2021

Theory of Computation

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Outline

Nondeterministic Pushdown Automata

Pushdown Automata and Context-Free Languages

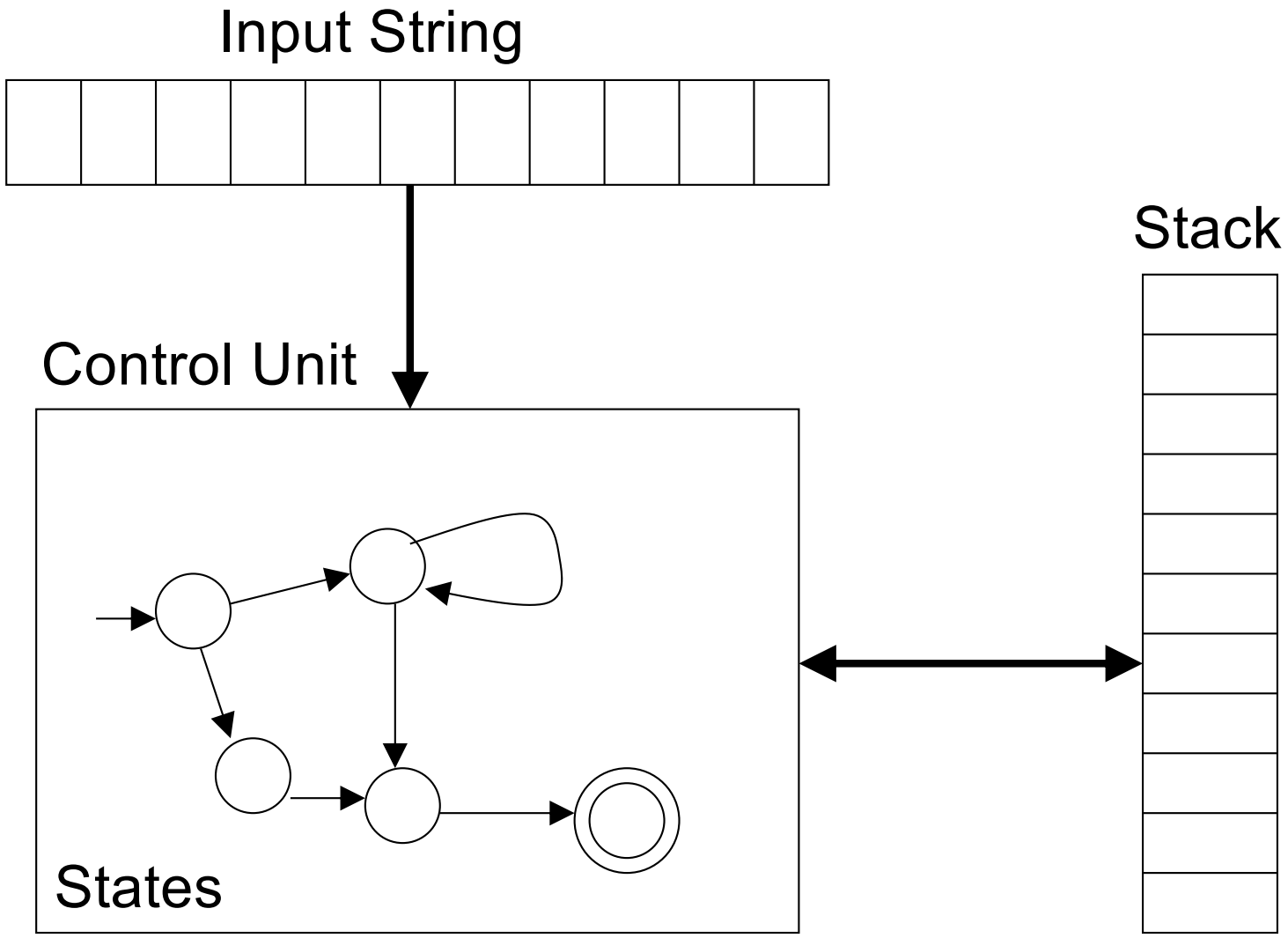
Deterministic Pushdown Automata and Deterministic CFLs

$$L = \{a^n b^n : n \geq 0\}$$

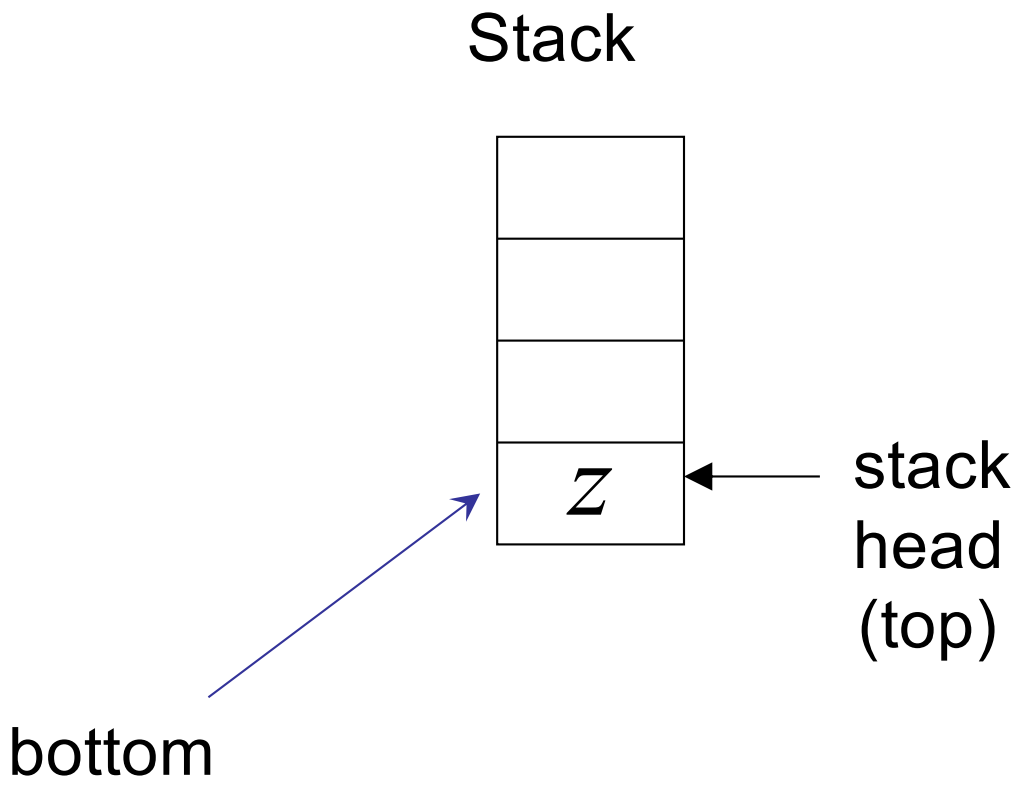
- Check all *a*'s precede the first *b*
- Count the number of *a*'s
- Count without limit
- Stack

PushDown Automata (PDA)

