Exercises

Section 2

2.1. For a commutative triangle of topological spaces and continuous maps,

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

if f and π are étale, so is h (in particular, h is open).

Let $y \in E$. Since f is étale, $\exists U$ an open neighborhood of $h(y)$ such that $f\mid_U$ is a homeomorphism. Likewise, since π is étale, $\exists W$ an open neighborhood of y such that $\pi\mid_W$ is a homeomorphism. Let $V = h^{-1}(U) \cap W$. Since h is continuous, $h^{-1}(U)$ is open, so V is open in E. Since $V \subseteq W$, $\pi\mid_V$ is a homeomorphism.

Since $h(V) \subseteq U$, and $f\mid_U$ is a homeomorphism, it only remains to show that $h(V)$ is open, since this will imply that $f\mid_{h(v)}$ is a homeomorphism, and in turn that $h\mid_V$ is a homeomorphism (since $\pi\mid_V$ is a homeomorphism, and the diagram commutes). To see that $h(V)$ is open, note that $\pi(V)$ is open in X, since π is étale and V is open in E. Further, $f(h(V)) = \pi(V)$ since the diagram commutes. Then since $h(V) \subseteq U$ and$f\mid_U$ is a homeomorphism, $h(V)$ is open in Z, so $h\mid_V$ is a homeomorphism and h is étale.

2.2. (a) Show that Corollary 2.10 determines the associated sheaf up to isomorphism. That is, if R is a sheaf and there exists a map $\psi : P \to R$ with the same property as η in Corollary 2.10, then $R \cong \Gamma Et(P)$.

Corollary. (2.10) For a map $\phi : P \to Q$ between presheaves on X, if Q is a sheaf then there exists a unique map of sheaves $\hat{\phi}$ such that $\hat{\phi} \circ \eta = \phi$.

$$\begin{tikzcd}
P \arrow[r, "\phi"] \arrow[d, "\eta"'] & Q \\
\Gamma EtP \arrow[ur, dashed, "!", "\hat{\phi}"']
\end{tikzcd}$$

The universal property of $(\Gamma EtP, \eta)$ implies a unique map $\hat{\psi}$ such that $\psi = \hat{\psi} \circ \eta$ and the universal property of (R, ψ) implies a unique map $\hat{\eta}$ such that $\eta = \hat{\eta} \circ \psi$, i.e. we have the following diagrams:

$$\begin{tikzcd}
P \arrow[r, "\psi"] \arrow[d, "\eta"'] & R \\
\Gamma EtP \arrow[ur, dashed, "!", "\hat{\psi}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "\eta"] \arrow[d, "\psi"'] & \Gamma EtP \\
R \arrow[ur, dashed, "!", "\hat{\eta}"']
\end{tikzcd}$$

This implies the following diagrams commute:

$$\begin{tikzcd}
 & \Gamma EtP \arrow[d, dashed, "!"', "\hat{\psi}"] \\
P \arrow[ur, "\eta"] \arrow[r, "\psi"] \arrow[d, "\eta"'] & R \arrow[dl, dashed, "!"', "\hat{\eta}"] \\
\Gamma EtP &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \arrow[d, dashed, "!"', "\hat{\eta}"] \\
P \arrow[ur, "\psi"] \arrow[r, "\eta"] \arrow[d, "\psi"'] & \Gamma EtP \arrow[dl, dashed, "!"', "\hat{\psi}"] \\
R &
\end{tikzcd}$$

In particular, $\hat{\eta} \circ \hat{\psi}$ is the unique map such that $\eta = \hat{\eta} \circ \hat{\psi} \circ \eta$ and $\hat{\psi} \circ \hat{\eta}$ is the unique map such that $\psi = \hat{\psi} \circ \hat{\eta} \circ \psi$. But then uniqueness implies $\hat{\eta} \circ \hat{\psi} = id_{\Gamma EtP}$ and $\hat{\psi} \circ \hat{\eta} = id_R$. Then $R \cong \Gamma EtP$.

2.2. (b) If P is a subpresheaf of R and R is a sheaf, then let

$$\tilde{P}(U) = \{r \in R(U) \mid \text{for each } x \in U \text{ there is a neighbourhood } W_x \subseteq U \text{ of } x \text{ such that } (r\mid_{W_x}) \in P(W_x)\}.$$

Show that $P \subseteq \tilde{P} \subseteq R, \tilde{P}$ is a sheaf and $P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10; hence $\tilde{P}$ is the associated sheaf for P.

<u>$P \subseteq \tilde{P} \subseteq R$:</u>
We want to show that for $U \in X$, open, $P(U) \subseteq \tilde{P}(U) \subseteq R(U)$. So let $U \subseteq X$ be open. By construction $\tilde{P}(U) \subseteq R(U)$, so we need only show $P(U) \subseteq \tilde{P}(U)$. Let $s \in P(U)$. Since P is a subpresheaf $s \in R(U)$ and for $x \in U$, U itself is a neighborhood of x such that, $s\mid_U$ is in $P(U)$. Then $s \in \tilde{P}(U)$.

