

# CPSC-354 Report

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## Abstract

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# 1 Introduction

## 2 Week by Week

### 2.1 Week 1

#### 2.1.1 MU Puzzle

It is impossible to solve the MU puzzle. The only way to change the amount of I's is to double them with rule 2, or subtract 3 with rule 3. In order to get rid of all the I's you would have to have them in groups of 3. A power of x is divisible by 3 only if x is a multiple of 3, because the prime factorization of a number must include the prime number 3 for the number to be divisible by 3. So, you would have to get an original group divisible by 3, which is impossible.

### 2.2 Week 2

#### 2.2.1 Rewriting

1.  $A = \{\}$

Empty ARS

Terminating: Yes

Confluent: Yes

Has Unique Normal Forms: Yes

2.  $A = \{a\}$  and  $R = \{\}$

$a$

Terminating: Yes

Confluent: Yes

Has Unique Normal Forms: Yes

3.  $A = \{a\}$  and  $R = \{(a,a)\}$

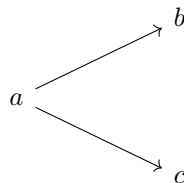


Terminating: No

Confluent: Yes

Has Unique Normal Forms: No

4.  $A = \{a,b,c\}$  and  $R = \{(a,b), (a,c)\}$



Terminating: Yes

Confluent: No

Has Unique Normal Forms: No

5.  $A = \{a,b\}$  and  $R = \{(a,a), (a,b)\}$

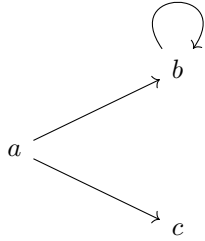


Terminating: No

Confluent: Yes

Has Unique Normal Forms: Yes

6.  $A = \{a, b, c\}$  and  $R = \{(a, b), (b, b), (a, c)\}$

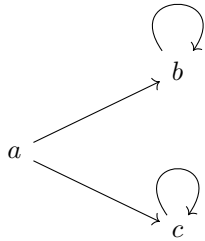


Terminating: No

Confluent: No

Has Unique Normal Forms: No

7.  $A = \{a, b, c\}$  and  $R = \{(a, b), (b, b), (a, c), (c, c)\}$



Terminating: No

Confluent: No

Has Unique Normal Forms: No

confluent	terminating	has unique normal forms	example
True	True	True	$a$
True	True	False	IMPOSSIBLE
True	False	True	$a \xrightarrow{\quad} b$
True	False	False	$a \xrightarrow{\quad} b$
False	True	False	$a \xrightarrow{\quad} b$
False	False	True	IMPOSSIBLE
False	False	False	$a \xrightarrow{\quad} b$
False	False	False	$a \xrightarrow{\quad} b$

## 2.3 Week 3

### 2.3.1 Homework

#### Exercise 5

$ab \rightarrow ba$   
 $ba \rightarrow ab$   
 $aa \rightarrow$   
 $b \rightarrow$

$abba \rightarrow aba \rightarrow aa \rightarrow$   
 $bababa \rightarrow ababa \rightarrow aaba \rightarrow aaa \rightarrow a$

The ARS isn't terminating because it has the loop  $ab \rightarrow ba, ba \rightarrow ab$ .

In the above example,  $abba \rightarrow$ ,  $bababa \rightarrow a$ , which makes them non-equivalent. There are infinitely many non-equivalent strings, because all strings with an even number of a's are only equivalent to each other, and all strings with an odd number of a's are only equivalent to each other.

There are 2 equivalence classes, all strings with an odd number of a's that have normal form a, and all strings with an even number of a's that have normal form of an empty string.

By getting rid of the first 2 rules, the ARS is terminating, with the same equivalence classes. There is no longer a loop, and you can still get rid of the same letters.

How would you have to modify the ARS so that there is only one equivalence class?

#### Exercise 5b

$ab \rightarrow ba$   
 $ba \rightarrow ab$   
 $aa \rightarrow a$   
 $b \rightarrow$

$abba \rightarrow aba \rightarrow aa \rightarrow a$   
 $bababa \rightarrow ababa \rightarrow aaba \rightarrow aaa \rightarrow aa \rightarrow a$

The ARS isn't terminating because it has the loop  $ab \rightarrow ba, ba \rightarrow ab$ .

bbb and aba are not equivalent. There are infinitely many non-equivalent strings, any string that has all b's is not equivalent to any string that has at least one a in it.

There are 2 equivalence classes, all strings that have no a's have the normal form of an empty string. All strings that have at least one a have the normal form a.

