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Theory of Computation

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Outline

1. Regular Expressions (RE)
2. Connection Between REs and Regular Languages
3. Regular Grammars

Specifying Language

How do we specify languages?

- If language is finite, you can list all of its strings.
  - $L = \{a, aa, aba, aca\}$
- Descriptive:
  - $L = \{x \mid n_a(x) = n_b(x)\}$
- Using basic Language operations
  - $L = \{aa, ab\}^* \cup \{b\}\{bb\}^*$
- Regular languages are described using the last method

Regular Expressions

Regular expressions describe regular languages and the notation involves a combination of:

- Strings of symbols from some alphabet Σ
- Parentheses ()
- Operators $+$, $\cdot$, $*$

Regular Expressions

Important thing to remember

- A regular expression is not a language
- A regular expression is used to describe a language.
- It is incorrect to say that for a language L, L = (a + b + c)*
- But it's okay to say that L is described by (a + b + c)*

Regular Expressions

All finite languages can be described by regular expressions

Example: $(a + b \cdot c)^* \Longleftrightarrow$ {{a} U {bc}}*

describes the language

$$\{a, bc\}^* = \{\lambda, a, bc, aa, abc, bca, ...\}$$

Definition 3.1

Let Σ be a given alphabet. Then

1. ϕ, λ, and $a \in \Sigma$ are all regular expressions. These are called primitive regular expressions.
2. If r_1 and r_2 are regular expressions, so are r_1+r_2, $r_1 \cdot r_2$, r_1^* and (r_1).
3. A string is a regular expression ***iff*** it can be derived from the primitive regular expressions by a finite number of applications of the rules in (2).

Example 3.1

A regular expression: $\left(a+b\cdot c\right)^{*}\cdot\left(c+\varnothing\right)$

Not a regular expression: (a + b +)

a^n

a^+

Languages of Regular Expressions

$L(r)$: language of regular expression r

Example

$$L((a+b\cdot c)^*)=\{\lambda,a,bc,aa,abc,bca,...\}$$

Definition 3.2

- For primitive regular expressions:

$$L(\varnothing) = \varnothing \qquad (1)$$

$$L(\lambda) = \{\lambda\} \qquad (2)$$

$$L(a) = \{a\} \qquad (3)$$

Definition (continued)

For regular expressions r_1 and r_2

$$L(r_1 + r_2) = L(r_1) \cup L(r_2) \qquad (4)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2) \qquad (5)$$

$$L(r_1 {}^*) = (L(r_1))^* \qquad (6)$$

$$L((r_1)) = L(r_1) \qquad (7)$$

Example 3.2

Regular expression: $(a+b)\cdot a^*$

$$
\begin{aligned}
L((a+b)\cdot a^*) &= L((a+b))\,L(a^*) \\
&= L(a+b)\,L(a^*) \\
&= (L(a)\cup L(b))\,(L(a))^* \\
&= (\{a\}\cup\{b\})\,(\{a\})^* \\
&= \{a,b\}\,\{\lambda,a,aa,aaa,\ldots\} \\
&= \{a,aa,aaa,\ldots,b,ba,baa,\ldots\}
\end{aligned}
$$

Priority of Operators

- Regular expression: $r = a \cdot b + c$

  $r_1 = a \cdot b \quad r_2 = c$ or $r_1 = a \quad r_2 = b + c$

  $L(r) = \{ab, c\} \neq \{ab, ac\}$

- Star closure (*) precedes
  concatenation (·) precedes
  union (+)

Example 3.3

$\Sigma=\{a,b\}$

- Regular expression $r = \underbrace{(a+b)^*}(a+bb)$

Stands for any string of a's and b's

$$L(r) = \{a, bb, aa, abb, ba, bbb, \ldots\}$$

L(r) is the set of all strings on {a,b}, terminated by either an a or a bb

Example 3.4

- Regular expression $r = (aa)^*(bb)^*b$

$$L(r) = \{a^{2n} b^{2m+1} : \quad n, m \geq 0\}$$

L(r) is the set of all strings with an even number of a's followed by an odd number of b's

Example 3.5

- For $\Sigma = \{0, 1\}$, give a regular expression r such that

$L(r) = \{ w \in \Sigma^* :$ w has at least one pair of consecutive 0 $\}$

$$00$$

Example 3.6

L(r) = { all strings with no pairs of consecutive 0s }

- Regular expression $\boxed{r = (1 + 01)^*(0 + \lambda)}$

$$r = \underbrace{(1^*011^*)^*}_{\text{Add 1 immediately after a 0}} \underbrace{}_{\text{String ending in 0}} \underbrace{}_{\text{String with all 1's}}$$

There are an unlimited number of REs for any given language!

Equivalent Regular Expressions

Definition:

Regular expressions r_1 and r_2

are **equivalent** if $L(r_1) = L(r_2)$

Example

L = { all strings without two consecutive 0 }

$$r_1 = (1 + 01)^*(0 + \lambda)$$

$$r_2 = (1^*011^*)^*(0 + \lambda) + 1^*(0 + \lambda)$$

$L(r_1) = L(r_2) = L$ $\Rightarrow$ r_1 and r_2 are equivalent regular expressions

More Examples

- L_1 = {a, aa, aba, aca}
- L_1 = {a} ∪ {aa} ∪ {aba} ∪ {aca}
- Regular expression describing L_1:

  (a + aa + aba + aca)

More Examples

- $L_2 = \{x \in \{0,1\}^* \mid |x| \text{ is even}\}$
- $L_2 = \{00, 01, 10, 11\}^*$
- Regular expressions describing L_2:

$(00 + 01 + 10 + 11)^*$

$((0 + 1)(0 + 1))^*$

More Examples

- $L_3 = \{x \in \{0,1\}^* \mid x \text{ does not end in } 01\ \}$

  If x does not end in 01, then either

  x ends in 00, 10, or 11

- A regular expression that describes L_3 is:

  $$(0 + 1)^*(00 + 10 + 11)$$

More Examples

- $L_4 = \{x \in \{0,1\}^* \mid x$ contains an odd number of 0s $\}$

  Express $x = yz$

  y is a string of the form $y=1^i01^j$

  In z, there must be an even number of 0's

  $z = (01^k01^m)^*$
- A regular expression that describes L_4 is:

  $(1^*01^*)(01^*01^*)^*$

Short Quiz

- Give regular expressions for the following language on $\Sigma = \{a, b, c\}$.
  - All strings containing exactly one a

$$r = (b+c)^*a(b+c)^*$$

Outline

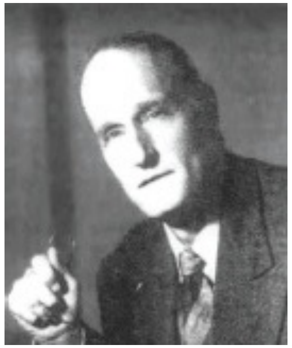
1. Regular Expressions (RE)
2. Connection Between REs and Regular Languages
3. Regular Grammars

Theorem

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Described by} \\ \text{Regular Expressions} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

For every regular language there is a regular expression
For every regular expression there is a regular language

Kleene Theorem:
Regular expressions and Finite Automata are equivalent (w.r.t. the languages they describe/accept)



Theorem - Part 1

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Described by}\\ \text{Regular Expressions}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

1. For any regular expression r the language $L(r)$ is regular

■ Theorem 3.1

Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Described by}\\ \text{Regular Expressions}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

2. For any regular language L there is a regular expression r with $L(r) = L$

■ Theorem 3.2

Proof - Part 1 $\left\{ \begin{array}{l} \text{Languages} \\ \text{Described by} \\ \text{Regular Expressions} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$

1. For any regular expression r
   the language $L(r)$ is regular

⇩

If we have any regular expression r,
we can construct an NFA(DFA) that accepts L(r)

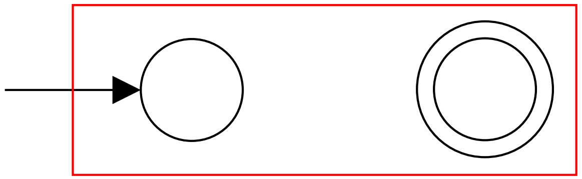
⇩

Proof by induction on the size of r

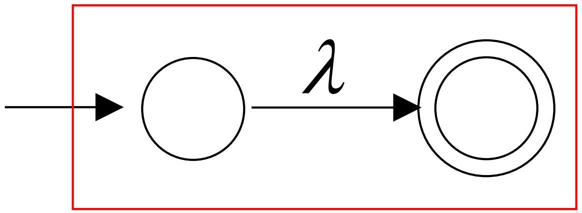
Induction Basis

- Primitive Regular Expressions: $\emptyset, \quad \lambda, \quad \alpha$

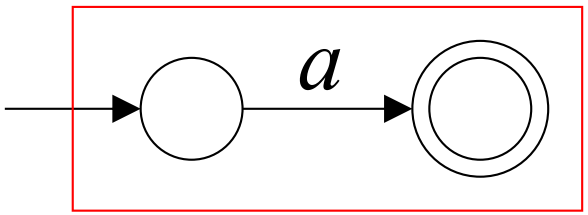
NFAs



$$L(M_1) = \emptyset = L(\emptyset)$$



$$L(M_2) = \{\lambda\} = L(\lambda)$$



$$L(M_3) = \{a\} = L(a)$$

regular languages

Inductive Hypothesis

Assume

for regular expressions r_1 and r_2

that

$L(r_1)$ and $L(r_2)$ are regular languages

Inductive Step

∵ REs are derived from these four rules:

We will prove:

$$\left.\begin{array}{l} L(r_1 + r_2) \\ \\ L(r_1 \cdot r_2) \\ \\ L(r_1 *) \\ \\ L((r_1)) \end{array}\right\}$$

Are regular Languages

- By definition of regular expressions:

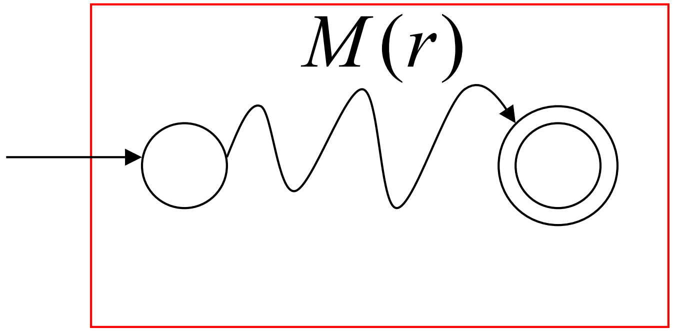
$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

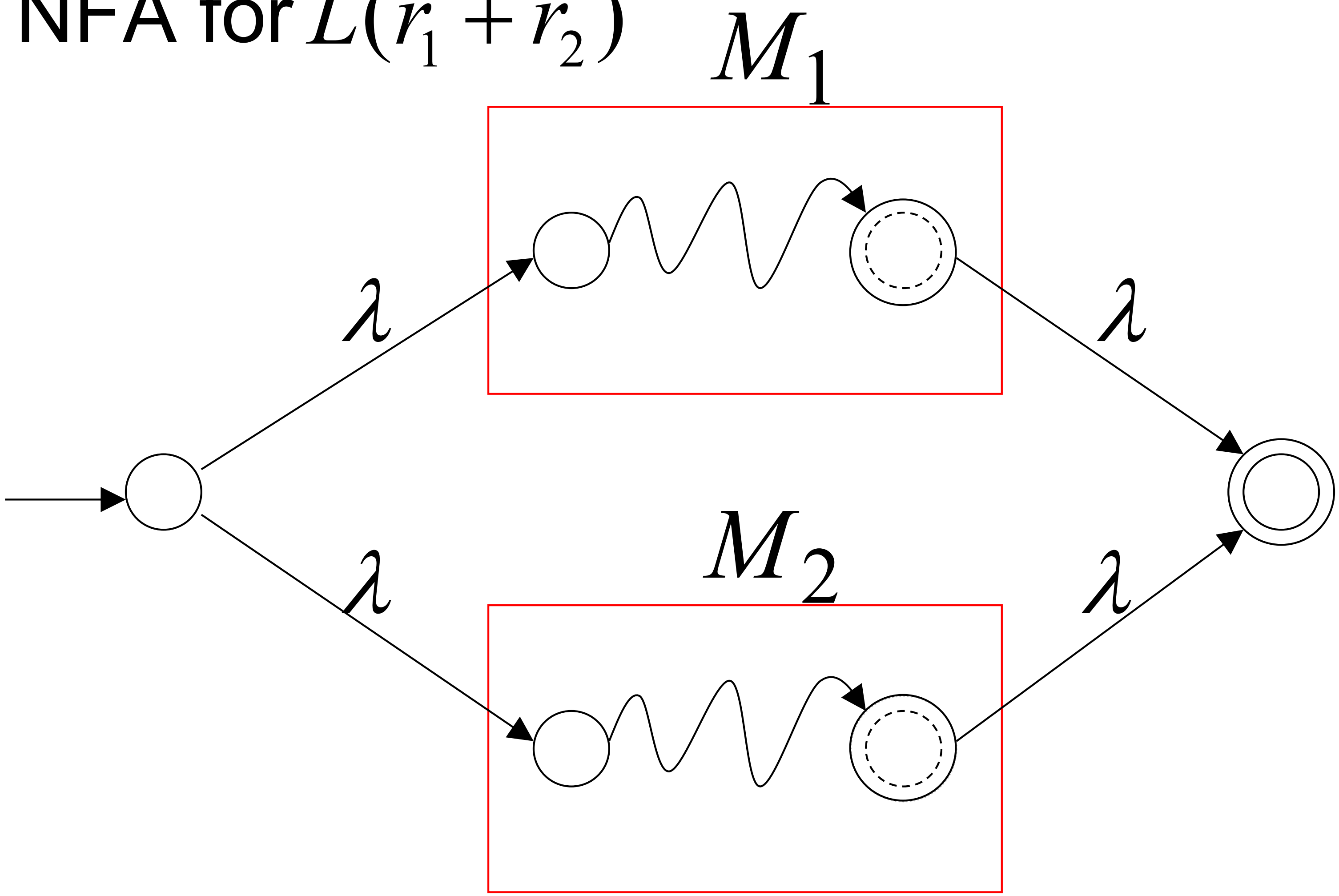
Schematic representation of an NFA (M(r)) accepting L(r)



We can claim that for every NFA there is only one final state (by exercise 7, section 2.3)