<u>$\tilde{P}$ is a sheaf:</u>
Let U be an open set in X and let $\cup U_i$, $i \in I$, be an open cover for U. Suppose $\{a_i\} \in \tilde{P}(U_i)$, for $i \in I$ is a compatible family for the U_i, i.e., $a_i\mid_{U_i \cap U_j} = a_j\mid_{U_i \cap U_j}$. For each $i \in I$ and each $x \in U_i$, choose W_x^i such that $(a_i\mid_{W_x^i}) \in P(W_x^i)$. Then $\cup_i \cup_{x \in U_i} W_x^i$ is an open cover for U (since $\cup_x W_x^i$ is an open cover for each U_i). Moreover, $a_i\mid_{W_x^i \cap W_y^j} = a_j\mid_{W_x^i \cap W_y^j}$ because $(a_i\mid_{U_i \cap U_j})\mid_{W_x^i \cap W_y^j} = (a_j\mid_{U_i \cap U_j})\mid_{W_x^i \cap W_y^j}$. So $\{a_i\} \in P(W_x^i)$ is a compatible family in R.

Since R is a sheaf, let a e the unique amalgamation. We want to show $a \in \tilde{P}(U)$, i.e., $\forall x \in U, \exists$ a neighborhood V_x of x such that $a\mid_{V_x} \in P(V_x)$. But since a is the amalgamation of the family $\{a_i\} \in P(W_x^i)$, if $x \in U$, then $x \in U_i$, for some i, so $a\mid_{W_X^i} = a_i$ and W_x was chosen so that $a_i\mid_{W_x^i} \in P(W_x^i)$. Then take V_x to be W_x^i (for appropriate i). Then $a \in \tilde{P}(U)$, so $\tilde{P}$ is a sheaf.

$P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10:

Let $i : P \hookrightarrow \tilde{P}$ be the map that injects $P(U)$ into $\tilde{P}(U)$. We must show that for a map between presheaves, $\phi : P \to Q$, where Q is a sheaf, there is a unique $\tilde{\phi}$ such that $\tilde{\phi} \circ i = \phi$:

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, hook, "i"'] & Q \\ \tilde{P} \arrow[ur, dashed, "\tilde{\phi}"', "!"] & \end{tikzcd}$$

So let Q be a sheaf and $\phi : P \to Q$ be a morphism. For each open set $U \subseteq X$, we define $\tilde{\phi}_U : \tilde{P}(U) \to Q(U)$ as follows: let $s \in \tilde{P}(U)$ and let $\cup_{x \in U} W_x$ be a cover of U such that $(s\ |_{W_x}) \in P(W_x)$ for each $x \in U$. Consider the family $\phi_{W_x}(s\ |_{W_x})$ for $x \in U$. This family is compatible because commutativity of the restriction maps implies $(s\ |_{W_x})\ |_{W_x \cap W_y} = (s\ |_{W_y})\ |_{W_x \cap W_y} = s\ |_{W_x \cap W_y}$ and naturality of ϕ implies for $x, y \in U$:

$$\phi(s\ |_{W_x})\ |_{W_x \cap W_y} = \phi((s\ |_{W_x})\ |_{W_x \cap W_y}) = \phi(s\ |_{W_x \cap W_y})$$

$$= \phi((s\ |_{W_y}|_{W_x \cap W_y})) = \phi(s\ |_{W_y})\ |_{W_x \cap W_y}\ .$$

Then since Q is a sheaf, let $t \in Q(U)$ be the unique amalgamation of the $\phi_{W_x}(s\ |_{W_x})$'s and define $\tilde{\phi}(s) = t$. In fact, by naturality of $\tilde{\phi}$, we must choose the amalgamation to ensure that the following commutes for each W_x:

$$\begin{tikzcd} s \in \tilde{P}(U) \arrow[r, "\tilde{\phi}_U"] \arrow[d, "\rho"'] & Q(U) \ni t \arrow[d, "\rho"] \\ (s\ |_{W_x}) \in \tilde{P}(W_x) \arrow[r, "\tilde{\phi}_{W_x}"'] & Q(W_x) \ni \tilde{\phi}(s\ |_{W_x}) = \phi(s\ |_{W_x}) \end{tikzcd}$$

since the bottom row must be $\tilde{\phi}(s\ |_{W_x}) = \phi(s\ |_{W_x})$ by the requirement that $\tilde{\phi} \circ i = \phi$. So then uniqueness of $\tilde{\phi}$ is implied so long as $\tilde{\phi}$ is well-defined.

To see that $\tilde{\phi}$ is well-defined, let $\cup_{x \in U} V_x$ be another cover of U with the property that $(s\ |_{V_x}) \in P(V_x)$, so that $\phi_{V_x}(s\ |_{V_x})$ is again a compatible family. We must show that t is also the amalgamation of this family, i.e., $t\ |_{V_x} = \phi_{V_x}(s\ |_{V_x})$ for each V_x.

Consider the cover of U formed by the refinement of the W_x's and V_x's, $U = \cup_{x\in U} W_x \cup_{x\in U} V_x$. This contains the compatible family

$$\{\{\phi(s\mid_{W_x})\}_{x\in U}, \{\phi(s\mid_{V_x})\}_{x\in U}\}$$

.

The amalgamation of this family, say $\bar{t}$, must have the property that $\bar{t}\mid_{W_x} = \phi(s\mid_{W_x})$, and since Q is a sheaf, hence separated, this implies that $\bar{t} = t$. Then $t\mid_{V_x} = \phi_{V_x}(s\mid_{V_x})$ for each V_x, so $\tilde{\phi}$ is well-defined. In particular, note that $\tilde{\phi} \circ i = \phi$, since $s \in P(U)$ implies that U is a cover of U for which $s\mid_U \in P(U)$ so $\tilde{\phi}(s) = \phi(s\mid_U) = \phi(s)$.

