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2.7 Pairwise independence does not imply mutual independence

Let X, Y, and Z be variables in $\{0, 1\}$ and define $P(X, Y, Z) = \dfrac{1}{4}(X \oplus Y \oplus Z)$. Then the joint probability distribution is

X	Y	Z	Pr
0	0	0	$\frac{1}{4}$
0	0	1	0
0	1	0	0
1	0	0	0
1	1	0	$\frac{1}{4}$
0	1	1	$\frac{1}{4}$
1	0	1	$\frac{1}{4}$
1	1	1	0

Marginalizing, we see that X takes on values $0, 1$ with probability $\frac{1}{2}$, and so does Y. And the joint probability over X and Y takes on each of the four possibilities with probability $\frac{1}{4}$, showing that X and Y are pairwise independent. By symmetry of the three variables, we can make the pairwise independence argument for any pair. However, it is clear from the table that joint probability distribution of the three variables does not factorize into the product of the marginals.