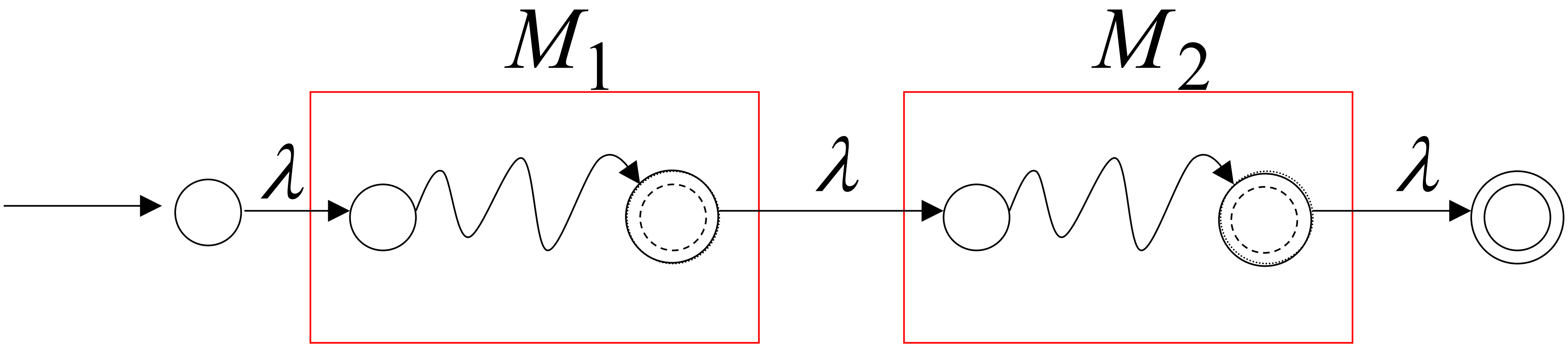
Union

- NFA for $L(r_1 + r_2)$



Concatenation

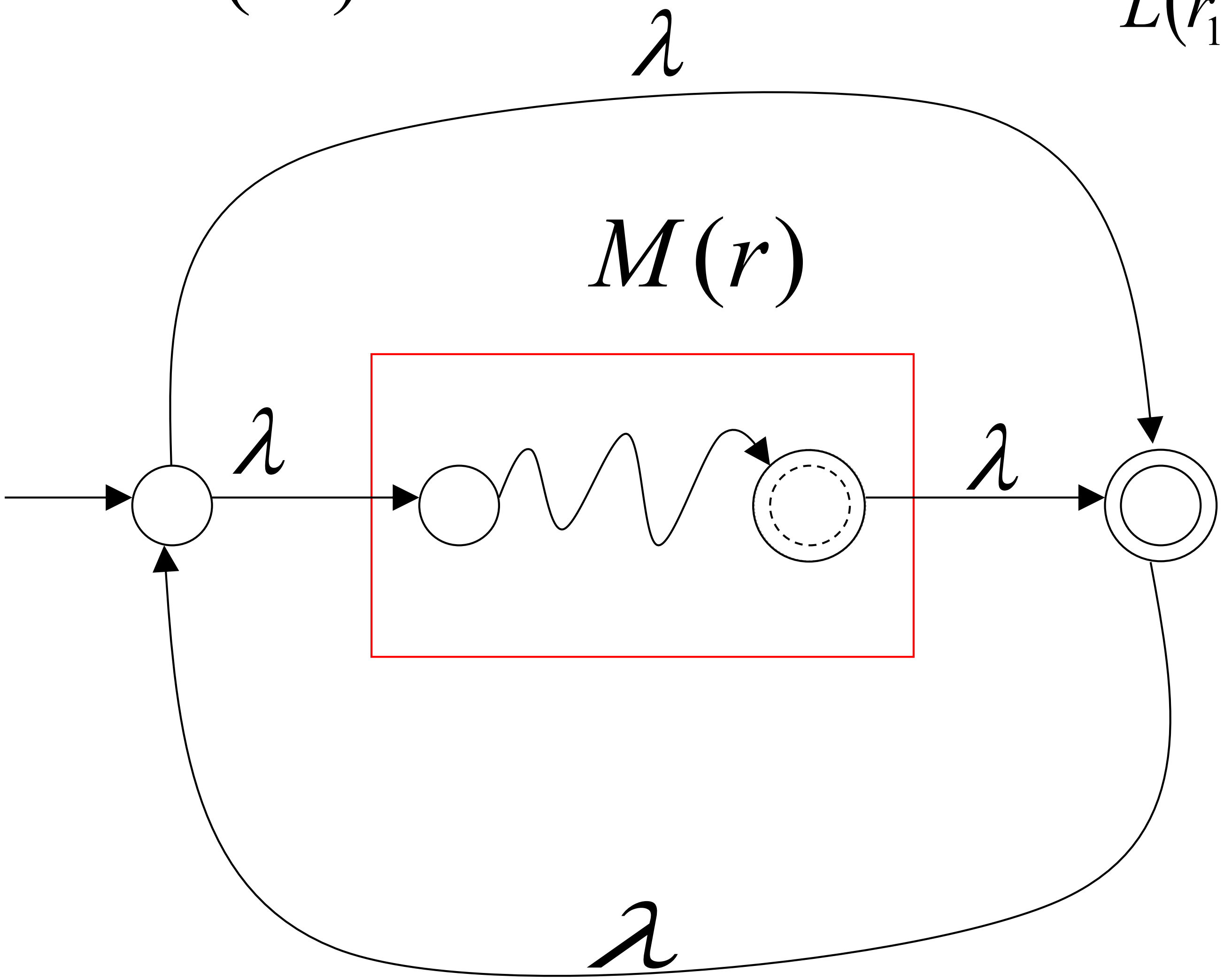
- NFA for $L(r_1 r_2)$



Star Operation

- NFA for $L(r^*)$

$$L(r_1^*) = (L(r_1))^*$$



By inductive hypothesis we know:

$L(r_1)$ and $L(r_2)$ are regular languages

We also know:

Regular languages are closed under:

Union	$L(r_1) \cup L(r_2)$
Concatenation	$L(r_1)\,L(r_2)$
Star	$(L(r_1))^*$

Therefore:

$$\left.\begin{aligned} L(r_1 + r_2) &= L(r_1) \cup L(r_2) \\[2em] L(r_1 \cdot r_2) &= L(r_1)\, L(r_2) \\[2em] L(r_1 *) &= (L(r_1))* \end{aligned}\right\} \text{Are regular languages}$$

And trivially:

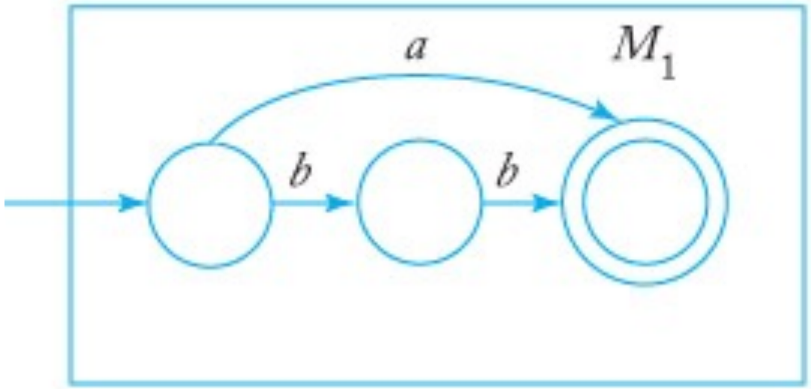
$L((r_1))$ is a regular language

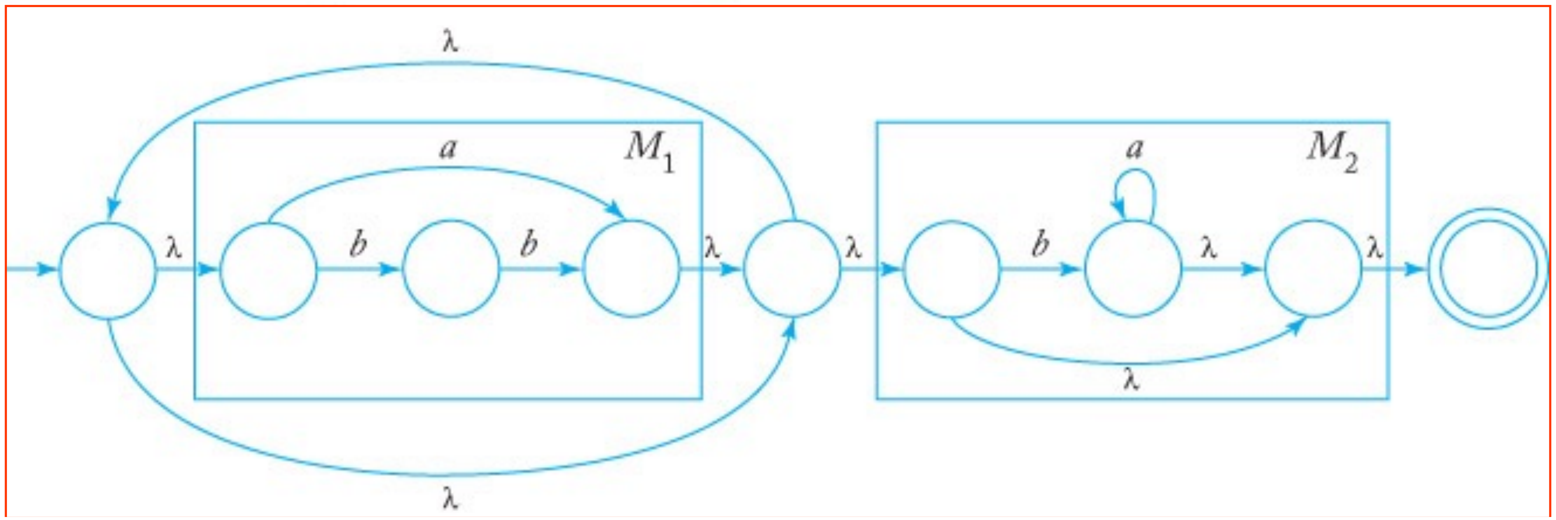
$\therefore$ For any regular expression r
the language $L(r)$ is regular

Example 3.7

- Find an NFA that accepts L(r), where

$$r = (a + bb)^*(ba^* + \lambda)$$





Proof – Part 2 $\left\{ \begin{array}{l} \text{Languages} \\ \text{Described by} \\ \text{Regular Expressions} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$

2. For any regular language L there is a regular expression r with $L(r) = L$

Since any regular language has an associated NFA and hence a transition graph, all we need to do is to find a regular expression capable of generating the labels of all the walks from q_0 to any final state.

Proof by construction of regular expression

Generalized Transition Graphs (GTG)

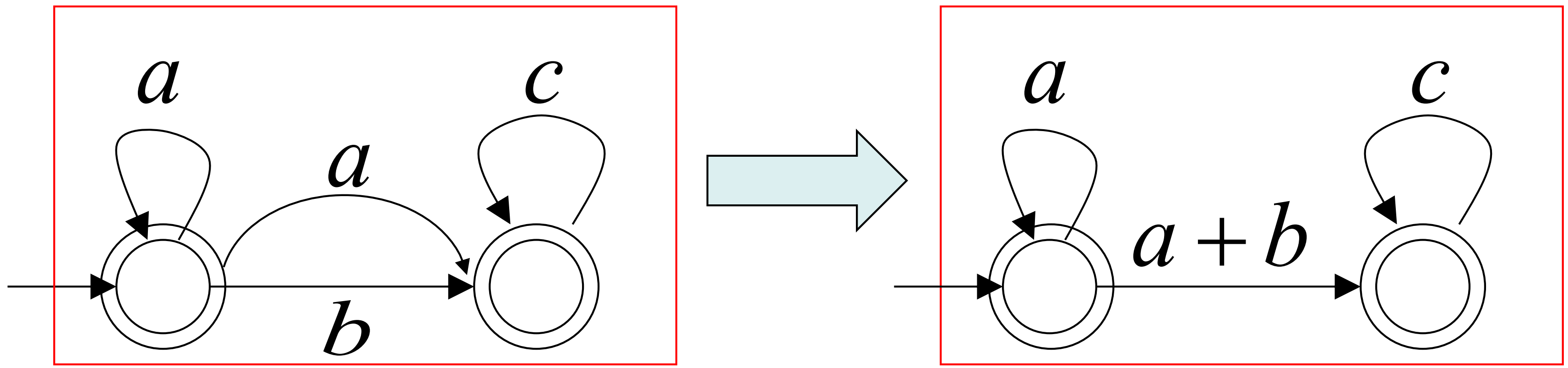
From M construct the equivalent

Generalized Transition Graph

in which transition labels are regular expressions

Example: M

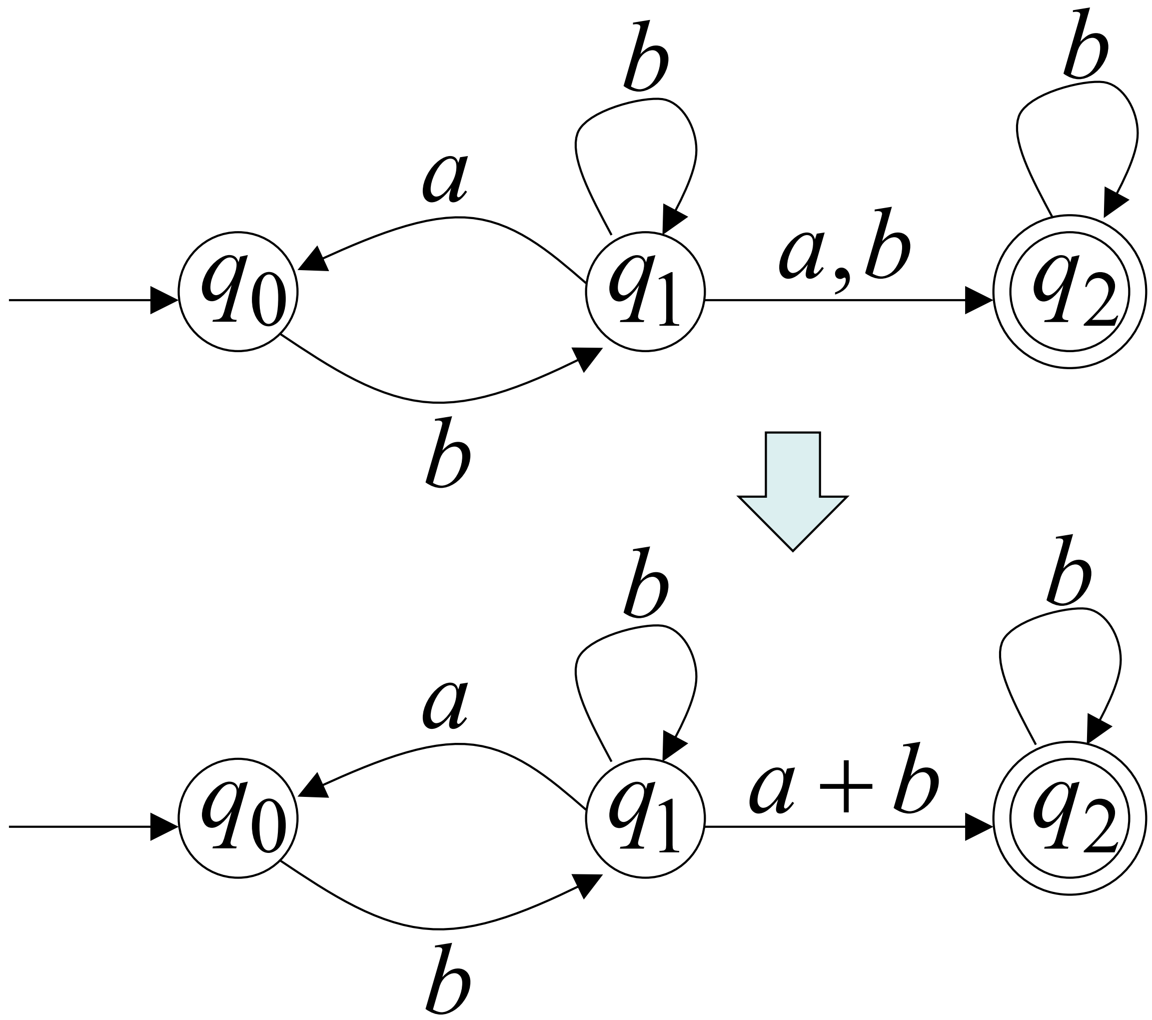
L(a* + a*(a + b)c*)



