

Chapter 1.

Introduction to differential equations

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Linear Differential Equations

- **Linear D.E.** 線性微分方程式，不具有下列任何一項

1. 因變數的自乘項
2. 因變數導數的自乘項
3. 因變數及其導數的互乘項

則稱為線性微分方程式；反之，若一微分方程式，具有上述**1,2,3**中任何一項，即稱為非線性**D.E.**。

Linear Differential Equations

例:

(1) $y'(x) + 5y''(x) = e^x \Rightarrow$ 線性2階1次O.D.E

(2) $\frac{\partial u(x, y)}{\partial x} + u(x, y) \frac{\partial u(x, y)}{\partial y} + 3\mu(x, y) = 0$

\Rightarrow 非線性(Disobey#3) , 1階1次P.D.E.

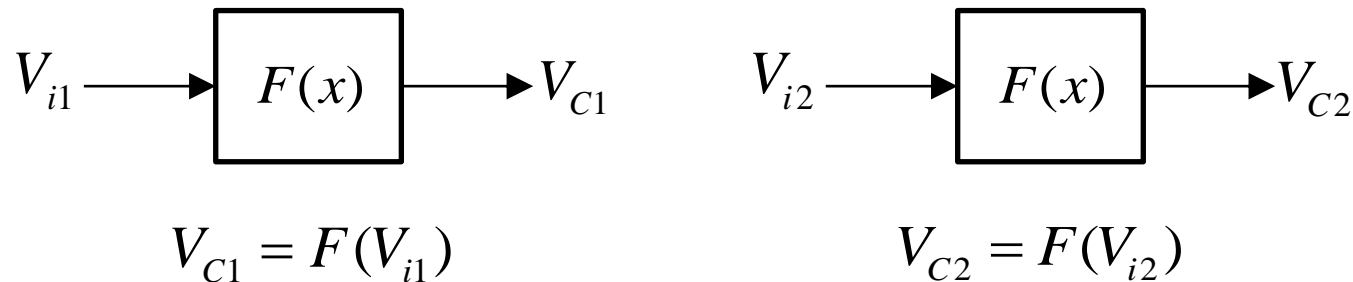
(3) $\frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} = 0 \quad (u = u(x, y))$

\Rightarrow 線性(x 是自變數) , 2階1次P.D.E.

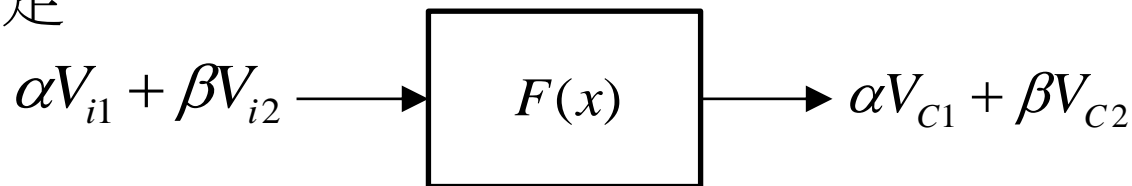
(4) $y'''(x) + 4y''(x) + 2y(x) = x^2 \Rightarrow$ 線性3階1次O.D.E.

Linear Differential Equations

- 一線性微分方程式會滿足重疊定理(Superposition)



若滿足



$$\alpha V_{C1} + \beta V_{C2} = F(\alpha V_{i1} + \beta V_{i2}) \quad \alpha, \beta \text{ 為任意純量}$$

則為線性。

Differential Equations

- 目的：分析微分方程式的解
⇒ 了解微分方程式由何而來

例：

$$(1) \ y(x) = C \quad C \in \text{常數 (Constant)}$$

如何消去C?

