# Chapter 4.
# Laplace Transform

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# Laplace Transform

- Review：

$$f(t) = e^{at} \xrightarrow{\mathscr{L}} \frac{1}{s-a}$$

$$f(t) = \cos at \xrightarrow{\mathscr{L}} \frac{s}{s^2 + a^2}$$

$$f(t) = \sin at \xrightarrow{\mathscr{L}} \frac{a}{s^2 + a^2}$$

$$f(t) = t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$f(t) = H(t) \xrightarrow{\mathscr{L}} \frac{1}{s}$$

$$f(t) = \delta(t) \xrightarrow{\mathscr{L}} 1$$

$$f(t) = t^n \xrightarrow{\mathscr{L}} \frac{n!}{s^{n+1}}$$

# Laplace Transform

- Laplace Transform之基本性質：

設$\mathscr{L}\{f(t)\}=F(s),\mathscr{L}\{g(t)\}=G(s)$已知

1. $\mathscr{L}\{k_1f(t)+k_2g(t)\}=k_1\mathscr{L}\{f(t)\}+k_2\mathscr{L}\{g(t)\}=k_1F(s)+k_2G(s)$

線性轉換 $k_1,k_2\in const$

$$pf:\mathscr{L}\{k_1f(t)+k_2g(t)\}=\int_0^\infty (k_1f(t)+k_2g(t))e^{-st}dt$$

$$=\int_0^\infty k_1f(t)e^{-st}dt+\int_0^\infty k_2g(t)e^{-st}dt$$

$$=k_1F(s)+k_2G(s)$$

# Laplace Transform

2. First shifting Thm.(第一移位定理)

$$\mathrm{f(t)} \xrightarrow{\mathscr{L}} \mathrm{F}(s) = \mathscr{L}\{f(t)\}$$

$$\mathrm{e}^{\mathrm{at}} f(t) \xrightarrow{\mathscr{L}} F(s-a)$$

$$pf : \mathscr{L}\{\mathrm{e}^{\mathrm{at}} f(t)\} = \int_0^{\infty} \mathrm{e}^{\mathrm{at}} f(t) e^{-st} dt$$

$$= \int_0^{\infty} f(t) e^{-(s-a)t} dt$$

$$= \int_0^{\infty} f(t) e^{-s' t} dt = F(s') = F(s-a)$$

- 例： $H(t) \xrightarrow{\mathscr{L}} \dfrac{1}{s}, e^{at} \xrightarrow{\mathscr{L}} \dfrac{1}{s-a}$

$$\Rightarrow e^{at} H(t) \xrightarrow{\mathscr{L}} H(s-a) = \frac{1}{s}\Big|_{s \to s-a} = \frac{1}{(s-a)}$$

# Laplace Transform

- 例： $\mathscr{L}\{e^{-2t}t\}$

$$\Rightarrow \text{已知} f(t) \to F(s) = \frac{1}{s^2}$$

$$\Rightarrow F(s-(-2)) = F(s+2) = \frac{1}{s^2}\Big|_{s\to s+2} = \frac{1}{(s+2)^2}$$

