Chapter 3.
Higher-Order Differential Equations

Chuan-Ching Sue

Dept. of Computer Science and Information Engineering,
National Cheng Kung University

Particular Solution

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = r(x)$$

其中 $a_0, a_1, a_2, \ldots, a_n \in Const,\ r(x) \neq 0$ 常係數非齊項O.D.E.

$$\text{特解求法} \begin{cases} \text{法一: undetermined coefficient} \\ \text{法二: 降階法} \rightarrow \text{聯想一階O.D.E.} \end{cases}$$

$$y = y_h + y_p$$

$$\Rightarrow \begin{cases} \text{法一: } \mathrm{e}^{\lambda \mathrm{x}} \rightarrow \text{找 } \lambda \text{ 求特性方程式} \\ \text{法二: D} \rightarrow \text{降階(重根)} \end{cases}$$

Differential Operator Method

- Method3:微分運算子法

$$D \equiv \frac{d}{dx} \qquad D^n \equiv \frac{d^n}{dx^n} \qquad D^{-1} \equiv \int dx$$

1. $y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = r(x)$

~~~~~~~~~~~~~~~~^~~~~~~~~~~~~~特性方程式

$\lambda^{(n)} + a_1 \lambda^{(n-1)} + \ldots + a_n = 0$ 求出 $\lambda_1, \lambda_2 \ldots, \lambda_n$

可以得到 $y_p = ?$

$$(D^n + a_1 D^{n-1} + \ldots + a_n) y = r(x) \ldots\ldots (*)$$

y_p 一定滿足(*)
~~~~~~~~~~~~~~~~

Differential Operator Method

$$(D^n + a_1 D^{n-1} + \ldots + a_n) y_p = r(x)$$

其中 $(D^n + a_1 D^{n-1} + \ldots + a_n)$

Linear Differential Operator (線性微分運算子)

定義為L(D)

$$\Rightarrow \; L(D) = D^n + a_1 D^{n-1} + \ldots + a_n$$

Differential Operator Method

- 特性1. $L(D)e^{ax} = L(a)e^{ax}$

例: $y''' + 6y'' + 11y' + 6y = e^{x}$

$$y_p = \frac{1}{24}e^{x}$$

$$(D^3 + 6D^2 + 11D + 6)y_p = e^{x}$$

$$\Rightarrow y_p = \frac{e^{x}}{D^3 + 6D^2 + 11D + 6}$$

$$= L(D)e^{x}$$

$$= L(1)e^{x} = \frac{e^{x}}{1+6+11+6} = \frac{1}{24}e^{x}$$

Differential Operator Method

- Pf:

$$L(D)e^{ax}$$

$$= (D^n + a_1 D^{n-1} + \dots + a_n)e^{ax}$$

$$= D^n e^{ax} + a_1 D^{n-1} e^{ax} + \dots + a_n e^{ax}$$

$$= a^n e^{ax} + a_1 a^{n-1} e^{ax} + \dots + a_n e^{ax}$$

$$(\because D^n = \frac{d^n}{dx^n} \therefore D^n e^{ax} = \frac{d^n e^{ax}}{dx^n} = a\frac{d^{n-1} e^{ax}}{dx^{n-1}} = \dots = a^n e^{ax})$$

$$= (a^n + a_1 a^{n-1} + \dots + a_n)e^{ax}$$

$$= L(a)e^{ax}$$

Differential Operator Method

例: $y'' + 3y' + 2y = e^{2x}$

$$y = y_h + y_p$$

$$y_h : \lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1, -2$$

$$y_h = C_1 e^{-x} + C_2 e^{-2x}$$

Differential Operator Method

- 求 y_p：

法一: 降階法

$$=> e^{-2x}\int e^{2x}[e^{-x}\int e^{x}e^{2x}dx]dx$$

$$\frac{1}{3}e^{3x}$$

$$\frac{1}{3}e^{4x}$$

$$\frac{1}{12}e^{4x}$$

$$=\frac{1}{12}e^{2x}$$

法二: 微分運算子法

$$(\mathrm{D}^2+3\mathrm{D}+2)y_p=e^{2x}$$

$$=> y_p=\frac{1}{\mathrm{D}^2+3\mathrm{D}+2}e^{2x}$$

$$=\frac{1}{2^2+3*2+2}e^{2x}=\frac{1}{12}e^{2x}$$

法三: 未定係數法

Differential Operator Method

例: $y' - 2y = e^{2x}$

$$y' - 2y = e^{2x}$$

$$y_h : \lambda = 2$$

$$y_h = Ce^{2x}$$

$$y_p = (D-2)y_p = e^{2x}$$

$$y_p = \frac{e^{2x}}{D-2} = \frac{e^{2x}}{2-2} = ?$$

(類似於特定係數法.當r(x)與 $e^{\lambda x}$ 相同時獲重根時的問題)

