Chapter 4.
Laplace Transform

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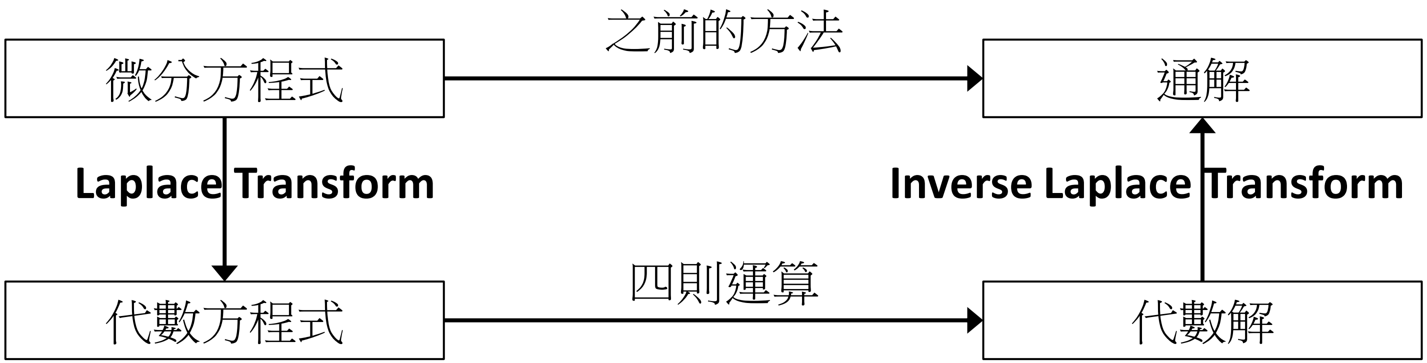
Laplace Transform

- 想法：



Laplace Transform

From 微分方程式 to 代數方程式必須具備:

$$y'' + 3y' + 5y = e^t$$

$$y'' + 3y' + 5y = \cos t$$

$$y'' + 3y' + 5y = \sin t$$

……

才可以進行 Laplace Transform

1. 基本函數
2. 一階、二階(y', y'')

Laplace Transform

- 定義：

$L\{f(t)\}= f(t)$函數的 Laplace Transform

$= \int_0^{\infty} f(t)e^{-st}dt$

$= F(s)$

$f(t) = L^{-1}\{F(s)\} \Rightarrow$ Inverse Laplace Transform

$= \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)e^{st}ds$ 複變(Complex Analysis)

*轉換與逆轉換要同時學習

Laplace Transform

例：$y^{'} + 2y = e^{t}, y(0) = 1$

$method(I)$

$$y_h = Ce^{-2t}$$

$$y_p = \frac{1}{D+2} e^{t} = \frac{1}{3} e^{t}$$

$$y(t) = Ce^{-2t} + \frac{1}{3} e^{t}, y(0) = 1, C = \frac{2}{3}$$

$$y(t) = \frac{2}{3} e^{-2t} + \frac{1}{3} e^{t}$$

Laplace Transform

例：$method(II)$

$$L\left\{y^{'}\right\}+2L\left\{y\right\}=L\left\{e^{t}\right\}\Rightarrow sY(s)-y(0)+2Y(s)=\frac{1}{s-1}$$

$$(s+2)Y(s)=\frac{s}{s-1}$$

$$Y(s)=\frac{s}{(s-1)(s+2)}=\frac{a}{s-1}+\frac{b}{s+2}$$

$$a=\frac{1}{3},b=\frac{2}{3}$$

$$Y(s)=\frac{1}{3(s-1)}+\frac{2}{3(s+2)}\Rightarrow$$ Inverse Laplace Transform

$$y(t)=\frac{1}{3}e^{t}+\frac{2}{3}e^{-2t}$$

Laplace Transform

(1) $f(t) = e^{at}, a \in const$

$$F(s) = L\{f(t)\} = \int_0^\infty e^{at} e^{-st} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^\infty$$

$$= -\frac{1}{s-a} e^{-(s-a)\infty} - (-\frac{1}{s-a}), s-a>0 \because \text{要收斂}$$

