Statistical Learning (5454) - Assignment 3

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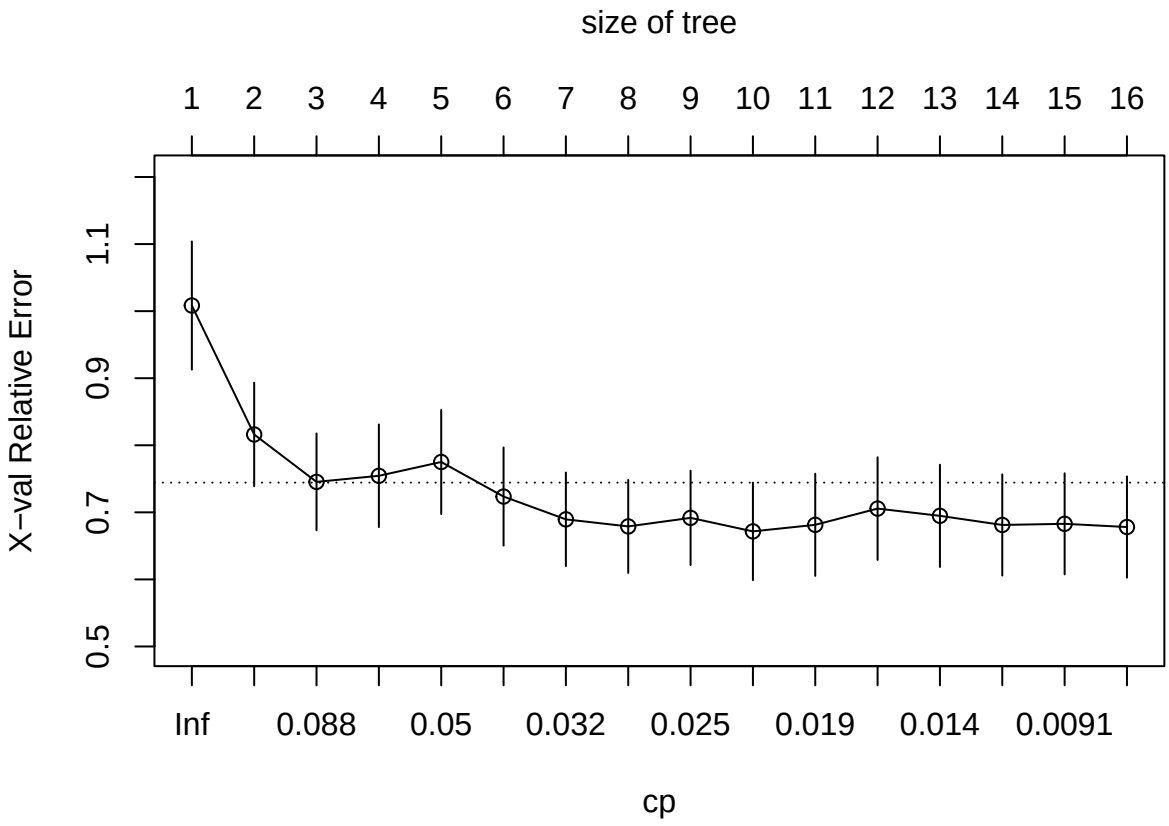
Exercise 1