Example 3.8

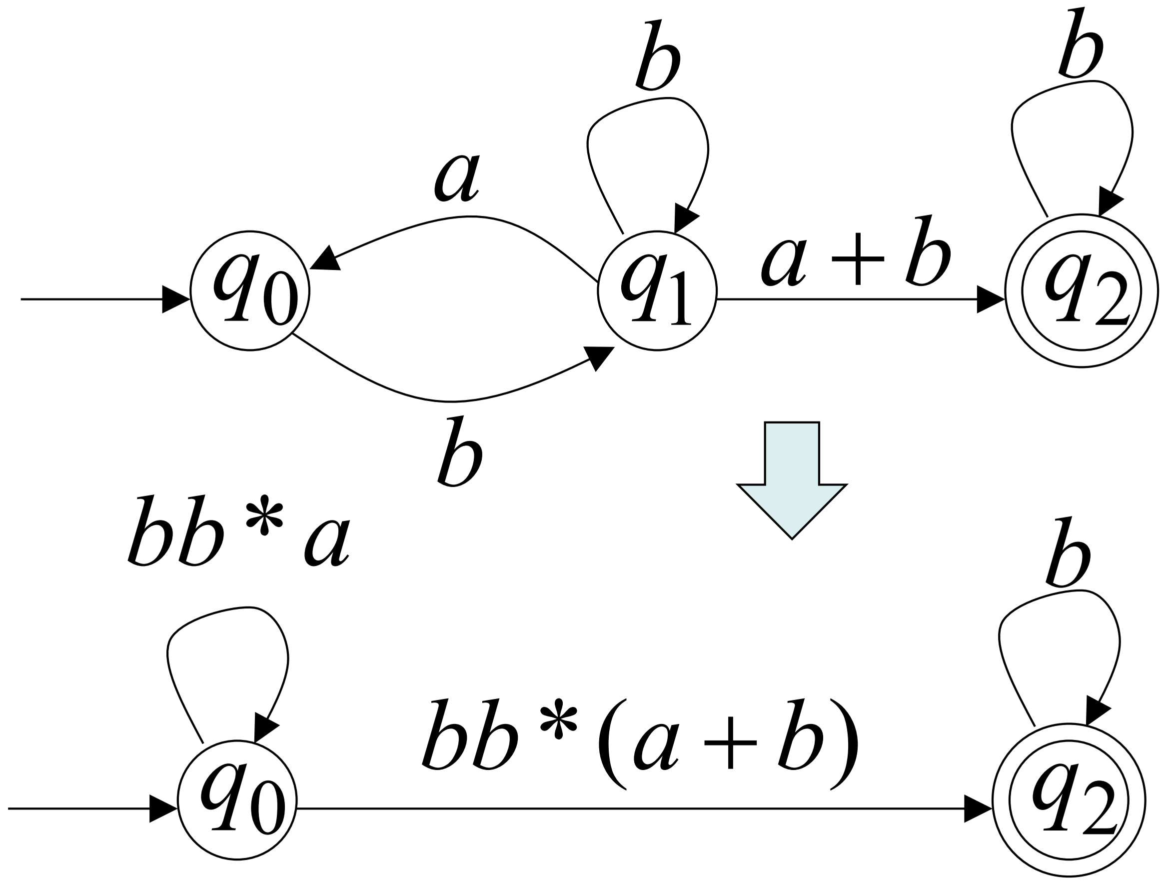
GTG may have many states

Enumerating all walks is time-consuming



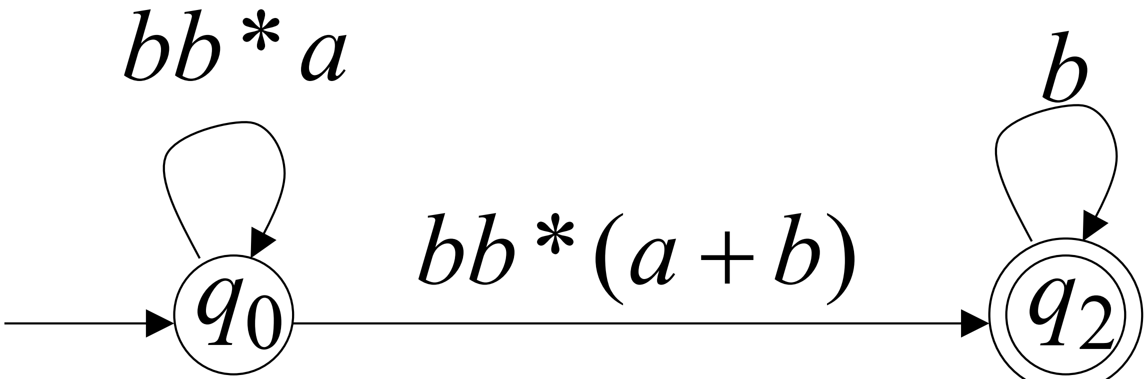
Reducing the states:

Ex. reduce q_1



Simple two-state GTG

Resulting Regular Expression:

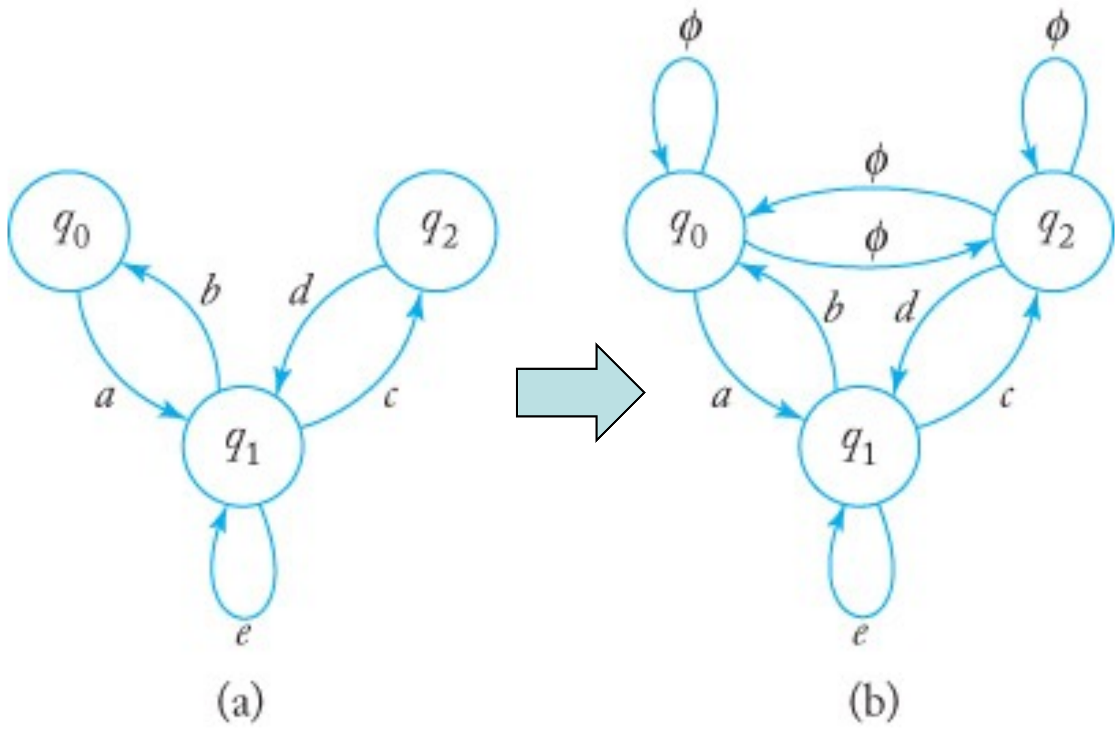


$$r = (bb^*a)^*bb^*(a+b)b^*$$

$$L(r) = L(M) = L$$

Complete GTG

GTG



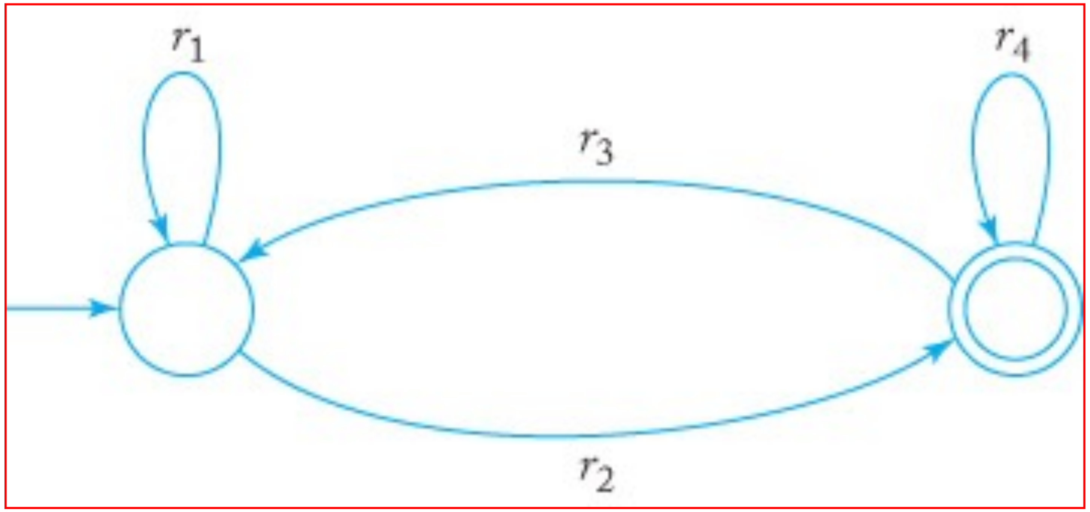
(a)

(b)

Complete GTG

- If a GTG, after conversion from an NFA, has some edges missing, we put them in and label them with ϕ
- A complete GTG with |V| vertices has exactly $|V|^2$ edges

Example 3.9



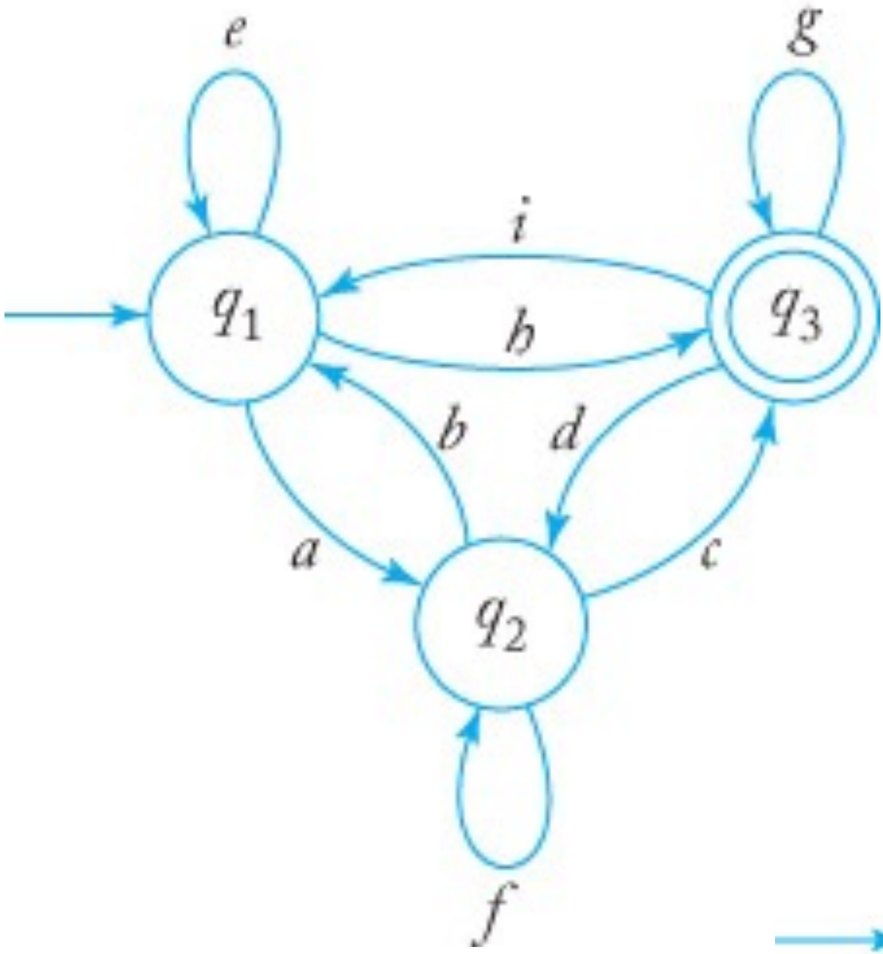
RE?

$r = r_1^* r_2 (r_4 + r_3 r_1^* r_2)^*$

How about a GTG with more than two states?

