Exercises

Section 2

2.1. For a commutative triangle of topological spaces and continuous maps,

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

if f and π are étale, so is h (in particular, h is open).

Let $y \in E$. Since f is étale, $\exists U$ an open neighborhood of $h(y)$ such that $f\mid_U$ is a homeomorphism. Likewise, since π is étale, $\exists W$ an open neighborhood of y such that $\pi\mid_W$ is a homeomorphism. Let $V = h^{-1}(U) \cap W$. Since h is continuous, $h^{-1}(U)$ is open, so V is open in E. Since $V \subseteq W$, $\pi\mid_V$ is a homeomorphism.

Since $h(V) \subseteq U$, and $f\mid_U$ is a homeomorphism, it only remains to show that $h(V)$ is open, since this will imply that $f\mid_{h(v)}$ is a homeomorphism, and in turn that $h\mid_V$ is a homeomorphism (since $\pi\mid_V$ is a homeomorphism, and the diagram commutes). To see that $h(V)$ is open, note that $\pi(V)$ is open in X, since π is étale and V is open in E. Further, $f(h(V)) = \pi(V)$ since the diagram commutes. Then since $h(V) \subseteq U$ and$f\mid_U$ is a homeomorphism, $h(V)$ is open in Z, so $h\mid_V$ is a homeomorphism and h is étale.

2.2. (a) Show that Corollary 2.10 determines the associated sheaf up to isomorphism. That is, if R is a sheaf and there exists a map $\psi : P \to R$ with the same property as η in Corollary 2.10, then $R \cong \Gamma Et(P)$.

Corollary. (2.10) For a map $\phi : P \to Q$ between presheaves on X, if Q is a sheaf then there exists a unique map of sheaves $\hat{\phi}$ such that $\hat{\phi} \circ \eta = \phi$.

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\eta"'] & Q \\ \Gamma EtP \arrow[ur, dashed, "!", "\hat{\phi}"'] & \end{tikzcd}$$

The universal property of $(\Gamma EtP, \eta)$ implies a unique map $\hat{\psi}$ such that $\psi = \hat{\psi} \circ \eta$ and the universal property of (R, ψ) implies a unique map $\hat{\eta}$ such that $\eta = \hat{\eta} \circ \psi$, i.e. we have the following diagrams:

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\eta"'] & R \\ \Gamma EtP \arrow[ur, dashed, "!", "\hat{\psi}"'] & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\psi"'] & \Gamma EtP \\ R \arrow[ur, dashed, "!", "\hat{\eta}"'] & \end{tikzcd}$$

This implies the following diagrams commute:

$$\begin{tikzcd} & \Gamma EtP \arrow[d, dashed, "!"', "\hat{\psi}"] \\ P \arrow[ur, "\eta"] \arrow[r, "\psi"] \arrow[d, "\eta"'] & R \arrow[dl, dashed, "!"', "\hat{\eta}"] \\ \Gamma EtP & \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[d, dashed, "!"', "\hat{\eta}"] \\ P \arrow[ur, "\psi"] \arrow[r, "\eta"] \arrow[d, "\psi"'] & \Gamma EtP \arrow[dl, dashed, "!"', "\hat{\psi}"] \\ R & \end{tikzcd}$$

In particular, $\hat{\eta} \circ \hat{\psi}$ is the unique map such that $\eta = \hat{\eta} \circ \hat{\psi} \circ \eta$ and $\hat{\psi} \circ \hat{\eta}$ is the unique map such that $\psi = \hat{\psi} \circ \hat{\eta} \circ \psi$. But then uniqueness implies $\hat{\eta} \circ \hat{\psi} = id_{\Gamma EtP}$ and $\hat{\psi} \circ \hat{\eta} = id_R$. Then $R \cong \Gamma EtP$.

2.2. (b) If P is a subpresheaf of R and R is a sheaf, then let

$$\tilde{P}(U) = \{r \in R(U) \mid \text{for each } x \in U \text{ there is a neighbourhood } W_x \subseteq U \text{ of } x \\ \text{such that } (r \mid_{W_x}) \in P(W_x)\}.$$

Show that $P \subseteq \tilde{P} \subseteq R, \tilde{P}$ is a sheaf and $P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10; hence $\tilde{P}$ is the associated sheaf for P.