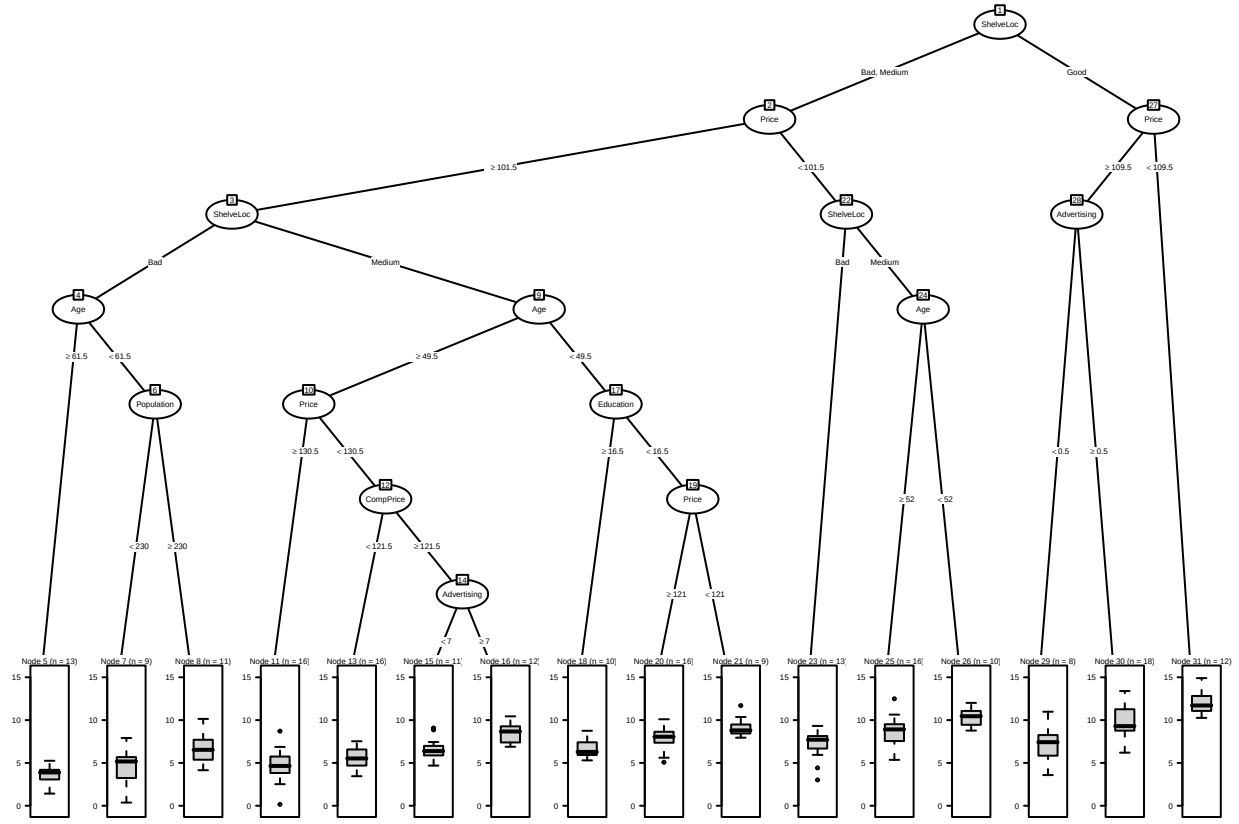
We load the data set *Carseats* from package *ISLR2*. It is a simulated data set containing sales of child car seats at 400 different stores. We then select 200 samples (50:50 split) as training data and use the remaining ones as test data.

We fit a regression tree to the training set. The stopping criterion is set to $cp = 10^{-4}$

```
##
## Regression tree:
## rpart(formula = Sales ~ ., data = train, method = "anova", parms = list(split = "gini"),
##     control = list(cp = 1e-04))
##
## Variables actually used in tree construction:
## [1] Advertising Age         CompPrice   Education
## [5] Population  Price       ShelveLoc
##
## Root node error: 1439/200 = 7.2
##
## n= 200
##
##        CP nsplit rel error xerror  xstd
## 1  0.1959      0      1.00   1.01 0.096
## 2  0.1160      1      0.80   0.82 0.077
## 3  0.0674      2      0.69   0.75 0.072
## 4  0.0559      3      0.62   0.75 0.077
## 5  0.0439      4      0.56   0.78 0.078
## 6  0.0323      5      0.52   0.72 0.073
## 7  0.0308      6      0.49   0.69 0.070
## 8  0.0276      7      0.46   0.68 0.069
## 9  0.0230      8      0.43   0.69 0.070
## 10 0.0225      9      0.41   0.67 0.073
## 11 0.0155     10      0.38   0.68 0.076
## 12 0.0147     11      0.37   0.71 0.077
## 13 0.0135     12      0.35   0.69 0.076
## 14 0.0118     13      0.34   0.68 0.075
## 15 0.0070     14      0.33   0.68 0.075
## 16 0.0001     15      0.32   0.68 0.076
```

