Trees

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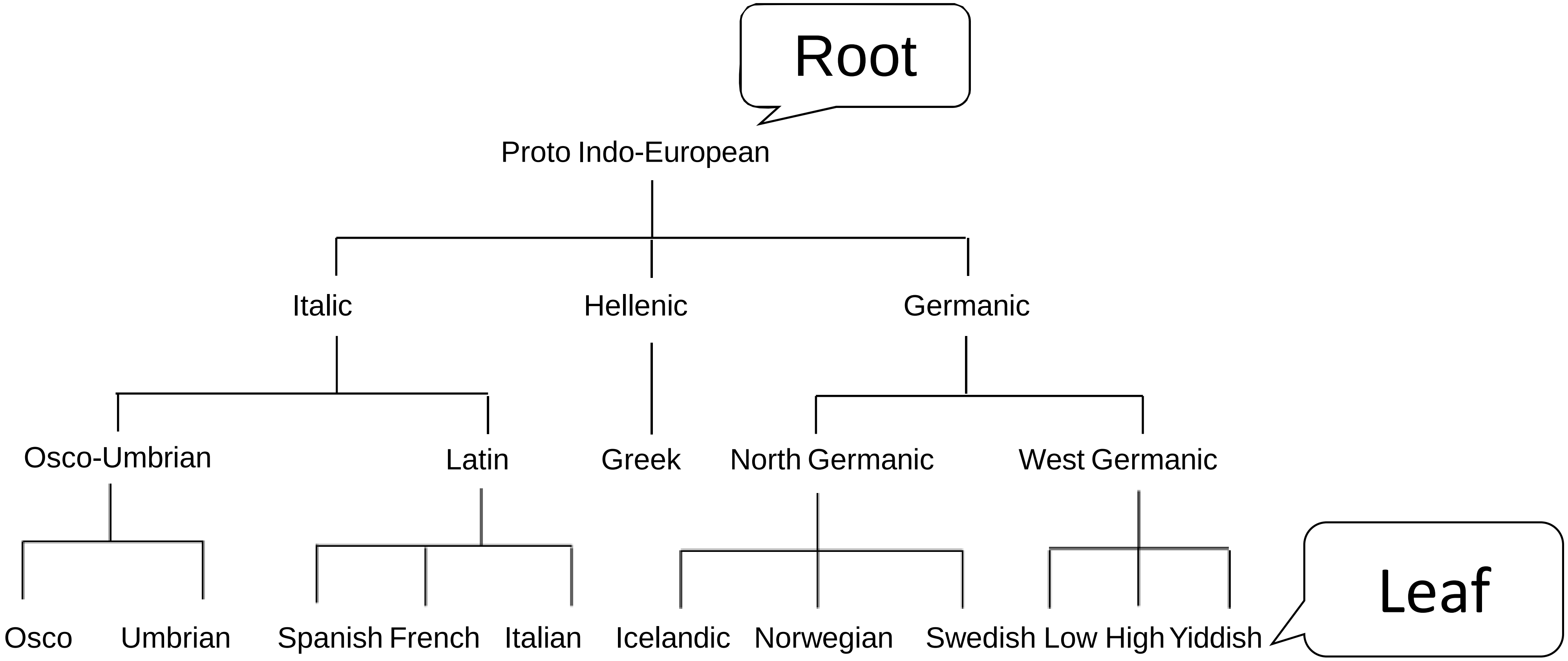
Outline

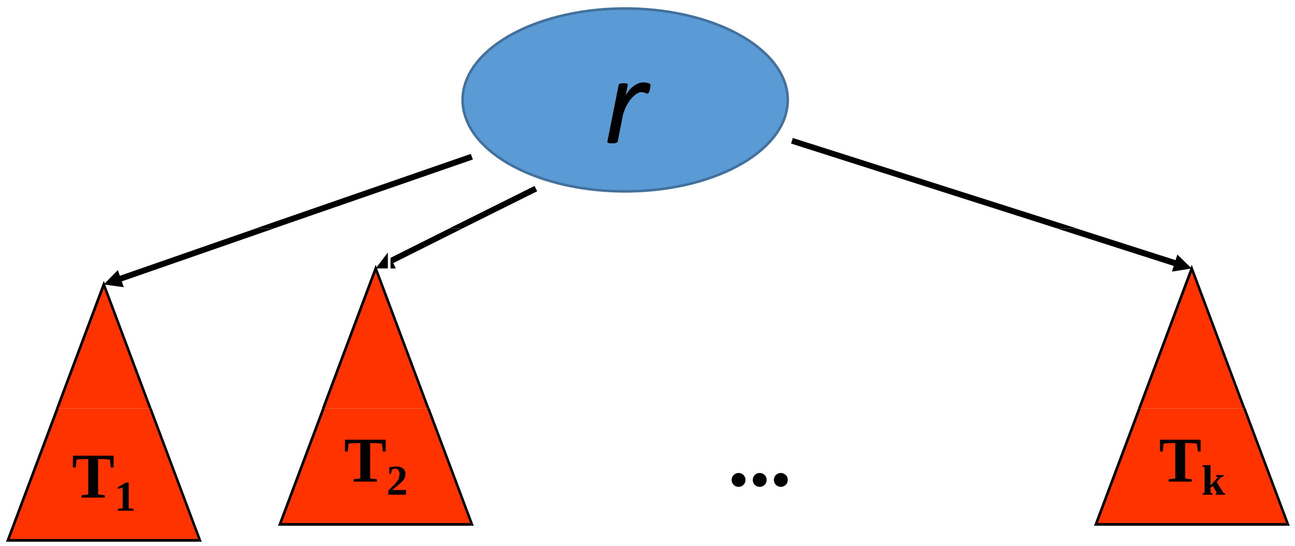
- Introduction
- Binary Trees
- Binary Tree Traversal and Tree Iterators
- Additional Binary Tree Operations
- Threaded Binary Trees
- Heaps
- Binary Search Trees
- Selection Trees
- Forests
- Representation of Disjoint Sets

Trees


Root
Proto Indo-European
Italic
Hellenic
Germanic
Osco-Umbrian
Latin
Greek
North Germanic
West Germanic
Osco
Umbrian
Spanish
French
Italian
Icelandic
Norwegian
Swedish
Low
High
Yiddish
Leaf

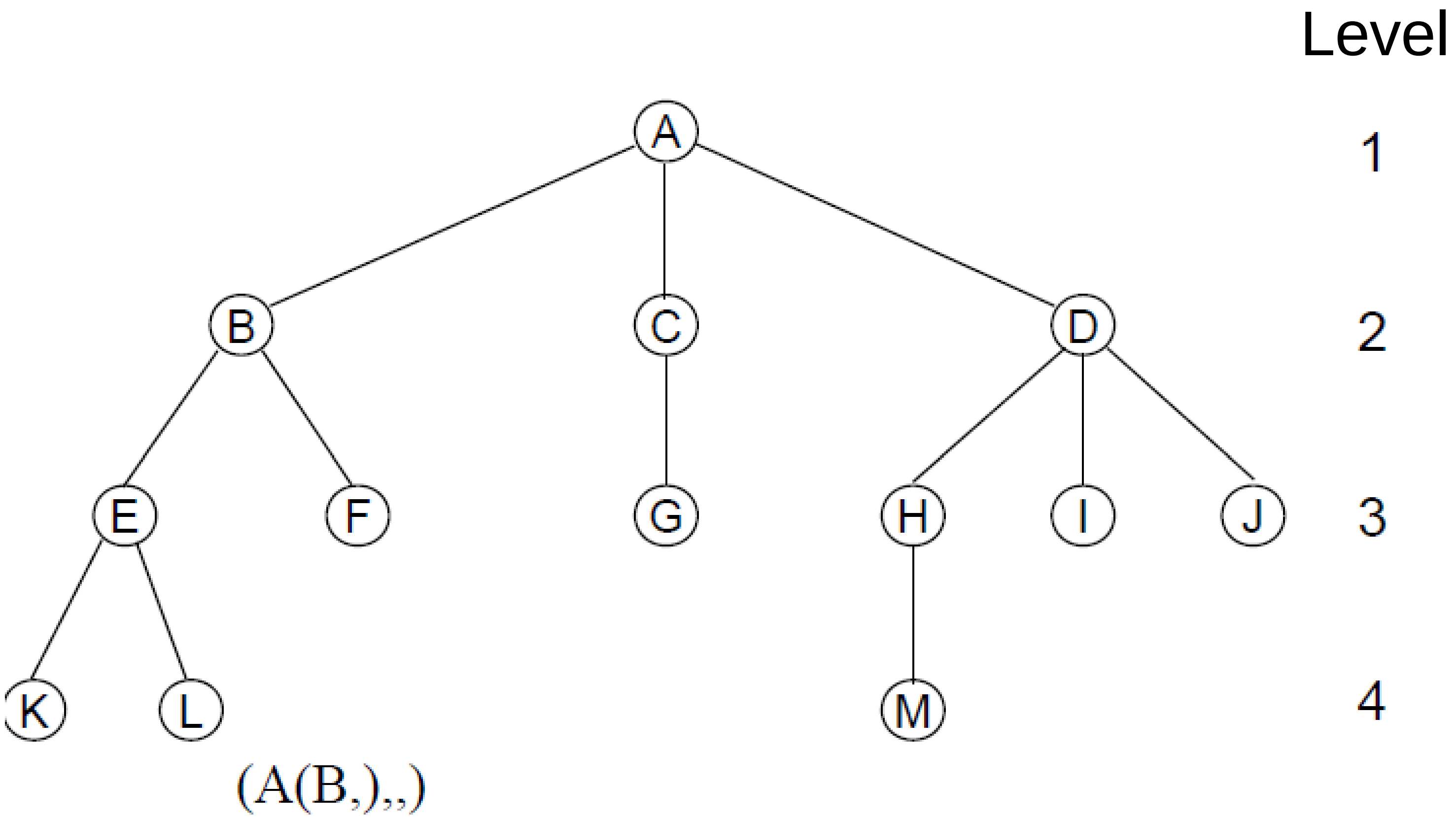
Trees (Cont'd)

- Tree: a finite set of one or more nodes such that
  - a distinguished node *r* (root)
  - zero or more nonempty (sub)trees $T_1, T_2, \ldots, T_k$ each of whose roots are connected by a directed edge from *r*



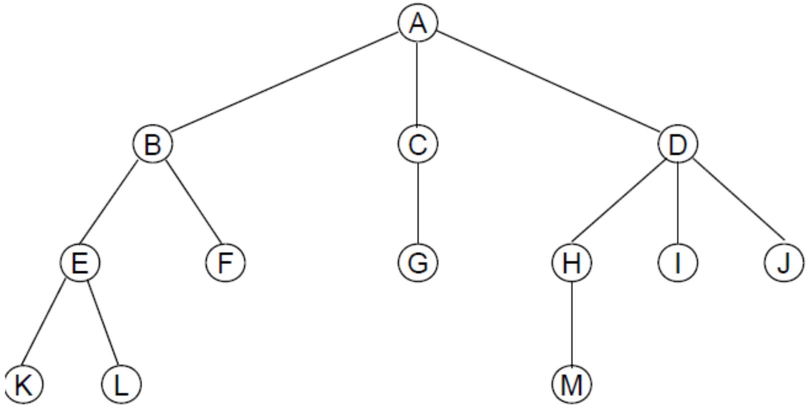
A Sample Tree

- Assume the root is at level 1, then the level of a node is the level of the node's parent plus one.
- The height (or the depth) of a tree is the maximum level



Terminology

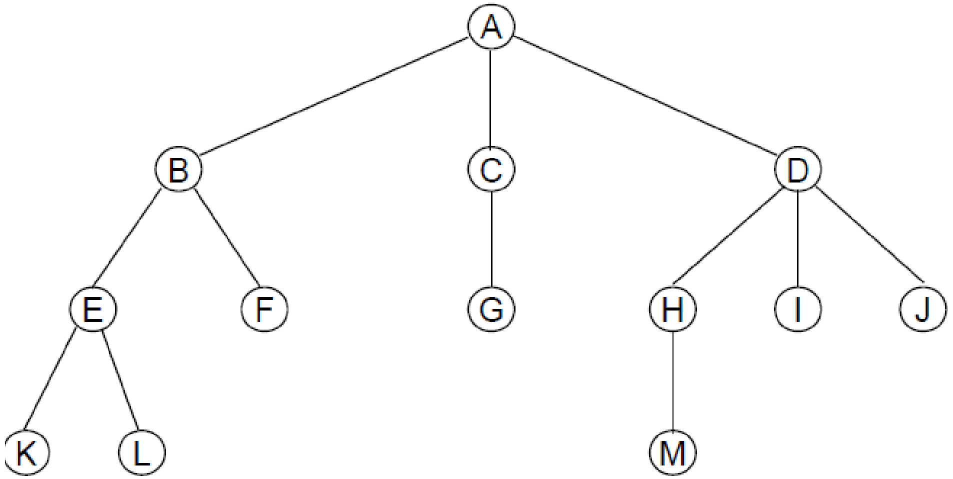
- The degree of a node is the number of subtrees of the node
  - The degree of A is 3; the degree of C is 1.
- The node with degree 0 is a leaf or terminal node.
- A node that has subtrees is the parent of the roots of the subtrees.
- The roots of these subtrees are the children of the node.
- Children of the same parent are siblings.
- The ancestors of a node are all the nodes long the path from the root to the node.



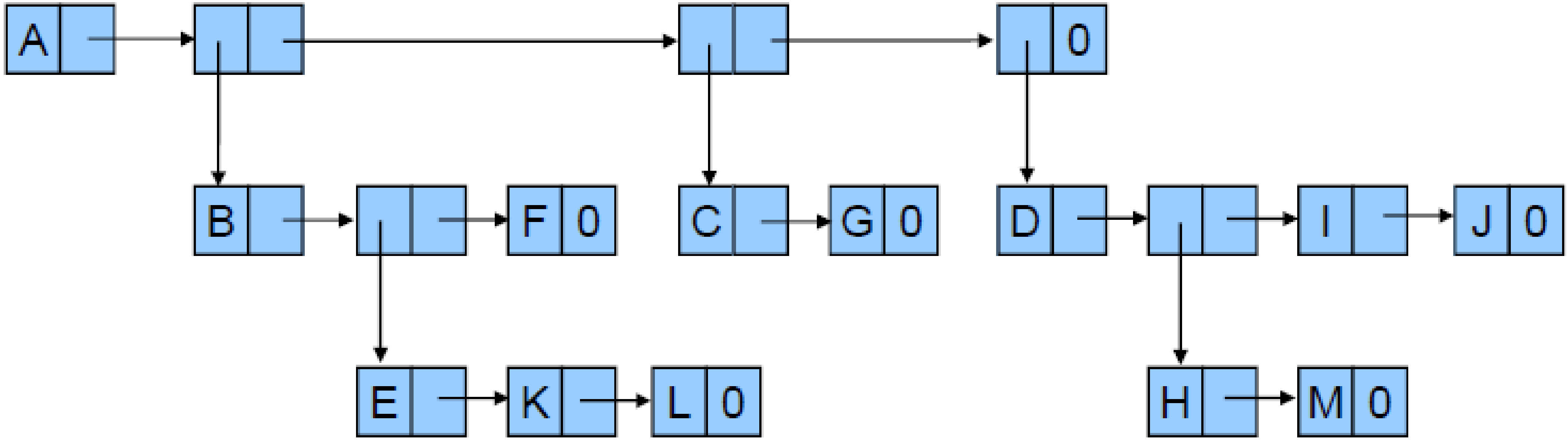
Representation of Trees

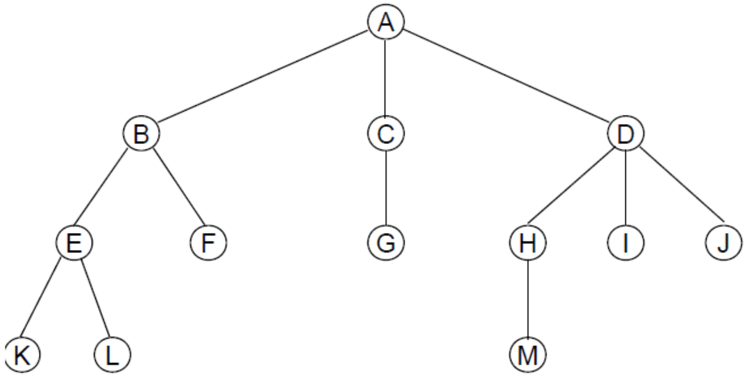
- List representation
  - The root comes first, followed by a list of sub-trees
  - $T=(root\ (T_1, T_2, \ldots, T_n))$

(A(B (E (K, L), F), C (G), D (H (M), I, J)))



List Representation of Trees





Possible Node Structure for a Tree of Degree *k*

- **Lemma 5.1:** If T is a *k*-ary tree (i.e., a tree of degree *k*) with n nodes, each having a fixed size as in Figure 5.4, then n(k-1) + 1 of the *nk* child fields are 0, $n \geq 1$.

The total number of child fields in a k-ary tree with n nodes

The number of non-zero child fields in a n-node tree

$$nk-(n-1)=n(k-1)+1$$

Data	Child 1	Child 2	Child 3	Child 4	…	Child k

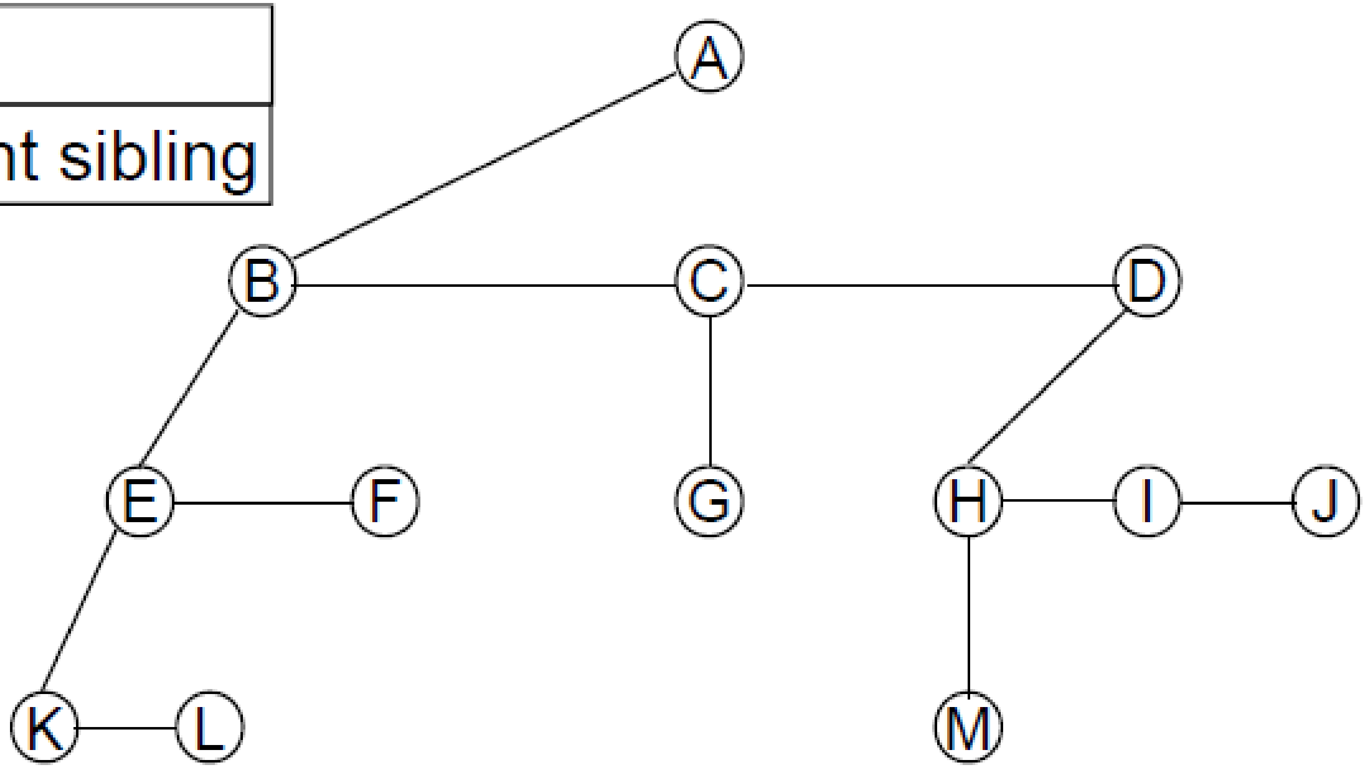
Wasting memory!

Representation of Trees

- Left Child-Right Sibling Representation
  - Each node has two links (or pointers).
  - Each node only has one leftmost child and one closest right sibling.

