Chapter 2
Probability

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2.6 Conditional Probability

- Conditional probability: $P(B|A)$
  - The probability of an event B occurring when it is known that some event A has occurred.
  - "the probability that B occurs given that A occurs"
  - "the probability of B, given A"
- E.g.P.62:
  - $S = \{1, 2, 3, 4, 5, 6\}$, $A=\{4,5,6\}$, $B=\{1,3,5\}$,
    => $P(B|A)$?
- Definition 2.10: $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Conditional Probability

- Example P.63:
  - Our sample space S is the population of adults in a small town who have completed the requirements for a college degree.
  - To investigate the advantage of establishing new industries in the town.
  - The concerned events
    - M: a man is chosen
    - E: the one chosen is employed
  - $$P(M \mid E) = \frac{460}{600} = \frac{23}{30}$$

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

$$P(M \mid E) = \frac{n(E \cap M)/n(S)}{n(E)/n(S)} = \frac{P(E \cap M)}{P(E)}$$

$$P(E) = \frac{600}{900} = \frac{2}{3},\ P(E \cap M) = \frac{460}{900} = \frac{23}{45},\ P(M \mid E) = \frac{23/45}{2/3} = \frac{23}{30}$$

Conditional Probability

- Example P.65
  - 2 cards are drawn in succession, with replacement
    - *A*: the first card is an ace
    - *B*: the second card is a spade
    - $$P(B \mid A) = \frac{13}{52} = \frac{1}{4} \text{ and } P(B) = \frac{13}{52} = \frac{1}{4}$$
- Definition 2.11:
  - Two events *A* and *B* are said to be independent if and only if *P*(*B*|*A*) = *P*(*B*).
  - Otherwise, *A* and *B* are dependent.
- The notion of conditional probability provides the capability of reevaluating the idea of probability of an event in light of additional information.

Multiplicative Rules

- Theorem 2.10:If in an experiment the events *A* and *B* can both occur, then

$$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$$

- Example 2.36
  - A fuse box contains 20 fuses, of which 5 are defective.
  - If 2 fuses are selected at random and removed from the box in succession without replacing the first.
  - What is the probability that both fuses are defective?
    - Let *A* be the event that the first fuse is defective
    - *B* be the event that the second fuse is defective

$$P(A \cap B) = P(A)P(B \mid A) = \left(\frac{5}{20}\right)\left(\frac{4}{19}\right) = \frac{1}{19}$$

Multiplicative Rules

- Example 2.37
  - One bag contains 4 white balls and 3 black balls.
  - A second bag contains 3 white balls and 5 black balls.
  - One ball is drawn from the first bag and placed unseen in the second bag.
  - What is the probability that a ball now drawn from the second bag is black?
    - Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1.
    - $$\begin{aligned} P[(B_1 \cap B_2) \cup (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2 \mid B_1) + P(W_1)P(B_2 \mid W_1) \\ &= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63} \end{aligned}$$