- 例： $\mathscr{L}\{e^{t}\cos 2t\}$

$$= F(s-1) \qquad \text{其中F(s)} = \frac{s}{s^2+4}$$

$$= \frac{s}{s^2+4}\Big|_{s\to s-1}$$

$$= \frac{s-1}{(s-1)^2+4} = \frac{s-1}{s-2s+5}$$

# Laplace Transform

- 例： $\mathscr{L}\{e^{3t}\sin 5t\}$

$$\mathscr{L}\{e^{3t}\sin 5t\}$$

$$= F(s-3) = \frac{5}{s^2+25}\Big|_{s\to s-3}$$

$$= \frac{5}{(s-3)^2+25} = \frac{5}{s^2-6s+34}$$

- 例： $\mathscr{L}^{-1}\{\frac{2}{s^2+2s+5}\}$

$$= \mathscr{L}^{-1}\{\frac{2}{(s+1)^2+2^2}\}$$

$$= \sin(2t)e^{-t}$$

# Laplace Transform

- 例：$\mathscr{L}^{-1}\{\frac{s}{(s+2)^2+4}\}$

$$= \mathscr{L}^{-1}\{\frac{s+2-2}{(s+2)^2+4}\}$$

$$= \mathscr{L}^{-1}\{\frac{s+2}{(s+2)^2+4}\} + \mathscr{L}^{-1}\{\frac{-2}{(s+2)^2+4}\}$$

$$\Rightarrow f(t) = e^{-2t}\cos 2t - e^{-2t}\sin 2t$$

# Laplace Transform

3. Second shifting Thm.(第二移位定理)

$$\mathrm{f(t)} \xrightarrow{\mathscr{L}} F(s)$$

$$f(t-a)H(t-a) \xrightarrow{\mathscr{L}} F(s)e^{-as}$$

$$pf: \mathscr{L}\{f(t-a)H(t-a)\}$$

$$= \int_0^\infty f(t-a)H(t-a)e^{-st}dt \qquad \because H(t-a) = \begin{cases} 1, t > a \\ 0, t < a \end{cases}$$

$$= \int_a^\infty f(t-a)e^{-st}dt \qquad 令 x = t-a, dx = dt$$

$$= \int_0^\infty f(x)e^{-sx}e^{-as}dx$$

$$= e^{-as}\int_0^\infty f(x)e^{-sx}dx$$

$$= e^{-as}F(s)$$

# Laplace Transform

- 例： $f(t) = e^{2t} \xrightarrow{\mathscr{L}} \dfrac{1}{s-2}$

  $\mathscr{L}\{e^{2(t-3)}\} = F(s)e^{-as} = \dfrac{1}{s-2}e^{-3s}$

- 例：$\mathscr{L}\{\cos(t-2)H(t-2)\}$

  $\because \mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1}$

  $= \dfrac{s}{s^2+1}e^{-2s}$

- 例：$G(s) = e^{-3s}\dfrac{s+1}{(s+1)^2+1}$

  $\Rightarrow g(t) = e^{-(t-3)}\cos(t-3)H(t-3)$

# Laplace Transform

- 例： $f(t)=t, \mathscr{L}\{f(t)\}=\dfrac{1}{s^2}$

$$\Rightarrow \mathscr{L}\{f(t-2)\}=\mathscr{L}\{t-2\}=\mathscr{L}\{t\}-\mathscr{L}\{2\}=\frac{1}{s^2}-\frac{2}{s}$$

$$\mathscr{L}\{f(t)H(t-2)\}$$
$$=\mathscr{L}\{tH(t-2)\}$$
$$=\mathscr{L}\{[(t-2)+2]H(t-2)\}$$
$$=\mathscr{L}\{(t-2)H(t-2)\}+L\{2H(t-2)\}$$
$$=e^{-2s}\frac{1}{s^2}+2\frac{1}{s}e^{-2s}$$

# Laplace Transform

- 例：$f(t)=t^2+3t+2$

$1.\mathscr{L}\{f(t)\}=\frac{2!}{s^3}+3\frac{1}{s^2}+2\frac{1}{s}$

$2.\mathscr{L}\{f(t-1)\}$

$=\mathscr{L}\{(t-1)^2+3(t-1)+2\}$

$=\mathscr{L}\{t^2+t\}$

$=\frac{2!}{s^3}+\frac{1}{s^2}$

$3.\mathscr{L}\{f(t)H(t-1)\}$

$=\mathscr{L}\{(t^2+3t+2)H(t-1)\}$

$=\mathscr{L}\{((t-1)^2+A(t-1)+B)H(t-1)\}$ $\quad A=5,B=6$

$=\frac{2}{s^3}e^{-s}+5\frac{1}{s^2}e^{-s}+6\frac{1}{s}e^{-s}$

$4.\mathscr{L}\{f(t-1)H(t-1)\}=[\frac{2!}{s^3}+3\frac{1}{s^2}+2\frac{1}{s}]e^{-s}$

