Chapter 2.
First-Order Ordinary Differential Equations

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Review

- 上周 $Mdx + Ndy = 0$
  若發現 $\dfrac{\partial N}{\partial x} \neq \dfrac{\partial M}{\partial y}$ 則檢查 $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$

  除以 $-N \rightarrow I(x)$

  除以 $M \rightarrow I(y)$

  除以 $-N + M \rightarrow I(x + y)$

  除以 $-y \times N + x \times M \rightarrow I(x \times y)$

Integrating Factor Examples

- 今天

例: $(xy+y^2+1)dx+(xy+x^2+1)dy=0$

$$\Rightarrow \quad x(ydx+xdy)+y(ydx+xdy)+dx+dy=0$$

$$xd(xy)+yd(xy)+d(x+y)=0$$

$$(x+y)d(xy)+d(x+y)=0 \Rightarrow d(xy)+\frac{1}{x+y}d(x+y)=0$$

$$\int d(xy)+\int\frac{1}{x+y}d(x+y)=\int 0$$

$$xy+\ln(x+y)=C$$

Integrating Factor Examples

例: $(y\cos x - \sin 2x)dx + dy = 0$

$$\frac{\partial M}{\partial y} = \cos x, \frac{\partial N}{\partial x} = 0$$

$$\frac{0-\cos x}{-N} = \frac{-\cos x}{-1} = \cos x$$

$$\cos x dx = \frac{dI}{I}$$

$$I = e^{\sin x}$$

$$(y(\cos x)e^{\sin x} - (\sin 2x)e^{\sin x})dx + e^{\sin x}dy = 0$$

Integrating Factor Examples

$$\frac{\partial u}{\partial x} = y(\cos x)e^{\sin x} - (\sin 2x)e^{\sin x}$$

$$\partial u = (y(\cos x)e^{\sin x} - (\sin 2x)e^{\sin x})dx$$

$$\int \partial u = \int y(\cos x)e^{\sin x}dx - \int (\sin 2x)e^{\sin x}dx$$

$$u = \int \underset{\substack{\uparrow \\ (1)}}{y(\cos x)e^{\sin x}dx} - \int \underset{\substack{\uparrow \\ (2)}}{(\sin 2x)e^{\sin x}dx}$$

Integrating Factor Examples

(1): $\int y(\cos x)e^{\sin x}dx$

$= y\int(\cos x)e^{\sin x}dx$

$= ye^{\sin x}$

(2): $\int(\sin 2x)e^{\sin x}dx$

$= \int(2\sin x\cos x)e^{\sin x}dx$

$= 2\int te^{t}dt$ (令t=sinx)

$= 2(te^{t} - e^{t})$

$$u = ye^{\sin x} - 2(\sin x)e^{\sin x} + 2e^{\sin x} + f(y)$$

$$\frac{\partial u}{\partial y} = e^{\sin x} \quad \partial u = e^{\sin x}dy$$

$$\int \partial u = \int e^{\sin x}dy$$

Integrating Factor Examples

$$u = \int e^{\sin x} dy + g(x)$$

$$= ye^{\sin x} + g(x)$$

$$f(y) = 0, g(x) = -2(\sin x)e^{\sin x} + 2e^{\sin x}$$

$$u = ye^{\sin x} - 2(\sin x)e^{\sin x} + 2e^{\sin x} = C$$

Integrating Factor Examples

例: $\dfrac{dy}{dx} = 3x^2 - 3x^2 y$ (分離變數法可解)

$$(3x^2 - 3x^2 y)dx - dy = 0$$

$$\frac{\partial M}{\partial y} = -3x^2, \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x^2$$

除以-N $\dfrac{3x^2}{-(-1)} = 3x^2$

Integrating Factor Examples

$$3x^2 dx = \frac{dI}{I}$$

$$I = e^{x^3}$$

$$e^{x^3}(3x^2 - 3x^2 y)dx - e^{x^3} dy = 0$$

$$\frac{\partial u}{\partial x} = e^{x^3}(3x^2 - 3x^2 y)$$

$$u = \int e^{x^3}(3x^2 - 3x^2 y)dx + f(y)$$

$$u = e^{x^3} - ye^{x^3} + f(y)$$

Integrating Factor Examples

$$\frac{\partial u}{\partial y} = -e^{x^3}$$

$$u = -ye^{x^3} + g(x)$$

$$g(x) = e^{x^3}, f(y) = 0$$

$$u = e^{x^3} - ye^{x^3} = C$$

Separation of Variables

- 2.2分離變數法(補充)