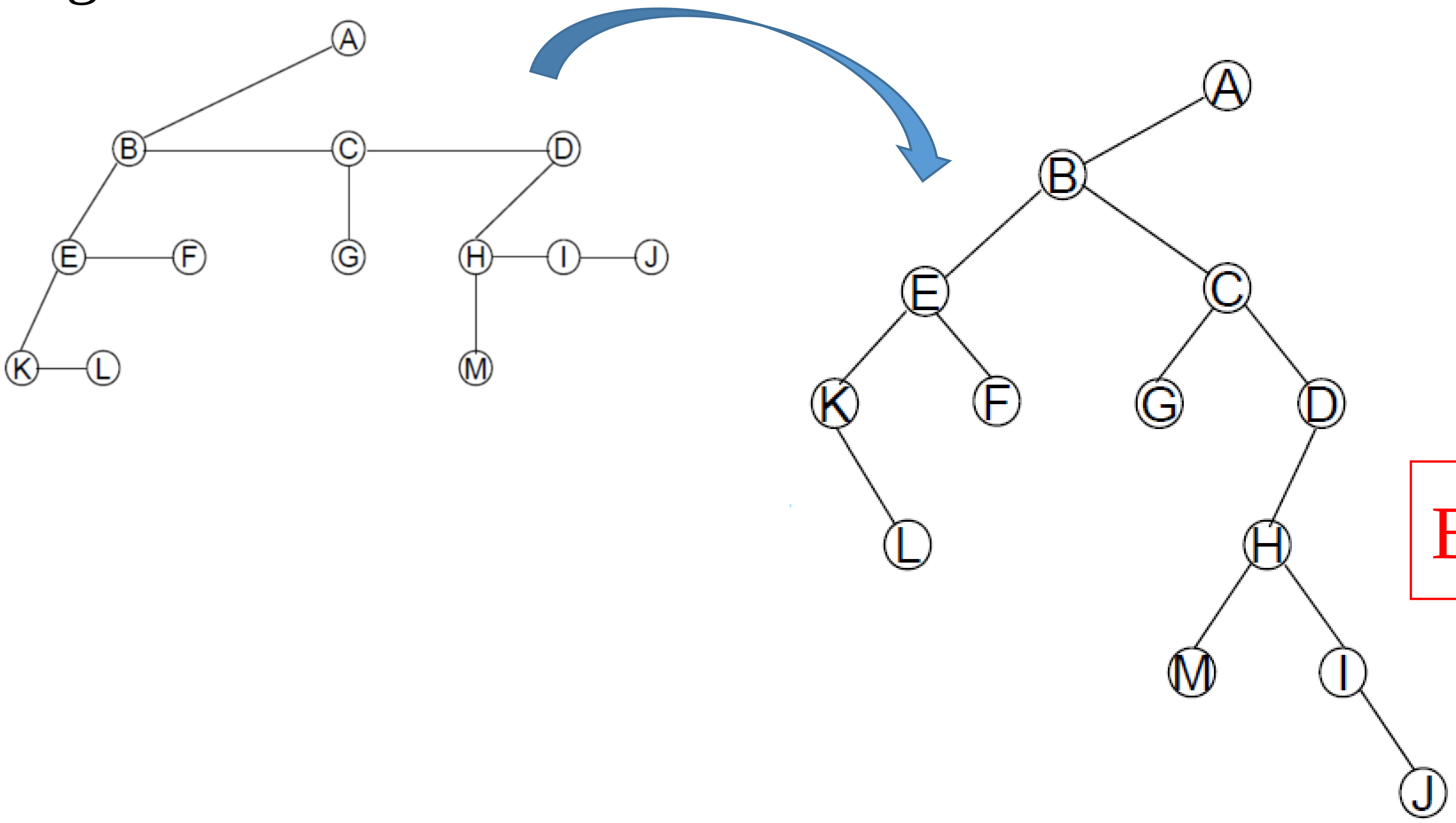
<table>
<tr><td colspan="2">data</td></tr>
<tr><td>left child</td><td>right sibling</td></tr>
</table>

The node structure has fixed number of fields



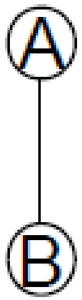
Degree-Two Tree Representation

- Rotate the right-sibling pointers in a left child-right sibling tree clockwise by 45 degrees

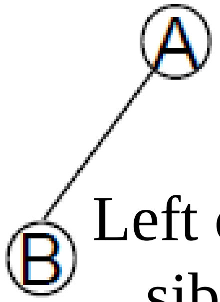


Binary Tree!

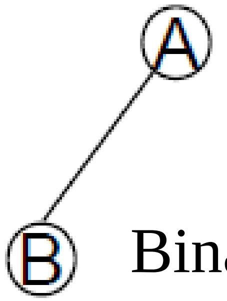
Tree Representations



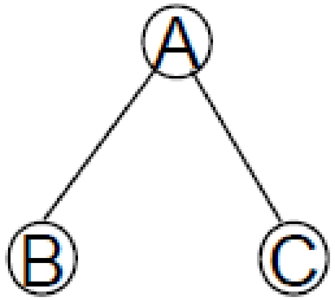
tree



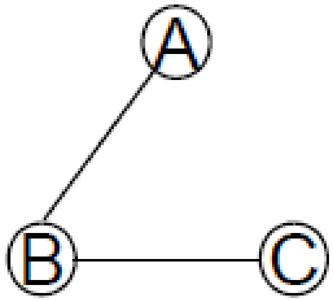
Left child-right sibling tree



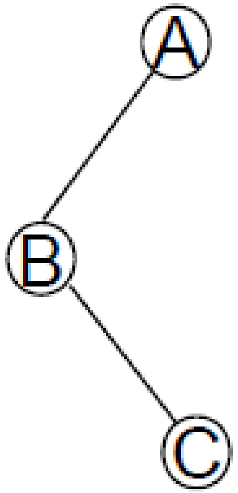
Binary tree



tree



Left child-right sibling



Binary tree

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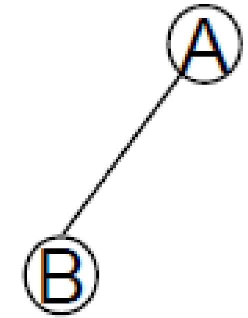
Binary Tree (B-Tree)

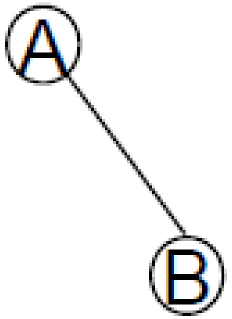
- Definition:
  - A binary tree is a finite set of nodes that is either empty or consists of a root and two disjoint binary trees called the left subtree and the right subtree.
- In a B-Tree, the degree of a node cannot exceed 2
- In a B-Tree, left subtree and right subtree are different
- There is no tree with zero nodes. But there is an empty binary tree.
- In a tree, the order of the subtrees is irrelevant.

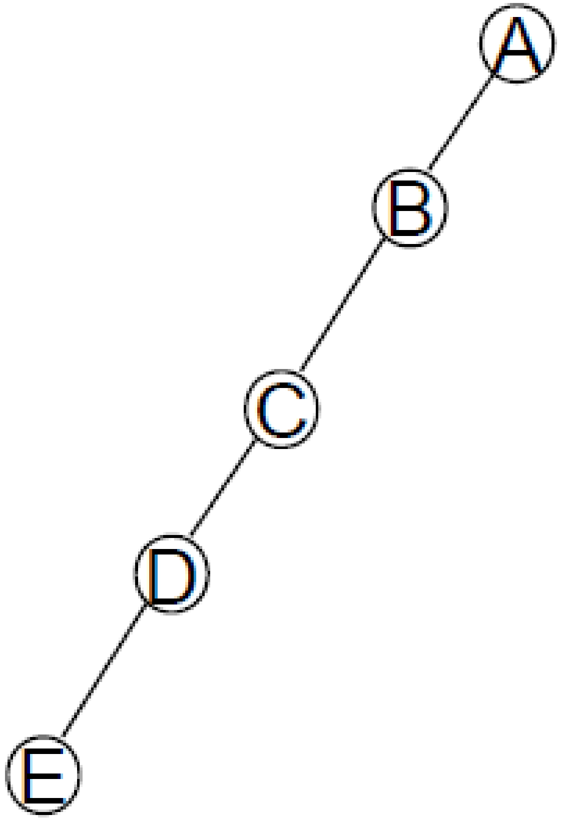
Distinctions between a Binary Tree and a Tree

	Binary tree	Tree
degree	$\leqq 2$	Not limited
order of the subtrees	ˇ	×
allow zero nodes	ˇ	×

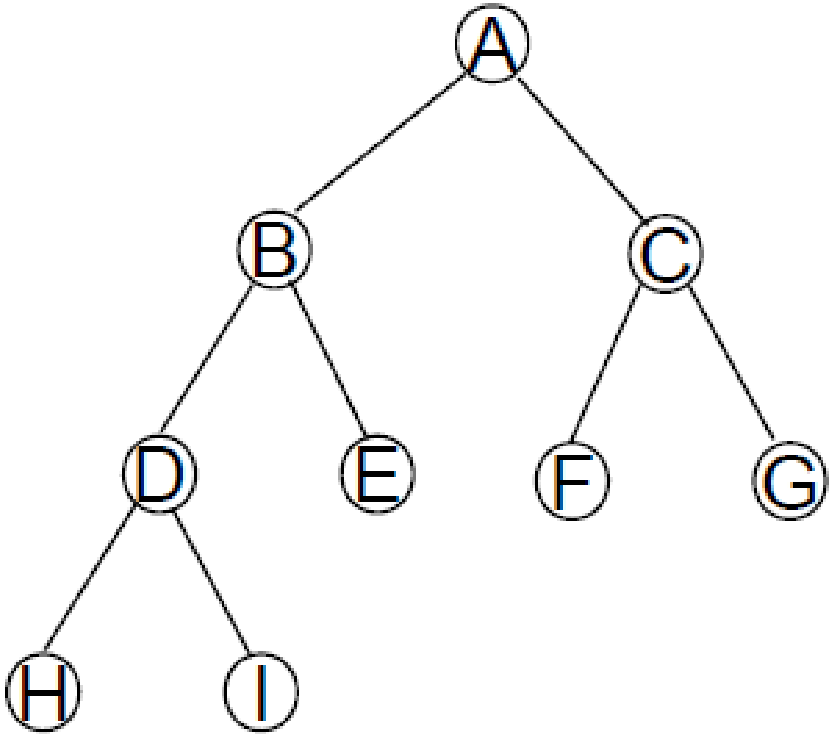
Binary Tree Examples







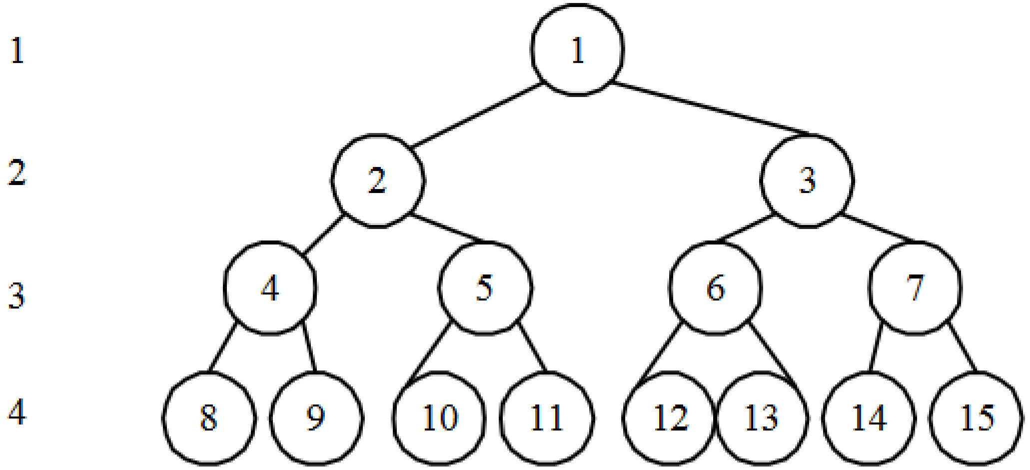
Skewed binary tree



Complete binary tree

The Properties of Binary Trees

- **Lemma 5.2** [Maximum number of nodes]
  1) The maximum number of nodes on level *i* of a binary tree is 2^{i-1}, $i \geq 1$.
  2) The maximum number of nodes in a binary tree of depth k is $2^k - 1$, $k \geq 1$.
- **Lemma 5.3** [Relation between number of leaf nodes and nodes of degree 2]: For any non-empty binary tree, T, if n_0 is the number of leaf nodes and n_2 the number of nodes of degree 2, then $n_0 = n_2 + 1$.
- **Definition: A full binary tree** of depth k is a binary tree of depth k having $2^k - 1$ nodes, $k \geq 0$.



Maximum Number of Nodes in Binary Trees

- The maximum number of nodes on level i of a binary tree is 2^{i-1}, i>=1.
- The maximum number of nodes in a binary tree of depth k is $2^k - 1$, k>=1.

- Prove by induction

$$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$$

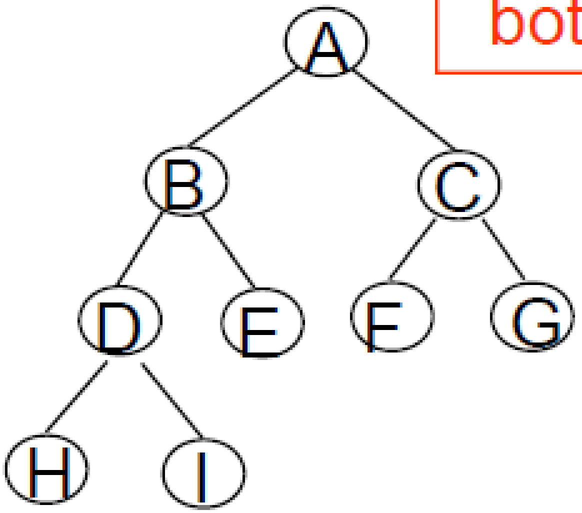
Relations between Number of Leaf Nodes and Nodes of Degree 2

- For any nonempty binary tree T, if n_0 is the number of leaf nodes and n_2 the number of nodes of degree 2, then $n_0 = n_2 + 1$
- Proof:
  - Let n and B denote the total number of nodes & branches in T.
  - Let n_0, n_1, n_2 represent the nodes with no children, single child, and two children respectively.
  - $n= n_0 + n_1 +n_2$, $B+1=n$, $B= n_1 +2n_2$ ==> $n_1 +2 n_2+1= n$,
  - $n_1 +2n_2 +1= n_0 + n_1 + n_2$ ==> $n_0 = n_2+1$

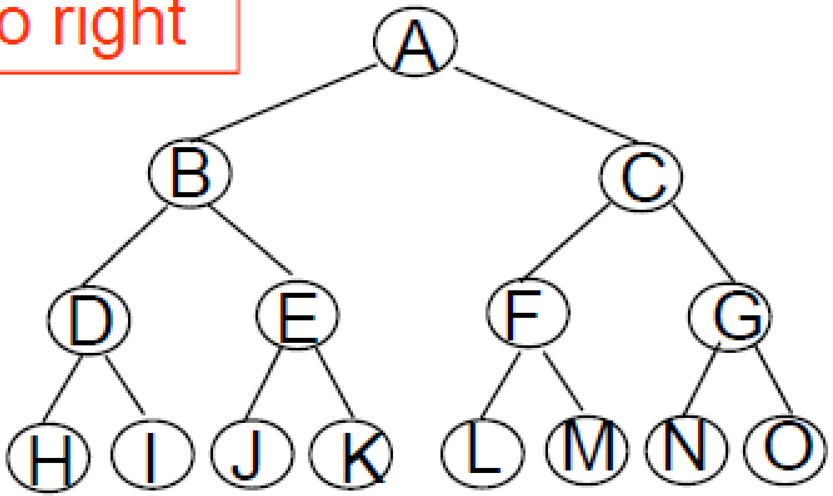
Full BT vs Complete BT

- A full binary tree of depth *k* is a binary tree of depth *k* having 2^k-1 nodes, *k*>=0.
- A binary tree with *n* nodes and depth *k* is complete *iff* its nodes correspond to the nodes numbered from 1 to *n* in the full binary tree of depth *k*.

Numbering from top to bottom, to left to right



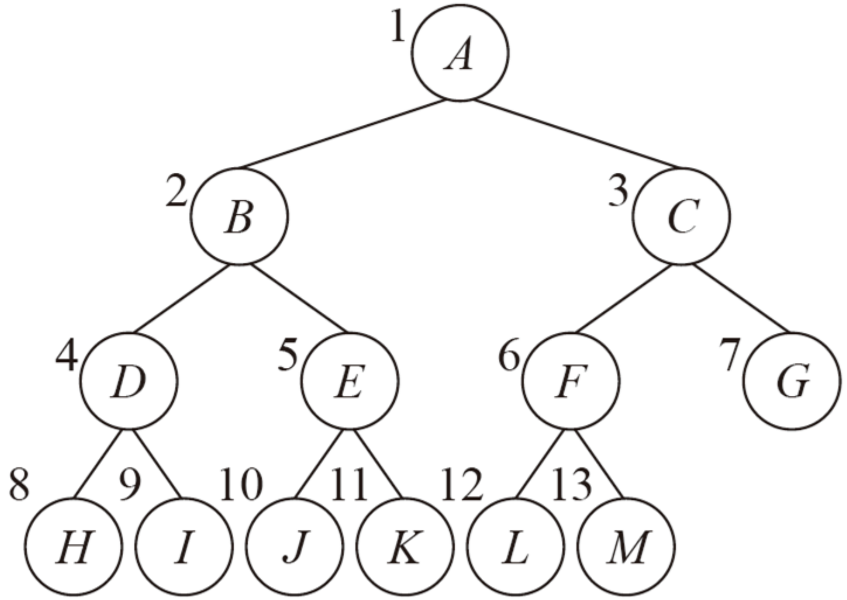
Complete binary tree



Full binary tree of depth 4

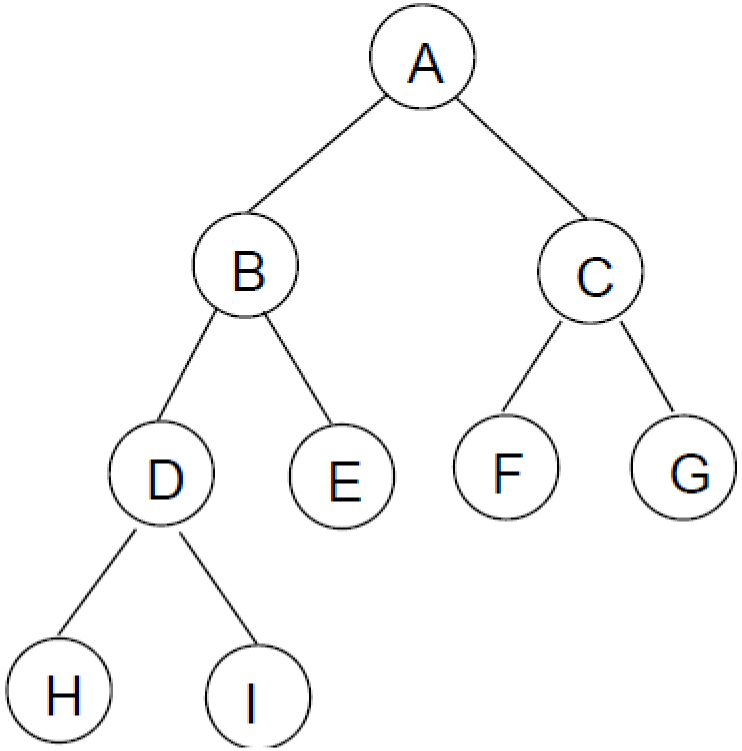
Array Representation of a Binary

- **Lemma 5.4:** If a complete binary tree with n nodes is represented sequentially, then for any node with index i, $1 \le i \le n$, we have:
  - parent(i) is at $\lfloor i/2 \rfloor$ if $i \ne 1$. If i = 1, i is at the root and has no parent.
  - left_child(i) is at 2i if 2$i \le n$. If 2i > n, then i has no left child.
  - right_child(i) is at 2i + 1 if 2i + 1 $\le n$. If 2i + 1 > n, then i has no right child.
- Position zero of the array is not used.

