1 Tests with $\log((y-x)^2)$

$$I = \int_{x=a}^{b} \int_{y=c}^{d} \log((y-x)^2) dy dx.$$

The exact solution to this integral is

1. $$I = -\frac{1}{2}((b-c)^2(3-\log((b-c)^2)) - (a-c)^2(3-\log((a-c)^2)) - (a-b)^2(3-\log((a-b)^2)))$$

   if $a = d$.

   Input:

   ```
   a=0; b=1; c=-1;
   vertexlist=[a b c];
   F = @(x,y) log((x-y).^2);
   [t,wt]=squad1d(0,nr, 2*nr,vertexlist);
   Q=sum(F(t(:,1), t(:,2)).*wt);
   ```

2. $$I = -(a-b)^2(3-\log((a-b)^2))$$

   if $a = c$ and $b = d$.

   Input:

   ```
   a=0; b=1;
   vertexlist=[a b];
   F = @(x,y) log((x-y).^2);
   [t,wt]=squad1d(0,nr, 2*nr,vertexlist);
   Q=sum(F(t(:,1), t(:,2)).*wt);
   ```

For the case of identical elements we used $a = c = 0$ and $b = d = 1$. Figure 1 shows the convergence of the quadrature to the expected value of $I = -(0-1)^2(3-\log((0-1)^2) = -3$.

For the case of elements with a common vertec we used $a = d = 0$, $b = 1$ and $c = -1$. Firgure 1 shows the convergence of the quadrature to the expected value of $I = -\frac{1}{2}((1+1)^2(3-\log((1+1)^2)) - (1)^2(3-\log((1)^2)) - (-1)^2(3-\log((-1)^2))) = -3+2\log(4)$.

2 Tests with $\log(\|y-x\|)y^2$ and $\log(\|y-x\|)$

1. Identical Elements $\tau^{(1)} = \tau^{(2)} = [0,1]$ (vertexlist 1):

   $$I = \int_{x=0}^{1} \int_{y=0}^{1} \log(|y-x|) y^2 dy dx.$$

   The exact solution is $I = -35/72$. The convergence against this value is plotted in Figure 2.

   Input:

   ```
   vertexlist=[0 1];
   F = @(x,y) log(abs(x-y)).*y.^2;
   [t,wt]=squad1d(0,nr, 2*nr,vertexlist);
   Q=sum(F(t(:,1), t(:,2)).*wt);
   ```

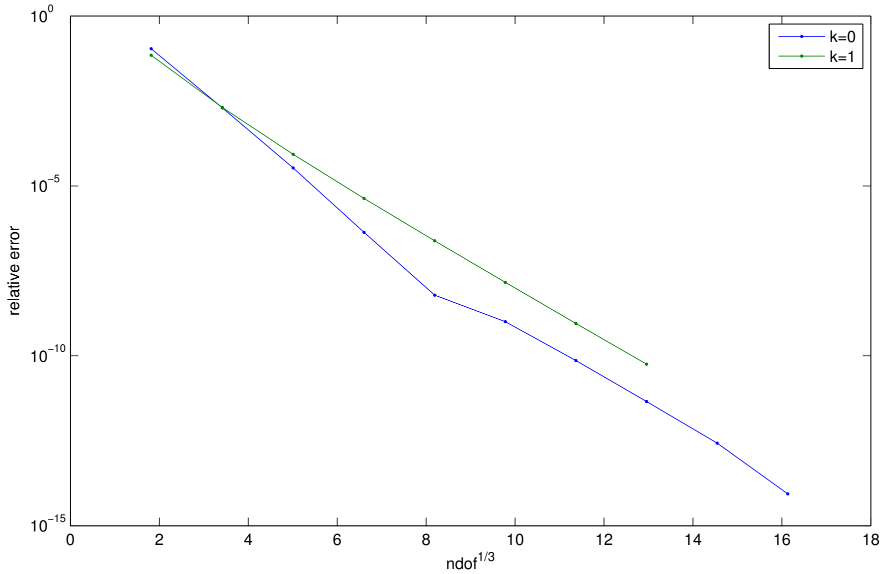

Figure 1: Convergence of the quadrature with the integrand $\log((y-x)^2)$ for identical elements and elements with a common vertex. ndof$^{1/2}$ is plotted against the logarithm of the relative error.

2. Common Vertex $\tau^{(1)} = [0,1]$ and $\tau^{(2)} = [-1,0]$ (vertexlist 2):

$$I = \int_{x=0}^{1}\int_{y=0}^{1} \log(|1+y-x|)(y+1)^2 dydx = \frac{1}{72}(-29 + 48\log(2)).$$

The convergence against this value is plotted in Figure 2.

Input:

```
vertexlist=[0 1 -1];
F = @(x,y) log(abs(x-y)).*y.^2;
[t,wt]=squad1d(0,nr, 2*nr,vertexlist);
Q=sum(F(t(:,1), t(:,2)).*wt);
```

3. Common Vertex $\tau^{(1)} = [0,1]\times\{0\}$ and $\tau^{(2)} = \{0\}\times[0,1]$ (vertexlist 3):

$$I = \int_{x=0}^{1}\int_{y=0}^{1} \log((x^2+y^2)^{1/2})dydx = \frac{1}{4}(-6 + \pi + \log(4)).$$

The convergence against this value is plotted in Figure 2.

