

8.2 Prove the following: let S^d be a d-dimensional sphere. For any Abelian group A,

$$H^n(S^d; A) = \begin{cases} A & \text{when } n = 0 \text{ or } n = d \text{ provided } d \neq 0 \\ A \oplus A & \text{when } d = n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Midterm

1(a). Let A be an Abelian group and ΔA the constant sheaf on a space X. Describe the structure on $Et(\Delta A)$ that corresponds to ΔA being a sheaf of Abelian groups.

You have the correct space; can you just give the abelian group structure on the space in terms of the group multiplication and unit(s)? What you write is not entirely clear to me, and once you write down the groups structure, you may agree that there are better ways to say this.

If $germ_y(s)$ and $germ_x(t)$ are both points in $Et(\Delta A)$, corresponding to $a, b \in A$, respectively, then we can use the structure of A to add them iff $x = y$. Define $germ_y(s) + germ_y(t) = germ_y(u)$, where $germ_y(u)$ corresponds to the element $a + b \in A$. This gives each stalk of $Et(\Delta A)$ an Abelian group structure, inherited directly from the group structure of A (and with unit $germ_y(e)$, where e corresponds to 0_A).

1(b). Let X be the two-point space with the discrete topology. Give the functor description of the sheaf ΔA on X in full detail and give the Abelian group structure on each $\Delta A(U)$ for U open in X.

1b is fine, but do read my comments.

2(a) Let E be the quotient space of $\mathbb{R} \amalg \mathbb{R}$ by the relation that identifies x in the first copy with x in the second when $x \leq 0$ and let $\pi_E : E \to \mathbb{R}$ be the projection map. Describe the presheaf of sections $\Gamma(\pi_E)$ on E.

I just added what I would have liked to see as final conclusions so that things becomes easier for the next part. Also, for any map $f : E \to X$, the presheaf of sections is always a sheaf. The associated etale space won't be homeomorphic to E though if f wasn't etale.

First note that π_E is not étale because every neighborhood of 0 contains $x_a > 0$ and $x_b > 0$ such that $\pi(x_a) = 0 = \pi(x_b)$, where $x_a$and x_b are distinct elements of E, occurring in each of the copies of $\mathbb{R}$. Then there is no neighborhood of 0 for which π_E restricts to a homeomorphism, so π_E is not étale (and thus $Et(\Gamma\pi_E)$ is not isomorphic to E).

Let $U \subseteq \mathbb{R}$ and let f be a section of $\Gamma\pi_E(U)$. If U is such that $x < 0$, $\forall x \in U$, then f must send $x \mapsto x_{ab}$ in E, where x_{ab} is the unique element in E such that $\pi_E(x_{ab}) = x$; that is, $\Gamma\pi_E(U) = \{f_{ab}\}$, where f_{ab} is the function $x \mapsto x_a b$, $\forall x \in U$.

If $x > 0$, $\forall x \in U$, there are two choices for f: either $f(x) = x_a$ or $f(x) = x_b$, where x_a and x_b are the two elements in E such that $\pi_E(x_a) = x = \pi_E(x_b)$, since f being continuous and $f(x) = x_a$ (or $f(x) = x_b$) for some $x \in U$ implies $f(x) = x_a$ (or $f(x) = x_b$) for all $x \in U$. Then $\Gamma\pi_E(U) = \{f_a, f_b\}$, where f_a is the function $x \mapsto x_a$ and f_b is the function $x \mapsto x_b$

If $0 \in U$, then $\forall x \in U$ such that $x \leq 0$, $f(x)$ must be x_{ab} and for $\forall x > 0$, either $f(x) = x_a$ or $f(x) = x_b$. Since f must be continuous, if $f(x) = x_a$ for any $x > 0$, then we must have $f(x) = x_a$ for all $x > 0$ and likewise if $f(x) = x_b$. Then $\Gamma\pi_E(U) = \{g_a, g_b\}$, where g_a is the function
$g_a(x) = \begin{cases} x_{ab} & x \leq 0 \\ x_a & x > 0 \end{cases}$, and g_b is the function $g_b(x) = \begin{cases} x_{ab} & x \leq 0 \\ x_b & x > 0 \end{cases}$.

