Chapter 24
Single-Source Shortest Paths

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Overview

- Given a directed graph $G = (V, E)$, weight function $w: E \rightarrow \Re$, $|V| = n$.
- Goal: create an $n \times n$ matrix of shortest-path distances $\delta(u, v)$.
- Could run BELLMAN-FORD once from each vertex:
  $O(V^2E)$ — which is $O(V^4)$ if the graph is **dense** ($E = \theta(V^2)$).
- If no negative-weight edges, could run Dijkstra's algorithm once from each vertex:
  $O(VE \lg V)$ with binary heap—$O(V^3 \lg V)$ if dense.
  $O(V^2 \lg V + VE)$ with Fibonacci heap — $O(V^3)$ if dense.
- We'll see how to do in $O(V^3)$ in all cases, with no fancy data structure.

Shortest paths and matrix multiplication

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Shortest paths and matrix multiplication

- Assume that G is given as adjacency matrix of weights: $W = (w_{ij})$, with vertices numbered 1 to n.

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \text{weight of } (i,j) & \text{if } i \neq j, (i,j) \in E, \\ \infty & \text{if } i \neq j, (i,j) \notin E. \end{cases}$$

- Output is matrix $D = (d_{ij})$, where $d_{ij} = \delta(i,j)$.
  Won't worry about predecessor—see book.
- Will use dynamic programming at first.
- ***Optimal substructure***: Recall: subpaths of shortest paths are shortest paths.
- ***Recursive solution***: Let $l_{ij}^{(m)}$ = weight of shortest path $i \leadsto j$ that contains $\leq m$ edges.

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- $m = 0$
  ⇒there is a shortest path $i \rightsquigarrow j$ with $\leq m$ edges if and only if $i = j$
  $$\Rightarrow l_{ij}^{(0)} = \begin{cases} 0 \text{ if } i = j \\ \infty \text{ if } i \neq j \end{cases}$$

- $m \geq 1$
  $$\Rightarrow \ l_{ij}^{(m)} = \min(l_{ij}^{(m-1)}, \min_{1 \leq k \leq n} \{l_{ik}^{(m-1)} + w_{kj}\}) \ (k \text{ is all predecessors of } j)$$
  $$= \min_{1 \leq k \leq n} \{l_{ik}^{(m-1)} + w_{kj}\} \ (\text{since } w_{jj} = 0 \text{ for all } j)$$

- Observer that when $m = 1$, must have $l_{ij}^{(1)} = w_{ij}$.
  Conceptually, when the path is restricted to at most 1 edge, the weight of the shortest path $i \rightsquigarrow j$ must be w_{ij}.
  And the math works out, too:

$$
\begin{aligned}
l_{ij}^{(1)} &= \min_{1\le k\le n}\{l_{ik}^{(0)} + w_{kj}\} \\
&= l_{ii}^{(0)} + w_{ij} \qquad (\mathrm{l}_{\mathrm{ii}}^{(0)} \text{ is the only non} - \infty \text{ among } l_{ik}^{(0)}) \\
&= w_{ij}.
\end{aligned}
$$

All simple shortest paths contain $\leq n - 1$ edges

$\Rightarrow \delta(i, j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n+1)} = \cdots$

Compute a solution bottom-up: : Compute $L^{(1)}, L^{(2)}, \dots, L^{(n-1)}$.

Start with $L^{(1)} = W$, since $l_{ij}^{(1)} = w_{ij}$.

Go from $L^{(m-1)}$ to $L^{(m)}$:

EXTEND-SHORTEST-PATH(L, W)

$n = L.rows$
let $L' = (l'_{ij})$ be a new $n \times n$ matrix
for $i = 1$ ***to*** n **do**
  for $j = 1$ ***to*** n **do**
    $l'_{ij} = \infty$
    for $k = 1$ ***to*** n **do**
      $l'_{ij} = \min(l'_{ij}, l_{ij} + w_{kj})$
return L'

SLOW-ALL-PAIRS-SHORTEST-PATH(W)

$n = W.rows$
$L^{(1)} = W$
for $m = 2$ ***to*** $n - 1$ **do**
    let $L^{(m)}$ be a new $n \times n$ matrix
    $L^{(m)} = \text{EXTEND-SHORTEST-PATH}(L^{(m-1)}, W)$
return $L^{(n-1)}$

***Time*:**

- EXTEND: $\Theta(n^3)$.
- SLOW-APSP: $\Theta(n^4)$.