Differential Operator Method

- 特性2. $L(D)[e^{ax} f(x)] = e^{ax} L(D+a)[f(x)]$

$$y_p = \frac{1}{D-2} e^{2x}$$

$$= e^{2x} \left. \frac{1}{D-2} \right|_{D = D+2} \times 1$$

$$= e^{2x} \frac{1}{D+2-2} \times 1$$

$$= e^{2x} \frac{1}{D} \times 1$$

$$= e^{2x} \int 1 dx$$

$$= e^{2x} x$$

Differential Operator Method

例: $y^{''} + 4y^{'} + 4y = e^{-2x}$

$$y_p = \frac{1}{D^2 + 4D + 4} e^{-2x}$$

$$= \frac{1}{(D+2)^2} e^{-2x} \times 1$$

$$= e^{-2x} \frac{1}{(D+(-2)+2)^2} \times 1$$

$$= e^{-2x} \frac{1}{D^2} \times 1$$

$$= e^{-2x} \int\int 1 dx dx = \frac{1}{2} x^2 e^{-2x}$$

$$y = y_p + y_h$$

$$y_h : \lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0$$

$$\lambda = \pm 2$$

$$y_h = C_1 e^{-2x} + C_2 x e^{-2x}$$

Differential Operator Method

- Pf 特性2:

$$L(D)e^{ax} f(x) = (D^n + a_1 D^{n-1} + \ldots + a_n)e^{ax} f(x)$$

$$= D^n e^{ax} f(x) + a_1 D^{n-1} e^{ax} f(x) + \ldots + a_n e^{ax} f(x)$$

$$De^{ax} f(x) = e^{ax} Df(x) + ae^{ax} f(x)$$

$$= e^{ax}(D+a) f(x)$$

$$D^2 e^{ax} f(x) = D(De^{ax} f(x))$$

$$= D(e^{ax} Df(x) + ae^{ax} f(x))$$

$$= e^{ax} D^2 f(x) + ae^{ax} Df(x) + ae^{ax} Df(x) + a^2 e^{ax} Df(x)$$

$$= e^{ax}\left(D^2 + 2aD + a^2\right) f\left(x\right)$$

$$= e^{ax}\left(D+a\right)^2 f\left(x\right)$$

$\ldots$

Differential Operator Method

(用數學歸納法或類推法)

$$D^n e^{ax} f(x) = e^{ax} (D+a)^n f(x)$$

$$\begin{aligned}
&L(D) e^{ax} f(x) \\
&= e^{ax} (D+a)^n f(x) + a_1 e^{ax} (D+a)^{n-1} f(x) + \cdots + a_{n-1} e^{ax} (D+a) f(x) + a_n e^{ax} f \\
&= e^{ax} \left[(D+a)^n + a_1 (D+a)^{n-1} + \cdots + a_{n-1} (D+a) + a_n \right] f(x) \\
&= e^{ax} L(D+a) f(x)
\end{aligned}$$

Differential Operator Method

- 特性3. $L\left(D^2\right)\sin ax = L\left(-a^2\right)\sin ax$

$$L\left(D^2\right)\cos ax = L\left(-a^2\right)\cos ax$$

例: $y'' + 4y = \cos 3x$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$\left(D^2 + 4\right) y_p = \cos 3x$$

$$y_p = \frac{1}{D^2 + 4}\cos 3x \quad (\because a = 3)$$

$$= \frac{1}{-3^2 + 4}\cos 3x$$

$$= -\frac{1}{5}\cos 3x$$

Differential Operator Method

Pf特性3：

$$D\cos ax = -a\sin ax$$

$$\begin{aligned} D^2\cos ax &= D(D\cos ax) \\ &= D(-a\sin ax) \\ &= -a^2\cos ax \end{aligned}$$

$$\therefore \mathrm{D}^2 \equiv -a^2$$

$$L(D^2) \equiv L(-a^2)$$

Differential Operator Method

例: $y'' + a^2 y = \cos ax$

$$\lambda^2 + a^2 = 0 \quad \lambda = \pm ai$$

$$y_h = C_1 \cos ax + C_2 \sin ax$$

$$\left(D^2 + a^2\right) y_p = \cos ax$$

$$\Rightarrow y_p = \frac{1}{D^2 + a^2} \cos ax$$

$$= \frac{1}{-a^2 + a^2} \cos ax$$

$=$？ (因此用極限)

