

# Chapter 2.

# First-Order Ordinary Differential Equations

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# Exact Examples

$$\text{例: } \underbrace{(e^x y + 6x + 5y)}_{\text{M}} dx + \underbrace{(e^x + 5x)}_{\text{N}} dy = 0$$

$$\frac{\partial \text{M}}{\partial y} = e^x + 5 \quad \Leftrightarrow \quad \frac{\partial \text{N}}{\partial x} = e^x + 5$$

(相等代表為正合)

$$\text{M} = \frac{\partial u}{\partial x}$$

$$\text{N} = \frac{\partial u}{\partial y}$$

$$\partial u = (e^x y + 6x + 5y) \partial x \quad \Leftrightarrow \quad \partial u = (e^x + 5x) \partial y$$

# Exact Examples

$$\begin{aligned} u = \int (e^x y + 6x + 5y) \partial x + f(y) &\Leftrightarrow u = \int (e^x + 5x) \partial y + g(x) \\ &= e^x y + 3x^2 + 5xy + f(y) \qquad \qquad \qquad = e^x y + 5xy + g(x) \end{aligned}$$

因此  $g(x) = 3x^2, f(y) = 0$

$$u(x, y) = e^x y + 5xy + 3x^2 = C$$

# Exact Examples

$$\text{例: } (\underbrace{\cos y + 8x}_M)dx + (\underbrace{-x \sin y + 3y^2}_N)dy = 0$$

$$\frac{\partial M}{\partial y} = -\sin y \quad \Leftrightarrow \quad \frac{\partial N}{\partial x} = -\sin y$$

(相等代表為正合)

$$M = \frac{\partial u}{\partial x}$$

$$N = \frac{\partial u}{\partial y}$$

$$\partial u = (\cos y + 8x) \partial x$$

$$\partial u = (-x \sin y + 3y^2) \partial y$$

$$u = \int (\cos y + 8x) \partial x \quad \Leftrightarrow$$

$$u = \int (-x \sin y + 3y^2) \partial y$$

# Exact Examples

$$\begin{aligned} u &= \int (\cos y + 8x) \partial x & \Leftrightarrow u &= \int (-x \sin y + 3y^2) \partial y \\ &= x \cos y + 4x^2 + f(y) \text{ ---(1)} & &= x \cos y + y^3 + g(x) \text{ ---(2)} \end{aligned}$$

比較(1)(2)

$$\text{得 } f(y) = y^3, g(x) = 4x^2$$

$$\therefore u(x, y) = x \cos y + y^3 + 4x^2 = C$$

# Non-Exact

- 若不為正合情況

$$u(x, y) = C \dots\dots (A)$$

↓

$$M(x, y)dx + N(x, y)dy = 0 \dots\dots (B)$$

若有乘法消去項,怎麼辦?

想法:

還它

# Non-Exact

方法：

解(B)式,應先將消去項,歸還回,使得(B)成為正合。

怎麼知道消去哪些項？

(1) 假設消去  $I(x, y)$

$$I(x, y)M(x, y)dx + I(x, y)N(x, y)dy = 0$$

若  $\frac{\partial M_1(x, y)}{\partial y} = \frac{\partial N_1(x, y)}{\partial x}$ , 則  $M_1 dx + N_1 dy = 0$  為正合

# Non-Exact

$$(2) \quad \frac{\partial}{\partial y}(I(x, y)M(x, y)) = \frac{\partial}{\partial x}(I(x, y)N(x, y))$$

$$M(x, y)\frac{\partial I(x, y)}{\partial y} + I(x, y)\frac{\partial M(x, y)}{\partial y} = N(x, y)\frac{\partial I(x, y)}{\partial x} + I(x, y)\frac{\partial N(x, y)}{\partial x}$$

$$-N(x, y)\frac{\partial I(x, y)}{\partial x} + M(x, y)\frac{\partial I(x, y)}{\partial y} = I(x, y)\left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}\right]$$



# Integrating Factor

- 目的：

解 $I(x, y)$ 所形成一階P.D.E. ;  $I(x, y)$ 稱為積分因子(Integrating Factor)。

考慮一階P.D.E.