$$= \frac{1}{s-a}$$

$$f(t) = e^{at} \Rightarrow F(s) = \frac{1}{s-a}$$

Laplace Transform

例：$f(t) = e^{-2t}$

$$\Rightarrow F(s) = \frac{1}{s+2}$$

例：$G(s) = \dfrac{2}{s+3}$

$$\Rightarrow g(t) = L^{-1}\left\{2\frac{1}{s-(-3)}\right\}$$

$$= 2e^{-3t}$$

Laplace Transform

(2) $$f(t) = \cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \cos(at)e^{-st}dt$$

$$L\{\cos at\} = L\left\{\frac{e^{iat} + e^{-iat}}{2}\right\}$$

$$= \frac{1}{2}L\{e^{iat}\} + \frac{1}{2}L\{e^{-iat}\} = \frac{1}{2}\frac{1}{(s-ia)} + \frac{1}{2}\frac{1}{(s+ia)}$$

$$= \frac{1}{2}\frac{2s}{(s-ia)(s+ia)} = \frac{s}{(s-ia)(s+ia)}$$

$$= \frac{s}{s^2 + a^2}$$

$$f(t) = \cos at \Rightarrow F(s) = \frac{s}{s^2 + a^2}$$

Laplace Transform

例：$\pounds\{\cos 3t\} = \dfrac{s}{s^2+3^2} = \dfrac{s}{s^2+9}$

完成$\int_0^\infty \cos at \cdot e^{-st} dt$

令$dv = e^{-st}dt, v = \dfrac{-1}{s}e^{-st}$

$u = \cos at, du = -a\sin at dt$

$\int_0^\infty \cos at e^{-st} dt$

$= \cos at \cdot \dfrac{-1}{s} e^{-st} \Big|_0^\infty - \int_0^\infty \dfrac{-1}{s} e^{-st}(-a\sin at)dt, s>0$

$= [0 - \dfrac{-1}{s}] - \dfrac{a}{s}\int_0^\infty \sin at e^{-st} dt$

Laplace Transform

$$令 u=\sin at, du = a\cos at dt$$

$$dv = e^{-st}dt, v=\frac{-1}{s}e^{-st}$$

$$=\frac{1}{s}-\frac{a}{s}[\sin at\cdot\frac{-1}{s}e^{-st}\Big|_0^\infty-\int_0^\infty(\frac{-1}{s}e^{-st})a\cos at dt], s>0$$

$$=\frac{1}{s}-\frac{a}{s}\cdot\frac{a}{s}\int_0^\infty \cos at\cdot e^{-st}dt$$

$$\Rightarrow F(s)=\frac{1}{s}-\frac{a^2}{s^2}F(s)$$

$$\Rightarrow F(s)=\frac{s}{s^2+a^2}$$

Laplace Transform

(3) $f(t) = \sin at$

$$F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty \sin at \cdot e^{-st} dt$$

利用 $\sin at = \dfrac{e^{iat} - e^{-iat}}{2i}$

$$\Rightarrow \mathscr{L}\{\sin at\} = \frac{1}{2i}\mathscr{L}\{e^{iat}\} - \frac{1}{2i}\mathscr{L}\{e^{-iat}\}$$

$$= \frac{1}{2i}\frac{1}{s-ai} - \frac{1}{2i}\frac{1}{s+ai} = \frac{a}{s^2+a^2}$$

Laplace Transform

例： $f(t) = \sin 2t$

$$\Rightarrow F(s) = \frac{2}{s^2+4}$$

例： $F(s) = \dfrac{s}{s^2+16}$

$$\Rightarrow f(t) = \mathscr{L}^{-1}\{F(s)\} = \cos 4t$$

例： $F(s) = \dfrac{3}{s^2+16}$

$$\Rightarrow f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{3}{4} * \frac{4}{s^2+16} = \frac{3}{4}\sin 4t$$