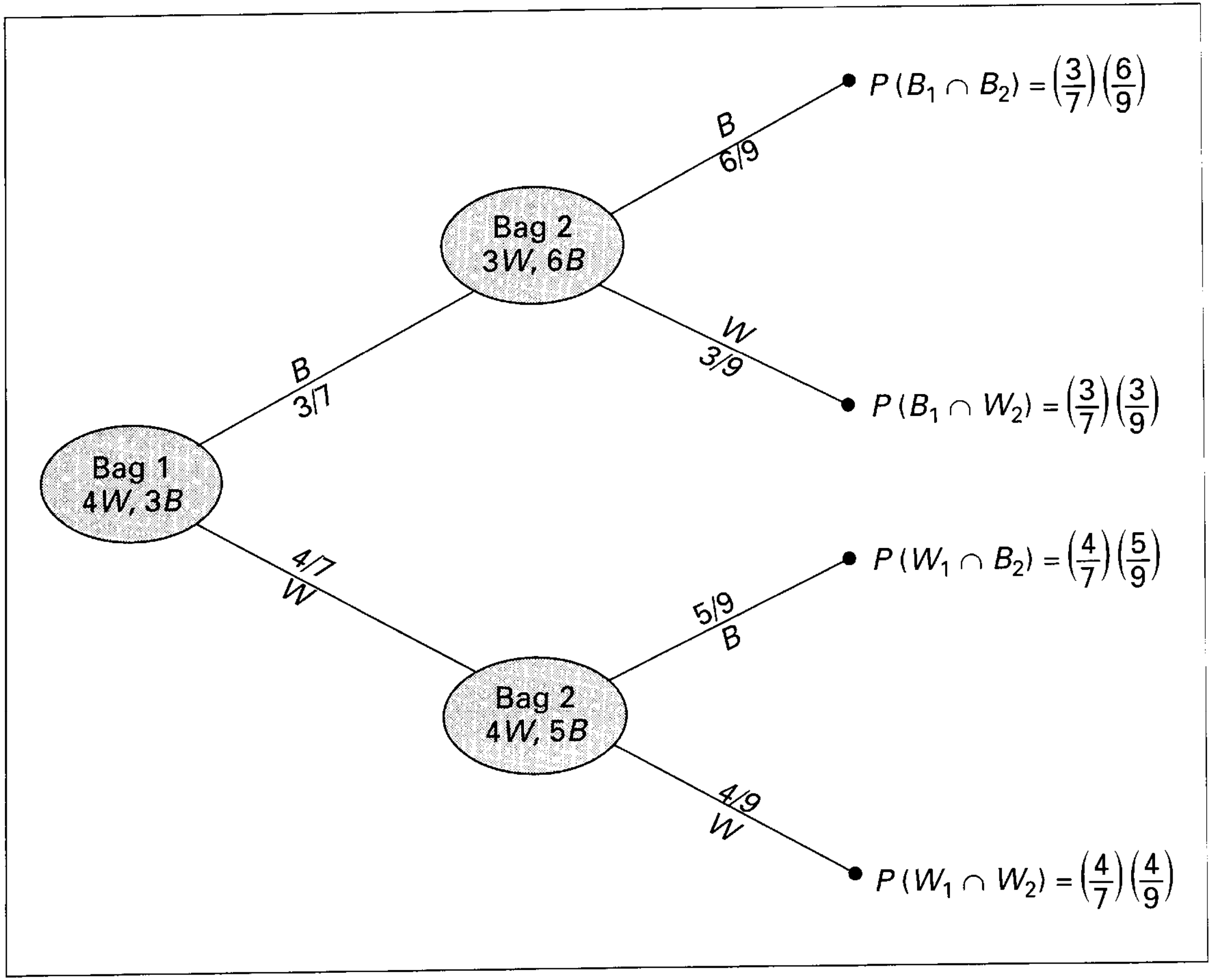

Figure 2.8 Tree diagram for Example 2.37 (2.36)

Multiplicative Rules

- Theorem 2.11: Two events A and B are <u>independent</u> if and only if $P(A \cap B) = P(A)P(B)$
- Example 2.38
  - A small town has one fire engine and one ambulance available for emergencies.
  - The probability that the fire engine is available is 0.98.
  - The probability that the ambulance is available is 0.92.
  - Find the probability that both the fire engine and the ambulance will be available when an event of an injury resulting from a burning building.
  - $P(A \cap B) = P(A)P(B) = 0.98 \times 0.92 = 0.9016$

Multiplicative Rules

- Example 2.39
  - Find the probability that
    a) the entire system works
    b) the component *C* does not work, given that the entire system works
  -

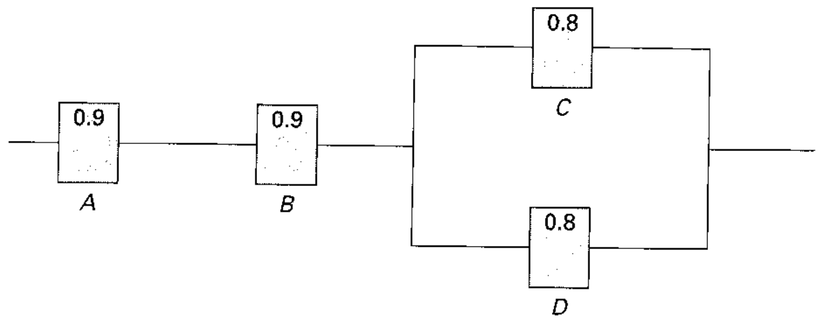

Figure 2.9 An electrical system for Example 2.35.

