Chapter 3.
Higher-Order Differential Equations

Chuan-Ching Sue

Dept. of Computer Science and Information Engineering,
National Cheng Kung University

Review

例: $y'' + 6y' + 5y = 0$

$$\lambda^2 + 6\lambda + 5 = 0$$

$$\lambda = -1, -5$$

$$\Rightarrow y = C_1 e^{-x} + C_2 e^{-5x} \quad \leftarrow \text{這個 y 真的是解嗎?}$$

Review

$$y'' + ay' + by = 0 \qquad a, b \in const.$$

$$\lambda^2 + a\lambda + b = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$\Rightarrow a = -(\lambda_1 + \lambda_2), \quad b = \lambda_1\lambda_2$$

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$

Differential Operator

- Def.

$$D \equiv \frac{d}{dx}(\text{微分運算子})$$

$$D^k \equiv \frac{d^k}{dx^k}$$

$$D^2 y - (\lambda_1 + \lambda_2) Dy + \lambda_1 \lambda_2 y = 0$$

$$\left(D^2 - (\lambda_1 + \lambda_2) D + \lambda_1 \lambda_2\right) y = 0$$

$$(D - \lambda_1)(D - \lambda_2) y = 0$$

Differential Operator

$$令(D-\lambda_2)y=z$$

$$(D-\lambda_1)z=0$$

$$z'-\lambda_1 z=0$$

$$z=k_1e^{\lambda_1 x}$$

$$\Rightarrow (D-\lambda_2)y=k_1e^{\lambda_1 x}$$

$$y'-\lambda_2 y=k_1e^{\lambda_1 x}$$

$$I=e^{\int -\lambda_2 dx}=e^{-\lambda_2 x}$$

Differential Operator

$$\Rightarrow y = k_2 I^{-1} + I^{-1}\int I k_1 e^{\lambda_1 x} dx$$

$$= k_2 e^{\lambda_2 x} + e^{\lambda_2 x}\int k_1 e^{\left(\lambda_1-\lambda_2\right)x} dx = k_2 e^{\lambda_2 x} + e^{\lambda_2 x}\,\frac{k_1}{\lambda_1-\lambda_2}\,e^{\left(\lambda_1-\lambda_2\right)x}$$

$$= k_2 e^{\lambda_2 x} + \frac{k_1}{\lambda_1-\lambda_2} e^{\lambda_1 x}$$

$$\mathrm{C}_2 \qquad\qquad C_1$$

Differential Operator

Case(3):

$\lambda_1 = \lambda_2 = \alpha$ (重根)

$\therefore (\lambda - \alpha)(\lambda - \alpha) = 0$ (特性方程式)

$\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$

$\Rightarrow y'' - 2\alpha y' + \alpha^2 y = 0$ (原D.E.)

$\mathrm{D}^2 y - 2\alpha D y + \alpha^2 y = 0$

$(D - \alpha)(D - \alpha) y = 0$

令$(D - \alpha) y = z$

$(D - \alpha) z = 0$

$z' - \alpha z = 0$

$z = C_1 e^{\alpha x}$

Differential Operator

$$\Rightarrow \left(D-\alpha\right)y = C_1 e^{\alpha x}$$

$$y' - \alpha y = C_1 e^{\alpha x}$$

$$I = e^{\int -\alpha dx} = e^{-\alpha x}$$

$$\Rightarrow y = C_2 e^{\alpha x} + e^{\alpha x} \int e^{-\alpha x} C_1 e^{\alpha x} dx$$

$$= C_2 e^{\alpha x} + C_1 x e^{\alpha x}$$

$$\Rightarrow y = C_2 e^{\alpha x} + C_1 x e^{\alpha x}$$

Differential Operator

例: $y'' - 2y' + y = 0$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda = 1,1$$