例： $\mathscr{L}^{-1}\{\dfrac{3}{s+6} + \dfrac{5s}{s^2+25} + \dfrac{2}{s^2+36}\}$

$$\Rightarrow 3e^{-6t} + 5\cos 5t + \frac{1}{3}\sin 6t$$

Laplace Transform

(4) $f(t) = t$

$$F(s) = \int_0^{\infty} te^{-st} dt$$

$$= t(\frac{-1}{s}e^{-st})\Big|_0^{\infty} - \int_0^{\infty} \frac{-1}{s}e^{-st} dt$$

$$= \lim_{t \to \infty} \frac{-t}{se^{st}} - 0 + \frac{1}{s}(\frac{-1}{s}e^{-st})\Big|_0^{\infty}$$

$$= \frac{1}{s^2}$$

$$\Rightarrow \mathscr{L}\{t\} = \frac{1}{s^2}$$

$$\Rightarrow f(t) = t \ \Rightarrow F(s) = \frac{1}{s^2}$$

Laplace Transform

例： $\mathscr{L}\{3t\}$

$$\Rightarrow F(s) = \frac{3}{s^2}$$

例： $G(s) = \dfrac{5}{s^2}$

$$\Rightarrow g(t) = 5t$$

例： $G(s) = \dfrac{5s+1}{s^2}$

$$\Rightarrow g(t) = 5 + t$$

Laplace Transform

(5) $f(t) = H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

單位步階函數=unit-step function

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{H(t)\} = \mathscr{L}\{U(t)\}$$

$$= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 1 \cdot e^{-st}dt = \frac{1}{s}$$

Laplace Transform

(a) High-pass filter

$$g(t) = \begin{cases} 1 & t > 3 \\ 0 & t < 3 \end{cases}$$

$$g(t) = H(t-3)$$

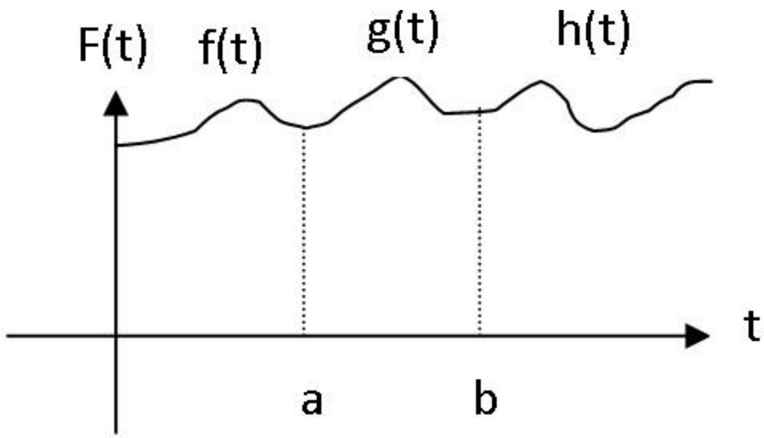
(b) Band-pass filter

$$g(t) = \begin{cases} 1 & -3 < t < 3 \\ 0 & t > 3 \text{或} t < -3 \end{cases}$$

$$g(t) = H(t+3) - H(t-3)$$

Laplace Transform

例：



$$F(t)=\begin{cases}0, t<0\\ f(t), 0<t<a\\ g(t), a<t<b\\ h(t), b<t\end{cases}$$

$$F(t)=f(t)\big[H(t)-H(t-a)\big]+g(t)\big[H(t-a)-H(t-b)\big]+h(t)[H(t-b)]$$

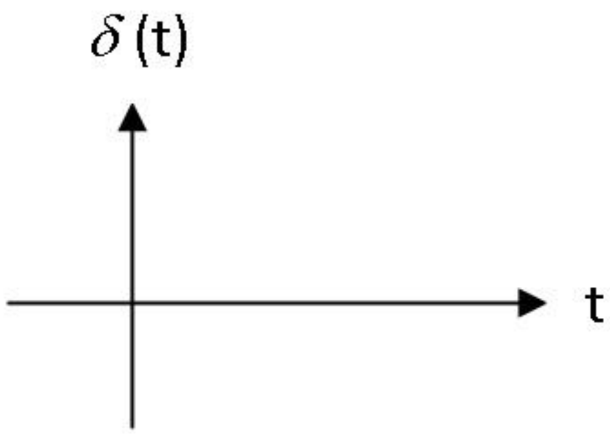
$$=f(t)+\big[g(t)-f(t)\big]H(t-a)+[h(t)-g(t)]H(t-b)$$

