The intervall $\tau^{(1)} = [0, x_1] \times \{0\}$ is parameterised by $t \in [0, x_1]$.
The intervall $\tau^{(2)}$ is parameterised by $s \in [0, 1]$ as follows

$$y \in \tau^{(2)} \Rightarrow \exists s \in [0,1] \text{ with } y = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix} s$$

Set $r = (x_3 - x_2)^2 + (y_3 - y_2)^2$.

$$\begin{aligned}
I &= \int_{y\in\tau^{(2)}} \int_{x\in\tau^{(1)}} \log(\|y - x\|) dx dy \\
&= \sqrt{r} \int_{s\in[0,1]} \int_{t\in[0,x_1]} \log\left(\left\| \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix} s - \begin{pmatrix} t \\ 0 \end{pmatrix} \right\| \right) dt ds \\
&= \frac{\sqrt{r}}{2} \int_{s\in[0,1]} \int_{t\in[0,x_1]} \log\left((x_2 + (x_3 - x_2)s - t)^2 + (y_2 + (y_3 - y_2)s)^2 \right) dt ds
\end{aligned}$$

Set $a = y_2 + (y_3 - y_2)s$ and $b = x_2 + (x_3 - x_2)s$ then we need to distinguish between two cases:

$a = 0$:

$$\begin{aligned}
I &= \frac{\sqrt{r}}{2} \int_{s\in[0,1]} \int_{t\in[0,x_1]} \log\left((b - t)^2 \right) dt ds \\
&= \frac{\sqrt{r}}{2} \int_{s\in[0,1]} b \log(b^2) - (b - 1) \log((b - 1)^2) - 2 ds \\
&= \frac{\sqrt{r}}{2} \left(\int_{s\in[0,1]} b \log(b^2) ds - \int_{s\in[0,1]} (b - 1) \log((b - 1)^2) ds \right) - \sqrt{r} \\
&= \frac{\sqrt{r}}{2(x_3 - x_2)} \left(\int_{b\in[x_2,x_3]} b \log(b^2) db - \int_{b\in[x_2,x_3]} (b - 1) \log((b - 1)^2) db \right) - \sqrt{r} \\
&= \frac{\sqrt{r}}{2(x_3 - x_2)} \left(\frac{1}{2} \left[b^2 \log(b^2) - (b - 1)^2 \log((b - 1)^2) + (1 - 2b) \right]_{x_2}^{x_3} \right) - \sqrt{r}
\end{aligned}$$

and $a \neq 0$:

$$\begin{aligned}
I &= \frac{\sqrt{r}}{2} \int_{s\in[0,1]} \int_{t\in[0,x_1]} \log\left((b - t)^2 + a^2 \right) dt ds \\
&= \frac{\sqrt{r}}{2} \int_{s\in[0,1]} \left[2a \arctan(\frac{t - b}{a}) + (t - b) \log((t - b)^2 + a^2) - 2(t - b) \right]_0^{x_1} ds \\
&= \frac{\sqrt{r}}{2} \int_{s\in[0,1]} \Big(2a \arctan(\frac{x_1 - b}{a}) + (x_1 - b) \log((x_1 - b)^2 + a^2) - 2(x_1 - b) \\
&\qquad - (-2a \arctan(\frac{b}{a}) - b \log(b^2 + a^2) + 2b) \Big) ds \\
&= \frac{\sqrt{r}}{2} \Big(\int_{s\in[0,1]} 2a \left(\arctan\left(\frac{b}{a}\right) + \arctan\left(\frac{x_1 - b}{a}\right) \right) ds + \int_{s\in[0,1]} (x_1 - b) \log((x_1 - b)^2 + a^2) ds \\
&\qquad + \int_{s\in[0,1]} b \log(b^2 + a^2) ds - 2x_1 \Big)
\end{aligned}$$

We solve for each summand individually:

- First we need the following two integrals for arbitrary $m, n \in \mathbb{R}$:

$$\begin{aligned}
\int_{s\in[0,1]} \log((s - m)^2 + n^2) ds &= \int_{u\in[-m,1-m]} \log(u^2 + n^2) du \\
&= \left[u \log(u^2 + n^2) - 2u + 2n \arctan\left(\frac{u}{n}\right) \right]_{-m}^{1-m} \\
&= (1 - m) \log((1 - m)^2 + n^2) - 2(1 - m) + 2n \arctan\left(\frac{1 - m}{n}\right) \\
&\qquad - (-m) \log(m^2 + k^2) + 2(-m) - 2n \arctan\left(\frac{-m}{n}\right) \\
&= (1 - m) \log((1 - m)^2 + n^2) - 2 + 2n \arctan\left(\frac{1 - m}{n}\right) + m \log(m^2 + n^2) + 2n \arctan\left(\frac{m}{n}\right)
\end{aligned}$$