$$\Rightarrow y = C_1 e^x + C_2 x e^x$$

例: $y'' + 4y' + 13y = 0$

$$\lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = -2 \pm 3i$$

$$\Rightarrow y = e^{-2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)$$

Differential Operator

例: $y'' + 6y' + 8y = 0$

$$\lambda^2 + 6\lambda + 8 = 0$$

$$\lambda = -4, -2$$

$$\Rightarrow y = C_1 e^{-4x} + C_2 e^{-2x}$$

例: $y'' + 10y' + 25y = 0$

$$\lambda^2 + 10\lambda + 25 = 0$$

$$\lambda = -5, -5$$

$$\Rightarrow y = C_1 e^{-5x} + C_2 x e^{-5x}$$

N-Order Constant Coefficients D.E

- 推廣N階常係數 O.D.E.

Def:

$$y^{(n)} \equiv \frac{d^n y}{dx^n}$$

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0$$

$$a_1, a_2, \cdots, a_n \in const.$$

N-Order Constant Coefficients D.E

Case(1) 相異實根

$$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n \in \Re$$

$$\Rightarrow y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}$$

Case(2) 相等實根

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \alpha \in \Re$$

$$\Rightarrow y = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \cdots + C_n x^{n-1} e^{\alpha x}$$

N-Order Constant Coefficients D.E

Case(3) 共軛複數根

$$\lambda_1, \lambda_2, \cdots, \lambda_{2k} \quad n = 2k$$

$$\alpha_j \pm \beta_j i \quad j = 1, 2, \cdots, k$$

$$\begin{aligned} \Rightarrow y = {} & e^{\alpha_1 x}\left(C_1 \cos \beta_1 x + C_2 \sin \beta_1 x\right) \\ & + e^{\alpha_2 x}\left(C_3 \cos \beta_2 x + C_4 \sin \beta_2 x\right) \\ & \cdots \\ & + e^{\alpha_k x}\left(C_{2k-1} \cos \beta_k x + C_{2k} \sin \beta_k x\right) \end{aligned}$$

N-Order Constant Coefficients D.E

Case(4)共軛複數根重根

$$\left(\alpha \pm \beta i\right)^k \qquad k\text{個重根}$$

$$\Rightarrow y = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right)$$

$$+ xe^{\alpha x}\left(C_3 \cos \beta x + C_4 \sin \beta x\right)$$

$$\cdots$$

$$+ x^{k-1}e^{\alpha x}\left(C_{2k-1} \cos \beta x + C_{2k} \sin \beta x\right)$$

N-Order Constant Coefficients D.E

例: 設一微分方程式的特性方程式的根分別為：

$$x_{1\sim16} = 1,\ 2,\ 3,\ 4,\ 4,\ 4,\ -2\pm3i,\ -3\pm2i,\ \left(-1\pm5i\right)^3$$

$$Y = ?$$

Sol:

$$\begin{aligned} y &= C_1e^{x} + C_2e^{2x} + C_3e^{3x} + C_4e^{4x} + C_5xe^{4x} \\ &\quad + C_6x^2e^{4x} + e^{-2x}\left(C_7\cos 3x + C_8\sin 3x\right) \\ &\quad + e^{-3x}\left(C_9\cos 2x + C_{10}\sin 2x\right) \\ &\quad + e^{-x}\left(C_{11}\cos 5x + C_{12}\sin 5x\right) \\ &\quad + xe^{-x}\left(C_{13}\cos 5x + C_{14}\sin 5x\right) \\ &\quad + x^2e^{-x}\left(C_{15}\cos 5x + C_{16}\sin 5x\right) \end{aligned}$$

Determine y_p

- 如何决定 y_p

例: $y' + 2y = e^{3x}$

$$y_h \Rightarrow y' + 2y = 0$$

$$\lambda + 2 = 0$$

$$\lambda = -2$$

$$y_h = Ce^{-2x}$$

Undetermined Coefficient

- Method 1 : Undetermined Coefficient (未定係數法)

$$y_p' + 2y_p = e^{3x}$$

猜 $y_p = ke^{3x}$ [依照 r (x) 函數的型式決定 y_p]