Pushdown Automaton -- PDA



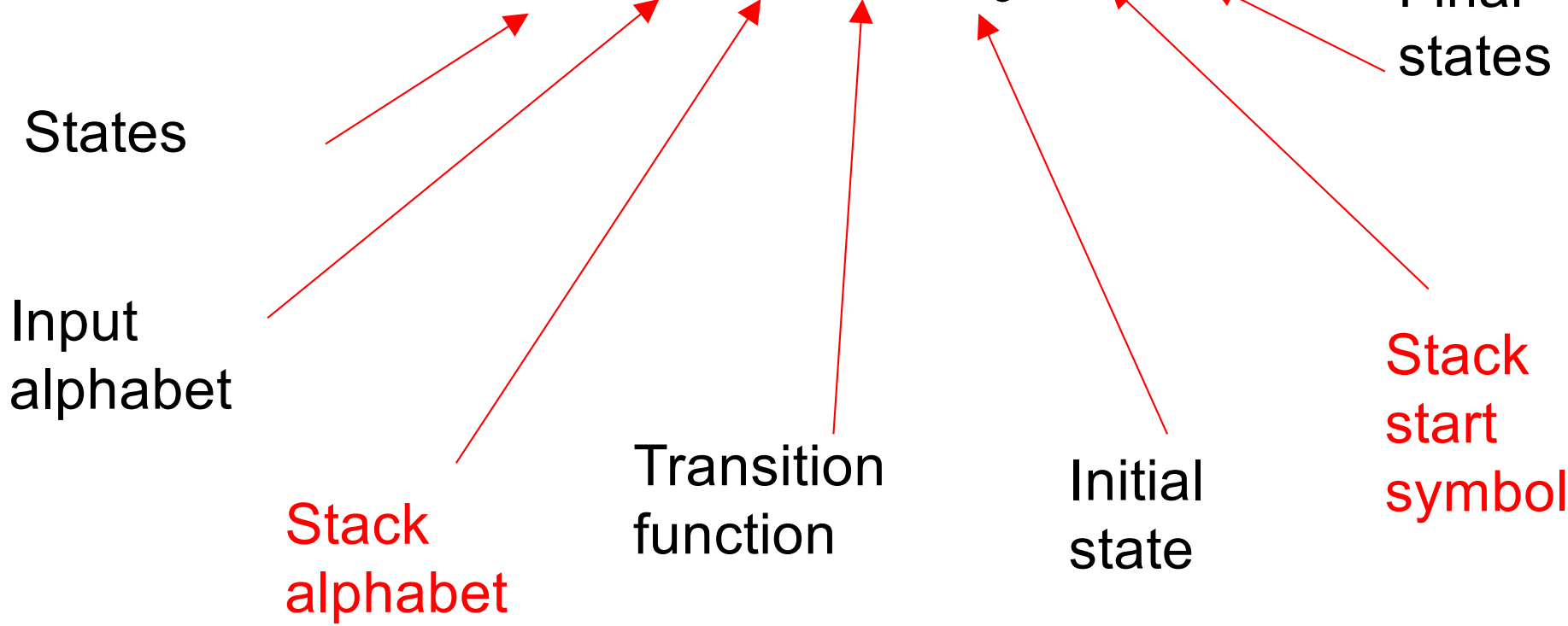
Initial Stack Symbol



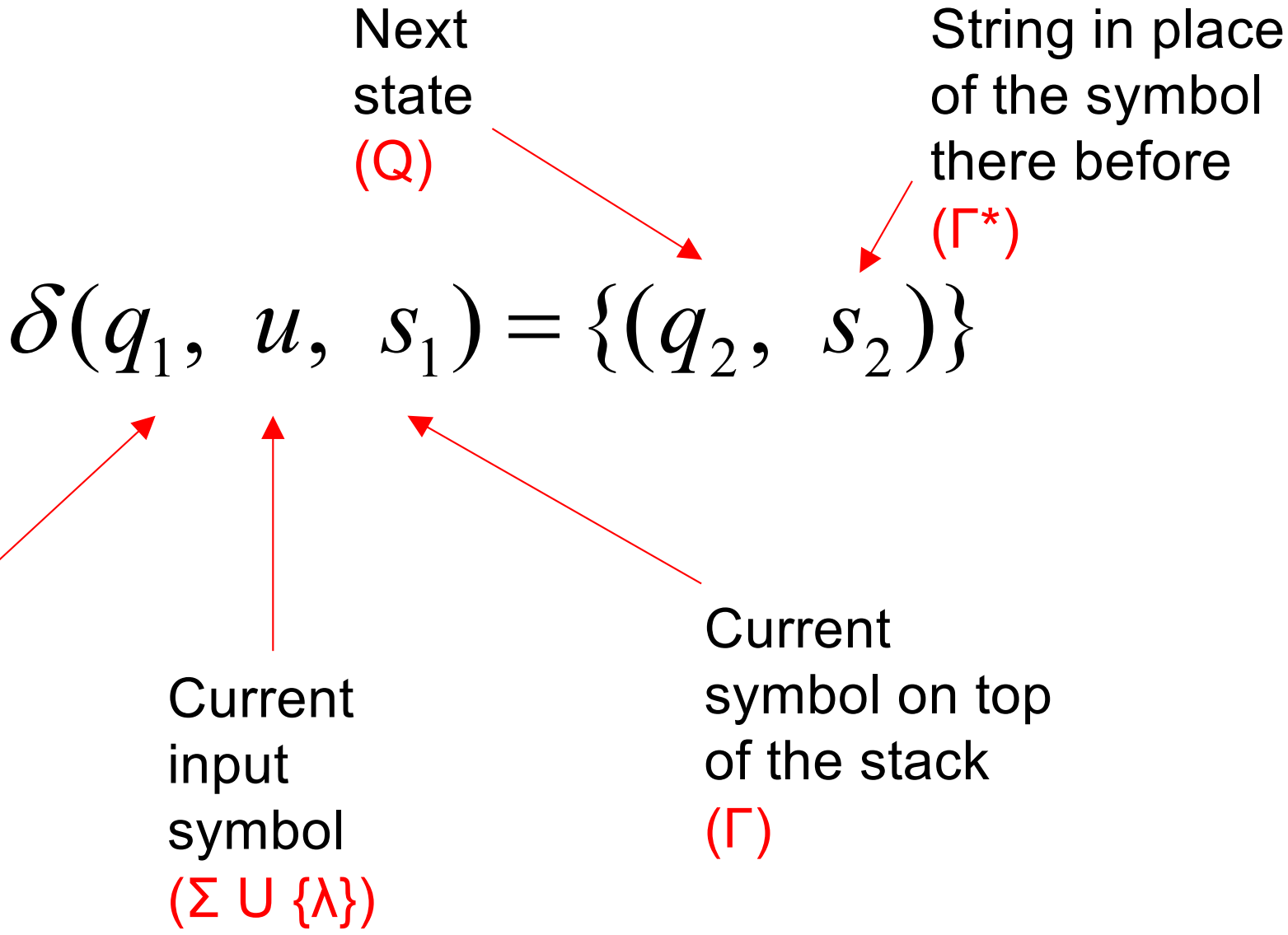
Definition 7.1

Non-Deterministic Pushdown Automaton (NPDA)

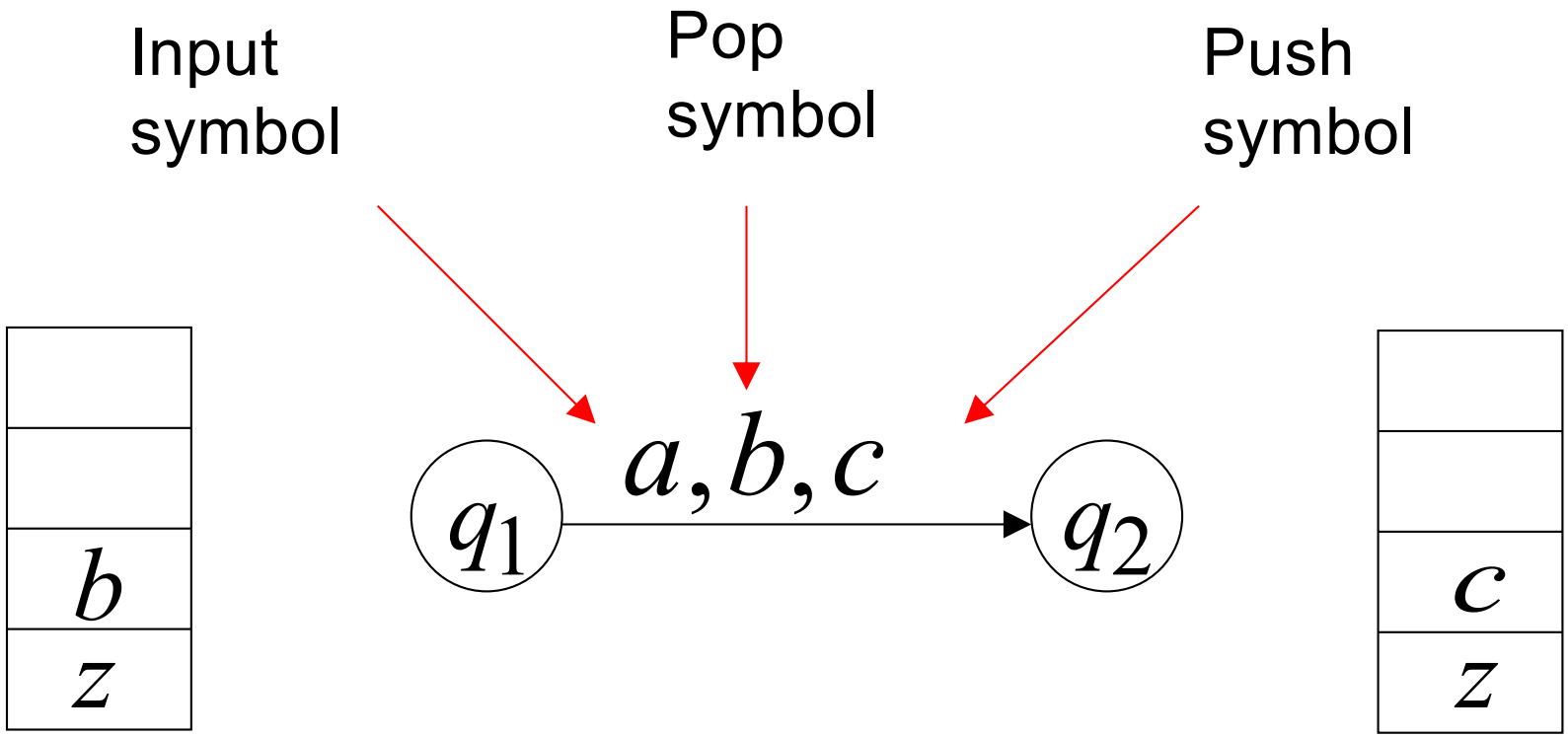
$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$



Transition Function

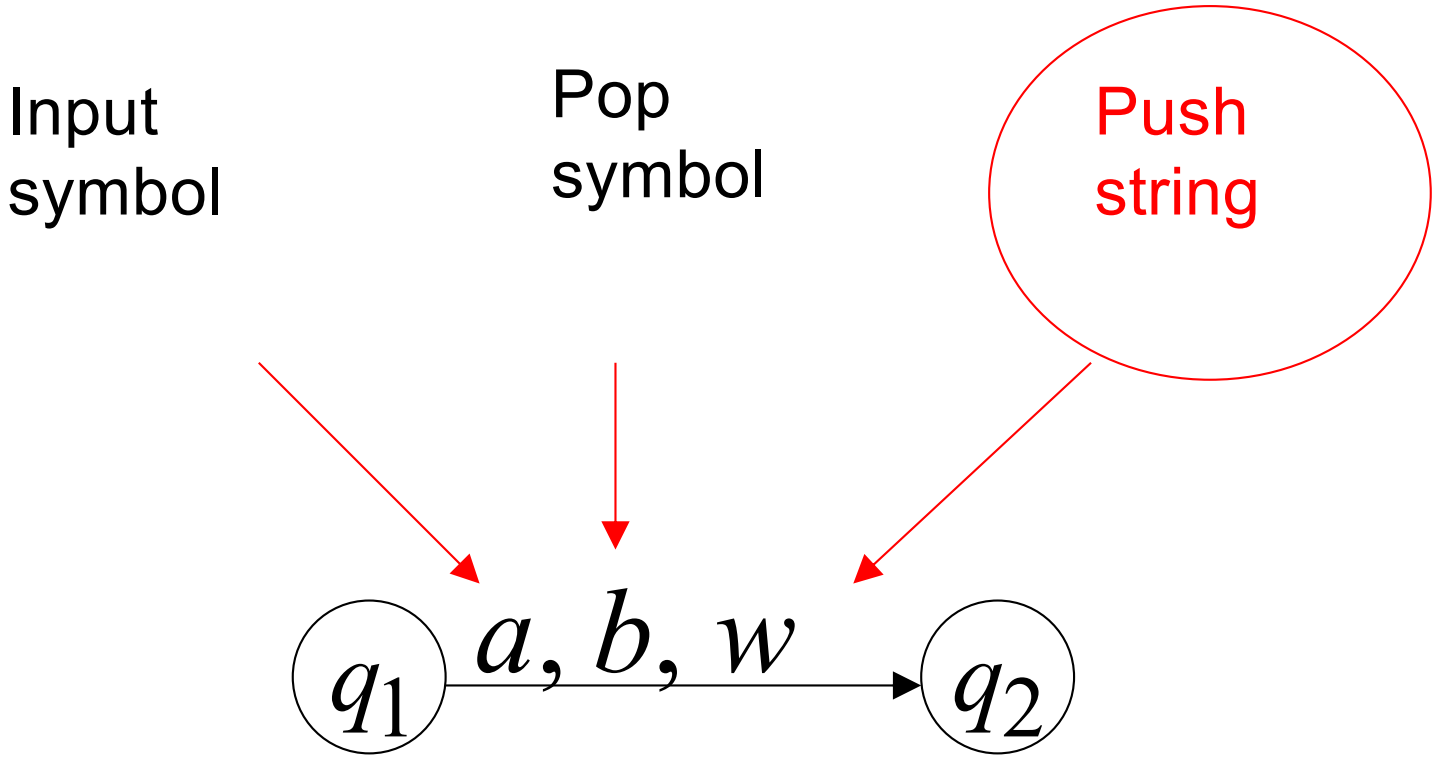


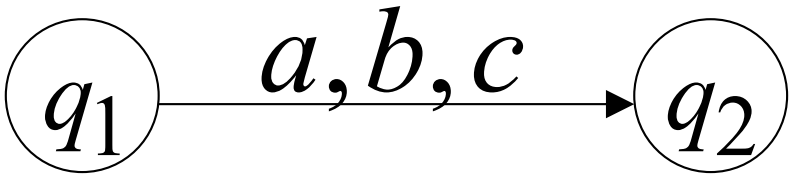
Labels on the Edges of Transition Graphs