2(b) Describe the corresponding étale space $Et(\Gamma(E))$ together with the counit of the adjunction, $Et \dashv \Gamma$ at the space.

just fix the case x=0 at the end

Let $x_0 \in \mathbb{R}$. If $x_0 < 0$, then ΓE_{x_0} has a single germ, represented by (f_{ab}, U) for some $U \ni x_0$, with $x < 0$, $\forall x \in U$. If $x_0 > 0$, then ΓE_{x_0} has two germs represented by (f_a, V) and (f_b, V), where V is some neighborhood of x_0 with $x > 0$ for every $x \in V$. If $x_0 = 0$, then again ΓE_{x_0} has two germs represented by (g_a, W) and (g_b, W), where W is some neighborhood of x.

The counit of the adjunction is the map $\epsilon_E : Et\Gamma_E \to E$ given by $\epsilon_E(germ_x(s)) = s(x)$. If $x < 0$, then $\epsilon_E(germ_x(f_{ab})) = x_{ab}$. If $x > 0$, then we have $\epsilon_E(germ_x(f_a)) = x_a$ and $\epsilon_E(germ_x(f_b)) = x_b$. If $x = 0$, then $\epsilon_E(germ_0(g_a)) = g_a(0) = 0_{ab}$ and $\epsilon_E(germ_0(g_b)) = g_b(0) = 0_{ab}$.

3. Give an example of a presheaf that is not a sheaf and find its associated sheaf together with the unit of adjunction at this presheaf.

I corrected your associated sheaf: if you dont have enough amalgamations, you need to add them in order to get a sheaf, but in your case, the problem was that you had too many amalgamations. In that case you need to identify them. In order to see this, you would need to think a bit more about the germs at a point x in X. I indicated how you can think about that. See whether this makes sense. You can redo the unit map on the next page. Your construction was correct, so just calculate what the result is in this case.

Let X be a topological group. Define a presheaf on X by $P(U) = \{0\}$ if U is an open set in X not equal to X and $P(X) = \mathbb{Z}$. Let $\rho_{XX} = id_{\mathbb{Z}}$ and all other restriction maps be constant (i.e., the zero map). Then P is not a sheaf because for any open cover $\cup_{i\in I}U_i$ of X, the family of elements given by $a_i = 0$, $\forall i \in I$ is compatible. However, $\forall z \in \mathbb{Z}$, $z\mid_{U_i}= a_i = 0$, so that P has too many amalgamations.

To describe the associated sheaf of P, we must first describe $Et(P)$. Let x be in X and suppose U is an open neighborhood of x. If $s \in P(U)$ and $U \neq X$ then $s = 0$. If $U = X$, then for all $t \in P(X)$, $t\mid_V= 0$, for all $V \subset X$, so that each stalk of P has only the zero section. The associated sheaf of P, ΓEtP, is defined for U open in X as $\Gamma EtP(U) = \{f : U \to Et(P) : f$ is continuous and $\pi_P \circ f = id_U\}$. For every U in X there is exactly one such function, namely $x \mapsto germ_x(0)$. For each open $U \subseteq X$, write f_U for the unique section of $\Gamma EtP(U)$. If $\cup_{i\in I}U_i$ is an open cover of $U \subseteq X$, then there is only family (since each $\Gamma EtP(U_i)$ only has one element), and it is compatible since $f_{U_i}\mid_{U_i\cap U_j}= f_{U_j}\mid_{U_i\cap U_j}= x \mapsto germ_x(0)$. Moreover, the unique section of $\Gamma EtP(U)$ is an amalgamation since $f_U\mid_{U_i}= f_{U_i}, \forall i$. Since $\Gamma EtP(U)$ has only one section, this amalgamation must be unique, so ΓEtP is a sheaf.

The unit of the adjunction $\eta_P : P \to \Gamma EtP$ is given by $\eta_{P,U}(s) = x \mapsto germ_x(s)$. For $U \neq X$, this is a bijection sending the unique section of $P(U)$ to the unique section of $\Gamma EtP(U)$, i.e. $\eta_{P,U}(0) = x \mapsto germ_x(0)$. For $U = X$, this is the map sending each section of $P(X)$, i.e. each $z \in \mathbb{Z}$, to $germ_x(0)$.

