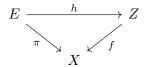
Exercises

Section 2

2.1. For a commutative triangle of topological spaces and continuous maps,



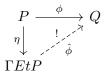
if f and π are étale, so is h (in particular, h is open).

Let $y \in E$. Since f is étale, $\exists U$ an open neighborhood of h(y) such that $f \mid_U$ is a homeomorphism. Likewise, since π is étale, $\exists W$ an open neighborhood of y such that $\pi \mid_W$ is a homeomorphism. Let $V = h^{-1}(U) \cap W$. Since h is continuous, $h^{-1}(U)$ is open, so V is open in E. Since $V \subseteq W$, $\pi \mid_V$ is a homeomorphism.

Since $h(V) \subseteq U$, and $f|_U$ is a homeomorphism, it only remains to show that h(V) is open, since this will imply that $f|_{h(v)}$ is a homeomorphism, and in turn that $h|_V$ is a homeomorphism (since $\pi|_V$ is a homeomorphism, and the diagram commutes). To see that h(V) is open, note that $\pi(V)$ is open in X, since π is étale and V is open in E. Further, $f(h(V)) = \pi(V)$ since the diagram commutes. Then since $h(V) \subseteq U$ and $f|_U$ is a homeomorphism, h(V) is open in Z, so $h|_V$ is a homeomorphism and h is étale.

2.2. (a) Show that Corollary 2.10 determines the associated sheaf up to isomorphism. That is, if R is a sheaf and there exists a map $\psi: P \to R$ with the same property as η in Corollary 2.10, then $R \cong \Gamma Et(P)$.

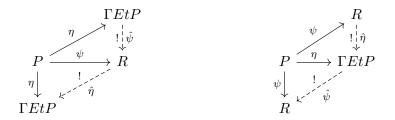
Corollary. (2.10) For a map $\phi: P \to Q$ between presheaves on X, if Q is a sheaf then there exists a unique map of sheaves $\hat{\phi}$ such that $\hat{\phi} \circ \eta = \phi$.



The universal property of $(\Gamma EtP, \eta)$ implies a unique map $\hat{\psi}$ such that $\psi = \hat{\psi} \circ \eta$ and the universal property of (R, ψ) implies a unique map $\hat{\eta}$ such that $\eta = \hat{\eta} \circ \psi$, i.e. we have the following diagrams:



This implies the following diagrams commute:



In particular, $\hat{\eta} \circ \hat{\psi}$ is the unique map such that $\eta = \hat{\eta} \circ \hat{\psi} \circ \eta$ and $\hat{\psi} \circ \hat{\eta}$ is the unique map such that $\psi = \hat{\psi} \circ \hat{\eta} \circ \psi$. But then uniqueness implies $\hat{\eta} \circ \hat{\psi} = id_{\Gamma EtP}$ and $\hat{\psi} \circ \hat{\eta} = id_R$. Then $R \cong \Gamma EtP$.

2.2. (b) If P is a subpresheaf of R and R is a sheaf, then let

 $\tilde{P}(U) = \{r \in R(U) \mid \text{ for each } x \in U \text{ there is a neighbourhood } W_x \subseteq U \text{ of } x \text{ such that } (r \mid_{W_x}) \in P(W_x) \}.$

Show that $P \subseteq \tilde{P} \subseteq R$, \tilde{P} is a sheaf and $P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10; hence \tilde{P} is the associated sheaf for P.

$P \subseteq \tilde{P} \subseteq R$:

We want to show that for $U \in X$, open, $P(U) \subseteq \tilde{P}(U) \subseteq R(U)$. So let $U \subseteq X$ be open. By construction $\tilde{P}(U) \subseteq R(U)$, so we need only show $P(U) \subseteq \tilde{P}(U)$. Let $s \in P(U)$. Since P is a subpresheaf $s \in R(U)$ and for $x \in U$, U itself is a neighborhood of x such that, $s \mid_{U}$ is in P(U). Then $s \in \tilde{P}(U)$.

\tilde{P} is a sheaf:

Let U be an open set in X and let $\cup U_i$, $i \in I$, be an open cover for U. Suppose $\{a_i\} \in \tilde{P}(U_i)$, for $i \in I$ is a compatible family for the U_i , i.e., $a_i \mid_{U_i \cap U_j} = a_j \mid_{U_i \cap U_j}$. For each $i \in I$ and each $x \in U_i$, choose W_x^i such that $(a_i \mid_{W_x^i}) \in P(W_x^i)$. Then $\cup_i \cup_{x \in U_i} W_x^i$ is an open cover for U (since $\cup_x W_x^i$ is an open cover for each U_i). Moreover, $a_i \mid_{W_x^i \cap W_y^j} = a_j \mid_{W_x^i \cap W_y^j}$ because $(a_i \mid_{U_i \cap U_j}) \mid_{W_x^i \cap W_y^j} = (a_j \mid_{U_i \cap U_j}) \mid_{W_x^i \cap W_y^j}$. So $\{a_i\} \in P(W_x^i)$ is a compatible family in \mathbb{R} .