We can find an equivalent graph by removing one state at a time

Example 3.10



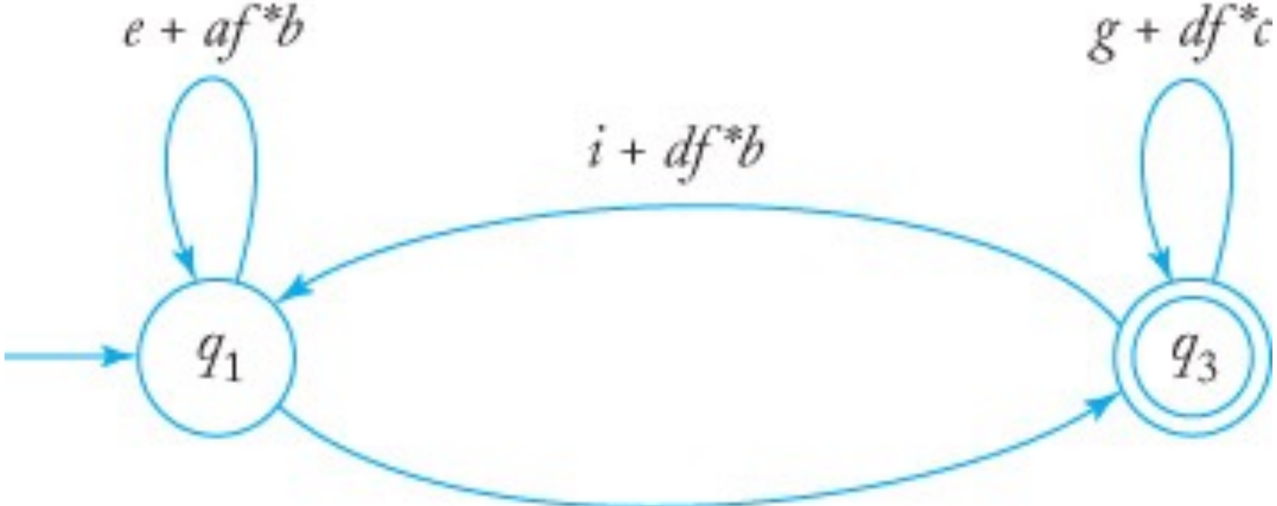
To remove q_2, we create edges as follows:

$\overrightarrow{q_1q_1}$ →

$\overrightarrow{q_1q_3}$ →

$\overrightarrow{q_3q_1}$ →

$\overrightarrow{q_3q_3}$ →



$r = r_1{}^*r_2(r_4 + r_3r_1{}^*r_2)^*$

$$\boxed{(e + af^*b)^*(h + af^*c)((g + df^*c) + (i + df^*b)(e + af^*b)^*(h + af^*c))^*}$$

NFA → RE

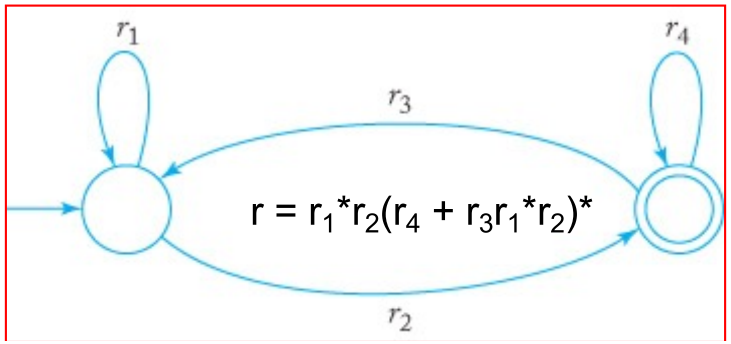
1. Convert the NFA (with single final state) into a complete GTG. Let r_{ij} stand for the label of the edge from q_i to q_j.
2. If the GTG has only two states with $q_i \in q_0$ and $q_j \in F$, as its associated RE is:

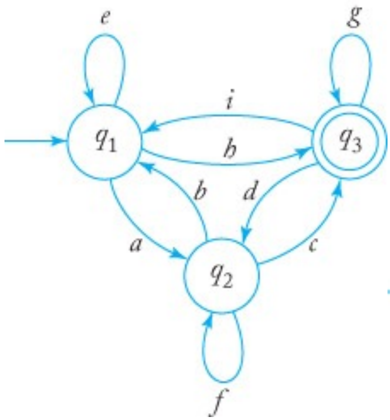
$$r = r_{ii}^* r_{ij} (r_{jj} + r_{ji} r_{ii}^* r_{ij})^*$$

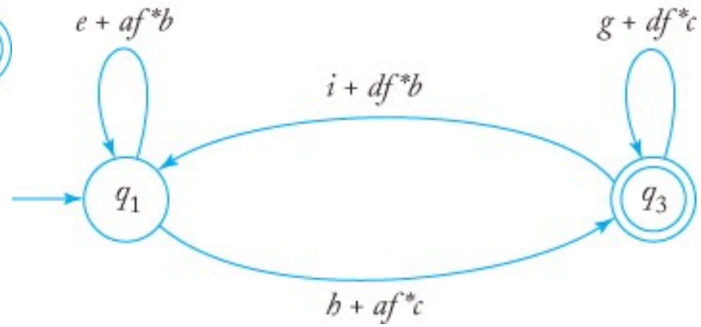
3. If the GTG has three states with $q_i \in q_0$, $q_j \in F$, and $q_k \in Q$, introduce new edges, labeled:

$$r_{pq} + r_{pk} r_{kk}^* r_{kq}$$

for p = i, j, q = i, j. When this is done, remove vertex q_k and its associated edges.







NFA → RE

4. If the GTG has four or more states, pick a state q_k to be removed. Apply rule 3 for all pairs of states (q_i, q_j), $i \neq k$, $j \neq k$. At each step apply the simplifying rules

$$r + \phi = r, \quad r \cdot \phi = \phi, \quad \phi^* = \lambda$$

   wherever possible. When this is done, remove state q_k.

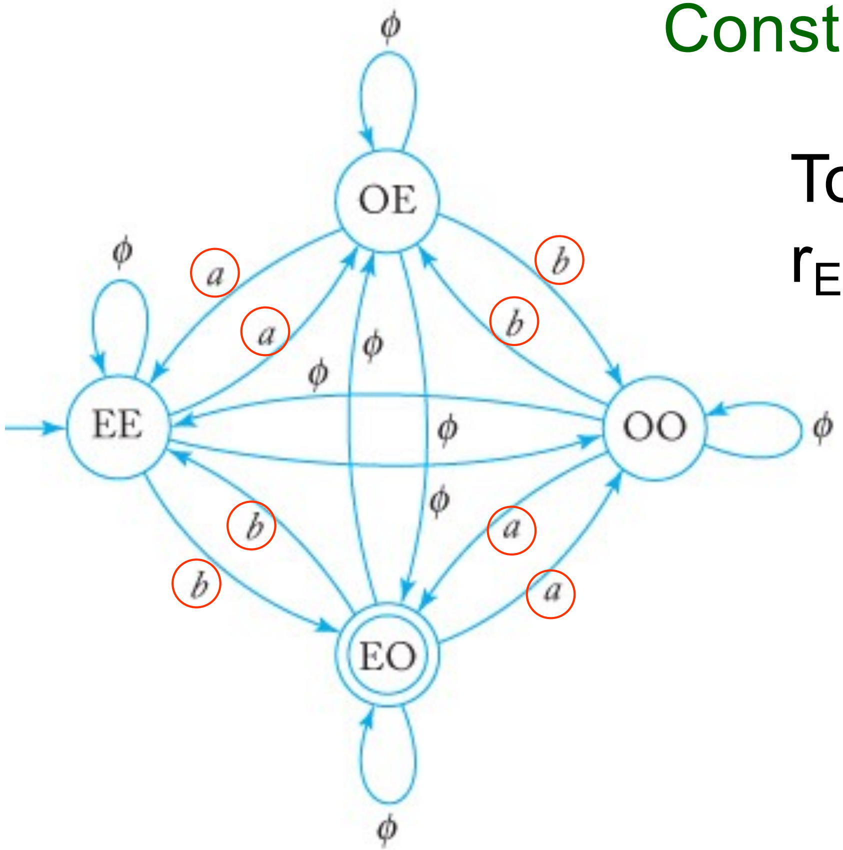
5. Repeat step 2 to 4 until the correct RE is obtained.

Example 3.11

- Find a RE for the language

$$L = \{w \in \{a,b\}^* : n_a(w) \text{ is even and } n_b(w) \text{ is odd}\}.$$

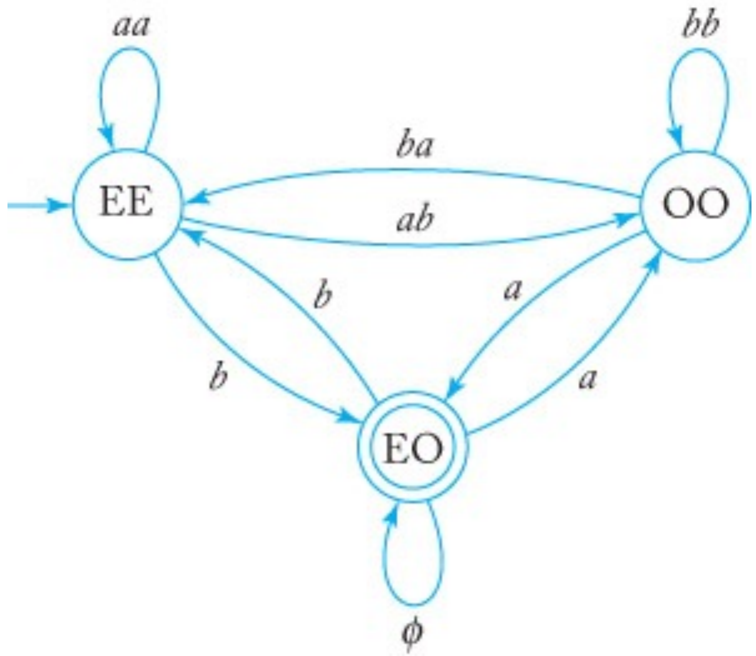
Construct NFA first!



To remove OE

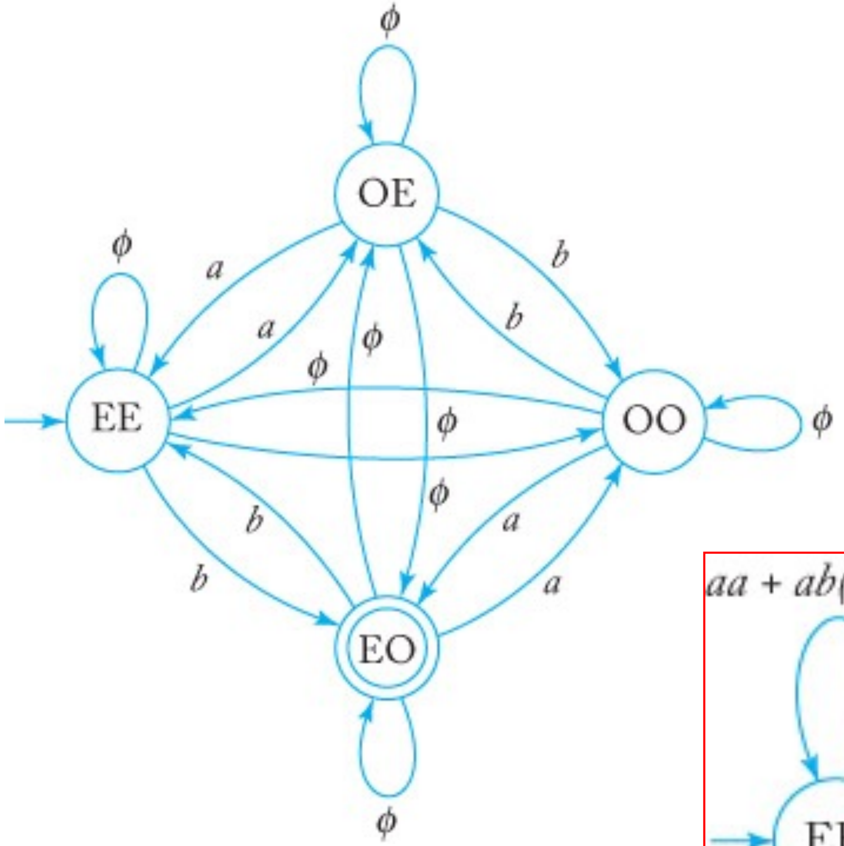
$r_{EE} = \phi + a\,\phi^* a$

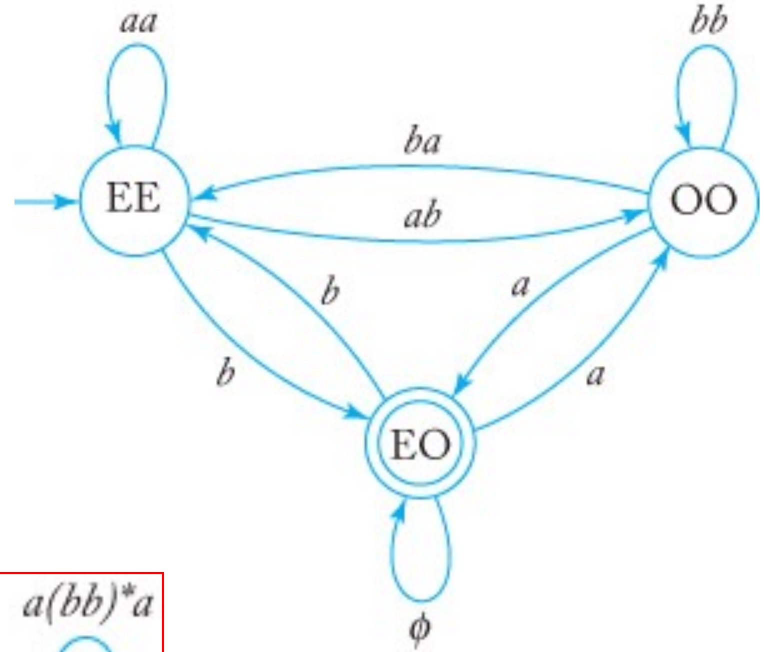
$= aa$

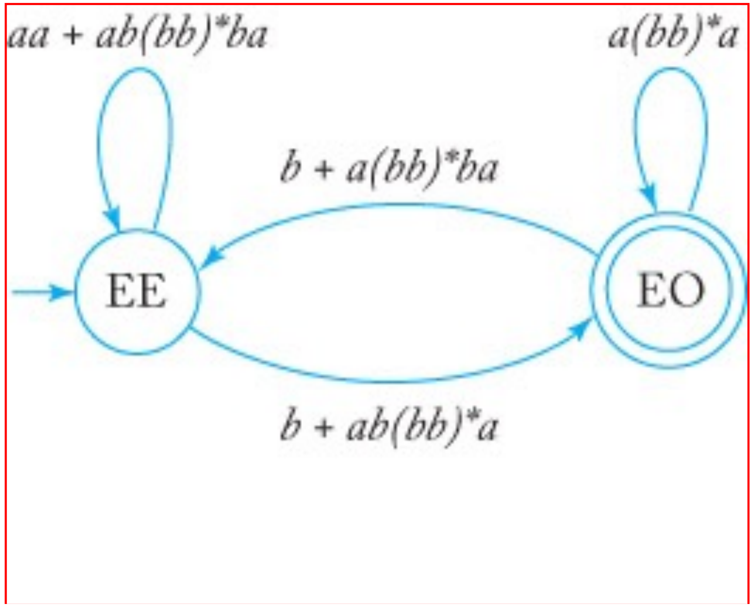


Example 3.11

$$L = \{w \in \{a,b\}^* : n_a(w) \text{ is even and } n_b(w) \text{ is odd}\}.$$







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3. Regular Grammars

Grammar Recap

- A grammar G is defined as a 4-tuple:

$$G = (V, T, S, P)$$

where

- V is a finite set of variables
- T is a finite set of terminals
- $S \in V$, called start variable
- P is a finite set of production rules

Grammar Recap

- Let $G = (V, T, S, P)$ be a grammar. Then the set

$$L(G) = \{w \in T^*: S \overset{*}{\Rightarrow} w\}$$

  is the language generated by G

- If $w \in L(G)$, then the sequence

$$S \Rightarrow w_1 \Rightarrow w_2 \Rightarrow \ldots \Rightarrow w_n \Rightarrow w$$

  is a derivation of the sentence w.
- $S, w_1, w_2, \ldots, w_n$ are called sentential forms

Linear Grammars

Grammars with

at most one variable at the right side

of a production

Examples:

$$S \to aSb$$
$$S \to \lambda$$

$$S \to Ab$$
$$A \to aAb$$
$$A \to \lambda$$

Another Linear Grammar

Grammar G :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

A Non-Linear Grammar

Grammar G:

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

$$L(G) = \{w: \quad n_a(w) = n_b(w)\}$$

Number of a in string w

Right-Linear Grammars

- All productions have form: $A \rightarrow xB$

or

$A \rightarrow x$

string of terminals

- Example: $S \rightarrow abS$

$S \rightarrow a$

Left-Linear Grammars

- All productions have form: $A \to Bx$

  or

  $A \to x$

  string of terminals

- Example: $S \to Aab$

  $A \to Aab \mid B$

  $B \to a$

Regular Grammars

A regular grammar is either right-linear or left-linear grammar

Examples:

G_1 ○

$$S \to abS$$

$$S \to a$$

G_2 ○

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

G_3 ✗

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

Observation

Regular grammars generate regular languages

A regular grammar is always linear, but
not all linear grammars are regular.