⇒ 應用微分，才可以消去C

$$\frac{dy(x)}{dx} = 0 \quad , \text{1階1次O.D.E.}$$

Differential Equations

$$(2) \quad y(x) = C_1 x + C_2 \quad C_1, C_2 \in \text{Constant}$$

$$\Rightarrow \frac{dy(x)}{dx} = C_1$$

$$\frac{d^2 y(x)}{dx} = 0 \quad \Rightarrow \text{線性, 2 階 1 次 O.D.E.}$$

$$(3) \quad y(x) = Cx + C^2 \quad C \in \text{constant}$$

$$\Rightarrow \frac{dy(x)}{dx} = C$$

一個常數，只能微分一次

(\because 一次不定積分，只會產生一個常數)

$$\Rightarrow y(x) = \frac{dy(x)}{dx} x + \left(\frac{dy(x)}{dx} \right)^2 \quad \Rightarrow \text{非線性, 1 階 2 次 O.D.E.}$$

Differential Equations

$$(4) \ y(x) = C_1 e^{2x} + C_2 e^x \quad - (a) \ , \ C_1, C_2 \in \text{Constant}$$

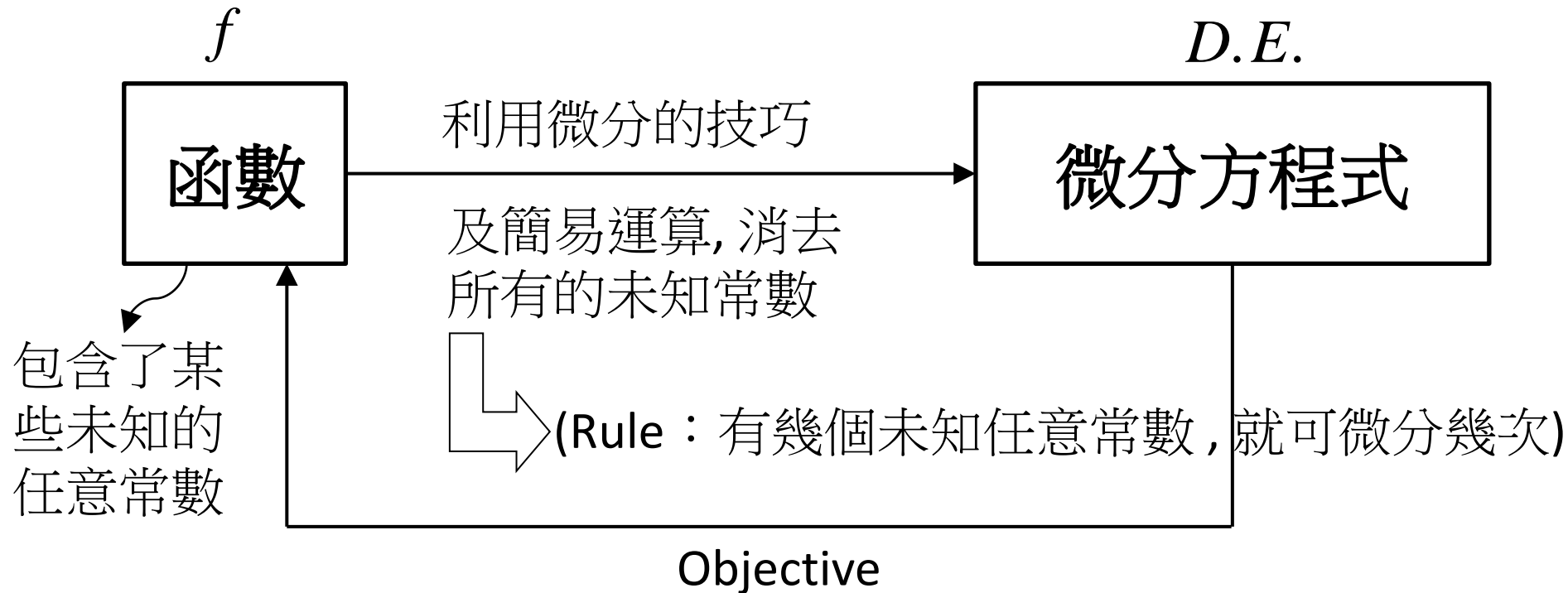
$$\frac{dy(x)}{dx} = 2C_1 e^{2x} + C_2 e^x \quad - (b)$$

