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2.16 Derive the mean, mode, and variance for the beta distribution

The beta distribution has support over $[0, 1]$ and is defined as

$$Beta(x|a, b) = \frac{1}{B(a, b)} x^{a-1}(1-x)^{b-1} \tag{1}$$

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2}$$

Let's begin by deriving the mean, using integration by parts

$$\mathbb{E}[X] = \int_0^1 x \cdot \frac{1}{B(a, b)} x^{a-1}(1-x)^{b-1} dx \tag{3}$$

Let $u = x^a$, $du = ax^{a-1}dx$, $dv = (1-x)^{b-1}$, and $v = \frac{-1}{b}(1-x)^b$.

$$\begin{aligned}
\frac{1}{B(a, b)} \int_0^1 x^a (1-x)^{b-1} dx &= \frac{1}{B(a, b)} [\frac{x^a}{b}(1-x)^b - \frac{a}{b} \int_0^1 (-1)(1-x)^b x^{a-1} dx] \\
&= \frac{1}{B(a, b)} [\frac{x^a}{b}(1-x)^b + \frac{a}{b} \int_0^1 (1-x)^{b-1}(1-x) x^{a-1} dx] \\
&= \frac{1}{B(a, b)} [(\frac{x^a}{b}(1-x)^b)\Big|_0^1 + \frac{a}{b} \int_0^1 x^{a-1}(1-x)^{(b-1)} dx - \frac{a}{b} \int_0^1 x \cdot x^{a-1}(1-x)^{b-1} dx] \\
&= \frac{a}{b}(1 - \mathbb{E}[X])
\end{aligned}$$

We have

$$\mathbb{E}[X] = (1 - \mathbb{E}[X]) \tag{4}$$

so

$$\mathbb{E}[X] = \frac{a}{a+b} \tag{5}$$

To derive the mode, we differentiate the pdf and set to zero.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\Big] &= \frac{\mathrm{d}}{\mathrm{d}x}\Big[x^{a-1}(1-x)^{b-1}\Big] \\ &= (a-1)x^{a-2}(1-x)^{b-1} - x^{a-1}(b-1)(1-x)^{b-2} \\ &= 0\end{aligned}$$

Solving we get

$$x = \frac{a-1}{a+b-2} \tag{6}$$

Recall that $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. We integrate by parts to find $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \frac{1}{B(a,b)}\int_0^1 x^2 x^{a-1}(1-x)^{b-1}dx$$

Let $u = x^{a+1}$, $du = (a+1)x^a$, $dv = (1-x)^{b-1}dx$, $v = \frac{-1}{b}(1-x)^b$.

$$\begin{aligned}\mathbb{E}[X^2] &= \frac{1}{B(a,b)}\Big[-x^{a+1}(1-x)^b\Big|_0^1 + \int_0^1 (a+1)x^a(1-x)^b dx\Big] \\ &= \frac{a+1}{b}\int_0^1 \frac{1}{B(a,b)}xx^{a-1}(1-x)^b\end{aligned}$$

If we distribute $(1-x)$ then we get two terms which look like the mean and second moment.

$$\begin{aligned}&= \frac{a+1}{b}\Big[\int_0^1 \frac{1}{B(a,b)}xx^{a-1}(1-x)^{b-1}dx - \int_0^1 \frac{1}{B(a,b)}x^2(1-x)^{b-1}dx\Big] \\ &= \frac{a+1}{b}\Big[\mathbb{E}[X] - \mathbb{E}[X^2]\Big]\end{aligned}$$

Solving we get

$$\mathbb{E}[X] = \frac{ab}{(a+b+1)(a+b)^2} \tag{7}$$