***Observation*:** EXTEND is like matrix multiplication:

$L \rightarrow A$
$W \rightarrow B$
$L' \rightarrow C$
$min \rightarrow +$
$+ \rightarrow \cdot$
$\infty \rightarrow 0$

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create C, an $n \times n$ matrix

for $i \leftarrow 1$ **to** n

  do for $j \leftarrow 1$ **to** n

    do $c_{ij} \leftarrow 0$

     for $k \leftarrow 1$ **to** n

      do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

So, we can view EXTEND as just like matrix multiplication!

Why do we care?

Because our goal is to compute $L^{(n-1)}$ as fast as we can.

Don't need to compute *all* the intermediate $L^{(1)}, L^{(2)}, L^{(3)}, \dots L^{(n-1)}$.

Suppose we had a matrix A and we wanted to compute A^{n-1} (like calling EXTEND $n-1$ times).

Could compute $A, A^2, A^4, A^8, \dots$

If we knew $A^m = A^{n-1}$ for all $m \geq n-1$, could just finish with A^r, where r is the smallest power of 2 that's $\geq n-1$. ($r = 2^{\lceil \lg(n-1) \rceil}$)

FASTER-ALL-PAIRS-SHORTEST-PATH(W)

$n = W.rows$
$L^{(1)} = W$
$m = 1$
while $m < n - 1$ **do**
    let $L^{(2m)}$ be a new $n \times n$ matrix
    $L^{(2m)} = \text{EXTEND-SHORTEST-PATH}(L^{(m)}, L^{(m)})$
    $m = 2m$
return $L^{(m)}$

OK to overshoot, since products don't change after $L^{(n-1)}$.

Time: $\Theta(n^3 \lg n)$

The Floyd-Warshall algorithm

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Floyd-Warshall algorithm

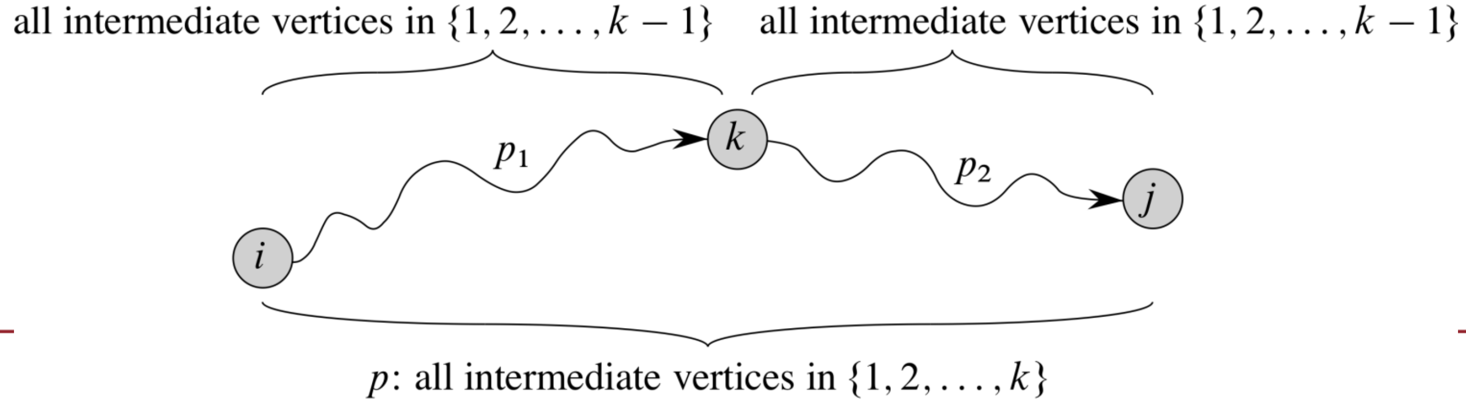
A different dynamic-programming approach.