$$\frac{d^2 y(x)}{dx^2} = 4C_1 e^{2x} + C_2 e^x \quad - (c)$$

$$(a) \times 2 + (b) \times -3 + (c) \times 1$$

$$\Rightarrow y''(x) - 3y'(x) + 2y(x) = 0$$

Differential Equations



⇒ Apply for O.D.E. Only

f 稱為 $D.E.$ 的通解(general solution) or 原函數

Differential Equations

例:

$$y(x) = C_1 \cos 3x + C_2 \sin 3x \quad , \quad C_1, C_2 \in \text{Constant}$$

$$\Rightarrow D.E. = ?$$

Sol:

$$\frac{dy}{dx} = -3C_1 \sin 3x + 3C_2 \cos 3x$$

$$\frac{d^2 y}{dx^2} = -9C_1 \cos 3x - 9C_2 \sin 3x$$

$$\Rightarrow y''(x) + 9y(x) = 0$$

Chapter 2.

First-Order Ordinary Differential Equations

First-Order Differential Equation

- 一般一階O.D.E., 可表成

1. $M(x, y)dx + N(x, y)dy = 0$

2. $y'(x) = f(x, y) \Rightarrow y'(x) = \frac{dy}{dx} = f(x, y)$

$$f(x, y)dx = dy \Rightarrow f(x, y)dx - dy = 0$$

$$\Rightarrow M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

$$y' = f(x, y)$$

First-Order Differential Equation

- 定理 P16 §1.2.1

對 $y' = f(x, y)$ 之 D.E. 有一個 Initial condition (I.C.)

$$y(x_0) = y_0$$

若 $f(x, y)$, $\frac{\partial f(x, y)}{\partial y}$ 於 (x_0, y_0) 之鄰域為連續,

則存在 $\varepsilon > 0$, 使得 $y(x)$ 於 $(x_0 - \varepsilon, x_0 + \varepsilon)$ 間有唯一解

First-Order Differential Equation

例: 下列各問題, 何者有唯一解

(1) $y' = e^{xy^2}$, $y(0) = 1$

(2) $y' = \sqrt{y}$, $y(0) = 0$

(3) $y' = \sqrt{y}$, $y(0) = 1$

(4) $y' = -\sqrt{1 - y^2}$, $y(0) = 0$

(5) $y' = -\sqrt{1 - y^2}$, $y(0) = 1$

First-Order Differential Equation

- Sol:

(1) $y' = e^{xy^2}$, $y(0) = 1$

$$f(x, y) = e^{xy^2} \quad (0,1)$$

$$\frac{\partial f(x, y)}{\partial y} = 2xye^{xy^2} \quad (0,1) \quad \Rightarrow \text{具唯一解}$$

(2) $y' = \sqrt{y}$, $y(0) = 0$

$$f(x, y) = \sqrt{y} \quad (0,0)$$

$$\frac{\partial f(x, y)}{\partial y} = \frac{1}{2\sqrt{y}} \quad (0,0) \quad \Rightarrow \text{不具唯一解}$$

First-Order Differential Equation

$$(3) \quad y' = \sqrt{y} \quad , \quad y(0) = 1$$

$(0,1) \Rightarrow$ 具唯一解

$$(4) \quad y' = -\sqrt{1-y^2} \quad , \quad y(0) = 0$$

$$f(x, y) = -\sqrt{1-y^2} \quad (0,0)$$

$$\frac{\partial f(x, y)}{\partial y} = \frac{y}{\sqrt{1-y^2}} \quad (0,0)$$

\Rightarrow 具唯一解

$$(5) \quad y' = -\sqrt{1-y^2} \quad , \quad y(0) = 1$$

$(0,1) \Rightarrow$ 不具唯一解

\Rightarrow (1) (3) (4) 具唯一解

Exact Equation

- 如何解 $M(x, y)dx + N(x, y)dy = 0 \dots (B)$

令 $u(x, y) = C \dots (A)$, $C \in \text{const}$.