例: $(1+x)dy - ydx = 0$

$\Rightarrow \quad -ydx + (1+x)dy = 0$

$$\frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$$

除以M $\Rightarrow \quad \frac{2}{-y}dy = \frac{dI}{I}$

Separation of Variables

$$I = y^{-2}$$

$$-y^{-1}dx + (1+x)y^{-2}dy = 0$$

$$\frac{\partial u}{\partial x} = -y^{-1} \qquad \frac{\partial u}{\partial y} = (1+x)y^{-2}$$

$$u = \int -y^{-1}dx + f(y) = -xy^{-1} + f(y)$$

$$u = \int (1+x)y^{-2}dy + g(x) = -(1+x)y^{-1} + g(x)$$

$$f(y) = -y^{-1}, g(x) = 0$$

$$u = -(1+x)y^{-1} = C$$

$$y = C(1+x) \quad \Leftarrow \quad \text{課本答案}$$

Separation of Variables

$$(1+x)dy - ydx = 0$$

除以 $(1+x)y$

$$\frac{dy}{y} - \frac{dx}{1+x} = 0 \qquad \int \frac{dy}{y} - \int \frac{dx}{1+x} = \int 0$$

$$\ln y = \ln(1+x) + C$$

$$y = (1+x)e^{c}$$

$$= (1+x)C$$

Separation of Variables Example

例: $\dfrac{dy}{dx} = y^2 - 4$

$$\frac{dy}{y^2 - 4} = dx \qquad (\frac{\frac{1}{4}}{y-2} - \frac{\frac{1}{4}}{y+2})dy = dx$$

$$\frac{1}{4}\ln(y-2) - \frac{1}{4}\ln(y+2) = x + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + C$$

$$\frac{y-2}{y+2} = \frac{e^{4x+C}}{1} \qquad \frac{2y}{4} = \frac{e^{4x+C}+1}{1-e^{4x+C}} \qquad y = 2\frac{1+e^{4x+C}}{1-e^{4x+C}},\ y = \pm 2$$

First-Order O.D.E.

- 2.3複習:一階O.D.E.(線性 OR 非線性)
  一階線性O.D.E.(常微分)

$$y'(x) + p(x)y(x) = r(x)$$

(1) $r(x) = 0$ Homogeneous 齊性

(2) $r(x) \neq 0$ Non- Homogeneous 非齊性

First-Order O.D.E. (homogeneous)

Case (1):

$$r(x) = 0 \Rightarrow y'(x) + p(x)y(x) = 0$$

$$\frac{dy(x)}{dx} = -p(x)y(x) \qquad \frac{dy(x)}{y(x)} = -p(x)dx$$

$$\int \frac{dy(x)}{y(x)} = \int -p(x)dx \qquad \ln y(x) = -\int p(x)dx + k$$

$$y(x) = e^{-\int p(x)dx} e^k$$

$$= Ce^{-\int p(x)dx} \quad (令\ \mathrm{C} = e^k)$$

First-Order O.D.E.(Non-homogeneous)

Case (2):

$$r(x) \neq 0 \Rightarrow y'(x) + p(x)y(x) = r(x)$$

$$\frac{dy}{dx} + p(x)y(x) - r(x) = 0$$

$$\left(p(x)y(x) - r(x)\right)dx + dy = 0$$

$$\frac{\partial M}{\partial y} = p(x), \qquad \frac{\partial N}{\partial x} = 0$$

$$\frac{0 - p(x)}{-N} = p(x) \qquad p(x)dx = \frac{dI}{I}$$

First-Order O.D.E.(Non-homogeneous)

$$I = e^{\int p(x)dx}$$

$$e^{\int p(x)dx}\left(p(x)y(x)-r(x)\right)dx + e^{\int p(x)dx}dy = 0$$

$$\frac{\partial u}{\partial x} = e^{\int p(x)dx}\left(p(x)y(x)-r(x)\right)$$

$$\partial u = e^{\int p(x)dx}\left(p(x)y(x)-r(x)\right)\partial x$$

(兩邊同積分)

$$u = \int e^{\int p(x)dx}\left(p(x)y(x)-r(x)\right)dx + f(y)$$

First-Order O.D.E.(Non-homogeneous)