Then $\tilde{\phi}$ is the required unique sheaf map satisfying $\tilde{\phi} \circ i = \phi$, so $\tilde{P}$ is the associated sheaf of P.

Section 3

3.1 The pullback of an étale map is also étale. That is, given a pullback diagram,

$$\begin{array}{ccc} X \times_Y E & \xrightarrow{\pi_2} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

then π_1 is étale if p is.

Let $(u, e) \in X \times_Y E$. Since p is étale, there exists a $U \subseteq E$, open neighborhood of $e = \pi_2(u, e)$, such that $p : U \to p(U)$ is a homeomorphism. Since $f : X \to Y$ is continuous (and p is open), the preimage of $p(U)$ under f is open in X. Let $W \subseteq X$ be the preimage of $p(U)$. Since $p(e) = f(u)$, $f(u) \in p(U)$ and so $u \in W$.

Since $W \times U$ is open in $X \times E$ under the product topology, $V = W \times U \cap X \times_Y E$ is an open neighborhood of (u, e) in $X \times_Y E$ under the subspace topology. I claim that $\pi_1 \mid_V$ is a homeomorphism. Since π_1 is a projection map, $\pi_1 \mid_V$ is continuous. We must show there exists a continuous $g : \pi_1(V) \to X \times_Y E$ such that $g \circ \pi_1 = id$. Since $p \mid_U$ is a homeomorphism, we have continuous $p^{-1} : f(\pi_1(V)) \to E$ such that $p^{-1} \circ p = id_E$. Then $p^{-1}(f(\pi_1(V))) = p^{-1}(p(\pi_2(V))) = \pi_2(V)$. Let $i : EtB \hookrightarrow X \times_Y EtB$ be the inclusion map. Then let $g : \pi_1(V) \to X \times_Y EtB$ be defined by $x \mapsto (x, i \circ p^{-1} \circ f(x))$. Then g is continuous and $g \circ \pi_1 \mid_V = id_{X \times_Y E}$, so that $\pi_1 \mid_V$ is a homeomorphism. Then π_1 is étale.

3.2. Verify the following equality of stalks:

$$(f^*B)_x \cong B_{f(x)}$$

for each $x \in X$.

First note that since $f^*(B) = \Gamma(X \times_Y EtB)$ and since π_1 is étale, $Et\Gamma(X \times_Y EtB) \cong X \times_Y EtB$ by the results in section 2. That is, $Et(f^*(B)) \cong X \times_Y EtB = \{(x, e) \in X \times EtB \mid f(x) = \pi_B(e)\}$.

Fix an $x_0 \in X$. The stalk of $f^*(B)$ at x_0 is $(f^*(B))_{x_0} = \{s \in Et(f^*(B)) \mid \pi_B(s) = x_0\} = \{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\}$. Then the map π_2 gives us a bijection $\{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\} \to \{e \in EtB \mid f(x_0) = \pi_B(e)\}$, which is exactly the stalk of $B_{f(x_0)}$.

3.3. Prove that

$$Hom_{Sh(Y)}(i_!A, B) \cong Hom_{Sh(X)}(A, i^*B)$$

where $i : X \hookrightarrow Y$ is an open inclusion.

First note that since $i \circ \pi_A$ is étale, $Et(i_!A) = Et(\Gamma EtA) = EtA$. Since B is a sheaf, 2.9 implies $\phi \in Hom_{Sh(Y)}(i_!A, B)$ corresponds to a map $g : EtA \to EtB$ such that the following commutes:

$$\begin{tikzcd}
EtA \arrow[rr, "g"] \arrow[dr, "i \circ \pi_A"'] & & EtB \arrow[dl, "\pi_B"] \\
& Y &
\end{tikzcd}$$

By the notes, $Et(i^*B) = Et(B \mid_X)$ and since i^*B is a sheaf, 2.9 implies $\psi \in Hom_{Sh(X)}(A, i^*B)$ corresponds to a map $h : EtA \to Et(B \mid_X)$ such that the follow commutes:

$$\begin{tikzcd}
EtA \arrow[rr, "h"] \arrow[dr, "\pi_A"'] & & Et(B \mid_X) \arrow[dl, "\pi_{i^*B}"] \\
& X &
\end{tikzcd}$$

To see the correspondance between g and h, note that $Et(B \mid_X) \cong X \times_Y EtB$, so that given $g : EtA \to EtB$, h is the unique map such that the following commutes (and given $h : EtA \to Et(B \mid_X) \cong X \times_Y EtB$, g is the map $\pi \circ h$):

$$\begin{tikzcd}
EtA \arrow[drr, bend left, "g"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "\pi_A"'] & & \\
& X \times_Y EtB \arrow[r, "\pi"] \arrow[d, "\pi_{Bi^*B}"'] & EtB \arrow[d, "\pi_B"] \\
& X \arrow[r, hook, "i"'] & Y
\end{tikzcd}$$

3.4 (a) Verify the adjunction formula involving j_* and $j^!$ in (7).

We must show that $Hom_{Ab(Y)}(j_*A, B) \cong Hom_{Ab(Z)}(A, j^!B)$.