$$\Delta u(x, y) = u(x + \Delta x, y + \Delta y) - u(x, y)$$

$$\begin{aligned}
 &= \underbrace{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)}_{\text{red star}} + \underbrace{u(x, y + \Delta y) - u(x, y)}_{\text{blue line}} \\
 &= \underbrace{\frac{\partial u(x, y)}{\partial x}}_{\text{red star}} (x + \Delta x - x) + \underbrace{\frac{\partial u(x, y)}{\partial y}}_{\text{blue line}} (y + \Delta y - y) = 0
 \end{aligned}$$

★ Mean-Value-Theorem 均值定理

$f(x)$, $a \leq x \leq b$, 一定存在一個 C , $a \leq C \leq b$

$$\text{St. } f'(C) = \frac{f(b) - f(a)}{b - a} \Rightarrow f'(C)(b - a) = f(b) - f(a)$$

Exact Equation

$$\Rightarrow \Delta u(x, y) = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y = 0$$

$$\Rightarrow du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$= M(x, y)dx + N(x, y)dy = 0$$

$$\Rightarrow M(x, y) = \frac{\partial u}{\partial x} \qquad N(x, y) = \frac{\partial u}{\partial y}$$

$$\partial u = M(x, y)\partial x \qquad \partial u = N(x, y)\partial y$$

$$\int \partial u = \int M(x, y)\partial x + f(y) \qquad \int \partial u = \int N(x, y)\partial y + g(x)$$

Exact Equation

$$\Rightarrow u = \begin{cases} \int M(x, y) \partial x + f(y) \cdots \cdots \cdots (1) \\ \int N(x, y) \partial y + g(x) \cdots \cdots \cdots (2) \end{cases}$$

經過比較(1)及(2)式，決定 $f(x)$ 和 $g(x)$

$$\Rightarrow u(x, y) = C$$

Exact Equation

- Q: 如何知道 (A) \rightarrow (B) 只有微分而已?

$$M(x, y) = \frac{\partial u}{\partial x}$$

$$N(x, y) = \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{M(x, y)}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{N(x, y)}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

假設 $u(x, y)$ 具有連續二階偏導數

$$\Rightarrow M(x, y)dx + N(x, y)dy = 0$$

若
$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

則稱此微分方程式為“正合”(Exact)

Exact Equation

例:

$$u(x, y) = x^2 y^3 = C$$

$$du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow 2xy^3 dx + x^2 3y^2 dy = 0$$

$$\text{Check: } \frac{\partial M(x, y)}{\partial y} = 2x3y^2 = \frac{\partial N(x, y)}{\partial x}$$

Exact Equation

例:

$$u(x, y) = xy^2 + 3x + 5y = C$$

$$du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow (y^2 + 3)dx + (2xy + 5)dy = 0$$

$$\frac{\partial M(x, y)}{\partial y} = 2y = \frac{\partial N(x, y)}{\partial x} \Rightarrow \text{正合}$$

$$\Rightarrow \frac{\partial u(x, y)}{\partial x} = y^2 + 3$$

$$\int \partial u(x, y) = \int (y^2 + 3)dx + f(y)$$

Exact Equation

$$\Rightarrow \frac{\partial u(x, y)}{\partial y} = 2xy + 5$$

$$\int \partial u(x, y) = \int (2xy + 5) dy + g(x)$$

$$\Rightarrow u = \begin{cases} xy^2 + 3x + f(y) \\ xy^2 + 5y + g(x) \end{cases}$$

$$\Rightarrow f(y) = 5y \quad , \quad g(x) = 3x$$

$$\Rightarrow u(x, y) = xy^2 + 3x + 5y = C$$