$$P(x, y, z) \frac{\partial z}{\partial x} + Q(x, y, z) \frac{\partial z}{\partial y} = R(x, y, z) \dots\dots (*)$$

⇒ 由下列等式決定出兩個獨立解

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{稱輔助方程組、lagrange方程組}$$

# Integrating Factor

其中，(\*)的通解可以是

$\varphi(u, v) = 0$       隱函數表示法

$v = f(u)$       顯函數表示法

比較前式

$$-N \frac{\partial I}{\partial x} + M \frac{\partial I}{\partial y} = I \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$\frac{dx}{-N} = \frac{dy}{M} = \frac{dI}{I \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$

⇒ 可以利用此式來求解  $I$ ，  
而非要求 (\*) 的通解

如何求  $\Rightarrow$  TRY !!

# Integrating Factor

- 招數：

(1) 猜  $I$  是  $x$  的函數，看看有沒有解。i.e.  $I(x)$

$$\frac{dx}{-N} = \frac{dI}{I \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)} \Rightarrow \frac{\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx}{-N} = \frac{dI}{I}$$

預達到希望(猜對)，則需  $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{-N} = f(x)$

# Integrating Factor

$$f(x)dx = \frac{dI}{I}$$

$$(\text{兩邊積分}) \Rightarrow \int f(x)dx = \ln I$$

$$\Rightarrow I = e^{\int f(x)dx}$$

(2) 猜  $I$  是  $y$  的函數。 i.e.  $I(y)$

$$\frac{dy}{M} = \frac{dI}{I \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)} \Rightarrow \frac{\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{M} dy = \frac{dI}{I}$$

# Integrating Factor

預達到希望(猜對)，則需  $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$

$$g(y)dy = \frac{dI}{I}$$

$$(\text{兩邊積分}) \Rightarrow \int g(y)dy = \ln I$$

$$\Rightarrow I = e^{\int g(y)dy}$$

# Integrating Factor

(3) 猜I是的  $(x + y)$  函數

$$\frac{d}{dx}(x + y) = 1 + \frac{dy}{dx}$$

$$\text{同乘} dx \Rightarrow d(x + y) = dx + dy$$

Hint: (和分比概念)

$$\frac{1}{2} = \frac{2}{4} = \frac{1+2}{2+4} = \frac{1 \times 2}{2 \times 2} = \frac{2 \times 3}{4 \times 3} = \frac{1 \times 2 + 2 \times 3}{2 \times 2 + 4 \times 3}$$

# Integrating Factor

$$\frac{dx + dy}{-N + M} = \frac{dI}{I \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)} \Rightarrow \frac{d(x + y)}{-N + M} = \frac{dI}{I \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$

預達到希望(猜對)，則需  $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{-N + M} = f(x + y)$

$$f(x + y) d(x + y) = \frac{dI}{I}$$

(兩邊積分)  $\Rightarrow \int f(x + y) d(x + y) = \ln I$

$$\Rightarrow I = e^{\int f(x+y) d(x+y)}$$

# Integrating Factor

(4) 猜  $I$  是  $xy$  的函數。

其中：

$$d(xy) = ?$$

$$\frac{d(xy)}{dx} = y \frac{dx}{dx} + x \frac{dy}{dx}$$

$$\frac{ydx + xdy}{y(-N) + xM} = \frac{dI}{I \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$



# Integrating Factor

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d(xy)}{-yN + xM} = \frac{dI}{I} \quad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{y(-N) + xM} = f(xy)$$

預達到希望(猜對)，則需

$$f(xy) d(xy) = \frac{dI}{I}$$

$$(\text{兩邊積分}) \Rightarrow \int f(xy) d(xy) = \ln I$$

$$\Rightarrow I = e^{\int f(xy) d(xy)}$$

# Integrating Factor

- Summary :

