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Theory of Computation

Kun-Ta Chuang
Department of Computer Science and Information Engineering
National Cheng Kung University

Outline

1 Two Pumping Lemmas

2 Closure Properties and Decision Algorithms for CFLs

The Pumping Lemma
for
Context-Free Languages

Consider now an infinite context-free language L

Let G be the grammar of $L-\{\lambda\}$

Take G so that L has no unit-productions
no λ -productions

Let p = (Number of productions) x (Largest right side of a production)

Let $m = p \underline{+1}$ (Largest number of states in NPDA)

Example : G

$$S \to AB$$
$$A \to aBb$$
$$B \to Sb$$
$$B \to b$$

$$p = 4 \times 3 = 12$$

$$m = p + 1 = 13$$

Take a string $w \in L(G)$
with length $|w| \geq m$

We will show:

in the derivation of w
a ~~variable~~ (production) of G is repeated

$$S \overset{*}{\Rightarrow} w$$

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$$S = v_1$$

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$$|\, v_{i+1} \,| \leq |\, v_i \,| + f$$

← maximum right hand side of any production

$$\Downarrow$$

$$|\, w \,| < k \cdot f$$

$$\Downarrow$$

$$m \leq |\, w \,| < k \cdot f \quad \Longrightarrow \quad p < k \cdot f$$

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$$p < k \cdot f$$

$$k > \frac{p}{f}$$

Number of productions
in grammar

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$k >$ Number of productions in grammar

Some production must be repeated

$$v_1 \Rightarrow \cdots \Rightarrow a_1 A a_2 \Rightarrow \cdots \Rightarrow a_3 A a_4 \Rightarrow \cdots \Rightarrow w$$

Repeated variable

$$S \to r_1$$

$$\boxed{A \to r_2}$$

$$B \to r_2$$

$$\cdots$$

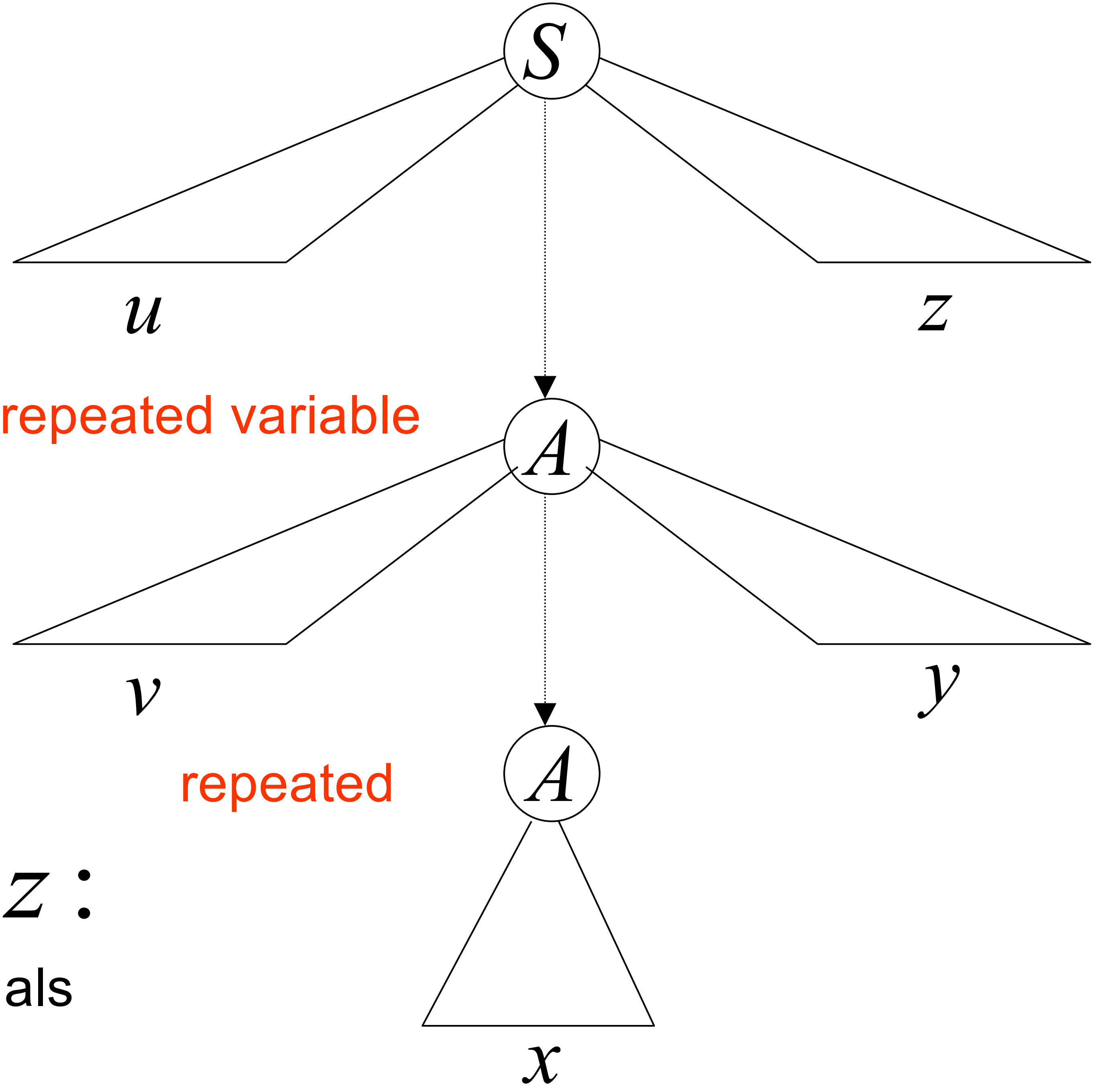
$w \in L(G)$ $|w| \geq m$

Derivation of string w

$$S \Rightarrow \cdots \Rightarrow a_1 A a_2 \Rightarrow \cdots \Rightarrow a_3 A a_4 \Rightarrow \cdots \Rightarrow w$$