# Laplace Transform

$$4. f(t) \xrightarrow{h} F(s)$$

$$tf(t) \xrightarrow{h} \frac{-dF(s)}{ds}$$

- 例：$1 \xrightarrow{\mathscr{L}} \frac{1}{s}$

$$t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$t \xrightarrow{\mathscr{L}} \frac{-d}{ds}(\frac{1}{s}) = \frac{-1}{-s^2} = \frac{1}{s^2}$$

# Laplace Transform

- 例：$t^n \xrightarrow{\mathscr{L}} \dfrac{n!}{s^{n+1}}$

$$t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$t^2 \xrightarrow{\mathscr{L}} -\left(\frac{d}{ds}\left(\frac{1}{s^2}\right)\right) = \frac{-(-2s)}{(s^2)^2} = \frac{2}{s^3}$$

$$\vdots$$

$$t^n \xrightarrow{\mathscr{L}} \frac{n!}{s^{(n+1)}}$$

# Laplace Transform

證明：$\mathscr{L}\{tf(t)\} = \int_0^\infty tf(t)e^{-st}dt$

$$\because \frac{d}{ds}e^{-st} = -te^{-st}$$

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(f(t)e^{-st})dt$$

$$= \int_0^\infty f(t)(-t)e^{-st}dt$$

$$= -\int_0^\infty f(t)te^{-st}dt$$

$$= -\mathscr{L}\{tf(t)\}$$

# Laplace Transform

- 例：$\mathscr{L}\{t\sin 2t\}$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\frac{0-2(2s)}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

- 例：$\mathscr{L}\{e^{2t}\cdot \underline{t\cdot\sin 2t}\} = F(s-2)$

$$\downarrow$$

$$\left.\frac{4s}{(s^2+4)^2}\right|_{s=s-2} = \frac{4\cdot(s-2)}{((s-2)^2+4)^2}$$

# Laplace Transform

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

證明：$1 \xrightarrow{h} \frac{1}{s}$

利用數學歸納法

$h=1 \quad \mathcal{L}\{t-1\} = \frac{-d}{ds} \cdot \frac{1}{s} = \frac{1}{s^2}$

設 $n=k-1$為真

$\Rightarrow \mathcal{L}\{t^{k-1}\} = \frac{(k-1)!}{s^k}$

欲證

$n=k$時亦為真

$\mathcal{L}\{t^k\} = \mathcal{L}\{t \cdot t^{k-1}\} = \frac{-d}{ds}\left(\frac{(k-1)!}{s^k}\right) = -(k-1)!(-k)s^{-k-1} = \frac{k!}{s^{k+1}}$ 故得證

# Laplace Transform

推廣：$t^n f(t) \xrightarrow{\mathscr{L}} \frac{-d}{ds} \cdots (\frac{-d}{ds} F(s))$

4. $tf(t) \xrightarrow{\mathscr{L}} \frac{-d}{ds} F(s)$

$$\frac{1}{t} f(t) \xrightarrow{\mathscr{L}} \int_s^{\infty} \int_0^{\infty} f(t) e^{-st} dt ds$$

$$\Rightarrow \int_0^{\infty} (1) ds = \int_s^{\infty} \int_0^{\infty} f(t) e^{-st} dt ds$$

$$= \int_0^{\infty} \int_s^{\infty} f(t) e^{-st} ds dt$$ (積分次序對調，要S .T在積分範圍獨立)

$$= \int_0^{\infty} f(t) \int_s^{\infty} e^{-st} ds dt$$

$$= \int_0^{\infty} f(t) \frac{1}{t} e^{-st} dt$$

$$= \mathscr{L}\{\frac{1}{t} f(t)\}$$