By getting rid of the first 2 rules, the ARS is terminating, with the same equivalence classes. There is no longer a loop, and you can still get rid of the same letters.

What would happen if  $aa \rightarrow a$  was changed to  $aaa \rightarrow a$ ?

## 2.4 Week 4

### 2.4.1 Homework

#### 4.1

```

while b != 0:
    temp = b
    b = a mod b
    a = temp
return a

```

This algorithm always terminates under the condition that  $a, b \in \mathbb{N}$ .

We can define this ARS as:  $A = (a, b) | a, b \in \mathbb{N}$ , with transition,  $(a, b) \rightarrow (b, a \bmod b)$ .

A measure function is:  $\phi(a, b) = b$ .

Since  $a \rightarrow b$  every iteration, if b is decreasing, so is a. In the transition,  $b \rightarrow a \bmod b$ ,  $a \bmod b$  will always be less than b, so,  $\phi(a, b) > \phi(b, a \bmod b)$ . Since this algorithm has a measure function, it terminates.

#### 4.2

```

function merge_sort(arr, left, right):
    if left >= right:
        return
    mid = (left + right) / 2
    merge_sort(arr, left, mid)
    merge_sort(arr, mid+1, right)
    merge(arr, left, mid, right)

```

Prove that  $\phi(left, right) = right - left + 1$  is a measure function.

For recursive calls,  $left \leq mid < right$ .

For the left side recursive call:

$merge\_sort(arr, left, mid) = mid - left + 1$

$right > mid$ , so,  $right - left + 1 > mid - left + 1$

$\phi(left, right) > \phi(left, mid)$

This call terminates

For the right side recursive call:

$merge\ sort(arr, mid + 1, right) = right - (mid + 1) + 1$

$mid \geq left$ , so,  $mid + 1 > left$

$right - left + 1 > right - (mid + 1) + 1$

$\phi(left, right) > \phi(mid + 1, right)$

This call terminates

Since both calls terminate,  $\phi(left, right) = right - left + 1$  is a measure function for merge sort.

## 2.5 Week 5

### 2.5.1 Homework

$(\lambda f. \lambda x. f(f(x))) (\lambda f. \lambda x. (f(f(f\ x))))$

alpha rule:

$(\lambda g. \lambda y. g(g(y))) (\lambda f. \lambda x. (f(f(f\ x))))$

beta rule:

$(\lambda y. (\lambda f. \lambda x. (f(f(f\ x)))) ((\lambda f. \lambda x. (f(f(f\ x))))(y)))$

beta rule:

$(\lambda y. (\lambda f. \lambda x. (f(f(f\ x)))) (\lambda x. (y(y(y\ x)))))$

alpha rule:

$(\lambda y. (\lambda f. \lambda z. (f(f(f\ z)))) (\lambda x. (y(y(y\ x)))))$

beta rule:

$(\lambda y. (\lambda z. ((\lambda x. (y(y(y\ x))))((\lambda x. (y(y(y\ x))))((\lambda x. (y(y(y\ x))))z))))))$

beta rule:

$(\lambda y. (\lambda z. ((\lambda x. (y(y(y\ x))))((\lambda x. (y(y(y\ x))))(y(y(y\ z)))))))$

beta rule:

$(\lambda y. (\lambda z. ((\lambda x. (y(y(y\ x))))(y(y(y\ (y(y(y\ z))))))))$

beta rule:

$(\lambda y. (\lambda z. (y(y(y(y(y(y(y\ z))))))))))$

## 2.6 Week 6

### 2.6.1 Homework

let rec fact =  $\lambda n. \text{ if } n=0 \text{ then } 1 \text{ else } n * \text{fact } (n-1)$  in fact 3

-> <def of let rec>

let fact = (fix ( $\lambda \text{fact}. \lambda n. \text{ if } n=0 \text{ then } 1 \text{ else } n * \text{fact } (n-1)$ )) in fact 3

-> <def of let>

( $\lambda \text{fact}. \text{fact } 3$ ) (fix ( $\lambda \text{fact}. \lambda n. \text{ if } n=0 \text{ then } 1 \text{ else } n * \text{fact } (n-1)$ ))

-> <beta rule: substitute fix F>

(fix ( $\lambda \text{fact}. \lambda n. \text{ if } n=0 \text{ then } 1 \text{ else } n * \text{fact } (n-1)$ )) 3