Some variable is repeated

Derivation tree of string w



Last (lowest) repeated variable

repeated

$$w = uvxyz$$

$$u, v, x, y, z:$$

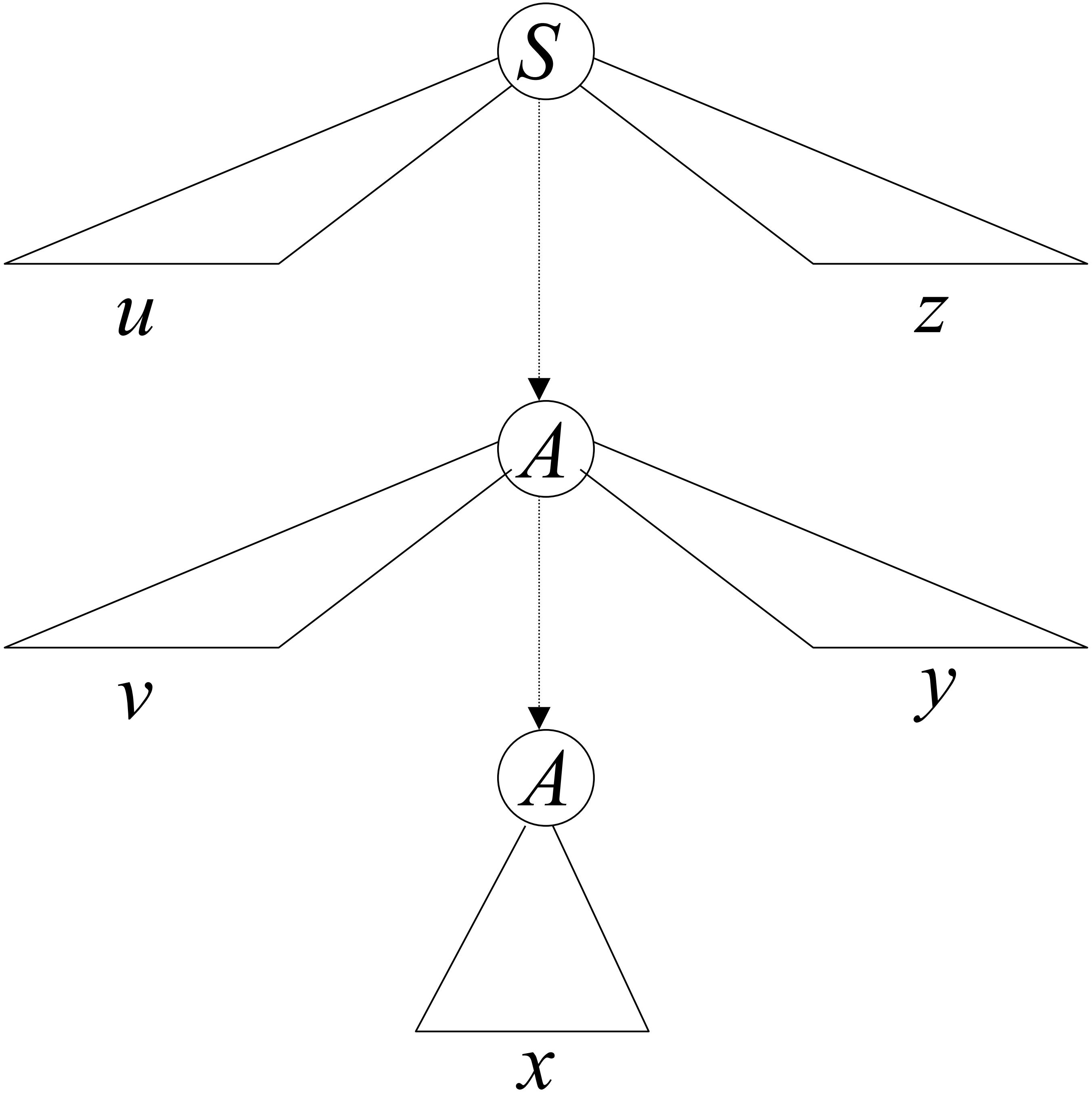
Strings of terminals

Possible derivations:

$$S \overset{*}{\Rightarrow} uAz$$

$$A \overset{*}{\Rightarrow} vAy$$

$$A \overset{*}{\Rightarrow} x$$



We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uxz$$

$$uv^0xy^0z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvxyz$$

The original $w = uv^1xy^1z$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow} uvvxyyz$$

$$uv^2xy^2z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \Rightarrow uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} uvvvxyyyz$$

$$uv^3xy^3z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} \ldots$$

$$\overset{*}{\Rightarrow} uvvv\cdots vAy\cdots yyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvv\cdots vxy\cdots yyyz$$