(1)(2)(3)(4)分子都是  $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

若發現  $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$

則算出  $(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$

除以  $-N \rightarrow I(x)$

除以  $-N+M \rightarrow I(x+y)$

除以  $M \rightarrow I(y)$

除以  $-y*N+x*M \rightarrow I(x*y)$

# Integrating Factor Examples

例:  $(x^2 + y^2 + x)dx + xydy = 0$

$M$   $N$

$$\frac{\partial M}{\partial y} = 2y \quad \leftarrow \text{不相等} \quad \rightarrow \quad \frac{\partial N}{\partial x} = y$$

$$\frac{dx}{x} = \frac{\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{-N} dx = \frac{dI}{I}$$

$$I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

# Integrating Factor Examples

$$\text{原式} \Rightarrow (x^3 + xy^2 + x^2)dx + x^2 y dy = 0$$

$$\frac{\partial M}{\partial y} = 2xy \quad = \quad \frac{\partial N}{\partial x} = 2xy$$

正合

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$\partial u = (x^3 + xy^2 + x^2)\partial x$$

$$u = \frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + \frac{1}{3}x^3 + f(y)$$

# Integrating Factor Examples

$$\partial u = (x^2 y) \partial y \qquad u = \frac{1}{2} x^2 y^2 + g(x)$$

$$\therefore f(y) = 0$$

$$g(x) = \frac{1}{4} x^4 + \frac{1}{3} x^3$$

$$\mu(x, y) = \frac{1}{2} x^2 y^2 + \frac{1}{4} x^4 + \frac{1}{3} x^3 = C$$

# Integrating Factor Examples

Sol2(用微積分分解) 不好解，但可以用來當驗算

題:  $(x^2 + y^2 + x)dx + xydy = 0$

$$y^2 dx + xydy + (x^2 + x)dx = 0$$

$$y(ydx + xdy) + (x^2 + x)dx = 0$$

$$yd(xy) + (x^2 + x)dx = 0$$

同乘x  $\Rightarrow xy d(xy) + (x^3 + x^2)dx = 0$

$$\Rightarrow \frac{1}{2}x^2 y^2 + \frac{1}{4}x^4 + \frac{1}{3}x^3 = C$$

# Integrating Factor Examples

例]:  $2\sin(y^2)dx + xy\cos(y^2)dy = 0$

$$\frac{\partial M}{\partial y} = 2\cos(y^2)2y = 4y\cos(y^2) \qquad \frac{\partial N}{\partial x} = y\cos(y^2)$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -3y\cos(y^2)$$

$$\Rightarrow \frac{-3y\cos(y^2)}{-N} dx = \frac{-3y\cos(y^2)}{-xy\cos(y^2)} dx = \frac{3}{x} dx = \frac{dI}{I}$$

$$\Rightarrow I = e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

# Integrating Factor Examples

得  $2x^3 \sin(y^2)dx + x^4 y \cos(y^2)dy = 0$

$$M = \frac{\partial u}{\partial x}$$

$$N = \frac{\partial u}{\partial y}$$

$$u = \int 2x^3 \sin(y^2)dx + f(y)$$

$$u = \int x^4 y \cos(y^2)dy + g(x)$$

$$= \frac{1}{2} x^4 \sin(y^2) + f(y)$$

$$t = y^2, dt = 2ydy, dy = \frac{dt}{2y}$$
$$= \frac{1}{2} x^4 \sin(y^2) + g(x)$$



# Integrating Factor Examples

$$\because g(x) = 0, f(y) = 0 \quad \therefore u = \frac{1}{2}x^4 \sin(y^2) = C$$

其中在  $u$  中沒有  $x$  的式

$$\int y \cos(y^2) dy = \int y \cos(t) \frac{dt}{2y} = \int \frac{1}{2} \cos(t) dt$$

$$= \frac{1}{2} \sin(t)$$

$$= \frac{1}{2} \sin(y^2)$$

# Integrating Factor Examples

例:  $xy \cos(x^2) dx + 2 \sin(x^2) dy = 0$

$$\frac{\partial M}{\partial y} = x \cos(x^2) \neq \frac{\partial N}{\partial x} = 2 \cos(x^2) 2x = 4x \cos(x^2)$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x \cos(x^2)$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xy \cos(x^2)} dy = \frac{3}{y} dy \quad I = e^{\int \frac{3}{y} dy} = y^3$$

