Exercises

Section 4

4.1 (a) Show that with the notation as above, $\Gamma(-,\pi)$ is indeed isomorphic to the sheaffication of P.

We first fix some notation. Suppose $U \subseteq X$ and let $s \in G(U)$ be a section. Let $\overline{s}$ be the equivalence class of s in $G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$, so that points in the étale space of P are denoted $germ_x(\overline{s})$. We denote by $\overline{germ_x(s)}$ the equivalence class of $germ_x(s)$ under the relation R given above, where $germ_x(s) \in Et(G)$.

To show $\Gamma(-,\pi) \cong P$, we define a map $\phi : \Gamma(-,\pi) \to P$ and show it induces an isomorphism on the stalks. For $U \subseteq X$ and $\sigma \in \Gamma(-,\pi)(U)$, define $\phi_U : \Gamma(-,\pi)(U) \to P(U)$ by $\phi_U(\sigma) = \tau$, where σ is a function $x \mapsto \overline{germ_x(s)}$ and τ is a function $x \mapsto germ_x(\overline{s})$ for some $s \in G(U)$. Naturality of ϕ is straightforward, since restriction maps are just restrictions of functions.

To see that ϕ is well defined, suppose $germ_x(t)$ is also a representative of $\overline{germ_x(s)}$. Then $\exists W \ni x$ and $f \in F(W)$ such that $s\,|_W - t\,|_W = \alpha_W(f)$. This however gives $germ_x(\overline{t}) = germ_x(\overline{s})$, since W is then a neighborhood for which $\overline{s\,|_W} = \overline{t\,|_W}$, so that they are equal as germs.

It remains to show that ϕ induces an isomorphism on the stalks. We show $\phi_x : \Gamma(-,\pi)_x \to P_x$ is injective and surjective. Fix an $x \in X$ and suppose $\phi(\overline{germ_x(a)}) = germ_x(\overline{a}) = \phi(\overline{germ_x(b)})$. We have to show $\exists W \ni x$ and $f \in F(W)$ such that $a\,|_W - b\,|_W = \alpha_W(f)$. To see that ϕ_x is surjective.

Let $\overline{germ_x(s)}$ be an element of $Et(G)/R$, and let $germ_x(s)$ be a representative, so that if $germ_x(t)$ is another representative of $\overline{germ_x(s)}$, then $\exists W \ni x$ and $f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$, where s and t are representatives of $germ_x(s)$ and $germ_x(t)$ respectively.

Let $germ_x(a)$ be an element of P_x, represented by (a, U) for some $U \ni x$ and $a \in P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U)) = G(U)/im(\alpha_U)$. If (b, V) is another representative of $germ_x(a)$, then there exists $W \ni x$ such that $a\mid_W = b\mid_W$, i.e., $a\mid_W$ and $b\mid_W$ are in the same equivalence class of $G(W)/im(\alpha_W)$, i.e., $\exists f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$. Then elements of G_x/R and P_x are both correspond to elements of G_x identified via the same relation.

Let $\overline{germ_x(s)}$ be an element of G_x/R, and let $germ_x(s)$ be a representative, so that if $germ_x(t)$ is another representative of $\overline{germ_x(s)}$, then $\exists W \ni x$ and $f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$, where s and t are representatives of $germ_x(s)$ and $germ_x(t)$ respectively.

Let $germ_x(a)$ be an element of P_x, represented by (a, U) for some $U \ni x$ and $a \in P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U)) = G(U)/im(\alpha_U)$. If (b, V) is another representative of $germ_x(a)$, then there exists $W \ni x$ such that $a\mid_W = b\mid_W$, i.e., $a\mid_W$ and $b\mid_W$ are in the same equivalence class of $G(W)/im(\alpha_W)$, i.e., $\exists f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$. Then elements of G_x/R and P_x are both correspond to elements of G_x identified via the same relation.

4.1(b) Show that the $coker(\alpha)$ has the following universal property: given any other sheaf H, a map $\chi : G \to H$ factors through $coker(\alpha)$ if and only if $\chi \circ \alpha = 0$.

