

# Review.

# Differential and Integral

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# Differential

- 極限

定義：

若  $\forall \varepsilon > 0, \exists \delta > 0$

st.  $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

則稱  $f(x)$  於  $x_0$  的極限為  $L$

記為  $\lim_{x \rightarrow x_0} f(x) = L$

# Differential

- 連續  $\Leftrightarrow$  極限 = 函數值

- (1)  $f(x)$  有意義

- (2)  $\lim_{x \rightarrow x_0} f(x) = L$  存在

- (3)  $L = f(x_0)$

則稱  $f(x_0)$  於  $x_0$  為連續

# Differential

- 有界

定義：

若  $f(x)$  於  $[a,b]$  為連續，則  $f(x)$  於  $[a,b]$  為有界，若為開區間  $(a,b)$  則不一定有界。

# Differential

- 分段(片段)連續 (piecewise continuous)

若  $\exists a < x_1 < x_2 < \dots < x_n < b$

(1)  $f(x)$  於  $(a, x_1)(x_2, x_3) \dots (x_n, b)$  為連續

(2)  $f(a^+)$ 、 $f(x_1^-)$ 、 $f(x_2^+)$ 、 $\dots$ 、 $f(x_n^+)$ 、 $f(b^-)$   
均存在

則稱  $f(x)$  於  $(a, b)$  分段連續

# Differential

- $f : V \rightarrow W$      $V、W$  : 集合  
定義域   對應域

$$f \in C_p([a, b])$$

$$C_p(V) \cong \{f \mid f \text{ 於 } V \text{ 為分段連續}\}$$

$$C(V) \cong \{f \mid f \text{ 於 } V \text{ 為連續}\}$$

# Differential

- 導數定義

$$f'(x_0) \cong \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

$f'(x_0)$  稱為  $f(x)$  於  $x_0$  的導數

若  $f'(x_0)$  存在，則稱  $f(x)$  於  $x_0$  為可微分

$$df = f'(x)dx$$

# Differential

- 對數微分法

$$\frac{df(x)}{dx} = f(x) \frac{d}{dx} \ln f(x)$$

例：

$$f(x) = \frac{(x-3)^2}{(x-2)^3(x+1)}$$

$$\Rightarrow \ln f(x) = 2\ln(x-3) - 3\ln(x-2) - \ln(x+1)$$

$$\Rightarrow \frac{d \ln f(x)}{dx} = \frac{2}{x-3} - \frac{3}{x-2} - \frac{1}{x+1}$$

$$\Rightarrow \frac{df(x)}{dx} = \frac{(x-3)^2}{(x-2)^3(x+1)} \left( \frac{2}{x-3} - \frac{3}{x-2} - \frac{1}{x+1} \right)$$



# Differential

$$\text{若 } f(x) = \frac{v_1(x)v_2(x)}{u_1(x)u_2(x)}$$

$$\Rightarrow \ln f(x) = \ln v_1(x) + \ln v_2(x) - \ln u_1(x) - \ln u_2(x)$$

$$\Rightarrow \frac{d \ln f(x)}{dx} = \frac{v_1'}{v_1} + \frac{v_2'}{v_2} - \frac{u_1'}{u_1} - \frac{u_2'}{u_1}$$

例：

$$f(x) = \frac{x^2 + 4x + 13}{(x^2 + 3x + 3)(x + 1)^4}$$

$$\Rightarrow \ln f(x) = \ln(x^2 + 4x + 13) - \ln(x^2 + 3x + 3) - 4 \ln(x + 1)$$

$$\Rightarrow \frac{df(x)}{dx} = f(x) \left( \frac{2x + 4}{x^2 + 4x + 13} - \frac{2x + 3}{x^2 + 3x + 3} - \frac{4(x + 1)^3}{(x + 1)^4} \right)$$

# Differential

- Formula

$$f, g \in C^n([a, b])$$

$$\Rightarrow \frac{d^n}{dx^n} (f * g) = \sum_{k=0}^n \frac{n!}{k!(n-k)!} f^{(n-k)} g^{(k)}$$