The complexity parameter table and the plot of the tree is seen below.


size of tree
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
X−val Relative Error
0.5 0.7 0.9 1.1
Inf 0.088 0.05 0.032 0.025 0.019 0.014 0.0091
cp

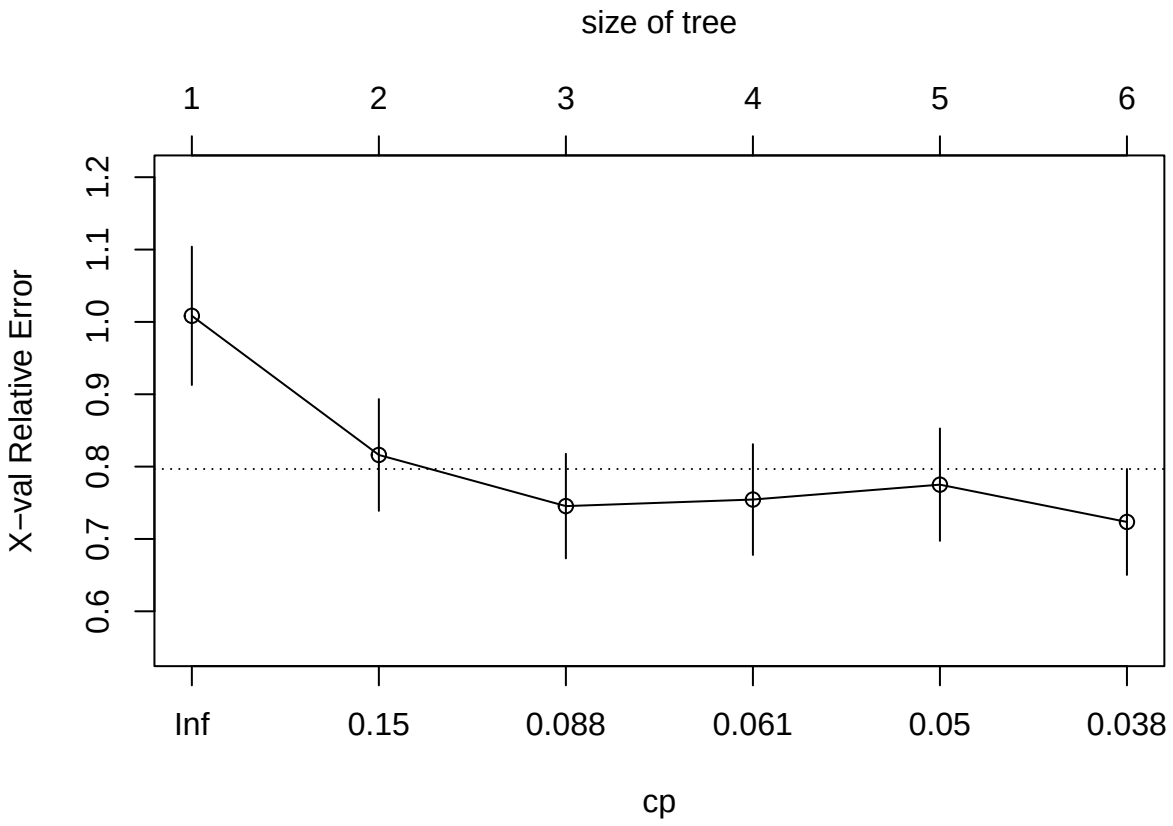


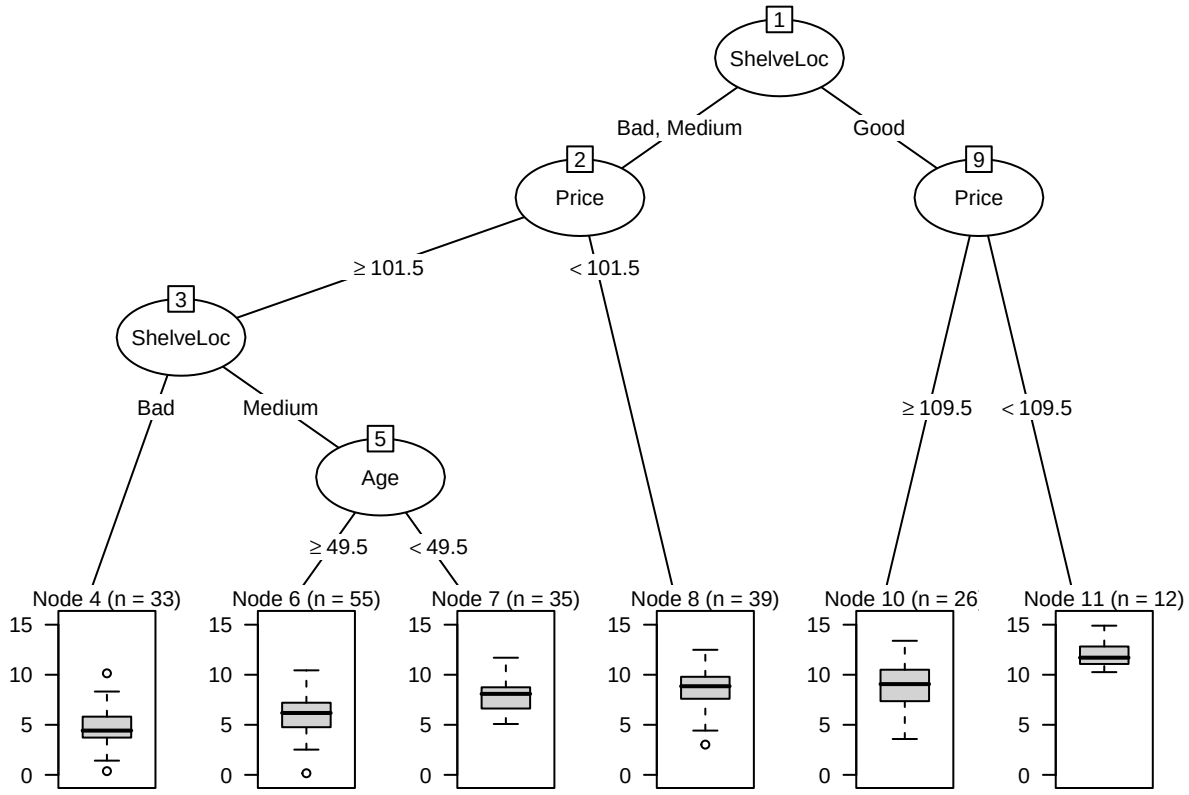
The tree has 16 terminal nods (15 splits). With the loose stopping criterion we are facing the risk of overiftting. In order to solve this problem we prune the tree later on. As he *xerrror* doesn't increase for high cp, I would already guess that overfitting isn't a big problem. Since the tree is rather long we do further interpretations with the pruned tree.