For path $p = \langle v_1, v_2, \dots, v_l \rangle$, an ***intermediate vertex*** is any vertex of p other than v_1 or v_l.

Let $d_{ij}^{(k)}$ = shortest-path weight of any path $i \rightsquigarrow j$ with all intermediate vertices in $\{1, 2, \dots, k\}$.

Consider a shortest path $i \overset{p}{\rightsquigarrow} j$ with all intermediate vertices in $\{1, 2, \dots, k\}$:

- If k is not an intermediate vertex, then all intermediate vertices of p are in $\{1, 2, \dots, k-1\}$.
- If k is an intermediate vertex:



Recursive formulation

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1. \end{cases}$$

(Have $d_{ij}^{(0)} = w_{ij}$ because can't have intermediate vertices $\Rightarrow \leq 1$ edges.)
Want $D^{(n)} = \left(d_{ij}^{(n)}\right)$, since all vertices numbered $\leq n$.

Compute bottom-up
Compute in increasing order of *k*

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FLOYD-WARSHALL(W)

$n = W.rows$

$D^{(0)} = W$

for $k = 1$ ***to*** n **do**

  let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

  for $i = 1$ ***to*** n **do**

    for $j = 1$ ***to*** n **do**

      $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

return $D^{(n)}$

Can drop superscripts. (See Exercise 25.2-4 in text.)

Time: $\boldsymbol{\Theta(n^3)}$.

Transitive closure

Given $G(V, E)$, directed.

Compute $G^* = (V, E^*)$.

- $E^* = \{(i, j)\text{: there is a path } i \leadsto j \text{ in } G\}$.

Could assign weight of 1 to each edge, then run FLOYD-WARSHALL.

- If $d_{ij} < n$, then there is a path $i \leadsto j$.
- Otherwise, $d_{ij} = \infty$ and there is no path.

Simpler way: Substitute other values and operators in FLOYD-WARSHALL.

- Use unweighted adjacency matrix
- $\min \to \vee$ (OR)
- $+ \to \wedge$ (AND)

- $d_{ij}^{(k)} = \begin{cases} 1 \text{ if there is path } i \rightsquigarrow j \text{ with alla intermediate vertices in } \{1,2,\dots,k\}, \\ 0 \text{ otherwise.} \end{cases}$

- $t_{ij}^{(0)} = \begin{cases} 0 \text{ if } i \neq j \text{ and } (i,j) \notin E, \\ 1 \text{ if } i = j \text{ or } (i,j) \in E. \end{cases}$

- $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$.

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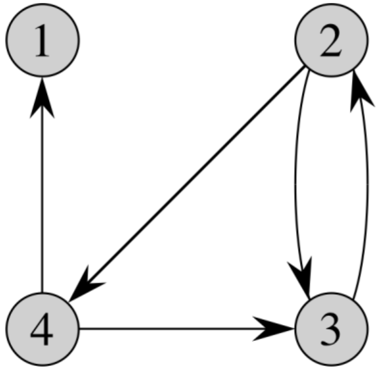
TRANSITIVE-CLOSURE(G)

```
n = |G.V|
let T^(0) = (t_ij^(0)) be a new n × n matrix
for i = 1 to n do
    for j = 1 to n do
        if i == j or (i, j) ∈ G.E then
            t_ij^(0) = 1
        else
            t_ij^(0) = 0
for k = 1 to n do
    let T^(k) = (t_ij^(k)) be a new n × n matrix
    for i = 1 to n do
        for j = 1 to n do
            t_ij^(k) = t_ij^(k-1) ∨ (t_ik^(k-1) ∧ t_kj^(k-1))
return T^(n)
```

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee \left(t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\right)$$

Time: $\Theta(n^3)$, but simpler operations than FLOYD-WARSHALL.



$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Johnoson’s algorithm for sparse graphs

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- ***Idea*:** If the graph is sparse, it pays to run Dijkstra's algorithm once from each vertex.
- If we use a Fibonacci heap for the priority queue, the running time is down to $O(V^2 \lg V + VE)$, which is better than FLOYD-WARSHALL's $\Theta(V^3)$ time if $E = o(V^2)$.
- But Dijkstra's algorithm requires that all edge weights be nonnegative.
- Donald Johnson figured out how to make an equivalent graph that *does* have all edge weights ≥ 0.