# Integrating Factor Examples

$$\text{得 } xy^4 \cos(x^2)dx + 2y^3 \sin(x^2)dy = 0$$

$$M = \frac{\partial u}{\partial x}$$

$$u = \int xy^4 \cos(x^2)dx + f(y)$$

$$= \frac{1}{2} y^4 \sin(x^2) + f(y)$$

$$N = \frac{\partial u}{\partial y}$$

$$u = \int 2y^3 \sin(x^2)dy + g(x)$$

$$= \frac{1}{2} y^4 \sin(x^2) + g(x)$$

$$\because f(y) = 0, g(x) = 0 \quad \therefore u = \frac{1}{2} y^4 \sin(x^2) = C$$

# Integrating Factor Examples

例:  $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$

$$\frac{\partial M}{\partial y} = x + 2y$$

$$\frac{\partial N}{\partial x} = y + 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - y$$

$$\frac{x - y}{-N + M} d(x + y)$$

$$= \frac{x - y}{y^2 - x^2} d(x + y) = \frac{-1}{x + y} d(x + y)$$

$$I = e^{\int \frac{-1}{x+y} d(x+y)} = e^{-\ln(x+y)} = e^{\ln \frac{1}{(x+y)}} = \frac{1}{x + y}$$

$$\frac{1}{x + y} (xy + y^2 + 1)dx + \frac{1}{x + y} (xy + x^2 + 1)dy = 0$$

# Integrating Factor Examples

$$M = \frac{\partial u}{\partial x}$$

$$u = \int \frac{xy + y^2 + 1}{x + y} dx + f(y)$$

$$= \int \left( y + \frac{1}{x + y} \right) dx + f(y)$$

$$= yx + \ln(x + y) + f(y)$$

$$N = \frac{\partial u}{\partial y}$$

$$u = \int \frac{xy + x^2 + 1}{x + y} dx + g(x)$$

$$= \int \left( x + \frac{1}{x + y} \right) dx + g(x)$$

$$= xy + \ln(x + y) + g(x)$$

$$\because g(x) = 0, f(y) = 0 \quad \therefore u(x, y) = xy + \ln(x + y) = C$$

# Integrating Factor Non-Unique

**Note:** 若取的不同，則積分因子可以不唯一，但答案一定相同。

如上題  $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - y$$

$$\frac{x - y}{-yN + xM} d(xy) = \frac{x - y}{-y(xy + x^2 + 1) + x(xy + y^2 + 1)} d(xy) \\ = d(xy)$$

$$I = e^{\int 1 d(xy)} = e^{xy} \quad \Leftarrow \quad \text{積分因子不唯一}$$

# Integrating Factor Non-Unique

$$\text{得 } e^{xy}(xy + y^2 + 1)dx + e^{xy}(xy + x^2 + 1)dy = 0$$

$$M = \frac{\partial \mu}{\partial x} \quad u = \int e^{xy}(xy + y^2 + 1)dx + f(y)$$

$$\text{其中 } \int e^{xy} xy dx = ? \dots (1)$$

$$y \int e^{xy} x dx = y \left( x \frac{1}{y} e^{xy} - \int \frac{1}{y} e^{xy} dx \right) = x e^{xy} - \int e^{xy} dx$$

$$\int e^{xy} y^2 dx \dots (2)$$

$$= \frac{1}{y} e^{xy} y^2 = y e^{xy}$$

$$\int e^{xy} dx \dots (3)$$

# Integrating Factor Non-Unique

$$\begin{aligned}\text{因此 } u &= (1) + (2) + (3) + f(y) \\ &= xe^{xy} + ye^{xy} + f(y)\end{aligned}$$

$$\text{同理 } N = \frac{\partial \mu}{\partial y}$$

$$u = xe^{xy} + ye^{xy} + g(x)$$

$$\because f(y) = 0, g(x) = 0 \quad \therefore u = xe^{xy} + ye^{xy} = C$$