$$uv^ixy^iz$$

Therefore, any string of the form

$$uv^i xy^i z \qquad i \geq 0$$

is generated by the grammar G

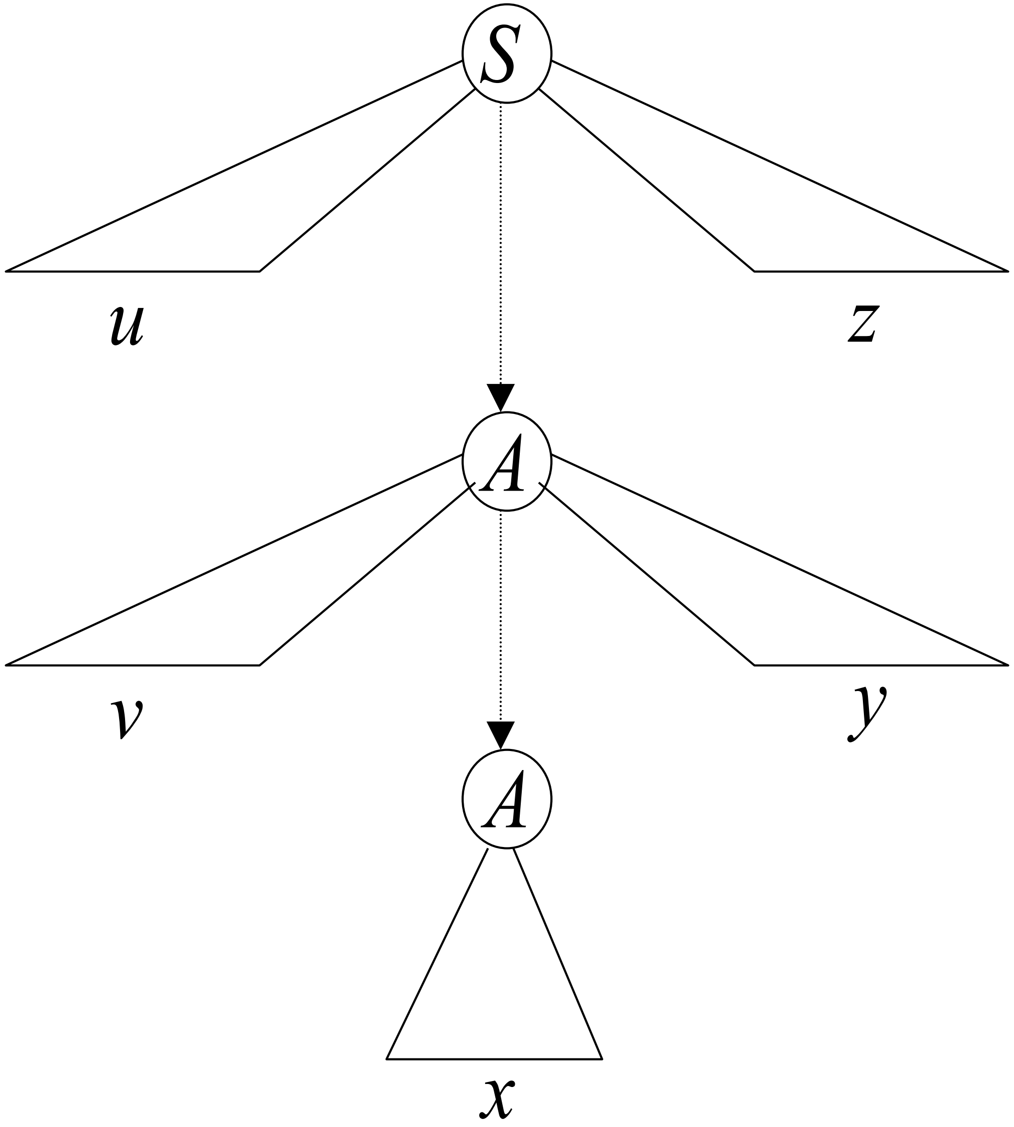
Therefore,

knowing that $uvxyz \in L(G)$

we also know that $uv^i xy^i z \in L(G)$

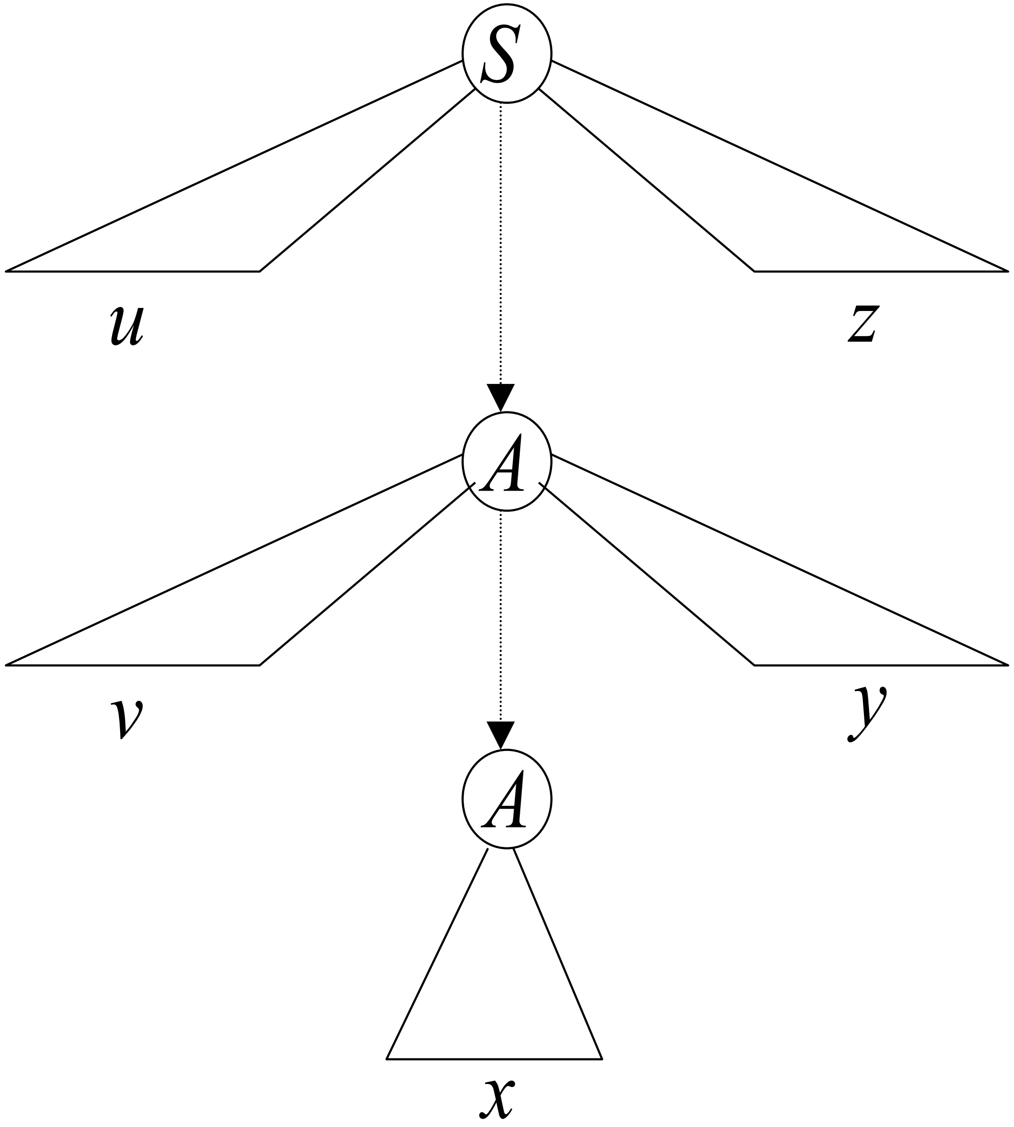
$$L(G) = L - \{\lambda\} \Big\downarrow$$

$$uv^i xy^i z \in L$$



Observation: $|vxy| \leq m$

Since A is the last repeated variable



Observation: $|vy| \geq 1$