and

$$\int_{s\in[0,1]} s\log((s-m)^2+n^2)ds = \int_{u\in[-m,1-m]} (u+m)\log(u^2+n^2)du$$
$$= \left[\frac{1}{2}((2mx+n^2+x^2)\log(n^2+x^2)+4mn\arctan(\frac{x}{n})-x(4m+x))\right]_{-m}^{1-m}$$
$$= \frac{1}{2}((2m(1-m)+n^2+(1-m)^2)\log(n^2+(1-m)^2)+4mn\arctan(\frac{1-m}{n})-(1-m)(3m+1))$$
$$-\left(\frac{1}{2}((-m^2+n^2)\log(n^2+m^2)-4mn\arctan(\frac{m}{n})+3m^2)\right)$$
$$= \frac{1}{2}(n^2-m^2+1)\log(n^2+(1-m)^2)-\frac{1}{2}(n^2-m^2)\log(n^2+m^2)$$
$$+2mn\left(\arctan(\frac{1-m}{n})+\arctan(\frac{m}{n})\right)-\frac{1}{2}(2m+1)$$

-
$$I_1 = \int_{s\in[0,1]} (x_1-b)\log((x_1-b)^2+a^2)ds$$
$$= (x_1-x_2)\int_{s\in[0,1]} \log\left(((x_1-x_2)^2+y_2^2)+(2(x_1-x_2)(x_2-x_3)+2y_2(y_3-y_2))s+rs^2\right)ds$$
$$+(x_2-x_3)\int_{s\in[0,1]} s\log((x_1-b)^2+a^2)ds$$

We set
$$m = -\frac{(x_1-x_2)(x_2-x_3)+y_2(y_3-y_2)}{r}$$
and
$$n = \frac{(x_1-x_2)(y_3-y_2)-y_2(x_2-x_3)}{r}$$
and get:

$$I_1 = (x_1-x_2)\int_{s=0}^{1}(\log(r)+\log((s-m)^2+n^2))ds-(x_3-x_2)\int_{s=0}^{1} s(\log(r)+\log((s-m)^2+n^2))ds$$
$$= (x_1-x_2)\left((1-m)\log((1-m)^2+n^2)-2+2n\arctan\left(\frac{1-m}{n}\right)+m\log(m^2+n^2)+2n\arctan\left(\frac{m}{n}\right)\right)$$
$$-(x_3-x_2)\left(\frac{1}{2}(n^2-m^2+1)\log(n^2+(1-m)^2)-\frac{1}{2}(n^2-m^2)\log(n^2+m^2)\right.$$
$$\left.+2mn\left(\arctan(\frac{1-m}{n})+\arctan(\frac{m}{n})\right)-\frac{1}{2}(2m+1)\right)+(x_1-x_2)\log(r)-\frac{1}{2}(x_3-x_2)\log(r)$$

if $n\neq 0$. In the case $n=0$ we get:

$$I_1 = (x_1-x_2)\int_{s=0}^{1}(\log(r)+2\log(s-m))ds-(x_3-x_2)\int_{s=0}^{1} s(\log(r)+2\log(s-m))ds$$
$$= (x_1-x_2)(\log(r)+2\int_{s=0}^{1}\log(s-m))ds)-(x_3-x_2)(\frac{1}{2}\log(r)+2\int_{s=0}^{1}\log(s-m))ds)$$
$$= (x_1-x_2)(\log(r)-2(m-1)\log(1-m)+2m\log(-m)-2)$$
$$-(x_3-x_2)(\frac{1}{2}\log(r)+\frac{1}{2}(2m^2\log(-m)-2(m^2-1)\log(1-m)-2m-1))$$

-
$$I_2 = \int_{s\in[0,1]} b\log(b^2+a^2)ds$$
$$= x_2\int_{s\in[0,1]}\log(b^2+a^2)ds+(x_2-x_3)\int_{s\in[0,1]} s\log(b^2+a^2)ds$$

