Chapter 3
Growth of Functions

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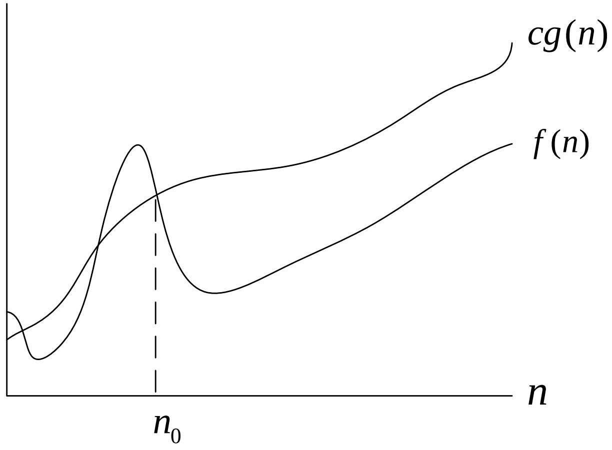
Asymptotic notation

$$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0$$
$$\text{s.t. } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \}$$



$g(n)$ is an asymptotic upper bound for $f(n)$

If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$

(will precisely explain this soon)

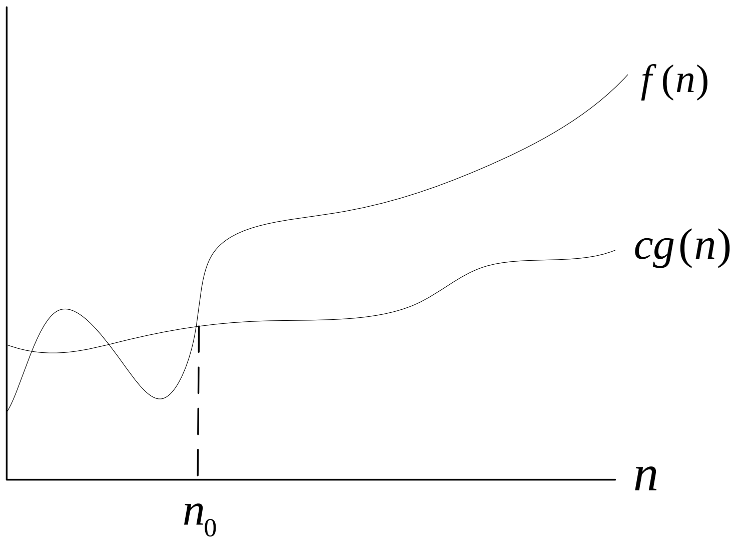
Asymptotic notation

- *O*-notation
- **Example:** $2n^2 = O(n^3)$, with $c = 1$ and $n_0 = 2$
- Examples of the functions in $O(n^2)$:

n^2	n
$n^2 + n$	$n / 1000$
$n^2 + 1000n$	$n^{1.99999}$
$1000n^2 + 1000n$	$n^2 / \lg \lg \lg n$

Asymptotic notation

$$\Omega(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ s.t. } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0 \}$$



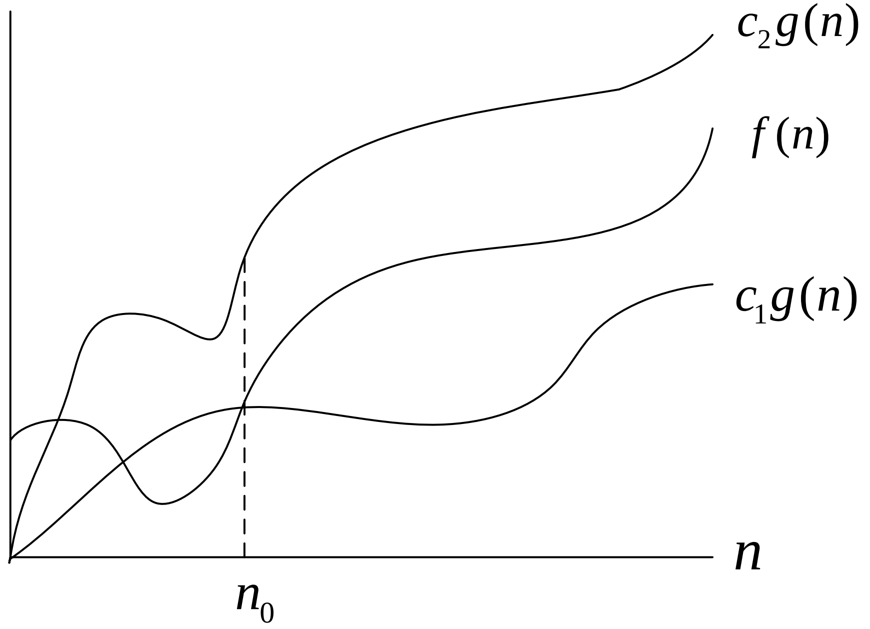
$g(n)$ is an asymptotic lower bound for $f(n)$