Let $\phi \in Hom_{Ab(Y)}(j_*A, B)$. Then ϕ is a collection of maps $\phi_U : j_*A(U) \to B(U)$ for every open set $U \subseteq Y$. Note that for $U \subseteq Y$ such that $U \cap Z = \emptyset$, $j_*A(U) = A(U \cap Z) = A(\emptyset) = \{0\}$, so that there is only one choice for ϕ_U. Likewise, if $U \subset Y$ with $U \cap \delta Z \neq \emptyset$, then there exists a $V \subset U$ such that $V \cap Z = \emptyset$ and naturality of ϕ implies:

$$\begin{tikzcd}
j_*A(U) \arrow[r, "\phi_U"] \arrow[d, "\rho"'] & B(U) \arrow[d, "\rho"] \\
j_*A(V) = A(V \cap Z) = \{0\} \arrow[r, "\phi_V"'] & B(V)
\end{tikzcd}$$

so that $\phi_U(j_*A(U))$ is the constant 0 map. Finally, suppose U is such that $U \cap Z = U$. For each such U, the choices for ϕ_U are the group homomorphisms, $A(U) \to B(U)$.

Now consider $\psi \in Hom_{Ab(Z)}(A, j^!B)$. ψ is a collection of maps ψ_V for each open V in Z. Suppose V is open in Z and $V \cap \delta Z \neq \emptyset$. Then $j^!B(V) = 0$ and so there is only one choice for ψ_V, the constant 0 map. On the other hand, if $V \cap \delta Z = \emptyset$, then the choices for ϕ_V are the group homomorphisms, $A(V) \to B(V)$.

Then elements of both $Hom_{Ab(Z)}(A, j^!B)$ and $Hom_{Ab(Y)}(j_*A, B)$ correspond to the group homorphisms from $A(U) \to B(U)$ for every $U \subseteq Y$ with $U \cap Z = U$.

3.4 (b) Show that $j^!B$ is isomorphic to a subsheaf of j^*B.

Since $j^!B$ is a sheaf,we know that $j^!B \cong \Gamma Etj^!B$. We show that $\Gamma Etj^!B$ is a subsheaf of j^*B and for this is suffices to show that $\Gamma Etj^!B$ is a sub-presheaf; that is, that for V open in Z, $\Gamma Etj^!B(V) \subseteq j^*B(V)$ and that the restrictions maps of $\Gamma Etj^!B$ agree with those of j^*B.

Let V be open in Y, i.e. $V = Z \cap U$ for some U open in Y. We have

$$j^*B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B) \text{ and } x \in V\}$$

and

$$\Gamma Etj^!B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B),\ x \in V \text{ and } supp(s) \subseteq Z\}$$

so that $\Gamma Etj^!B(V) \subseteq j^*B(V)$. Moreover, the restriction maps agree since they are both restrictions of functions. Then $\Gamma Etj^!B \cong j^!B$ is a subsheaf of j^*B.

Section 4

4.1 (a) Show that with the notation as above, $\Gamma(-, \pi)$ is indeed isomorphic to the sheaffication of P.

We first fix some notation. Suppose $U \subseteq X$ and let $s \in G(U)$ be a section. Let $\overline{s}$ be the equivalence class of s in $G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$, so that points in the étale space of P are denoted $germ_x(\overline{s})$. We denote by $\overline{germ_x(s)}$ the equivalence class of $germ_x(s)$ under the relation R given above, where $germ_x(s) \in Et(G)$.

To show $\Gamma(-, \pi) \cong P$, we define a map $\phi : \Gamma(-, \pi) \to P$ and show it induces an isomorphism on the stalks. For $U \subseteq X$ and $\sigma \in \Gamma(-, \pi)(U)$, define $\phi_U : \Gamma(-, \pi)(U) \to P(U)$ by $\phi_U(\sigma) = \tau$, where σ is a function $x \mapsto \overline{germ_x(s)}$ and τ is a function $x \mapsto germ_x(\overline{s})$ for some $s \in G(U)$. Naturality of ϕ is straightforward, since restriction maps are just restrictions of functions.

To see that ϕ is well defined, suppose $germ_x(t)$ is also a representative of $\overline{germ_x(s)}$. Then $\exists W \ni x$ and $f \in F(W)$ such that $s\,|_W - t\,|_W = \alpha_W(f)$. This however gives $germ_x(\overline{t}) = germ_x(\overline{s})$, since W is then a neighborhood for which $\overline{s\,|_W} = \overline{t\,|_W}$, so that they are equal as germs.

It remains to show that ϕ induces an isomorphism on the stalks. We show $\phi_x : \Gamma(-, \pi)_x \to P_x$ is injective and surjective. Fix an $x \in X$ and suppose $\phi(\overline{germ_x(a)}) = germ_x(\overline{a}) = \phi(\overline{germ_x(b)})$. We have to show $\exists W \ni x$ and $f \in F(W)$ such that $a\,|_W - b\,|_W = \alpha_W(f)$. To see that ϕ_x is surjective.

Let $\overline{germ_x(s)}$ be an element of $Et(G)/R$, and let $germ_x(s)$ be a representative, so that if $germ_x(t)$ is another representative of $\overline{germ_x(s)}$, then $\exists W \ni x$ and $f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$, where s and t are representatives of $germ_x(s)$ and $germ_x(t)$ respectively.

