

Chapter 4.

Laplace Transform

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Laplace Transform

例: $3e^{2t} \xrightarrow{\mathcal{L}} \frac{3}{s-2}$

$$2\cos 3t \xrightarrow{\mathcal{L}} \frac{2s}{s^2+9}$$

$$3\sin 2t \xrightarrow{\mathcal{L}} \frac{6}{s^2+4}$$

$$t \xrightarrow{\mathcal{L}} \frac{1}{s^2}$$

$$H(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$$

$$\delta(t) \xrightarrow{\mathcal{L}} 1$$

Laplace Transform

- Properties :

1. Linear

2. 第一移位

$$f(t) \rightarrow F(s)$$

$$e^{at} f(t) \rightarrow F(s - a)$$

EX :

$$e^{2t} \cos 3t \xrightarrow{\mathcal{L}} \frac{s}{s^2 + 9} \Big|_{s = s - 2} = \frac{s - 2}{(s - 2)^2 + 9}$$

Laplace Transform

3. 第二移位

$$f(t) \rightarrow F(s)$$

$$H(t-a)f(t-a) \rightarrow e^{-as}F(s)$$

EX :

$$F(s) = \frac{2}{s^2 + 4} e^{-2s}, f(t) = H(t-2) \sin 2(t-2)$$

Laplace Transform

4. $f(t) \rightarrow F(s)$

$$tf(t) \rightarrow \frac{-dF(s)}{ds}$$

$$t^2 f(t) \rightarrow \frac{-d}{ds} \left(\frac{-dF(s)}{ds} \right)$$

5. $f(t) \rightarrow F(s)$

$$\frac{1}{t} f(t) \rightarrow \int_s^\infty F(s) ds$$

$$\frac{1}{t^2} f(t) \rightarrow \int_s^\infty \int_s^\infty F(s) ds ds$$

Laplace Transform

EX: $\frac{1}{t} \sin t \xrightarrow{\mathcal{L}} ?$

$$\sin t \xrightarrow{\mathcal{L}} \frac{1}{s^2 + 1}$$

$$\frac{1}{t} \sin t \xrightarrow{\mathcal{L}} \int_s^\infty \frac{1}{s^2 + 1} ds = \tan^{-1} s \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

微積分內瑕積分

Laplace Transform

$$\int_0^{\infty} \frac{\sin t}{t} dt$$

$$= \lim_{t \rightarrow \infty} \int_0^t \frac{\sin t}{t} dt$$

$$\lim_{t \rightarrow 0} \int_0^{\infty} \frac{\sin t}{t} e^{-st} dt$$

$$= \lim_{s \rightarrow 0} \mathcal{L}\left\{\frac{\sin t}{t}\right\} = \lim_{s \rightarrow 0} \left(\frac{\pi}{2} - \tan^{-1} s\right) = \frac{\pi}{2}$$

$$\int_0^{\infty} \lim_{s \rightarrow 0} \frac{\sin t}{t} e^{-st} dt$$

$$= \int_0^{\infty} \frac{\sin t}{t} dt = \text{原式}$$

Laplace Transform

結合性質4.和性質5.

$$\begin{array}{ccccc}
 G(s) & \xleftarrow{\int_s^\infty F(s)ds} & F(s) & \xrightarrow{\frac{-dF(s)}{ds}} & G(s) \\
 \mathcal{L}^{-1} \downarrow & & \downarrow \mathcal{L}^{-1} & & \downarrow \mathcal{L}^{-1} \\
 g(t) & \xrightarrow{\times t} & f(t) & \xleftarrow{\div t} & g(t) \\
 \\
 \mathcal{L}^{-1} \left(\int_s^\infty \mathcal{L} \{ g(t) \cdot t \} ds \right) & = & g(t) & &
 \end{array}$$

Laplace Transform

EX: $F(s) = \ln \frac{s+1}{s+2}, f(t) = ?$

$$\begin{aligned}\frac{dF(s)}{ds} &= \frac{d}{ds} \ln \frac{s+1}{s+2} = \frac{s+2}{s+1} \left(\frac{d}{ds} \cdot \frac{s+1}{s+2} \right) \\&= \frac{s+2}{s+1} \cdot \frac{(s+2) - (s+1)}{(s+2)^2} = \frac{1}{(s+2)(s+1)} = \frac{a}{s+1} + \frac{b}{s+2} \\&= \frac{1}{s+1} + \frac{-1}{s+2} \\&= \mathcal{L}(e^{-t} - e^{-2t}) \\&\Rightarrow \mathcal{L}^{-1}\left(\frac{-dF(s)}{ds}\right) = e^{-2t} - e^{-t} \\f(t) &= \frac{1}{t} (e^{-2t} - e^{-t})\end{aligned}$$