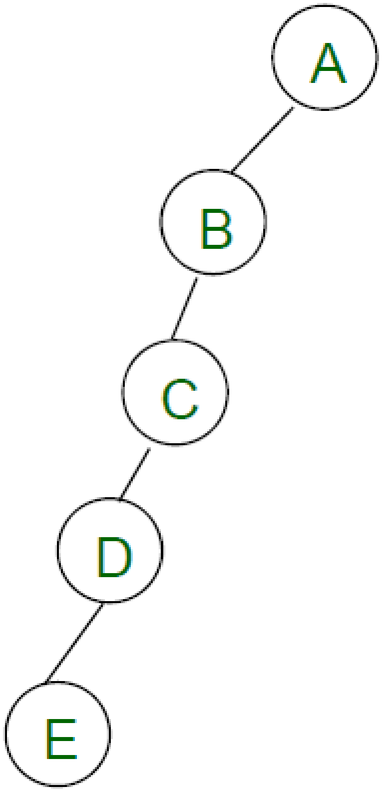


Sequential Representation

- Waste space
- Insertion/deletion problem

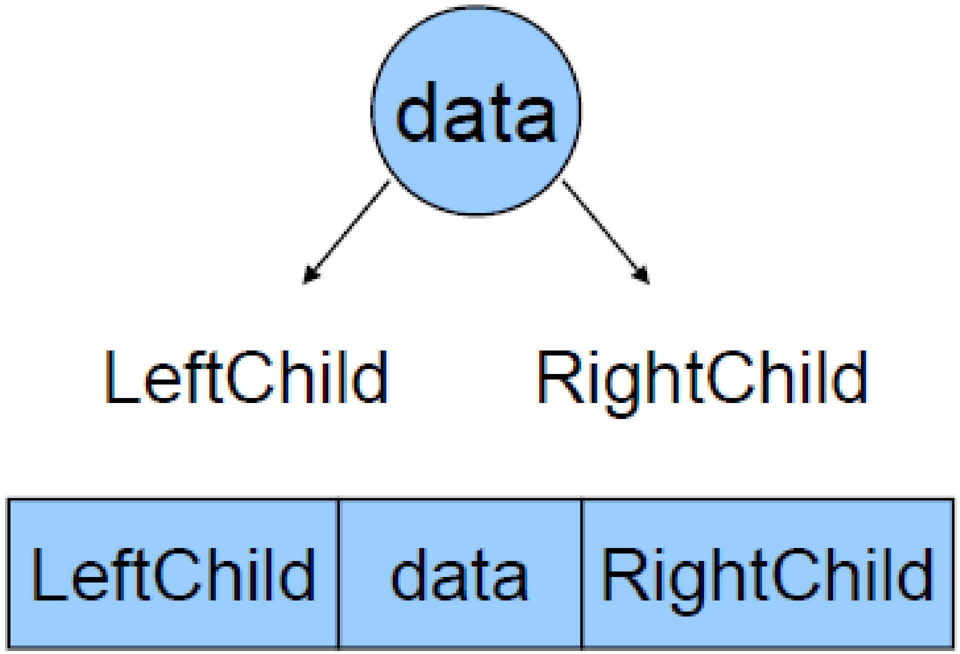


[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I



[1]	A
[2]	B
[3]	--
[4]	C
[5]	--
[6]	--
[7]	--
[8]	D
[9]	--
.	.
[16]	E

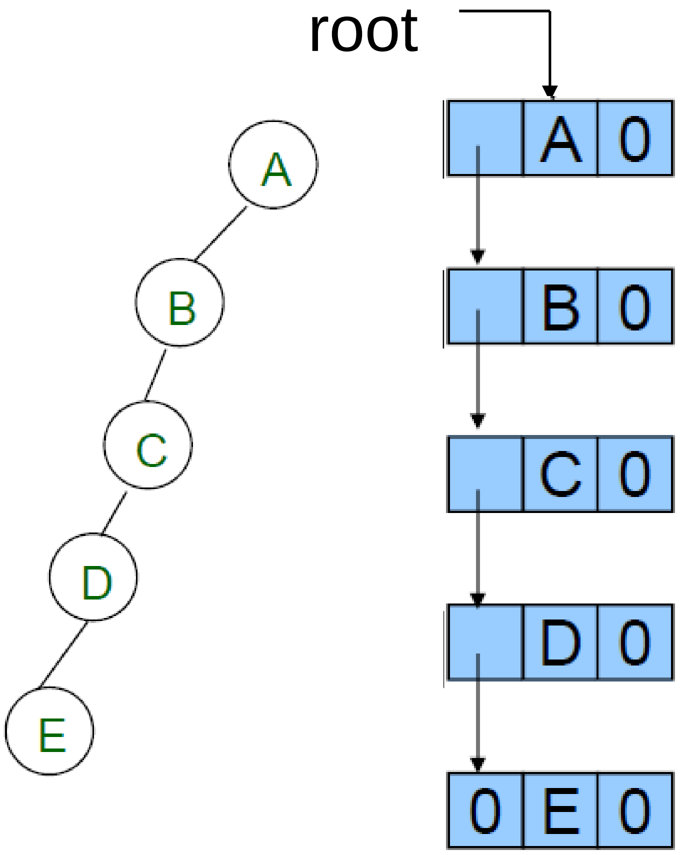
Node Representation



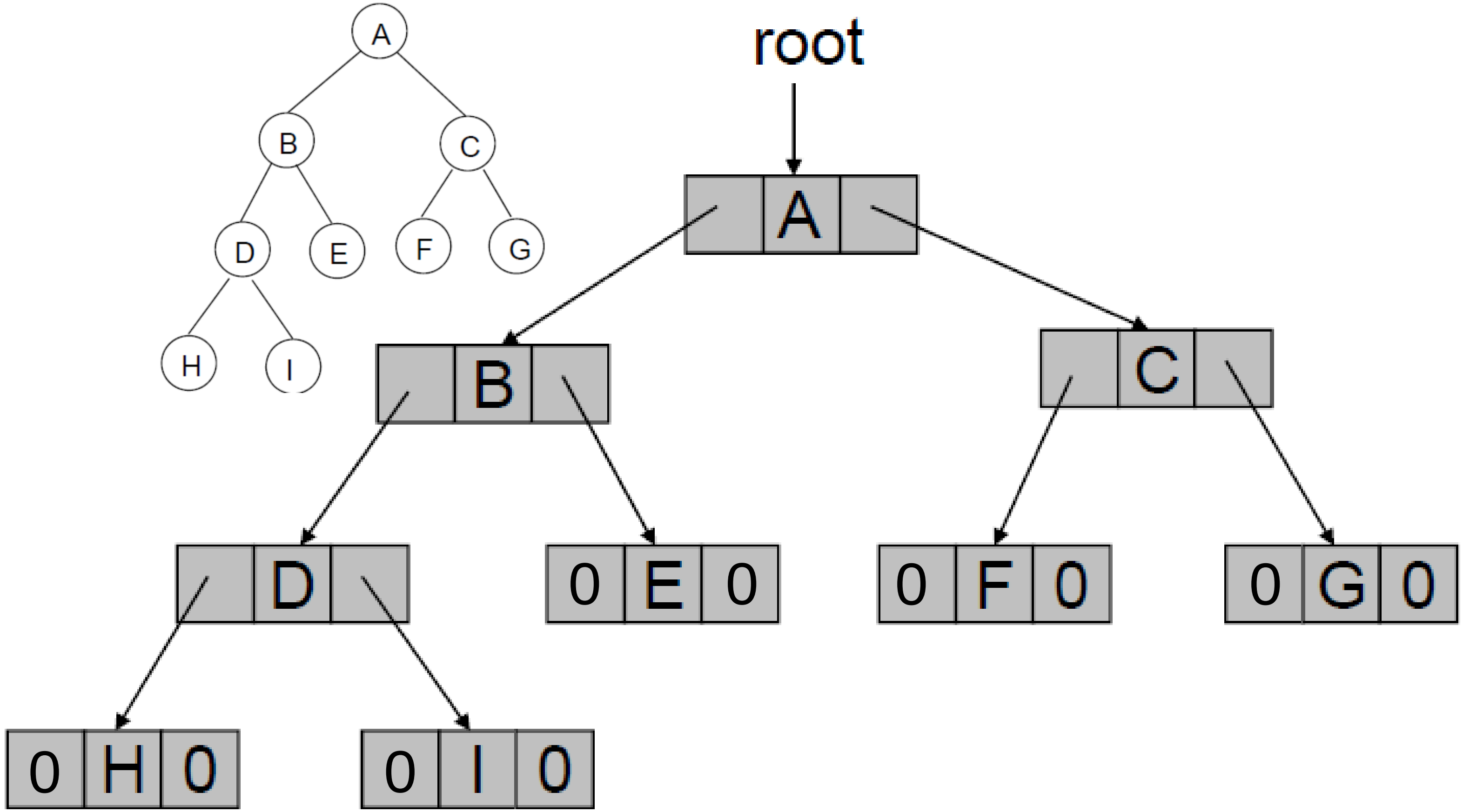
Linked Representation

```
class Tree;
class TreeNode {
friend class Tree;
private:
  char data;
  TreeNode *LeftChild;
  TreeNode *RightChild;
};
class Tree {
public:
// Tree operations
private:
  TreeNode *root;
};
```

Linked List Representation For The Binary Trees



For the tree in
Figure 5.10(a)



For the tree in Figure 5.10(b)

Compare Two Binary Tree Representations

	Array representation	Linked representation
Determination the locations of the parent, left child and right child	Easy	Difficult
Space overhead	Much	Little
Insertion and deletion	Difficult	Easy

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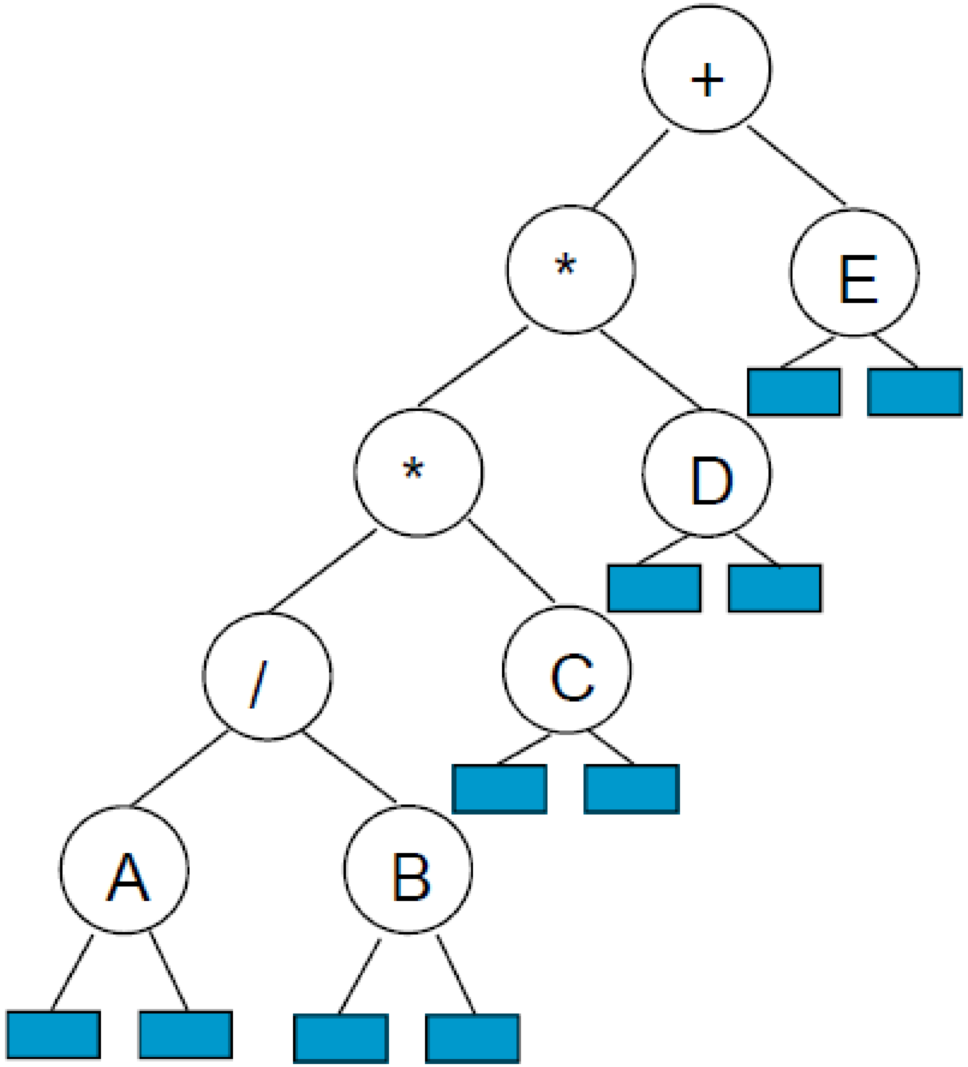
Binary Tree Traversals

- Let L, V, and R stand for moving left, visiting the node, and moving right.
- There are six possible combinations of traversal
  - LVR, LRV, VLR, VRL, RVL, RLV
- Adopt convention that we traverse left before right, only 3 traversals remain
  - LVR, VLR, LRV
  - inorder, preorder, postorder

Arithmetic Expression Using BT



inorder traversal

A / B * C * D + E

preorder traversal

+ * * / A B C D E

postorder traversal

A B / C * D * E +

level order traversal

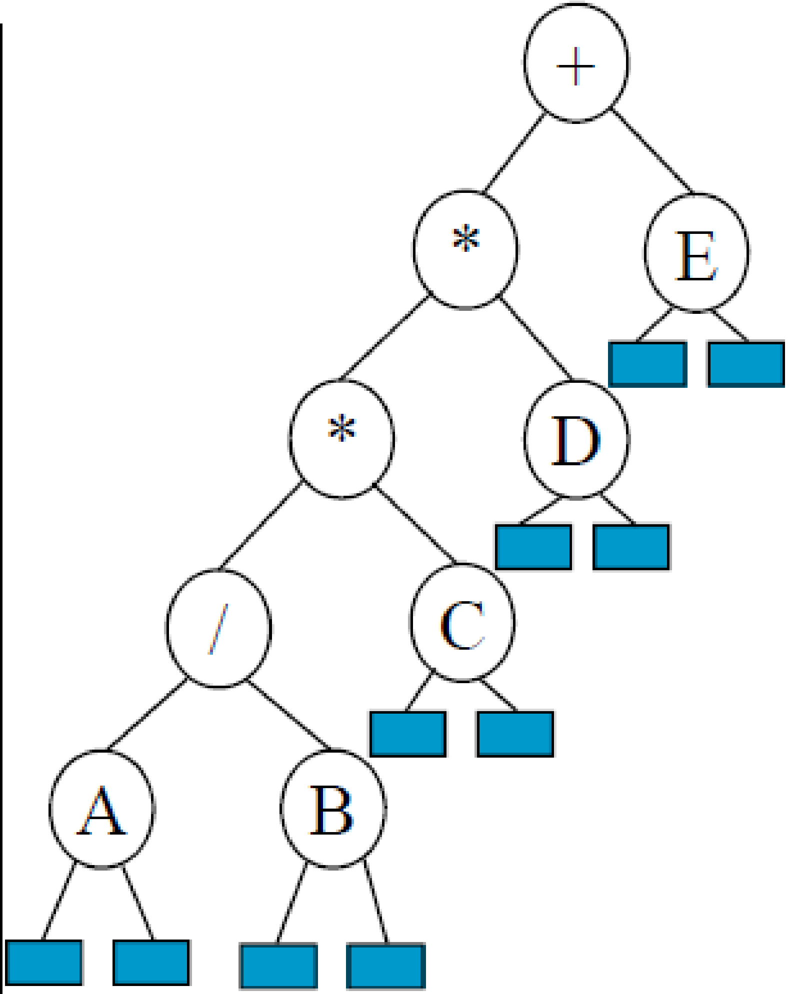
+ * E * D / C A B

Program 5.1

```
void Tree::inorder()
// Driver calls workhorse for traversal of entire tree. The driver is declared as a public member function of Tree.
{
  inorder(root);
}
void Tree::inorder(TreeNode *CurrentNode)
/* Workhorse traverses the subtree rooted at CurrentNode  (which is a pointer to a node in a binary tree). The workhorse is declared as a private member function of Tree. */
{
  if(CurrentNode){
      inorder(CurrentNode -> LeftChild);
      cout << CurrentNode -> data;
      inorder(CurrentNode-> RightChild);
      }
}
```

Trace Operations of Inorder Traversal

Call of inorder	Value in root	Action	Call of inorder	Value in root	Action
1	+		11	C	
2	*		12	NULL	
3	*		11	C	cout
4	/		13	NULL	
5	A		2	*	cout
6	NULL		14	D	
5	A	cout	15	NULL	
7	NULL		14	D	cout
4	/	cout	16	NULL	
8	B		1	+	cout
9	NULL		17	E	
8	B	cout	18	NULL	
10	NULL		17	E	cout
3	*	cout	19	NULL	



Program 5.2

```
void Tree::preorder(){
  // Driver calls workhorse for traversal of entire tree. The
  // driver is declared as a public member function of Tree.
  preorder(root);
}
void Tree::preorder(TreeNode *CurrentNode)
  // Workhorse traverses the subtree rooted at CurrentNode (which is a pointer to a node
  // in a binary tree). The workhorse is declared as a private member function of Tree.
{
if(CurrentNode){
   cout << CurrentNode -> data;
   preorder(CurrentNode -> LeftChild);
   preorder(CurrentNode-> RightChild);
  }
}
```

Program 5.3

```
void Tree::postorder
// Driver calls workhorse for traversal of entire tree. The driver is declared as a public member function of Tree.
{
    postorder(root);
}
void Tree::postorder(TreeNode *CurrentNode)
  // Workhorse traverses the subtree rooted at CurrentNode (which is a pointer to a node in a binary tree).
  // The workhorse is declared as a private member function of Tree.
{
   if(CurrentNode){
       postorder(CurrentNode -> LeftChild);
       postorder(CurrentNode-> RightChild);
       cout << CurrentNode -> data;
   }
}
```

Program 5.4 Iterative Inorder Traversal

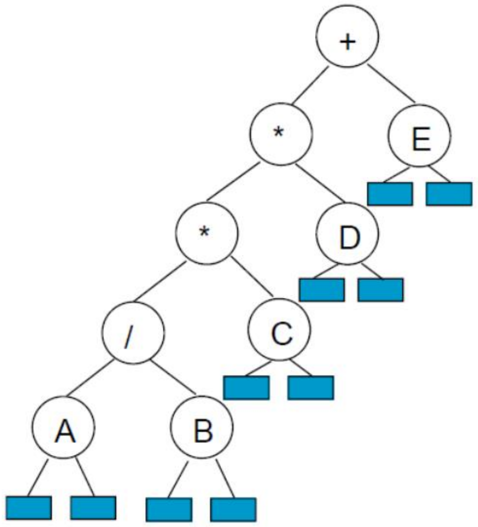
```
void Tree::NonrecInorder()
// nonrecursive inorder traversal using a stack
{
  Stack<TreeNode *> s; // declare and initialize stack
  TreeNode *CurrentNode = root;
  while (1) {
        while (CurrentNode) { // move down LeftChild fields
        s.Add(CurrentNode); // add to stack
        CurrentNode = CurrentNode->LeftChild;
      }
        if (!s.IsEmpty()) {                // stack is not empty
        CurrentNode = *s.Delete(CurrentNode);
        cout << CurrentNode->data << endl;
        CurrentNode = CurrentNode->RightChild;
      }
      else break;
  }
}
```

Level-Order Traversal

- All previous mentioned schemes use stacks
- Level-order traversal uses a queue
- Level-order scheme visit the root first, then the root's left child, followed by the root's right child
- All the nodes at a level are visited before moving down to another level

Level-Order Traversal of A Binary Tree

```
void Tree::LevelOrder()
// Traverse the binary tree in level order
{
  Queue<TreeNode *> q;
  TreeNode*CurrentNode = root;
  while (CurrentNode) {
     cout << CurrentNode->data<<endl;
     if (CurrentNode->LeftChild) q.Add(CurrentNode->LeftChild);
     if (CurrentNode->RightChild) q.Add(CurrentNode->RightChild);
     CurrentNode = *q.Delete();
  }
}
```



+ * E * D / C A B

+*E*D/CAB

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Some Other Binary Tree Functions

- With the inorder, postorder, or preorder mechanisms, we can implement all needed binary tree functions. e.g.,
  - Copying Binary Trees
  - Testing Equality
    - Two binary trees are equal if their topologies are the same and the information in corresponding nodes is identical.

Program 5.9 Copying a binary tree

```
//Copy constructor
Tree::Tree(const Tree& s) //driver
{
  root = copy(s.root);
}
TreeNode* Tree::copy(TreeNode *orignode)
//Workhorse
//This function returns a pointer to an exact copy of the binary tree rooted at orignode.
{
  if(orignode) {
     TreeNode *temp = new TreeNode;
     temp->data = orignode->data;
     temp->LeftChild = copy(orignode->LeftChild);
     temp->RightChild = copy(orignode->RightChild);
     return temp;
  }
  else return 0;
}
```

Program 5.10 Binary tree equivalence

```
//Driver-assumed to be a friend of class Tree.
int operator = = (const Tree& s, const Tree& t)
{
   return equal(s.root, t.root);
//Workhorse-assumed to be a friend of TreeNode.
int equal(TreeNode *a, TreeNode *b)
//This function returns 0 if the subtrees at a and b are not equivalent. Otherwise, it will return 1.
{
   if((!a)&&(!b)) return 1; //both a and b are 0
   if(a && b   // both a and b are non-0
                 && (a->data == b->data) //data is the same
          && equal(a->LeftChild, b->LeftChild)
         //left subtrees are the same
         && equal(a->RightChild, b->RightChild))
         //right subtrees are the same
                 return 1;
   return 0;
}
```

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Threaded Binary Trees

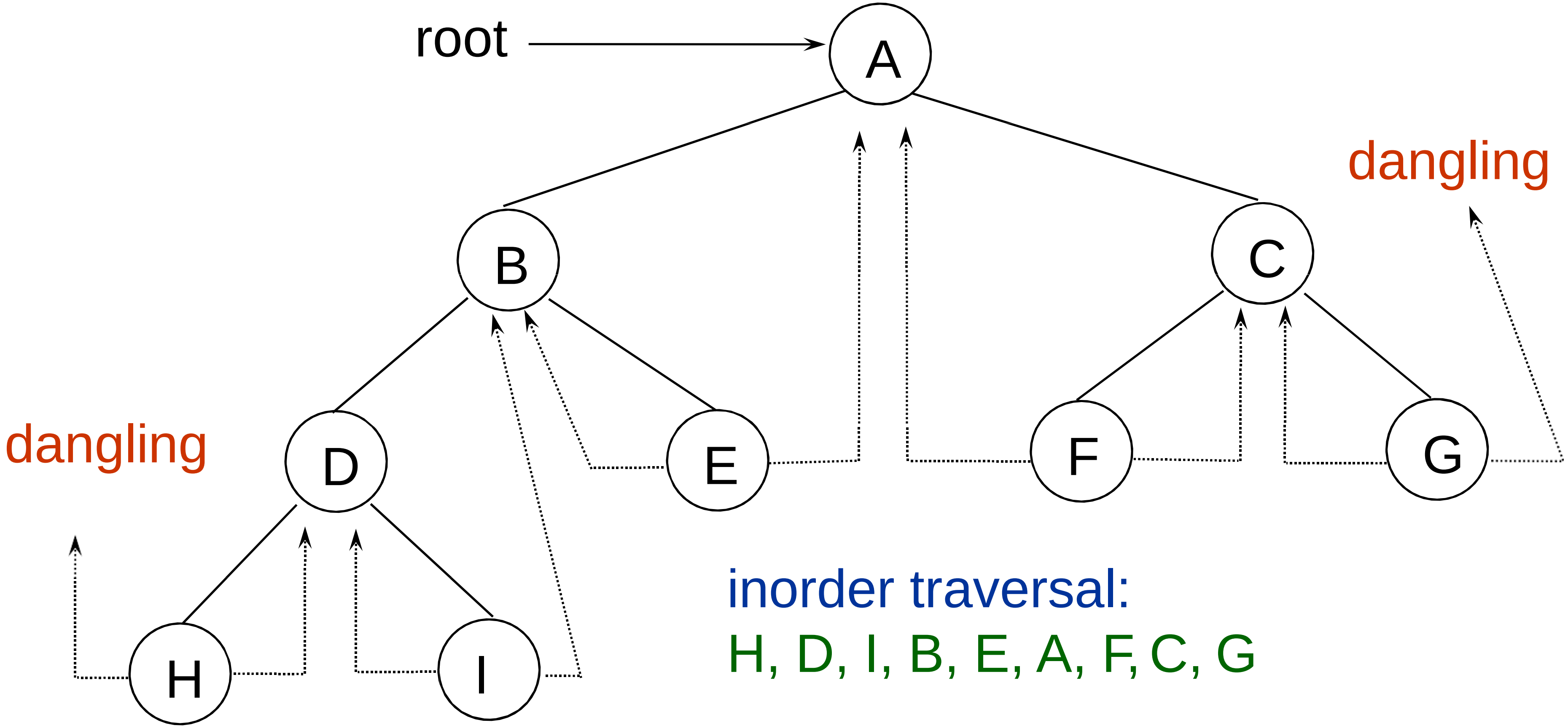
- Too many null pointers in current representation of binary trees
  - n: number of nodes
  - number of non-null links: n-1
  - total links: 2n
  - null links: 2n-(n-1)=n+1
- Replace these null pointers with some useful “threads”.