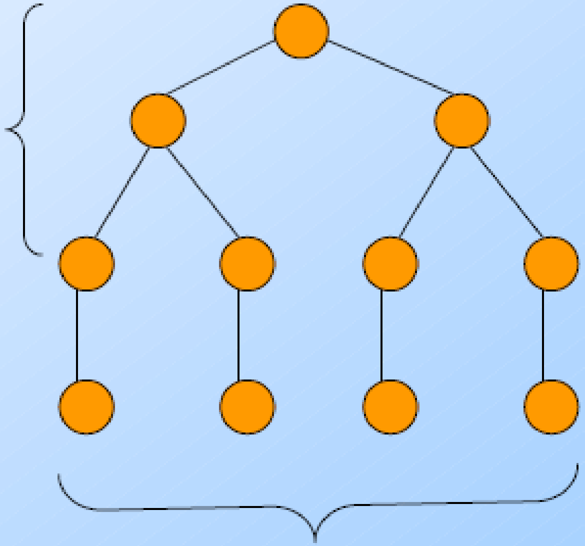
Since there are no unit or λ-productions

Proof of the Pumping Lemma

- Start with a CNF grammar for L – {ϵ}.
- Let the grammar have m variables.
- Pick n = 2^m.
- Let |z| $\geq$ n.
- We claim ("*Lemma 1* ") that a parse tree with yield z must have a path of length m+2 or more.

Proof of Lemma 1

- If all paths in the parse tree of a CNF grammar are of length $\leq m+1$, then the longest yield has length 2^{m-1}, as in:



m variables

one terminal

2^{m-1} terminals

The Pumping Lemma I:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

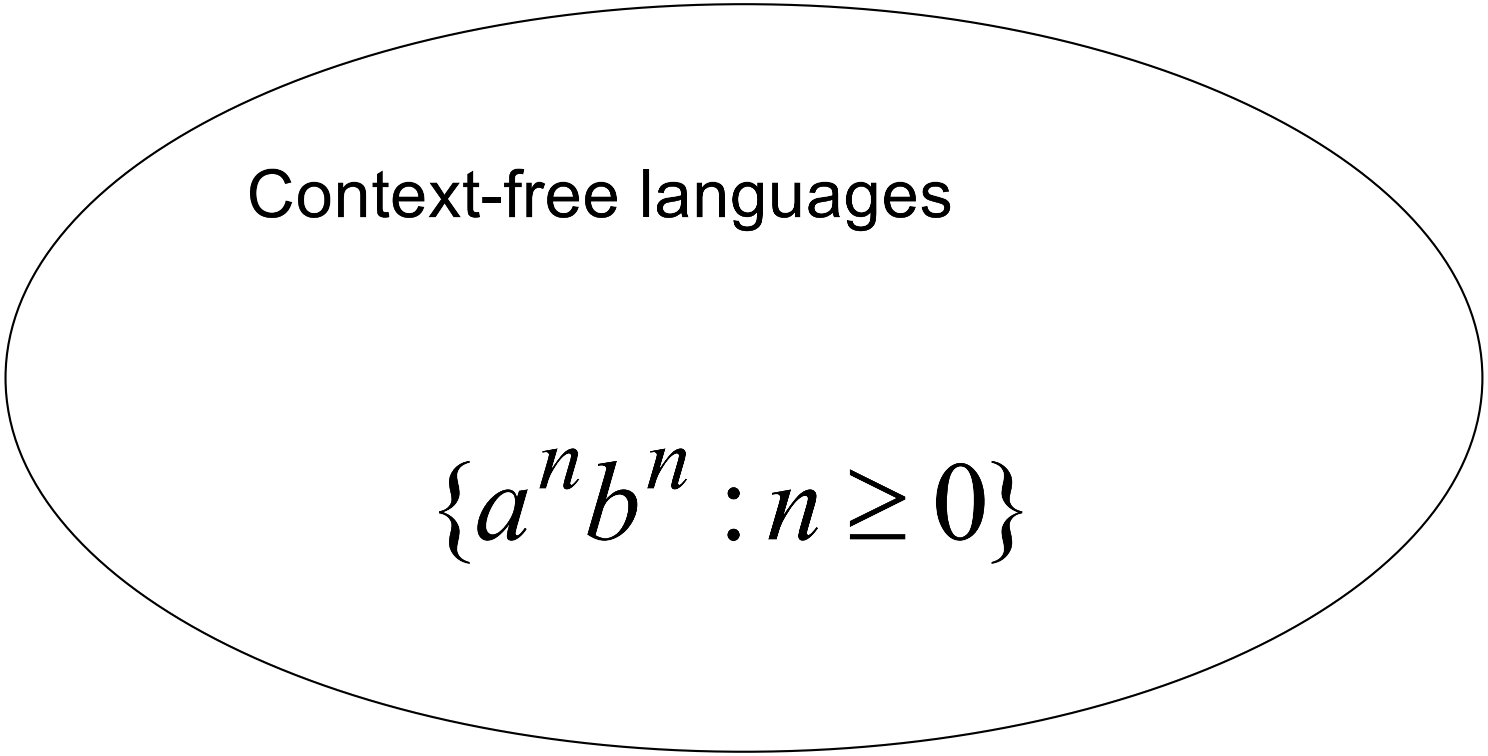
and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Applications
of
The Pumping Lemma

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$



Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Example 8.1: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma
for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^n b^n c^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $w = a^m b^m c^m$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

We can write: $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Pumping Lemma says:

$$uv^i xy^i z \in L \quad \text{for all} \quad i \geq 0$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

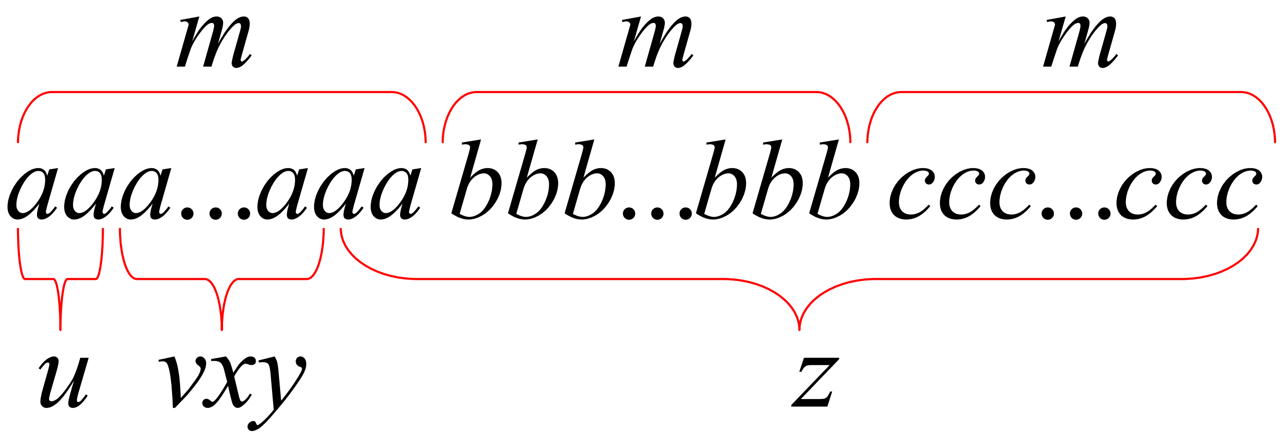
We examine **all** the possible locations of string vxy in w

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within a^m



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: v and y consist from only a

$$\overbrace{\underbrace{a}_{u}\underbrace{aa\ldots aa}_{vxy}}^{m}\underbrace{a\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: Repeating v and y

$$k \geq 1$$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aaaaa...aaaaa}_{v^2xy^2}\underbrace{a\,bbb...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

$$\overbrace{\underbrace{a}_{u}\underbrace{aaaaa\ldots aaaaa}_{v^2xy^2}}^{m+k}\underbrace{a\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

However: $uv^2xy^2z = a^{m+k} b^m c^m \notin L$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: vxy is within b^m