Input:

```
vertexlist=[0 0; 1 0; 0 1]';
F = @(x,y) log(sqrt((y(:,1)-x(:,1)).^2+(y(:,2)-x(:,2)).^2));
[t,wt]=squad1d(0,nr, 2*nr,vertexlist);
Q=sum(F(t(:,1:2), t(:,3:4)).*wt);
```

4. Common Vertex $\tau^{(1)} = [0,1] \times \{0\}$ and $\tau^{(2)} = \{(0,0), (\cos(\beta), \sin(\beta))\}$ (vertexlist 4):

$$
\begin{aligned}
I &= \int_{x\in\tau^{(1)}} \int_{y\in\tau^{(2)}} \log(\|y-x\|)dydx \\
&= \frac{1}{2}\int_{s=0}^{1}\int_{t=0}^{1} \log(s^2+t^2-2st\cos(\beta))dtds \\
&= \int_{s=0}^{1}\int_{t=0}^{s} \log(s^2+t^2-2st\cos(\beta))dtds \\
&= 2\int_{s=0}^{1} s\log(s)ds + \int_{s=0}^{1}\int_{t=0}^{s} \log\left(1+\frac{t^2}{s^2}-2\frac{t}{s}\cos(\beta)\right)dtds \\
&= -\frac{1}{2} + \int_{s=0}^{1}\int_{z=0}^{1} \log(1+z^2-2z\cos(\beta))sdzds \\
&= -\frac{1}{2} + \frac{1}{2}\int_{z=0}^{1} \log(1+z^2-2z\cos(\beta))dz \\
&= -\frac{1}{2} + \frac{1}{2}\int_{z=0}^{1} \log((z-\cos(\beta))^2+\sin^2(\beta))dz \\
&= -\frac{1}{2} + \frac{1}{2}\int_{z=-\cos(\beta)}^{1-\cos(\beta)} \log(u^2+\sin^2(\beta))du \\
&= -\frac{1}{2} + \frac{1}{2}\left[u\log(u^2+\sin^2(\beta)) - 2u + 2\sin(\beta)\arctan\left(\frac{u}{\sin(\beta)}\right)\right]_{-\cos(\beta)}^{1-\cos(\beta)} \\
&-\frac{1}{2} + \frac{1}{2}\left((1-\cos(\beta))(\log(2-2\cos(\beta))) - 2 + 2\sin(\beta)\left(\arctan(\frac{1-\cos(\beta)}{\sin(\beta)}) - \arctan(-\frac{\cos(\beta)}{\sin(\beta)})\right)\right)
\end{aligned}
$$

We choose $\beta = 32^\circ = 8\pi/45$ and get

$$I = -0.906076846593465$$

The convergence against this value is plotted in Figure 2.

Input:

```
vertexlist=[0 0; 1 0; cosd(32) sind(32)];
F = @(x,y) log(sqrt((y(:,1)-x(:,1)).^2+(y(:,2)-x(:,2)).^2));
[t,wt]=squad1d(0,nr, 2*nr,vertexlist);
Q=sum(F(t(:,1:2), t(:,3:4)).*wt);
```

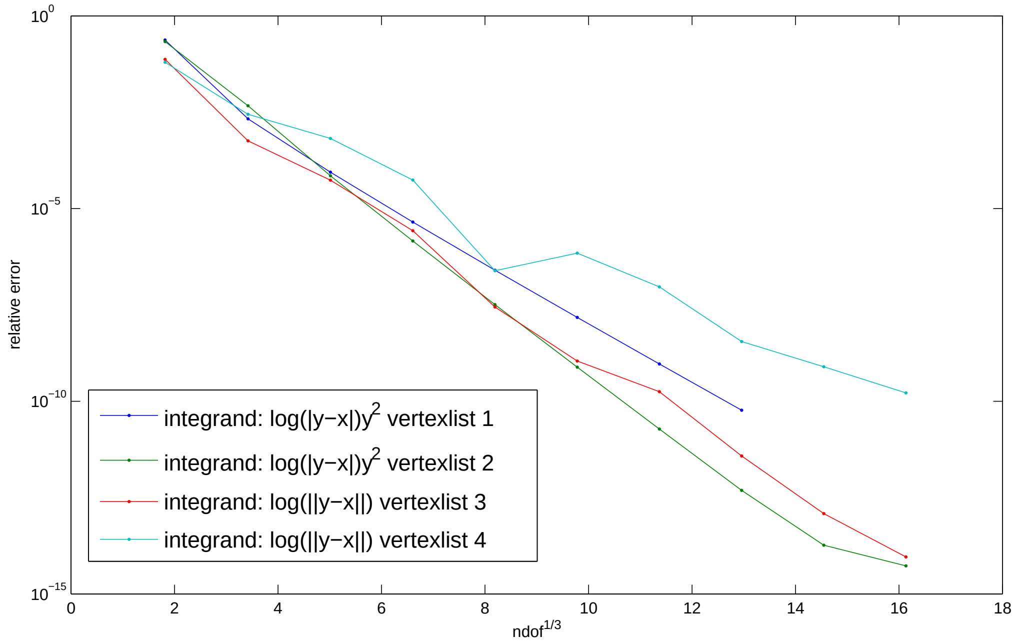

Figure 2: Convergence of the quadrature for the integrands $\log(\|y-x\|)y^2$ and $\log(\|y-x\|)$ for identical elements and elements with a common vertex. ndof$^{1/2}$ is plotted against the logarithm. of the relative error.