110-1 ENGINEERING MATHEMATICS PRACTICE

(考前練習題)(有附課本頁碼及題號)

Solve the given differential equation by separation of variables.(2-2)

1. $\frac{dy}{dx} = (x+1)^2$ (p48.2) (p49.2)

From $dy = (x+1)^2\, dx$ we obtain $y = \frac{1}{3}(x+1)^3 + c$.

2. $dy - (y-1)^2 dx = 0$ (p48.4) (p49.4)

From $\frac{1}{(y-1)^2}\, dy = dx$ we obtain $-\frac{1}{y-1} = x + c$ or $y = 1 - \frac{1}{x+c}$.

3. $\frac{dy}{dx} + 2xy^2 = 0$ (p48.6) (p49.6)

From $\frac{1}{y^2}\, dy = -2x\, dx$ we obtain $-\frac{1}{y} = -x^2 + c$ or $y = \frac{1}{x^2 + c_1}$.

Find the general solution of the given differential equation.(2-3)

4. $\frac{dy}{dx} + 2y = 0$ (p57.2) (p59.2)

For $y' - 5y = 0$ an integrating factor is $e^{-\int 5\, dx} = e^{-5x}$ so that $\frac{d}{dx}\left[e^{-5x}y\right] = 0$ and $y = ce^{5x}$ for $-\infty < x < \infty$.

5. $3\frac{dy}{dx} + 12y = 4$ (p57.4) (p59.4)

For $y' + 4y = \frac{4}{3}$ an integrating factor is $e^{\int 4\, dx} = e^{4x}$ so that $\frac{d}{dx}\left[e^{4x}y\right] = \frac{4}{3}e^{4x}$ and $y = \frac{1}{3} + ce^{-4x}$ for $-\infty < x < \infty$. The transient term is ce^{-4x}.

6. $y' + 2xy = x^3$ (p57.6) (p59.6)

For $y' + 2xy = x^3$ an integrating factor is $e^{\int 2x\, dx} = e^{x^2}$ so that $\frac{d}{dx}\left[e^{x^2}y\right] = x^3e^{x^2}$ and $y = \frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$ for $-\infty < x < \infty$. The transient term is ce^{-x^2}.

Determine whether the given differential equation is exact. If it is exact, solve it.(2-4)

7. $(2x+y)dx - (x+6y)dy = 0$ (p64.2) (p66.2)

Let $M = 2x + y$ and $N = -x - 6y$. Then $M_y = 1$ and $N_x = -1$, so the equation is not exact.

8. $(5x+4y)dx+(4x-8y^3)dy=0$ (p64.3) (p66.3)

Let $M=5x+4y$ and $N=4x-8y^3$ so that $M_y=4=N_x$. From $f_x=5x+4y$ we obtain $f=\frac{5}{2}x^2+4xy+h(y)$, $h'(y)=-8y^3$, and $h(y)=-2y^4$. A solution is $\frac{5}{2}x^2+4xy-2y^4=c$.

9. $(2xy^2-3)dx+(2x^2y+4)dy=0$ (p64.5) (p66.5)

Let $M=2y^2x-3$ and $N=2yx^2+4$ so that $M_y=4xy=N_x$. From $f_x=2y^2x-3$ we obtain $f=x^2y^2-3x+h(y)$, $h'(y)=4$, and $h(y)=4y$. A solution is $x^2y^2-3x+4y=c$.

D.E. Solve by substitution(2-5)

10. $(x+y)dx+xdy=0$ (p68.2) (p71.2)

Letting $y=ux$ we have

$$\begin{aligned}(x+ux)\,dx+x(u\,dx+x\,du)&=0\\(1+2u)\,dx+x\,du&=0\\\frac{dx}{x}+\frac{du}{1+2u}&=0\\\ln|x|+\frac{1}{2}\ln|1+2u|&=c\\x^2\left(1+2\frac{y}{x}\right)&=c_1\\x^2+2xy&=c_1.\end{aligned}$$

11. $ydx=2(x+y)dy$ (p68.4) (p71.4)

Letting $x=vy$ we have

$$\begin{aligned}y(v\,dy+y\,dv)-2(vy+y)\,dy&=0\\y\,dv-(v+2)\,dy&=0\\\frac{dv}{v+2}-\frac{dy}{y}&=0\\\ln|v+2|-\ln|y|&=c\\\ln\left|\frac{x}{y}+2\right|-\ln|y|&=c\\x+2y&=c_1y^2.\end{aligned}$$

12. $(y^2 + yx)dx + x^2dy = 0$ (p68.6) (p71.6)