G_3 is linear grammar
but not regular grammar

$$G_3$$

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

Example 3.13

Regular grammars generate regular languages

G_1

$S \to abS$

$S \to a$

G_2

$S \to Aab$

$A \to Aab \mid B$

$B \to a$

$L(G_1) = (ab)^* a$

$L(G_2) = aab(ab)^*$

Theorem

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Theorem - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular grammar generates
a regular language

- Theorem 3.3

Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Any regular language is generated by a regular grammar

- Theorem 3.4

Proof – Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

The language $L(G)$ generated by any regular grammar G is regular

The case of Right-Linear Grammars

Let G be a right-linear grammar

We will prove: $L(G)$ is regular

Proof idea: We will construct NFA M with $L(M) = L(G)$

- Grammar G is right-linear

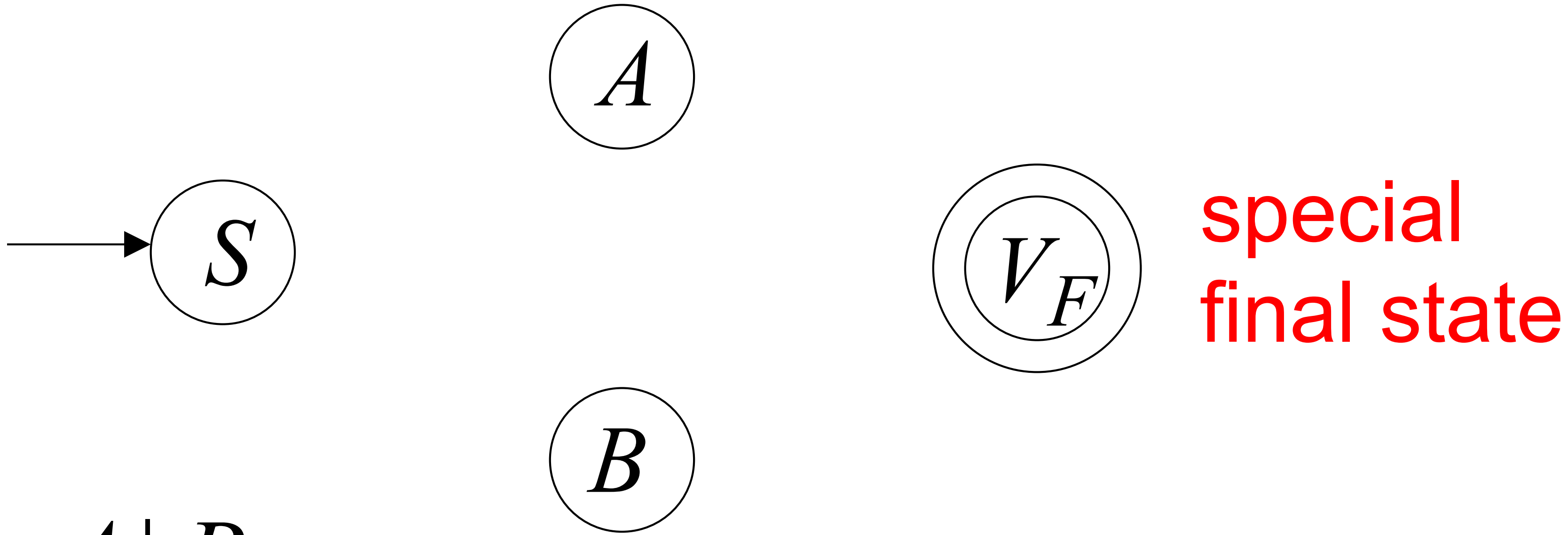
Example:

$$S \to aA \mid B$$

$$A \to aa\, B$$

$$B \to b\, B \mid a$$

Construct NFA M such that every state is a grammar variable:

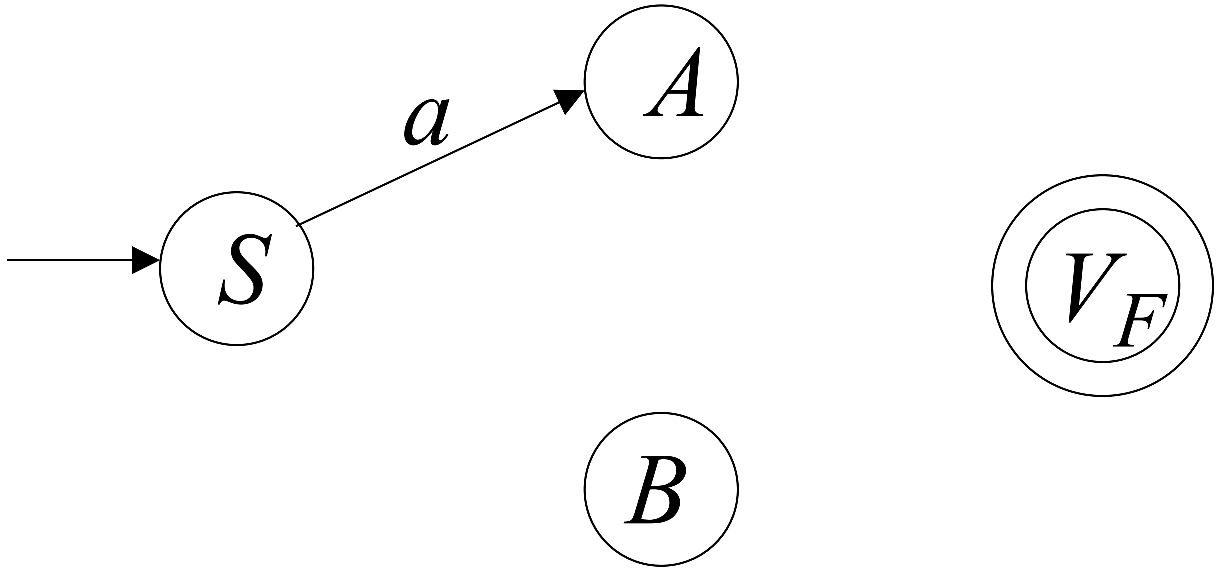


$$S \to aA \mid B$$

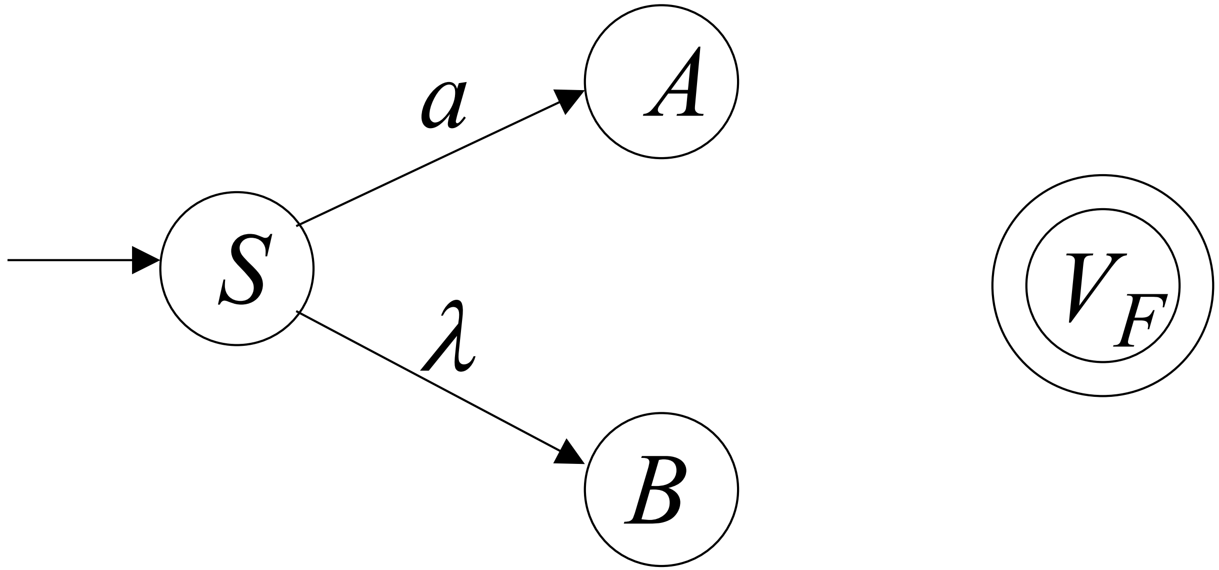
$$A \to aa\,B$$

$$B \to b\,B \mid a$$

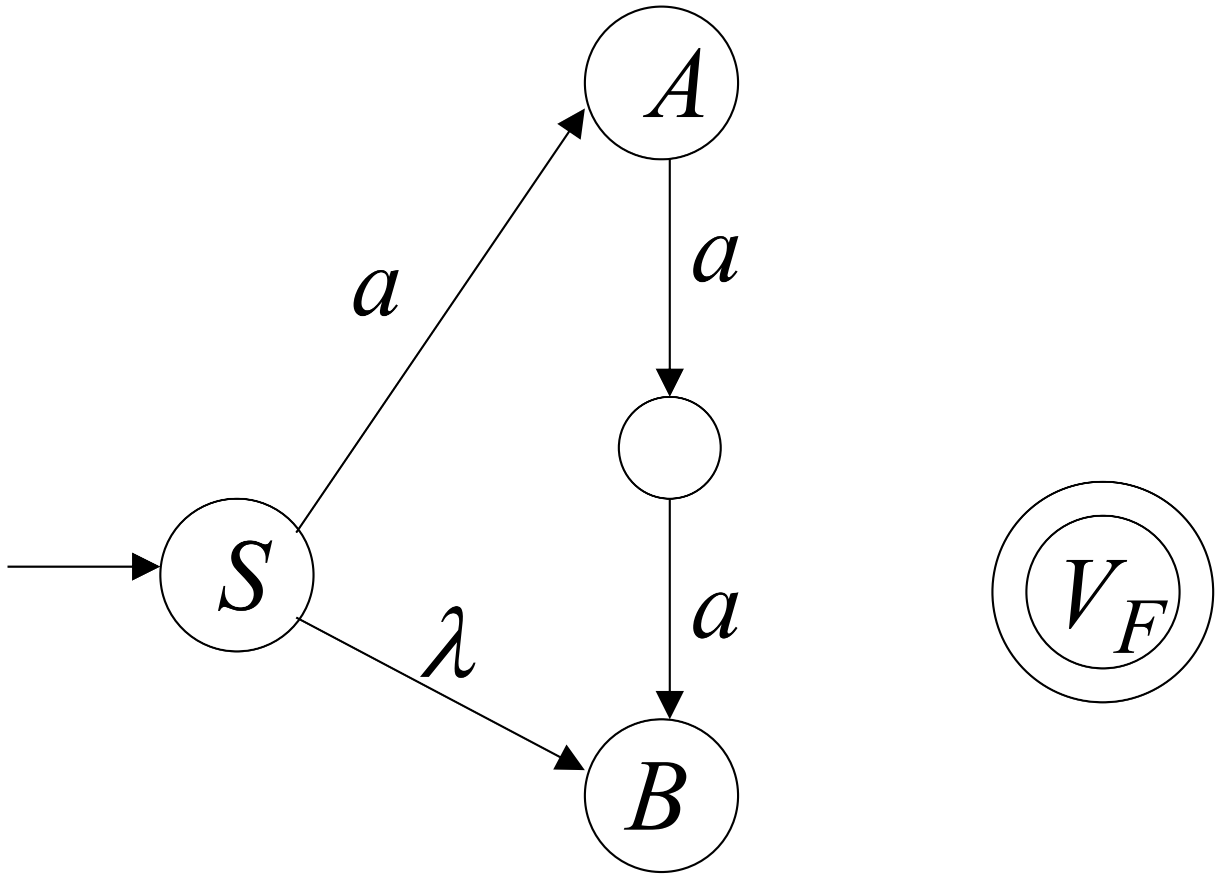
- Add edges for each production:



$$S \rightarrow aA$$

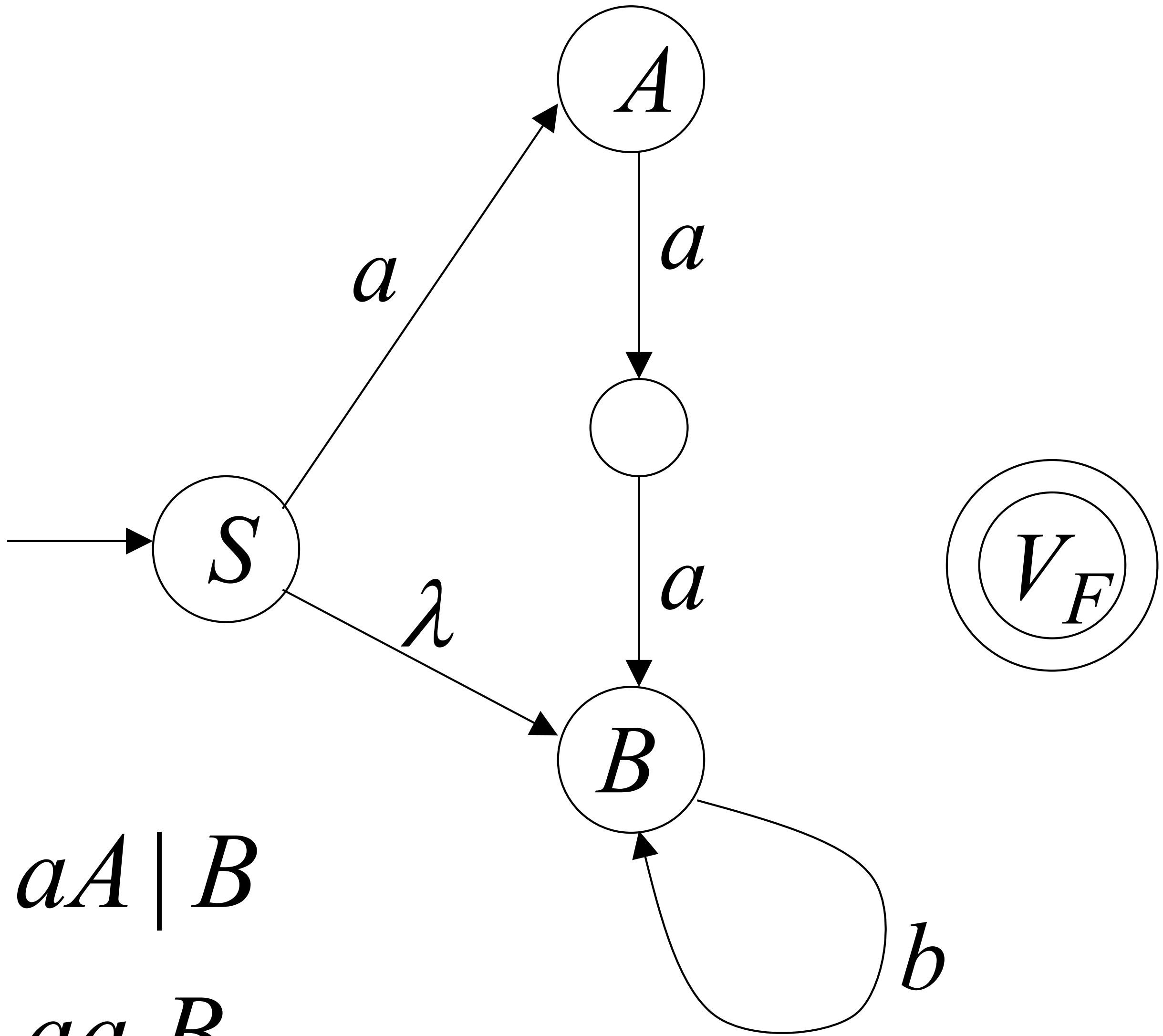


$$S \to aA \mid B$$



$$S \to aA \mid B$$

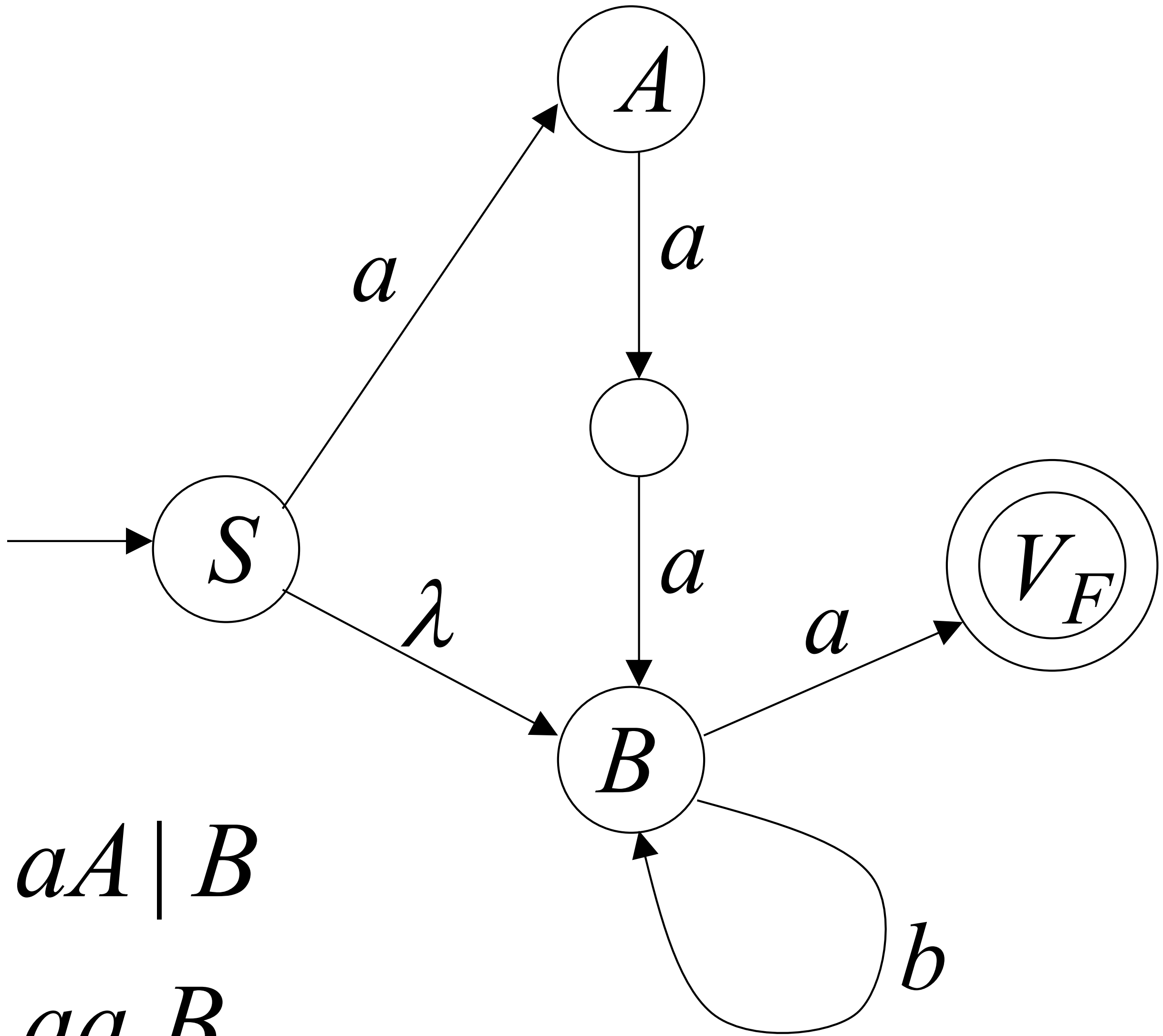
$$A \to aa\ B$$



$$S \to aA \mid B$$

$$A \to aa\ B$$

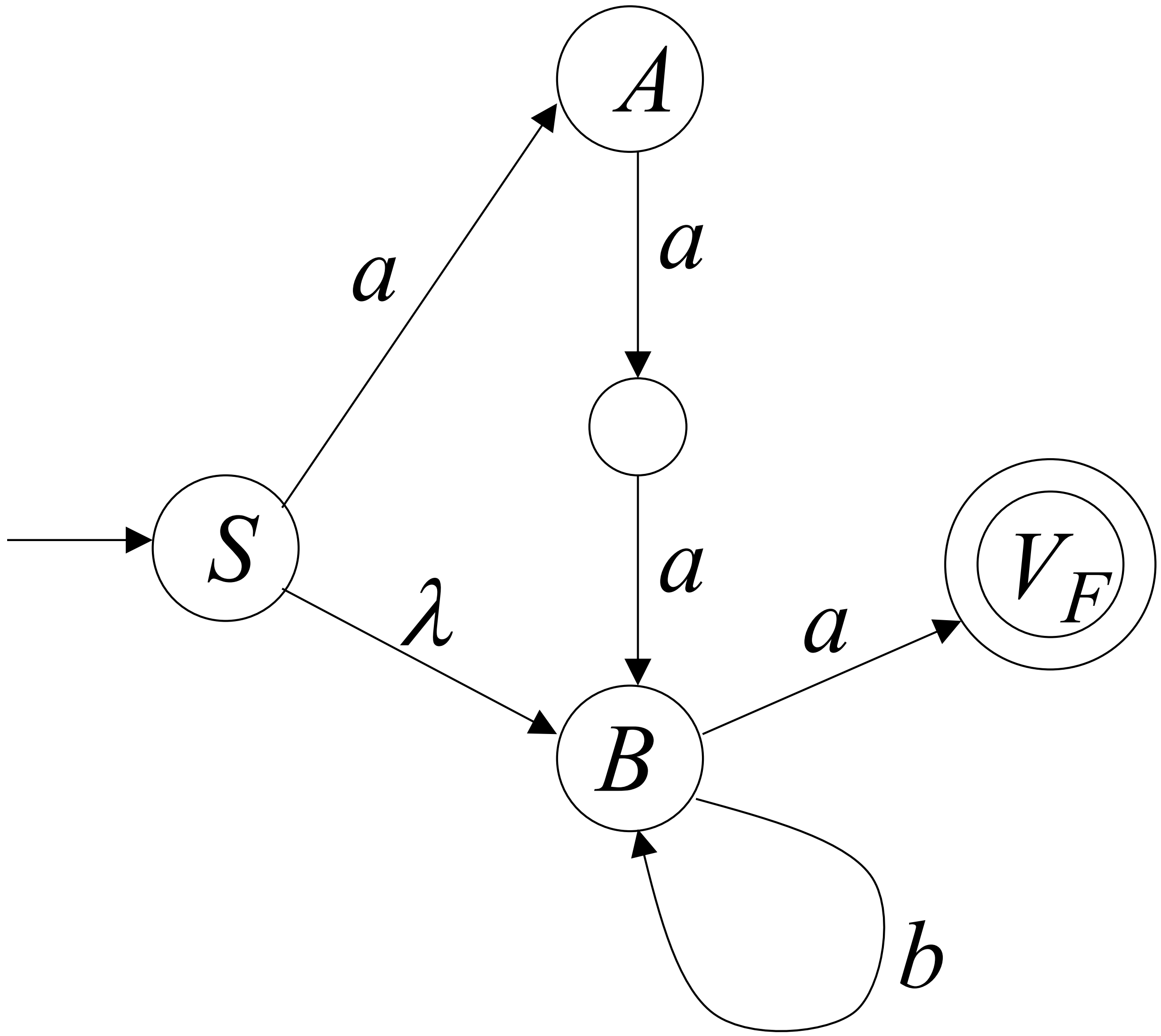
$$B \to bB$$



$$S \to aA \mid B$$

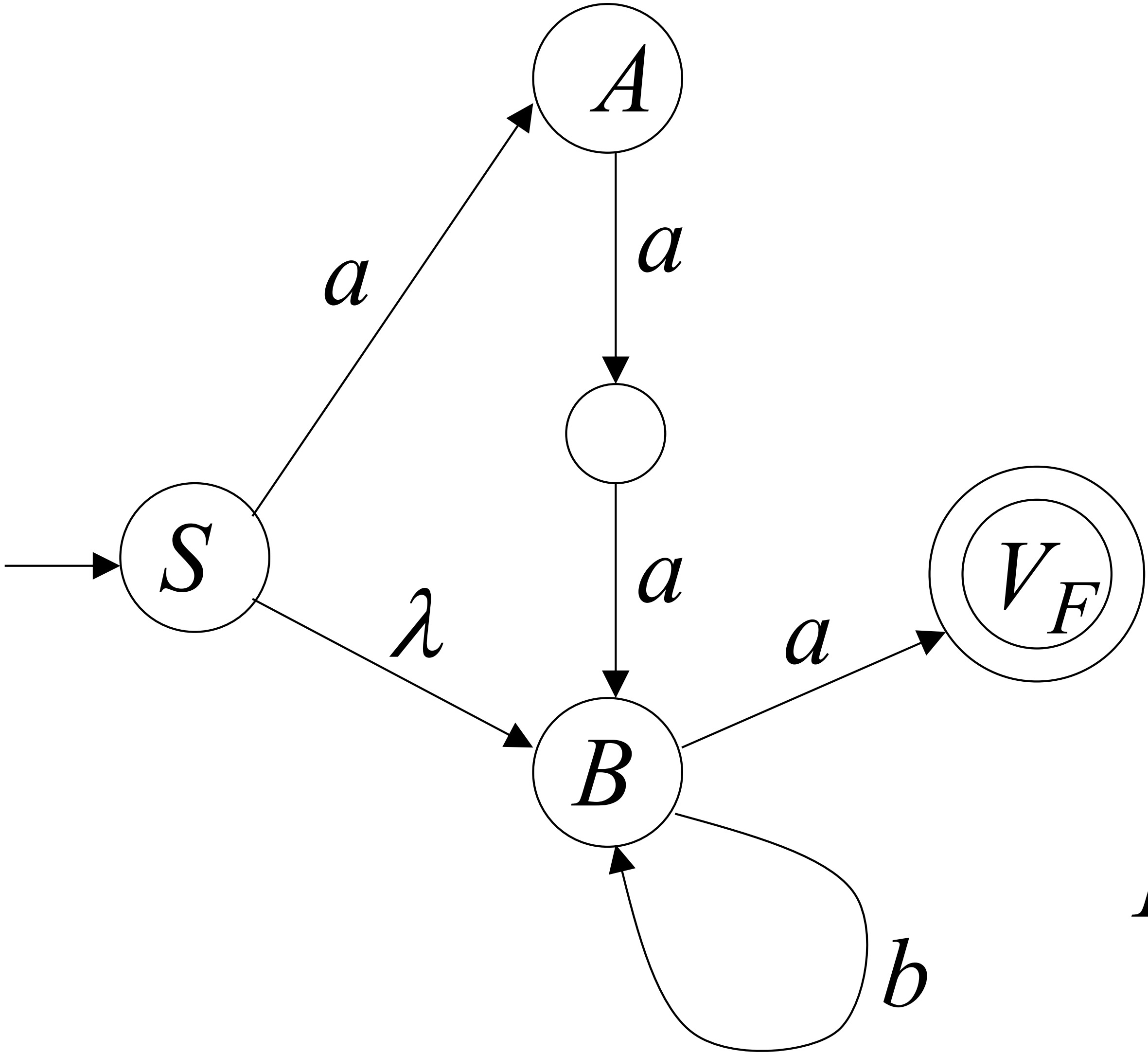
$$A \to aa\ B$$

$$B \to bB \mid a$$



$$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$$

NFA M



Grammar G

$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\ B$$

$$B \rightarrow bB \mid a$$

$$L(M) = L(G) =$$

In General

A right-linear grammar G

has variables: $V_0, V_1, V_2, \ldots$

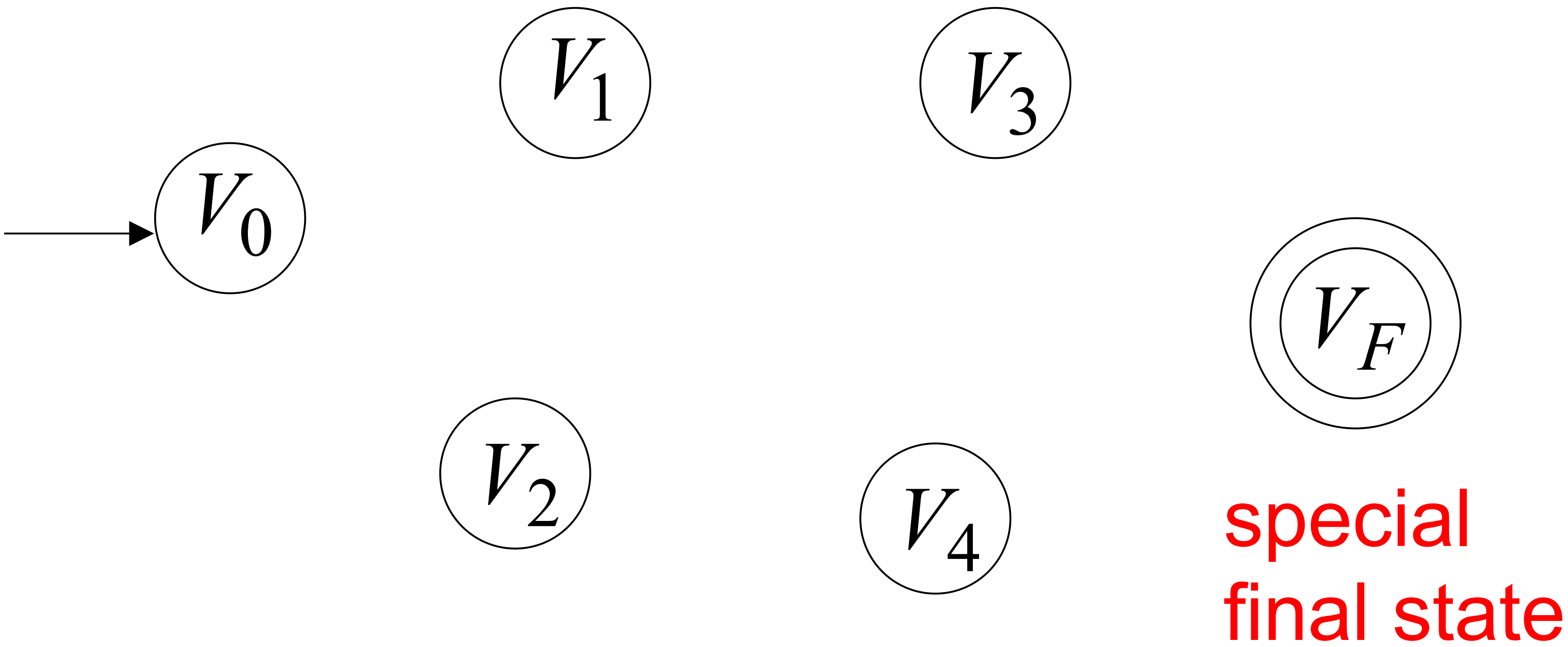
and productions: $V_i \to a_1 a_2 \cdots a_m V_j$

or

$$V_i \to a_1 a_2 \cdots a_m$$

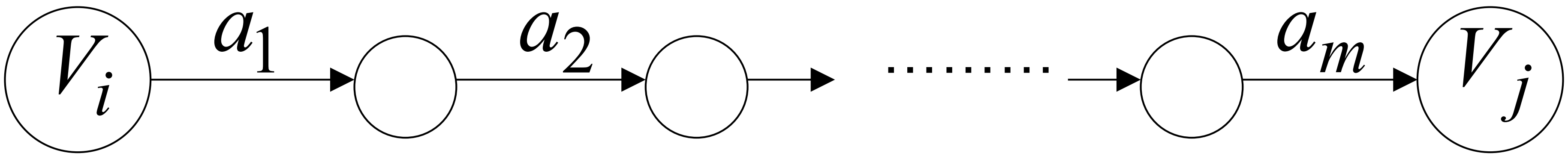
We construct the NFA M such that:

each variable V_i corresponds to a node:



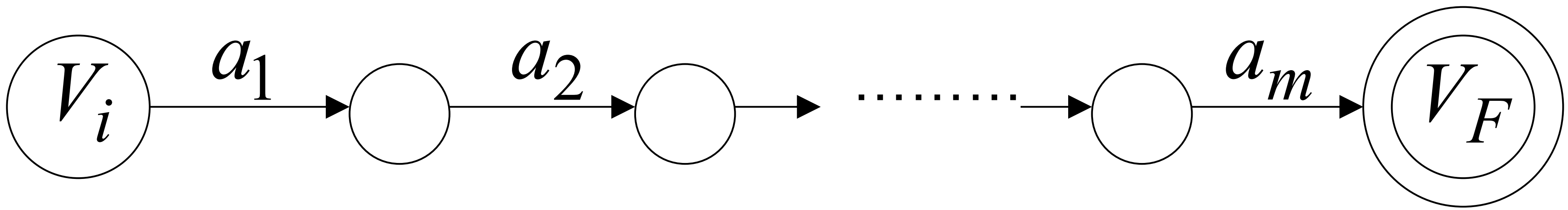
For each production: $V_i \to a_1 a_2 \cdots a_m V_j$

we add transitions and intermediate nodes



For each production: $V_i \to a_1 a_2 \cdots a_m$

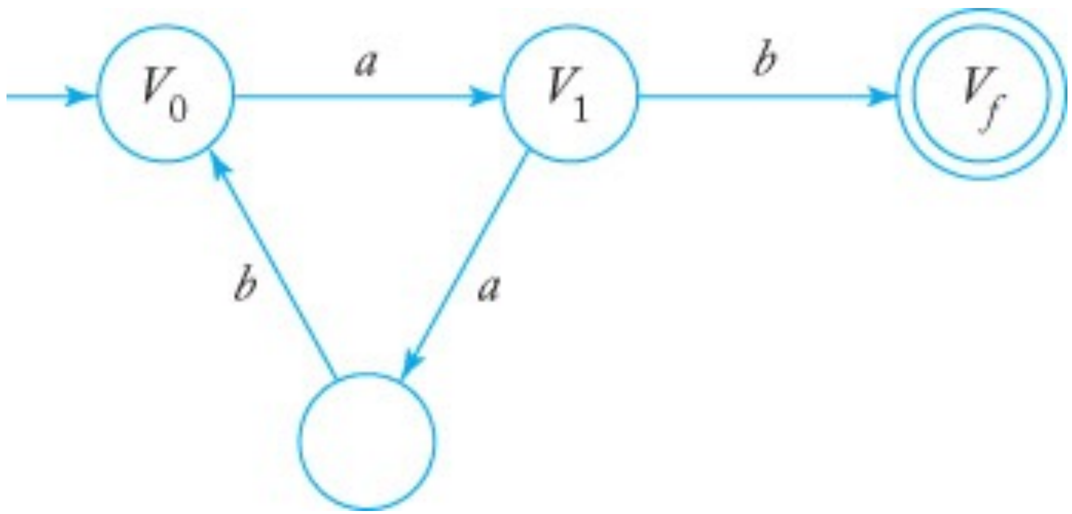
we add transitions and intermediate nodes



Example 3.15

- Construct a FA that accepts the language generated by the grammar

$$V_0 \rightarrow aV_1,$$