- **Ω-notation**
- **Example:** $\sqrt{n} = \Omega(\lg n)$, with $c = 1$ and $n_0 = 16$
- Examples of the functions in $\Omega(n^2)$:

n^2	n^3
$n^2 + n$	$n^{2.00001}$
$n^2 - n$	$n^2 \lg\lg\lg n$
$n^2 + 1000n$	2^{2^n}
$n^2 - 1000n$	

Asymptotic notation

$$\Theta(g(n)) = \{ f(n) : \text{there exist positive constants } c_1, c_2, \text{and } n_0$$
$$\text{s.t. } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0 \}$$



g (*n*) is an asymptotic tight bound for *f* (*n*)

Asymptotic notation

- Θ-notation
- Example: $\frac{n^2}{2} - 3n = \Theta(n^2)$, with $c_1 = \frac{1}{14}$, $c_2 = \frac{1}{2}$, and $n_0 = 7$
- Theorem

$$f(n) = \Theta(g(n)) \text{ if and only if } f(n) = O(g(n)) \text{ and } \Omega(g(n))$$

- Leading constants and low-order terms don’t matter.

Asymptotic notation in equations

- **When on the right-hand side:**

$O(n^2)$ stands fot some anonymous function in the set $O(n^2)$

$2n^2+3n+1=2n^2+\Theta(n)$ means $2n^2+3n+1=2n^2+f(n)$ for some $f(n)\in\Theta(n)$

In particular, $f(n)=3n+1$

- We interpret # of anonymous functions as = # of times the asymptotic notation appears:

$\sum_{i=1}^{n} O(i)$ OK: 1 anonymous function

$O(1)+O(2)+\cdots+O(n)$ not OK: n hidden constants => no clean interpretation

Asymptotic notation in equations

- **When on the left-hand side:**

No matter how the anonymous functions are chosen on the left-hand side, there is a way to choose the anonymous functions on the right-hand side to make the equation valid.

Interpret $2n^2 + \Theta(n) = \Theta(n^2)$ as meaning

for all functions $f(n) \in \Theta(n)$,

there exists a function $g(n) \in \Theta(n^2)$ such that $2n^2 + f(n) = g(n)$

Can chain together:

$$\begin{aligned} 2n^2 + 3n + 1 &= 2n^2 + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

Asymptotic notation in equations

- Interpretation

$$2n^2+3n+1=2n^2+\Theta(n)=\Theta(n^2)$$

First equation : There exist $f(n)\in\Theta(n)$ such that

$$2n^2+3n+1=2n^2+f(n)$$

Second equation : For all $g(n)\in\Theta(n)$ (such as the $f(n)$ used to make the first equation hold), there exists $h(n)\in\Theta(n^2)$ such that $2n^2+g(n)=h(n)$

Asymptotic notation in equations

o -notation

$$o(g(n)) = \{ f(n) : \text{for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0 \}$$

Another vi ew, probably easier to use : $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

$$n^{1.9999} = o(n^2)$$

$$n^2 / \lg n = o(n^2)$$

$$n^2 \ne o(n^2) \text{(just like } 2 < 2)$$

$$n^2 / 1000 \ne o(n^2)$$

Asymptotic notation in equations

ω -notation

$$\omega(g(n)) = \{ f(n) : \text{for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0 \}$$

Another vi ew, again, probably easier to use : $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

$$n^{2.0001} = \omega(n^2)$$

$$n^2 \lg n = \omega(n^2)$$

$$n^2 \ne \omega(n^2)$$

Comparisons of functions

- Relational properties:
  - **Transitivity:**

$$f(n)=\Theta(g(n)) \text{ and } g(n)=\Theta(h(n)) \Rightarrow f(n)=\Theta(h(n))$$

Same at O, Ω, o, and ω.

  - **Reflexivity:** $f(n)=\Theta(f(n))$

Same for O and Ω.

Comparisons of functions

- Relational properties:
  - Symmetry:

$$f(n)=\Theta(g(n)) \text{ if and only if } g(n)=\Theta(f(n))$$

  - Transpose symmetry:

$$f(n)=O(g(n)) \text{ if and only if } g(n)=\Omega(f(n))$$

$$f(n)=o(g(n)) \text{ if and only if } g(n)=\omega(f(n))$$

- Comparisons:

  $f(n)$ is asymptotically smaller than $g(n)$ if $f(n) = o(g(n))$

  $f(n)$ is asymptotically larger than $g(n)$ if $f(n) = \omega(g(n))$

No trichotomy. Although intuitively, we can liken O to $\leq$, Ω to $\geq$, etc., unlike real numbers, where $a < b$, $a = b$, or $a > b$ we might not be able to compare functions.

Example: $n^{1+\sin n}$ and n, since $1+\sin n$ oscillates between 0 and 2.