Reweighting

Compute a new weight function $\widehat{w}$ such that

1. For all $u, v \in V$, p is a shortest path $u \rightsquigarrow v$ using w if and only if p is a shortest path $u \rightsquigarrow v$ using $\widehat{w}$.
2. For all $(u, v) \in E$, the new weight $\widehat{w}(u, v)$ is nonnegative.

- Property(1) says that it suffices to find shortest paths with $\widehat{w}$.
- Property(2) says we can do so by running Dijkstra's algorithm from each vertex.

- How to come up with $\widehat{w}$?
- Lemma 25.1 shows it's easy to get property(1):

Lemma (Rewighting doesn't change shortest paths)

Given a directed, weighted graph $G = (V, E)$, $w: E \to \boldsymbol{R}$. Let h be any function such that $h: V \to \boldsymbol{R}$.
For all $(u, v) \in E$, define

$$\widehat{w}(u, v) = w(u, v) + h(u) - h(v)$$

Let $p = \langle v_0, v_1, \dots, v_k \rangle$ be any path $v_0 \rightsquigarrow v_k$.

Then, p is a shortest path $v_0 \rightsquigarrow v_k$ with w if and only if p is a shortest path $v_0 \rightsquigarrow v_k$ with $\widehat{w}$.

Also, G has a negative-weight cycle with weight w iff G has a negative-weight cycle with weight $\widehat{w}$.

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Proof

- First, we'll show that $\widehat{w}(p) = w(p) + h(v_0) - h(v_k)$:

$$\begin{aligned}\widehat{w}(p) &= \sum_{i=1}^{k} \widehat{w}(v_{i-1}, v_i) \\ &= \sum_{i=1}^{k} (w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i)) \\ &= \sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_0) - h(v_k) \quad \text{(sum telescopes)} \\ &= w(p) + h(v_0) - h(v_k).\end{aligned}$$

- Therefore, any path $v_0 \rightsquigarrow v_k$ has $\widehat{w}(p) = w(p) + h(v_0) - h(v_k)$.
  Since $h(v_0)$ and $h(v_k)$ don't depend on the path from v_0 to v_k , if one path $v_0 \rightsquigarrow v_k$ is shorter than another with w, it's also shorter with $\widehat{w}$.
- Now show there exists a negative-weight cycle with w if and only if there exists a negative-weight cycle with $\widehat{w}$:

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- Let cycle $C = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = v_k$.
- Then $\widehat{w}(C) = w(C) + h(v_0) - h(v_k) = w(C)$ (since $v_0 = v_k$).

Therefore, C has a negative-weight cycle with w if and only if it has a negative-weight cycle with $\widehat{w}$. ■ (lemma)

So, now to get property(2), we just need to come up with a function

$h: V \to \boldsymbol{R}$ such that when we compute $\widehat{w}(u, v) = w(u, v) + h(u) - h(v)$, it's ≥ 0..

Do what we did for difference constraints:

- $G' = (V', E')$
- $V' = V \cup \{s\}$, where *s* is a new vertex.
- $E' = E \cup \{(s, v): v \in V\}$.
- $w(s, v) = 0$ for all $v \in V$.