Let

$$e = \frac{x_2(x_3 - x_2) + y_2(y_3 - y_2)}{r}$$

and

$$f = \frac{y_2 x_3 - x_2 y_3}{r}.$$

Then

$$\begin{aligned}
I_2 &= x_2 \int_{s=0}^{1} (\log(r) + \log((s-e)^2 + f^2))ds + (x_3 - x_2) \int_{s=0}^{1} s(\log(r) + \log((s-e)^2 + f^2))ds \\
&= x_2 \left((1-e)\log((1-e)^2 + f^2) - 2 + 2f \arctan\left(\frac{1-e}{f}\right) + e\log(e^2 + f^2) + 2e \arctan\left(\frac{e}{f}\right) \right) \\
&\quad + (x_3 - x_2) \Bigg(\frac{1}{2}(f^2 - e^2 + 1)\log(f^2 + (1-e)^2) - \frac{1}{2}(f^2 - e^2)\log(f^2 + e^2) \\
&\quad + 2fe\left(\arctan(\frac{1-e}{f}) + \arctan(\frac{e}{f})\right) - \frac{1}{2}(2e+1) \Bigg) + \frac{1}{2}(x_3 + x_2)\log(r)
\end{aligned}$$

if $f \neq 0$. In the case $f = 0$ we get:

$$\begin{aligned}
I_1 &= x_2 \int_{s=0}^{1} (\log(r) + 2\log(s-e))ds + (x_3 - x_2) \int_{s=0}^{1} s(\log(r) + 2\log(s-e))ds \\
&= x_2(\log(r) + 2\int_{s=0}^{1} \log(s-e))ds) + (x_3 - x_2)(\frac{1}{2}\log(r) + 2\int_{s=0}^{1} \log(s-e))ds) \\
&= x_2(\log(r) - 2(e-1)\log(1-e) + 2e\log(-e) - 2) \\
&\quad + (x_3 - x_2)(\frac{1}{2}\log(r) + \frac{1}{2}(2e^2\log(-e) - 2(e^2-1)\log(1-e) - 2e - 1))
\end{aligned}$$

- For the next two integrals we need the formulas

$$\int \arctan(\frac{a+bs}{c+ds})ds = \frac{1}{d}\int_{c}^{c+d} \arctan((a - bc/d)t^{-1} + b/d)dt$$

$$\begin{aligned}
\int_{s=0}^{1} s\arctan(\frac{a+bs}{c+ds})ds &= \frac{1}{d^2}\int_{t=c}^{c+d} (t-c)\arctan\left((a-bc/d)t^{-1} + b/d\right)dt \\
&= \frac{1}{d^2}\int_{t=c}^{c+d} (t-c)\arctan\left((a-bc/d)t^{-1} + b/d\right)dt
\end{aligned}$$

We use

$$\begin{aligned}
&\int \arctan\left(as^{-1} + b\right)ds \\
&\qquad = s\arctan(h + g/s) - \frac{2gh}{2(h^2+1)}\arctan(h + \frac{(h^2+1)}{g}s) \\
&\qquad\quad + \frac{g}{2(h^2+1)}\log\left(s^2 + \frac{2gh}{(h^2+1)}s + \frac{g^2}{(h^2+1)}\right)
\end{aligned}$$

$$\begin{aligned}
&\int \arctan\left(as^{-1} + b\right)sds \\
&\qquad = \frac{1}{2}s^2\arctan(h + g/s) - \frac{g(g - gh^2)}{(2(h^2+1)^2)}\arctan(h + \frac{(h^2+1)}{g}s) \\
&\qquad\quad - \frac{g^2 h}{(2(h^2+1)^2)}\log((h^2+1)s^2 + 2ghs + g^2) + \frac{g}{(2(h^2+1))}s
\end{aligned}$$

Further

$$\int \arctan\left(\frac{a+bs}{c}\right)ds = \frac{1}{2b}\left(2(a+bs)\arctan(\frac{a+bs}{c}) - c\log(a^2 + 2abs + b^2s^2 + c^2)\right)$$

and

$$\begin{aligned}
&\int s\arctan\left(\frac{a+bs}{c}\right)sds \\
&= \frac{1}{2b^2}(ac\log(a^2 + 2abs + b^2s^2 + c^2) + (a^2 - c^2)\arctan(\frac{c}{a+bs}) + b^2s^2\arctan(\frac{a+bs}{c}) - bcs)
\end{aligned}$$

-
$$
\begin{aligned}
I_3 &= 2\int_{s=0}^{1} a\arctan(\frac{x_1-b}{a})ds \\
&= 2\left(y_2\int_{s=0}^{1} \arctan(\frac{x_1-b}{a})ds + (y_3-y_2)\int_{s=0}^{1} s\arctan(\frac{x_1-b}{a})ds\right)
\end{aligned}
$$

  and

$$
\begin{aligned}
I_4 &= 2\int_{s=0}^{1} a\arctan(\frac{b}{a})ds \\
&= 2\left(y_2\int_{s=0}^{1} \arctan(\frac{b}{a})ds + (y_3-y_2)\int_{s=0}^{1} s\arctan(\frac{b}{a})ds\right)
\end{aligned}
$$

  Now we can apply the equations given above.

It follows:

$$
I = \frac{\sqrt{r}}{2}(I_1 + I_2 + I_3 + I_4 - 2x_1)
$$