For now, we calculate the test MSE, which is

```
## [1] 4.21
```

Let's use cross-validation in order to determine the optimal level of tree complexity. To do this, we use the first entry where the *xerror* is smaller than the minimum error plus one standard deviation as complexity parameter.


size of tree
1
2
3
4
5
6
X−val Relative Error
0.6
0.7
0.8
0.9
1.0
1.1
1.2
Inf
0.15
0.088
0.061
0.05
0.038
cp



The pruned tree has 6 terminal nods (5 splits). The fist split is done according to the quality of the shelving location for the car seats. It splits into *Bad & Medium* and *Good*. The second split is the car seat price after a *Bad & Medium* shelving location. In the third it's again split by the shelving location. This time between *Bad* and *Medium*. The next is after a Medium shelving location according to the average age of the local population. And lastly by Price after a *Good* shelving location. Looking at the boxplots, it seems that even the last split was still important.

The test MSE with pruning is

```
## [1] 4.669
```

The test MSE with pruning is higher than without. Therefore pruning the tree doesn't improve the test MSE. Hence, overfitting was acutally not too much of a problem.

Exercise 2

We draw 100 observations from four independent variables $X_1, \ldots, X_4$ where

- X_1 follows a uniform distribution,
- X_2 follows a standard normal distribution,
- X_3 follows a Bernoulli distribution with success probability $\pi = 0.5$,
- X_4 follows a Bernoulli distribution with success probability $\pi = 0.1$.

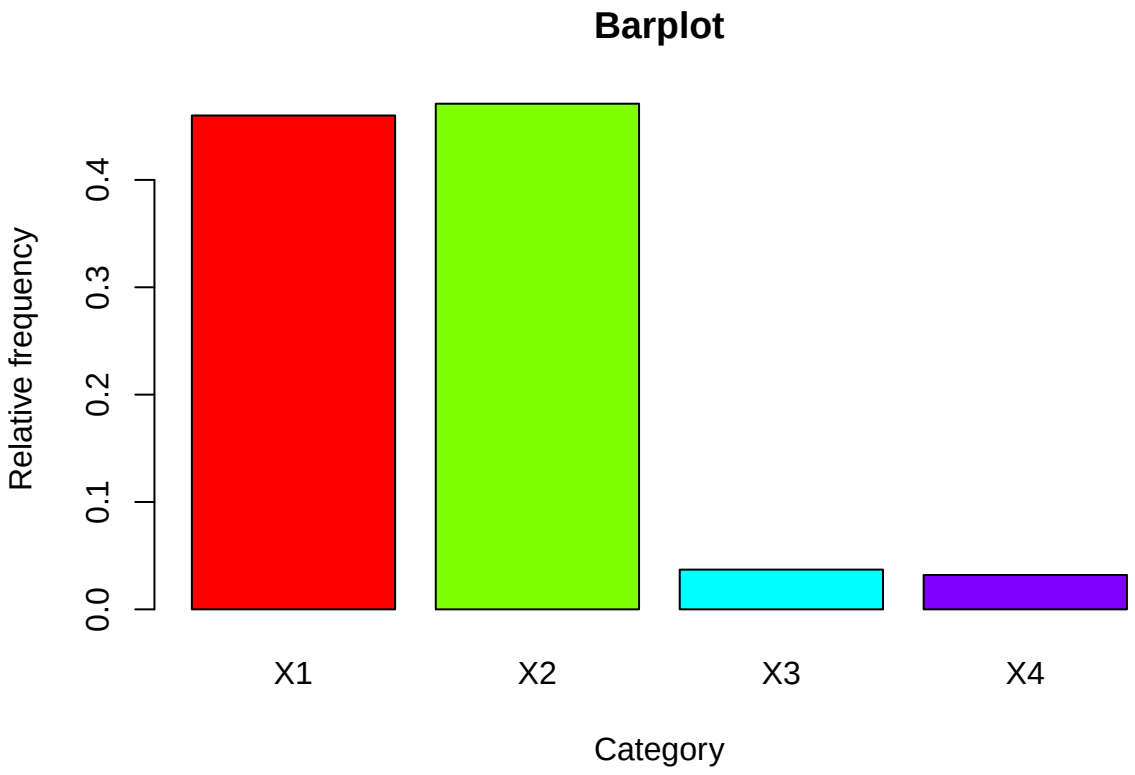
We repeat 1000 times the following:

- Draw a dependent variable y from a standard normal distribution which is independent of the four independent variables.