Since R is a sheaf, let a e the unique amalgamation. We want to show $a \in \tilde{P}(U)$, i.e., $\forall x \in U, \exists$ a neighborhood V_x of x such that $a \mid_{V_x} \in P(V_x)$. But since a is the amalgamation of the family $\{a_i\} \in P(W_x^i)$, if $x \in U$, then $x \in U_i$, for some i, so $a \mid_{W_x^i} = a_i$ and W_x was chosen so that $a_i \mid_{W_x^i} \in P(W_x^i)$. Then take V_x to be W_x^i (for appropriate i). Then $a \in \tilde{P}(U)$, so \tilde{P} is a sheaf.

$P \hookrightarrow \tilde{P}$ has the unique universal property of Corollary 2.10:

Let $i: P \hookrightarrow \tilde{P}$ be the map that injects P(U) into $\tilde{P}(U)$. We must show that for a map between presheaves, $\phi: P \to Q$, where Q is a sheaf, there is a unique $\tilde{\phi}$ such that $\tilde{\phi} \circ i = \phi$:

So let Q be a sheaf and $\phi: P \to Q$ be a morphism. For each open set $U \subseteq X$, we define $\tilde{\phi}_U: \tilde{P}(U) \to Q(U)$ as follows: let $s \in \tilde{P}(U)$ and let $\cup_{x \in U} W_x$ be a cover of U such that $(s \mid_{W_x}) \in P(W_x)$ for each $x \in U$. Consider the family $\phi_{W_x}(s \mid_{W_x})$ for $x \in U$. This family is compatible because commutativity of the restriction maps implies $(s \mid_{W_x}) \mid_{W_x \cap W_y} = (s \mid_{W_y}) \mid_{W_x \cap W_y} = s \mid_{W_x \cap W_y}$ and naturality of ϕ implies for $x, y \in U$:

$$\phi(s \mid W_x) \mid_{W_x \cap W_y} = \phi((s \mid W_x) \mid_{W_x \cap W_y}) = \phi(s \mid_{W_x \cap W_y})$$
$$= \phi((s \mid_{W_y} \mid_{W_x \cap W_y})) = \phi(s \mid_{W_y}) \mid_{W_x \cap W_y}.$$

Then since Q is a sheaf, let $t \in Q(U)$ be the unique amalgamation of the $\phi_{W_x}(s|_{W_x})$'s and define $\tilde{\phi}(s) = t$. In fact, by naturality of $\tilde{\phi}$, we must choose the amalgamation to ensure that the following commutes for each W_x :

$$s \in \tilde{P}(U) \xrightarrow{\tilde{\phi}_{U}} Q(U) \ni t$$

$$\downarrow^{\rho} \qquad \qquad \downarrow^{\rho}$$

$$(s \mid_{W_{x}}) \in \tilde{P}(W_{x}) \xrightarrow{\tilde{\phi}_{W_{x}}} Q(W_{x}) \ni \tilde{\phi}(s \mid_{W_{x}}) = \phi(s \mid_{W_{x}})$$

since the bottom row must be $\tilde{\phi}(s|_{W_x}) = \phi(s|_{W_x})$ by the requirement that $\tilde{\phi} \circ i = \phi$. So then uniqueness of $\tilde{\phi}$ is implied so long as $\tilde{\phi}$ is well-defined.

To see that $\tilde{\phi}$ is well-defined, let $\cup_{x\in U}V_x$ be another cover of U with the property that $(s\mid_{V_x})\in P(V_x)$, so that $\phi_{V_x}(s\mid_{V_x})$ is again a compatible family. We must show that t is also the amalgamation of this family, i.e., $t\mid_{V_x}=\phi_{V_x}(s\mid_{V_x})$ for each V_x .

Consider the cover of U formed by the refinement of the W_x 's and V_x 's, $U = \bigcup_{x \in U} W_x \bigcup_{x \in U} V_x$. This contains the compatible family

$$\{\{\phi(s\mid_{W_x})\}_{x\in U}, \{\phi(s\mid_{V_x})\}_{x\in U}\}$$

.

The amalgamation of this family, say \bar{t} , must have the property that $\bar{t}\mid_{W_x} = \phi(s\mid_{W_x})$, and since Q is a sheaf, hence separated, this implies that $\bar{t} = t$. Then $t\mid_{V_x} = \phi_{V_x}(s\mid_{V_x})$ for each V_x , so $\tilde{\phi}$ is well-defined. In particular, note that $\tilde{\phi} \circ i = \phi$, since $s \in P(U)$ implies that U is a cover of U for which $s\mid_{U} \in P(U)$ so $\tilde{\phi}(s) = \phi(s\mid_{U}) = \phi(s)$.

Then $\tilde{\phi}$ is the required unique sheaf map satisfying $\tilde{\phi} \circ i = \phi$, so \tilde{P} is the associated sheaf of P.

Section 3

3.1 The pullback of an étale map is also étale. That is, given a pullback diagram,

$$\begin{array}{ccc} X \times_Y E & \xrightarrow{\pi_2} & E \\ \downarrow^{\pi_1} & & \downarrow^p \\ X & \xrightarrow{f} & Y \end{array}$$

then π_1 is étale if p is.