Let $germ_x(a)$ be an element of P_x, represented by (a, U) for some $U \ni x$ and $a \in P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U)) = G(U)/im(\alpha_U)$. If (b, V) is another representative of $germ_x(a)$, then there exists $W \ni x$ such that $a\mid_W = b\mid_W$, i.e., $a\mid_W$ and $b\mid_W$ are in the same equivalence class of $G(W)/im(\alpha_W)$, i.e., $\exists f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$. Then elements of G_x/R and P_x are both correspond to elements of G_x identified via the same relation.

Let $\overline{germ_x(s)}$ be an element of G_x/R, and let $germ_x(s)$ be a representative, so that if $germ_x(t)$ is another representative of $\overline{germ_x(s)}$, then $\exists W \ni x$ and $f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$, where s and t are representatives of $germ_x(s)$ and $germ_x(t)$ respectively.

Let $germ_x(a)$ be an element of P_x, represented by (a, U) for some $U \ni x$ and $a \in P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U)) = G(U)/im(\alpha_U)$. If (b, V) is another representative of $germ_x(a)$, then there exists $W \ni x$ such that $a\mid_W = b\mid_W$, i.e., $a\mid_W$ and $b\mid_W$ are in the same equivalence class of $G(W)/im(\alpha_W)$, i.e., $\exists f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$. Then elements of G_x/R and P_x are both correspond to elements of G_x identified via the same relation.

4.1(b) Show that the $coker(\alpha)$ has the following universal property: given any other sheaf H, a map $\chi : G \to H$ factors through $coker(\alpha)$ if and only if $\chi \circ \alpha = 0$.

(Forward direction) Let $\alpha : F \to G$ and $\chi : G \to H$ and suppose $\chi \circ \alpha = 0$. Then for $U \in X$, $ker(\chi_U) = im(\alpha_U)$. By the first isomorphism theorem, $\chi_U(G) \cong G(U)/ker(\chi_U) = G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$, so that $\chi = g \circ f$, where $f : G \to coker(\alpha)$ is the projection map and $g : coker(\alpha) \to H$ is an isomorphism.

(Backwards direction) Let $\alpha : F \to G$ and $\chi : G \to H$ and suppose $\chi = g \circ f$ for some $f : G \to coker(\alpha)$ and $g : coker(\alpha) \to H$. Note that $f \circ \alpha = 0$, since f_U has codomain $coker(F(U) \xrightarrow{\alpha_U} G(U)$ for $U \in X$. Then $\chi \circ \alpha = g \circ f \circ \alpha = g \circ 0 = 0$.

4.1(c) Show that $coker(\alpha)_x = coker(\alpha_x)$.

Let $\alpha : F \to G$ for F and G Abelian sheaves on X and let P be the presheaf given by $P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$. Since $\pi_P : Et(P) \to X$ is étale, $Et\Gamma EtP \cong EtP$, so the left-hand side is isomorphic to P_x. On the right-hand side, note that $coker(\alpha_x) = coker(F_x \xrightarrow{\alpha_x} G_x) = G_x/Im(\alpha_x)$, so that two points are equivalent in $coker(\alpha_x)$ iff they are both points in G_x, say $germ_x(s)$ and $germ_y(t)$, with $germ_x(s) - germ_y(t) \in Im(\alpha_X)$, ie $\exists W \ni x$ and $f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$. This is exactly the relation given by R in the notes, so that the right-hand side is isomorphic to G_x/R. Then exercise 4.1(a) implies $coker(\alpha)_x \cong coker(\alpha_x)$.

4.2 Show that $\mathbf{Ab(X)}$ is an Abelian category with kernel, cokernel and sum defined as above.

We first show that $\mathbf{Ab(X)}$ has a 0 object. Let $\Delta\{*\}$ be the constant sheaf on a singleton set. Then for $U \subseteq X$, $\Delta\{*\}(U)$ is the trivial group and every restriction map is the identity, so $\Delta\{*\} \in \mathbf{Ab(X)}$. Moreover, for $A \in \mathbf{Ab(X)}$, there is exactly one map from $A(U) \to \Delta\{*\}(U)$ for every open $U \subseteq X$, namely the constant 0-map, and there is also exactly one map from $\Delta\{*\}(U) \to A(U)$ since group homomorphism must send identity to identity. Then $\Delta\{*\}$ is both initial and terminal, and so serves as the 0 element in $\mathbf{Ab(X)}$. We can now define the 0 morphism in $\mathbf{Ab(X)}$ as the map defined for $A, B \in \mathbf{Ab(X)}$ and $U \in X$, open, by $a \in A(U) \mapsto 0_{B(U)}$.

Let us define the sum of two Abelian sheaves A and B by $A(U) \oplus B(U)$ for U open in X, as in the notes. Then this gives the biproduct of A and B in $\mathbf{Ab(X)}$. We know from the notes that $A \oplus B$ is a sheaf and we observe that for $U \subseteq X$, $A(U) \oplus B(U)$ has an Abelian group structure: let (a, b) and (a', b') be elements of $A(U) \oplus B(U)$ for some U. Then $(a, b) + (a', b') = (a + a', b + b') = (a' + a, b' + b) = (a', b') + (a, b) \in A(U) \oplus B(U)$, since $A(U)$ and $B(U)$ are Abelian groups. This also implies $(a, b) + (0_{A(U)}, 0_{B(U)}) = (a, b) = (0_{A(U)}, 0_{B(U)}) + (a, b)$. To see that this is a biproduct, let $p_j : A_1 \oplus A_2 \to A_j$ be the map defined on $U \subseteq X$ as the projection map $p_{U,j} : A_1(U) \oplus A_2(U) \to A_j(U)$ and i_j be the map defined on $U \subseteq X$ as the injection map $i_j(U) : A_j(U) \to A_1(U) \oplus A_2(U)$ for $j \in \{1, 2\}$ and $A_j \in \mathbf{Ab(X)}$. Then since $p_{U,j}$ and $i_{U,j}$ give the product and coproduct on each $A_1(U) \oplus A_2(U)$, viewed as elements in the category of Abelian groups, their collection gives the product and coproduct on $A_1 \oplus A_2$ (given a map into or out of the biproduct, find a new map that factors through p_j or i_j respectively, by taking the collection of maps implied by the universal property of each $p_{U,j}$ or $i_{U,j}$).