(Forward direction) Let $\alpha : F \to G$ and $\chi : G \to H$ and suppose $\chi \circ \alpha = 0$. Then for $U \in X$, $ker(\chi_U) = im(\alpha_U)$. By the first isomorphism theorem, $\chi_U(G) \cong G(U)/ker(\chi_U) = G(U)/im(\alpha_U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$, so that $\chi = g \circ f$, where $f : G \to coker(\alpha)$ is the projection map and $g : coker(\alpha) \to H$ is an isomorphism.

(Backwards direction) Let $\alpha : F \to G$ and $\chi : G \to H$ and suppose $\chi = g \circ f$ for some $f : G \to coker(\alpha)$ and $g : coker(\alpha) \to H$. Note that $f \circ \alpha = 0$, since f_U has codomain $coker(F(U) \xrightarrow{\alpha_U} G(U)$ for $U \in X$. Then $\chi \circ \alpha = g \circ f \circ \alpha = g \circ 0 = 0$.

4.1(c) Show that $coker(\alpha)_x = coker(\alpha_x)$.

Let $\alpha : F \to G$ for F and G Abelian sheaves on X and let P be the presheaf given by $P(U) = coker(F(U) \xrightarrow{\alpha_U} G(U))$. Since $\pi_P : Et(P) \to X$ is étale, $Et\Gamma EtP \cong EtP$, so the left-hand side is isomorphic to P_x. On the right-hand side, note that $coker(\alpha_x) = coker(F_x \xrightarrow{\alpha_x} G_x) = G_x/Im(\alpha_x)$, so that two points are equivalent in $coker(\alpha_x)$ iff they are both points in G_x, say $germ_x(s)$ and $germ_y(t)$, with $germ_x(s) - germ_y(t) \in Im(\alpha_X)$, ie $\exists W \ni x$ and $f \in F(W)$ such that $s\mid_W - t\mid_W = \alpha_W(f)$. This is exactly the relation given by R in the notes, so that the right-hand side is isomorphic to G_x/R. Then exercise 4.1(a) implies $coker(\alpha)_x \cong coker(\alpha_x)$.

4.2 Show that $\mathbf{Ab(X)}$ is an Abelian category with kernel, cokernel and sum defined as above.

We first show that $\mathbf{Ab(X)}$ has a 0 object. Let $\Delta\{*\}$ be the constant sheaf on a singleton set. Then for $U \subseteq X$, $\Delta\{*\}(U)$ is the trivial group and every restriction map is the identity, so $\Delta\{*\} \in \mathbf{Ab(X)}$. Moreover, for $A \in \mathbf{Ab(X)}$, there is exactly one map from $A(U) \to \Delta\{*\}(U)$ for every open $U \subseteq X$, namely the constant 0-map, and there is also exactly one map from $\Delta\{*\}(U) \to A(U)$ since group homomorphism must send identity to identity. Then $\Delta\{*\}$ is both initial and terminal, and so serves as the 0 element in $\mathbf{Ab(X)}$. We can now define the 0 morphism in $\mathbf{Ab(X)}$ as the map defined for $A, B \in \mathbf{Ab(X)}$ and $U \in X$, open, by $a \in A(U) \mapsto 0_{B(U)}$.

Let us define the sum of two Abelian sheaves A and B by $A(U) \oplus B(U)$ for U open in X, as in the notes. Then this gives the biproduct of A and B in $\mathbf{Ab(X)}$. We know from the notes that $A \oplus B$ is a sheaf and we observe that for $U \subseteq X$, $A(U) \oplus B(U)$ has an Abelian group structure: let (a, b) and (a', b') be elements of $A(U) \oplus B(U)$ for some U. Then $(a, b) + (a', b') = (a + a', b + b') = (a' + a, b' + b) = (a', b') + (a, b) \in A(U) \oplus B(U)$, since $A(U)$ and $B(U)$ are Abelian groups. This also implies $(a, b) + (0_{A(U)}, 0_{B(U)}) = (a, b) = (0_{A(U)}, 0_{B(U)}) + (a, b)$. To see that this is a biproduct, let $p_j : A_1 \oplus A_2 \to A_j$ be the map defined on $U \subseteq X$ as the projection map $p_{U,j} : A_1(U) \oplus A_2(U) \to A_j(U)$ and i_j be the map defined on $U \subseteq X$ as the injection map $i_j(U) : A_j(U) \to A_1(U) \oplus A_2(U)$ for $j \in \{1, 2\}$ and $A_j \in \mathbf{Ab(X)}$. Then since $p_{U,j}$ and $i_{U,j}$ give the product and coproduct on each $A_1(U) \oplus A_2(U)$, viewed as elements in the category of Abelian groups, their collection gives the product and coproduct on $A_1 \oplus A_2$ (given a map into or out of the biproduct, find a new map that factors through p_j or i_j respectively, by taking the collection of maps implied by the universal property of each $p_{U,j}$ or $i_{U,j}$).