Let $(u, e) \in X \times_Y E$. Since p is étale, there exists a $U \subseteq E$, open neighborhood of $e = \pi_2(u, e)$, such that $p : U \to p(U)$ is a homeomorphism. Since $f : X \to Y$ is continuous (and p is open), the preimage of p(U) under f is open in X. Let $W \subseteq X$ be the preimage of p(U). Since p(e) = f(u), $f(u) \in p(U)$ and so $u \in W$.

Since $W \times U$ is open in $X \times E$ under the product topology, $V = W \times U \cap X \times_Y E$ is an open neighborhood of (u,e) in $X \times_Y E$ under the subspace topology. I claim that $\pi_1 \mid_V$ is a homeomorphism. Since π_1 is a projection map, $\pi_1 \mid_V$ is continuous. We must show there exists a continuous $g: \pi_1(V) \to X \times_Y E$ such that $g \circ \pi_1 = id$. Since $p \mid_U$ is a homeomorphism, we have continuous $p^{-1}: f(\pi_1(V)) \to E$ such that $p^{-1} \circ p = id_E$. Then $p^{-1}(f(\pi_1(V))) = p^{-1}(p(\pi_2(V))) = \pi_2(V)$. Let $i: EtB \hookrightarrow X \times_Y EtB$ be the inclusion map. Then let $g: \pi_1(V) \to X \times_Y EtB$ be defined by $x \mapsto (x, i \circ p^{-1} \circ f(x))$. Then g is continuous and $g \circ \pi_1 \mid_V = id_{X \times_Y E}$, so that $\pi_1 \mid_V$ is a homeomorphism. Then π_1 is étale.

3.2. Verify the following equality of stalks:

$$(f^*B)_x \cong B_{f(x)}$$

for each $x \in X$.

First note that since $f^*(B) = \Gamma(X \times_Y EtB)$ and since π_1 is étale, $Et\Gamma(X \times_Y EtB) \cong X \times_Y EtB$ by the results in section 2. That is, $Et(f^*(B)) \cong X \times_Y EtB = \{(x, e) \in X \times EtB \mid f(x) = \pi_B(e)\}.$

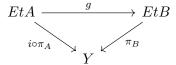
Fix an $x_0 \in X$. The stalk of $f^*(B)$ at x_0 is $(f^*(B))_{x_0} = \{s \in Et(f^*(B)) \mid \pi_B(s) = x_0\} = \{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\}$. Then the map π_2 gives us a bijection $\{(x_0, e) \in X \times EtB \mid f(x_0) = \pi_B(e)\} \rightarrow \{e \in EtB \mid f(x_0) = \pi_B(e)\}$, which is exactly the stalk of $B_{f(x_0)}$.

3.3. Prove that

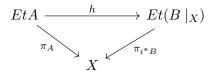
$$Hom_{Sh(Y)}(i_!A, B) \cong Hom_{Sh(X)}(A, i^*B)$$

where $i: X \hookrightarrow Y$ is an open inclusion.

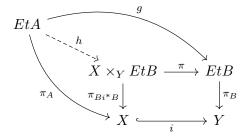
First note that since $i \circ \pi_A$ is étale, $Et(i_!A) = Et(\Gamma EtA) = EtA$. Since B is a sheaf, 2.9 implies $\phi \in Hom_{Sh(Y)}(i_!A, B)$ corresponds to a map $g : EtA \to EtB$ such that the following commutes:



By the notes, $Et(i^*B) = Et(B \mid_X)$ and since i^*B is a sheaf, 2.9 implies $\psi \in Hom_{Sh(X)}(A, i^*B)$ corresponds to a map $h : EtA \to Et(B \mid_X)$ such that the follow commutes:



To see the correspondence between g and h, note that $Et(B \mid_X) \cong X \times_Y EtB$, so that given $g: EtA \to EtB$, h is the unique map such that the following commutes (and given $h: EtA \to Et(B \mid_X) \cong X \times_Y EtB$, g is the map $\pi \circ h$):



3.4 (a) Verify the adjunction formula involving j_* and $j^!$ in (7).

We must show that $Hom_{Ab(Y)}(j_*A, B) \cong Hom_{Ab(Z)}(A, j!B)$.

Let $\phi \in Hom_{Ab(Y)}(j_*A, B)$. Then ϕ is a collection of maps $\phi_U : j_*A(U) \to B(U)$ for every open set $U \subseteq Y$. Note that for $U \subseteq Y$ such that $U \cap Z = \emptyset$, $j_*A(U) = A(U \cap Z) = A(\emptyset) = \{0\}$, so that there is only one choice for ϕ_U . Likewise, if $U \subset Y$ with $U \cap \delta Z \neq \emptyset$, then there exists a $V \subset U$ such that $V \cap Z = \emptyset$ and naturality of ϕ implies:

$$j_*A(U) \xrightarrow{\phi_U} B(U)$$

$$\downarrow^{\rho} \qquad \qquad \downarrow^{\rho}$$

$$j_*A(V) = A(V \cap Z) = \{0\} \xrightarrow{\phi_V} B(V)$$

so that $\phi_U(j_*A(U))$ is the constant 0 map. Finally, suppose U is such that $U \cap Z = U$. For each such U, the choices for ϕ_U are the group homomorphisms, $A(U) \to B(U)$.