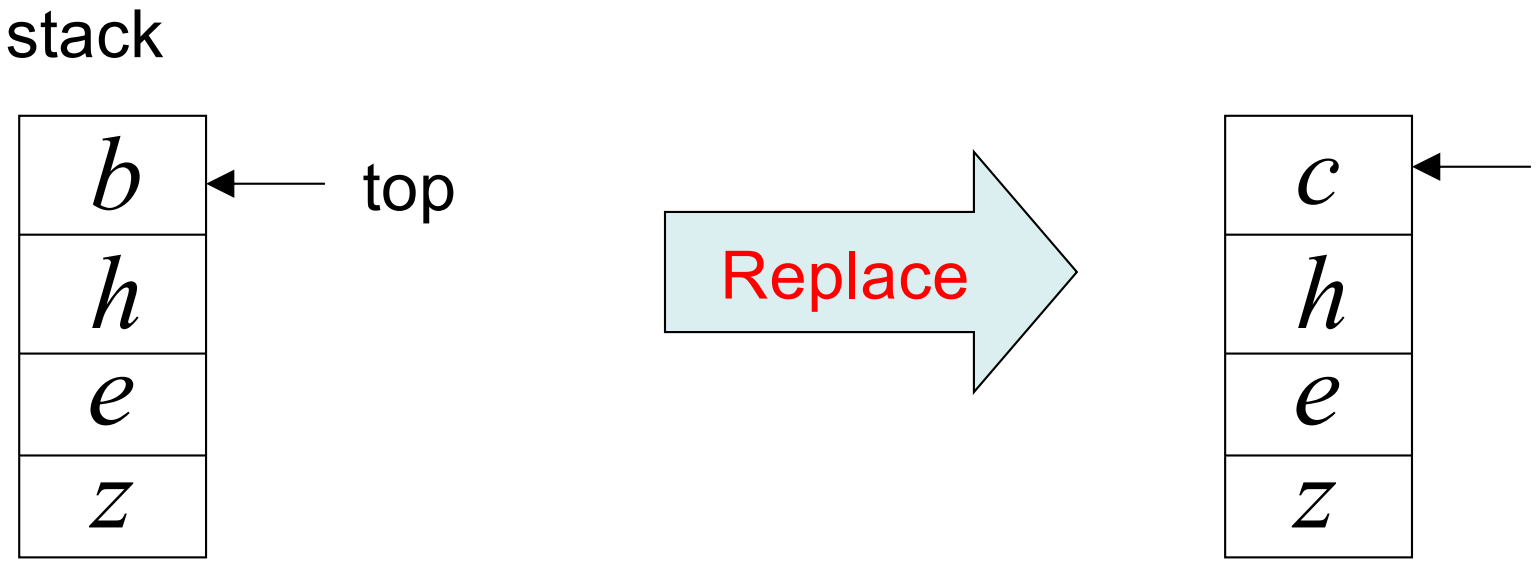
$$\delta(q_1,\ a,\ b) = \{(q_2,\ c)\}$$

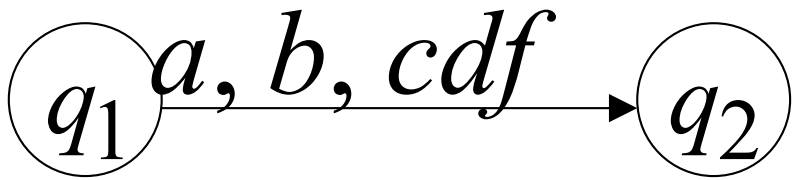
Pushing Strings

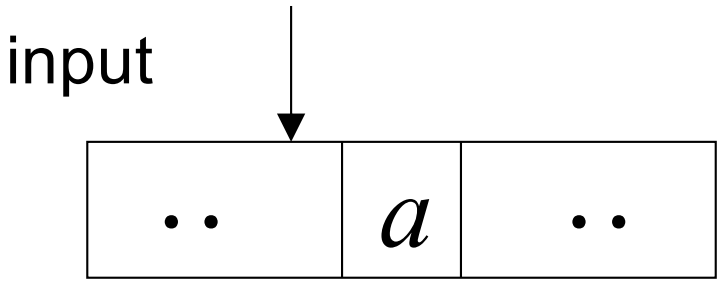


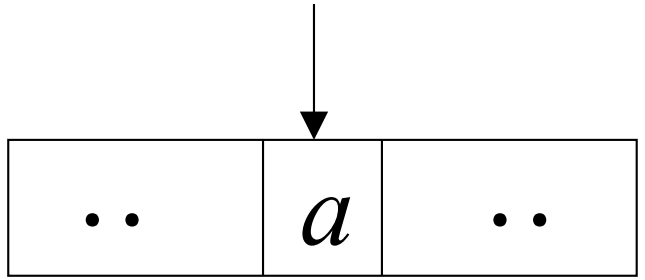

q_1
a, b, c
q_2

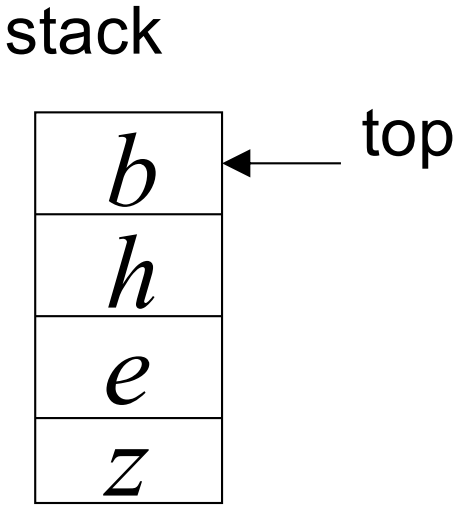
input
$\ldots$
a
$\ldots$
$\ldots$
a
$\ldots$

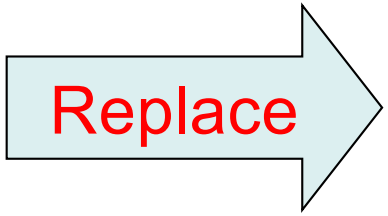

stack
b
top
h
e
z
Replace
c
h
e
z

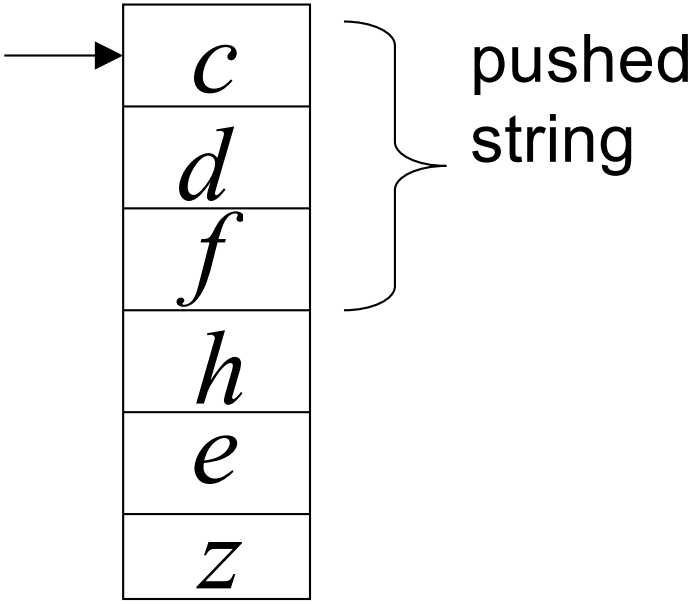

q_1
a, b, cdf
q_2

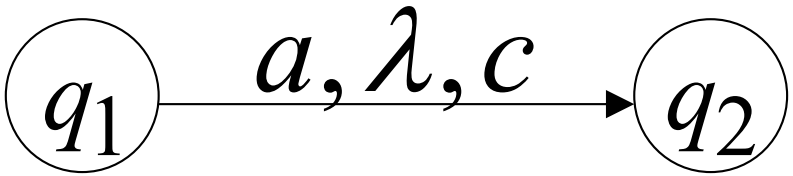

input
. .
a
. .


. .
a
. .