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

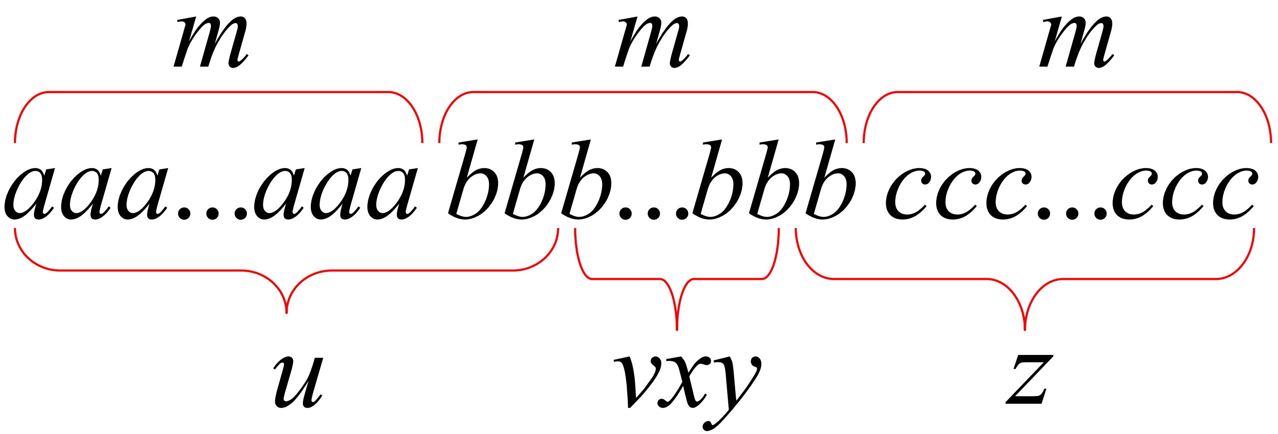
$$\underbrace{aaa\ldots aaa\,bb}_{u}\underbrace{b\ldots bb}_{vxy}\underbrace{b\,ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: Similar analysis with case 1



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: vxy is within c^m

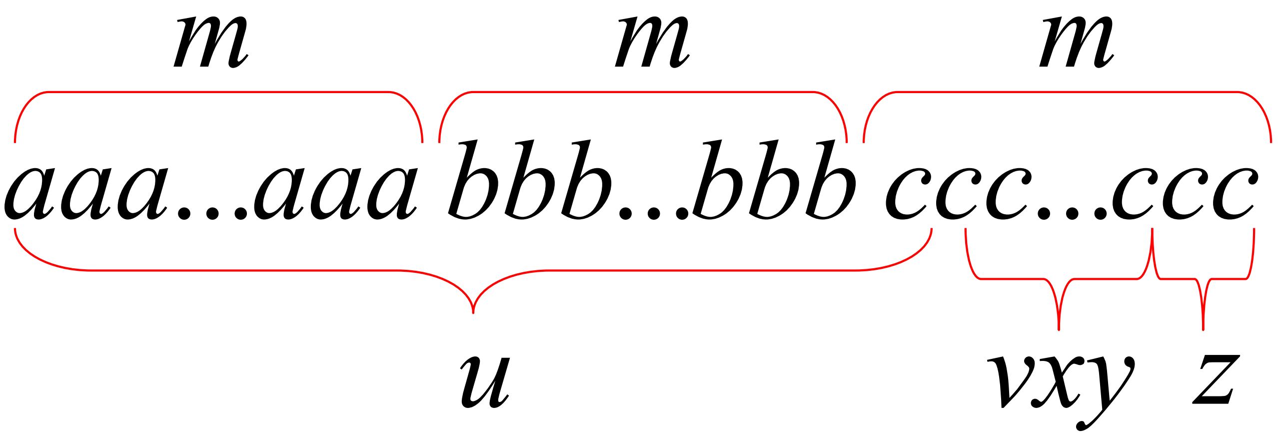
$$\underbrace{\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}\;\mathrlap{\overbrace{}^{m}}cc}_{u}\,\underbrace{c\ldots c}_{vxy}\,\underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: Similar analysis with case 1

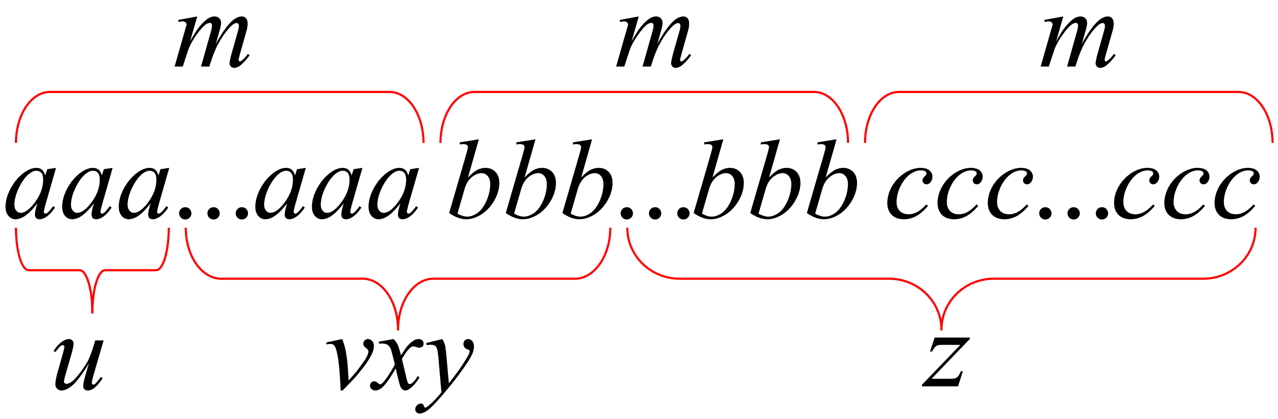


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: vxy overlaps a^m and b^m



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b

$$\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaa\;bbb}_{vxy}\underbrace{\ldots bbb\;ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b

$$k_1 + k_2 \geq 1$$

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\,\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaaaa\,bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

$$\overbrace{\underbrace{aaa}_{u}\mathrlap{\underbrace{}_{v^2xy^2}}\ldots aaaaaaa}^{m+k_1}\,\overbrace{bbbbbbb\mathrlap{\underbrace{}_{z}}\ldots bbb}^{m+k_2}\,\overbrace{ccc\ldots ccc}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