Threaded Binary Trees (Cont'd)

- If ptr->left_child is null,
  - replace it with a pointer to the node that would be visited *before* ptr in an *inorder traversal* (inorder predecessor)

- If ptr->right_child is null,
  - replace it with a pointer to the node that would be visited *after* ptr in an *inorder traversal* (inorder successor)

A Threaded Binary Tree



inorder traversal:
H, D, I, B, E, A, F, C, G

Threads

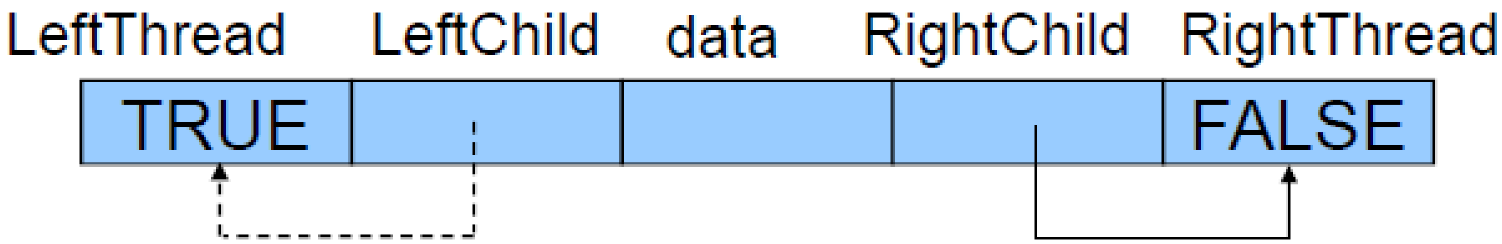
- To distinguish between normal pointers and threads, two boolean fields, LeftThread and RightThread, are added to the record in memory representation.
  - t->LeftThread = TRUE

    => t->LeftChild is a **thread**
  - t->LeftThread = FALSE

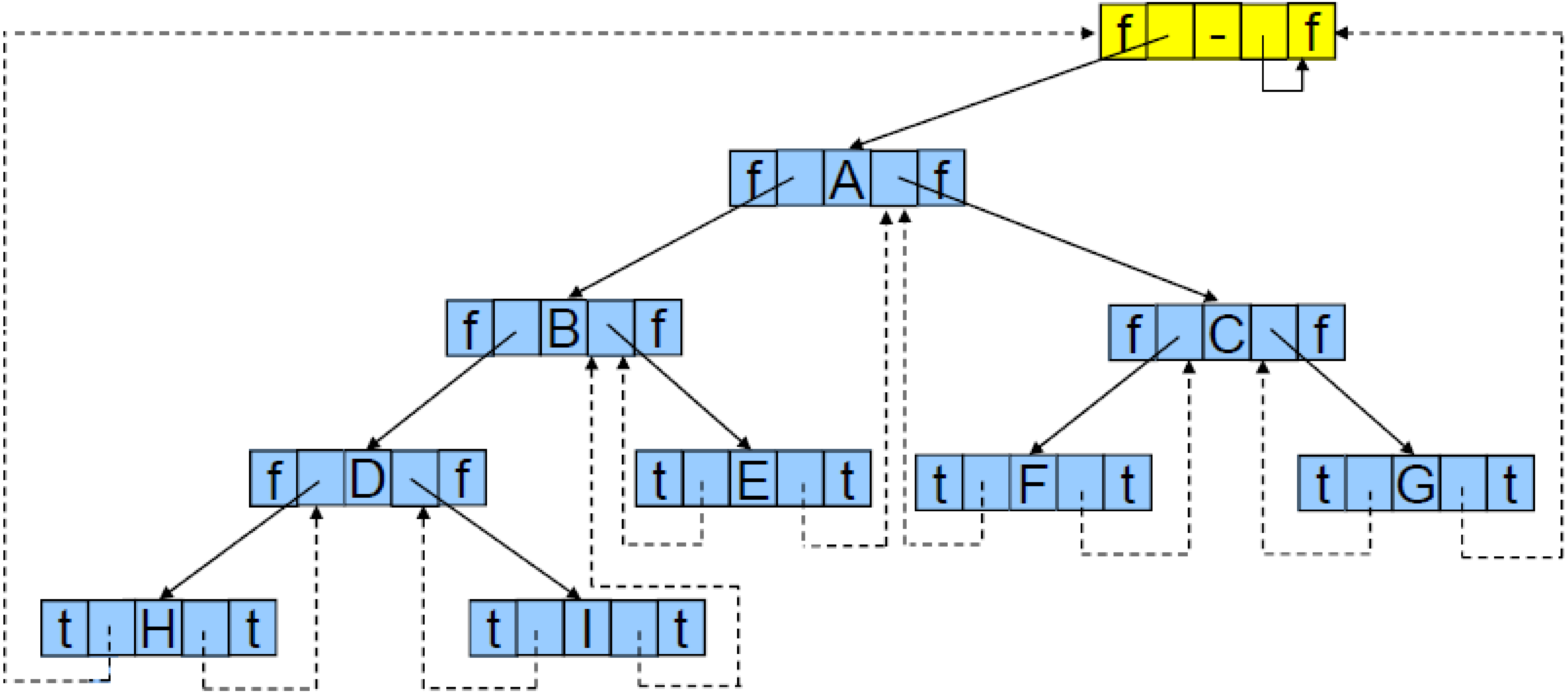
    => t->LeftChild is a **pointer** to the left child.

Threads (Cont'd)

- To avoid dangling threads, a head node is used in representing a binary tree.
- The original tree becomes the left subtree of the head node.
- Empty Binary Tree



Memory Representation of Threaded Tree of Figure 5.20



Program 5.14

```
char* ThreadedInorderIterator::Next()
// Find the inorder successor of CurrentNode in a threaded binary tree
{
    ThreadedNode *temp = CurrentNode -> RightChild;
    if(!CurrentNode -> RightThread)
        while(!temp->LeftThread)
            temp = temp->LeftChild;
    CurrentNode = temp;
    if(CurrentNode == t.root) return 0;
    else return &CurrentNode ->data;
}
```

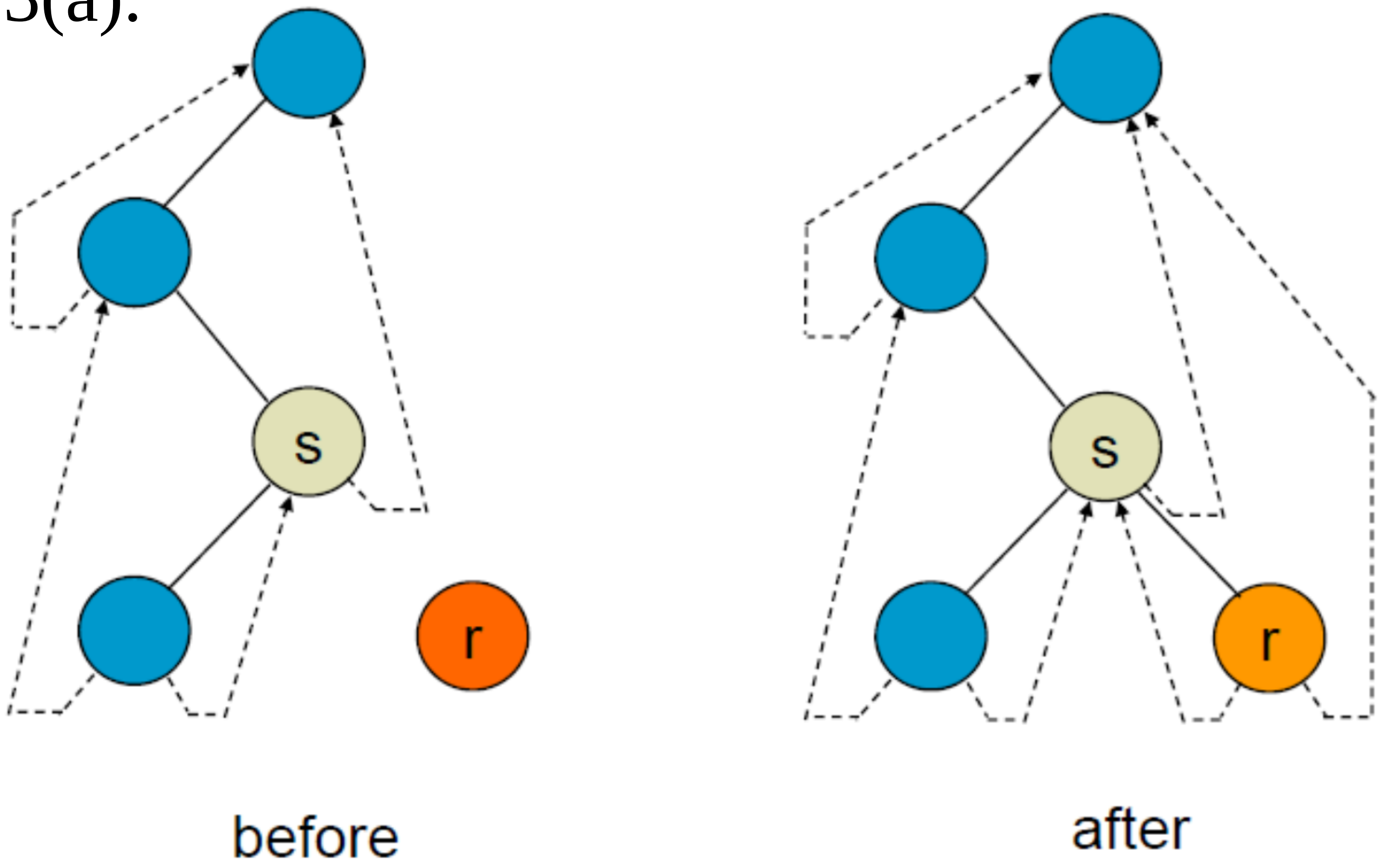
Inorder traversal can be performed without stack

Inserting a Node to a Threaded Binary Tree

- Inserting a node **r** as the right child of a node **s**.
  - If s has an empty right subtree, then the insertion is simple and diagram in Figure 5.23(a).
  - If the right subtree of s is not empty, then this right subtree is made the right subtree of r after insertion. When this is done, r becomes the inorder predecessor of a node that has a LdeftThread==TRUE field, and consequently there is an thread which has to be updated to point to r. The node containing this thread was previously the inorder successor of s. Figure 5.23(b) illustrates the insertion for this case.

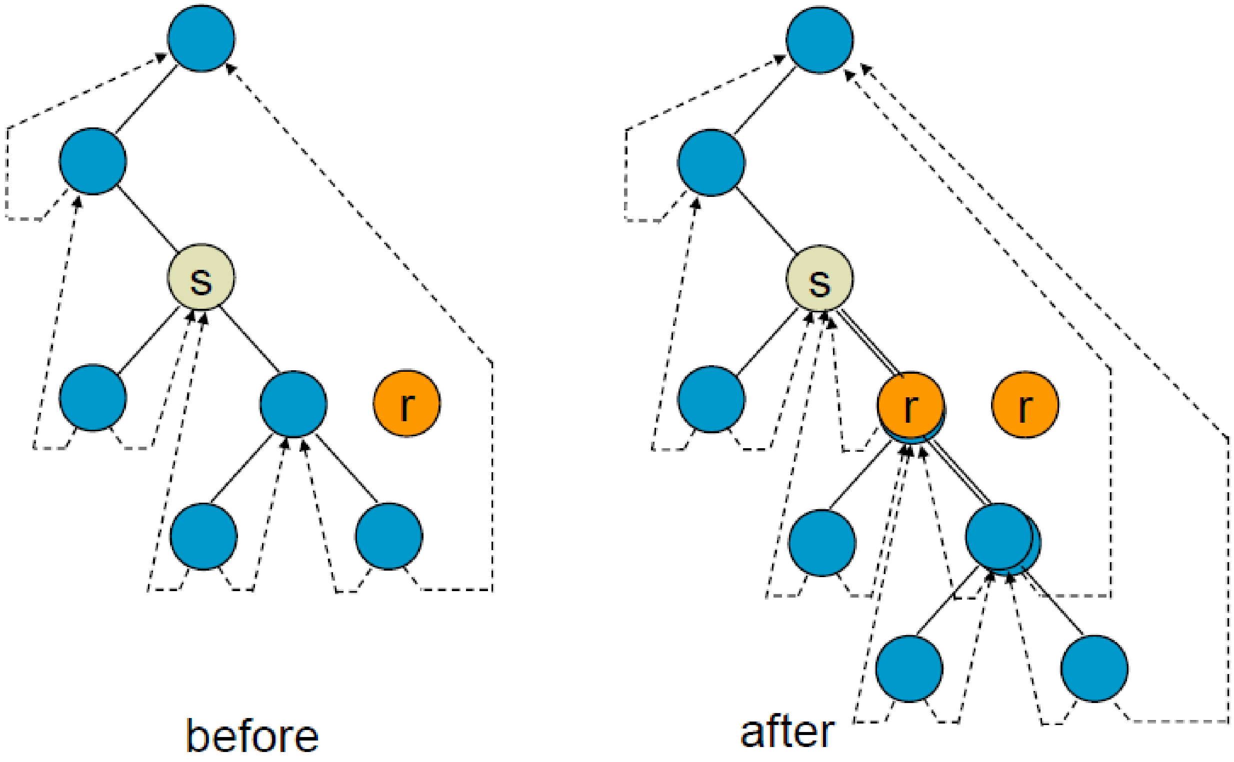
Inserting a Node to a Threaded Binary Tree (Cont’d)

- Inserting a node **r** as the right child of a node **s**.
  - If s has an empty right subtree, then the insertion is simple and diagram in Figure 5.23(a).



Inserting a Node to a Threaded Binary Tree (Cont'd)

- Inserting a node **r** as the right child of a node **s**.
  - If the right subtree of s is not empty, then this right subtree is made the right subtree of r after insertion. When this is done, r becomes the inorder predecessor of a node that has a LeftThread==TRUE field, and consequently there is an thread which has to be updated to point to r. The node containing this thread was previously the inorder successor of s. Figure 5.23(b) illustrates the insertion for this case.



Program 5.16 Inserting r As the Right Child of s

```
void ThreadedTree::InsertRight(ThreadNode *s, ThreadedNode *r)
// Insert r as the right child of s
{
   r->RightChild = s->RightChild;
   r->RightThread = s->RightThread;
   r->LeftChild = s;
   r->LeftThread = TRUE;       // LeftChild is a thread
   s->RightChild = r;    // attach r to s
   s->RightThread = FALSE;
   if (!r->RightThread) {
        ThreadedNode *temp = InorderSucc(r); // returns the inorder  successor of r
        temp->LeftChild = r;
   }
}
```

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Priority Queues

- In a priority queue, the element to be deleted is the one with highest (or lowest) priority.
- An element with arbitrary priority can be inserted into the queue according to its priority.
- A data structure supports the above two operations is called max (min) priority queue.

Application: priority queue

- machine service
    - amount of time (min heap)
    - amount of payment (max heap)

Data Structures

- Unordered linked list
- Unordered array
- Sorted linked list
- Sorted array
- Heap

Priority Queue Representations

Representation	Insertion	Deletion
Unordered array	$\Theta(1)$	$\Theta(n)$
Unordered linked list	$\Theta(1)$	$\Theta(n)$
Sorted array	$O(n)$	$\Theta(1)$
Sorted list	$O(n)$	$\Theta(1)$
Max heap	$O(\log_2 n)$	$O(\log_2 n)$

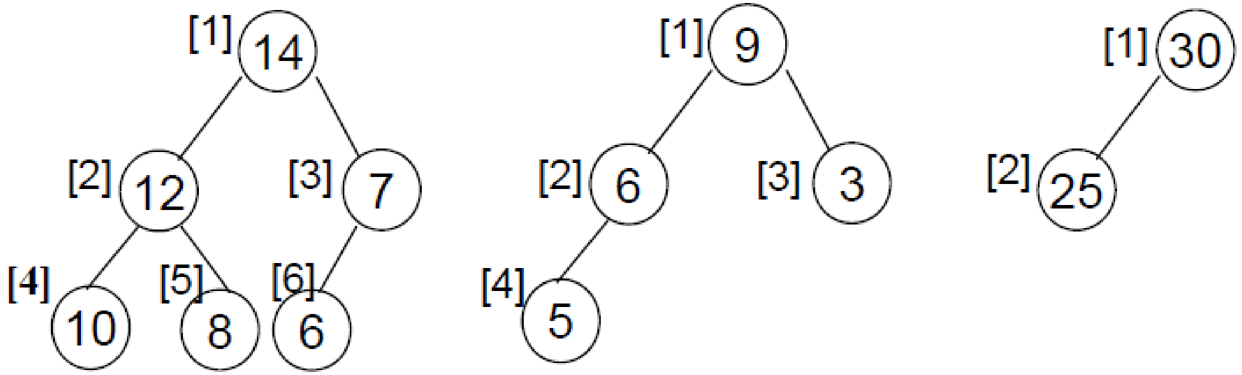
Max (Min) Heap

- Heaps are frequently used to implement priority queues. The complexity is **O(log n)**.
- Definition:
  - A max (min) tree is a tree in which the key value in each node is no smaller (larger) than the key values in its children (if any).
  - A max heap is a complete binary tree that is also a max tree.
  - A min heap is a complete binary tree that is also a min tree.

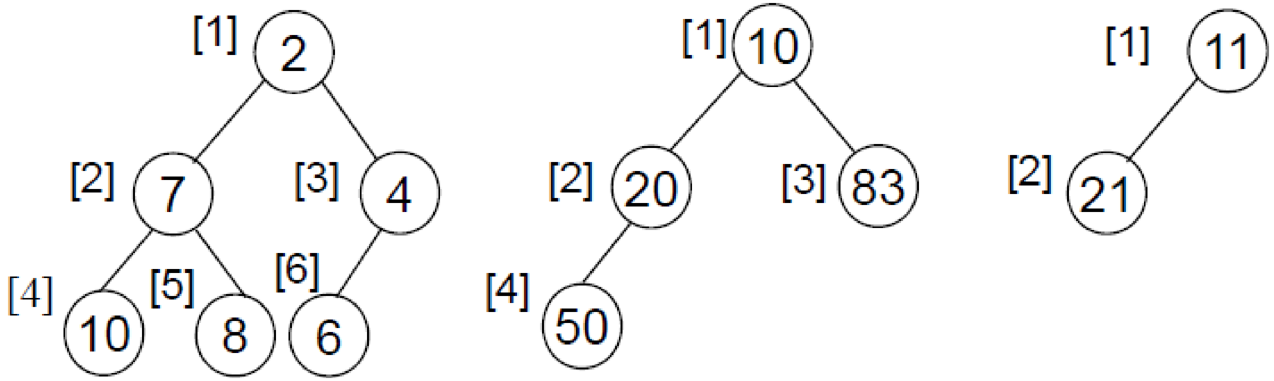
Max Heap Examples

- Property:
  - The root of max heap (min heap) contains the largest (smallest).