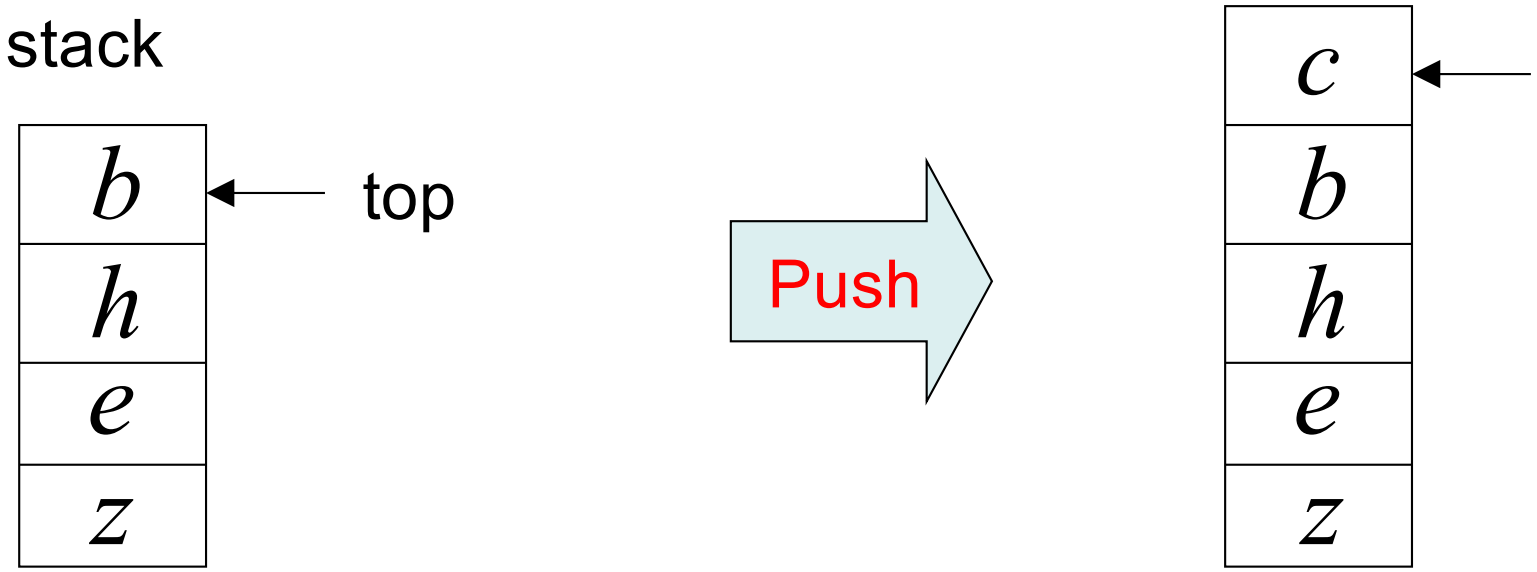

stack
top
b
h
e
z

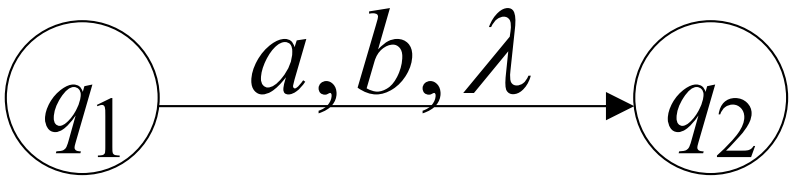

Replace


c
d
f
h
e
z
pushed
string

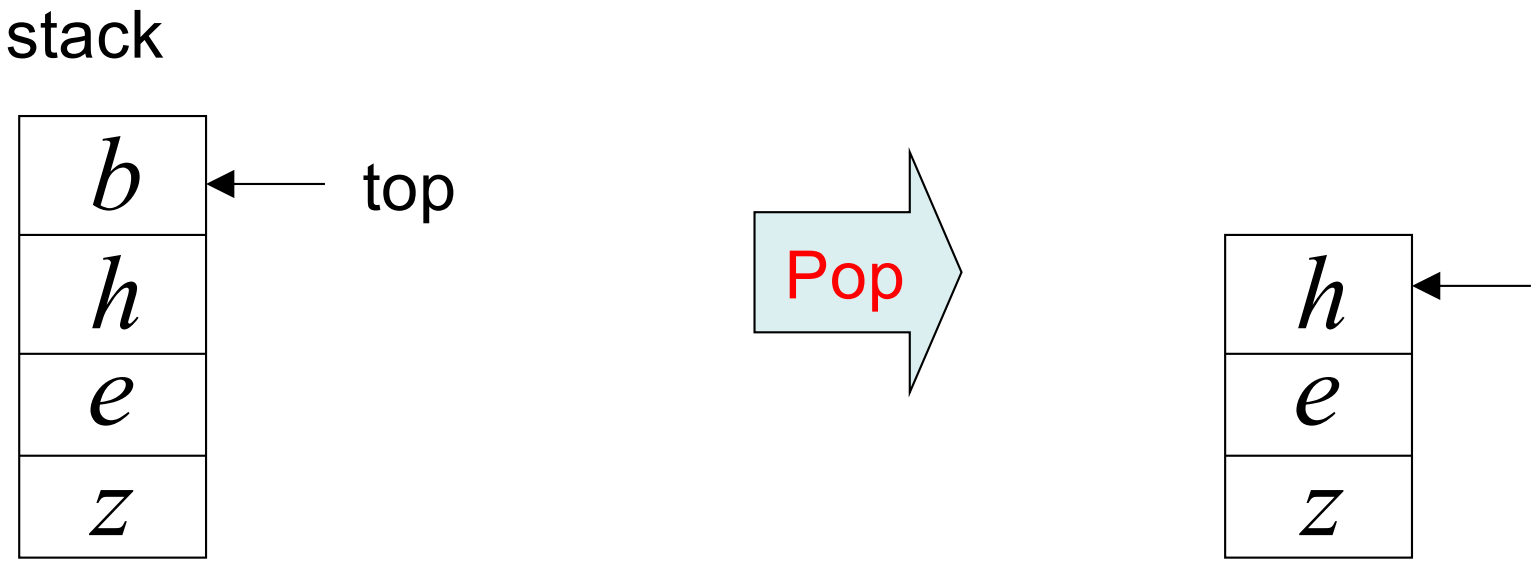

q_1
a, λ, c
q_2

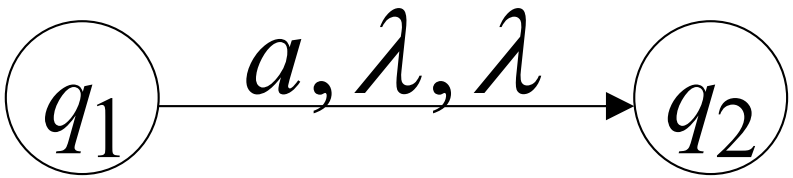
input
$\ldots$
a
$\ldots$
$\ldots$
a
$\ldots$


stack
b
top
h
e
z
Push
c
b
h
e
z

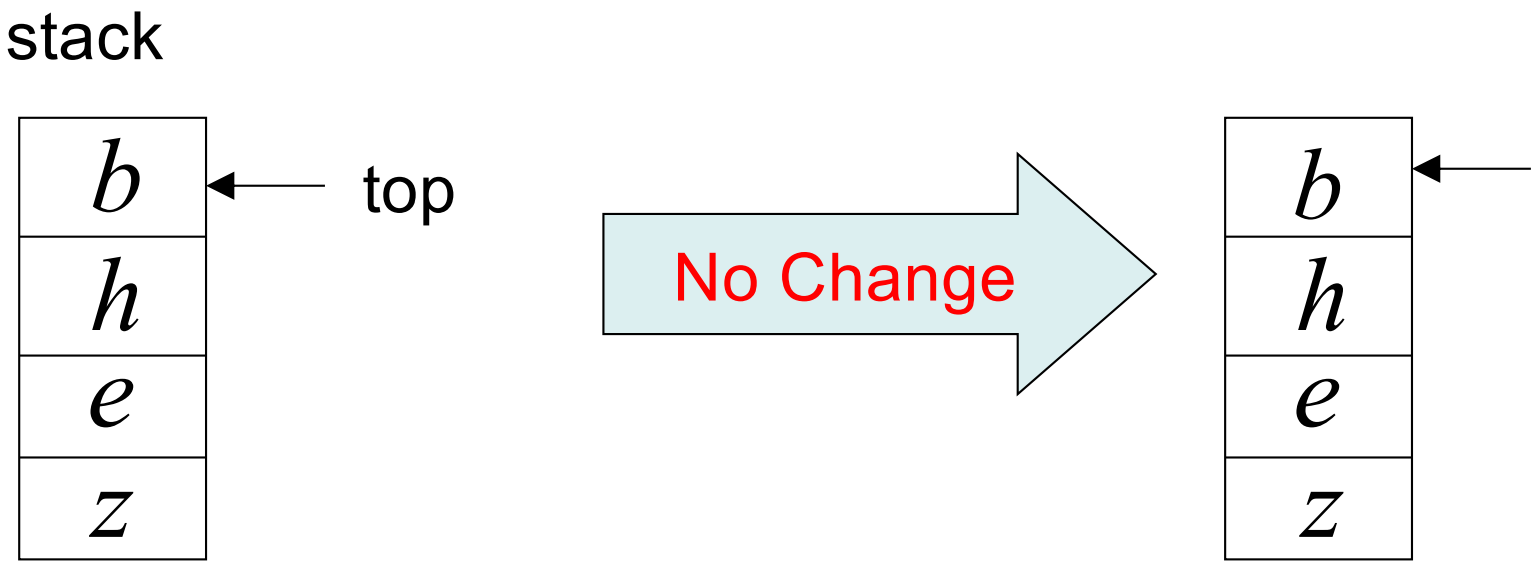

q_1
a, b, λ
q_2

input
$\ldots$
a
$\ldots$
$\ldots$
a
$\ldots$

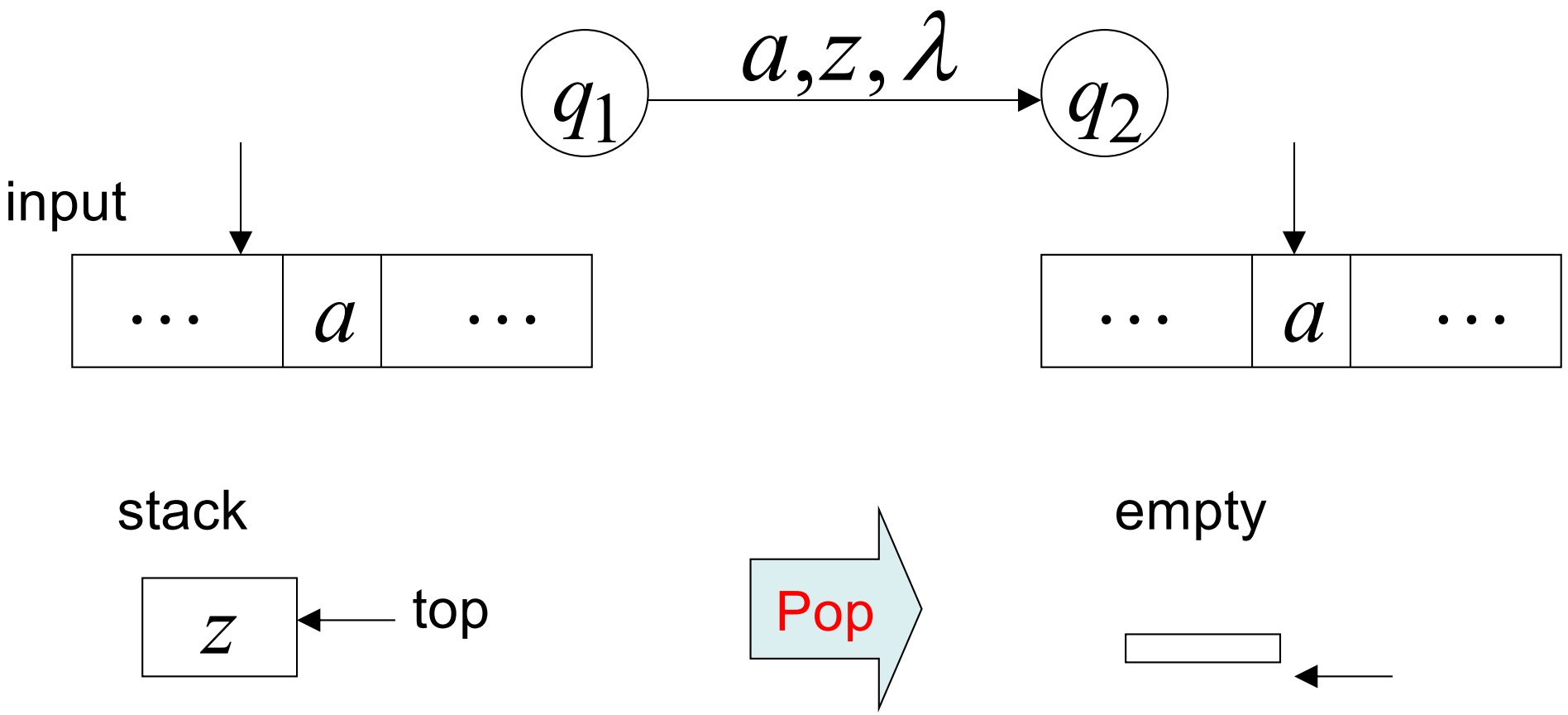

stack
b
top
h
e
z
Pop
h
e
z


q_1
a, λ, λ
q_2

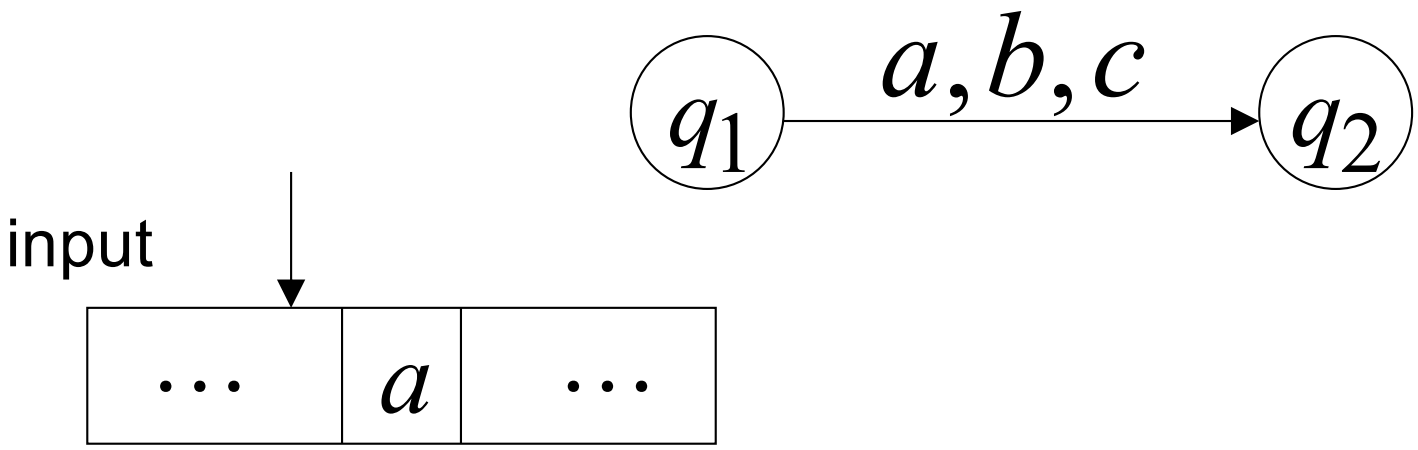
input
$\cdots$
a
$\cdots$
$\cdots$
a
$\cdots$