- Fit a tree stump, i.e., a tree which contains only one split.
- Determine which variable was used for splitting.

Below is the table of relative frequencies, how often each of the variables was selected for splitting. We also provide a barplot.

```
##
##    X1    X2    X3    X4
## 0.460 0.471 0.037 0.032
```



The probability of including a particular independent variable is not the same. X_2 has the highest probability, closely followed by X_1. The inclusion probabilities for X_3 and X_4 are much lower. The different probabilities are a result of the distributions of the independent variables. It is no great surprise that the highest inclusion probability occurs for X_2 with its standard normal distribution when the dependent variable is also standard normally distributed. The uniform distribution also covers it quite well. The Bernoulli distribution bears little resemblance to the standard normal distribution. Therefore, the low inclusion probabilities.

Exercise 3

Let's assume the following data generating process

$$Y = X + \epsilon,$$

with $X \sim N(0, 1)$ and $\epsilon \sim N(0, 1)$ independent. In addition 20 covariates $Z_1, ..., Z_{20}$ are given with

$$Z_i \sim \sqrt{0.9}X + \epsilon Z_i,$$

where $\epsilon Z_i \sim N(0, 0.1)$.

We draw a training data with 30 observations and a test data with 10,000 observations from the data generating process including the additional covariates **Z**.

We then sample 100 bootstrap samples of size 30 from the training data by drawing with replacement.

To each bootstrap sample we fit:

1. a regression tree,
2. the null model with predicted value equal to the observed empirical mean of Y ,
3. a linear model including linear effects for X and all Z variables and
4. a linear model potentially including linear effects for X and all Z variables, but using model selection with the AIC to select a suitable model starting from the null model.

We determine the predicted values on the test data for the bagged model estimator by calculating the average predictions over the 100 trees fitted to the bootstrap samples, the 100 null models, the 100 linear models including all linear effects and the 100 linear models based on model selection.

Last, we determine the mean squared error (MSE) of the four bagged model estimators on the test sample of size 10,000.

```
##   Model     MSE
## 1  Tree   2.098
## 2  Null   2.114
## 3    LM 377.018
## 4   AIC   2.680
```

The lowest MSE is achieved by the regression tree, flowed by the null model and the model selected by AIC. The highest MSE results for the linear model including X and all Z.

Exercise 4

We load the dataset icu in package **aplore3** which contains information on patients who were admitted to an adult intensive care unit (ICU). We develop a predictive model for the probability of survival to hospital discharge of these patients. To fit a predictive model to the data we use random forests.

We select a suitable number of bootstrap iterations.

Assess the influence of varying the hyperparameter m on the out-of-bag error obtained and select a suitable value.

Inspect the variable importance measures. Compare the mean decrease Gini and the mean decrease accuracy measures and assess if the observed differences in relative importance assigned might be related to the predictor variable being numeric or not.

Exercise 5

Assume that there are four predictor variables which have the following distributions:

$$X_1 \sim N(0,1), \qquad X_2 \sim U(0,1), \tag{1}$$

$$X_3 \sim M(1,(0.5,0.5)), \qquad X_4 \sim M(1,(0.2,0.2,0.2,0.2,0.2)). \tag{2}$$

This means we have two continuous variables which follow either a standard normal or a standard uniform distribution ($U(0,1)$ and two categorical variables with balanced categories with either 2 or 5 categories, i.e., $M(N,\pi)$ is the multinomial distribution for N trials and success probability vector π. The dependent variable y is assumed to be a binary categorical variable with equal-sized classes.The sample size is set to N = 200.

We generate 100 datasets for each setting and fit a random forest to each dataset and determine the mean decrease Gini and mean decrease accuracy values for each of the predictor variables.

Let's suitably visualize the results and interpret them.