

An Overview of the equations used in Astroresearch 2019

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1 Runge Kutta

Suppose we had a system of coupled first order Differential Equations in the following form:

$$\begin{aligned}\frac{du}{dx} &= f(u, v, x) \\ \frac{dv}{dx} &= g(u, v, x)\end{aligned}$$

with the given inition condition of $x = x_0$. Now suppose we wanted to know the values of u and v at a given value $x + h$. Provided with the step size of h , we can calculate that values using the 4th order Runge Kutta integration method:

$$\begin{aligned}u_{n+1} &= u_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ v_{n+1} &= v_n + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)\end{aligned}$$

Where we have:

$$\begin{aligned}K_1 &= h * f(u_0, v_0, x_0) \\ K_2 &= h * f(u_0 + \frac{1}{2}K_1, v_0 + \frac{1}{2}L_1, x_0 + \frac{1}{2}h) \\ K_3 &= h * f(u_0 + \frac{1}{2}K_2, v_0 + \frac{1}{2}L_2, x_0 + \frac{1}{2}h) \\ K_4 &= h * f(u_0 + K_3, v_0 + L_3, x_0 + h)\end{aligned}$$

And:

$$\begin{aligned}
L_1 &= h * g(u_0, v_0, x_0) \\
L_2 &= h * g(u_0 + \frac{1}{2}K_1, v_0 + \frac{1}{2}L_1, x_0 + \frac{1}{2}h) \\
L_3 &= h * g(u_0 + \frac{1}{2}K_2, v_0 + \frac{1}{2}L_2, x_0 + \frac{1}{2}h) \\
L_4 &= h * g(u_0 + K_3, v_0 + L_3, x_0 + h)
\end{aligned}$$

2 Potentials

The system that we are dealing with currently is a 3 body system with masses m_1 , m_2 , and m_3 . The motion of these bodies is given by the following potentials. The units that we are operating in are solar radii, solar mass, and days.

Note: to simplify these equations, I have used the following shorthands:

$$r_{12} = ((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{1}{2}} \quad (1)$$

$$r_{23} = ((x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2)^{\frac{1}{2}} \quad (2)$$

$$r_{13} = ((x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2)^{\frac{1}{2}} \quad (3)$$

$$G = 2945.49 R_{\odot}^3 M_{\odot}^{-1} \text{days}^{-2} \quad (4)$$

Motion of m_1 :

$$\begin{cases}
x'_1 = u_1 \\
y'_1 = v_1 \\
z'_1 = w_1 \\
v'_1 = G * \left[\frac{m_2(x_2 - x_1)}{r_{12}^3} + \frac{m_3(x_3 - x_1)}{r_{13}^3} \right] \\
u'_1 = G * \left[\frac{m_2(y_2 - y_1)}{r_{12}^3} + \frac{m_3(y_3 - y_1)}{r_{13}^3} \right] \\
w'_1 = G * \left[\frac{m_2(z_2 - z_1)}{r_{12}^3} + \frac{m_3(z_3 - z_1)}{r_{13}^3} \right]
\end{cases} \quad (5)$$

Motion of m_2 :

$$\begin{cases}
x'_2 = u_2 \\
y'_2 = v_2 \\
z'_2 = w_2 \\
v'_2 = G * \left[\frac{m_1(x_1 - x_2)}{r_{12}^3} + \frac{m_3(x_3 - x_2)}{r_{23}^3} \right] \\
u'_2 = G * \left[\frac{m_1(y_1 - y_2)}{r_{12}^3} + \frac{m_3(y_3 - y_2)}{r_{23}^3} \right] \\
w'_2 = G * \left[\frac{m_1(z_1 - z_2)}{r_{12}^3} + \frac{m_3(z_3 - z_2)}{r_{23}^3} \right]
\end{cases} \quad (6)$$

Motion of m_3 :

$$\begin{cases} x_3' = u_3 \\ y_3' = v_3 \\ z_3' = w_3 \\ v_3' = G * \left[\frac{m_2(x_2 - x_3)}{r_{23}^3} + \frac{m_1(x_1 - x_3)}{r_{13}^3} \right] \\ u_3' = G * \left[\frac{m_2(y_2 - y_3)}{r_{23}^3} + \frac{m_1(y_1 - y_3)}{r_{13}^3} \right] \\ w_3' = G * \left[\frac{m_2(z_2 - z_3)}{r_{23}^3} + \frac{m_1(z_1 - z_3)}{r_{13}^3} \right] \end{cases} \quad (7)$$

3 Positions and Velocities

The above potentials of our system would be simple to use if we were given initial positions and initial velocities of the 3 bodies of interest. However, that is not really reflective of the systems we are interested in. The systems that we are interested are systems in which m_1 and m_2 orbit each other in a close binary with the third mass orbiting this inner binary at a further distance. Essentially, what we have is a system of nested binaries with m_1 and m_2 making up the inner binary and the outer binary consisting of m_3 orbiting essentially a point mass of $m_1 + m_2$.

Therefore, the most natural initial conditions of our study are explicit initial positions and initial velocities. Instead, we will be inputting the initial separation of the inner binary and the eccentricity of that orbit, as well as the initial separation of the center of mass of the inner binary from the third mass and the eccentricity of that orbit. These values are denoted a_1 , e_1 , a_2 , and e_2 respectively. Further, the third body may be elevated relative to the inner binary by an angle i and may be out of line with the inner binary by an angle θ . The initial positions and the initial velocities can then be computed if we set the origin to be the global center of mass of the entire system.