To see that $\mathbf{Ab(X)}$ has kernels, note that if $A, B \in \mathbf{Ab(X)}$ and $\phi : A \to B$, then ϕ has a kernel when everything is viewed in the category $\mathbf{Sh(X)}$. It remains only to check that $ker(\phi)(U) = \{f \in A(U) \mid \phi_U(f) = 0_{B(U)}\}$ is an Abelian group for all U, open in X. But $ker(\phi)(U) = ker(\phi_U)$ is the kernel of the group homomorphism ϕ_U, and so must itself be a subgroup of $A(U)$ for all $U \in X$. Then $ker(\phi) \in \mathbf{Ab(X)}$. Likewise, the cokernel of ϕ exists in $\mathbf{Sh(X)}$, and so we need only show $coker(\phi)(U)$ is an Abelian group, $\forall U$.

4.3 Let $0 \to A \hookrightarrow I^0 \to I^1 \to ...$ and $0 \to B \to B \hookrightarrow J^0 \to J^1 \to J^2 \to ...$ be injective resolutions of A and B respectively. Show that a map $\phi : A \to B$ extends to a map of complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "\phi"] & I^0 \arrow[r] \arrow[d, dashed, "\phi^0"] & I^1 \arrow[r] \arrow[d, dashed, "\phi^1"] & I^2 \arrow[r] \arrow[d, dashed, "\phi^2"] & ... \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & ...
\end{tikzcd}$$

Since $0 \to A \hookrightarrow I^0 \to I^1 \to ...$ is a resolution, in particular exact, for each n, we have an injective map from $A \to I^n$ given by d^n. Since $\phi : A \to B$, we also have for each n, a map from $A \to J^n$ given by $d^n \circ \phi$. Then injectivity of J^n implies the required map $\phi^n : I^n \to J^n$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "d^n"] \arrow[d, "d^n \circ \phi"'] & I^n \arrow[dl, dashed, "\phi^n"] \\
 & J^n &
\end{tikzcd}$$

4.3 (b) Show that a map of complexes as above induces a homomorphism of cohomology groups

$$H^n(X; A) \to H^n(X, B)$$

Note that the above map of chain complexes implies a map for $U \in X$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A(U) \arrow[r] \arrow[d, "\phi_U"] & I^0(U) \arrow[r] \arrow[d, dashed, "\phi_U^0"] & I^1(U) \arrow[r] \arrow[d, dashed, "\phi_U^1"] & I^2(U) \arrow[r] \arrow[d, dashed, "\phi_U^2"] & ... \\
0 \arrow[r] & B(U) \arrow[r] & J^0(U) \arrow[r] & J^1(U) \arrow[r] & J^2(U) \arrow[r] & ...
\end{tikzcd}$$

which for $U = X$ gives the map:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \Gamma A \arrow[r] \arrow[d, "\Gamma\phi"] & \Gamma I^0 \arrow[r] \arrow[d, dashed, "\Gamma\phi^0"] & \Gamma I^1 \arrow[r] \arrow[d, dashed, "\Gamma\phi^1"] & \Gamma I^2 \arrow[r] \arrow[d, dashed, "\Gamma\phi^2"] & ... \\
0 \arrow[r] & \Gamma B \arrow[r] & \Gamma J^0 \arrow[r] & \Gamma J^1 \arrow[r] & \Gamma J^2 \arrow[r] & ...
\end{tikzcd}$$

Since $H^n(X; A) = ker(\Gamma I^n \to \Gamma I^{n+1}))/im(\Gamma I^{n-1} \to \Gamma I^n) \subseteq \Gamma I^n$ and likewise $H^n(X; B) = ker(\Gamma J^n \to \Gamma J^{n+1}))/im(\Gamma J^{n-1} \to \Gamma J^n) \subseteq \Gamma J^n$, the maps $\Gamma\phi^n : \Gamma I^n \to \Gamma J^n$ implies a homomorphism $\Gamma\phi^n \mid_{H^n(X,A)}: H^n(X; A) \to \Gamma J^n$. It remains only to check that $im(\Gamma\phi^n \mid_{H^n(X,A)}) \subseteq H^n(X, B)$. Let $\alpha \in H^n(X; A)$. Then $\alpha \in ker(d^n : \Gamma I^n \to \Gamma I^{n+1})$, so $\Gamma\phi^{n+1} \circ d^n(\alpha) = 0$. Since the above diagram commutes, this implies $d^n \circ \Gamma\phi^n(\alpha) = 0$, so that $\Gamma\phi^n(\alpha) \in ker(d^n : \Gamma J^n \to \Gamma J^{n+1})$, which implies $\Gamma\phi^n(\alpha) \in H^n(X; B)$.