Multiplicative Rules

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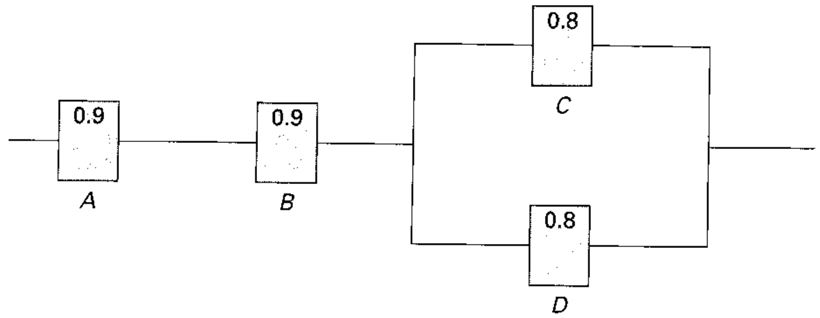

Figure 2.9 An electrical system for Example 2.35.

  - a) $P(A \cap B \cap (C \cup D)) = P(A)P(B)P(C \cup D) = P(A)P(B)(1 - P(C' \cap D'))$

$$= P(A)P(B)(1 - P(C')P(D')) = 0.9 \times 0.9 \times (1 - (1 - 0.8)(1 - 0.8)) = 0.7776$$

b) $P = \dfrac{P(\text{the system works but } C \text{ does not work})}{P(\text{the system works})}$

$$= \frac{P(A \cap B \cap C' \cap D)}{P(A \cap B \cap (C \cup D))} = \frac{0.9 \times 0.9 \times (1 - 0.8) \times 0.8}{0.7776} = 0.1667$$

Multiplicative Rules

- Theorem 2.12 : If the events A_1, A_2, A_3,..., A_k can occur, then
- $$P(A_1 \cap A_2 \cap \cdots \cap A_k)$$
  $$= P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)\cdots P(A_k \mid A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$
  - If the events A_1, A_2, A_3,..., A_k are independent, then
    $$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2)\cdots P(A_k).$$
- Example 2.40
  - Three cards are drawn in succession without replacement. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs
    - A_1 : the first card is red ace
    - A_2 : the second card is a 10 or jack
    - A_3 : the third card is greater than 3 but less than 7

Multiplicative Rules

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    - A_3 : the third card is greater than 3 but less than 7

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) = \left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right) = \frac{8}{5525}$$

Multiplicative Rules

- Example: Several Chinese characters, such as 機率統計 (微分方程, 線性代數…), are written down successively. Find the probability that the string occurs in your conversation, text book, Web,…

Multiplicative Rules

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  - $(C_1 \cap C_2 \cap C_3 \cap C_4)$
  - C_1: 機, C_2: 率, C_3 : 統, C_4 : 計
  - $$P(C_1 \cap C_2 \cap C_3 \cap C_4) = \begin{cases} P(C_1)P(C_2 \mid C_1)P(C_3 \mid C_1 \cap C_2)P(C_4 \mid C_1 \cap C_2 \cap C_3) = ? \\ P(C_1)P(C_2)P(C_3)P(C_4) = ? \end{cases}$$

Multiplicative Rules

- Definition 2.12: A collection of events $A = \{A_1, A_2, ..., A_n\}$ are mutually independent if for any subset of A $(A_i = \{A_{i1}, A_{i2}, ..., A_{ik}\})$, for $k \leq n$, we have

$$P(A_{i1} \cap A_{i2} \cap \cdots \cap A_{ik}) = P(A_{i1})P(A_{i2})\cdots P(A_{ik})$$

2.7 Bayes' Rules

- In the example of employment status (Section 2.6).
  - Give the additional information that 36 of those employed and 12 of those unemployed are members of the Rotary Club.
  - Find the probability of the event A that individual selected is a member of the Rotary Club.