Quick shorthand:

$$M = m_1 + m_2 + m_3 \quad (8)$$

Magnitude of the Velocity of the center of mass of the Inner Binary:

$$\mathbf{V} = \sqrt{(G * m_3^2 * (1 - e_2)) / (a_2 * M)} \quad (9)$$

Initial Positions of m_1 :

$$\begin{cases} x_1 = -(a_1 * m_2) / (m_1 + m_2) - \cos(\theta) * \cos(i) * a_2 * (m_3 / M) \\ y_1 = -\sin(\theta) * \cos(i) * a_2 * (m_3 / M) \\ z_1 = -\sin(i) * a_2 * (m_3 / M) \\ u_1 = -\mathbf{V} * \sin(\theta) \\ v_1 = -\sqrt{G * m_2^2 * (1 - e_1) / (a_1 * (m_2 + m_1))} - \mathbf{V} * \cos(\theta) \\ w_1 = 0 \end{cases} \quad (10)$$

Initial Positions of m_2 :

$$\begin{cases} x_2 = (m_1 * a_1)/(m_1 + m_2) - \cos(\theta) * \cos(i) * a_2 * (M_3/M) \\ y_2 = -\sin(\theta) * \cos(i) * a_2 * (m_3/M) \\ z_2 = -\sin(i) * a_2 * (m_3/M) \\ u_2 = -\mathbf{V} * \sin(\theta) \\ v_2 = \sqrt{G * m_1^2 * (1 - e_1)/(a_1 * (m_2 + m_1))} - \mathbf{V} * \cos(\theta) \\ w_2 = 0 \end{cases} \quad (11)$$

Initial Positions of m_3 :

$$\begin{cases} x_3 = \cos(\theta) * a_2 * \cos(i) * (m_1 + m_2)/M \\ y_3 = \sin(\theta) * a_2 * \cos(i) * (m_1 + m_2)/M \\ z_3 = a_2 * \sin(i) * (m_1 + m_2)/M \\ u_3 = \sqrt{G * (m_1 + m_2)^2 * (1 - e_2)/(a_2 * M)} * \sin(\theta) \\ v_3 = \sqrt{G * (m_1 + m_2)^2 * (1 - e_2)/(a_2 * M)} * \cos(\theta) \\ w_3 = 0 \end{cases} \quad (12)$$

4 Energies and Angular Momentum

The total energy and angular momentum of all three bodies is given by:

$$E = \frac{1}{2}m_1\mathbf{V}_1^2 + \frac{1}{2}m_2\mathbf{V}_2^2 + \frac{1}{2}m_3\mathbf{V}_3^2 - \frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} \quad (13)$$

$$L = \vec{r}_1 \times m_1 * \vec{V}_1 + \vec{r}_2 \times m_2 * \vec{V}_2 + \vec{r}_3 \times m_3 * \vec{V}_3 \quad (14)$$

Where we are using the short hands:

$$G = 2945.49 R_\odot^3 M_\odot^{-1} \text{days}^{-2} \quad (15)$$

$$r_{12} = ((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{1}{2}} \quad (16)$$

$$r_{23} = ((x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2)^{\frac{1}{2}} \quad (17)$$

$$r_{13} = ((x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2)^{\frac{1}{2}} \quad (18)$$

$$\mathbf{V}_1 = \sqrt{u_1^2 + v_1^2 + w_1^2} \quad (19)$$

$$\mathbf{V}_2 = \sqrt{u_2^2 + v_2^2 + w_2^2} \quad (20)$$

$$\mathbf{V}_3 = \sqrt{u_3^2 + v_3^2 + w_3^2} \quad (21)$$

$$\vec{r}_1 = [x_1, y_1, z_1] \quad (22)$$

$$\vec{r}_2 = [x_2, y_2, z_2] \quad (23)$$

$$\vec{r}_3 = [x_3, y_3, z_3] \quad (24)$$

$$\vec{V}_1 = [u_1, v_1, w_1] \quad (25)$$

$$\vec{V}_2 = [u_2, v_2, w_2] \quad (26)$$

$$\vec{V}_3 = [u_3, v_3, w_3] \quad (27)$$

5 Energies and Angular Momentum of Individual Binaries

We can also calculate the energy and angular momentum of the inner binary (m_1 and m_2) and the outer binary (m_3 and $m_1 + m_2$).

$$E_1 = \frac{1}{2}m_1||\vec{V}_1 - \vec{V}_{CM}||^2 + \frac{1}{2}m_2||\vec{V}_2 - \vec{V}_{CM}||^2 - \frac{Gm_1m_2}{r_{12}} \quad (28)$$

$$E_2 = \frac{1}{2}(m_1 + m_2)||\vec{V}_{CM}||^2 + \frac{1}{2}m_3||\vec{V}_3||^2 - \frac{G(m_1 + m_2)m_3}{||\vec{r}_3 - \vec{r}_{CM}||} \quad (29)$$

$$L_1 = (\vec{r}_1 - \vec{r}_{CM}) \times m_1(\vec{V}_1 - \vec{V}_{CM}) + (\vec{r}_2 - \vec{r}_{CM}) \times m_2(\vec{V}_2 - \vec{V}_{CM}) \quad (30)$$

$$L_2 = \vec{r}_{CM} \times (m_1 + m_2)\vec{V}_{CM} + \vec{r}_3 \times m_3\vec{V}_3 \quad (31)$$

Where in addition to the short hands above we are using:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \quad (32)$$

$$\vec{V}_{CM} = \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1 + m_2} \quad (33)$$

Where \vec{r}_{CM} is the position of the center of mass of the inner binary and \vec{V}_{CM} is the velocity of that center of mass.