-> <def of fix>

(( $\lambda \text{fact}. \lambda n. \text{ if } n=0 \text{ then } 1 \text{ else } n * \text{fact } (n-1)$ ) (fix ( $\lambda \text{fact}. \lambda n. \text{ if } n=0 \text{ then } 1 \text{ else } n * \text{fact } (n-1)$ ))) 3

```

-> <beta rule: substitute fix F>
( $\lambda n.$  if  $n=0$  then 1 else  $n * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) ( $n-1$ ) 3

-> <beta rule: substitute 3>
if  $3=0$  then 1 else  $3 * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) ( $3-1$ )

-> <def of if>
 $3 * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) 2

-> <def of fix>
 $3 * ((\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1)) (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) 2

-> <beta rule>
 $3 * (\lambda n.$  if  $n=0$  then 1 else  $n * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) ( $n-1$ ) 2

-> <beta rule>
 $3 * (\text{if } 2=0 \text{ then } 1 \text{ else } 2 * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) ( $2-1$ )

-> <def of if>
 $3 * (2 * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) 1)

-> <def of fix>
 $3 * (2 * ((\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1)) (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ )) 1)

-> <beta rule>
 $3 * (2 * (\lambda n.$  if  $n=0$  then 1 else  $n * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ )( $n-1$ )) 1)

-> <beta rule>
 $3 * (2 * (\text{if } 1=0 \text{ then } 1 \text{ else } 1 * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ )( $1-1$ )))

-> <def of if>
 $3 * (2 * (1 * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) 0))

-> <def of fix>
 $3 * (2 * (1 * ((\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1)) (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ )) 0))

-> <beta rule>
 $3 * (2 * (1 * (\lambda n.$  if  $n=0$  then 1 else  $n * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) ( $n-1$ )) 0))

-> <beta rule>
 $3 * (2 * (1 * (\text{if } 0=0 \text{ then } 1 \text{ else } 0 * (\text{fix } (\lambda \text{fact}. \lambda n.$  if  $n=0$  then 1 else  $n * \text{fact } (n-1))$ ) ( $0-1$ ))))

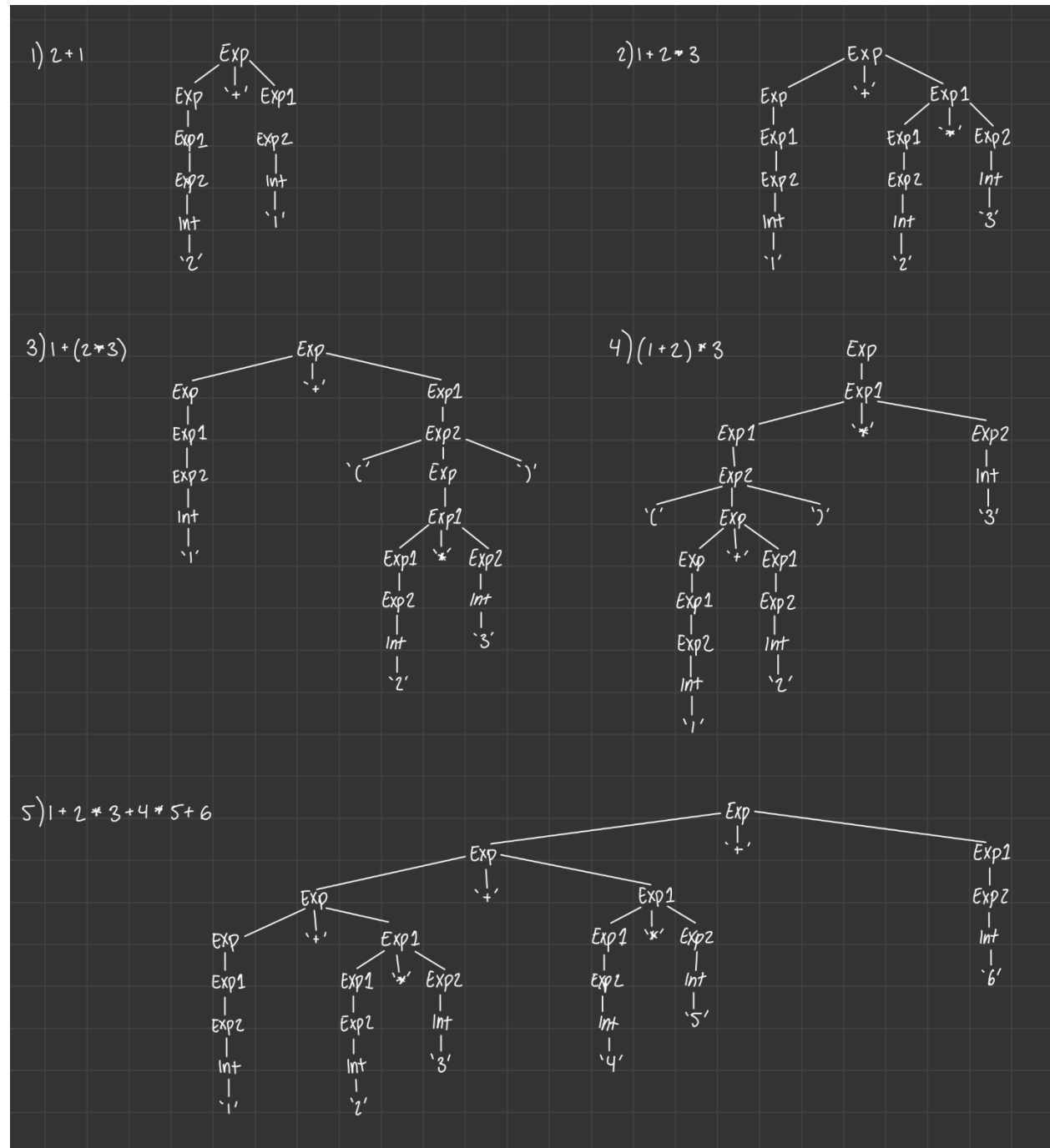
-> <def of if>
 $3 * (2 * (1 * (1)))$ 

-> <multiplication>
6