令 $t = \int p(x)dx$

$$dt = p(x)dx \quad , \quad dx = \frac{dt}{p(x)}$$

$$\begin{aligned} u &= \int e^t y(x)dt - \int e^{\int p(x)dx} r(x)dx + f(y) \\ &= y(x)e^t - \int e^{\int p(x)dx} r(x)dx + f(y) \\ &= y(x)e^{\int p(x)dx} - \int e^{\int p(x)dx} r(x)dx + f(y) \end{aligned}$$

First-Order O.D.E.(Non-homogeneous)

$$\frac{\partial u}{\partial y} = e^{\int p(x)dx}$$

$$\partial u = e^{\int p(x)dx}\partial y + g(x)$$

(兩邊同積分)

$$u = \int e^{\int p(x)dx}dy + g(x)$$

$$= ye^{\int p(x)dx} + g(x)$$

$$g(x) = -\int e^{\int p(x)dx}r(x)dx \quad , \quad f(y) = 0$$

$$u = ye^{\int p(x)dx} - \int e^{\int p(x)dx}r(x)dx = C \ \#$$

First-Order O.D.E.(Non-homogeneous)

$$y = Ce^{-\int p(x)dx} + e^{-\int p(x)dx} \int e^{\int p(x)dx} r(x)dx$$

$$= \ y_h + y_p$$

y_h ： $r(x) = 0 \quad homogeneous\ solution$
y_p ： 特解$(particular\ solution)$,
互補解 $(complementary\ solution)$
記法：$y' + p(x)y(x) = r(x)$ 之解

$$y = CI^{-1} + I^{-1} \int Ir(x)dx$$

First-Order O.D.E.(Non-homogeneous)

例: $y' + 2xy = 3x$

$$y = Ce^{-x^2} + e^{-x^2}\int e^{x^2} 3x dx$$

$$= Ce^{-x^2} + \frac{3}{2} \quad \#$$

$$y_h \qquad y_p$$

(1) $y_h' + p(x) y_h = 0$

(2) Theorem:

又$y_p(x)$滿足非齊性方程式$\Leftrightarrow y_p' + p(x) y_p(x) = r(x)$

First-Order O.D.E.(Non-homogeneous)

Proof:

$$\because y = y_h + y_p$$ 代入原方程式

$$y' + Py = r$$

$$\left(y_h + y_p\right)' + P\left(y_h + y_p\right) = r$$

$$y_h' + y_p' + Py_h + Py_p = r$$

$$\because y_h' + Py_h = 0$$

$$\therefore y_p' + Py_p = r$$

Non-homogeneous Example

例: $y' + 2xy = x$

$$I = e^{\int 2x\,dx} = e^{x^2}$$

$$y = Ce^{-x^2} + e^{-x^2}\int e^{x^2} x\,dx$$

$$= Ce^{-x^2} + \frac{1}{2}$$

Chapter 3.
Higher-Order Differential Equations

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First-Order O.D.E.

$$y' + ay = r(x) \quad a \in const.$$

Case (1):

$$r(x) = 0 \qquad \text{Homogeneous}$$

$$y' + ay = 0$$

$$y = Ce^{-\int a dx}$$

$$= Ce^{-ax}$$

$$= y_h(x)$$

常係數 $\Rightarrow$ $y_h(x)$ 的部分必為指數函數 $e^{\lambda x}$ $\lambda \in const.$

First-Order O.D.E.

例: $y' + 2y = 0$

$y = Ce^{-2x}$

$apply \quad y = e^{\lambda x}$代入 $y' + 2y = 0$

$\lambda e^{\lambda x} + 2e^{\lambda x} = 0$

$(\lambda + 2)e^{\lambda x} = 0 \quad , \lambda + 2 = 0 \Rightarrow$ 特性方程式

$\lambda = -2$

$y = \mathrm{C}e^{-2x}$

Higher-Order O.D.E.