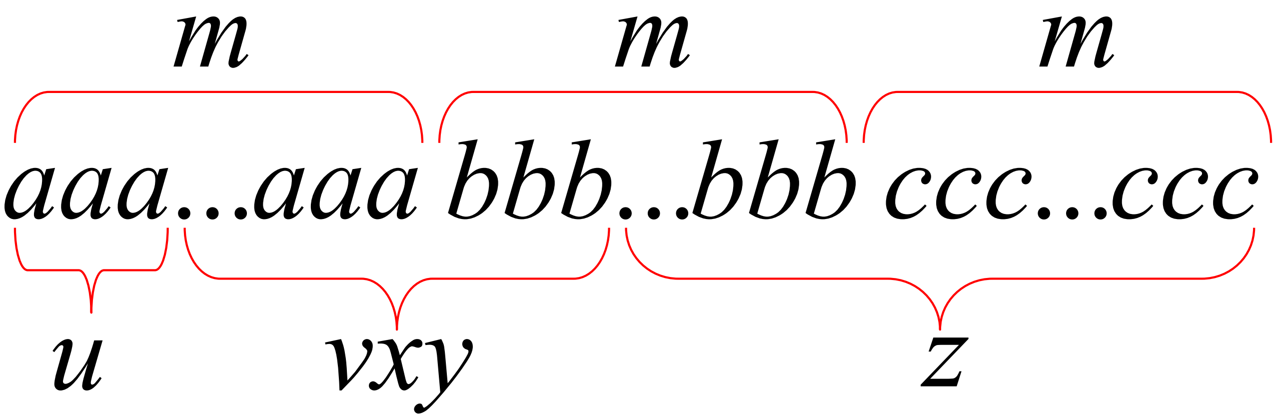
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 2: v contains a and b

y contains only b



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 2: v contains a and b

y contains only b

$$k_1 + k_2 + k \geq 1$$

$$\overbrace{aaa\ldots aaaaa}^{m}\overbrace{bb}^{k_1}\overbrace{aa}^{k_2}\overbrace{bb\ bbbbbbb\ldots bbb}^{m+k}\ \overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaabbaabb\ bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\ ccc\ldots ccc}_{z}$$

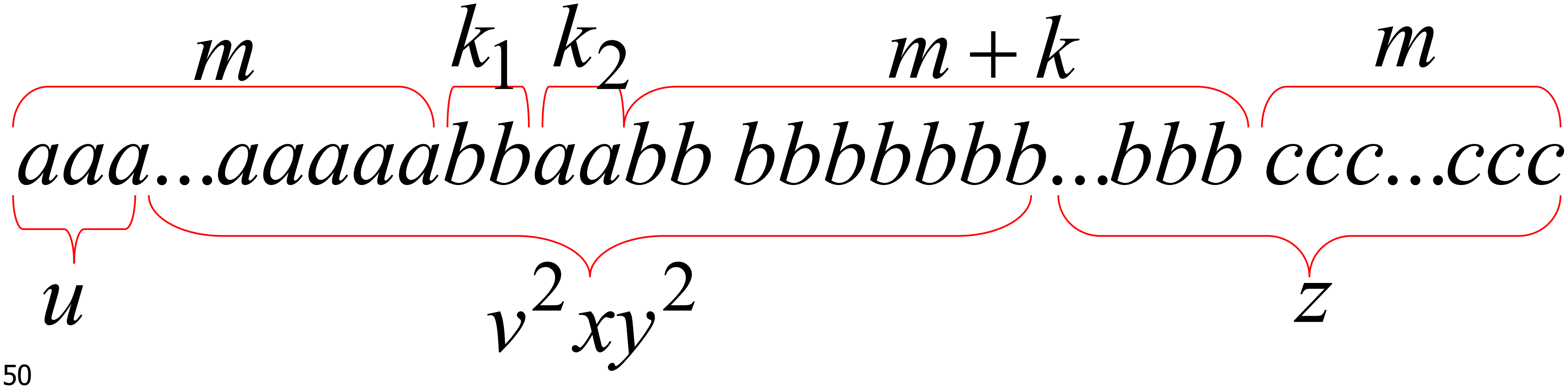
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 + k \geq 1$$



$$aaa\ldots aaaaabbaabb\ bbbbbbb\ldots bbb\ ccc\ldots ccc$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$w = uvxyz$ $\qquad |vxy| \leq m$ $\qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $\qquad k_1 + k_2 + k \geq 1$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

Contradiction!!!

