Chapter 3.
Higher-Order Differential Equations

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Linear Independent & Dependent

$W(y_1, y_2)$

補充說明

一組函數集合 $\{u_1(x), u_2(x), ..., u_n(x)\}$

考慮 $C_1u_1(x) + C_2u_2(x) + ... + C_nu_n(x) = 0$ -----------(1)

其中 $C_1, C_2, ..., C_n \in const$

(A)若(1)成立 $\Leftrightarrow$ $C_1 = C_n = ... = C_n = 0$

則 $u_1(x), u_2(x), ..., u_n(x)$ 為線性獨立(Linear Independent)

(B) $\exists C_i \neq 0$ s.t. (1)成立

則 $u_1(x), u_2(x), ..., u_n(x)$ 為線性相依(Linear Dependent)

Linear Independent & Dependent

例: $u_1(x) = x^2, u_2(x) = x$

$$C_1 x^2 + C_2 x = 0 \Leftrightarrow C_1 = C_2 = 0$$

$\therefore u_1(x), u_2(x)$線性獨立

例: $u_1(x) = 3x, u_2(x) = -2x$

$$C_1(3x) + C_2(\text{-}2x) = 0$$

$$(3C_1\text{-}2C_2) = 0$$

$\therefore 3C_1 = 2C_2$ 取$C_1 = 1, C_2 = \dfrac{3}{2}$

$u_1(x), u_2(x)$線性相依

Linear Independent & Dependent

Pf:

設$C_i \neq 0$

$$C_1 u_1 + C_2 u_2 + \ldots\ldots + C_n u_n = 0$$

$$u_i = \frac{-C_1}{C_i} u_1 + \frac{-C_2}{C_i} u_2 + \ldots\ldots + \frac{-C_n}{C_i} u_n$$

$$= k_1 u_1 + k_2 u_2 + \ldots\ldots + k_n u_n$$

上述例子只有兩個函數容易判斷 C_1, C_2

Linear Independent & Dependent

Extend to n 個函數?

$$\Rightarrow \begin{cases} C_1u_1 + C_2u_2 + \ldots\ldots + C_nu_n = 0 \\ C_1u_1^{'} + C_2u_2^{'} + \ldots\ldots + C_nu_n^{'} = 0 \\ \ldots\ldots \\ C_1u_1^{(n-1)} + C_2u_2^{(n-1)} + \ldots\ldots + C_nu_n^{(n-1)} = 0 \end{cases}$$

回顧: $\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$

Linear Independent & Dependent

$$x = \frac{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = 0, y = \frac{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = 0$$

$$\Rightarrow (1) x = y = 0 \Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0. (2) x, y \text{具有非零解} \Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

Linear Independent & Dependent

上述聯立方程式之行列式為

$$\begin{vmatrix} u_1 & u_2 & \dots & u_n \\ u_1' & u_2' & \dots & u_n' \\ . & . & \dots & . \\ u_1^{(n-1)} & u_2^{(n-1)} & \dots & u_n^{(n-1)} \end{vmatrix} = wronski(u_1 \quad u_2 \quad \dots \quad u_n)$$

Linear Independent & Dependent

$(A)w(u_1 \quad u_2 \quad \ldots \quad u_n) = 0$

$\Leftrightarrow$ 至少存在$C_1 C_2 \ldots C_n$有非零解

$\Leftrightarrow u_1 \quad u_2 \quad \ldots \quad u_n$ 線性相依

$(B)w(u_1 \quad u_2 \quad \ldots \quad u_n) \neq 0$

$\Leftrightarrow C_1 = C_2 = \ldots = C_n = 0$

$\Leftrightarrow u_1 \quad u_2 \quad \ldots \quad u_n$ 線性獨立

Linear Independent & Dependent

例: $3x, -2x$

$$w = \begin{vmatrix} 3x & -2x \\ 3 & -2 \end{vmatrix} \qquad L.D.$$

例: x, x^2

$$w = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0, \forall x \qquad L.I.$$

Linear Independent & Dependent

例: e^{x}, e^{2x}, e^{3x}

$$w = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0 \qquad L.I.$$

Euler-Cauchy Differential Equations

變係數微分方程式
Euler-Cauchy Differential Eqs.
尤拉科西 or 等維

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_{n-1} x^1 y' + a_n y = r(x)$$