Now consider $\psi \in Hom_{Ab(Z)}(A, j^!B)$. ψ is a collection of maps ψ_V for each open V in Z. Suppose V is open in Z and $V \cap \delta Z \neq \emptyset$. Then $j^!B(V) = 0$ and so there is only one choice for ψ_V , the constant 0 map. On the other hand, if $V \cap \delta Z = \emptyset$, then the choices for ϕ_V are the group homomorphisms, $A(V) \to B(V)$.

Then elements of both $Hom_{Ab(Z)}(A, j^!B)$ and $Hom_{Ab(Y)}(j_*A, B)$ correspond to the group homorphisms from $A(U) \to B(U)$ for every $U \subseteq Y$ with $U \cap Z = U$.

3.4 (b) Show that $j^!B$ is isomorphic to a subsheaf of j^*B .

Since $j^!B$ is a sheaf,we know that $j^!B \cong \Gamma Etj^!B$. We show that $\Gamma Etj^!B$ is a subsheaf of j^*B and for this is suffices to show that $\Gamma Etj^!B$ is a subpresheaf; that is, that for V open in Z, $\Gamma Etj^!B(V) \subseteq j^*B(V)$ and that the restrictions maps of $\Gamma Etj^!B$ agree with those of j^*B .

Let V be open in Y, i.e. $V = Z \cap U$ for some U open in Y. We have

$$j^*B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B) \text{ and } x \in V\}$$

and

$$\Gamma Etj^!B(V) = \{x \mapsto germ_x(s) \mid germ_x(s) \in Et(B), x \in V \text{ and } supp(s) \subseteq Z\}$$

so that $\Gamma Etj^!B(V)\subseteq j^*B(V)$. Moreover, the restriction maps agree since they are both restrictions of functions. Then $\Gamma Etj^!B\cong j^!B$ is a subsheaf of j^*B .

Section 4

4.1 (a) Show that with the notation as above, $\Gamma(-,\pi)$ is indeed isomorphic to the sheaffication of P.

We first fix some notation. Suppose $U \subseteq X$ and let $s \in G(U)$ be a section. Let \overline{s} be the equivalence class of s in $G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$, so that points in the étale space of P are denoted $germ_x(\overline{s})$. We denote by $\overline{germ_x(s)}$ the equivalence class of $germ_x(s)$ under the relation R given above, where $germ_x(s) \in Et(G)$.

To show $\Gamma(-,\pi) \cong P$, we define a map $\phi: \Gamma(-,\pi) \to P$ and show it induces an isomorphism on the stalks. For $U \subseteq X$ and $\sigma \in \Gamma(-,\pi)(\underline{U})$, define $\phi_U: \Gamma(-,\pi)(U) \to P(U)$ by $\phi_U(\sigma) = \tau$, where σ is a function $x \mapsto \overline{germ_x(s)}$ and τ is a function $x \mapsto \overline{germ_x(s)}$ for some $s \in G(U)$. Naturality of ϕ is straightforward, since restriction maps are just restrictions of functions.

To see that ϕ is well defined, suppose $germ_x(t)$ is also a representative of $\overline{germ_x(s)}$. Then $\exists W \ni x$ and $f \in F(W)$ such that $s \mid_W - t \mid_W = \alpha_W(f)$. This however gives $germ_x(\overline{t}) = germ_x(\overline{s})$, since W is then a neighborhood for which $\overline{s \mid_W = t \mid_W}$, so that they are equal as germs.

It remains to show that ϕ induces an isomorphism on the stalks. We show $\phi_x : \Gamma(-,\pi)_x \to P_x$ is injective and surjective. Fix an $x \in X$ and suppose $\phi(\overline{germ_x(a)}) = \overline{germ_x(a)} = \phi(\overline{germ_x(b)})$. We have to show $\exists W \ni x$ and $f \in F(W)$ such that $a \mid_W - b \mid_W = \alpha_W(f)$. To see that ϕ_x is surjective.

Let $germ_x(s)$ be an element of Et(G)/R, and let $germ_x(s)$ be a representative, so that if $germ_x(t)$ is another representative of $germ_x(s)$, then $\exists W \ni x$ and $f \in F(W)$ such that $s \mid_W - t \mid_W = \alpha_W(f)$, where s and t are representatives of $germ_x(s)$ and $germ_x(t)$ respectively.

Let $germ_x(a)$ be an element of P_x , represented by (a,U) for some $U \ni x$ and $a \in P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U)) = G(U)/im(\alpha_U)$. If (b,V) is another representative of $germ_x(a)$, then there exists $W \ni x$ such that $a \mid_W = b \mid_W$, i.e., $a \mid_W$ and $b \mid_W$ are in the same equivalence class of $G(W)/im(\alpha_W)$, i.e., $\exists f \in F(W)$ such that $s \mid_W -t \mid_W = \alpha_W(f)$. Then elements of G_x/R and P_x are both correspond to elements of G_x identified via the same relation.