Standard notations and common functions

- Monotonicity

$f(n)$ is monotonically increasing if $m \le n \Rightarrow f(m) \le f(n)$

$f(n)$ is monotonically decreasing if $m \le n \Rightarrow f(m) \ge f(n)$

$f(n)$ is strictly increasing if $m < n \Rightarrow f(m) < f(n)$

$f(n)$ is strictly decreasing if $m < n \Rightarrow f(m) > f(n)$

- # Exponentials

Userful identities :

$$a^{-1} = 1/a,$$

$$\left(a^m\right)^n = a^{mn}$$

$$a^m a^n = a^{m+n}$$

Can relate rates of growth of polynomial s and expoonenti als : for all real constants a and b such that $a > 1$,

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0,$$

which implies that $n^b = o\left(a^n\right)$.

A suprosongl y useful inequality : for all real x,

$$e^x \geq 1 + x.$$

As x gets colsers to 0, e^x gets colser to $1 + x$.

e = Euler's number ≈ 2.71828

- # Logarithms(1)

Notations :

$\lg n = \log_2 n$ (binary logarithm) ,

$\ln n = \log_e n$ (natural logarithm) ,

$\lg^k n = (\lg n)^k$ (exponenti ation),

$\lg \lg n = \lg(\lg n)$ (compositi on),

Logarithm functions apply only to the next term in the formula,

so the $\lg n + k$ means $(\lg n) + \mathrm{k}$, and *not* $\lg(n + k)$

In the expression $\log_b a$:

– If we hole b constant, then the expression is strictly increasing as a increases.

– If we hold a constant, then the expression is strictly decreasing as b increases.

- # Logarithms(2)

Usegful identities for all real $a > 0, b > 0, c > 0$, and n, and where logarothm bases are not 1:

$$a = b^{\log_b a},$$

$$\log_c (ab) = \log_c a + \log_c b,$$

$$\log_b a^n = n \log_b a,$$

$$\log_b a = \frac{\log_c a}{\log_c b},$$

$$\log_b (1/a) = -\log_b a,$$

$$\log_b a = \frac{1}{\log_a b},$$

$$a^{\log_b c} = c^{\log_b a}.$$

- # Logarithms(3)

Changing the base of a logarithm from one constant to another only changes the value by a constant factor, so we usually don't worry about logarithm bases in asymptotic notation. Covention is to use lg within asymptotic notation, unless the base actually matters.

Just as polynomials grow more slowly than exponentials, logarithms grow more slowly than polynomials.

In $\lim_{n\to\infty} \frac{n^b}{a^n} = 0$, substitute $\lg n$ for n and 2^a for a:

$$\lim_{n\to\infty} \frac{\lg^b n}{(2^a)^{\lg n}} = \lim_{n\to\infty} \frac{\lg^b n}{n^a} = 0,$$

implying that $\lg^b n = o(n^a)$.

- Factorials

$$n! = 1 \cdot 2 \cdot 3 \cdots n. \text{ Special case}: 0! = 1.$$

Can use Stirling's approximation,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right),$$

$$\text{to derive that } \lg(n!) = \Theta(n \lg n)$$

- Functional iteration
  - $f^{(i)}(n)$: $f(n)$ iteratively applied i times to an initial value of n.

$$f^{(i)}(n) = \begin{cases} n & \text{if } i = 0 \\ f(f^{(i-1)}(n)) & \text{if } i > 0 \end{cases}$$

ex. If $f(n) = 2n$, then $f^{(i)}(n) = 2^i n$.

- The iterated logarithm function
  - $\lg^* n = \min\{i \geq 0 : \lg^{(i)} n \leq 1\}$

  - ex. $\lg^* 2 = 1$,
    $\lg^* 4 = 2$,
    $\lg^* 16 = 3$,
    $\lg^* 65536 = 4$,
    $\lg^*(2^{65536}) = 5$.

- Fibonacci numbers

$F_0 = 0,$

$F_1 = 1,$

$F_i = F_{i-1} + F_{i-2} \quad \text{for } i \geq 2.$

golden ratio

$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803...$$

$$\hat{\phi} = \frac{1-\sqrt{5}}{2} = -.61803...$$

$$\Rightarrow \quad F_i = \frac{\phi^i + \hat{\phi}^i}{\sqrt{5}}$$

Function	Name	Value					
$O(1)$	constant						
$O(\log n)$	logarithm	0	1	2	3	4	5
$O(n)$	linear	1	2	4	8	16	32
$O(n \log n)$	$n \log n$	0	2	8	24	64	160
$O(n^2)$	square	1	4	16	64	256	1,024
$O(n^3)$	cube	1	8	64	512	4,096	32,768
$O(2^n)$	exponential	2	4	16	256	65,536	4,294,967,296
$O(n!)$	factorial						