To see that $\mathbf{Ab(X)}$ has kernels, note that if $A, B \in \mathbf{Ab(X)}$ and $\phi : A \to B$, then ϕ has a kernel when everything is viewed in the category $\mathbf{Sh(X)}$. It remains only to check that $ker(\phi)(U) = \{f \in A(U) \mid \phi_U(f) = 0_{B(U)}\}$ is an Abelian group for all U, open in X. But $ker(\phi)(U) = ker(\phi_U)$ is the kernel of the group homomorphism ϕ_U, and so must itself be a subgroup of $A(U)$ for all $U \in X$. Then $ker(\phi) \in \mathbf{Ab(X)}$. Likewise, the cokernel of ϕ exists in $\mathbf{Sh(X)}$, and so we need only show $coker(\phi)(U)$ is an Abelian group, $\forall U$.

4.3 Let $0 \to A \hookrightarrow I^0 \to I^1 \to ...$ and $0 \to B \to B \hookrightarrow J^0 \to J^1 \to J^2 \to ...$ be injective resolutions of A and B respectively. Show that a map $\phi : A \to B$ extends to a map of complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "\phi"] & I^0 \arrow[r] \arrow[d, dashed, "\phi^0"] & I^1 \arrow[r] \arrow[d, dashed, "\phi^1"] & I^2 \arrow[r] \arrow[d, dashed, "\phi^2"] & ... \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & ...
\end{tikzcd}$$

Since $0 \to A \hookrightarrow I^0 \to I^1 \to ...$ is a resolution, in particular exact, for each n, we have an injective map from $A \to I^n$ given by d^n. Since $\phi : A \to B$, we also have for each n, a map from $A \to J^n$ given by $d^n \circ \phi$. Then injectivity of J^n implies the required map $\phi^n : I^n \to J^n$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "d^n"] \arrow[d, "d^n \circ \phi"'] & I^n \arrow[dl, dashed, "\phi^n"] \\
& J^n &
\end{tikzcd}$$

4.3 (b) Show that a map of complexes as above induces a homomorphism of cohomology groups

$$H^n(X;A) \to H^n(X,B)$$

Note that the above map of chain complexes implies a map for $U \in X$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A(U) \arrow[r] \arrow[d, "\phi_U"] & I^0(U) \arrow[r] \arrow[d, dashed, "\phi_U^0"] & I^1(U) \arrow[r] \arrow[d, dashed, "\phi_U^1"] & I^2(U) \arrow[r] \arrow[d, dashed, "\phi_U^2"] & ... \\
0 \arrow[r] & B(U) \arrow[r] & J^0(U) \arrow[r] & J^1(U) \arrow[r] & J^2(U) \arrow[r] & ...
\end{tikzcd}$$

which for $U = X$ gives the map:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \Gamma A \arrow[r] \arrow[d, "\Gamma\phi"] & \Gamma I^0 \arrow[r] \arrow[d, dashed, "\Gamma\phi^0"] & \Gamma I^1 \arrow[r] \arrow[d, dashed, "\Gamma\phi^1"] & \Gamma I^2 \arrow[r] \arrow[d, dashed, "\Gamma\phi^2"] & ... \\
0 \arrow[r] & \Gamma B \arrow[r] & \Gamma J^0 \arrow[r] & \Gamma J^1 \arrow[r] & \Gamma J^2 \arrow[r] & ...
\end{tikzcd}$$