Let $\overline{germ_x(s)}$ be an element of G_x/R , and let $\underline{germ_x(s)}$ be a representative, so that if $\underline{germ_x(t)}$ is another representative of $\overline{germ_x(s)}$, then $\exists W \ni x$ and $f \in F(W)$ such that $s \mid_W -t \mid_W = \alpha_W(f)$, where s and t are representatives of $\underline{germ_x(s)}$ and $\underline{germ_x(t)}$ respectively.

Let $germ_x(a)$ be an element of P_x , represented by (a,U) for some $U\ni x$ and $a\in P(U)=coker(F(U)\xrightarrow{\alpha_U}G(U))=G(U)/im(\alpha_U)$. If (b,V) is another representative of $germ_x(a)$, then there exists $W\ni x$ such that $a\mid_W=b\mid_W$, i.e., $a\mid_W$ and $b\mid_W$ are in the same equivalence class of $G(W)/im(\alpha_W)$, i.e., $\exists f\in F(W)$ such that $s\mid_W-t\mid_W=\alpha_W(f)$. Then elements of G_x/R and P_x are both correspond to elements of G_x identified via the same relation.

4.1(b) Show that the $coker(\alpha)$ has the following universal property: given any other sheaf H, a map $\chi: G \to H$ factors through $coker(\alpha)$ if and only if $\chi \circ \alpha = 0$.

(Forward direction) Let $\alpha: F \to G$ and $\chi: G \to H$ and suppose $\chi \circ \alpha = 0$. Then for $U \in X$, $ker(\chi_U) = im(\alpha_U)$. By the first isomorphism theorem, $\chi_U(G) \cong G(U)/ker(\chi_U) = G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$, so that $\chi = g \circ f$, where $f: G \to coker(\alpha)$ is the projection map and $g: coker(\alpha) \to H$ is an isomorphism.

(Backwards direction) Let $\alpha: F \to G$ and $\chi: G \to H$ and suppose $\chi = g \circ f$ for some $f: G \to coker(\alpha)$ and $g: coker(\alpha) \to H$. Note that $f \circ \alpha = 0$, since f_U has codomain $coker(F(U) \xrightarrow{\alpha_U} G(U))$ for $U \in X$. Then $\chi \circ \alpha = g \circ f \circ \alpha = g \circ 0 = 0$.

4.1(c) Show that $coker(\alpha)_x = coker(\alpha_x)$.

Let $\alpha: F \to G$ for F and G Abelian sheaves on X and let P be the presheaf given by $P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$. Since $\pi_P: Et(P) \to X$ is étale, $Et\Gamma EtP \cong EtP$, so the left-hand side is isomorphic to P_x . On the right-hand side, note that $coker(\alpha_x) = coker(F_x \xrightarrow{\alpha_x} G_x) = G_x/Im(\alpha_x)$, so that two points are equivalent in $coker(\alpha_x)$ iff they are both points in G_x , say $germ_x(s)$ and $germ_y(t)$, with $germ_x(s) - germ_y(t) \in Im(\alpha_X)$, ie $\exists W \ni x$ and $f \in F(W)$ such that $s \mid_W - t \mid_W = \alpha_W(f)$. This is exactly the relation given by R in the notes, so that the right-hand side is isomorphic to G_x/R . Then exercise 4.1(a) implies $coker(\alpha)_x \cong coker(\alpha_x)$.

4.2 Show that $\mathbf{Ab}(\mathbf{X})$ is an Abelian category with kernel, cokernel and sum defined as above.

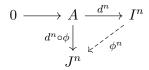
We first show that $\mathbf{Ab}(\mathbf{X})$ has a 0 object. Let $\Delta\{*\}$ be the constant sheaf on a singleton set. Then for $U \subseteq X$, $\Delta\{*\}(U)$ is the trivial group and every restriction map is the identity, so $\Delta\{*\} \in \mathbf{Ab}(\mathbf{X})$. Moreover, for $A \in \mathbf{Ab}(\mathbf{X})$, there is exactly one map from $A(U) \to \Delta\{*\}(U)$ for every open $U \subseteq X$, namely the constant 0-map, and there is also exactly one map from $\Delta\{*\}(U) \to A(U)$ since group homomorphism must send identity to identity. Then $\Delta\{*\}$ is both initial and terminal, and so serves as the 0 element in $\mathbf{Ab}(\mathbf{X})$. We can now define the 0 morphism in $\mathbf{Ab}(\mathbf{X})$ as the map defined for $A, B \in \mathbf{Ab}(\mathbf{X})$ and $U \in X$, open, by $a \in A(U) \mapsto 0_{B(U)}$.