Letting $y = ux$ and using partial fractions, we have

$$\begin{aligned}
\left(u^2x^2 + ux^2\right) dx + x^2(u\,dx + x\,du) &= 0 \\
x^2\left(u^2 + 2u\right) dx + x^3\,du &= 0 \\
\frac{dx}{x} + \frac{du}{u(u+2)} &= 0 \\
\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| &= c \\
\frac{x^2u}{u+2} &= c_1 \\
x^2\frac{y}{x} &= c_1\left(\frac{y}{x} + 2\right) \\
x^2y &= c_1(y + 2x).
\end{aligned}$$

Find a second solution $y_2(x)$(3-2)

13. $y'' + 2y' + y = 0,\ y_1 = xe^{-x}$ (p119.2) (p124.2)

Define $y = u(x)xe^{-x}$ so

$$y' = (1-x)e^{-x}u + xe^{-x}u', \quad y'' = xe^{-x}u'' + 2(1-x)e^{-x}u' - (2-x)e^{-x}u,$$

and

$$y'' + 2y' + y = e^{-x}(xu'' + 2u') = 0 \quad \text{or} \quad u'' + \frac{2}{x}u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + \frac{2}{x}w = 0$ which has the integrating factor $e^{2\int dx/x} = x^2$. Now

$$\frac{d}{dx}[x^2w] = 0 \quad \text{gives} \quad x^2w = c.$$

Therefore $w = u' = c/x^2$ and $u = c_1/x$. A second solution is $y_2 = \frac{1}{x}xe^{-x} = e^{-x}$.

14. $y'' + 9y = 0,\ y_1 = sin3x$ (p119.4) (p124.4)

Define $y = u(x)\sin 3x$ so

$$y' = 3u\cos 3x + u'\sin 3x, \quad y'' = u''\sin 3x + 6u'\cos 3x - 9u\sin 3x,$$

and

$$y'' + 9y = (\sin 3x)u'' + 6(\cos 3x)u' = 0 \quad \text{or} \quad u'' + 6(\cot 3x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 6(\cot 3x)w = 0$ which has the integrating factor $e^{6\int \cot 3x\,dx} = \sin^2 3x$. Now

$$\frac{d}{dx}[(\sin^2 3x)w] = 0 \quad \text{gives} \quad (\sin^2 3x)w = c.$$

Therefore $w = u' = c\csc^2 3x$ and $u = c_1\cot 3x$. A second solution is $y_2 = \cot 3x\sin 3x = \cos 3x$.

15. $y'' - 25y = 0,\ y_1 = e^{5x}$ (p119.6) (p124.6)

Define $y = u(x)e^{5x}$ so

$$y' = 5e^{5x}u + e^{5x}u', \quad y'' = e^{5x}u'' + 10e^{5x}u' + 25e^{5x}u$$

and

$$y'' - 25y = e^{5x}(u'' + 10u') = 0 \quad \text{or} \quad u'' + 10u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 10w = 0$ which has the integrating factor $e^{10\int dx} = e^{10x}$. Now

$$\frac{d}{dx}[e^{10x}w] = 0 \quad \text{gives} \quad e^{10x}w = c.$$

Therefore $w = u' = ce^{-10x}$ and $u = c_1e^{-10x}$. A second solution is $y_2 = e^{-10x}e^{5x} = e^{-5x}$.

Find the general solution (3-3,3-4,3-5)

16. $\dfrac{d^3x}{dt^3} - \dfrac{d^2x}{dt^2} - 4x = 0$ (p125.20) (p130.20)

From $m^3 - m^2 - 4 = 0$ we obtain $m = 2$ and $m = -1/2 \pm \sqrt{7}\,i/2$ so that

$$x = c_1e^{2t} + e^{-t/2}[c_2\cos(\sqrt{7}t/2) + c_3\sin(\sqrt{7}t/2)].$$

17. $y''' - 6y'' + 12y' - 8y = 0$ (p125.22) (p130.22)

From $m^3 - 6m^2 + 12m - 8 = 0$ we obtain $m = 2$, $m = 2$, and $m = 2$ so that

$$y = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}.$$

18. $y^{(4)} - 2y'' + y = 0$ (p125.24) (p130.24)

From $m^4 - 2m^2 + 1 = 0$ we obtain $m = 1$, $m = 1$, $m = -1$, and $m = -1$ so that

$$y = c_1e^{x} + c_2xe^{x} + c_3e^{-x} + c_4xe^{-x}.$$

19. $y'' - 8y' + 20y = 100x^2 - 26xe^x$ (p135.6) (p139.6)

From $m^2 - 8m + 20 = 0$ we find $m_1 = 4 + 2i$ and $m_2 = 4 - 2i$. Then $y_c = e^{4x}(c_1\cos 2x + c_2\sin 2x)$ and we assume $y_p = Ax^2 + Bx + C + (Dx + E)e^x$. Substituting into the differential equation we obtain

$$\begin{aligned} 2A - 8B + 20C &= 0 \\ -6D + 13E &= 0 \\ -16A + 20B &= 0 \\ 13D &= -26 \\ 20A &= 100. \end{aligned}$$