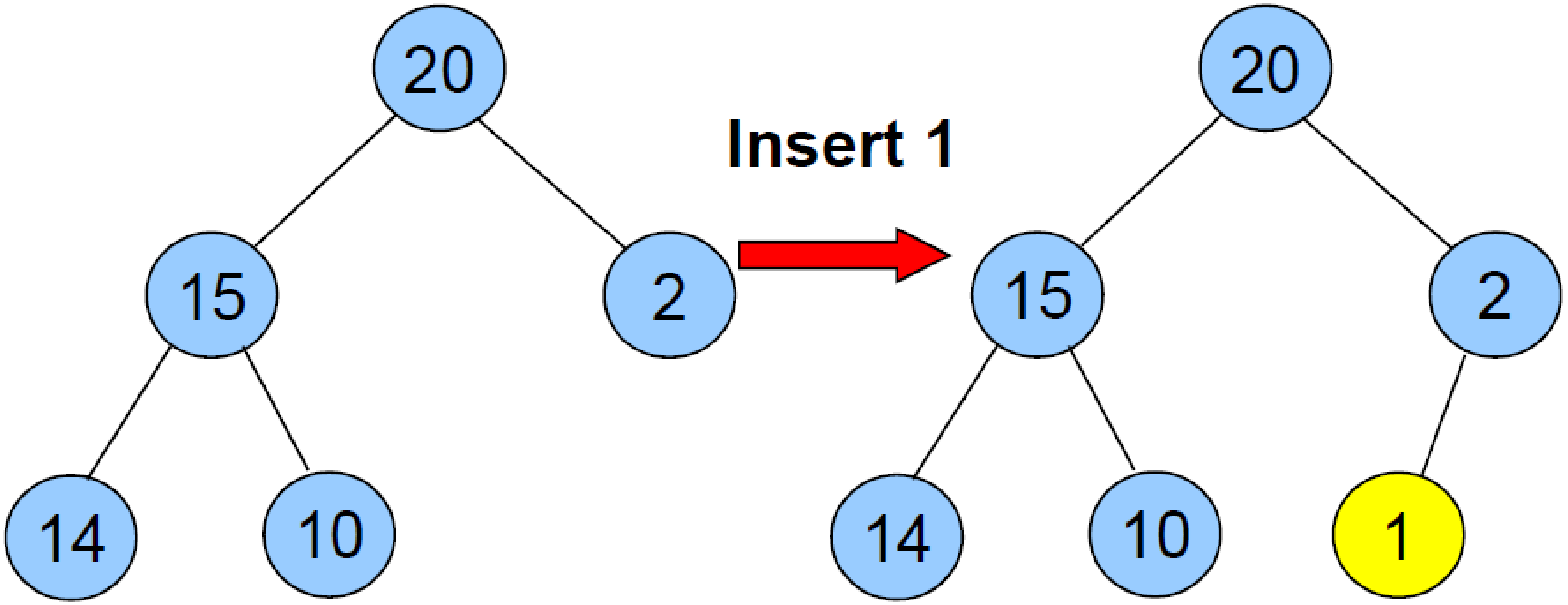
max heap



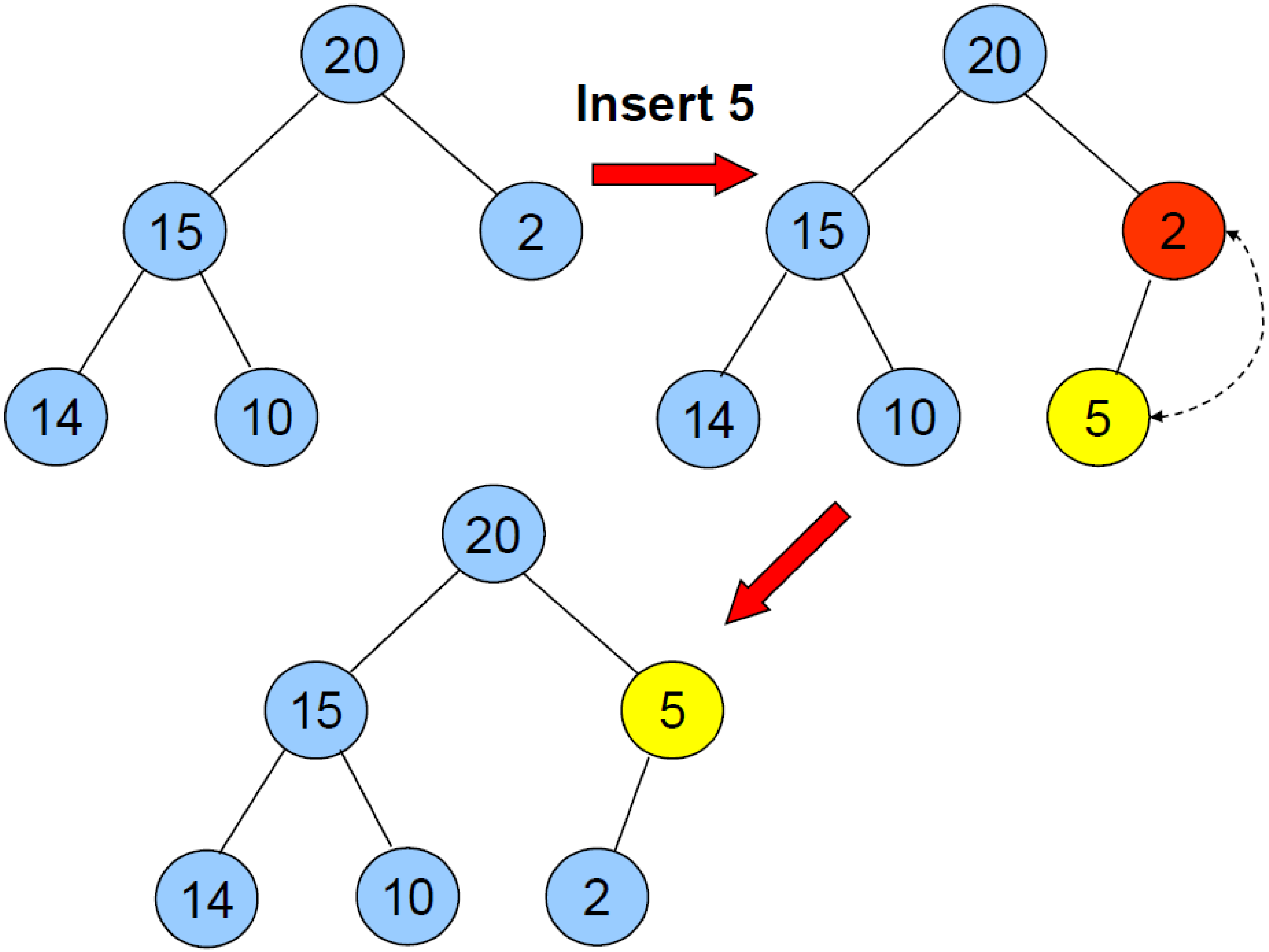
min heap



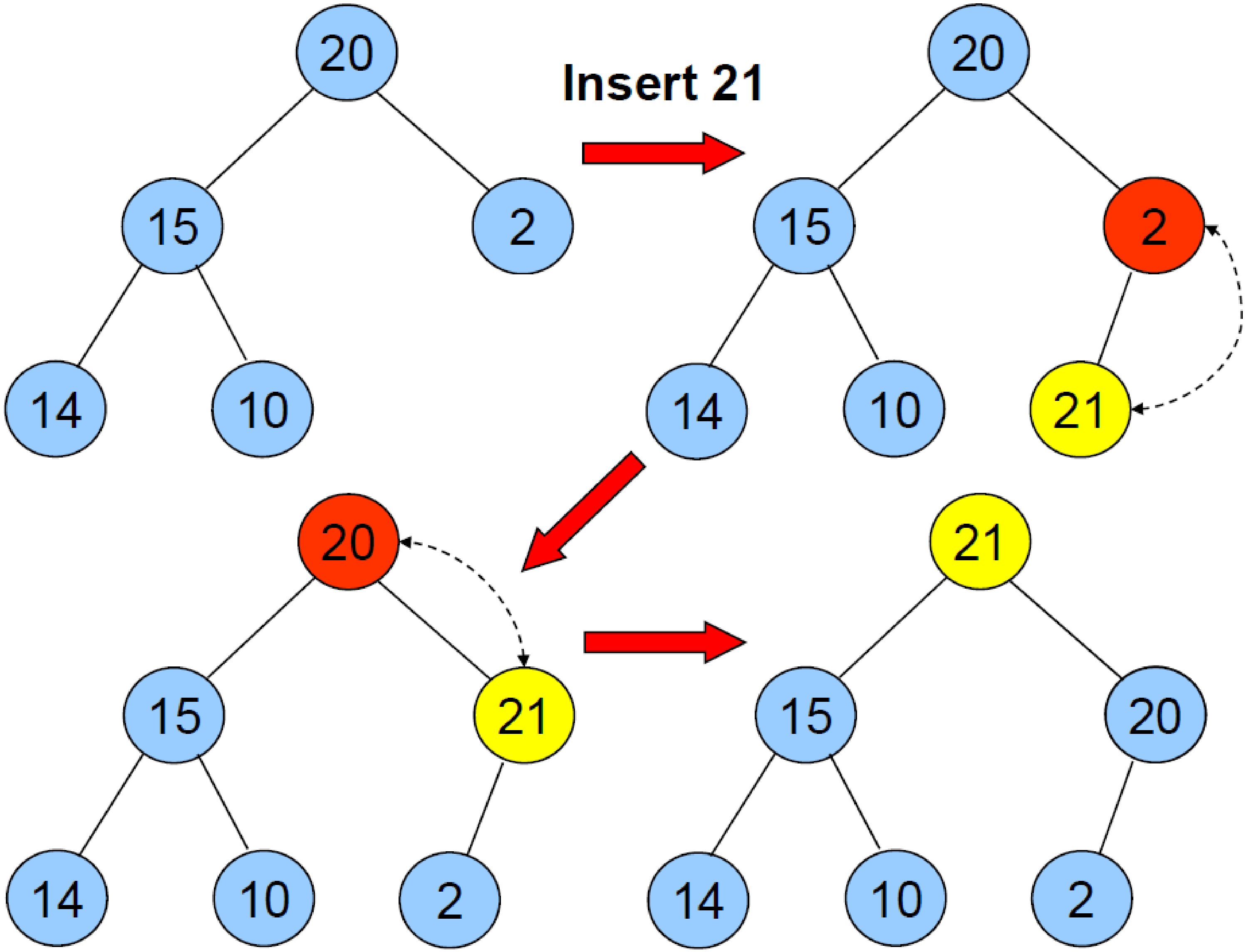
Insertion Into a Max Heap



Insertion Into a Max Heap (Cont’d)



Insertion Into a Max Heap (Cont'd)



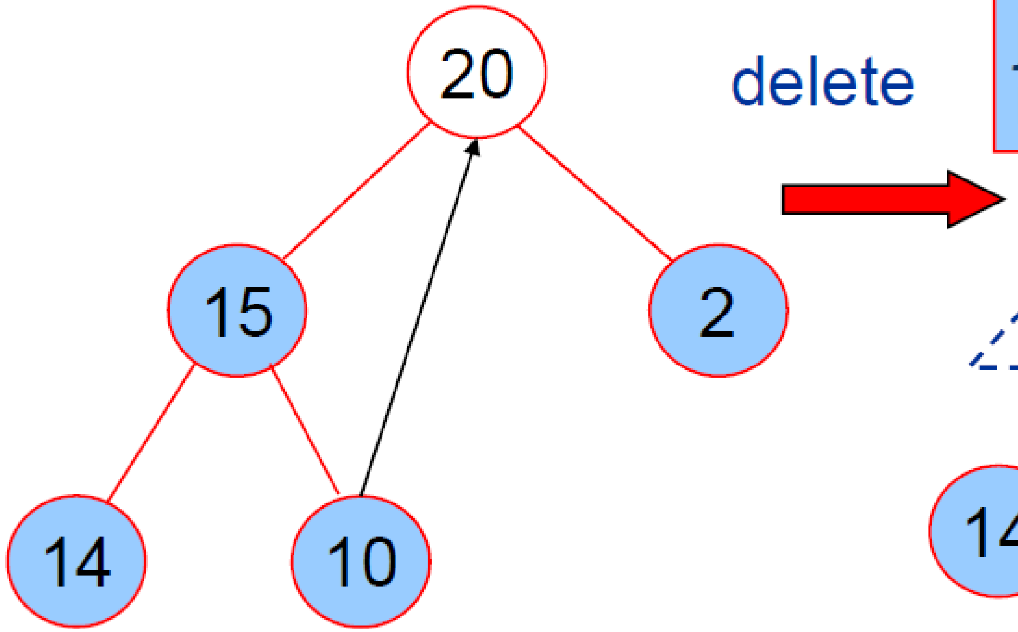
Program 5.16 Insertion into a max heap

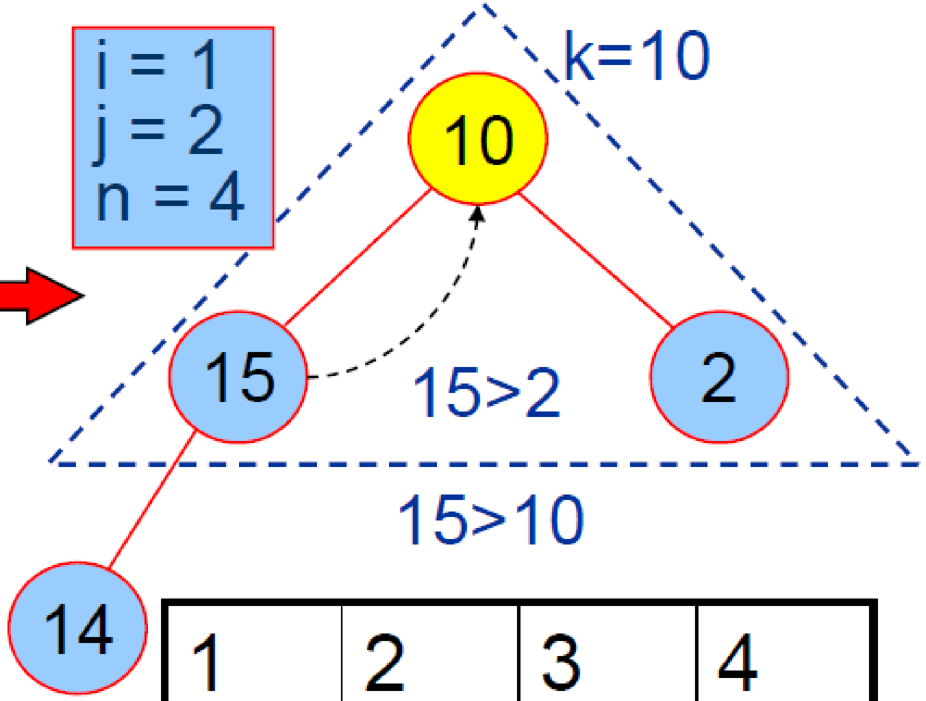
```
template <class Type>
void MaxHeap<Type>::Insert(const Element <Type> &x)
// insert x into the max heap
{
  if(n == MaxSize) {HeapFull(); return;}
  n++;
  for(int i = n;1;){
     if(i == 1) break; // at root
     if(x.key <= heap[i/2].key) break;
     // move from parent to i
     heap[i] = heap[i/2];
     i/=2;
  }
  heap[i] = x;
}
```

Deletion from a Max Heap

1	2	3	4	5
20	15	2	14	10

1	2	3	4
20	15	2	14



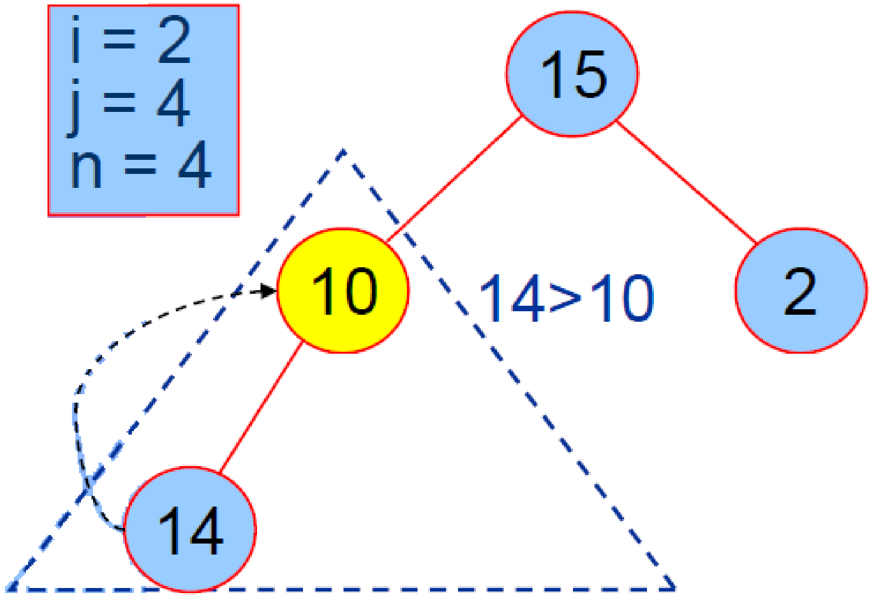


1	2	3	4
15	15	2	14

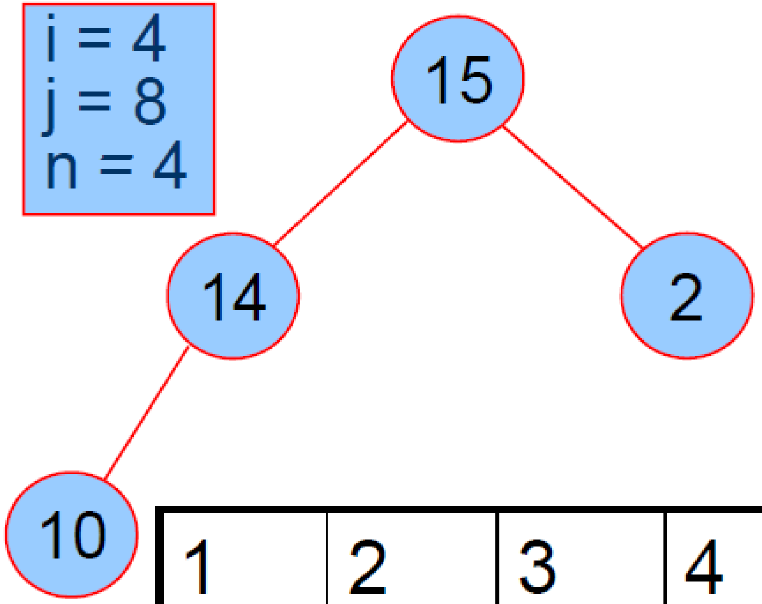
Deletion from a Max Heap (Cont'd)

1	2	3	4
15	15	2	14

1	2	3	4
15	14	2	14



1	2	3	4
15	14	2	14



1	2	3	4
15	14	2	10

Program 5.17 Deletion from a max heap

```
template <class Type>
Element <Type>* MaxHeap <Type>::DeleteMax(Element<Type>& x)
// Delete from the max heap
{
    if (!n) {HeapEmpty(); return 0;}
    x = heap[1];
      Element <Type> k = heap[n];
      n--;
    for (int i = 1, j =2; j < =n;) {
       if (j < n) {
            if (heap[j].key < heap[j+1].key)
                j++;
          }
         // j points to the larger child
        if (k.key >= heap[j].key) break;
        heap[i] = heap[j];
        i = j; j *= 2;
    }
    heap[i] = k;
    return &x;
}
```

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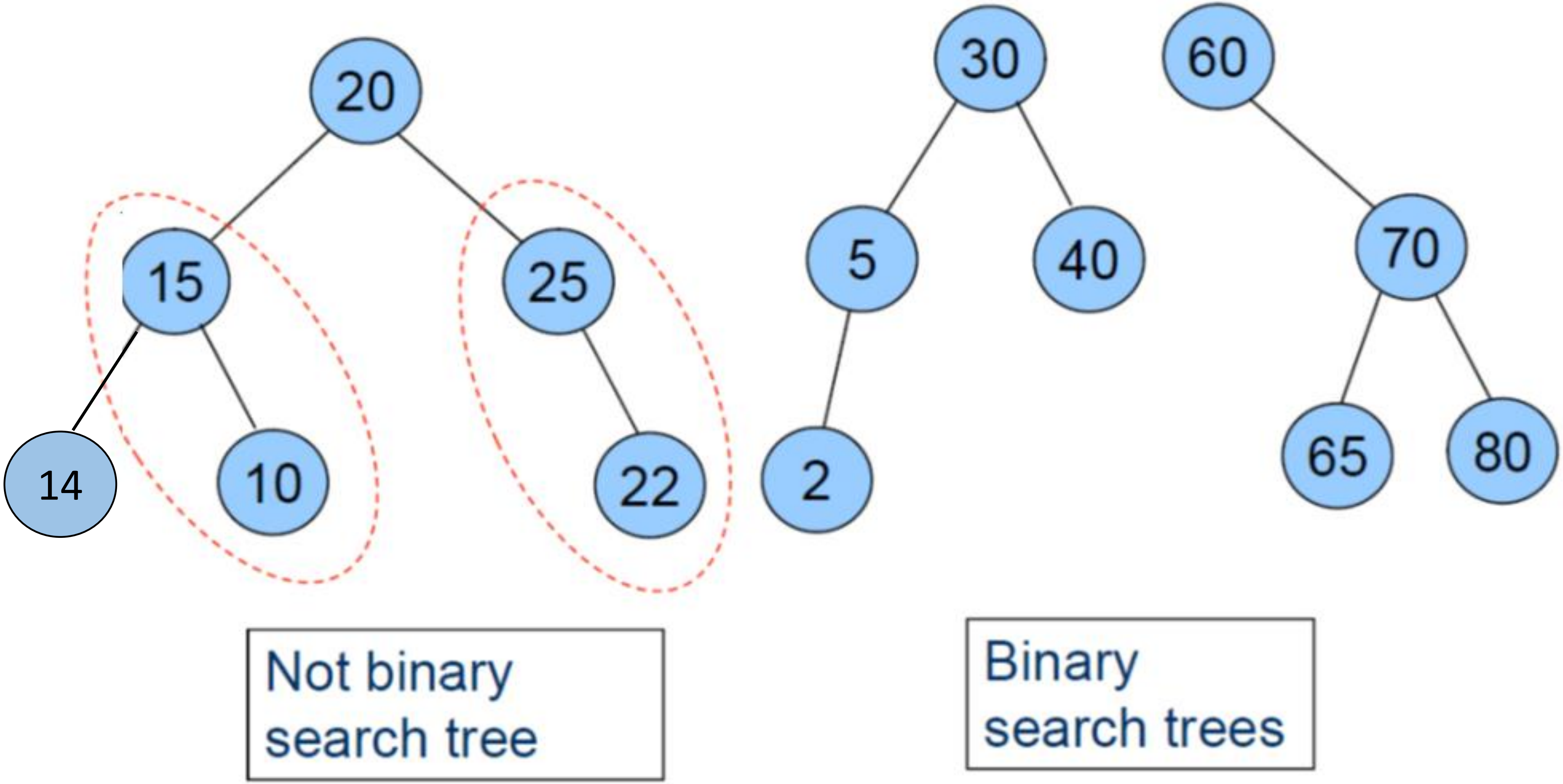
Binary Search Tree (BST)

- Heap
  - a min (max) element is deleted $O(\log_2 n)$
  - deletion of an arbitrary element $O(n)$
  - search for an arbitrary element $O(n)$
- Binary search tree
  - Every element has a unique key.
  - The keys in a nonempty left subtree (right subtree) are smaller (larger) than the key in the root of subtree.
  - The left and right subtrees are also binary search trees.