Let us define the sum of two Abelian sheaves A and B by $A(U) \oplus B(U)$ for U open in X, as in the notes. Then this gives the biproduct of Aand B in Ab(X). We know from the notes that $A \oplus B$ is a sheaf and we observe that for $U \subseteq X$, $A(U) \oplus B(U)$ has an Abelian group structure: let (a,b) and (a',b') be elements of $A(U) \oplus B(U)$ for some U. Then $(a,b) + (a',b') = (a+a',b+b') = (a'+a,b'+b) = (a',b') + (a,b) \in$ $A(U) \oplus B(U)$, since A(U) and B(U) are Abelian groups. This also implies $(a,b) + (0_{A(U)}, 0_{B(U)}) = (a,b) = (0_{A(U)}, 0_{B(U)}) + (a,b)$. To see that this is a biproduct, let $p_j: A_1 \oplus A_2 \to A_j$ be the map defined on $U \subseteq X$ as the projection map $p_{U,i}: A_1(U) \oplus A_2(U) \to A_i(U)$ and i_i be the map defined on $U \subseteq X$ as the injection map $i_j(U): A_j(U) \to A_1(U) \oplus A_2(U)$ for $j \in \{1, 2\}$ and $A_i \in \mathbf{Ab}(\mathbf{X})$. Then since $p_{U,i}$ and $i_{U,i}$ give the product and coproduct on each $A_1(U) \oplus A_2(U)$, viewed as elements in the category of Abelian groups, their collection gives the product and coproduct on $A_1 \oplus A_2$ (given a map into or out of the biproduct, find a new map that factors through p_i or i_i respectively, by taking the collection of maps implied by the universal property of each $p_{U,j}$ or $i_{U,j}$).

To see that $\mathbf{Ab}(\mathbf{X})$ has kernels, note that if $A, B \in \mathbf{Ab}(\mathbf{X})$ and $\phi : A \to B$, then ϕ has a kernel when everything is viewed in the category $\mathbf{Sh}(\mathbf{X})$. It remains only to check that $\ker(\phi)(U) = \{f \in A(U) \mid \phi_U(f) = 0_{B(U)}\}$ is an Abelian group for all U, open in X. But $\ker(\phi)(U) = \ker(\phi_U)$ is the kernel of the group homomorphism ϕ_U , and so must itself be a subgroup of A(U) for all $U \in X$. Then $\ker(\phi) \in \mathbf{Ab}(\mathbf{X})$. Likewise, the cokernel of ϕ exists in $\mathbf{Sh}(\mathbf{X})$, and so we need only show $\operatorname{coker}(\phi)(U)$ is an Abelian group, $\forall U$.

4.3 Let $0 \to A \hookrightarrow I^0 \to I^1 \to \dots$ and $0 \to B \to B \hookrightarrow J^0 \to J^1 \to J^2 \to \dots$ be injective resolutions of A and B respectively. Show that a map $\phi: A \to B$ extends to a map of complexes:

Since $0 \to A \hookrightarrow I^0 \to I^1 \to \dots$ is a resolution, in particular exact, for each n, we have an injective map from $A \to I^n$ given by d^n . Since $\phi : A \to B$, we also have for each n, a map from $A \to J^n$ given by $d^n \circ \phi$. Then injectivity of J^n implies the required map $\phi^n : I^n \to J^n$:



4.3 (b) Show that a map of complexes as above induces a homomorphism of cohomology groups

$$H^n(X;A) \to H^n(X,B)$$

Note that the above map of chain complexes implies a map for $U \in X$:

which for U = X gives the map:

$$0 \longrightarrow \Gamma A \longrightarrow \Gamma I^0 \longrightarrow \Gamma I^1 \longrightarrow \Gamma I^2 \longrightarrow \dots$$

$$\downarrow \Gamma \phi \qquad \downarrow \Gamma \phi^0 \qquad \downarrow \Gamma \phi^1 \qquad \downarrow \Gamma \phi^2$$

$$0 \longrightarrow \Gamma B \longrightarrow \Gamma J^0 \longrightarrow \Gamma J^1 \longrightarrow \Gamma J^2 \longrightarrow \dots$$

Since $H^n(X;A) = \ker(\Gamma I^n \to \Gamma I^{n+1}))/im(\Gamma I^{n-1} \to \Gamma I^n) \subseteq \Gamma I^n$ and likewise $H^n(X;B) = \ker(\Gamma J^n \to \Gamma J^{n+1}))/im(\Gamma J^{n-1} \to \Gamma J^n) \subseteq \Gamma J^n$, the maps $\Gamma \phi^n : \Gamma I^n \to \Gamma J^n$ implies a homomorphism $\Gamma \phi^n \mid_{H^n(X,A)}: H^n(X;A) \to \Gamma J^n$. It remains only to check that $im(\Gamma \phi^n \mid_{H^n(X,A)}) \subseteq H^n(X,B)$. Let $\alpha \in H^n(X;A)$. Then $\alpha \in \ker(d^n : \Gamma I^n \to \Gamma I^{n+1})$, so $\Gamma \phi^{n+1} \circ d^n(\alpha) = 0$. Since the above diagram commutes, this implies $d^n \circ \Gamma \phi^n(\alpha) = 0$, so that $\Gamma \phi^n(\alpha) \in \ker(d^n : \Gamma J^n \to \Gamma J^{n+1})$, which implies $\Gamma \phi^n(\alpha) \in H^n(X;B)$.

