110-1 ENGINEERING MATHEMATICS HW3_SOL

1. (5-1) 11

$$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} = 2\cdot 1\cdot c_1 x^0 + \underbrace{\sum_{n=2}^{\infty} 2nc_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty} 6c_n x^{n+1}}_{k=n+1}$$

$$= 2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} 6c_{k-1}x^k$$

$$= 2c_1 + \sum_{k=1}^{\infty} [2(k+1)c_{k+1} + 6c_{k-1}]x^k$$

2. (5-1) 21

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - 2xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (2k-1)c_k]x^k = 0.$$

Thus

$$2c_2 + c_0 = 0$$
$$(k+2)(k+1)c_{k+2} - (2k-1)c_k = 0$$

and

$$c_2 = -\frac{1}{2}c_0$$
$$c_{k+2} = \frac{2k-1}{(k+2)(k+1)}c_k, \quad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2} \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = -\frac{1}{8} \qquad c_6 = -\frac{7}{240}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = \frac{1}{6} \qquad c_5 = \frac{1}{24} \qquad c_7 = 1112$$

and so on. Thus two solutions are

$$y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \cdots \qquad \text{and} \qquad y_2 = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots .$$

3.(5-2) 15.

Substituting $y=\sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}2xy''-y'+2y &= \left(2r^2-3r\right)c_0x^{r-1}+\sum_{k=1}^{\infty}\left[2(k+r-1)(k+r)c_k-(k+r)c_k+2c_{k-1}\right]x^{k+r-1}\\ &=0,\end{aligned}$$

which implies

$$2r^2-3r=r(2r-3)=0$$

and

$$(k+r)(2k+2r-3)c_k+2c_{k-1}=0.$$

The indicial roots are $r=0$ and $r=3/2$. For $r=0$ the recurrence relation is

$$c_k=-\frac{2c_{k-1}}{k(2k-3)},\qquad k=1,2,3,\ldots,$$

and

$$c_1=2c_0,\qquad c_2=-2c_0,\qquad c_3=\frac{4}{9}c_0,$$

and so on. For $r=3/2$ the recurrence relation is

$$c_k=-\frac{2c_{k-1}}{(2k+3)k},\qquad k=1,2,3,\ldots,$$

and

$$c_1=-\frac{2}{5}c_0,\qquad c_2=\frac{2}{35}c_0,\qquad c_3=-\frac{4}{945}c_0,$$

and so on. The general solution on $(0,\infty)$ is

$$y=C_1\left(1+2x-2x^2+\frac{4}{9}x^3+\cdots\right)+C_2x^{3/2}\left(1-\frac{2}{5}x+\frac{2}{35}x^2-\frac{4}{945}x^3+\cdots\right).$$

4. (5-3) 11.

If $y=x^{-1/2}v(x)$ then

$$\begin{aligned}y' &= x^{-1/2}v'(x)-\frac{1}{2}x^{-3/2}v(x),\\ y'' &= x^{-1/2}v''(x)-x^{-3/2}v'(x)+\frac{3}{4}x^{-5/2}v(x),\end{aligned}$$

and

$$x^2y''+2xy'+\alpha^2x^2y=x^{3/2}v''(x)+x^{1/2}v'(x)+\left(\alpha^2x^{3/2}-\frac{1}{4}x^{-1/2}\right)v(x)=0.$$

Multiplying by $x^{1/2}$ we obtain

$$x^2v''(x)+xv'(x)+\left(\alpha^2x^2-\frac{1}{4}\right)v(x)=0,$$

whose solution is $v=c_1J_{1/2}(\alpha x)+c_2J_{-1/2}(\alpha x)$.
Then $\boxed{y=c_1x^{-1/2}J_{1/2}(\alpha x)+c_2x^{-1/2}J_{-1/2}(\alpha x).}$

$$J_{\frac{1}{2}}(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\frac{1}{2}} \cdot n! \, \Gamma(n+\frac{1}{2}+1)} (\alpha x)^{2n+\frac{1}{2}}$$

$$J_{-\frac{1}{2}}(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n-\frac{1}{2}} \cdot n! \cdot \Gamma(n-\frac{1}{2}+1)} (\alpha x)^{2n-\frac{1}{2}}$$