Then $A = 5$, $B = 4$, $C = \frac{11}{10}$, $D = -2$, $E = -\frac{12}{13}$, $y_p = 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x$ and

$$y = e^{4x}(c_1\cos 2x + c_2\sin 2x) + 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x.$$

20. $4y'' - 4y' - 3y = cos2x$ (p135.8) (p139.8)

From $4m^2 - 4m - 3 = 0$ we find $m_1 = \frac{3}{2}$ and $m_2 = -\frac{1}{2}$. Then $y_c = c_1 e^{3x/2} + c_2 e^{-x/2}$ and we assume $y_p = A\cos 2x + B\sin 2x$. Substituting into the differential equation we obtain $-19 - 8B = 1$ and $8A - 19B = 0$. Then $A = -\frac{19}{425}$, $B = -\frac{8}{425}$, $y_p = -\frac{19}{425}\cos 2x - \frac{8}{425}\sin 2x$, and

$$y = c_1 e^{3x/2} + c_2 e^{-x/2} - \frac{19}{425}\cos 2x - \frac{8}{425}\sin 2x.$$

21. $y'' + 2y' = 2x + 5 - e^{-2x}$ (p135.10) (p139.10)

From $m^2 + 2m = 0$ we find $m_1 = -2$ and $m_2 = 0$. Then $y_c = c_1 e^{-2x} + c_2$ and we assume $y_p = Ax^2 + Bx + Cxe^{-2x}$. Substituting into the differential equation we obtain $2A + 2B = 5$, $4A = 2$, and $-2C = -1$. Then $A = \frac{1}{2}$, $B = 2$, $C = \frac{1}{2}$, $y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$, and

$$y = c_1 e^{-2x} + c_2 + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}.$$

22. $y'' + y = \sec\theta\tan\theta$ (p140.4) (p144.4)

The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec x\tan x$ we obtain

$$u_1' = -\sin x(\sec x\tan x) = -\tan^2 x = 1 - \sec^2 x$$
$$u_2' = \cos x(\sec x\tan x) = \tan x.$$

Then $u_1 = x - \tan x$, $u_2 = -\ln|\cos x|$, and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x + x\cos x - \sin x - \sin x\ln|\cos x| \\ &= c_1\cos x + c_3\sin x + x\cos x - \sin x\ln|\cos x|. \end{aligned}$$

23. $y'' + y = \sec^2 x$ (p140.6) (p144.6)

The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec^2 x$ we obtain

$$u_1' = -\frac{\sin x}{\cos^2 x}$$
$$u_2' = \sec x.$$

Then

$$u_1 = -\frac{1}{\cos x} = -\sec x$$
$$u_2 = \ln|\sec x + \tan x|$$

and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x - \cos x\sec x + \sin x\ln|\sec x + \tan x| \\ &= c_1\cos x + c_2\sin x - 1 + \sin x\ln|\sec x + \tan x|. \end{aligned}$$

24. $y'' - y = \cosh x$ (p140.7) (p144.7)

The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1 e^x + c_2 e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Identifying $f(x) = \cosh x = \frac{1}{2}(e^{-x} + e^x)$ we obtain

$$u_1' = \frac{1}{4}e^{-2x} + \frac{1}{4}$$
$$u_2' = -\frac{1}{4} - \frac{1}{4}e^{2x}.$$

Then

$$u_1 = -\frac{1}{8}e^{-2x} + \frac{1}{4}x$$
$$u_2 = -\frac{1}{8}e^{2x} - \frac{1}{4}x$$

and

$$\begin{aligned} y &= c_1 e^x + c_2 e^{-x} - \frac{1}{8}e^{-x} + \frac{1}{4}xe^x - \frac{1}{8}e^x - \frac{1}{4}xe^{-x} \\ &= c_3 e^x + c_4 e^{-x} + \frac{1}{4}x(e^x - e^{-x}) \\ &= c_3 e^x + c_4 e^{-x} + \frac{1}{2}x \sinh x. \end{aligned}$$

Find a homogeneous Cauchy–Euler differential equation whose general solution is given.(3-6)

25. $y = c_1 x^4 + c_2 x^{-2}$ (p146.33) (p151.33)

The solution $y = c_1 x^4 + c_2 x^{-2}$ suggests that the auxiliary equation has the roots $m = 4$ and $m = -2$ therefore the auxiliary equation itself has the form

$$\begin{aligned} am^2 + (b-a)m + c &= 0 \\ (m-4)(m+2) &= 0 \\ m^2 - 2m - 8 &= 0 \quad \Rightarrow \quad a = 1,\ b - a = -2,\ c = -8 \end{aligned}$$

Now if $a = 1$ then $b - a = -2$ means that $b = -1$ and therefore the Cauchy-Euler equation is

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \quad \Rightarrow \quad x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 8y = 0$$