$$V_1 \rightarrow abV_0 | b$$



L(G)=

The case of Left-Linear Grammars

Let G be a left-linear grammar

We will prove: $L(G)$ is regular

Proof idea:

We will construct a right-linear grammar G' with $L(G) = L(G')^R$

Since G is left-linear grammar
the productions look like:

$$A \to Ba_1a_2\cdots a_k$$

$$A \to a_1a_2\cdots a_k$$

- Construct right-linear grammar G'

Left linear G

$$A \to B a_1 a_2 \cdots a_k$$

$$A \to B v$$

Right linear G'

$$A \to a_k \cdots a_2 a_1 B$$

$$A \to v^R B$$

- Construct right-linear grammar G'

Left linear G

$$A \to a_1 a_2 \cdots a_k$$

$$A \to v$$

Right linear G'

$$A \to a_k \cdots a_2 a_1$$

$$A \to v^R$$

It is easy to see that: $L(G) = L(G')^R$

Since G' is right-linear, we have:

$L(G') \Rightarrow L(G')^R \Rightarrow L(G)$

Regular Language

Regular Language

Regular Language

Proof - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language L is generated by some regular grammar G

Any regular language L is generated by some regular grammar G

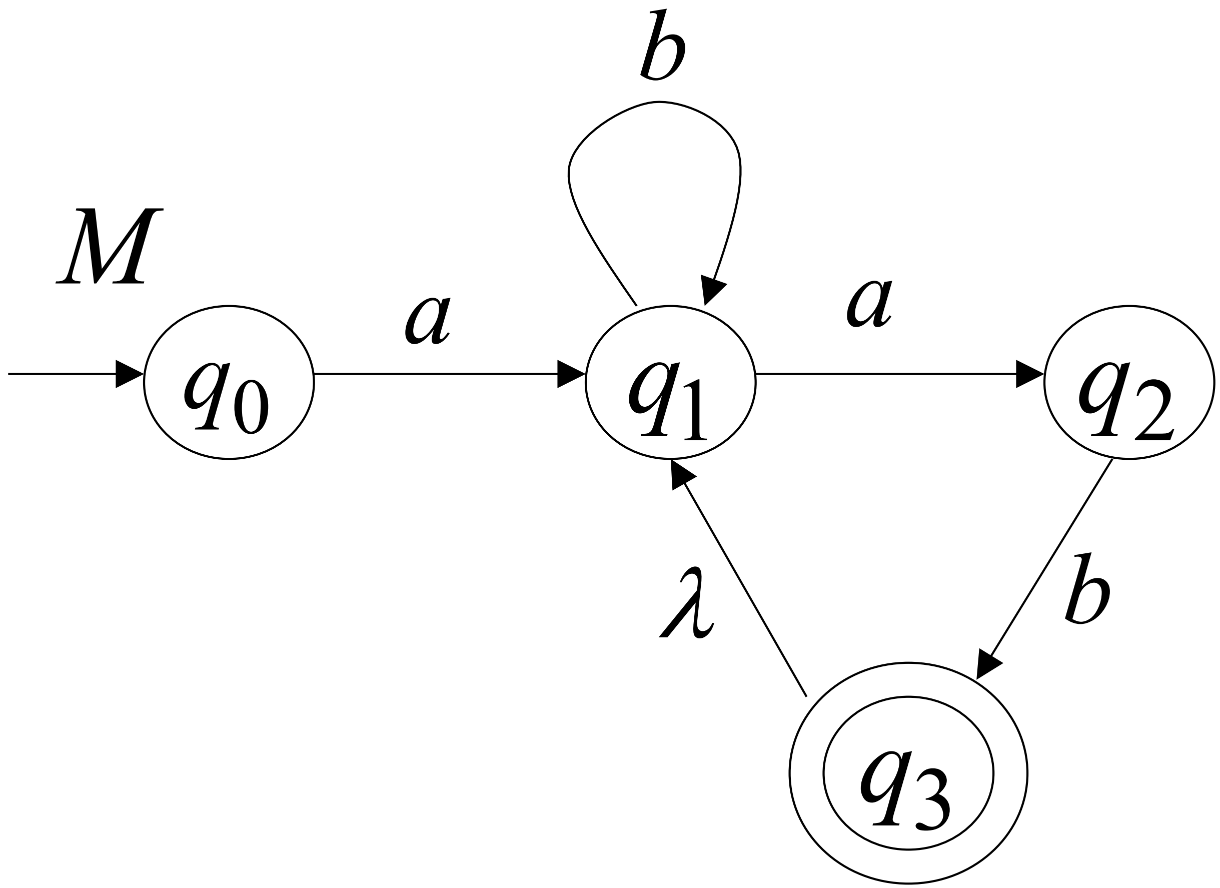
Proof idea:

Let M be the NFA with $L = L(M)$.

Construct from M to a regular grammar G such that $L(M) = L(G)$

Since L is regular
there is an NFA M such that $L = L(M)$

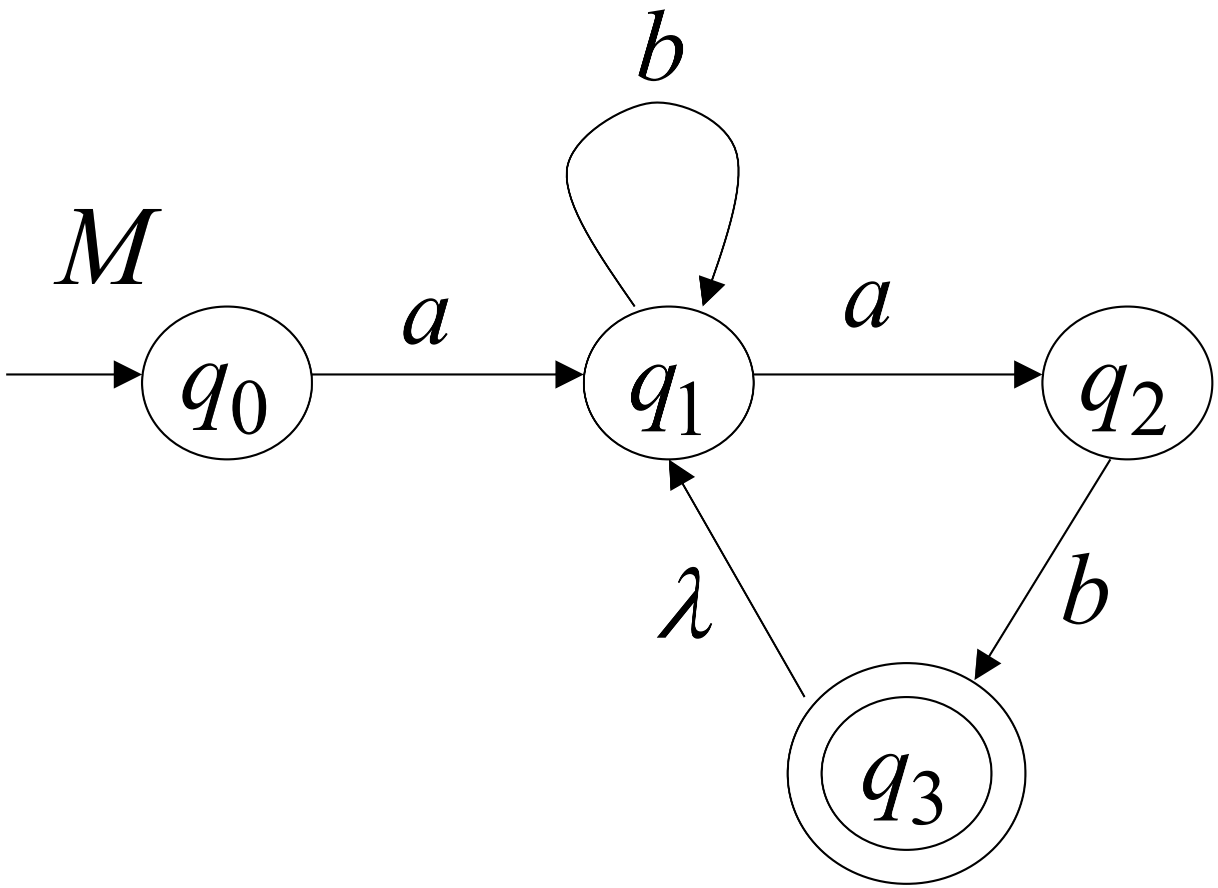
Example:



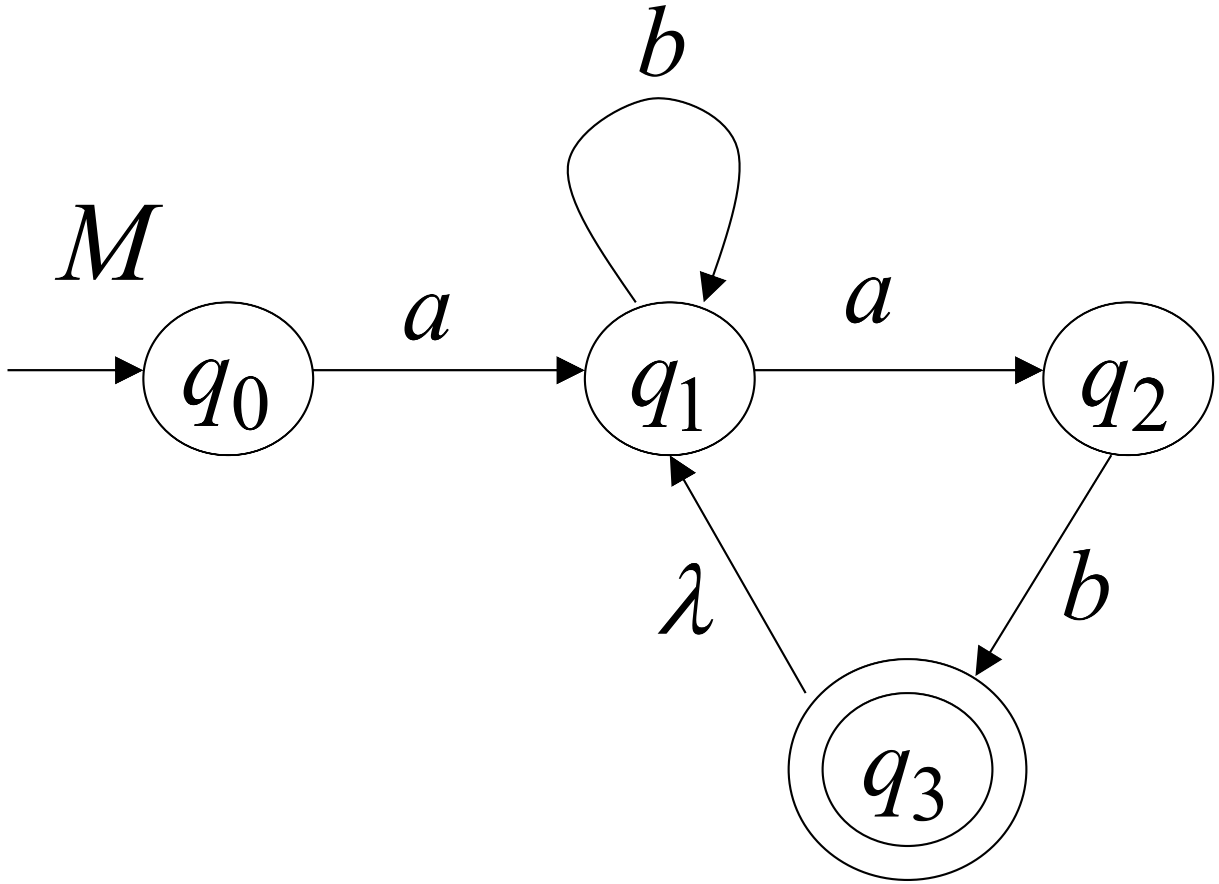
$L =$

$L = L(M)$

Convert M to a right-linear grammar (BFS-like)



$$q_0 \to aq_1$$



$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

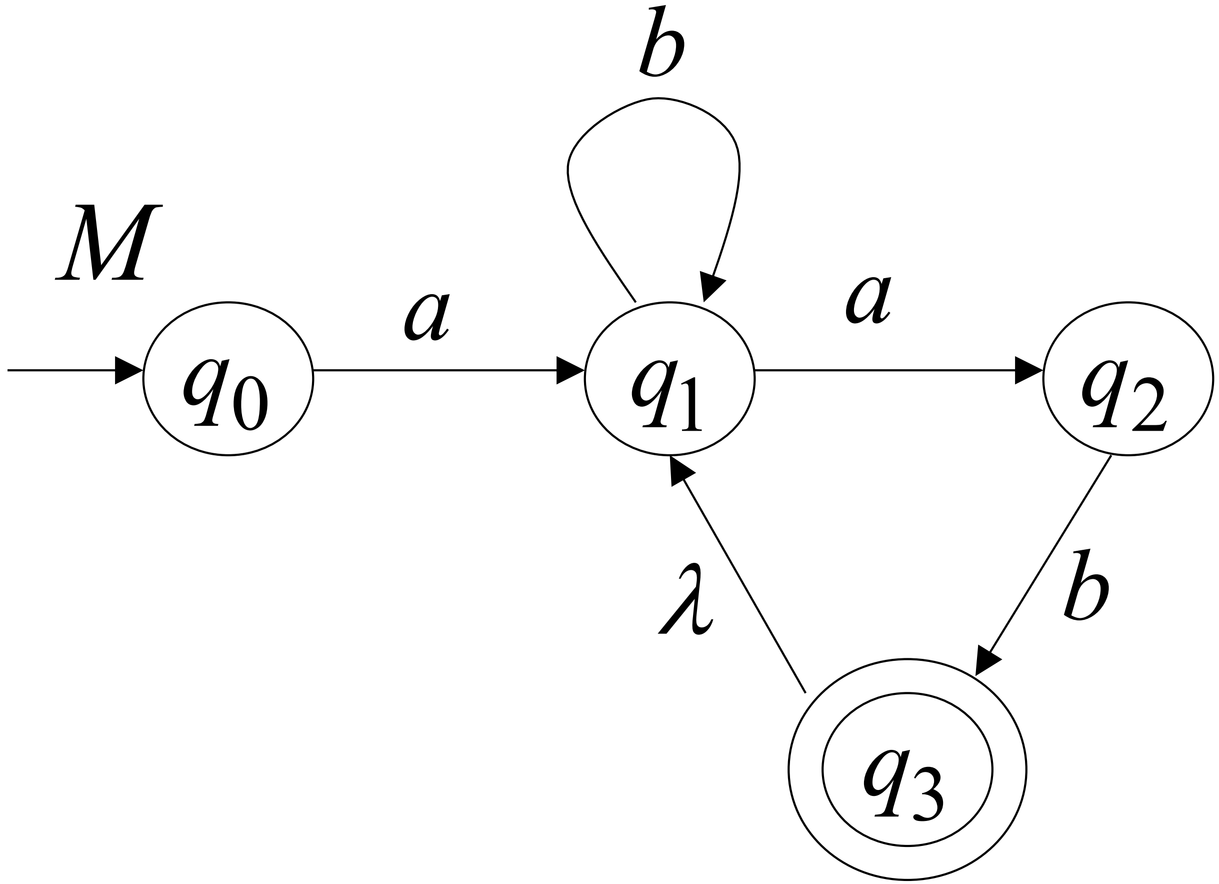
$$q_1 \to aq_2$$

$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

$$q_1 \to aq_2$$

$$q_2 \to bq_3$$



$$L(G) = L(M) = L$$

G

$$q_0 \to aq_1$$

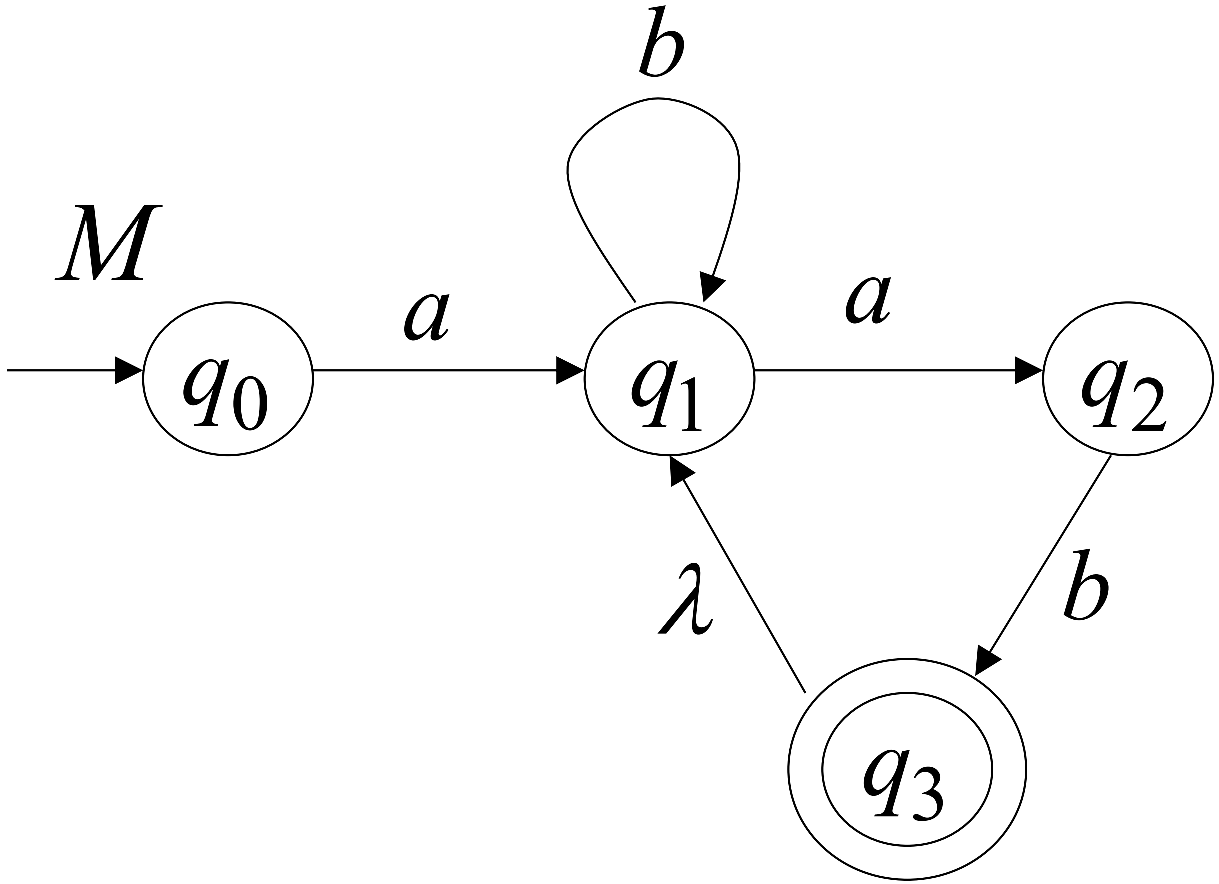
$$q_1 \to bq_1$$

$$q_1 \to aq_2$$

$$q_2 \to bq_3$$

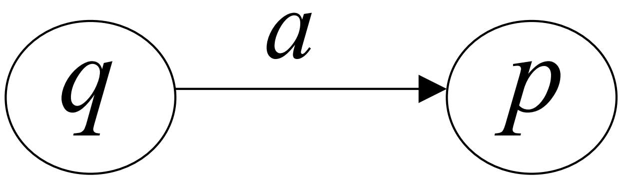
$$q_3 \to q_1$$

$$q_3 \to \lambda$$



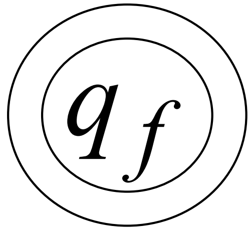
In General

For any transition:



Add production: $q \rightarrow ap$

variable terminal variable

For any final state: 

Add production: $q_f \to \lambda$

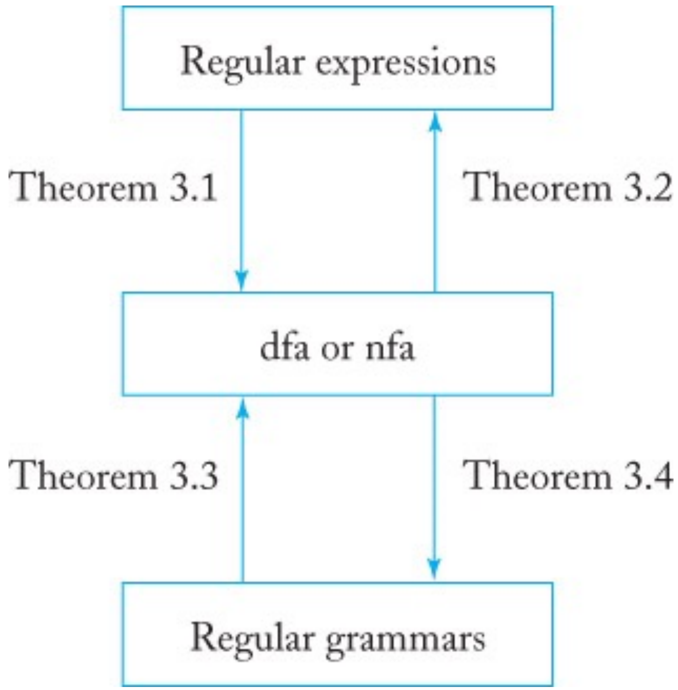
Since G is right-linear grammar

G is also a regular grammar

with $L(G) = L(M) = L$

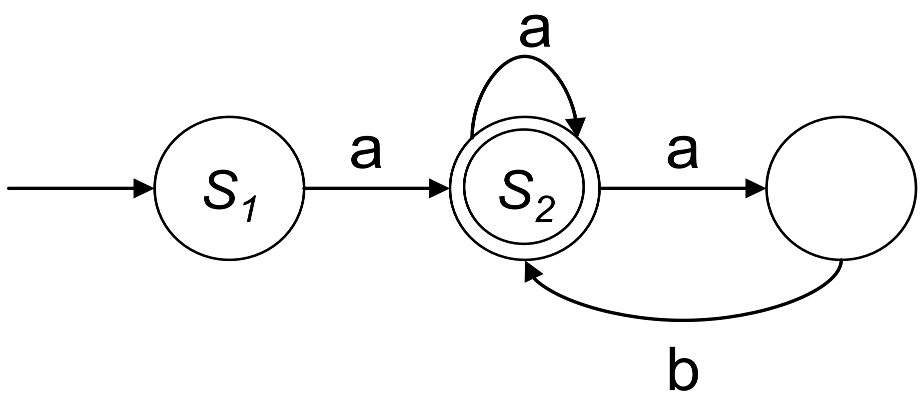
Summary

- We now have several ways of describing regular languages:
  - DFA
  - NFA
  - RE
  - RG



Short Quiz

- Find a regular grammar that generates the language L(aa*(ab+a)*).



$$S_1 \rightarrow aS_2$$

$$S_2 \rightarrow aS_2 \mid abS_2 \mid \lambda$$

Questions?