```

## 2.7 Week 7

### 2.7.1 Homework





## 2.8 Week 8

### 2.8.1 Homework

Level 5:  $a + (b + 0) + (c + 0) = a + b + c$

rw[add\_zero]

rw[add\_zero]

rfl

Level 6:  $a + (b + 0) + (c + 0) = a + b + c$

rw[add\_zero c]

rw[add\_zero]

rfl

Level 7: For all natural numbers  $a$ , we have  $\text{succ}(a) = a + 1$

rw[one\_eq\_succ\_zero]

rw[add\_succ]

rw[add\_zero]

rfl

Level 8:  $2 + 2 = 4$

rw[four\_eq\_succ\_three]

rw[three\_eq\_succ\_two]

rw[two\_eq\_succ\_one]

rw[one\_eq\_succ\_zero]

rw[add\_succ]

rw[add\_succ]

rw[add\_zero]

rfl

Natural Language Proof of Level 5:

$a + (b + 0) + (c + 0) = a + b + c$

$a + b + c = a + b + c$

by algorithm 1: addition ( $m + 0 = 0$ )

True by reflexivity

## 2.9 Week 9

### 2.9.1 Homework

ADDITION WORLD LEVEL 5

WITH INDUCTION:  $a + b + c = a + c + b$

induction b

rw[add\_zero]

rw[add\_zero]

rfl

rw[add\_succ]

rw[succ\_add]

rw[add\_succ]

rw[n\_ih]

rfl

$$a + b + c = a + c + b$$

Induction on b:

Basis:  $a + 0 + c = a + c + 0$

$a + c = a + c + 0$  def of +

$a + c = a + c$  def of +

True by reflexivity

Inductive Hypothesis:  $a + n + c = a + c + n$

Inductive Step:  $a + n + c = a + c + n$

$(a + n) + c = a + c + n$  def of +

$(a + n + c) = a + c + n$  def of +

$(a + n + c) = a + (c + n)$  def of +

$(a + n + c) = (a + c + n)$  def of +

$(a + c + n) = (a + c + n)$  inductive hypothesis

True by reflexivity

WITHOUT INDUCTION:  $a + b + c = a + c + b$

rw[add\_assoc]

rw[add\_comm b]

rw[add\_assoc]

rfl

$(a + b) + c = (a + c) + b$

$a + (b + c) = (a + c) + b$  associativity

$a + (c + b) = (a + c) + b$  commutativity

$(a + c) + b = (a + c) + b$  associativity

True by reflexivity

## 2.10 Week 10

### 2.10.1 Homework

PARTY INVITES

LEVEL 6: example (C D S: Prop) (h : C → D → S) : C → D → S := by

exact c d h <c, d>

LEVEL 7: example (C D S: Prop) (h : C → D → S) : C → D → S := by

exact c h c.left c.right

LEVEL 8: example (C D S : Prop) (h : (S → C) → (S → D)) : S → C → D := by

exact s <h.left s, h.right s>

LEVEL 9: example (R S : Prop) : R → (S → R) → (¬S → R) := by

exact r <\_ r, \_ r>

### 3 Essay

### 4 Evidence of Participation

### 5 Conclusion

### References

[BLA] Author, [Title](#), Publisher, Year.