<u>$P \subseteq \tilde{P} \subseteq R$:</u>
We want to show that for $U \in X$, open, $P(U) \subseteq \tilde{P}(U) \subseteq R(U)$. So let $U \subseteq X$ be open. By construction $\tilde{P}(U) \subseteq R(U)$, so we need only show $P(U) \subseteq \tilde{P}(U)$. Let $s \in P(U)$. Since P is a subpresheaf $s \in R(U)$ and for $x \in U$, U itself is a neighborhood of x such that, $s \mid_U$ is in $P(U)$. Then $s \in \tilde{P}(U)$.

<u>$\tilde{P}$ is a sheaf:</u>
Let U be an open set in X and let $\cup U_i$, $i \in I$, be an open cover for U. Suppose $\{a_i\} \in \tilde{P}(U_i)$, for $i \in I$ is a compatible family for the U_i, i.e., $a_i \mid_{U_i \cap U_j} = a_j \mid_{U_i \cap U_j}$. For each $i \in I$ and each $x \in U_i$, choose W_x^i such that $(a_i \mid_{W_x^i}) \in P(W_x^i)$. Then $\cup_i \cup_{x \in U_i} W_x^i$ is an open cover for U (since $\cup_x W_x^i$ is an open cover for each U_i). Moreover, $a_i \mid_{W_x^i \cap W_y^j} = a_j \mid_{W_x^i \cap W_y^j}$ because $(a_i \mid_{U_i \cap U_j}) \mid_{W_x^i \cap W_y^j} = (a_j \mid_{U_i \cap U_j}) \mid_{W_x^i \cap W_y^j}$. So $\{a_i\} \in P(W_x^i)$ is a compatible family in R.

Since R is a sheaf, let a e the unique amalgamation. We want to show $a \in \tilde{P}(U)$, i.e., $\forall x \in U, \exists$ a neighborhood V_x of x such that $a \mid_{V_x} \in P(V_x)$. But since a is the amalgamation of the family $\{a_i\} \in P(W_x^i)$, if $x \in U$, then $x \in U_i$, for some i, so $a \mid_{W_X^i} = a_i$ and W_x was chosen so that $a_i \mid_{W_x^i} \in P(W_x^i)$. Then take V_x to be W_x^i (for appropriate i). Then $a \in \tilde{P}(U)$, so $\tilde{P}$ is a sheaf.

$P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10:

Let $i : P \hookrightarrow \tilde{P}$ be the map that injects $P(U)$ into $\tilde{P}(U)$. We must show that for a map between presheaves, $\phi : P \to Q$, where Q is a sheaf, there is a unique $\tilde{\phi}$ such that $\tilde{\phi} \circ i = \phi$.
The universal property of (P, η_P) implies that for a map $\phi : P \to Q$, there is a unique map $\hat{\phi} : \Gamma EtP \to Q$ such that the following commutes:

$$\begin{tikzcd}
P \arrow[r, "\phi"] \arrow[d, "\eta_P"'] & Q \\
\Gamma EtP \arrow[ur, dashed, "!", "\hat{\phi}"']
\end{tikzcd}$$

Then it suffices to find a unique map, f, from $\tilde{P} \to \Gamma EtP$ such that $\eta_p = f \circ i$, since that will imply the follow diagram commutes (in particular, $\phi = \hat{\phi} \circ f \circ i$, so that $\tilde{\phi} = \hat{\phi} \circ f$ obtains the result):

$$\begin{tikzcd}
\tilde{P} \arrow[dr, dashed, "!", "f"'] & P \arrow[l, "i"'] \arrow[r, "\phi"] \arrow[d, "\eta_p"'] & Q \\
& \Gamma EtP \arrow[ur, dashed, "!", "\hat{\phi}"'] &
\end{tikzcd}$$

Section 3

3.1 The pullback of an étale map is also étale. That is, given a pullback diagram,

$$\begin{array}{ccc} X \times_Y E & \xrightarrow{\pi_2} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

then π_1 is étale if p is.