stack
b
h
e
z
top
No Change
b
h
e
z

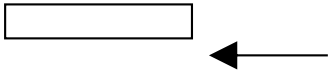
A Possible Transition

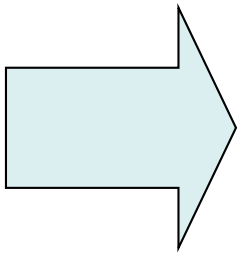


A Bad Transition



Empty stack

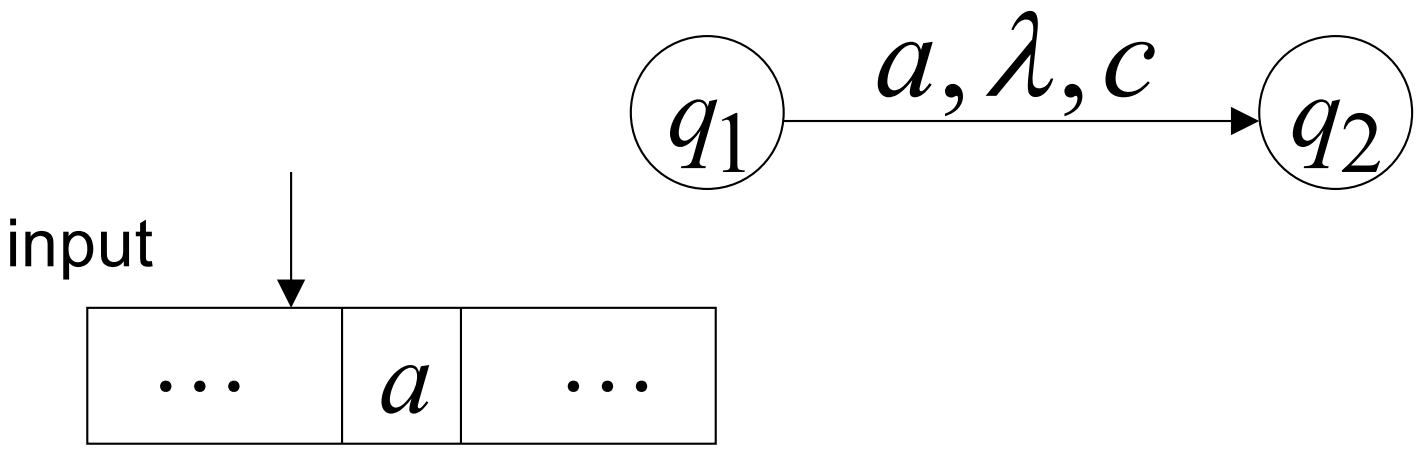




HALT

The automaton **Halts** in state q_1
and **Rejects** the input string

A Bad Transition

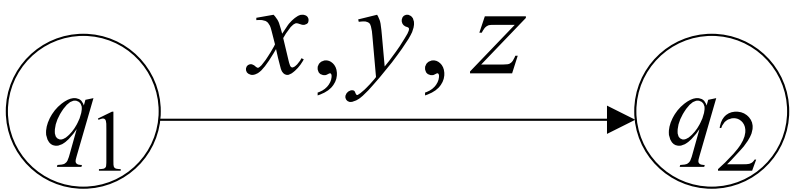


Empty stack

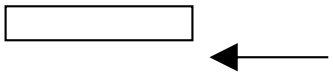
HALT

The automaton **Halts** in state q_1
and **Rejects** the input string

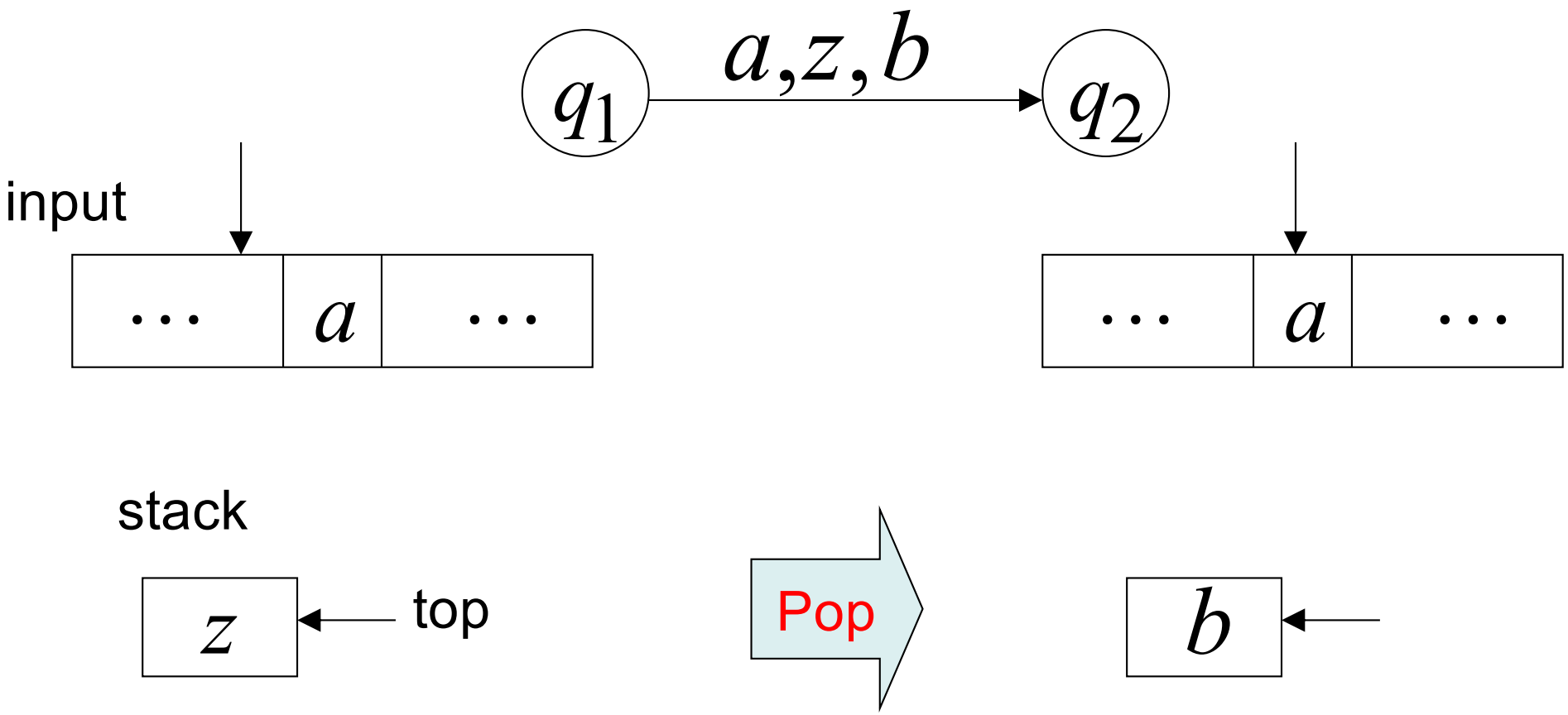
No transition is allowed to be followed
When the stack is empty