Since $H^n(X;A) = ker(\Gamma I^n \to \Gamma I^{n+1}))/im(\Gamma I^{n-1} \to \Gamma I^n) \subseteq \Gamma I^n$ and likewise $H^n(X;B) = ker(\Gamma J^n \to \Gamma J^{n+1}))/im(\Gamma J^{n-1} \to \Gamma J^n) \subseteq \Gamma J^n$, the maps $\Gamma\phi^n : \Gamma I^n \to \Gamma J^n$ implies a homomorphism $\Gamma\phi^n \mid_{H^n(X,A)}: H^n(X;A) \to \Gamma J^n$. It remains only to check that $im(\Gamma\phi^n \mid_{H^n(X,A)}) \subseteq H^n(X,B)$. Let $\alpha \in H^n(X;A)$. Then $\alpha \in ker(d^n : \Gamma I^n \to \Gamma I^{n+1})$, so $\Gamma\phi^{n+1} \circ d^n(\alpha) = 0$. Since the above diagram commutes, this implies $d^n \circ \Gamma\phi^n(\alpha) = 0$, so that $\Gamma\phi^n(\alpha) \in ker(d^n : \Gamma J^n \to \Gamma J^{n+1})$, which implies $\Gamma\phi^n(\alpha) \in H^n(X;B)$.

4.4(a) Let $f : Y \to X$ be a map of topological spaces. Show that there is a natural isomorphism $\Gamma_Y \to \Gamma_X \circ f_*$,

$$\begin{tikzcd} Ab(Y) \arrow[rr, "f_*"] \arrow[dr, "\Gamma_Y"'] & & Ab(X) \arrow[dl, "\Gamma_X"] \\ & \textbf{Abelian Groups} & \end{tikzcd}$$

We must find η_A and η_B, isomorphisms, such that the following commutes for all $A, B \in Ab(Y)$ and $\phi : A \to B$:

$$\begin{tikzcd} \Gamma_Y(A) \arrow[r, "\Gamma_Y \phi"] \arrow[d, "\eta_A"'] & \Gamma_Y(B) \arrow[d, "\eta_B"] \\ \Gamma_X \circ f_*(A) \arrow[r, "\Gamma_x \circ f_* \phi"'] & \Gamma_X \circ f_*(B) \end{tikzcd}$$

By the definition of $\Gamma_{\_}$ and f_*, this is the same as finding isomorphisms η_A and η_B such that the following commutes:

$$\begin{tikzcd} A(Y) \arrow[r, "\phi_Y"] \arrow[d, "\eta_A"'] & B(Y) \arrow[d, "\eta_B"] \\ A(f^{-1}(X)) \arrow[r, "\phi_{f^{-1}(X)}"'] & B(f^{-1}(X)) \end{tikzcd}$$

But $f^{-1}(X) = Y$, so taking $\eta_A = \rho_{Y,Y} = id_{A(Y)}$ (likewise for η_B) implies the result, since ϕ is a map $A \to B$ and so must be natural wrt the restriction maps.

4.4 (b) Show that a natural isomorphism between left exact functors $\tau : T_1 \to T_2$ induces a natural isomorphism $R^nT_1 \to R^nT_2$ for each n.

Let T_1 and T_2 be left exact functors from $\mathcal{C} \to \mathcal{D}$ and let $\tau : T_1 \to T_2$ be a natural isomorphism. Let $0 \to C \to I^0 \to I^1 \to ...$ be an injective resolution of C for some $C \in \mathcal{C}$. Naturality of τ implies that the following commutes:

$$\begin{tikzcd}
0 \arrow[r] & T_1(C) \arrow[r] \arrow[d, "\wr"'] & T_1(I^0) \arrow[r] \arrow[d, "\wr"'] & T_1(I^1) \arrow[r] \arrow[d, "\wr"'] & T_1(I^2) \arrow[r] \arrow[d, "\wr"'] & ... \\
0 \arrow[r] & T_2(C) \arrow[r] & T_2(I^0) \arrow[r] & T_2(I^1) \arrow[r] & T_2(I^2) \arrow[r] & ...
\end{tikzcd}$$

Moreover, since T_1 and T_2 are both left exact, each row in the above diagram is exact. Then, by analogous argument to 4.3(b), τ restricts to a natural isomorphism $R^nT_1 \to R^nT_2$ for each n.