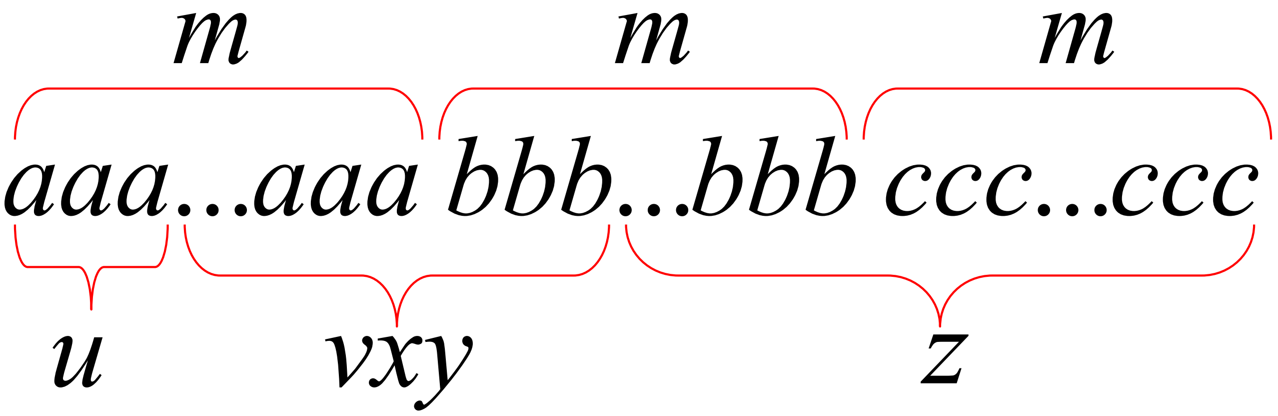
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b

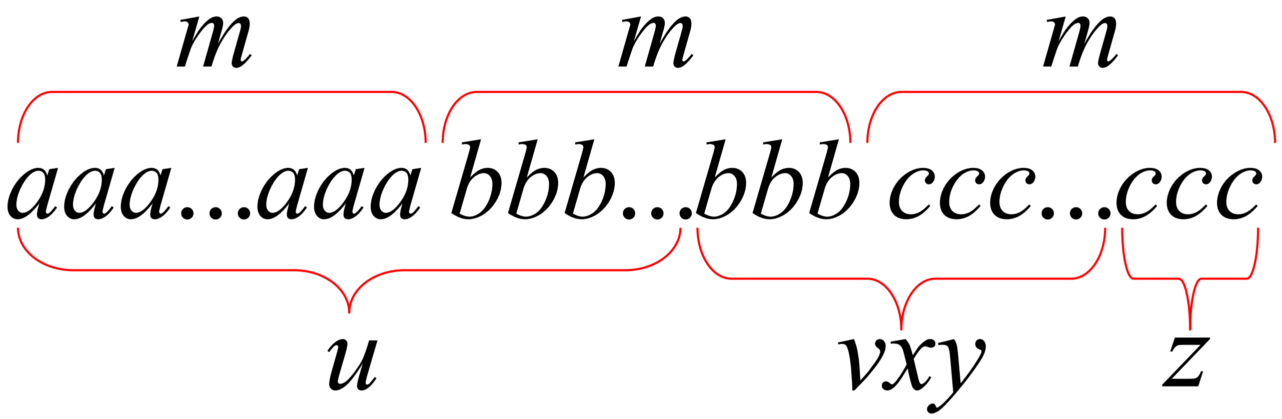
Similar analysis with Possibility 2

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: vxy overlaps b^m and c^m

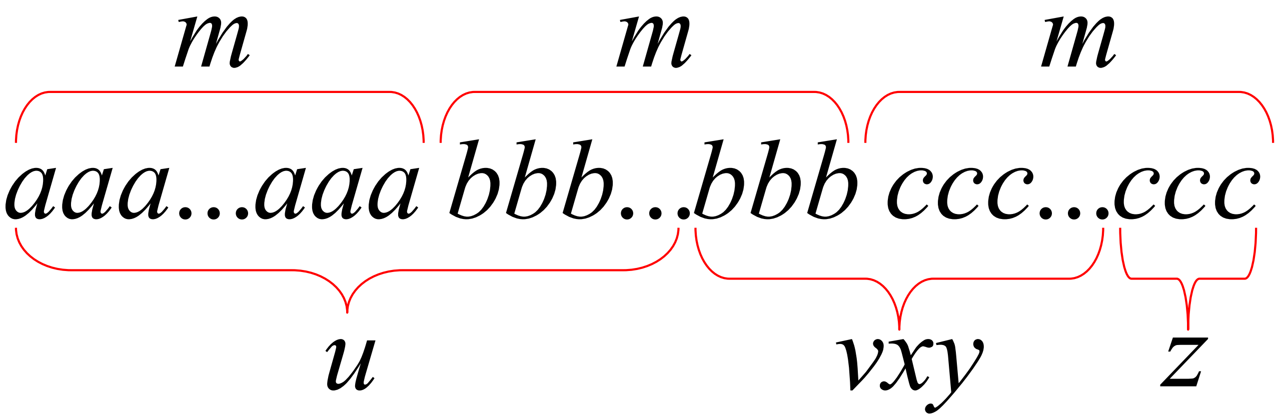


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: Similar analysis with case 4



There are no other cases to consider

(since $|vxy| \leq m$, string vxy cannot

overlap a^m, b^m and c^m at the same time)

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

$$\{ww : w \in \{a,b\}\}$$

$$\{a^{n^2} b^n : n \geq 0\}$$

$$\{a^{n!} : n \geq 0\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}^*\}$$

The Pumping Lemma II:

For infinite linear language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $\underline{|uvyz| \leq m \quad \text{and} \quad |vy| \geq 1}$

and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Example 8.6

- Show the following language

$L = \{w : n_a(w) = n_b(w)\}$ is not linear

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

O Given m

S Picks $w = a^m b^{2m} a^m$

O Picks any *uvyz* s.t. $uv = a^k$, $yz = a^l$ and $k, l \geqq 1$

S Picks i = 2 → $w_2 = a^{m+k} b^{2m} a^{m+l}$ is not in L