4(i). Let $\gamma : \mathbb{R} \to S^1$ be the étale map defined by $\gamma(r) = (cos(2\pi r), sin(2\pi r))$.

I don't think you had realized what the map gamma really is. This is where you miss the point set topology course. I should have given you a bit more of a picture with it.I have now drawn the picture in the margin. The idea is that you map the real line to the circle by sending each interval [a,a+1) 1-1 to the circle and then just keep repeating so as a fiber bundle it looks like a spiral.With the image and info added, have another look at this problem. You really need to keep this image in mind; otherwise you can make mistakes with the algebra.So redo this...

Sine γ is étale, $Et\Gamma Et\gamma \cong Et(\gamma)$, so it suffices to describe the stalks of $Et(\gamma)$. Fix a point x in S^1 and suppose x is given by $(cos\theta, sin\theta)$ for $\theta \in [0, 2\pi)$. The fiber of γ over x is $\gamma^{-1}(x) = \{r \in \mathbb{R} \mid (cos(2\pi r), sin(2\pi r)) = (cos\theta, sin\theta)\}$. Note that if r_1 and r_2 are both in $\gamma^{-1}(x)$, then $r_1 - r_2 = m$ for some $min\mathbb{Z}$.

4iii). Let $\delta : S^1 \to S^1$ be the étale map given by $\delta(cos(\theta), sin(\theta)) = (cos(2\theta), sin(2\theta))$.

Again we describe the stalks of $Et(\delta)$. If $x_0 = (cosx, sinx) \in S^1$, then the fiber of δ over x_0 is $\{(cos\theta, sin\theta) \mid (cos(2\theta), sin(2\theta)) = (cos(x), sin(x)\} = \{(cos\frac{x}{2}, sin\frac{x}{2}), (cos\frac{-x}{2}, sin\frac{x}{2})$.

4iv) Let $\phi : S^1 \coprod S^1 \to S^1$ be the étale map that is equal to the identity in the first component and $(cos2\theta, sin2\theta)$ in the second.

5. *I will still send you my feedback on the rest of question 5 as well, but that part was OK, so you won't need to work on it. 5(c) is OK.*

6. Give an example of a closed inclusion $j : X \hookrightarrow Y$ with an Abelian sheaf B on Y such that j^*B is not equal to $j^!B$.

Complete your proof that these two sheaves are not the same, by showing that something is in one and not in the other.

Let $Y = \mathbb{R}$ and let B be the sheaf that assigns to $U \subseteq \mathbb{R}$ the set of continuous functions from $U \to \mathbb{R}$ (with restriction maps being the normal restriction of functions). Let X be the compact subset $[0,1]$ of $\mathbb{R}$. Then B is an Abelian sheaf since the set of continous functions on an open set into $\mathbb{R}$ form an algebra, hence an Abelian group. Let $j : X \hookrightarrow Y$ be the inclusion map.

Then the inverse image sheaf, j^*B on X is just the restriction of B to X, that is, $j^*B(U) = \{f : U \to \mathbb{R} \mid f \text{ is continuous }\}$ for U, an open subset of X. The sheaf $j^!B(U)$ however is given by $j^!B(U) = \{f \in B(W) \mid supp(f) \subseteq [0,1]\}$ for U an open subset of X and $W = U \cup \mathbb{R} \setminus [0,1]$.

Let $U = X = [0,1]$. Then any $g \in j^!B(U)$ must be continuous on $W = \mathbb{R}$ with support in $[0,1]$. Let $g : [0,1] \to \mathbb{R}$ be the function $x \mapsto 1$, the constant 1 function. Then f is continuous on $U = [0,1]$, so $f \in j^*B(U)$. But any extension of f to $W = \mathbb{R}$ with support in $[0,1]$ would not be continuous, so $f \notin j^!B(U)$. Then $j^*B(U) \neq j^!B(U)$.