4.4(a) Let $f: Y \to X$ be a map of topological spaces. Show that there is a natural isomorphism $\Gamma_Y \to \Gamma_X \circ f_*$,

$$Ab(Y) \xrightarrow{\Gamma_X} Ab(X)$$
Abelian Groups

We must find η_A and η_B , isomorphisms, such that the following commutes for all $A, B \in Ab(Y)$ and $\phi : A \to B$:

$$\Gamma_{Y}(A) \xrightarrow{\Gamma_{Y}\phi} \Gamma_{Y}(B)$$

$$\eta_{A} \downarrow \qquad \qquad \downarrow \eta_{B}$$

$$\Gamma_{X} \circ f_{*}(A) \xrightarrow{\Gamma_{x} \circ f_{*}\phi} \Gamma_{X} \circ f_{*}(B)$$

By the definition of Γ and f_* , this is the same as finding isomorphisms η_A and η_B such that the following commutes:

$$A(Y) \xrightarrow{\phi_Y} B(Y)$$

$$\eta_A \downarrow \qquad \qquad \downarrow \eta_B$$

$$A(f^{-1}(X)) \xrightarrow{\phi_{f^{-1}(X)}} B(f^{-1}(X))$$

But $f^{-1}(X) = Y$, so taking $\eta_A = \rho_{Y,Y} = id_{A(Y)}$ (likewise for η_B) implies the result, since ϕ is a map $A \to B$ and so must be natural wrt the restriction maps.

4.4 (b) Show that a natural isomorphism between left exact functors $\tau: T_1 \to T_2$ induces a natural isomorphism $R^n T_1 \to R^n T_2$ for each n.

Let T_1 and T_2 be left exact functors from $\mathcal{C} \to \mathcal{D}$ and let $\tau: T_1 \to T_2$ be a natural isomorphism. Let $0 \to C \to I^0 \to I^1 \to ...$ be an injective resolution of C for some $C \in \mathcal{C}$. Naturality of τ implies that the following commutes:

$$0 \longrightarrow T_1(C) \longrightarrow T_1(I^0) \longrightarrow T_1(I^1) \longrightarrow T_1(I^2) \longrightarrow \dots$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow T_2(C) \longrightarrow T_2(I^0) \longrightarrow T_2(I^1) \longrightarrow T_2(I^2) \longrightarrow \dots$$

Moreover, since T_1 and T_2 are both left exact, each row in the above diagram is exact. Then, by analogous argument to 4.3(b), τ restricts to a natural isomorphism $R^nT_1 \to R^nT_2$ for each n.

Section 7

7.1 Prove Corollary 7.4 (Proper Base Change): For a pullback diagram as in (26) with f and f' proper with Hausdorff fibers, the canonical map

$$p^*(R^n f_* B) \to R^n f'_*(q^* B)$$

is an isomorphism for any sheaf B on Y.

We show $p^*(R^n f_* B)_{x'} \cong (R^n f'_*(q^* B))_{x'}$ for $x' \in X'$. Let $x' \in X'$, B be a sheaf on Y and let the following be a pullback diagram:

$$(f')^{-1}(x') \in Y' \xrightarrow{q} Y \ni f^{-1}(p(x'))$$

$$f' \downarrow \qquad \qquad \downarrow f$$

$$x' \in X' \xrightarrow{p} X \ni p(x')$$

Note that since the above is a pullback diagram, q gives an isomorphism $q:(f')^{-1}(x')\xrightarrow{\sim} f^{-1}(p(x'))$, which by Prop (4.3) induces an isomorphism $H^n(f^{-1}(p(x')); B\mid_{f^{-1}p(x')})\xrightarrow{\sim} H^n((f')^{-1}(x'); q^*B\mid_{(f')^{-1}(x')})$. Then we have:

$$p^{*}(R^{n}f_{*}B)_{x'} \cong (R^{n}f_{*}B)_{p(x')} \qquad \text{exercise } 3.2$$

$$\cong H^{n}(f^{-1}(p(x')); B \mid_{f^{-1}(p(x'))}) \qquad (7.3)$$

$$\cong H^{n}((f')^{-1}(x'); q^{*}B \mid_{(f')^{-1}(x')}) \qquad \text{pullback and } (4.3)$$

$$\cong (R^{n}f'_{*}(q^{*}B))_{x'} \qquad (7.3)$$

Section 8

8.1 Let X be connected and locally simply connected, and choose a base point x_0 . Show that for any locally constant sheaf (of sets) A, the group $\pi_1(X, x_0)$ acts on the stalk A_{x_0} . Show that this gives a functor $A \mapsto A_{x_0}$ (which is an equivalence of categories between the category of locally constant sheaves on X and the category of sets with a $\pi_1(X, x_0)$ -action.)