If r(x) = 0 homogeneous case

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_{n-1} x^1 y' + a_n y = 0$$

想法:常係數時積 $y = e^{\lambda x}$ 求λ

想辦法變成常系數
想法:令 $x = e^t$

Euler-Cauchy Differential Equations

例: $x^2 y'' - 2xy' + 2y = 0$

$$令 x = e^t$$

$$\frac{dx}{dt} = e^t = x$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} (chain\ rule)$$

$$= \frac{dy}{dx} x - - - - - - (1)$$

Euler-Cauchy Differential Equations

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(\frac{dy}{dt}) = \frac{d}{dx} \times (\frac{dy}{dt}) \times \frac{dx}{dt} (chain\ rule)$$

$$= \frac{d}{dx} \times (x\frac{dy}{dx}) \times x = [x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1]x$$

$$= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - - - - - - - -(2)$$

$$xy^{'} = x\frac{dy}{dx} = \frac{dy}{dt} - - - - - - - -(1)$$

$$x^2y^{''} = x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - x\frac{dy}{dx} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Euler-Cauchy Differential Equations

原式$x^2 y'' - 2xy' + 2y = 0$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = 0$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0 \quad \text{常係數}$$

$$(1) x\frac{dy}{dx} = \frac{dy}{dt} \Rightarrow xDy = \wp y$$

$$(2) x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} \Rightarrow x^2D^2y = \wp^2 y - \wp y = (\wp^2 - \wp)y$$

$$(3) x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} \Rightarrow x^3D^3y = \wp^3 y - 3\wp^2 y + 2\wp y$$

$$= (\wp^3 - 3\wp^2 + 2\wp)y = \wp(\wp - 1)(\wp - 2)y$$

Euler-Cauchy Differential Equations

$$\text{其中}\frac{d^3y}{dt^3}=\frac{d}{dt}(\frac{d^2y}{dt^2})$$

$$=\frac{d}{dx}(\frac{d^2y}{dt^2})\frac{dx}{dt}$$

$$=\frac{d}{dx}(x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx})x$$

$$=[x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+x\frac{d^2y}{dx^2}+\frac{dy}{dx}]x$$

$$=x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}$$

Euler-Cauchy Differential Equations

$$x^3 \frac{d^3 y}{dx^3} = \frac{d^3 y}{dt^3} - 3x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx}$$

$$= \frac{d^3 y}{dt^3} - 3(\frac{d^2 y}{dt^2} - \frac{dy}{dt}) - \frac{dy}{dt}$$

$$= \frac{d^3 y}{dt^3} - 3\frac{d^2 y}{dt^2} + 2\frac{dy}{dt}$$

Euler-Cauchy Differential Equations

例: $x^2 y'' + 4xy' + 2y = 0$

令$x = e^t \quad \wp \equiv \dfrac{d}{dt}$

$$\wp(\wp - 1)y + 4\wp y + 2y = 0$$

$$(\wp^2 + 3\wp + 2)y = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$$y = C_1 e^{-t} + C_2 e^{-2t}$$

$$= C_1 x^{-1} + C_2 x^{-2}$$

Euler-Cauchy Differential Equations

例: $x^3 y''' + 4x^2 y'' - 5xy' - 15y = 0$

令$x = e^t, \wp \equiv \dfrac{d}{dt}$

$\wp(\wp - 1)(\wp - 2)y + 4\wp(\wp - 1)y - 5\wp y - 15y = 0$

$(\wp^3 + \wp^2 - 7\wp - 15)y = 0$

$\lambda^3 + \lambda^2 - 7\lambda - 15 = 0$

$(\lambda - 3)(\lambda^2 + 4\lambda + 5) = 0$

$\lambda = 3, -2 \pm i$

$y(t) = C_1 e^{3t} + e^{-2t}(C_2 \cos t + C_3 \sin t)$

$y(x) = C_1 x^3 + x^{-2}(C_2 \cos(\ln x) + C_3 \sin(\ln x))$

Euler-Cauchy Differential Equations

例: $x^2 y'' - xy' - 3y = 4x$

$y = y_h + y_p$

$y_h : x^2 y_h'' - x y_h' - 3y_h = 0$

令$x = e^t, \wp = \dfrac{d}{dt}$

$\wp(\wp - 1)y - \wp y - 3y = 0$

$(\wp^2 - 2\wp - 3)y = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda = 3, -1$