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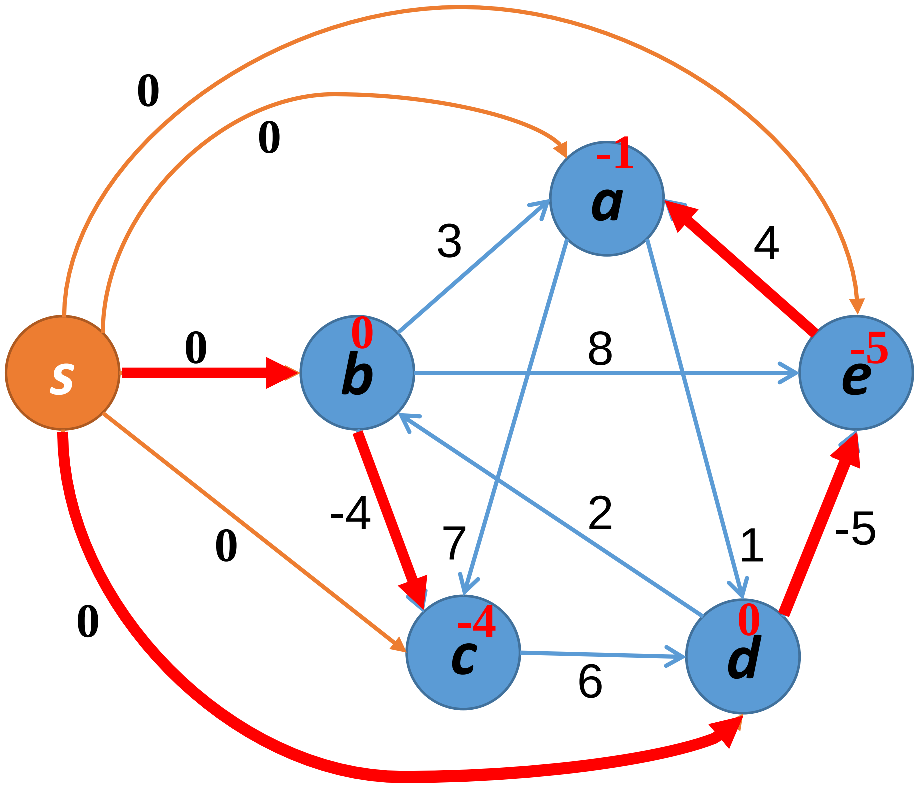

Figure 25.6 Johnson's all-pairs shortest-paths algorithm

(a) The graph *G'* with the original weight function *w*. The new vertex ***s*** is black.

- Since no edges enter s, G' has the same set of cycles as G.
  In particular, G' has a negative-weight cycle if and only if G does.

Define $h(v) = \delta(s, v)$ for all $v \in V$.

Claim $\hat{w}(u, v) = w(u, v) + h(u) - h(v) \geq 0$

Proof By the triangle inequality,

$$\delta(s, v) \leq \delta(s, u) + w(u, v)$$

$$h(v) \leq h(u) + w(u, v) .$$

Therefore, $w(u, v) + h(u) - h(v) \geq 0.$ ■(claim)

JOHNSON(G)

compute G', where $G'.V = G.V \cup \{s\}$,
$G'.E = G.E \cup \{(s,v) : v \in G.V\}$, and $w(s,v) = 0$ for all $v \in G.V$

if *BELLMAN-FORD*(G', w, s) $==$ *FALSE* **then**

  print "the input graph contains a negative-weight cycle"

else

  for *each vertex* $v \in G'V$ **do**

    set $h(v)$ to the value of $\delta(s,v)$ computed by the Bellman-Ford algorithm

  for *each edge* $(u,v) \in G'.E$ **do**

    $\hat{w}(u,v) = w(u,v) + h(u) - h(v)$

  let $D = (d_{uv})$ be a new $n \times n$ matrix

  for *each vertex* $u \in G.V$ **do**

    run DIJKSTRA(G, $\hat{w}$, u) to compute $\hat{\delta}(u,v)$ for all $v \in G.V$

    for *each vertex* $v \in G.V$ **do**

      $d_{uv} = \hat{\delta}(u,v) + h(v) - h(u)$

return D

▷ Compute entry d_{uv} in matrix D

$$d_{uv} = \underbrace{\hat{\delta}(u,v) + \delta(s,v) - \delta(s,u)}_{\substack{\text{because if } p \text{ is a path } u \to v, \\ \text{then } \hat{w}(p) = w(p) + h(u) - h(v)}}$$

Time:

- $\Theta(V + E)$ to compute G' .
- $o(VE)$ to run BELLMAN-FORD.
- $\Theta\ (E)$ to compute $\widehat{w}$.
- $O(V^2 \lg V + VE)$ to run Dijkstra's algorithm $|V|$ times (using Fibonacci heap).
- $\Theta(V^2)$ to compute *D* matrix.

Total: $O(V^2 \lg V + VE)$.

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