Differential Operator Method

$$\Rightarrow y_p = \frac{1}{D^2 + a^2} \cos ax$$

$$= \lim_{\Delta \to 0} \frac{1}{-(a+\Delta)^2 + a^2} \cos(a+\Delta)x$$

$$= \lim_{\Delta \to 0} \frac{1}{-2a\Delta - \Delta^2} \cos(a+\Delta)x$$

Differential Operator Method

$$\left(\begin{array}{l} f(t)\text{於 } t = \mathrm{a} \text{ 之 } \mathit{Taylor} \text{ 級數展開式} \\ f(t) = f(a) + f'(a)(t-a) + \dfrac{1}{2\,!} f''(a)(t-a)^2 + \cdots + \\ \qquad \dfrac{1}{n\,!} f^{(n)}(a)(t-a)^n + \cdots \\ \cos t \text{ 於 } t = ax \text{ 之 Taylor 展開} \\ \cos t = \cos ax - \sin ax(t-ax) - \dfrac{1}{2\,!}\cos ax(t-ax)^2 + \dfrac{1}{3\,!}\sin ax(t-ax)^3 + \cdots \end{array}\right)$$

Differential Operator Method

其中 令 $t=(a+\Delta)x$

$$\cos(a+\Delta)x=\cos ax-\sin ax\left((a+\Delta)x-ax\right)-\frac{1}{2!}\cos ax\left((a+\Delta)x-ax\right)^2+$$

$$\frac{1}{3!}\sin ax\left((a+\Delta)x-ax\right)^3+\cdots$$

$$=\cos ax-\sin ax\Delta x-\frac{1}{2!}\cos ax(\Delta x)^2+\frac{1}{3!}\sin ax(\Delta x)^3+\cdots$$

Differential Operator Method

$$\Rightarrow y_p = \lim_{\Delta\to 0}\frac{1}{-(2a\Delta+\Delta^2)}\left[\cos ax - \Delta x\sin ax - \frac{1}{2!}(\Delta x)^2\cos ax + \frac{1}{3!}(\Delta x)^3\sin ax + \cdots\right]$$

$$\left(\begin{array}{l}\text{因解不下去,想一想}\cos(ax)\text{是否可以不考慮?}\\ \text{YES 因為 } y_h \text{ 已含}\cos(ax)\text{, 可以消去}\end{array}\right)$$

$$= \lim_{\Delta\to 0}\frac{1}{-(2a+\Delta)}\left[-x\sin ax - \frac{1}{2!}\Delta x^2\cos ax + \frac{1}{3!}\Delta^2 x^3\sin ax + \cdots\right]$$

$$= \frac{1}{-2a} - x\sin ax = \frac{1}{2a}x\sin ax$$

$$\Rightarrow y'' + a^2 y = \cos ax$$

$$y = C_1\cos ax + C_2\sin ax + \frac{1}{2a}x\sin ax$$

Example Practice

例: $y'' + 6y' + 9y = x^2$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

$$\lambda = -3, -3$$

$$y_h = C_1 e^{-3x} + C_2 x e^{-3x}$$

Example Practice

- [法1] 未定係數法

$$y_p = K_1 x^2 + K_2 x + K_3$$

代入 $y_p' = ?$

$y_p'' = ?$

代入原式求 $\mathrm{K}_1\mathrm{K}_2\mathrm{K}_3$

Example Practice

- [法2] 降階法

$$\left(D^2+6D+9\right)y_p=x^2$$

$$\Rightarrow \left(\mathrm{D}+3\right)\left(\mathrm{D}+3\right)y_p=x^2$$

$$\Rightarrow e^{-3x}\int e^{3x}\left[e^{-3x}\int e^{3x}x^2dx\right]dx=?$$

Example Practice

- [法3] 微分運算子法

$$\left(D^2+6D+9\right)y_p = x^2$$

$$y_p = \frac{1}{\left(D^2+6D+9\right)}x^2$$

$$= \frac{1}{9(1+\frac{D^2+6D}{9})}x^2$$

$$= \frac{1}{9}\left[1-\frac{D^2+6D}{9}+\left(\frac{D^2+6D}{9}\right)^2-\left(\frac{D^2+6D}{9}\right)^3+\cdots\right]x^2$$

Example Practice

$$= \frac{1}{9}\left[1 - \frac{D^2+6D}{9} + \frac{D^4+12D^3+36D^2}{81} + \cdots\right]x^2$$

$$= \frac{1}{9}\left[x^2 - \frac{1}{9}(2+12x) + \frac{1}{81}\cdot 36\cdot 2\right]$$

$$= \frac{1}{9}\left(x^2 - \frac{2}{9} - \frac{12}{9}x + \frac{8}{9}\right)$$

$$= \frac{1}{9}\left(x^2 - \frac{4}{3}x + \frac{2}{3}\right)$$