Binary Trees



Searching a Binary Search Tree

- If the root is null, then this is an empty tree. No search is needed.
- If the root is not null, compare the x with the key of root.
  - If x is equal to the key of the root, then it's done.
  - If x is less than the key of the root, then no elements in the right subtree have key value x. We only need to search the left subtree.
  - If x is larger than the key of the root, only the right subtree is to be searched.

Program 5.18 Recursive search of a BST

```
template <class Type> //Driver
BstNode <Type>* BST <Type>::Search(const Element<Type>& x)
/* Search the binary search tree (*this) for an element with key x. If such an element
is found, return a pointer to the node that contains it. */
{
 return Search(root, x);
}
template <class Type> //Workhorse
BstNode <Type>* BST
 <Type>::Search(BstNode<Type>*b,  const Element <Type>&x)
{
 if(!b) return 0;
        if(x.key == b->data.key) return b;
 if(x.key < b->data.key) return Search(b->LeftChild, x);
 return Search(b->RightChild, x);
}
```

Recursive version

Program 5.19 Iterative search of a BST

```
template <class Type>
BstNode <Type>*BST<Type>::IterSearch(const  Element<Type>& x)
/* Search the binary search tree for an element  with key x */
{
 for(BstNode<Type> *t = root; t;)
 {
        if(x.key == t->data.key) return t;
        if(x.key < t->data.key) t = t->LeftChild;
        else t = t->RightChild;
 }
 return 0;
}
```

Iterative version

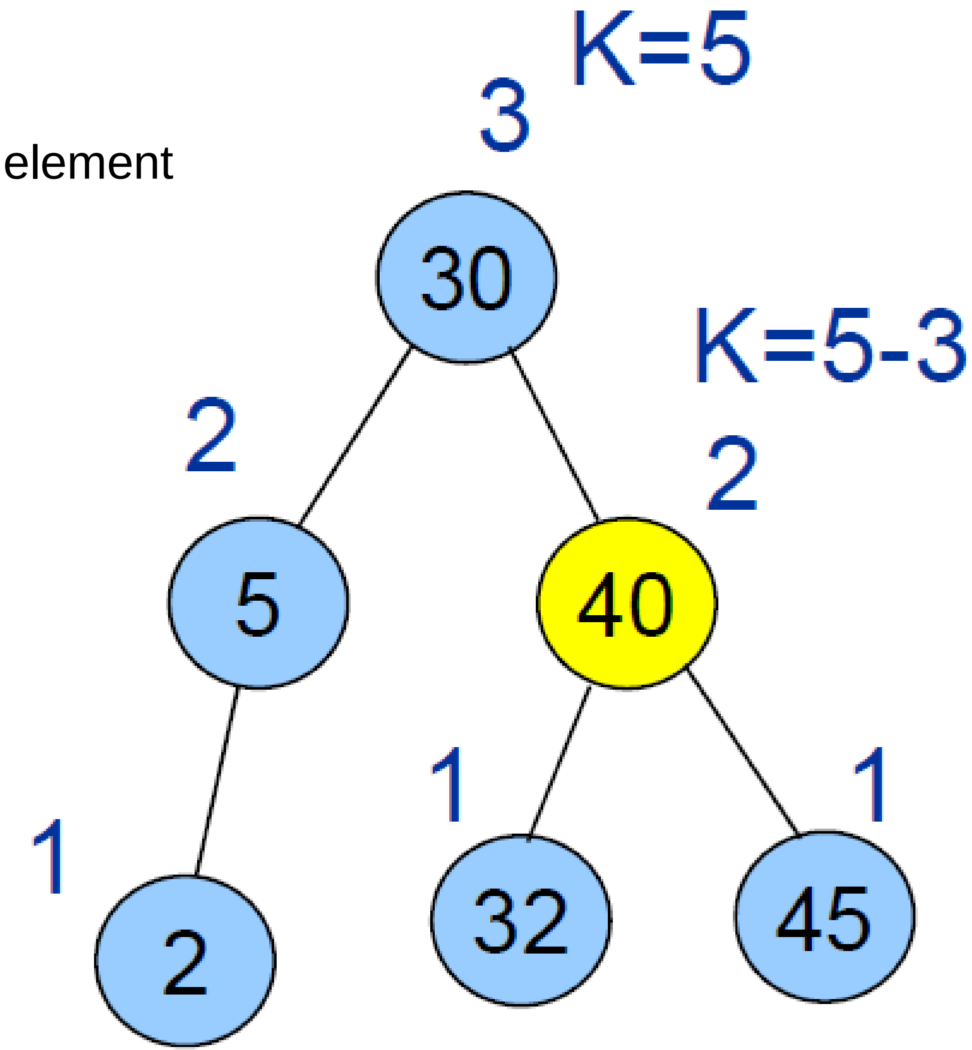
Search Binary Search Tree by Rank

- To search a binary search tree by the ranks of the elements in the tree, we need additional field *LeftSize*.
- LeftSize is **the number of the elements** in the left subtree of a node plus one.
- It is obvious that a binary search tree of height h can be searched by key as well as by rank in O(h) time.

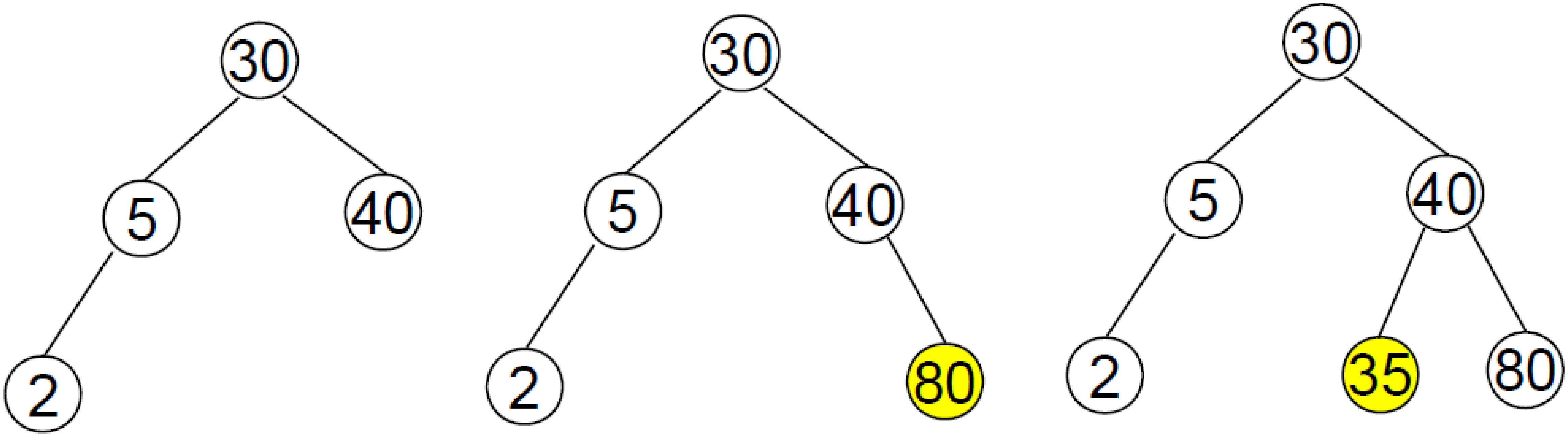
Search Binary Search Tree by Rank

```
template <class Type>

BstNode <Type>* BST<Type>::Search(int k)
// Search the binary search tree for the kth smallest element
{
   BstNode<Type> *t = root;
   while(t)
   {
           if (k == t->LeftSize) return t;
           if (k < t->LeftSize) t = t->LeftChild;
           else {
             k -=t-> LeftSize;
            t = t->RightChild;
          }
     }
     return 0;
}
```



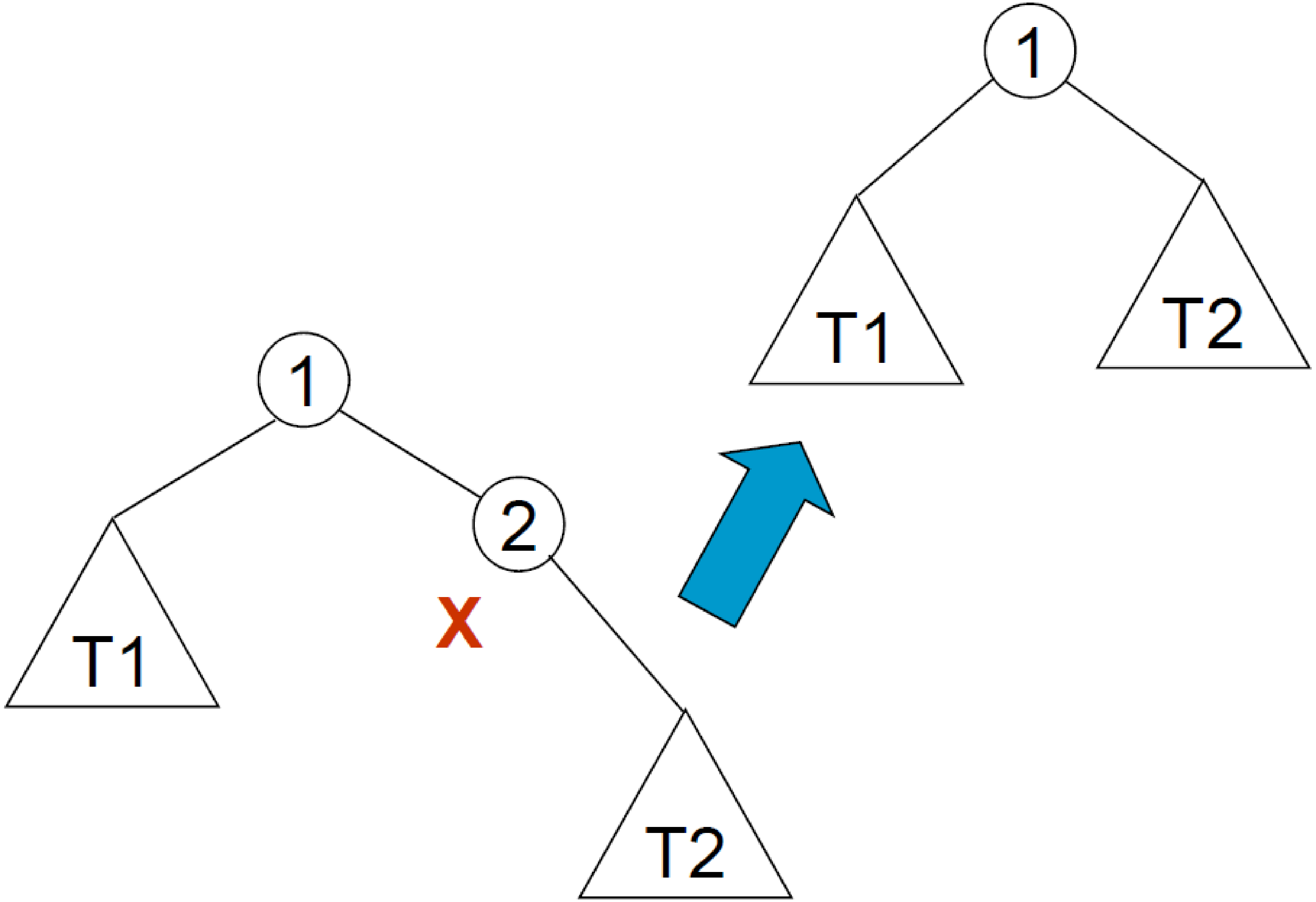
Inserting a Node into a BST



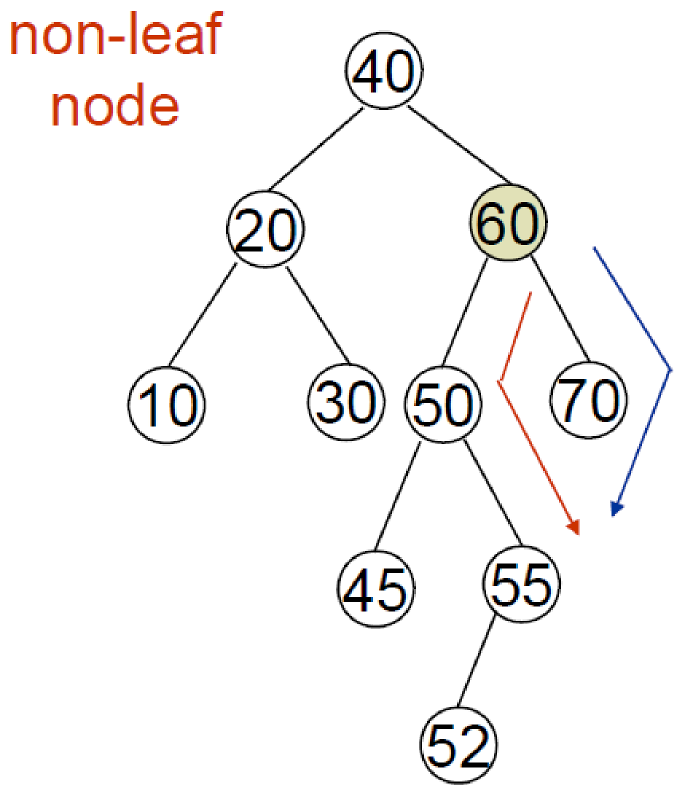
Insert 80

Insert 35

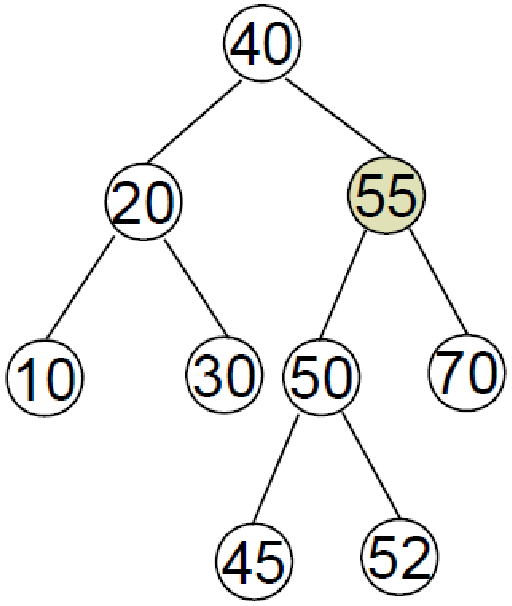
Deletion from a BST



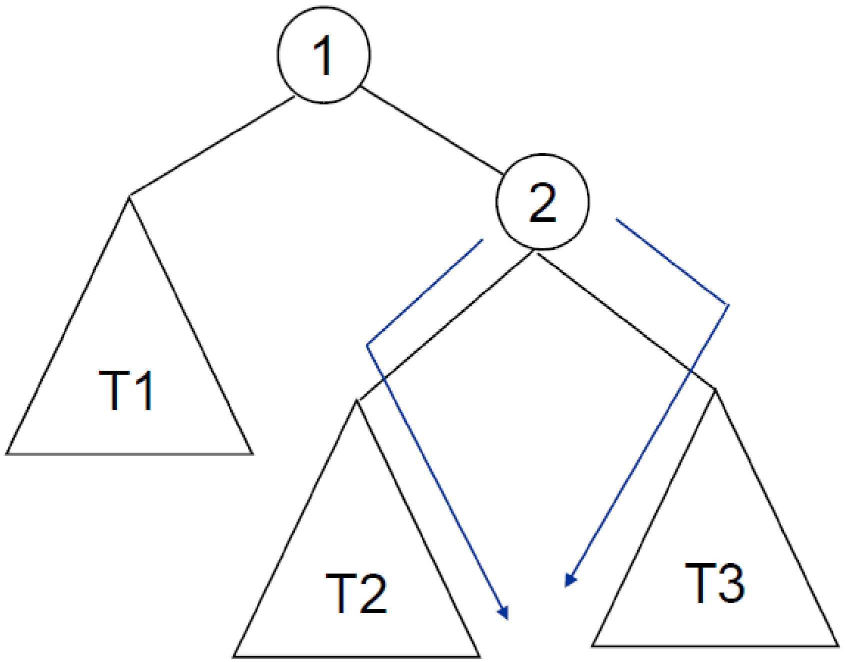
Deletion from a BST (Cont'd)



Before deleting 60



After deleting 60



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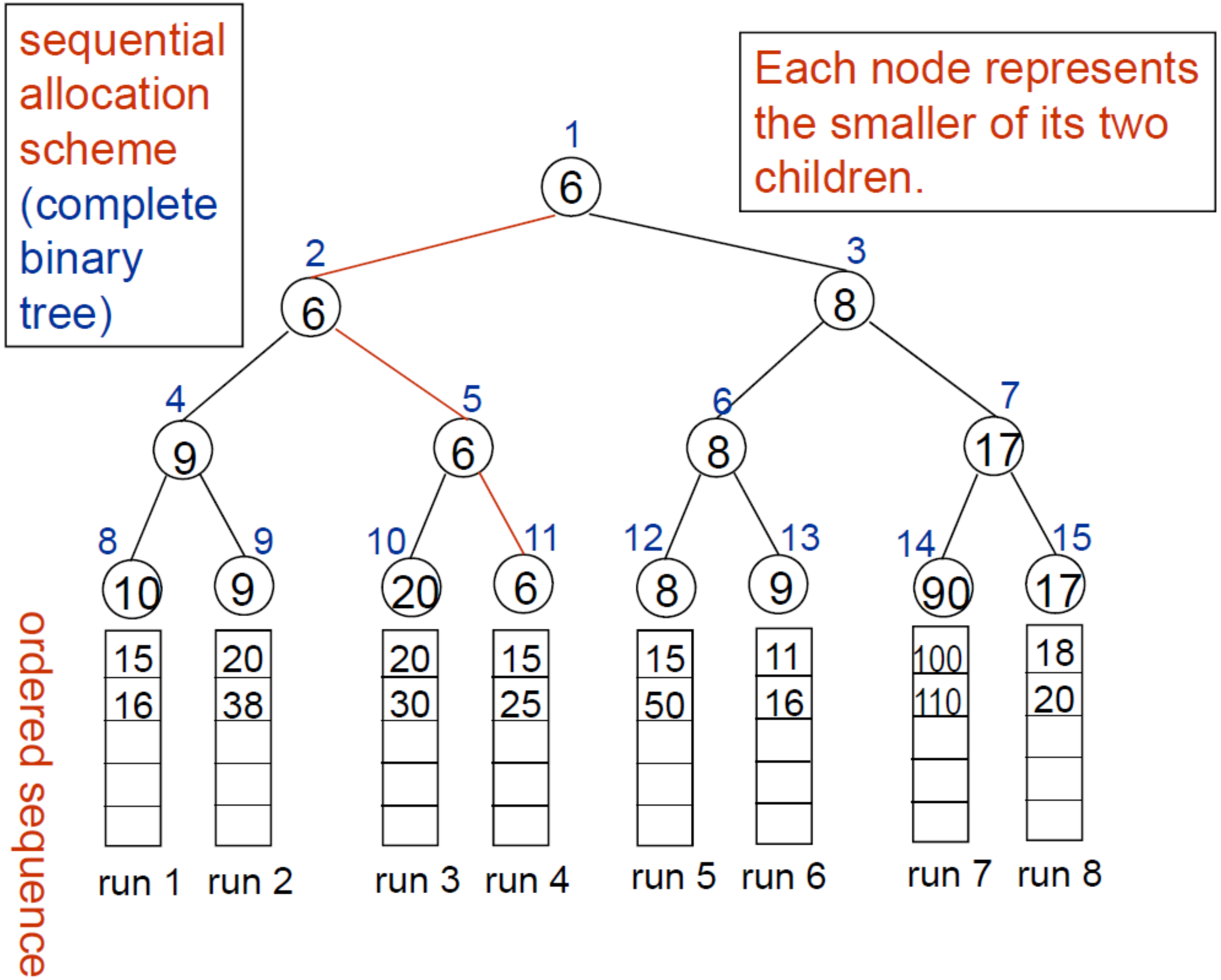
Selection Trees