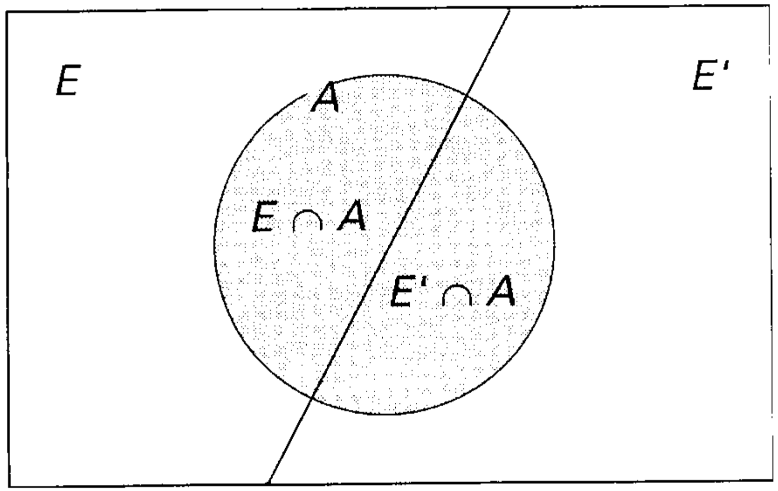

Figure 2.12 Venn diagram for the events A, E, and E'.

2.7 Bayes' Rules

– Event A is the union of the two mutually exclusive events E∩A and E'∩A. Hence,

$$A = (E \cap A) \cup (E' \cap A)$$

$$\begin{aligned} P(A) &= P[(E \cap A) \cup (E' \cap A)] \\ &= P(E \cap A) + P(E' \cap A) \\ &= P(E)P(A \mid E) + P(E')P(A \mid E') \end{aligned}$$

$$P(E) = \frac{600}{900} = \frac{2}{3}, \; P(A \mid E) = \frac{36}{600} = \frac{3}{50},$$

$$P(E') = \frac{1}{3}, \; P(A \mid E') = \frac{12}{300} = \frac{1}{25}$$

$$P(A) = \left(\frac{2}{3}\right)\left(\frac{3}{50}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{25}\right) = \frac{4}{75}$$

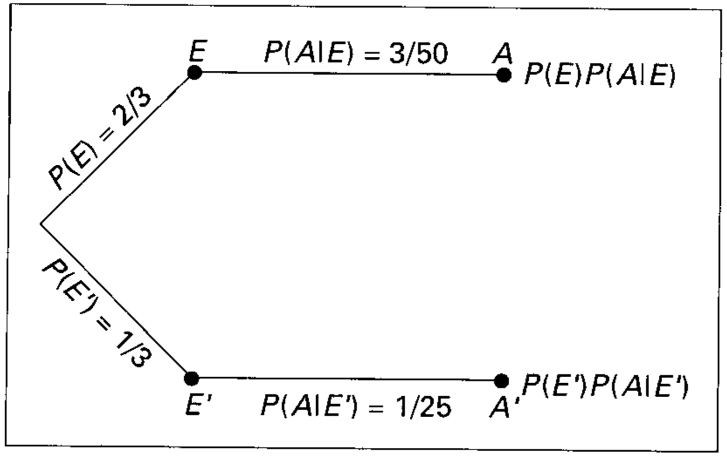

Figure 2.13 Tree diagram for the data on page 48.

Total Probability (Rule of Elimination)

- Theorem 2.13: If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then $$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i) P(A \mid B_i).$$

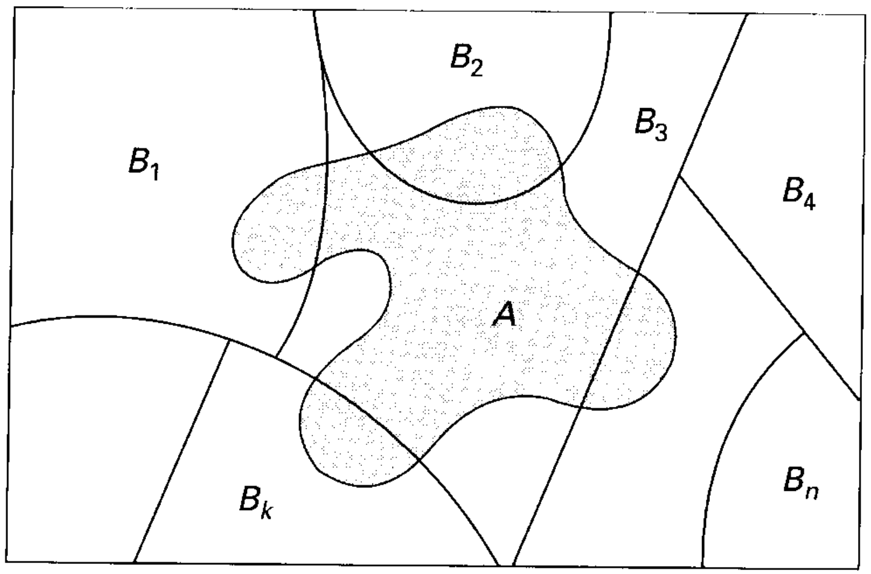

Figure 2.14 Partitioning the sample space S.