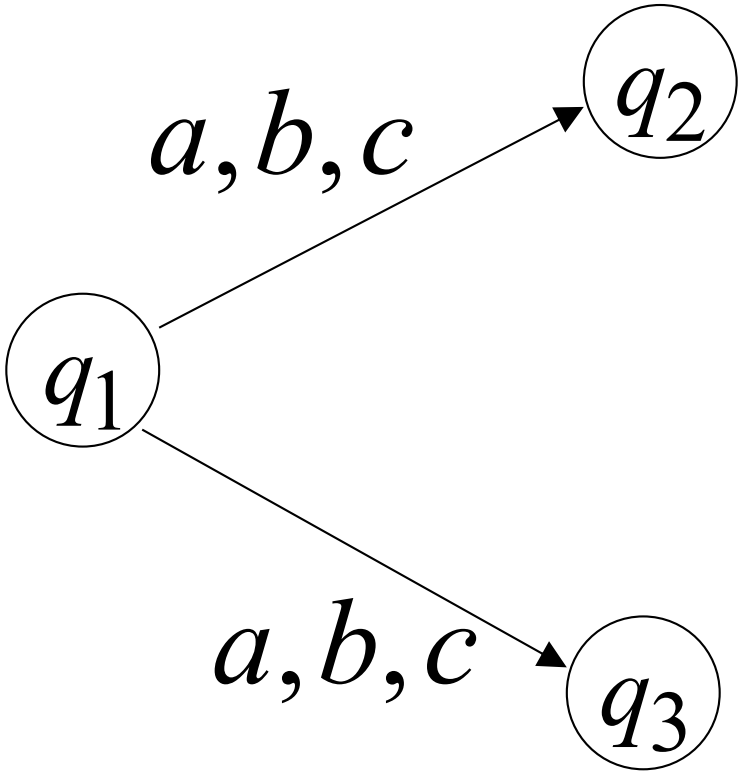
Empty stack

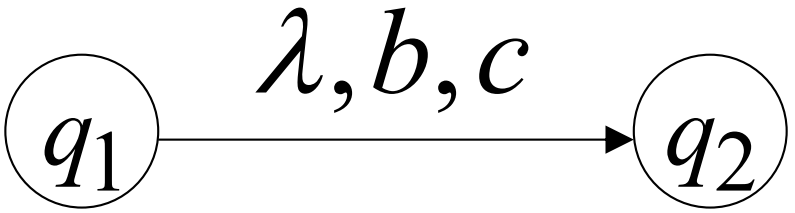


A Good Transition



Non-Determinism



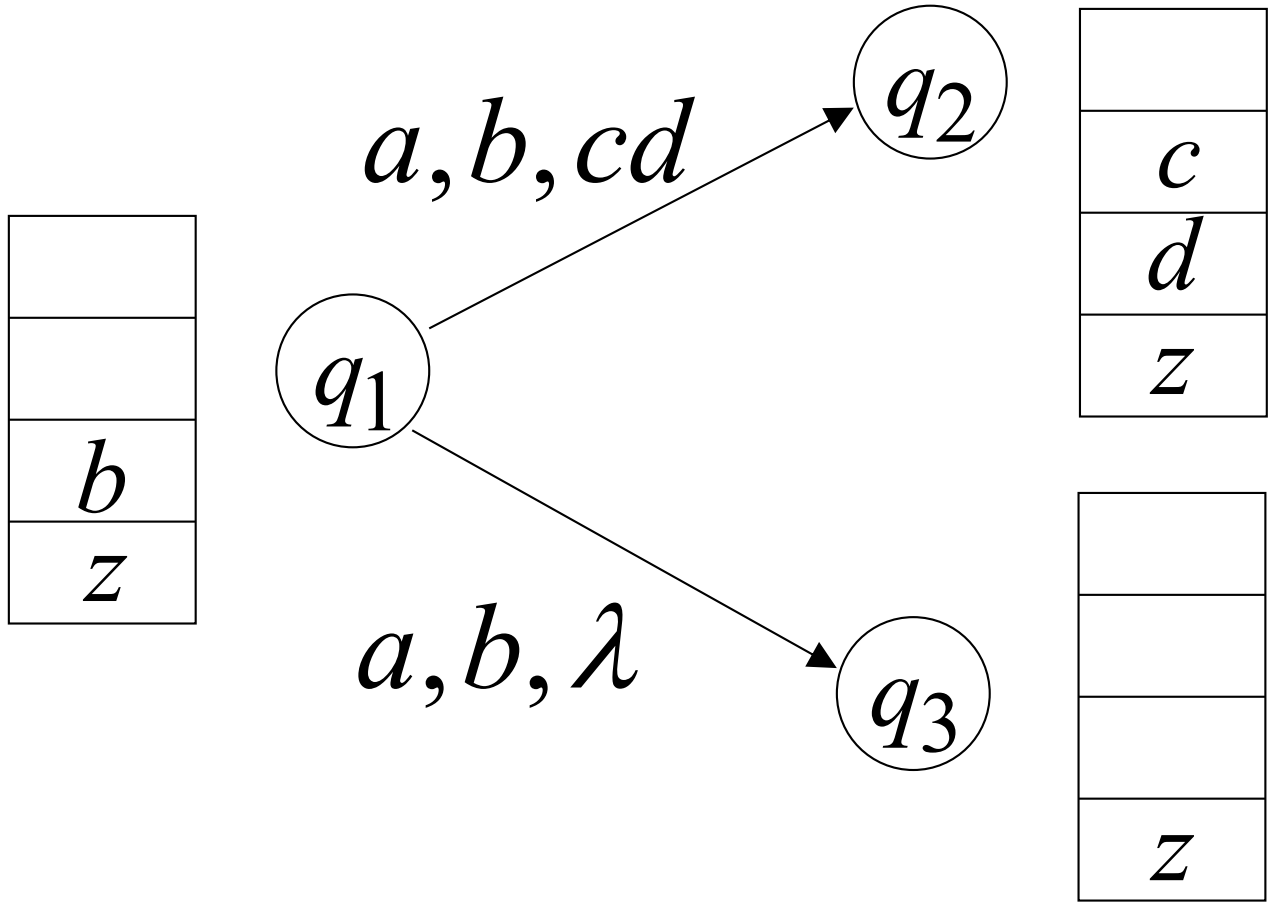


λ – transition

These are allowed transitions in a
Non-deterministic PDA (NPDA)

Example 7.1

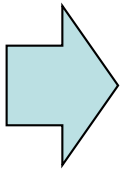
$$\delta(q_1, a, b) = \{(q_2, cd), (q_3, \lambda)\},$$



Example 7.2

$$L = \{a^n b^n : n \geq 0\} \cup \{a\}$$

$$Q = \{q_0, q_1, q_2, q_3\},$$
$$\Sigma = \{a, b\},$$
$$\Gamma = \{0,1\},$$
$$z = 0,$$
$$F = \{q_3\}$$



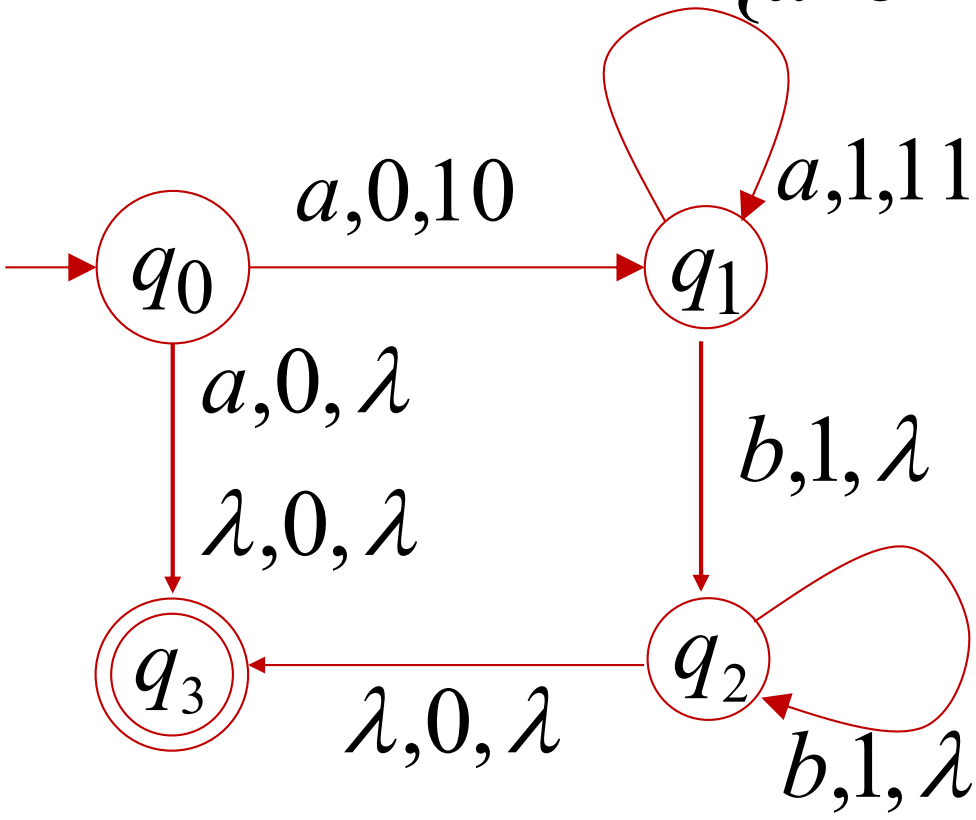
$$\delta(q_0, a, 0) = \{(q_1, 10), (q_3, \lambda)\}$$
$$\delta(q_0, \lambda, 0) = \{(q_3, \lambda)\}$$
$$\delta(q_1, a, 1) = \{(q_1, 11)\}$$
$$\delta(q_1, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, \lambda, 0) = \{(q_3, \lambda)\}$$

$$\delta(q_0, b, 0)?$$

Nondeterministic

Example 7.3

$$L = \{a^n b^n : n \geq 0\} \cup \{a\}$$



$$\delta(q_0, a, 0) = \{(q_1, 10), (q_3, \lambda)\}$$

$$\delta(q_0, \lambda, 0) = \{(q_3, \lambda)\}$$

$$\delta(q_1, a, 1) = \{(q_1, 11)\}$$

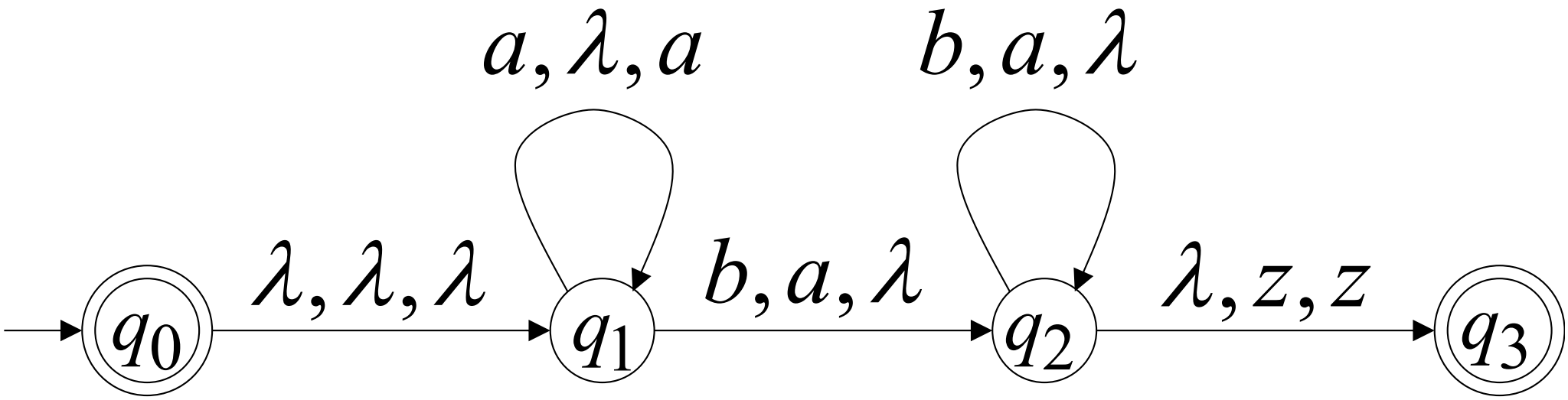
$$\delta(q_1, b, 1) = \{(q_2, \lambda)\}$$

$$\delta(q_2, b, 1) = \{(q_2, \lambda)\}$$

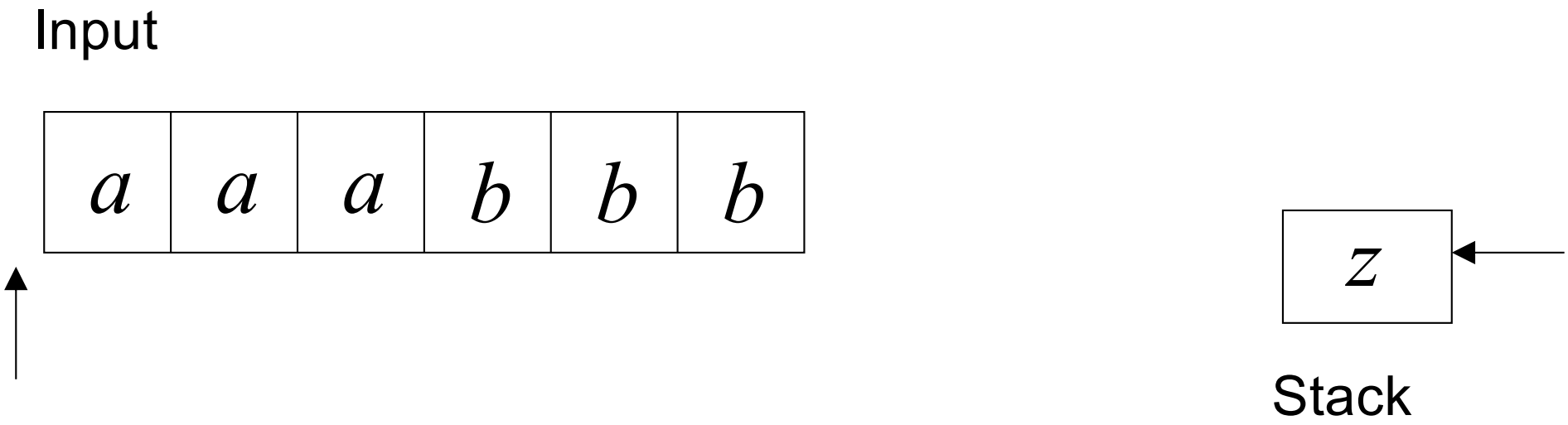
$$\delta(q_2, \lambda, 0) = \{(q_3, \lambda)\}$$