Let $(u, e) \in X \times_Y E$. Since p is étale, there exists a $U \subseteq E$, open neighborhood of $e = \pi_2(u, e)$, such that $p : U \to p(U)$ is a homeomorphism. Since $f : X \to Y$ is continuous (and p is open), the preimage of $p(U)$ under f is open in X. Let $W \subseteq X$ be the preimage of $p(U)$. Since $p(e) = f(u)$, $f(u) \in p(U)$ and so $u \in W$.

Since $W \times U$ is open in $X \times E$ under the product topology, $V = W \times U \cap X \times_Y E$ is an open neighborhood of (u, e) in $X \times_Y E$ under the subspace topology. I claim that $\pi_1 |_V$ is a homeomorphism. Since π_1 is a projection map, $\pi_1 |_V$ is continuous. We must show there exists a continuous $g : \pi_1(V) \to X \times_Y E$ such that $g \circ \pi_1 = id$. Since $p |_U$ is a homeomorphism, we have continuous $p^{-1} : f(\pi_1(V)) \to E$ such that $p^{-1} \circ p = id_E$. Then $p^{-1}(f(\pi_1(V))) = p^{-1}(p(\pi_2(V))) = \pi_2(V)$. Let $i : EtB \hookrightarrow X \times_Y EtB$ be the inclusion map. Then let $g : \pi_1(V) \to X \times_Y EtB$ be defined by $x \mapsto (x, i \circ p^{-1} \circ f(x))$. Then g is continuous and $g \circ \pi_1 |_V = id_{X \times_Y E}$, so that $\pi_1 |_V$ is a homeomorphism. Then π_1 is étale.

3.2. Verify the following equality of stalks:

$$(f^*B)_x \cong B_{f(x)}$$

for each $x \in X$.

First note that since $f^*(B) = \Gamma(X \times_Y EtB)$ and since π_1 is étale, $Et\Gamma(X \times_Y EtB) \cong X \times_Y EtB$ by the results in section 2. That is, $Et(f^*(B)) \cong X \times_Y EtB = \{(x, e) \in X \times EtB \mid f(x) = \pi_B(e)\}$.

Fix an $x_0 \in X$. The stalk of $f^*(B)$ at x_0 is $(f^*(B))_{x_0} = \{s \in Et(f^*(B)) \mid \pi_B(s) = x_0\} = \{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\}$. Then the map π_2 gives us a bijection $\{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\} \to \{e \in EtB \mid f(x_0) = \pi_B(e)\}$, which is exactly the stalk of $B_{f(x_0)}$.

3.3. Prove that

$$Hom_{Sh(Y)}(i_!A, B) \cong Hom_{Sh(X)}(A, i^*B)$$

where $i : X \hookrightarrow Y$ is an open inclusion.

First note that since $i \circ \pi_A$ is étale, $Et(i_!A) = Et(\Gamma EtA) = EtA$. Since B is a sheaf, 2.9 implies $\phi \in Hom_{Sh(Y)}(i_!A, B)$ corresponds to a map $g : EtA \to EtB$ such that the following commutes:

$$\begin{tikzcd} EtA \arrow[rr, "g"] \arrow[dr, "i \circ \pi_A"'] & & EtB \arrow[dl, "\pi_B"] \\ & Y & \end{tikzcd}$$

By the notes, $Et(i^*B) = Et(B \mid_X)$ and since i^*B is a sheaf, 2.9 implies $\psi \in Hom_{Sh(X)}(A, i^*B)$ corresponds to a map $h : EtA \to Et(B \mid_X)$ such that the follow commutes:

$$\begin{tikzcd} EtA \arrow[rr, "h"] \arrow[dr, "\pi_A"'] & & Et(B \mid_X) \arrow[dl, "\pi_{i^*B}"] \\ & X & \end{tikzcd}$$

To see the correspondance between g and h, note that $Et(B \mid_X) \cong X \times_Y EtB$, so that given $g : EtA \to EtB$, h is the unique map such that the following commutes (and given $h : EtA \to Et(B \mid_X) \cong X \times_Y EtB$, g is the map $\pi \circ h$):

$$\begin{tikzcd} EtA \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "\pi_A"'] & & \\ & X \times_Y EtB \arrow[r, "\pi"] \arrow[d, "\pi_{Bi^*B}"'] & EtB \arrow[d, "\pi_B"] \\ & X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

3.4 (a) Verify the adjunction formula involving j_* and $j^!$ in (7).