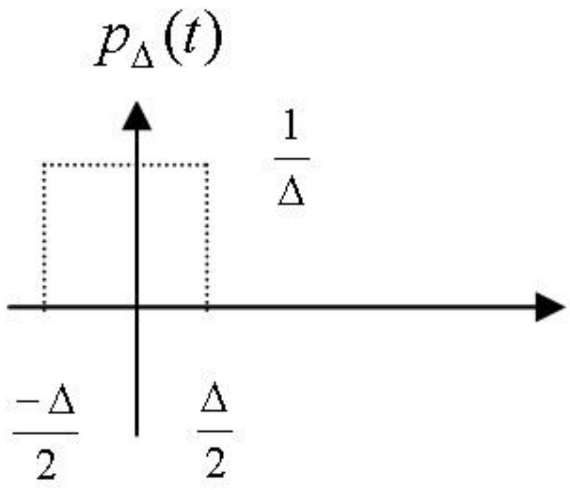
Laplace Transform

(6) $\delta(t)\begin{cases}1, t=0\\0, t\neq 0\end{cases}$,其中1為面積值

$\delta(t)=unit-impulse\ function$ 單位脈衝函數

自然界沒有這函數，可用近似函數來取代





Laplace Transform

$$\lim_{\Delta \to 0} P_\Delta(t) = \begin{cases} 1, t = 0 \\ 0, t \neq 0 \end{cases}$$

$$\mathscr{L}\{\delta(\mathrm{t})\}$$

$$= \int_0^\infty \delta(t) e^{-st} dt$$

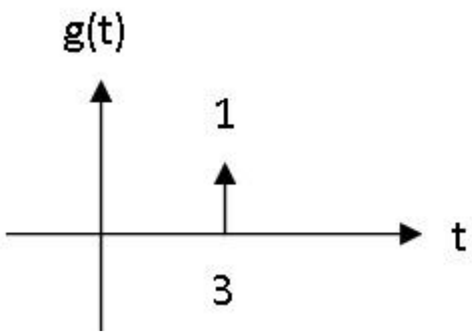
在 t 的積分範圍內，只有在$t = 0$時$\delta(\mathrm{t})$才有值

$$= \int_0^\infty \delta(\mathrm{t}) e^{-0} dt$$

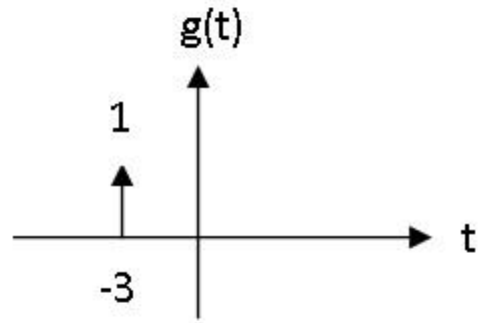
$$= 1$$

Laplace Transform

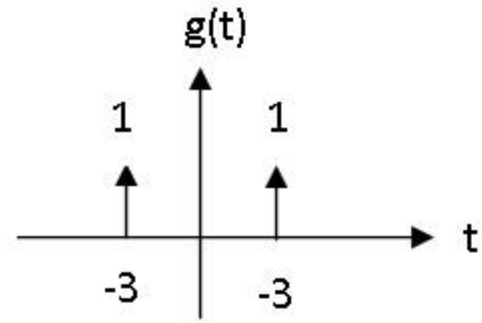
例：



$$g(t)=\delta(t-3)$$



$$g(t)=\delta(t+3)$$



$$g(t)=\delta(t+3)+\delta(t-3)$$

Laplace Transform

(7) $f(t^n) \quad \Rightarrow \quad F(s) = \dfrac{n!}{s^{n+1}}$

例: $\mathscr{L}\left\{e^t + 3\sin 2t + 5\cos 3t + 3t^2 + 5\delta(t)\right\}$

$$= \frac{1}{s-1} + \frac{2}{s^2+4}\cdot 3 + \frac{s}{s^2+9}\cdot 5 + \frac{2!}{s^3}\cdot 3 + 5$$

例: $\mathscr{L}^{-1}\left\{7 + \dfrac{2}{s^5} + \dfrac{2}{s^2+1} + \dfrac{s}{s^2+49} + \dfrac{3}{s+5}\right\}$

$$= 7\delta(t) + \frac{2}{4!}t^4 + 2\sin t + \cos 7t + 3e^{-5t}$$