4.4(a) Let $f : Y \to X$ be a map of topological spaces. Show that there is a natural isomorphism $\Gamma_Y \to \Gamma_X \circ f_*$,

$$\begin{tikzcd} Ab(Y) \arrow[rr, "f_*"] \arrow[dr, "\Gamma_Y"'] & & Ab(X) \arrow[dl, "\Gamma_X"] \\ & \textbf{Abelian Groups} & \end{tikzcd}$$

We must find η_A and η_B, isomorphisms, such that the following commutes for all $A, B \in Ab(Y)$ and $\phi : A \to B$:

$$\begin{tikzcd} \Gamma_Y(A) \arrow[r, "\Gamma_Y \phi"] \arrow[d, "\eta_A"'] & \Gamma_Y(B) \arrow[d, "\eta_B"] \\ \Gamma_X \circ f_*(A) \arrow[r, "\Gamma_x \circ f_* \phi"'] & \Gamma_X \circ f_*(B) \end{tikzcd}$$

By the definition of $\Gamma_{\_}$ and f_*, this is the same as finding isomorphisms η_A and η_B such that the following commutes:

$$\begin{tikzcd} A(Y) \arrow[r, "\phi_Y"] \arrow[d, "\eta_A"'] & B(Y) \arrow[d, "\eta_B"] \\ A(f^{-1}(X)) \arrow[r, "\phi_{f^{-1}(X)}"'] & B(f^{-1}(X)) \end{tikzcd}$$

But $f^{-1}(X) = Y$, so taking $\eta_A = \rho_{Y,Y} = id_{A(Y)}$ (likewise for η_B) implies the result, since ϕ is a map $A \to B$ and so must be natural wrt the restriction maps.

4.4 (b) Show that a natural isomorphism between left exact functors $\tau : T_1 \to T_2$ induces a natural isomorphism $R^n T_1 \to R^n T_2$ for each n.

Let T_1 and T_2 be left exact functors from $\mathcal{C} \to \mathcal{D}$ and let $\tau : T_1 \to T_2$ be a natural isomorphism. Let $0 \to C \to I^0 \to I^1 \to ...$ be an injective resolution of C for some $C \in \mathcal{C}$. Naturality of τ implies that the following commutes:

$$\begin{tikzcd}
0 \arrow[r] & T_1(C) \arrow[r] \arrow[d, "\wr"'] & T_1(I^0) \arrow[r] \arrow[d, "\wr"'] & T_1(I^1) \arrow[r] \arrow[d, "\wr"'] & T_1(I^2) \arrow[r] \arrow[d, "\wr"'] & ... \\
0 \arrow[r] & T_2(C) \arrow[r] & T_2(I^0) \arrow[r] & T_2(I^1) \arrow[r] & T_2(I^2) \arrow[r] & ...
\end{tikzcd}$$

Moreover, since T_1 and T_2 are both left exact, each row in the above diagram is exact. Then, by analogous argument to 4.3(b), τ restricts to a natural isomorphism $R^n T_1 \to R^n T_2$ for each n.

Section 7

7.1 Prove Corollary 7.4 (Proper Base Change): For a pullback diagram as in (26) with f and f' proper with Hausdorff fibers, the canonical map

$$p^*(R^n f_* B) \to R^n f'_*(q^* B)$$

is an isomorphism for any sheaf B on Y.

We show $p^*(R^n f_* B)_{x'} \cong (R^n f'_*(q^* B))_{x'}$ for $x' \in X'$. Let $x' \in X'$, B be a sheaf on Y and let the following be a pullback diagram:

$$\begin{tikzcd}
(f')^{-1}(x') \in Y' \arrow[r, "q"] \arrow[d, "f'"'] & Y \ni f^{-1}(p(x')) \arrow[d, "f"] \\
x' \in X' \arrow[r, "p"'] & X \ni p(x')
\end{tikzcd}$$

Note that since the above is a pullback diagram, q gives an isomorphism $q : (f')^{-1}(x') \xrightarrow{\sim} f^{-1}(p(x'))$, which by Prop (4.3) induces an isomorphism $H^n(f^{-1}(p(x')); B\mid_{f^{-1}p(x')}) \xrightarrow{\sim} H^n((f')^{-1}(x'); q^*B\mid_{(f')^{-1}(x')})$. Then we have:

$$\begin{aligned}
p^*(R^n f_* B)_{x'} &\cong (R^n f_* B)_{p(x')} && \text{exercise 3.2} \\
&\cong H^n(f^{-1}(p(x')); B\mid_{f^{-1}(p(x'))}) && (7.3) \\
&\cong H^n((f')^{-1}(x'); q^*B\mid_{(f')^{-1}(x')}) && \text{pullback and (4.3)} \\
&\cong (R^n f'_*(q^* B))_{x'} && (7.3)
\end{aligned}$$

Section 8

8.1 Let X be connected and locally simply connected, and choose a base point x_0. Show that for any locally constant sheaf (of sets) A, the group $\pi_1(X, x_0)$ acts on the stalk A_{x_0}. Show that this gives a functor $A \mapsto A_{x_0}$ (which is an equivalence of categories between the category of locally constant sheaves on X and the category of sets with a $\pi_1(X, x_0)$-action.)