代入 $\left(ke^{3x}\right)' + 2\left(ke^{3x}\right) = e^{3x}$

$$3ke^{3x} + 2ke^{3x} = e^{3x}$$

$$5ke^{3x} = e^{3x}$$

$$\Rightarrow k = \frac{1}{5} \quad \therefore y_p = \frac{1}{5}e^{3x}$$

Undetermined Coefficient

例: $y'' + 3y' + 2y = e^x$

$$y_h \Rightarrow y_h'' + 3y_h' + 2y_h = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow y_h = C_1 e^{-x} + C_2 e^{-2x}$$

$y_p \Rightarrow$ 猜 $y_p = ke^x$

$$\Rightarrow y_p = ke^x$$

Undetermined Coefficient

$$y_p'' + 3y_p' + 2y_p = e^x$$

$$y_p' = ke^x$$

$$y_p'' = ke^x$$

$$\Rightarrow ke^x + 3ke^x + 2ke^x = e^x$$

$$6ke^x = e^x \qquad k = \frac{1}{6}$$

$$\therefore y = y_h + y_p = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{6} e^x$$

Undetermined Coefficient

*考慮二階O.D.E. $\xleftrightarrow[\text{推廣}]{}$ n階 O.D.E.

$$y'' + ay' + by = r(x) \qquad\qquad y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = r(x)$$

I. 依照前述 $e^{\lambda x}$，D 的結果，y_h 可以先決定

II. $r(x)$ 決定 y_p

Undetermined Coefficient

r(x)的函數型式

(1) $e^{\alpha x} <-> y_p = ke^{\alpha x}$

(2) $\cos \beta x \,\&\, \sin \beta x <-> y_p = k_1 \cos \beta x + k_2 \sin \beta x$

(3) $x^k <-> y_p = k_0 x^k + k_1 x^{k-1} + ... + k_k$

(4) $k <-> y_p = k$

(5) $e^{\alpha x} \cos \beta x \,\&\, e^{\alpha x} \sin \beta x <->$

$$y_p = e^{\alpha x}(k_1 \cos \beta x + k_2 \sin \beta x)$$

Undetermined Coefficient

$$(6)\ e^{\alpha x}x^k <-> y_p = e^{\alpha x}(k_0x^k + k_1x^{k-1} + ... + k_k)$$

$$(7)\ (\cos\beta x)x^k \,\&\, (\sin\beta x)x^k <->$$

$$y_p = \cos\beta x(A_0x^k + A_1x^{k-1} + ... + A_k) +$$

$$\sin\beta x(B_0x^k + B_1x^{k-1} + ... + B_k)$$

Undetermined Coefficient

- 觀察

$$y' + ay = e^{-ax} \qquad y = CI^{-1} + I^{-1}\int Irdx$$

$$y_h = Ce^{-ax} \qquad = Ce^{-ax} + xe^{-ax}$$

猜 $y_p = ke^{-ax}$ => 失效

$$\Rightarrow y_p = kxe^{-ax} \quad k = 1$$

=>當我們依r(x)的函數型式決定y_p後，將$y_p(x)$與$y_h(x)$比較是否有相同項。若有，必須將相同的部分乘上x的最低冪次，使其不再相同為止，之後再將修正後y_p代入，決定未定的係數。