Laplace Transform

$$\begin{aligned}& \int_0^{\infty} \frac{1}{t} (e^{-2t} - e^{-t}) dt \\&= \int_0^{\infty} \lim_{s \rightarrow 0} \frac{1}{t} (e^{-2t} - e^{-t}) e^{-st} dt = \lim_{s \rightarrow 0} \int_0^{\infty} \frac{1}{t} (e^{-2t} - e^{-t}) e^{-st} dt \\&= \lim_{s \rightarrow 0} \int_0^{\infty} \frac{1}{t} (e^{-2t} - e^{-t}) e^{-st} dt \\&= \lim_{s \rightarrow 0} \mathcal{L} \left\{ \frac{1}{t} (e^{-2t} - e^{-t}) \right\} = \lim_{s \rightarrow 0} \ln \frac{s+1}{s+2} = -\ln 2\end{aligned}$$

Laplace Transform

EX: $F(s) = \frac{2s}{(s^2 + 4)^2}, f(t) = ?$

越微分越複雜

$$\int_0^\infty F(s)ds = \int_s^\infty \frac{2s}{(s^2 + 4)^2} ds$$

$$\text{令 } t = s^2 + 4, dt = 2sds, ds = \frac{1}{2s} dt$$

$$= \int_{s^2+4}^\infty \frac{2s}{t^2} \frac{1}{2s} dt$$

$$= \int_{s^2+4}^\infty \frac{2s}{t^2} \frac{1}{2s} dt$$

$$= \int_{s^2+4}^\infty \frac{1}{t^2} dt$$

$$= -\frac{1}{t} \Big|_{s^2+4}^\infty$$

Laplace Transform

$$= -\frac{1}{t} \Big|_{s^2+4}^{\infty}$$

$$= 0 - \left(-\frac{1}{s^2+4} \right)$$

$$= \frac{1}{s^2+4}$$

$$\Rightarrow g(t) = \frac{1}{2} \sin 2t$$

$$\Rightarrow f(t) = \frac{1}{2} t \sin 2t$$

$$\int_0^{\infty} \frac{1}{2} t \sin 2t dt = 0$$

Laplace Transform

6.

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$f'(t) \xrightarrow{\mathcal{L}} sF(s) - f(0)$$

Laplace Transform

EX:

$$y' + 2y = e^t, y(0) = 0$$

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{e^t\}$$

$$\mathcal{L}\{y'\} + \mathcal{L}\{2y\} = \mathcal{L}\{e^t\}$$

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s-1}$$

$$(s+2)Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} + \frac{-\frac{1}{3}}{s+2}$$

$$y(t) = \frac{-1}{3}e^{-2t} + \frac{1}{3}e^t$$

Laplace Transform

Pf:

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^{\infty} f'(t)e^{-st}dt \\&= \int_0^{\infty} e^{-st}df(t) \\&= e^{-st}f(t)\Big|_0^{\infty} - \int_0^{\infty} f(t)(-se^{-st})dt \\&= (0 - f(0)) + s\int_0^{\infty} f(t)e^{-st}dt \\&= sF(s) - f(0)\end{aligned}$$

Laplace Transform

性質6.1

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= L\{(f'(t))'\} \quad \text{令 } g(t) = f'(t) \\ &= \mathcal{L}\{g'(t)\} \\ &= s\mathcal{L}\{g(t)\} - g(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Laplace Transform

性質6.2

$$\begin{aligned} & \mathcal{L}\{f^{(n)}(t)\} \\ &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \end{aligned}$$

*用數學歸納法可證

$$y'' + 3y' + 2y = e^t, y(0) = 0, y'(0) = 0$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0) \quad \mathcal{L}\{y'\} = sY(s) - y(0)$$

$$s^2 Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s+1)(s+2)} = \frac{\frac{1}{6}}{s-1} + \frac{\frac{-1}{2}}{s+1} + \frac{\frac{1}{3}}{s+2}$$

$$y(t) = \frac{1}{6}e^t - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}$$

Laplace Transform

性質7

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$\int_0^t f(t)dt \xrightarrow{\mathcal{L}} \frac{F(s)}{s}$$

$$1 \xrightarrow{\mathcal{L}} \frac{1}{s}$$

$$t \xrightarrow{\mathcal{L}} \frac{1}{s^2}$$

$$\int_0^t 1dt \xrightarrow{\mathcal{L}} \frac{1}{s^2}$$

$$\mathcal{L} \left\{ \int_0^t \cdots \int_0^t f(t)dt \cdots dt \right\} = \frac{1}{s^k} F(s)$$

Laplace Transform

pf :

$$\begin{aligned}& \mathcal{L} \left\{ \int_0^t f(t) dt \right\} \\&= \int_0^\infty \int_0^t f(t) dt e^{-st} dt \\&= \int_0^\infty \int_0^t f(\lambda) d\lambda e^{-st} dt \quad u = \int_0^t f(\lambda) d\lambda \quad du = f(t) dt \quad dv = e^{-st} dt \quad v = \frac{-1}{s} e^{-st} \\&= uv \Big|_0^\infty - \int_0^\infty v du \\&= \left(\int_0^t f(\lambda) d\lambda \right) \left(\frac{-1}{s} e^{-st} \right) \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st} f(t) dt \\&= (0 - 0) + \frac{1}{s} \int_0^\infty f(t) e^{-st} dt \\&= \frac{1}{s} F(s)\end{aligned}$$