Bayes' Rules

- Example 2.41: In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%,45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?
  - $$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$$
    $$= 0.3 \times 0.02 + 0.45 \times 0.03 + 0.25 \times 0.02 = 0.0245$$

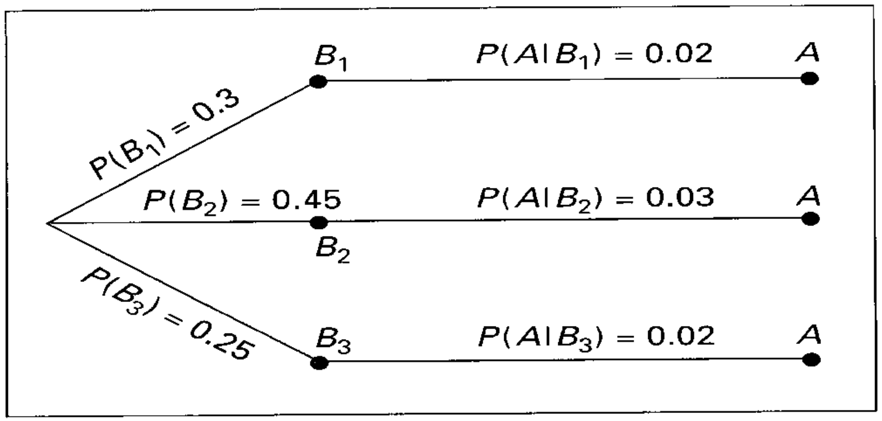

Figure 2.15 Tree diagram for Example 2.41.

Bayes' Rules

- Theorem 2.14: (Bayes's Rule) If the events B_1, B_2,..., B_k constitute a partition of the sample space S such that $P(B_i) \neq 0$ for i = 1, 2,…, k, then

$$P(B_r \mid A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A \mid B_r)}{\sum_{i=1}^{k} P(B_i)P(A \mid B_i)}.$$

for $r = 1, 2, \ldots, k$

Bayes' Rules

- Example 2.42: With reference to Example 2.41, if a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3?
  –

Bayes' Rules

- Example 2.42: With reference to Example 2.41, if a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3?
  - $$P(B_3 \mid A) = \frac{P(B_3)P(A \mid B_3)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)}.$$

$$P(B_3 \mid A) = \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{0.005}{0.0245} = \frac{10}{49}.$$

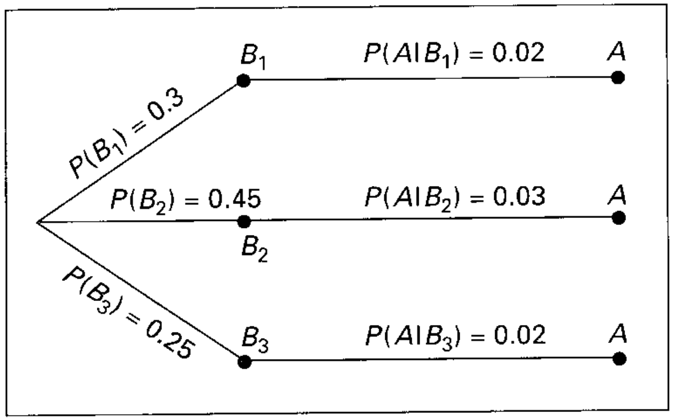

Figure 2.13 Tree diagram for Example 2.38.

Basic Formula

$$P(x) = \sum_y P(x, y) \qquad P(y) = \sum_x P(x, y)$$

$$P(x \mid y) = \sum_h P(x, h \mid y)$$

$$P(x \mid y) = \sum_h P(h \mid y) P(x \mid y, h)$$

$$P(x, h \mid y) = P(h \mid y) P(x \mid y, h)$$

$$P(x \mid y) \cong \sum_h P(h \mid y) P(x \mid h)$$

$$P(x \mid y) = \frac{P(x, y)}{P(y)} = \frac{P(x, y)}{\sum_x P(x, y)}$$

Exercise

- 2.81, 2.93, 2.100