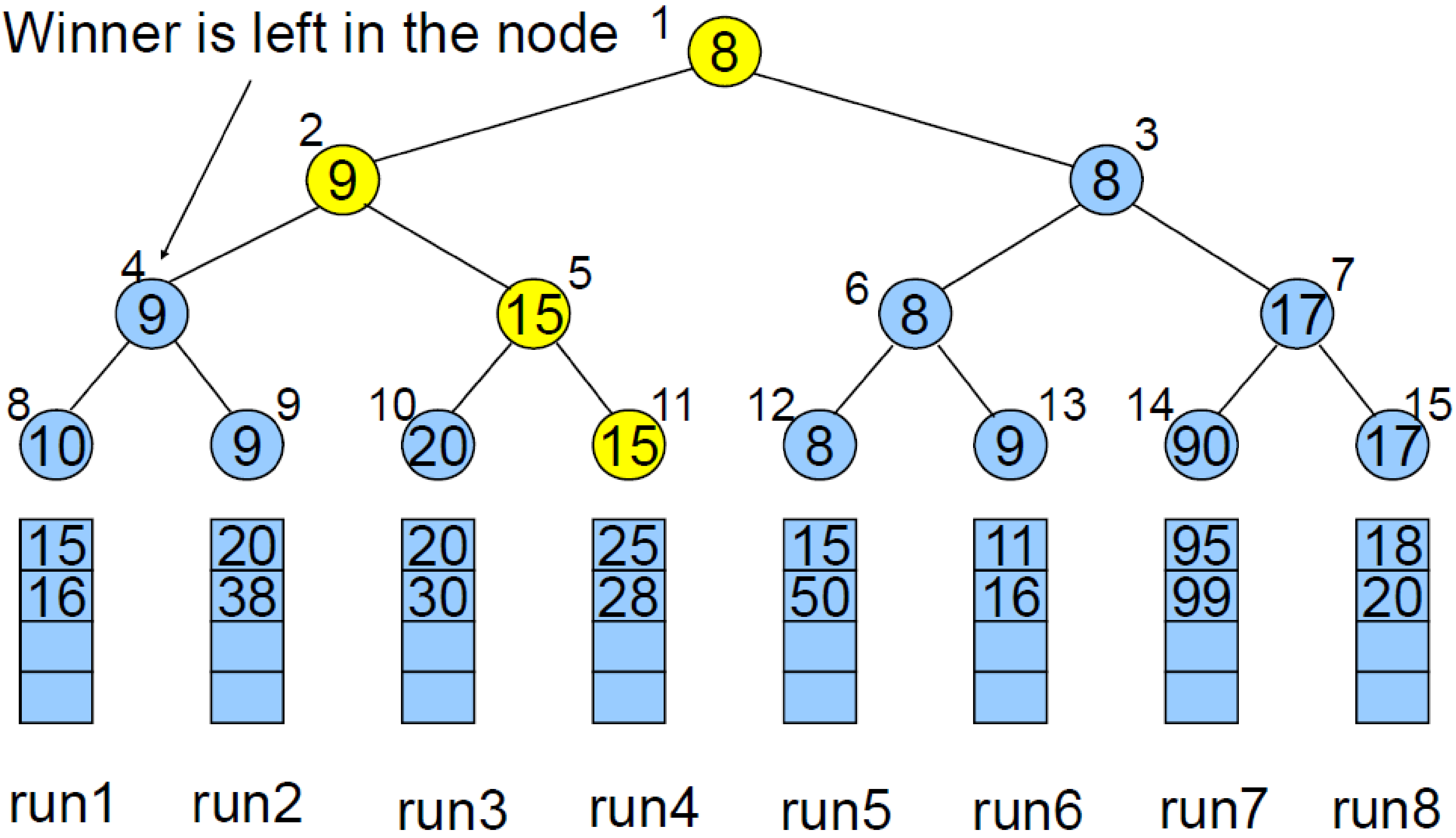
- Winner tree
- Loser tree

Winner Tree



sequential allocation scheme (complete binary tree)
Each node represents the smaller of its two children.
1
6
2
6
3
8
4
9
5
6
6
8
7
17
8
10
9
9
10
20
11
6
12
8
13
9
14
90
15
17
15
16
20
38
20
30
15
25
15
50
11
16
100
110
18
20
run 1
run 2
run 3
run 4
run 5
run 6
run 7
run 8
ordered sequence

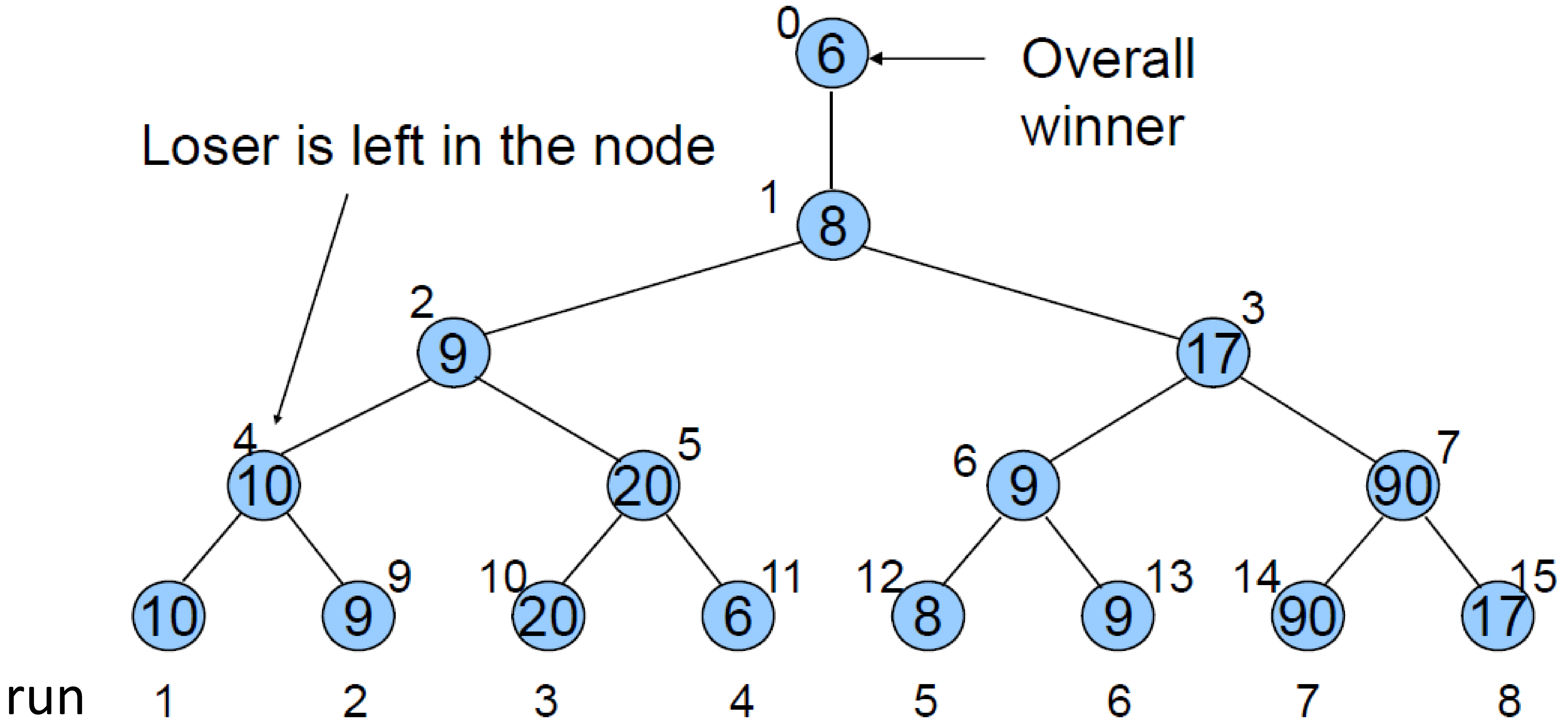
Winner Tree for $k = 8$



Analysis

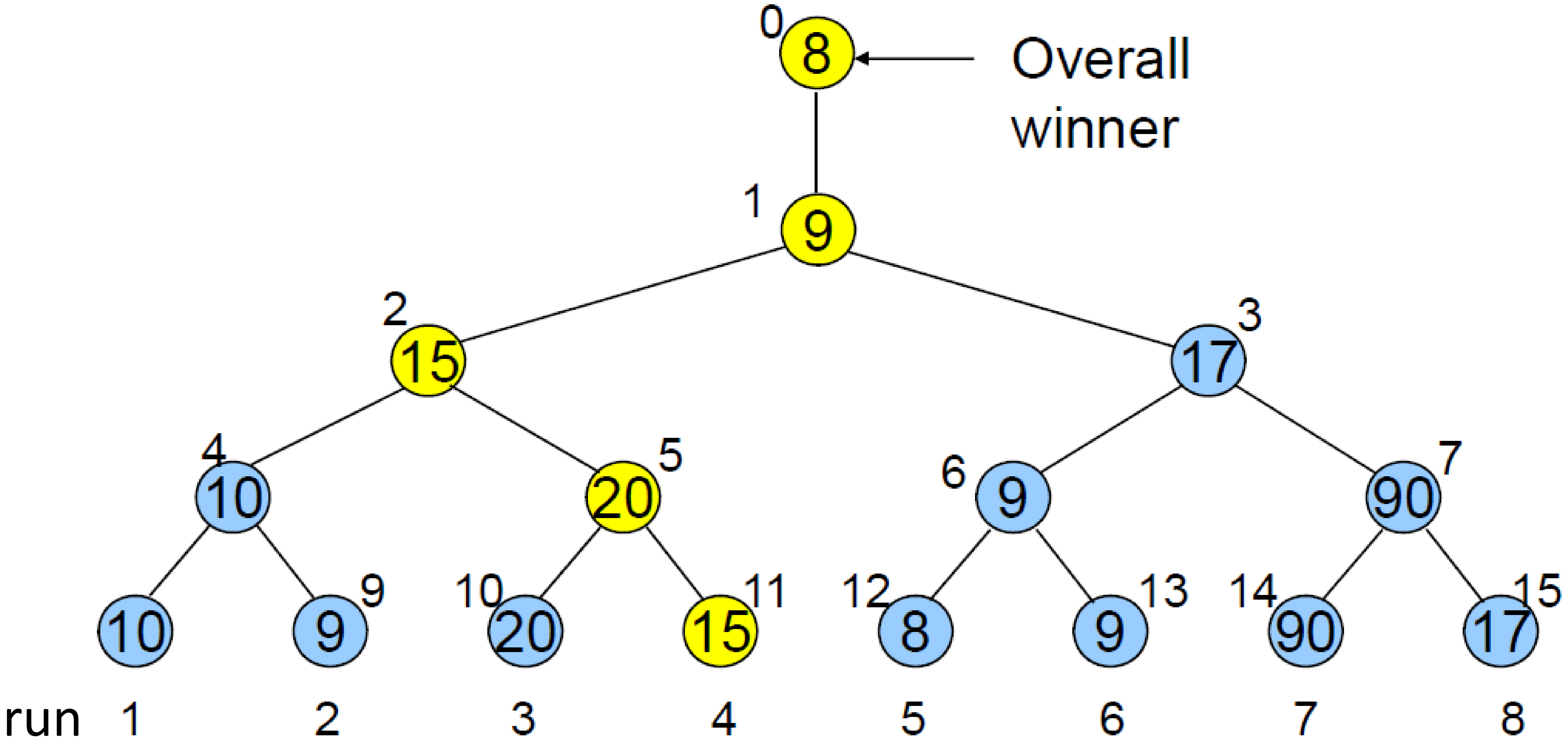
- K: # of runs
- n: # of records
- setup time: O(K) (K-1)
- restructure time: O(log2K) $\lceil \log_2(K+1) \rceil$
- merge time: O(nlog2K)
- **slight** modification: tree of loser
  - consider the parent node only (vs. sibling nodes)

Loser Tree



Overall winner
Loser is left in the node
0
6
1
8
2
9
3
17
4
10
5
20
6
9
7
90
10
9
9
20
10
6
11
8
12
9
13
90
14
17
15
run
1
2
3
4
5
6
7
8

Loser Tree (Cont'd)



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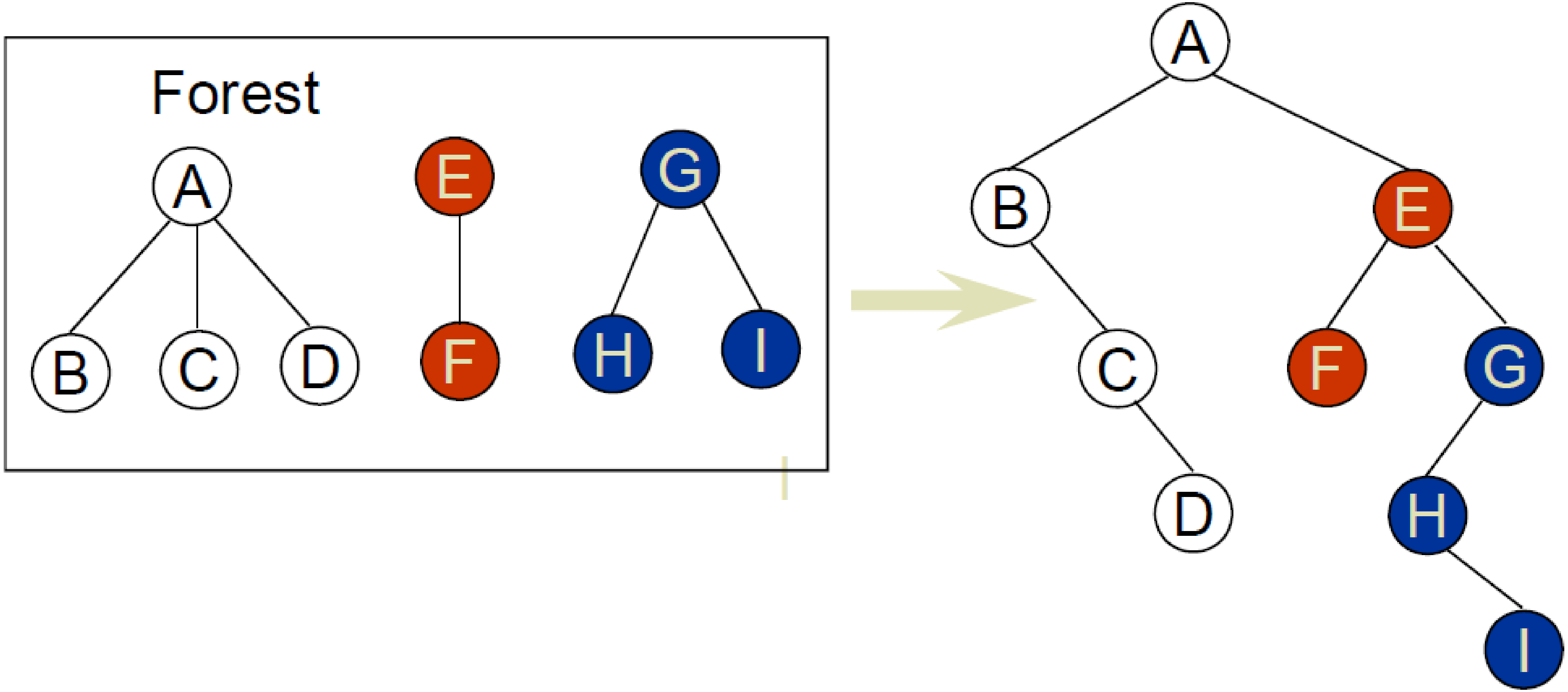
Forest

- A forest is a set of n>=0 disjoint trees

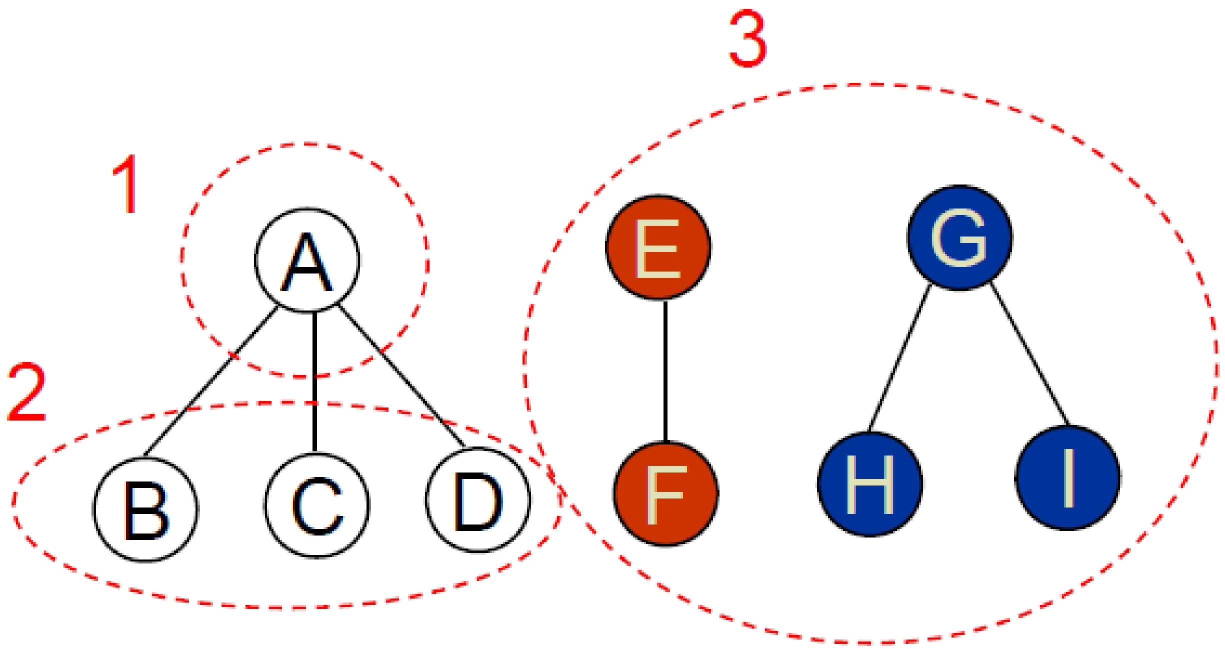


Transform a Forest into a Binary Tree

- T1, T2, …, Tn: a forest of trees
- B(T1, T2, …, Tn): a binary tree corresponding to this forest
- Algorithm
  - empty, if n=0
  - has root equal to root(T1);
  - has left subtree equal to B(T11,T12,…,T1m); where B(T11,T12,…,T1m) are subtrees of root(T1);
  - and has right subtree equal to B(T2,T3,…,Tn)

Forest Traversals

- Preorder
  - If *F* is empty, then return
  - Visit the root of the first tree of *F*
  - Traverse the subtrees of the first tree in forest preorder
  - Traverse the remaining trees of *F* in forest preorder



Forest Traversals (Cont’d)

- Inorder
  - If *F* is empty, then return
  - Traverse the subtrees of the first tree in forest inorder
  - Visit the root of the first tree
  - Traverse the remaining trees of F in forest indorer
- Postorder
  - If *F* is empty, then return
  - Traverse the subtrees of the first tree in forest postorder
  - Traverse the remaining trees of F in forest indorer
  - Visit the root of the first tree

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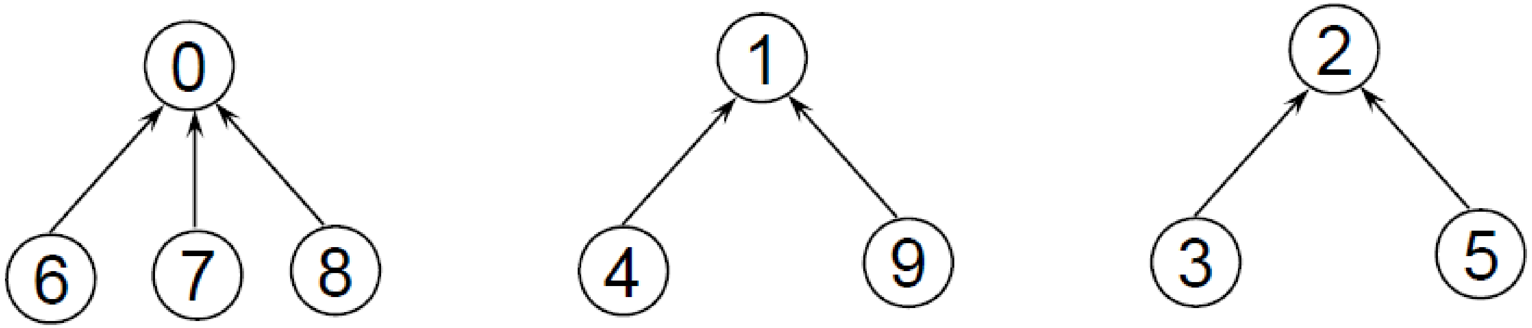
Set Representation

- S1={0, 6, 7, 8}, S2={1, 4, 9}, S3={2, 3, 5}

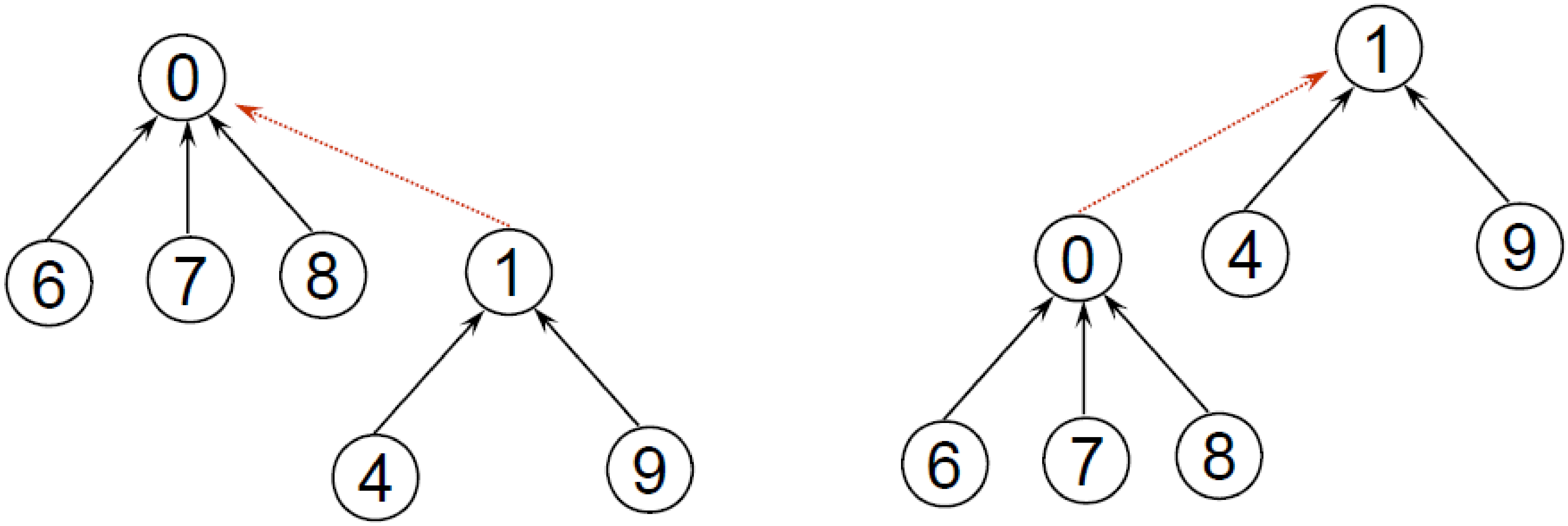


$S_i \cap S_j = \phi$

- Two operations considered here
  - *Disjoint set union* $S_1 \cup S_2 = \{0,6,7,8,1,4,9\}$
  - *Find*(i): Find the set containing the element i.
    - $3 \in S_3$, $8 \in S_1$