We must show that $Hom_{Ab(Y)}(j_*A, B) \cong Hom_{Ab(Z)}(A, j^!B)$.

Let $\phi \in Hom_{Ab(Y)}(j_*A, B)$. Then ϕ is a collection of maps $\phi_U : j_*A(U) \to B(U)$ for every open set $U \subseteq Y$. Note that for $U \subseteq Y$ such that $U \cap Z = \emptyset$, $j_*A(U) = A(U \cap Z) = A(\emptyset) = \{0\}$, so that there is only one choice for ϕ_U. Likewise, if $U \subset Y$ with $U \cap \delta Z \neq \emptyset$, then there exists a $V \subset U$ such that $V \cap Z = \emptyset$ and naturality of ϕ implies:

$$\begin{array}{ccc} j_*A(U) & \xrightarrow{\phi_U} & B(U) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ j_*A(V) = A(V \cap Z) = \{0\} & \xrightarrow[\phi_V]{} & B(V) \end{array}$$

so that $\phi_U(j_*A(U))$ is the constant 0 map. Finally, suppose U is such that $U \cap Z = U$. For each such U, the choices for ϕ_U are the group homomorphisms, $A(U) \to B(U)$.

Now consider $\psi \in Hom_{Ab(Z)}(A, j^!B)$. ψ is a collection of maps ψ_V for each open V in Z. Suppose V is open in Z and $V \cap \delta Z \neq \emptyset$. Then $j^!B(V) = 0$ and so there is only one choice for ψ_V, the constant 0 map. On the other hand, if $V \cap \delta Z = \emptyset$, then the choices for ϕ_V are the group homomorphisms, $A(V) \to B(V)$.

Then elements of both $Hom_{Ab(Z)}(A, j^!B)$ and $Hom_{Ab(Y)}(j_*A, B)$ correspond to the group homorphisms from $A(U) \to B(U)$ for every $U \subseteq Y$ with $U \cap Z = U$.

3.4 (b) Show that $j^! B$ is isomorphic to a subsheaf of $j^* B$.

We must show that for V open in Z, $j^! B(V) \subseteq j^* B(V)$ and for $U \supseteq V$, the restrictions maps of $j^! B$ agree with those of $j^* B$.

So let V be open in Z and choose U open in Y such that $V = Z \cap U$. Let $s \in j^! B(V)$. Then $s \in B(U)$ and $s \mid_{U \setminus Z} = 0$. Note that elements of $j^* B(V)$ are continuous functions $\sigma : V \to Z \times_Y EtB$ such that $\pi_1 \circ \sigma = id_V$. Then sections of $j^* B(V)$ are functions defined by $x \mapsto (x, germ_x(s))$ for some s, a section of $B(U)$ for some $U \ni x$.

3.5 Let $h :\hookrightarrow X$ be an inclusion map.
(a) Show that there are natural isomorphisms

- $h_*h^* \cong id \cong h_!h^*$ if h is open;
- $h_*h^* \cong h_*h^!$ is h is closed.

3.5 (b) If $h : Z \hookrightarrow X$ is locally closed, prove that the definitions of $h_!$ and $h^!$ do not depend on the choice of the factorization $h = i \circ j$. Also prove that $h_!h^! \cong id$.
3.5 (c) Conclude that for a locally closed subspace $h : Z \hookrightarrow X$,

- $h_! \cong h_!$ and $h^! \cong h^*$ if h is open;
- $h_! \cong h_*$ and $h^! \cong h^!$ if h is closed.

3.5 (d) For two composable locally closed inclusions $W \overset{k}{\hookrightarrow} Z \overset{h}{\hookrightarrow} X$, show that $h_!k_! \cong (hk)_!$ and $k^!h^! \cong (hk)^!$.

Section 4

4.1 (a) Show that with the notation as above, $\Gamma(-, \pi)$ is indeed isomorphic to the sheaffication of P.
4.1(b) Show that the $coker(\alpha)$ has the following universal property: given any other sheaf H, a map $\chi : G \to H$ factors through $coker(\alpha)$ if and only if $\chi \circ \alpha = 0$.
4.1(c) Show that $coker(\alpha)_x = coker(\alpha_x)$.