Euler-Cauchy Differential Equations

$$y_h = C_1 e^{3t} + C_2 e^{-t}$$

$$= C_1 x^3 + C_2 x^{-1}$$

$$y_p : x^2 y_p'' - x y_p' - 3 y_p = 4x$$

[法1] *Undetermined Coefficient*

$$y_p'' - \frac{1}{x} y_p' - \frac{3}{x^2} y_p = \frac{4}{x}$$

Euler-Cauchy Differential Equations

[法2] $Order\ \ Reduction$

$$(\wp^2\text{-}2\wp\text{-}3)y_p = 4e^t$$

$$(\wp-3)(\wp+1)y_p = 4e^t, z(t) = (\wp+1)y_p$$

$$z^{'}(t) - 3z(t) = 4e^t$$

$$z_0(t) = I_1^{-1}\int I_1 r dt$$

$$I_1 = e^{-3t}, r = 4e^t$$

$$(\wp+1)y_p = z_p = I_1^{-1}\int I_1 r dt$$

Euler-Cauchy Differential Equations

$$y_p^{'} + y_p = I_1^{-1} \int I_1 r dt$$

$$y_p = C I_2^{-1} + I_2^{-1} \int I_2 r^{'} dt$$

$$I_2 = e^t$$

$$r^{'} = I_1^{-1} \int I_1 r dt$$

$$y_p = I_2^{-1} \int I_2 I_1^{-1} \int I_1 r dt dt$$

$$= e^{-t} \int e^t e^{3t} \int e^{-3t} 4e^t dt dt$$

$$= e^{-t} \int -2e^{2t} dt$$

$$= e^{-t} (-1) e^{2t} = -e^t = -x$$

Euler-Cauchy Differential Equations

[法3] *Differential Operator*

$$(\wp^2\text{-}2\wp\text{-}3) = 4e^t$$

$$y_p = \frac{1}{\wp^2\text{-}2\wp\text{-}3} \times 4e^t$$

$$= \frac{1}{1-2-3} 4e^t = -e^t = x$$

Euler-Cauchy Differential Equations

[法4] *Variation of Variable*

$$y_1 = e^{3t}, y_2 = e^{-t}$$

$$w(y_1, y_2) = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} = -4e^{2t}$$

$$y_p = e^{3t}\int \frac{-4e^{t} \times e^{-t}}{-4e^{2t}} dt + e^{-t}\int \frac{e^{3t} \times 4e^{t}}{-4e^{2t}} dt$$

$$= e^{3t}\int e^{-2t} dt + e^{-t}\int -e^{2t} dt$$

$$= e^{3t}(\frac{-1}{2}e^{-2t}) + e^{-t}(-\frac{1}{2}e^{2t}) = \frac{-1}{2}e^{t} - \frac{1}{2}e^{t} = -e^{t} = -x$$

Euler-Cauchy Differential Equations

*注意 若是用 $y_h = C_1 x^3 + C_2 x^{-1}$

$$w(y_1, y_2) = \begin{vmatrix} x^3 & x^{-1} \\ 3x^2 & -x^{-2} \end{vmatrix} = -4x$$

$$y_p = y_1 \int \frac{-y_2 r}{w} dx + y_2 \int \frac{y_1 r}{w} dx$$

$$= x^3 \int \frac{-x^{-1}(r(x))}{-4x} dx + x^{-1} \int \frac{x^3 (r(x))}{-4x} dx$$

$$= -x$$

Euler-Cauchy Differential Equations

例: $(2x-3)^2 y'' - 6(2x-3)y' + 12y = 0$

令$u = 2x-3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} (chain\ rule)$$

$$= \frac{dy}{du} \times 2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\frac{dy}{du}\right)$$

$$= \frac{d}{du}[\ldots]\frac{du}{dx}$$

$$= \frac{d}{du}\left[2\frac{dy}{du}\right] \times 2 = 4\frac{d^2y}{du^2}$$

Euler-Cauchy Differential Equations

$$原式 = u^2 \times 4 \frac{d^2 y}{du^2} - 6u \times 2 \frac{dy}{du} + 12y = 0$$

$$4u^2 \frac{d^2 y}{du^2} - 12u \frac{dy}{du} + 12y = 0$$

$$令 u = e^t, \wp = \frac{d}{dt}$$

$$[4\wp(\wp - 1) - 12\wp + 12]y = 0$$

$$[4\wp^2 - 16\wp + 12]y = 0$$

$$[\wp^2 - 4\wp + 3]y = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1\ ,\ 3$$

Euler-Cauchy Differential Equations

$$y(t) = C_1 e^t + C_2 e^{3t}$$

$$\Rightarrow y(u) = C_1 u + C_2 u^3$$

$$y(x) = C_1(2x-3) + C_2(2x-3)^3$$

例: $(3x+4)^2 y'' - 6(3x+4)y' + 18y = 9\ln(3x+4)$

令$u = 3x+4$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3\frac{dy}{du}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\frac{dy}{dx}\frac{du}{dx}$$