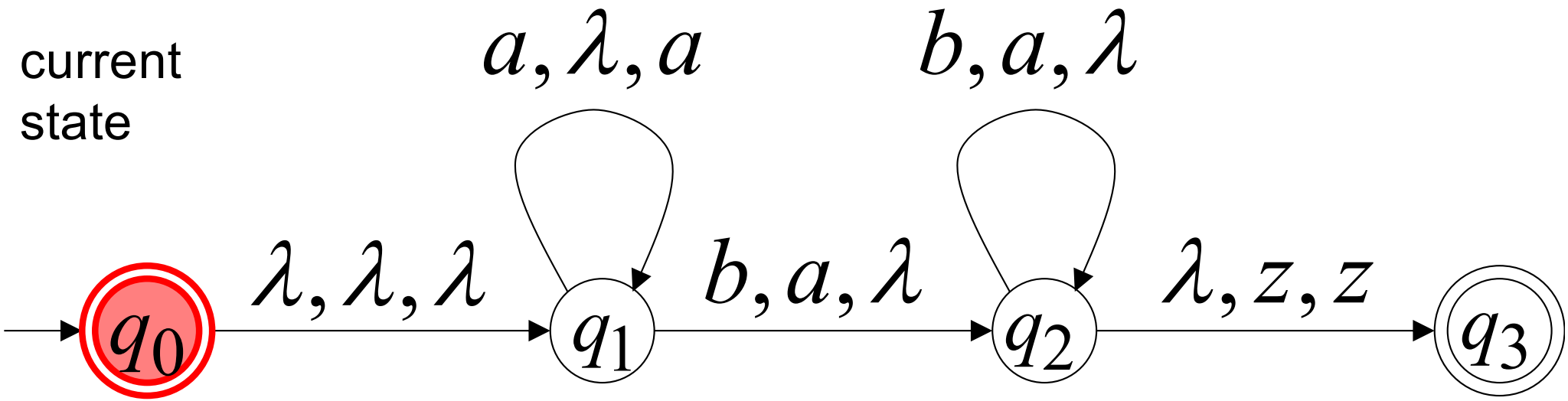
Another Example

Example: $L = \{a^n b^n : n \geq 0\}$

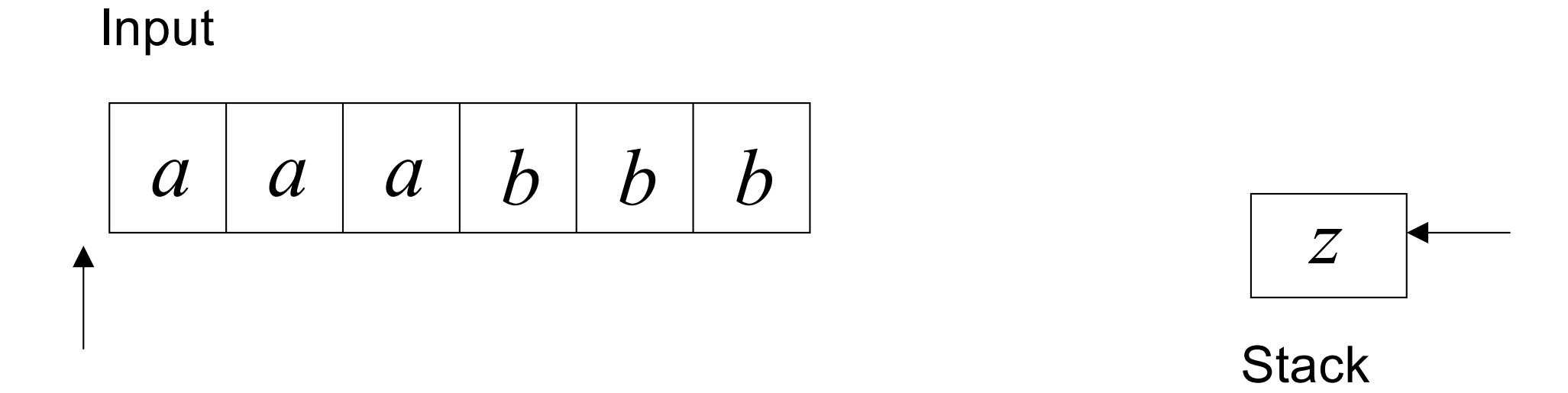


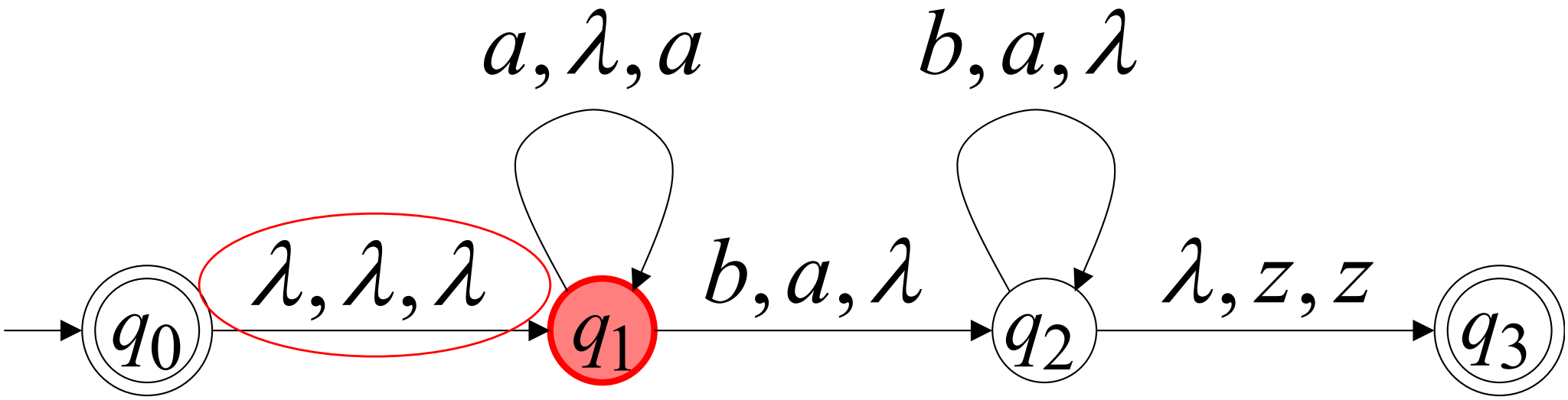
Execution Example: Time 0





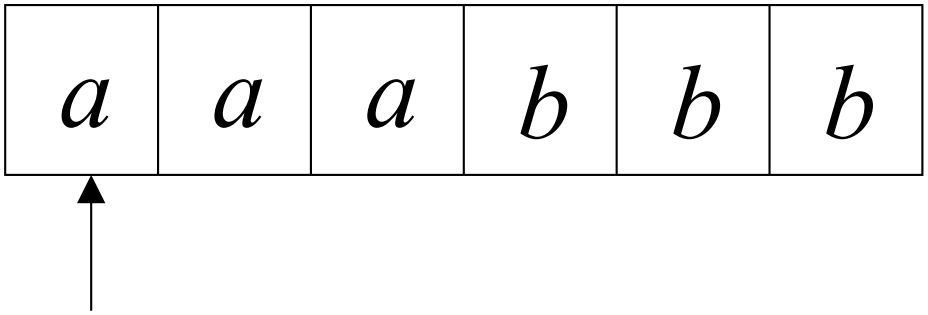
Time 1

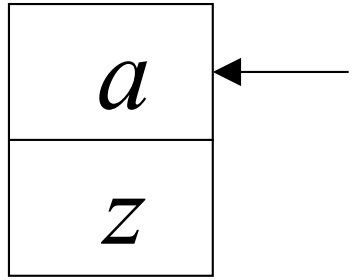




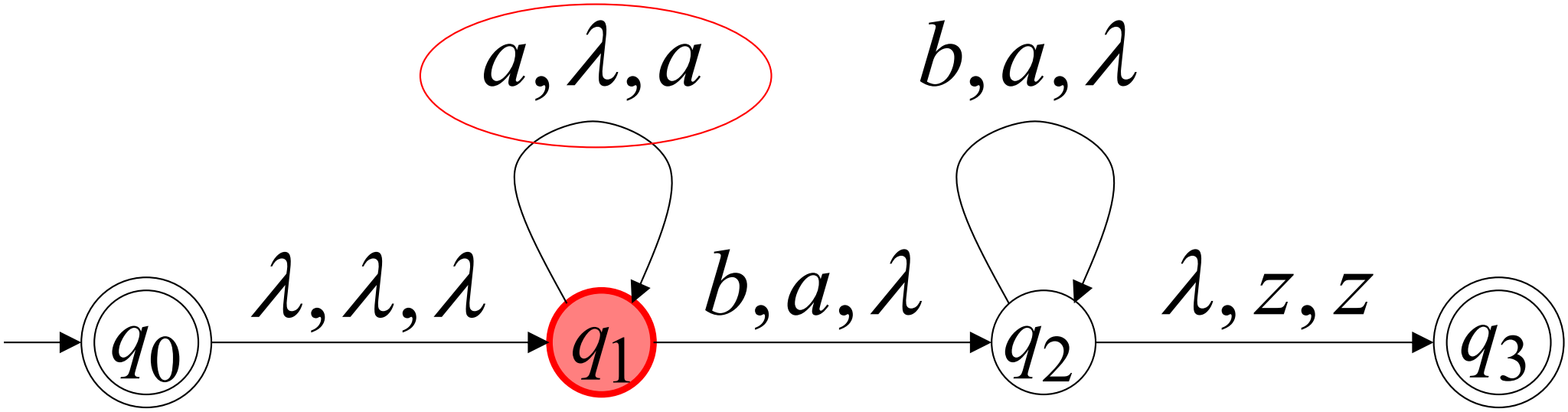
Time 2

Input