# Integral

- 不定積分 = 反(逆)導數

若存在  $F(x)$  使得  $F'(x) = f(x)$

則  $\int f(x)dx = F(x) + c$

例：  $\int \cos x dx = \sin x + c$

# Integral

- 微分運算子

$$D(f(x)) = \frac{d}{dx} f(x)$$

$$\Rightarrow \frac{1}{D} f(x) = D^{-1} f(x) = \int f(x) dx$$

# Integral

- 分部積分法

$$d(uv) = vdu + udv \quad \text{同取積分}$$

$$\Rightarrow \int d(uv) = \int (vdu + udv)$$

$$\Rightarrow uv = \int vdu + \int udv$$

$$\Rightarrow \int udv = uv - \int vdu$$

# Integral

$$\int uv' dx = uv - \int u'v dx \quad u, v \rightarrow u(x), v(x)$$

若定義  $f^{(1)} = \frac{d}{dx} f(x) \quad f_{(1)} = \int f(x) dx$

$$f^{(2)} = \frac{d^2}{dx^2} f(x) \quad f_{(2)} = \iint f(x) dx dx$$

$$u \rightarrow f, v' \rightarrow g$$

$$\Rightarrow \int fg dx = fg_{(1)} - \int f^{(1)} g_{(1)} dx$$

$$\begin{array}{ccc} f & \searrow \oplus & g \\ f^{(1)} & \xrightarrow{\ominus} & g_{(1)} \end{array}$$

# Integral

$$\int f^{(1)} g_{(1)} dx = f^{(1)} g_{(2)} - \int f^{(2)} g_{(2)} dx$$

$$\begin{array}{ccc} f^{(1)} & \begin{array}{c} \oplus \\ \diagdown \end{array} & g_{(1)} \\ f^{(2)} & \begin{array}{c} \ominus \\ \hline \end{array} & g_{(2)} \end{array}$$

例:  $\int (x^2 + 3x + 3) \cos x dx = ?$

*sol :*

$$\begin{array}{ccc} x^2 + 3x + 3 & \begin{array}{c} \oplus \\ \diagdown \end{array} & \cos x \\ 2x + 3 & \begin{array}{c} \ominus \\ \diagdown \end{array} & \sin x \\ 2 & \begin{array}{c} \oplus \\ \diagdown \end{array} & -\cos x \\ 0 & \begin{array}{c} \ominus \\ \hline \end{array} & -\sin x \end{array}$$

$$\Rightarrow (x^2 + 3x + 3) \sin x - (2x + 3)(-\cos x) + 2(-\sin x) + c$$

# Integral

- Euler 公式

$$\begin{cases} e^{i\theta} = \cos \theta + i \sin \theta \\ e^{-i\theta} = \cos \theta - i \sin \theta \end{cases}$$

$$\Rightarrow \begin{cases} \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{cases}$$



# Integral

$$\begin{cases} \int e^{ax} \cos(bx) dx = ? \dots\dots\dots (1) & \text{實} \\ \int e^{ax} \sin(bx) dx = ? \dots\dots\dots (2) & \text{虛} \end{cases}$$

$$\Rightarrow (1) + i(2)$$

$$= \int e^{ax} * e^{ibx} dx$$

$$= \frac{1}{a + ib} e^{(a+ib)x} = \frac{a - ib}{a^2 + b^2} e^{ax} (\cos bx + i \sin bx)$$

$$\text{實} = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$\text{虛} = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

# Integral

- 定積分

$$\int_a^b f(x)dx$$

$$= \lim_{(\Delta x)_{\max} \rightarrow 0} \sum_k f(x_k^*) \times \Delta x_k$$

$$\Delta x_k = x_k - x_{k-1}$$

$x_k^*$  : 為任一個在  $[x_k, x_{k-1}]$  之點

# Integral

- Thm：微積分第一基本定理

若  $f(x)$  於  $[a,b]$  連續，則存在  $F(x) \in C'([a,b])$

使得  $F'(x) = f(x)$  ,  $a \leq x \leq b$

$$\Rightarrow \int_a^b f(x)dx = F(b) - F(a)$$

$$\text{則 } \int_a^x f(u)du = F(x) - F(a)$$

$$\therefore \frac{d}{dx} \int_a^x f(u)du = F'(x) = f(x)$$

# Integral

例:  $\frac{d}{dx} \int_0^{x^3} e^{-u^2} du = ?$

Sol:

$$\text{令 } x^3 = \zeta$$

$$J(x) = \int_0^{\zeta} e^{-u^2} du$$

$$\Rightarrow \frac{dJ(x)}{dx} = \frac{dJ(x)}{d\zeta} \times \frac{d\zeta}{dx}$$

$$= e^{-\zeta^2} \times 3x^2$$

$$= e^{-x^6} \times 3x^2$$

# Introduction to differential equations

- 例：RC電路

$$i(t) = C \times \frac{dV_c(t)}{dt}$$

歐姆定律  $V_R = iR$

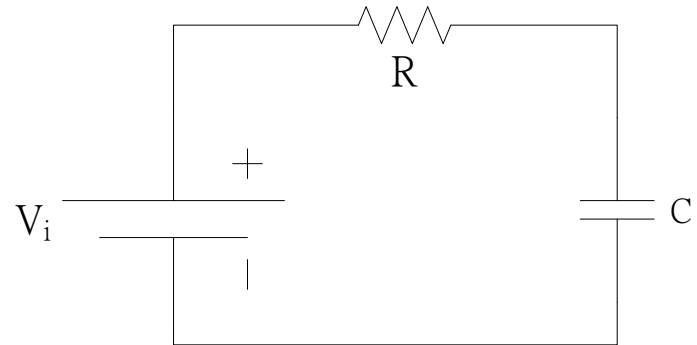
$$= CR \times \frac{dV_c(t)}{dt}$$

$$V_i = V_R + V_C$$

$$\Rightarrow V_i(t) = CR \times \frac{dV_c(t)}{dt} + V_c(t)$$

$$V_i : \text{given}$$

$$\Rightarrow V_c(t) = ?$$



# Definitions

- What is differential equations(微分方程式) ?  
一個方程式包含因變數及因變數之導數項

$$\frac{dy^2}{dt^2} + 3\frac{dy}{dt} + 5y = 3e^t \Rightarrow y(t)$$

$$CR \times \frac{dV_c(t)}{dt} + V_c(t) = V_i(t)$$

$\Rightarrow y(t), V_i(t)$  這樣，只有一個自變數 $t$ ，此微分方程式又稱**常微分方程式(Ordinary DE)**，若包含二項變數以上，則稱**偏微分方程式(Partial DE)**

# Definitions

$$\text{D.E.} \begin{cases} \text{O.D.E.} \\ \text{P.D.E.} \end{cases}$$

例:

$$\frac{\partial^2 v(x, y)}{\partial x^2} + \frac{\partial^2 v(x, y)}{\partial y^2} = 0 \Rightarrow \text{P.D.E.}$$

$$y'' + 4y' + 8y = 0 \Rightarrow \text{O.D.E.}$$

# Terminology

- Terminology 術語

1. 階數(Order):

一個微分方程式中，最高階導數項的微分次數即稱此微分方程式的階數

例:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + 5u(x,y) = 0 \Rightarrow \text{二階P.D.E}$$

$$\frac{d^3 y(x)}{dx^3} + 4(y(x))^5 = e^x \Rightarrow \text{三階O.D.E}$$



# Terminology

## 2.Degree(次數)：

**Pre:** 須將所有項有理化(ie：沒有  $x^{\frac{1}{2}}$  項)

一個微分方程式中，每項均為有理項，最高階導數項的次數，即稱為此方程式之次數

例：

$$y' = \sqrt{y} + 5y$$

$$\Rightarrow (y' - 5y)^2 = y$$

$$\Rightarrow (y')^2 - 25yy' + 25y^2 = y$$

$$\Rightarrow \text{一階二次O.D.E.}$$

$$y'' + 3y^2 = e^x$$

$$\Rightarrow \text{二階一次O.D.E.}$$