Let X be connected and locally simply connected. Let A be a locally constant sheaf, so that $\pi : Et(A) \to X$ is a covering map. Fix x_0 in X and let $p : [0,1] \to X$ be a loop in X based at x_0. Consider $\pi^{-1}(x_0) \in Et(A)$. Since π is a covering map, there exists a neighborhood U_0 of x_0 in X and a set $S = \pi^{-1}(x_0)$ such that U_0 is evenly covered by π with $\pi^{-1}(U_0) = U_0 \times S$, where S has the discrete topology. (i.e., each $x \in X$ has a neighborhood whose preimage under π is a stack of pancakes with one pancake for each $s \in S = A_x$).

Note that for each $s \in S$ (that is, each point in the fiber of x_0 or equivalently each point in A_{x_0}), the fact that π is a covering map implies that $p : [0,1] \to X$ has a unique lifting, $\tilde{p}_s$, to a path in $Et(A)$ beginning at s and ending at some $t \in S$ (Munkres Lemma 54.1). Moreover, if p_1 and p_2 are both loops based at x_0 and are homotopic, then the induced paths in $Et(A)$ are homotopic with identical endpoints (Munkres Theorem 54.3). Now we can define an action of $\pi_1(X, x_0)$ on $A_{x_0} = \pi^{-1}(x_0)$ by $p \cdot s = \tilde{p}_s(1)$, where p is a loop based at x_0, s is in the fiber of x_0 (ie s in the stalk of A_{x_0}) and $\tilde{p}_s$ is the unique lift of p for s implied by the fact that π is a covering map. Note that if e is the constant loop, then $\tilde{e}_s(1) = s$, $\forall s$ so that $e \cdot s = \tilde{e}_s(1) = s$. Since the group operation, $*$, of $\pi(X, x_0)$ is composition of loops, we also have $p \cdot (q \cdot s) = p \cdot \tilde{q}_s(1) = \tilde{p}_{\tilde{q}_s(1)}(1) = (p * q) \cdot s$. Then $\pi(X, x_0)$ gives an action on A_{x_0}.

Let **LocConSh(X)** be the category of locally constant sheaves on X. Define a functor F : **LocConSh(X)** $\to$ **Set** by $A \mapsto A_{x_0}$. Note that since X is (path-) connected, this does not depend on the choice of base point, x_0. Suppose $\phi : A \to B$ is an arrow in **LocConSh(X)**. We must show:

$$\begin{array}{ccc} A(U) & \xrightarrow{F} & A_{x_0} \\ {\scriptstyle \phi_U}\big\downarrow & & \big\downarrow{\scriptstyle F\phi_U} \\ B(U) & \xrightarrow[F]{} & B_{x_0} \end{array}$$

8.2 Let S^d be a d-dimensional sphere. For any Abelian group A, prove that,

$$H^n(S^d; A) = \begin{cases} A & \text{when } n = 0 \text{ or } n = d \text{ provided } d \neq 0; \\ A \oplus A & \text{when } d = n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Write S^d as the union of closed subspaces, N and S, the closed northern and southern hemispheres, $S^d = N \cup S$. Note that $S^{d-1} = N \cap S$. We proceed by induction of d and consider the cases $n \neq d$ and $n = d$ separately. The cases $n = 0$ and $d = 0$ are obvious (since S^d is connected for $d > 0$ and S^0 has two components).

Suppose $d > 1$ and the result holds for $d = 0$. By inductive hypothesis, for $n \neq d$ and $n > 0$, $H^n(S^{d-1}; A\,|_{S^{d-1}}) = 0$. Then the Mayer-Vietoris sequence:

$$\ldots \to H^{n-1}(S^{d-1}; A\,|_{s^{d-1}}) \to$$

$$H^n(S^d; A) \to H^n(N; A\,|_N) \oplus H^n(S; A\,|_S) \to H^n(S^{d-1}; A\,|_{S^{d-1}}) \to \ldots$$

becomes :

$$\ldots \to 0 \to H^n(S^d; A) \to H^n(N; A\,|_N) \oplus H^n(S; A\,|_S) \to 0 \to \ldots$$

And so by exactness, we have

$$\ldots \to 0 \to H^n(S^d; A) \xrightarrow{\sim} H^n(N; A\,|_N) \oplus H^n(S; A\,|_S) \to 0 \to \ldots$$

It remains to show that $H^n(N; A\,|_N) \oplus H^n(S; A\,|_S) = 0$.*** Then $H^n(S^d; A) = 0$ for all $n \neq d$ and $n > 0$ and for all d.

We now consider the case $n = d$. By inductive hypothesis, we have $H^{n-1}(S^{d-1}; A) = H^{d-1}(S^{d-1}; A) = A$. So the Mayer-Vietoris sequence:

$$\ldots \to H^{d-1}(N; A\,|_N) \oplus H^{d-1}(S; A\,|_S) \to H^{d-1}(S^{d-1}; A\,|_{S^{d-1}}) \to H^d(S^d; A) \to$$

becomes

$$\ldots \to H^{d-1}(N; A\,|_N) \oplus H^{d-1}(S; A\,|_S) \to A \to H^d(S^d; A) \to \ldots$$

But by above, this is

$$\ldots \to 0 \to A \to H^d(S^d; A) \to \ldots$$

which by exactness and the first isomorphism theorem implies $H^d(S^d; A) \cong A$ and so the result follows.