Undetermined Coefficient

例: $y'' + 3y' + 2y = e^{-2x}$

$$y = y_h + y_p$$

$$\Rightarrow y_h = C_1 e^{-x} + C_2 e^{-2x}$$

$$y_p = ke^{-2x} \to y_p = kxe^{-2x}$$

$$y_p' = ke^{-2x} - 2kxe^{-2x}$$

$$y_p'' = -2ke^{-2x} - 2ke^{-2x} + 4kxe^{-2x}$$

Undetermined Coefficient

代入原O.D.E

$$\Rightarrow (2-6+4)kxe^{-2x} + (3k-4k)e^{-2x} = e^{-2x}$$

$$k = -1$$

$$\Rightarrow y = C_1e^{-x} + C_2e^{-2x} - xe^{-2x}$$

Undetermined Coefficient

例: $y'' + 4y' + 4y = 3e^{-2x}$

$$y_h = C_1 e^{-2x} + C_2 x e^{-2x}$$

$$y_p = ke^{-2x} \rightarrow y_p = kx^2 e^{-2x}$$

(練習)

$$\Rightarrow k = \frac{3}{2}$$

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{3}{2} x^2 e^{-2x}$$

Undetermined Coefficient

例: $y'' + 4y = \cos 2x$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = (k_1 \cos 2x + k_2 \sin 2x)x \qquad y_p' = \ldots \qquad y_p'' = \ldots$

代入原O.D.E求出 k_1、k_2

Note:

1. 未定係數法，道理簡單卻費時
2. 有些函數，不知道如何猜 y_p $(ex : \csc x)$

Order Reduction Method

- Method 2: Order Reduction Method(降階法)

例: $y'' + 3y' + 2y = e^x$

$$(D^2 + 3D + 2)y = e^x$$

$$(D+1)(D+2)y = e^x$$

令$(D+2)y = Z_p$

$$(D+1)Z_p = e^x$$

$$Z_p^{'} + Z_p = e^x, I = e^x$$

$$\Rightarrow Z_p = I^{-1}\int Ie^x dx$$

Order Reduction Method

$$(D+2)y_p = I^{-1}\int Ie^x dx$$

$$y_p^{'} + 2y_p = I^{-1}\int Ie^x dx, \quad I_{new} = e^{2x}$$

$$\Rightarrow y_p = CI_{new}^{-1} + I_{new}^{-1}\int I_{new}(I^{-1}\int Ie^x dx)dx$$

$$= e^{-2x}\int e^{2x}(e^{-x}\int e^x e^x dx)dx$$

$$= \frac{1}{6}e^x$$

Order Reduction Method

例: $y'' + 4y' + 4y = e^{-2x}$

$$y'' + 4y' + 4y = e^{-2x}$$

$$y_h = C_1 e^{-2x} + C_2 x e^{-2x}$$

$$y_p = kx^2 e^{-2x}$$

$$(D+2)(D+2)y = e^{-2x}$$

$$I_1 = e^{2x}$$

$$I_2 = e^{2x}$$

Order Reduction Method

$$y_p = I_2^{-1}\int I_2(I_1^{-1}\int I_1 e^{-2x}dx)dx$$

$$= e^{-2x}\int e^{2x}(e^{-2x}\int e^{2x}e^{-2x}dx)dx$$

$$= \frac{1}{2}e^{-2x}x^2$$

$$\Rightarrow y = C_1e^{-2x} + C_2xe^{-2x} + \frac{1}{2}e^{-2x}x^2$$

Note:

降階的順序是否會影響 y_p?

Ans: NO!

Order Reduction Method

例: 設某個微分方程式的特性方程式的根為

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

$$(D-\lambda_1)(D-\lambda_2)\ldots(D-\lambda_n)y_p = r(x)$$

$$y_p(x) = ?$$

Sol:

$$y_p(x) = e^{\lambda_n x}\int e^{-\lambda_n x}(\ldots e^{\lambda_2 x}\int e^{-\lambda_2 x}(e^{\lambda x}\int e^{-\lambda x}r(x)dx)dx\ldots)dx$$

Order Reduction Method

例: $y''' + 6y'' + 11y' + 6y = e^x$

$$y_h = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x}$$

$$y_p = e^{-3x} \int e^{3x} (e^{-2x} \int e^{2x} (e^{-x} \int e^x e^x dx) dx) dx$$

$$= \frac{1}{24} e^x$$

$$\Rightarrow y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x} + \frac{1}{24} e^x$$