Disjoint Set Union

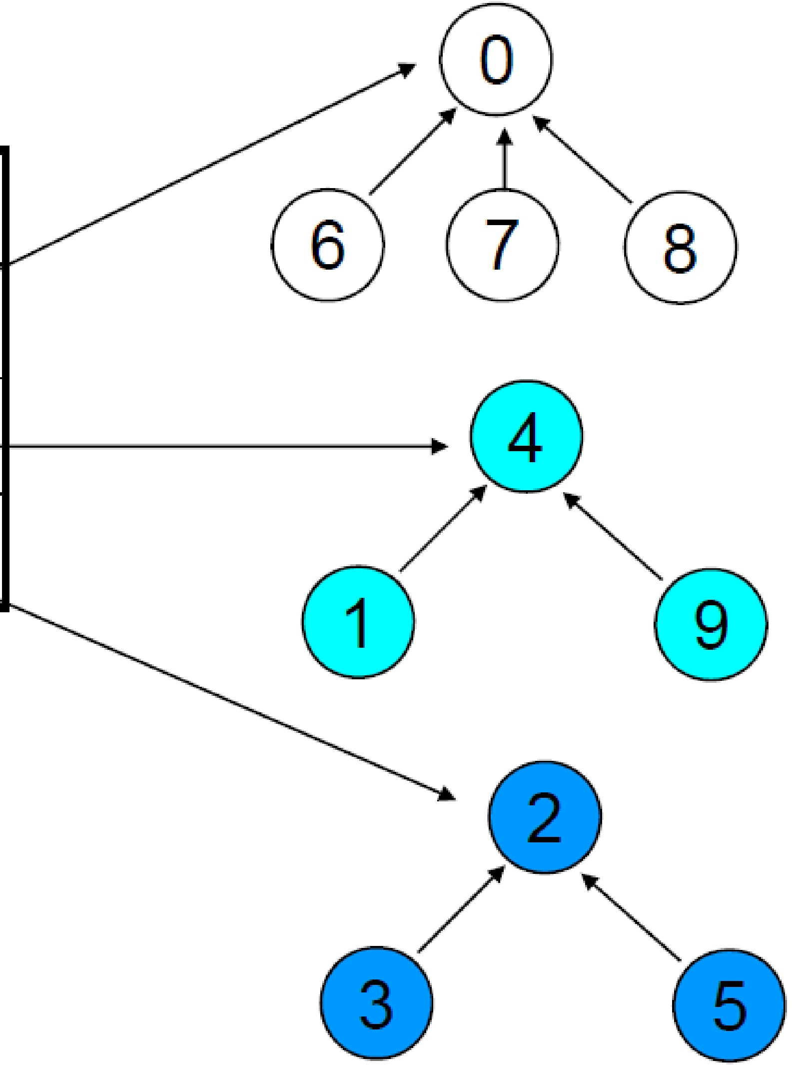
Make one of trees a subtree of the other



Possible representation for S_1 *union* S_2

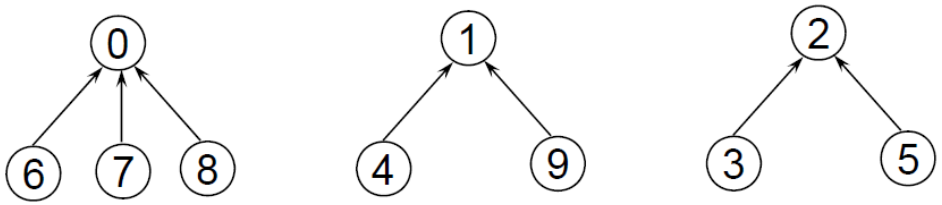
Data Representation of S_1, S_2 and S_3

Set name	pointer
S_1	
S_2	
S_3	



Array Representation of S_1, S_2 and S_3

- We could use an array for the set name.
- Or the set name can be an element at the root.
- Assume set elements are numbered 0 through n-1.



i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

root

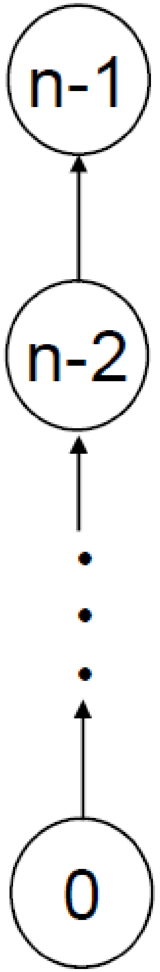
SimpleFind

```
SimpleFind(int i)
// Find the root of the tree
// containing element i
{
    while(parent[i]>=0)
        i=parent[i];
    return;
}
```

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

Degenerate Tree

union(0,1),
union(1,2),
.
.
.
union(n-2,n-1)
find(0),
find(1),
.
.
.
find(n-1)



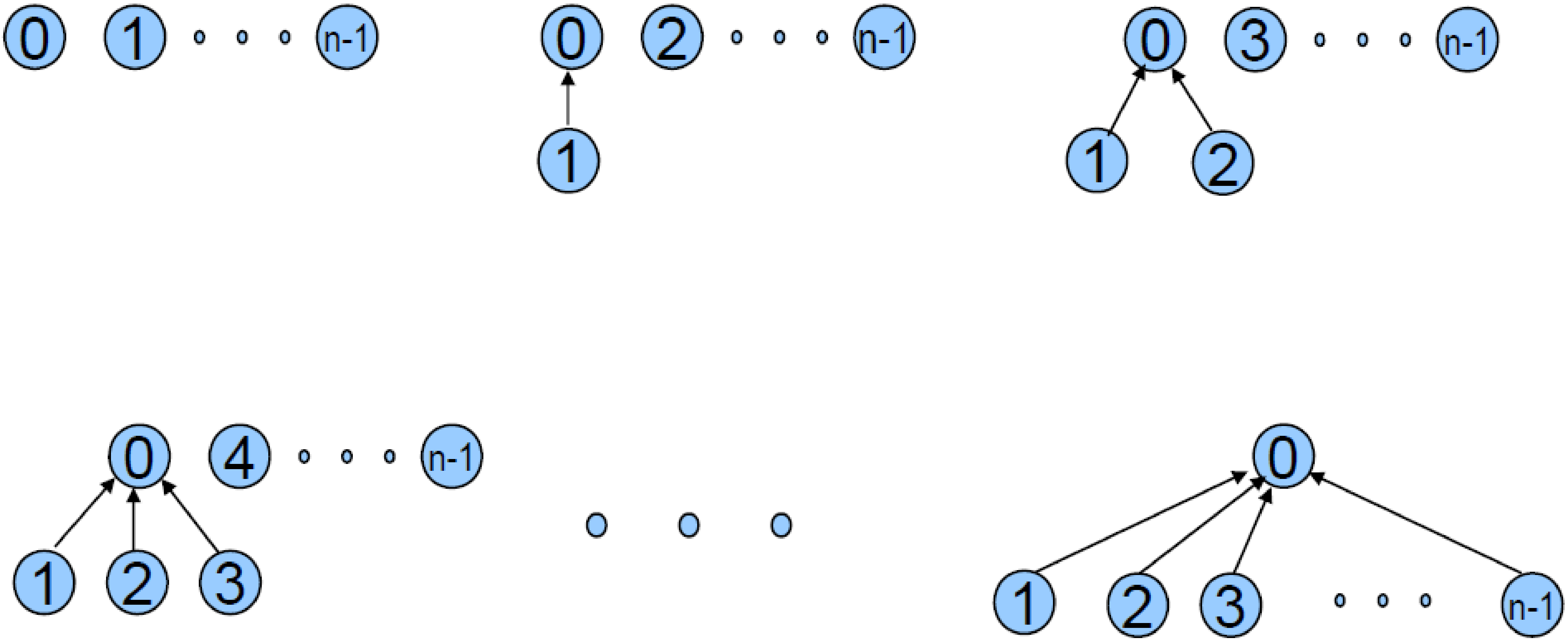
degenerate tree

- n-1 union operations
  - O(n)
- One union operation
  - O(1)
- n find operations
  - $O(n^2)$ $\sum_{i=2}^{n} i$
- One find operations
  - O(n)

Weighting Rule

- Weighting rule for *union(i, j)*
  - If the number of nodes in the tree with root *i* is less than the number in the tree with root *j*, then make *j* the parent of *i*; otherwise make *i* the parent of *j*.
- Use the weighting rule on the union operation to avoid the creation of degenerate trees.

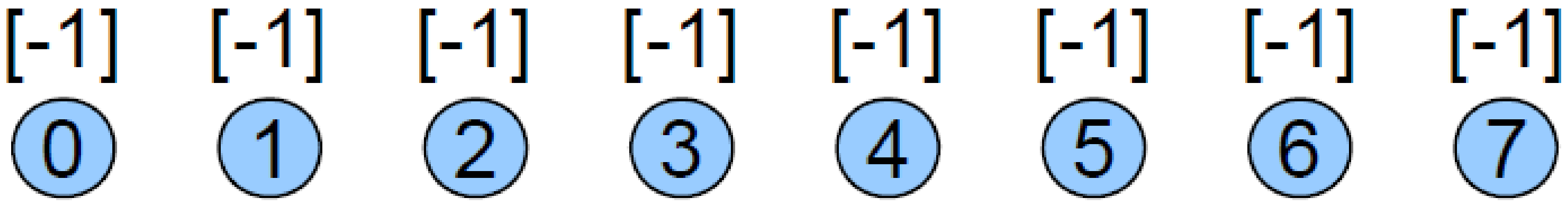
Trees Obtained Using The Weighting Rule



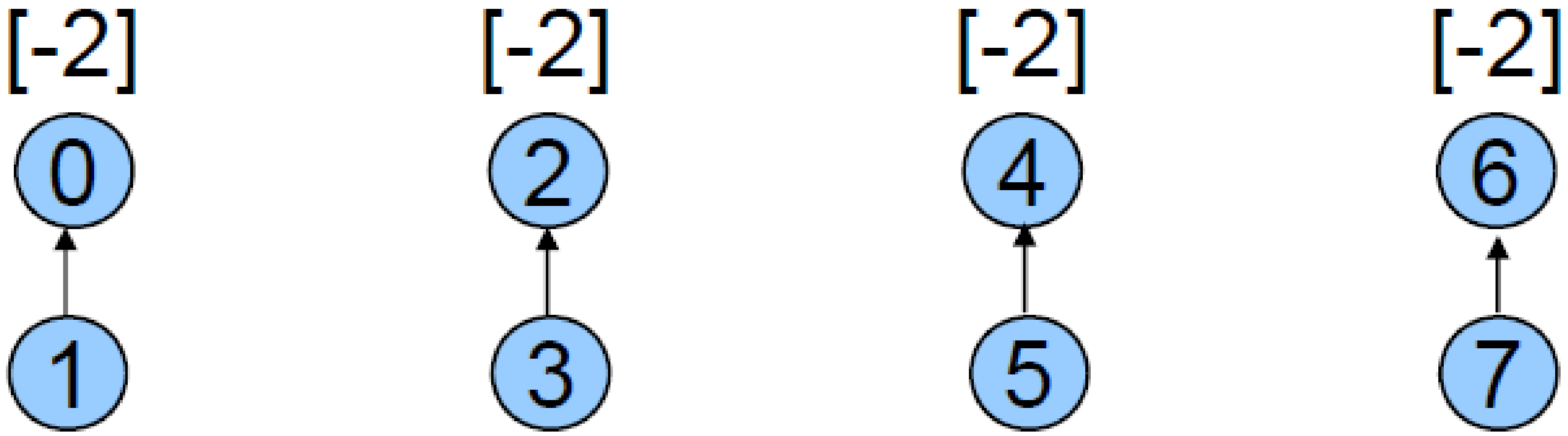
Weighted Union

- **Lemma 5.5**: Assume that we start with a forest of trees, each having one node. Let *T* be a tree with m nodes created as a result of a sequence of unions each performed using function WeightedUnion. The height of *T* is no greater than $\lfloor \log_2 m \rfloor + 1$.
- For the processing of an intermixed sequence of *u*–1 unions and *f* find operations, the time complexity is O(*u*+*f**log *u*).
  - No ree has more than u nodes in it.
  - We need O(n) additional time to initialize the n-tree forest

Trees Achieving Worst-Case Bound

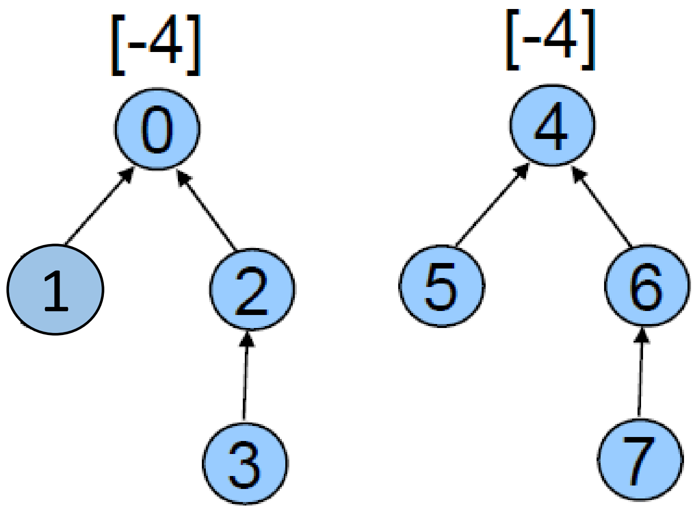


(a) Initial height trees

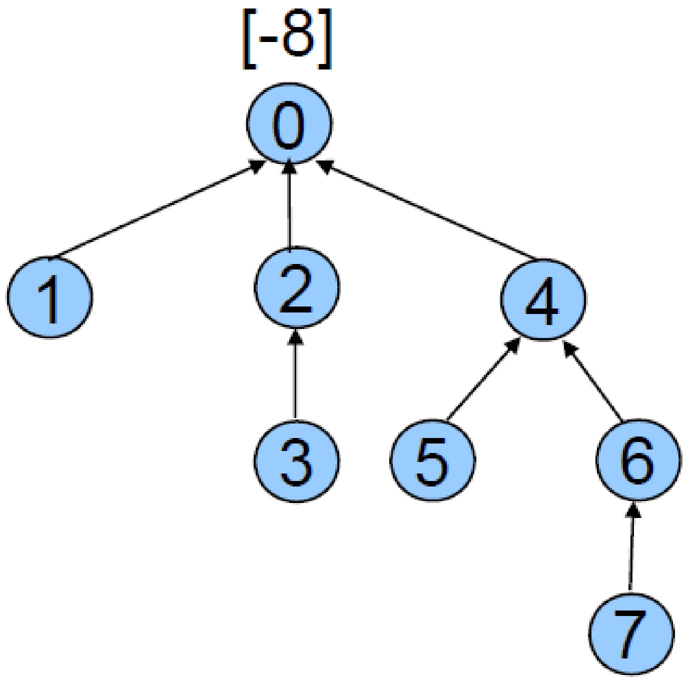


(b) Height-2 trees following union (0, 1), (2, 3), (4, 5), and (6, 7)

Trees Achieving Worst-Case Bound (Cont'd)



(c) Height-3 trees following union (0, 2), (4,6)

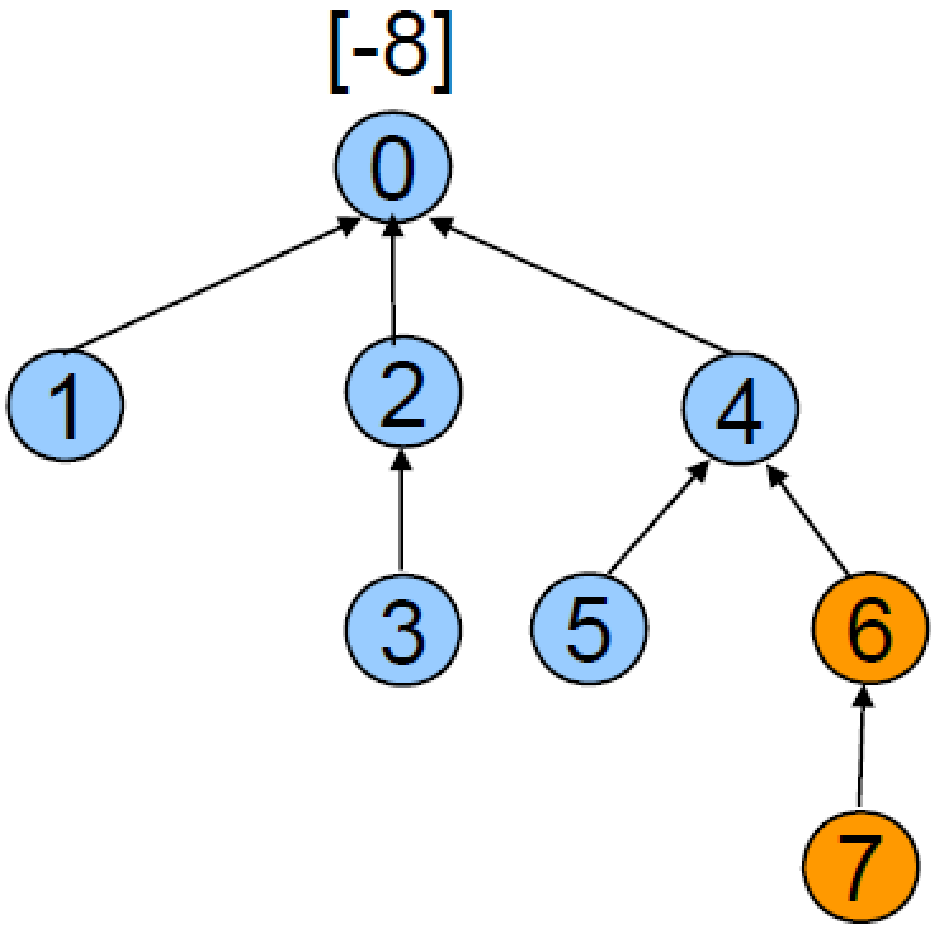


(d) Height-4 trees following union (0, 4)

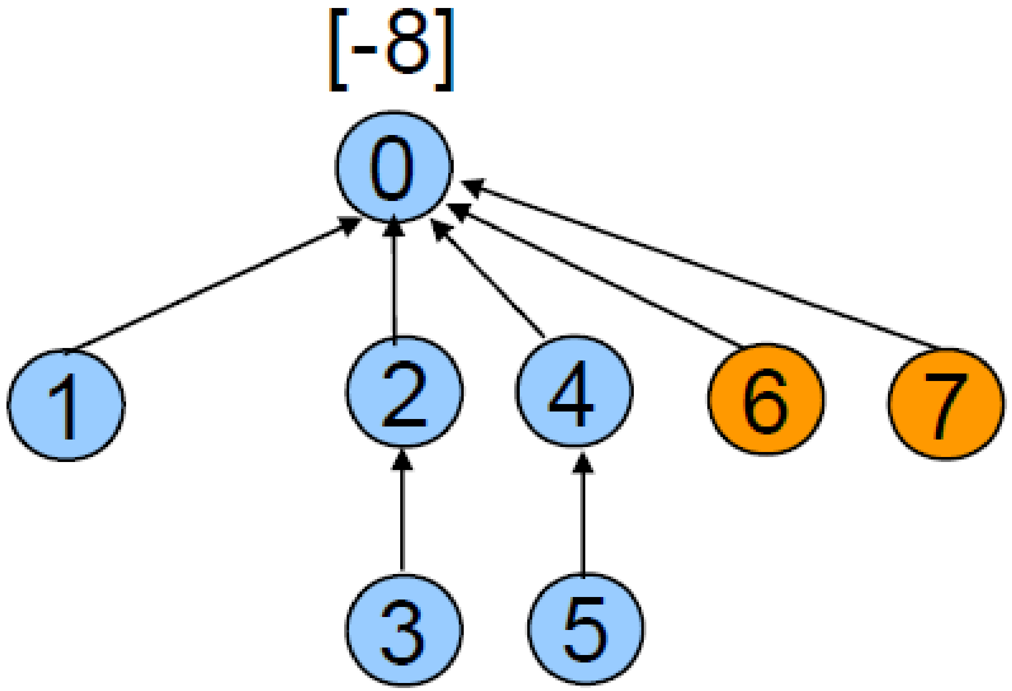
Collapsing Rule

- Collapsing rule:
  - If *j* is a node on the path from *i* to its root and *parent[i]≠root(i)*, then set *parent[j]* to *root(i)*.
- The first run of find operation will collapse the tree. Therefore, each following find operation of the same element only goes up one link to find the root.

CollapsingFind



Before collapsing



After collapsing