例: $y' - 3y = 0$

$y = Ce^{3x}$ #

推廣 $y'' + ay' + by = 0$, $a, b \in const.$

猜 $e^{\lambda x}$ 是解

$$\left(e^{\lambda x}\right)'' + a\left(e^{\lambda x}\right)' + b\left(e^{\lambda x}\right) = 0$$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0 \qquad \because \forall x \ , \ e^{\lambda x} \neq 0$$

$$e^{\lambda x}\left(\lambda^2 + a\lambda + b\right) = 0 \qquad \therefore \lambda^2 + a\lambda + b = 0$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Higher-Order O.D.E.

Case (1):

$$\lambda_1 \neq \lambda_2 \in \Re$$ 相異實根

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

例: $y'' + 3y' + 2y = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} \ \#$$

例: $y'' + 6y' + 5y = 0$

$$\lambda = -5, -1$$

$$y = C_1 e^{-x} + C_2 e^{-5x}$$

Higher-Order O.D.E.

Case (2):

$$\lambda_1 = \alpha + \beta\ i\ \ ,\ \ \lambda_2 = \alpha - \beta\ i \qquad \text{共軛複數根}$$

$$y = k_1 e^{(\alpha+\beta\ i)x} + k_2 e^{(\alpha-\beta\ i)x}$$

$$= e^{\alpha x}\left(k_1\left(\cos\beta x + i\sin\beta x\right) + k_2\left(\cos\beta x - i\sin\beta x\right)\right)$$

$$(\text{令} k_2 = ik_2')$$

$$= e^{\alpha x}\left(k_1\left(\cos\beta x + i\sin\beta x\right) + ik_2'\left(\cos\beta x - i\sin\beta x\right)\right)$$

$$= \mathrm{e}^{\alpha \mathrm{x}}\left(\left(k_1 + ik_2'\right)\cos\beta x + \left(k_2' + ik_1\right)\sin\beta x\right)$$

$$= e^{\alpha x}\left(C_1\cos\beta x + C_2\sin\beta x\right)$$

$$\Rightarrow C_1 = k_1 + ik_2'\ \ ,\ \ C_2 = k_2' + ik_1$$

Higher-Order O.D.E. Examples

例: $y'' + 2y' + 10y = 0$

$$\lambda = -1 \pm 3\,i$$

$$y = e^{-x}\left(C_1 \cos 3x + C_2 \sin 3x\right) \#$$

例: $y'' + 4y = 0$

$$\lambda = \pm 2i$$

$$y = C_1 \cos 2x + C_2 \sin 2x \#$$

Higher-Order O.D.E.

- 補充 :

$$y'' + ay' + by = 0$$

$$\Rightarrow \lambda^2 + a\lambda + b = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$a = -(\lambda_1 + \lambda_2) \qquad b = \lambda_1\lambda_2$$

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$

Higher-Order O.D.E.

- 定義:

$$\mathrm{D} \equiv \frac{d}{dx}$$ ：微分運算子

$\mathrm{D}x^2 = \dfrac{d}{dx}x^2$ D只對右邊函式作運算 note: $\mathrm{D}x^2 \neq x^2 D$

$$\mathrm{D}^k = \frac{d^k}{dx^k}$$

Higher-Order O.D.E.

$$y'' - (\lambda_1 + \lambda_2) y' + \lambda_1 \lambda_2 y = 0$$

$$\Rightarrow D^2 y - (\lambda_1 + \lambda_2) Dy + \lambda_1 \lambda_2 y = 0$$

$$\left(D^2 - (\lambda_1 + \lambda_2) D + \lambda_1 \lambda_2\right) y = 0$$

$$(D - \lambda_1)(D - \lambda_2) y = 0$$

$$\circ 令 (D - \lambda_2) y = z(x)$$

$$\Rightarrow (D - \lambda_1) z(x) = 0$$

$$z'(x) - \lambda_1 z(x) = 0$$

$$z(x) = k_1 e^{-\lambda_1 x}$$

$$(D - \lambda_2) y = z(x) = k_1 e^{-\lambda_1 x}$$

$$y' - \lambda_2 y = k_1 e^{-\lambda_1 x}$$

$$y = CI^{-1} + I^{-1} \int Ir dx$$

$$I = e^{\int \lambda_2 dx} = e^{\lambda_2 x}$$