Exercises

Section 8

8.1 Let X be connected and locally simply connected, and choose a base point x_0. Show that for any locally constant sheaf (of sets) A, the group $\pi_1(X, x_0)$ acts on the stalk A_{x_0}. Show that this gives a functor $A \mapsto A_{x_0}$ (which is an equivalence of categories between the category of locally constant sheaves on X and the category of sets with a $\pi_1(X, x_0)$-action.)

Let X be connected and locally simply connected. Let A be a locally constant sheaf, so that $\pi : Et(A) \to X$ is a covering map. Fix x_0 in X and let $p : [0,1] \to X$ be a loop in X based at x_0. Consider $\pi^{-1}(x_0) \in Et(A)$. Since π is a covering map, there exists a neighborhood U_0 of x_0 in X and a set $S = \pi^{-1}(x_0)$ such that U_0 is evenly covered by π with $\pi^{-1}(U_0) = U_0 \times S$, where S has the discrete topology. (i.e., each $x \in X$ has a neighborhood whose preimage under π is a stack of pancakes with one pancake for each $s \in S = A_x$).

Note that for each $s \in S$ (that is, each point in the fiber of x_0 or equivalently each point in A_{x_0}), the fact that π is a covering map implies that $p : [0,1] \to X$ has a unique lifting, $\tilde{p}_s$, to a path in $Et(A)$ beginning at s and ending at some $t \in S$ (Munkres Lemma 54.1). Moreover, if p_1 and p_2 are both loops based at x_0 and are homotopic, then the induced paths in $Et(A)$ are homotopic with identical endpoints (Munkres Theorem 54.3). Now we can define an action of $\pi_1(X, x_0)$ on $A_{x_0} = \pi^{-1}(x_0)$ by $p \cdot s = \tilde{p}_s(1)$, where p is a loop based at x_0, s is in the fiber of x_0 (ie s in the stalk of A_{x_0}) and $\tilde{p}_s$ is the unique lift of p for s implied by the fact that π is a covering map. Note that if e is the constant loop, then $\tilde{e}_s(1) = s$, $\forall s$ so that $e \cdot s = \tilde{e}_s(1) = s$. Since the group operation, $*$, of $\pi(X, x_0)$ is composition of loops, we also have $p \cdot (q \cdot s) = p \cdot \tilde{q}_s(1) = \tilde{p}_{\tilde{q}_s(1)}(1) = (p * q) \cdot s$. Then $\pi(X, x_0)$ gives an action on A_{x_0}.

Let **LocConSh(X)** be the category of locally constant sheaves on X. Define a morphism $F : \textbf{LocConSh(X)} \to \textbf{Set}$ by $A \mapsto A_{x_0}$. If ϕ is a morphism from A to B, define $F\phi : A_{x_0} \to B_{x_0}$ by using the induced map: if $s \in A(U)$ for some $U \subseteq X$, $F(\phi)(germ_{y_0}(s)) = germ_{y_0}(\phi(s))$ for some $y_0 \in U$ (this makes sense since $A_{y_0} \cong A_{x_0}$ and likewise for the stalks of B). Then ϕ is natural and hence gives a functor $\textbf{LocConSh(X)} \to \textbf{Set}$.

8.2 Let S^d be a d-dimensional sphere. For any Abelian group A, prove that,

$$H^n(S^d;A) = \begin{cases} A & \text{when } n = 0 \text{ or } n = d \text{ provided } d \neq 0; \\ A \oplus A & \text{when } d = n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Write S^d as the union of closed subspaces, N and S, the closed northern and southern hemispheres, $S^d = N \cup S$. Note that $S^{d-1} = N \cap S$. We proceed by induction of d and consider the cases $n \neq d$ and $n = d$ separately. The cases $n = 0$ and $d = 0$ are obvious (since S^d is connected for $d > 0$ and S^0 has two components).

Suppose $d \geq 1$ and the result holds for $d = 0$. By inductive hypothesis, for $n \neq d$ and $n > 0$, $H^n(S^{d-1}; A\,|_{S^{d-1}}) = 0$. Then the Mayer-Vietoris sequence:

$$\ldots \to H^{n-1}(S^{d-1}; A\,|_{s^{d-1}}) \to$$

$$H^n(S^d;A) \to H^n(N;A\,|_N) \oplus H^n(S;A\,|_S) \to H^n(S^{d-1};A\,|_{S^{d-1}}) \to \ldots$$

becomes :

$$\ldots \to 0 \to H^n(S^d;A) \to H^n(N;A\,|_N) \oplus H^n(S;A\,|_S) \to 0 \to \ldots$$

And so by exactness, we have

$$\ldots \to 0 \to H^n(S^d;A) \xrightarrow{\sim} H^n(N;A\,|_N) \oplus H^n(S;A\,|_S) \to 0 \to \ldots$$

It remains to show that $H^n(N;A\,|_N) \oplus H^n(S;A\,|_S) = 0$. Notice that N and S are both contractible; that is, they are both homotopic to the 1-point space. Then since $H^n(\{*\};F) = 0, \forall n > 0$, we must have
$H^n(N;A\,|_N) \oplus H^n(S;A\,|_S) \cong 0$, for $n > 0$. Then $H^n(S^d;A) = 0$ for all $n \neq d$ and $n > 0$ and for all d.

We now consider the case $n = d$. By inductive hypothesis, we have $H^{n-1}(S^{d-1};A) = H^{d-1}(S^{d-1};A) = A$. So the Mayer-Vietoris sequence:

$$\ldots \to H^{d-1}(N;A\,|_N) \oplus H^{d-1}(S;A\,|_S) \to H^{d-1}(S^{d-1};A\,|_{S^{d-1}}) \to H^d(S^d;A) \to$$

becomes

$$\ldots \to H^{d-1}(N;A\,|_N) \oplus H^{d-1}(S;A\,|_S) \to A \to H^d(S^d;A) \to \ldots$$

But by above, this is

$$\ldots \to 0 \to A \to H^d(S^d;A) \to \ldots$$

which by exactness and the first isomorphism theorem implies $H^d(S^d;A) \cong A$ and so the result follows.