Let X be connected and locally simply connected. Let A be a locally constant sheaf, so that $\pi: Et(A) \to X$ is a covering map. Fix x_0 in X and let $p:[0,1] \to X$ be a loop in X based at x_0 . Consider $\pi^{-1}(x_0) \in Et(A)$. Since π is a covering map, there exists a neighborhood U_0 of x_0 in X and a set $S = \pi^{-1}(x_0)$ such that U_0 is evenly covered by π with $\pi^{-1}(U_0) = U_0 \times S$, where S has the discrete topology. (i.e., each $x \in X$ has a neighborhood whose preimage under π is a stack of pancakes with one pancake for each $s \in S = A_x$).

Note that for each $s \in S$ (that is, each point in the fiber of x_0 or equivalently each point in A_{x_0}), the fact that π is a covering map implies that $p:[0,1]\to X$ has a unique lifting, \tilde{p}_s , to a path in Et(A) beginning at s and ending at some $t\in S$ (Munkres Lemma 54.1). Moreover, if p_1 and p_2 are both loops based at x_0 and are homotopic, then the induced paths in Et(A) are homotopic with identical endpoints (Munkres Theorem 54.3). Now we can define an action of $\pi_1(X,x_0)$ on $A_{x_0}=\pi^{-1}(x_0)$ by $p\cdot s=\tilde{p}_s(1)$, where p is a loop based at x_0 , s is in the fiber of x_0 (ie s in the stalk of A_{x_0}) and \tilde{p}_s is the unique lift of p for s implied by the fact that π is a covering map. Note that if e is the constant loop, then $\tilde{e}_s(1)=s$, $\forall s$ so that $e\cdot s=\tilde{e}_s(1)=s$. Since the group operation, *, of $\pi(X,x_0)$ is composition of loops, we also have $p\cdot (q\cdot s)=p\cdot \tilde{q}_s(1)=\tilde{p}_{\tilde{q}_s(1)}(1)=(p*q)\cdot s$. Then $\pi(X,x_0)$ gives an action on A_{x_0} .

Let **LocConSh(X)** be the category of locally constant sheaves on X. Define a functor $F : \mathbf{LocConSh(X)} \to \mathbf{Set}$ by $A \mapsto A_{x_0}$. Note that since X is (path-) connected, this does not depend on the choice of base point, x_0 . Suppose $\phi : A \to B$ is an arrow in $\mathbf{LocConSh(X)}$. We must show:

$$A(U) \xrightarrow{F} A_{x_0}$$

$$\phi_U \downarrow \qquad \qquad \downarrow F \phi_U$$

$$B(U) \xrightarrow{F} B_{x_0}$$

8.2 Let S^d be a d-dimensional sphere. For any Abelian group A, prove that,

$$H^{n}(S^{d};A) = \begin{cases} A & \text{when } n = 0 \text{ or } n = d \text{ provided } d \neq 0; \\ A \oplus A & \text{when } d = n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Write S^d as the union of closed subspaces, N and S, the closed northern and southern hemispheres, $S^d = N \cup S$. Note that $S^{d-1} = N \cap S$. We proceed by induction of d and consider the cases $n \neq d$ and n = d separately. The cases n = 0 and d = 0 are obvious (since S^d is connected for d > 0 and S^0 has two components).

Suppose d > 1 and the result holds for d = 0. By inductive hypothesis, for $n \neq d$ and n > 0, $H^n(S^{d-1}; A|_{S^{d-1}}) = 0$. Then the Mayer-Vietoris sequence:

$$\dots \to H^{n-1}(S^{d-1}; A\mid_{s^{d-1}}) \to$$

$$H^{n}(S^{d}; A) \to H^{n}(N; A \mid_{N}) \oplus H^{n}(S; A \mid_{S}) \to H^{n}(S^{d-1}; A \mid_{S^{d-1}}) \to \dots$$

becomes:

$$\dots \to 0 \to H^n(S^d; A) \to H^n(N; A \mid_N) \oplus H^n(S; A \mid_S) \to 0 \to \dots$$

And so by exactness, we have

$$\dots \to 0 \to H^n(S^d; A) \xrightarrow{\sim} H^n(N; A \mid_N) \oplus H^n(S; A \mid_S) \to 0 \to \dots$$

It remains to show that $H^n(N; A|_N) \oplus H^n(S; A|_S) = 0.***$ Then $H^n(S^d; A) = 0$ for all $n \neq d$ and n > 0 and for all d.

We now consider the case n = d. By inductive hypothesis, we have $H^{n-1}(S^{d-1}; A) = H^{d-1}(S^{d-1}; A) = A$. So the Mayer-Vietoris sequence:

$$\ldots \to H^{d-1}(N;A\mid_N) \oplus H^{d-1}(S;A\mid_S) \to H^{d-1}(S^{d-1};A\mid_{S^{d-1}}) \to H^d(S^d;A) \to H^d(S^d;A) \to H^d(S^d;A)$$

becomes

$$\ldots \to H^{d-1}(N; A\mid_N) \oplus H^{d-1}(S; A\mid_S) \to A \to H^d(S^d; A) \to \ldots$$

But by above, this is

$$\dots \to 0 \to A \to H^d(S^d; A) \to \dots$$

which by exactness and the first isomorphism theorem implies $H^d(S^d;A)\cong A$ and so the result follows.