4.2 Show that $\mathbf{Ab(X)}$ is an Abelian category with kernel, cokernel and sum defined as above.

4.3 Let $0 \to A \hookrightarrow I^0 \to I^1 \to ...$ and $0 \to B \to B \hookrightarrow J^0 \to J^1 \to J^2 \to ...$ be injective resolutions of A and B respectively. Show that a map $\phi : A \to B$ extends to a map of complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, dashed, "\phi"] & I^0 \arrow[r] \arrow[d, dashed, "\phi^0"] & I^1 \arrow[r] \arrow[d, dashed, "\phi^1"] & I^2 \arrow[r] \arrow[d, dashed, "\phi^2"] & ... \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & ...
\end{tikzcd}$$

4.3 (b) Show that a map of complexes as above induces a homomorphism of cohomology groups

$$H^n(X; A) \to H^n(X, B)$$

4.4(a) Let $f : Y \to X$ be a map of topological spaces. Show that there is a natural isomorphism $\Gamma_Y \to \Gamma_X \circ f_*$,

$$\begin{tikzcd}
Ab(Y) \arrow[rr, "f_*"] \arrow[dr, "\Gamma_Y"'] & & Ab(X) \arrow[dl, "\Gamma_X"] \\
& \textbf{Abelian Groups} &
\end{tikzcd}$$

4.4 (b) Show that a natural isomorphism between left exact functors $\tau : T_1 \to T_2$ induces a natural isomorphism $R^nT_1 \to R^nT_2$ for each n.

Section 7

7.1 Prove Corollary 7.4 (Proper Base Change): For a pullback diagram as in (26) with f and f' proper with Hausdorff fibers, the canonical map

$$p^*(R^n f_* B) \to R^n f'_*(q^* B)$$

is an isomorphism for any sheaf B on Y.

Section 8

8.1 Let X be connected and locally simply connected, and choose a base point x_0. Show that for any locally constant sheaf (of sets) A, the group $\pi_1(X, x_0)$ acts on the stalk A_{x_0}. Show that this gives a functor $A \mapsto A_{x_0}$ which is an equivalence of categories between the category of locally constant sheaves on X, and the category of sets with a $\pi_1(X, x_0) - action$.

8.2 Prove the following: let S^d be a d-dimensional sphere. For any Abelian group A,

$$H^n(S^d; A) = \begin{cases} A & \text{when } n = 0 \text{ or } n = d \text{ provided } d \neq 0 \\ A \oplus A & \text{when } d = n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Midterm

1(a) You have the correct space; can you just give the abelian group structure on the space in terms of the group multiplication and unit(s)? What you write is not entirely clear to me, and once you write down the groups structure, you may agree that there are better ways to say this.

1(b) is fine, but do read my comments.

2(a) I just added what I would have liked to see as final conclusions os that things becomes easier for the next part. Also, for any map $f : E \to X$, the presheaf of sections is always a sheaf. The associated etale space won't be homeomorphic to E though if f wasn't etale.

2(b) just fix the case x=0 at the end

3. I corrected your associated sheaf: if you don't have enough amalgamations, you need to add them in order to get a sheaf, but in your case, the problem was that you had too many amalgamations. In that case you need to identify them. In order to see this, you would need to think a bit more about the germs at a point x in X. I indicated how you can think about that. See whether this makes sense. You can redo the unit map on the next page. Your construction was correct, so just calculate what the result is in this case.

4(i). I don't think you had realized what the map gamma really is. This is where you miss the point set topology course. I should have given you a bit more of a picture with it.I have now drawn the picture in the margin. The idea is that you map the real line to the circle by sending each interval [a,a+1) 1-1 to the circle and then just keep repeating so as a fiber bundle it looks like a spiral.With the image and info added, have another look at this problem. You really need to keep this image in mind; otherwise you can make mistakes with the algebra.So redo this...

5. I will still send you my feedback on the rest of question 5 as well, but that part was OK, so you won't need to work on it. 5(c) is OK.

6. complete your proof that these two sheaves are not the same, by showing that something is in one and not in the other.