Euler-Cauchy Differential Equations

$$= \frac{d}{du} \times 3 \times \frac{dy}{du} \times 3 = 9 \frac{d^2 y}{du^2}$$

$$原式: u^2 \times 9 \frac{d^2 y}{du^2} - 6u \times 3 \frac{dy}{du} + 18y = 9 \ln(u)$$

$$u^2 \frac{d^2 y}{du^2} - 2u \frac{dy}{du} + 2y = \ln(u)$$

$$令 u = e^t, \wp = \frac{d}{dt}$$

$$(\wp(\wp - 1) - 2\wp + 2) y = \ln(u) = t$$

$$(\wp^2 - 3\wp + 2) y = t$$

Euler-Cauchy Differential Equations

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1,2$$

$$y_h = C_1 e^t + C_2 e^{2t}$$

$$= C_1 u + C_2 u^2$$

$$= C_1(3x+4) + C_2(3x+4)^2$$

Euler-Cauchy Differential Equations

$$y_p = \frac{1}{D^2 - 3D + 2} \cdot t$$

$$= \frac{1}{2} \cdot \frac{1}{1 + \dfrac{D^2 - 3D}{2}} \cdot t$$

$$= \frac{1}{2}\left[1 - \frac{D^2 - 3D}{2} + \left(\frac{D^2 - 3D}{2}\right)^2 - \cdots\right] t$$

$$= \frac{1}{2} t - \frac{1}{4} \cdot 0 + \frac{3}{4}$$

Euler-Cauchy Differential Equations

$$= \frac{1}{2}t + \frac{3}{4}$$

$$= \frac{1}{2}\ln(3x+4) + \frac{3}{4}$$

$$y = y_h + y_p$$

$$= C_1(3x+4) + C_2(3x+4)^2 + \frac{1}{2}\ln(3x+4) + \frac{3}{4}$$

Bernoulli Equation

- Bernoulli equation [非線性]

$$y'(x)+p(x)y=r(x)y^n$$

$$n=1,\ y'(x)+(p-r)y=0$$

$$n\neq 0, n\neq 1, y^{-n}y'+py^{1-n}=r\text{-------}(*)$$

令 $z=y^{1-n}$

$$\frac{dz}{dx}=(1-n)y^{1-n-1}\frac{dy}{dx}$$

$$=(1-n)y^{-n}\frac{dy}{dx}$$

$$y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dz}{dx}$$

Bernoulli Equation

$$原式(*) = \frac{1}{1-n}\frac{dz(x)}{dx} + p(x)z(x) = r(x)$$

$$\frac{dz(x)}{dx} + (1-n)p(x)z(x) = (1-n)r(x)$$

$$\frac{dz}{dx} + p'z = r'$$

$$z = CI^{-1} + I^{-1}\int Ir'dx$$

$$I = e^{\int p'dx}$$

$$z = Ce^{-\int(1-n)pdx} + e^{-\int(1-n)pdx}\int e^{\int(1-n)pdx}(1-n)r(x)dx = y^{1-n}$$

Bernoulli Equation

例: $x\dfrac{dy}{dx}+y=x^2y^2$

$$\frac{dy}{dx}+\frac{1}{x}y=xy^2=xy^n$$

$$\Rightarrow n=2$$

令 $z=y^{1-n}=y^{-1}$

$$\frac{dz}{dx}-\frac{1}{x}z(x)=-x \quad I=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=\frac{1}{x}$$

$$z(x)=C_1(x^{-1})^{-1}+(x^{-1})^{-1}\int x^{-1}(-x)dx$$

$$=Cx-x^2=y^{-1},\quad y=\frac{1}{Cx-x^2}$$

Bernoulli Equation

- 非線性→線性

  型式(1)

$$f'(y)\frac{dy}{dx}+p(x)f(y)=g(x)$$

$$\text{令}z=f(y)$$

$$\frac{dz}{dx}=\frac{dz}{dy}\times\frac{dy}{dx}$$

$$=\frac{df(y)}{dy}\times\frac{dy}{dx}=f'(y)\times\frac{dy}{dx}$$

$$\text{原式}:\frac{dz}{dx}+p(x)z=q(x)$$

Bernoulli Equation

例: $x^2 \cos y \dfrac{dy}{dx} = 2x \sin y - 1$

令 $z = \sin y$ 則 $\dfrac{dz}{dx} = \cos y \cdot \dfrac{dy}{dx}$

$$\frac{dz}{dx} - \frac{2}{x} z = \frac{-1}{x^2}$$

$$z = CI^{-1} + I^{-1} \int Ir dx$$

對於型式(1)

$f(y)$同常等於 $y^2, y^3, \cdots, \sin(y) \cdots e^y$

Bernoulli Equation

型式(2)

Bernoulli

型式(3)

Riccati

$$\frac{dy}{dx} + p(x)y = q(x) + y^2 r(x)$$

若 y_1 為上式之一特解

則令 $y = y_1 + \frac{1}{z}$ 得Z的線性D.E.