Laplace Transform

$$\text{EX : } \mathcal{L} \left\{ e^{2t} \int_0^t e^{3t}(t)(\sin t) dt \right\}$$

$$\mathcal{L} \{ \sin t \} = \frac{1}{s^2 + 1}$$

$$\mathcal{L} \{ t \sin t \} = \frac{-d}{ds} \frac{1}{s^2 + 1} = \frac{2s}{(s^2 + 1)^2}$$

$$\mathcal{L} \{ e^{3t} t \sin t \} = \frac{2s}{(s^2 + 1)^2} \Big|_{s=s-3} = \frac{2(s-3)}{((s-3)^2 + 1)^2} = \frac{2s-6}{(s^2 - 6s + 10)^2}$$

$$\mathcal{L} \left\{ \int_0^t e^{3t} \times (t) \times (\sin t) dt \right\} = \frac{1}{s} \frac{2s-6}{(s^2 - 6s + 10)^2}$$

$$\mathcal{L} \left\{ e^{2t} \int_0^t e^{3t} \times (t) \times (\sin t) dt \right\} = \frac{2s-6}{s(s^2 - 6s + 10)^2} \Big|_{s=s-2}$$

$$= \frac{2(s-2)-6}{(s-2)((s-2)^2 - 6(s-2) + 10)^2}$$

$$= \frac{2s-10}{(s-2)(s^2 - 10s + 26)^2}$$

Laplace Transform

性質8 Convolution theorem $\mathcal{L}\{f \otimes g\} = F(s)G(s)$

$$f(t) \otimes g(t) \triangleq \int_0^t f(\lambda)g(t-\lambda)d\lambda$$

EX : $f(t) = e^{2t}, g(t) = t$

$$f(t) \otimes g(t) = \int_0^t g(\lambda)f(t-\lambda)d\lambda \quad \text{令 } x = t - \lambda, dx = -d\lambda$$

$$= \int_t^0 f(t-x)g(t)(-dx)$$

$$= \int_0^t f(t-x)g(x)dx \quad \text{令 } x = \lambda$$

$$= \int_0^t f(t-\lambda)g(\lambda)d\lambda = g(t) \otimes f(t)$$

$$e^{2t} \otimes t = \int_0^t e^{2\lambda}(t-\lambda)d\lambda = \int_0^t \lambda e^{2(t-\lambda)}d\lambda$$

Laplace Transform

$$\text{EX : } \int_0^t \cos \lambda e^{2(t-\lambda)} d\lambda$$

$$= \cos t \otimes e^{2t}$$

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$g(t) \xrightarrow{\mathcal{L}} G(s)$$

$$f(t) \otimes g(t) \xrightarrow{\mathcal{L}} F(s)G(s)$$

Laplace Transform

$$\text{EX : } \mathcal{L} \left\{ \int_0^t e^{2\lambda} (t - \lambda) d\lambda \right\}$$

$$= \frac{1}{s-2} \frac{1}{s^2}$$

$$\text{EX : } f(t) = e^{2t}, g(t) = e^{3t}$$

$$f(t) \otimes g(t) = \int_0^t e^{2\lambda} e^{3(t-\lambda)} d\lambda$$

$$= e^{3t} \int_0^t e^{-\lambda} d\lambda$$

$$= e^{3t} (-e^{-\lambda} \Big|_0^t)$$

$$= e^{3t} (-e^{-t} - (-1))$$

$$= -e^{2t} + e^{3t}$$

$$\mathcal{L} \{ e^{3t} - e^{2t} \}$$

Laplace Transform

$$= \frac{1}{s-3} - \frac{1}{s-2}$$

$$= \frac{1}{(s-3)(s-2)}$$

$$= \frac{1}{s-3} \bullet \frac{1}{s-2}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s-3} \bullet \frac{1}{s-2} \right\} = e^{3t} \otimes e^{2t}$$

Laplace Transform

pf :

$$\begin{aligned} & L\{f(t) \otimes g(t)\} \\ &= \int_0^\infty [f(t) \otimes g(t)] e^{-st} dt \\ &= \int_0^\infty \left(\int_0^t f(\lambda) g(t-\lambda) d\lambda \right) e^{-st} dt \\ &= \int_0^\infty \int_0^t f(\lambda) g(t-\lambda) e^{-st} d\lambda dt \\ &= \int_0^\infty \int_\lambda^\infty f(\lambda) g(t-\lambda) e^{-st} dt d\lambda \end{aligned}$$

λ 和 t 積分範圍相關, 想對調

要在積分區域相同條件下

Laplace Transform

$$= \int_0^\infty f(\lambda) \int_0^\infty g(t-\lambda) e^{-st} dt d\lambda \quad \text{令 } x = t - \lambda, dx = dt$$

$$= \int_0^\infty f(\lambda) \int_0^\infty g(x) e^{-s(\lambda+x)} dx d\lambda$$

$$= \int_0^\infty f(\lambda) e^{-s\lambda} \int_0^\infty g(x) e^{-sx} dx d\lambda$$

$$= G(s) \int_0^\infty f(\lambda) e^{-s\lambda} d\lambda$$

$$= G(s)F(s)$$