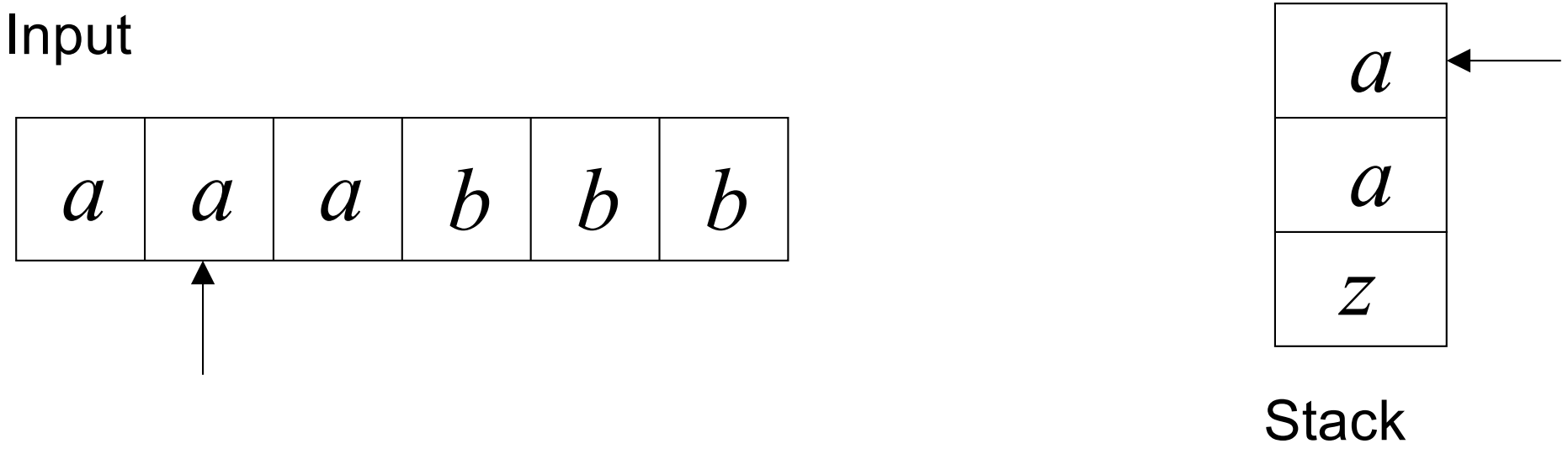


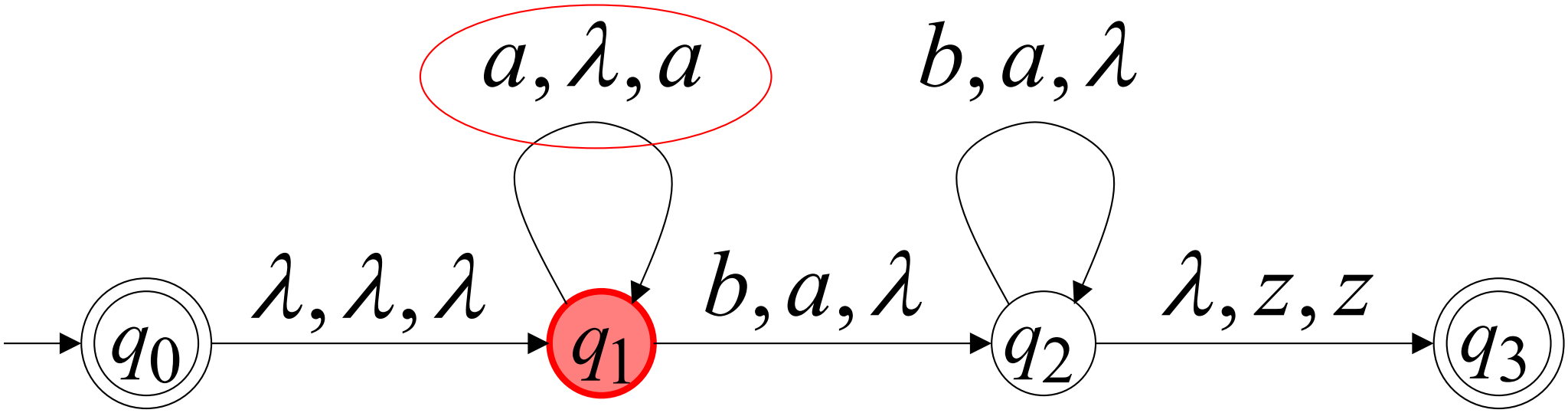


Stack



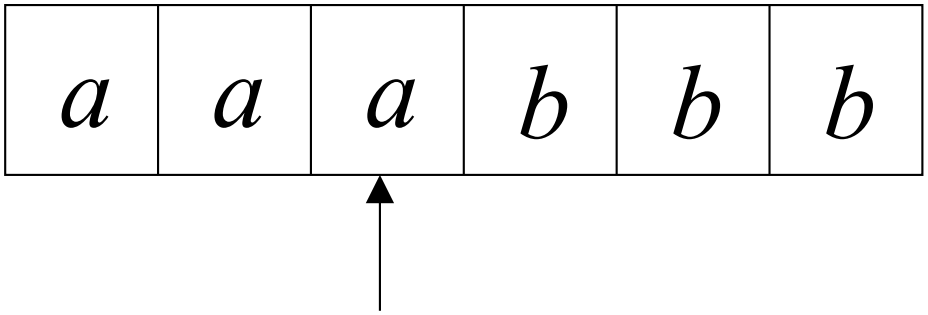
Time 3

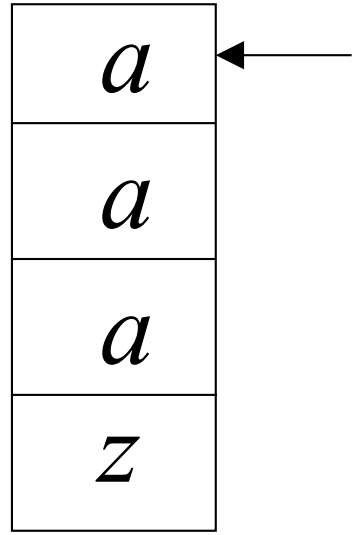




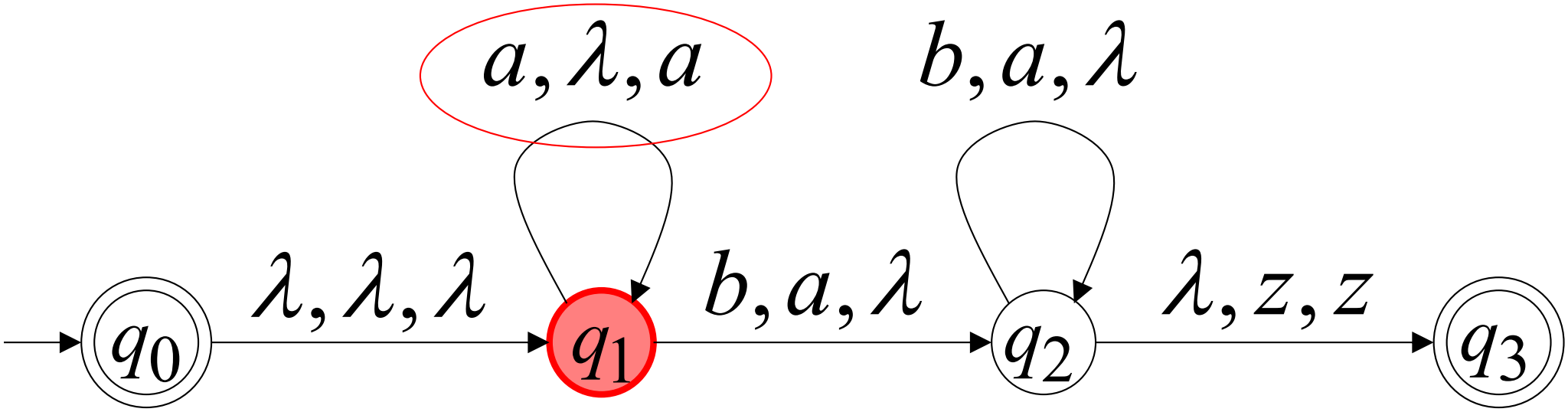
Time 4

Input

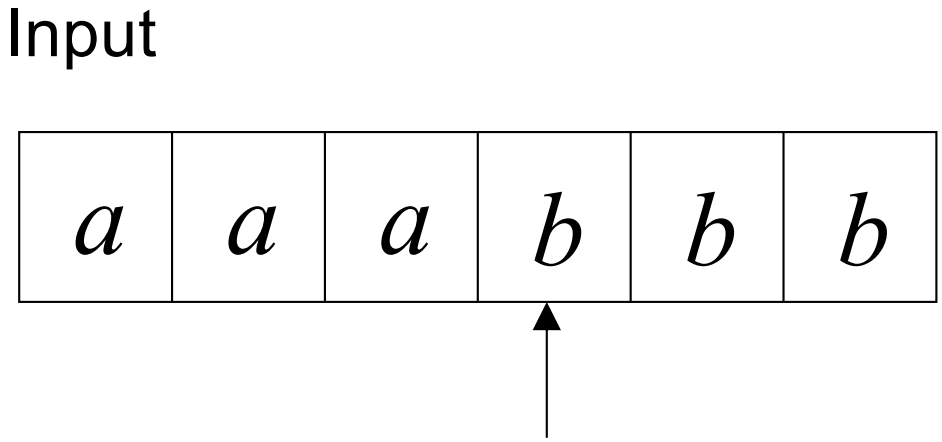


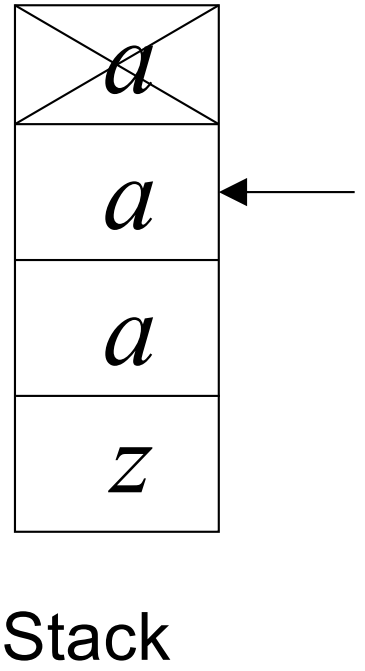


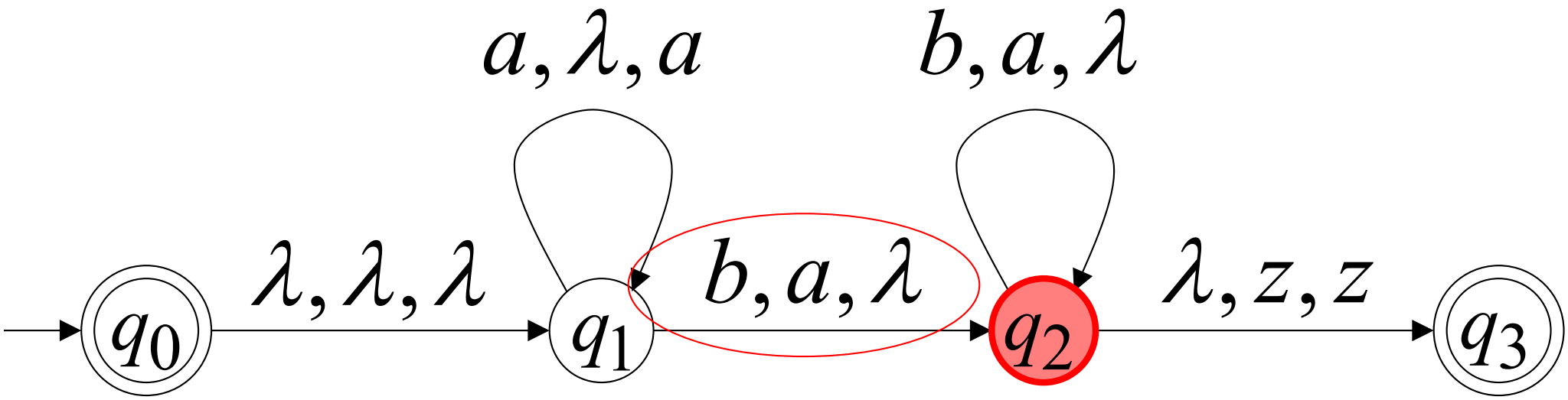
Stack



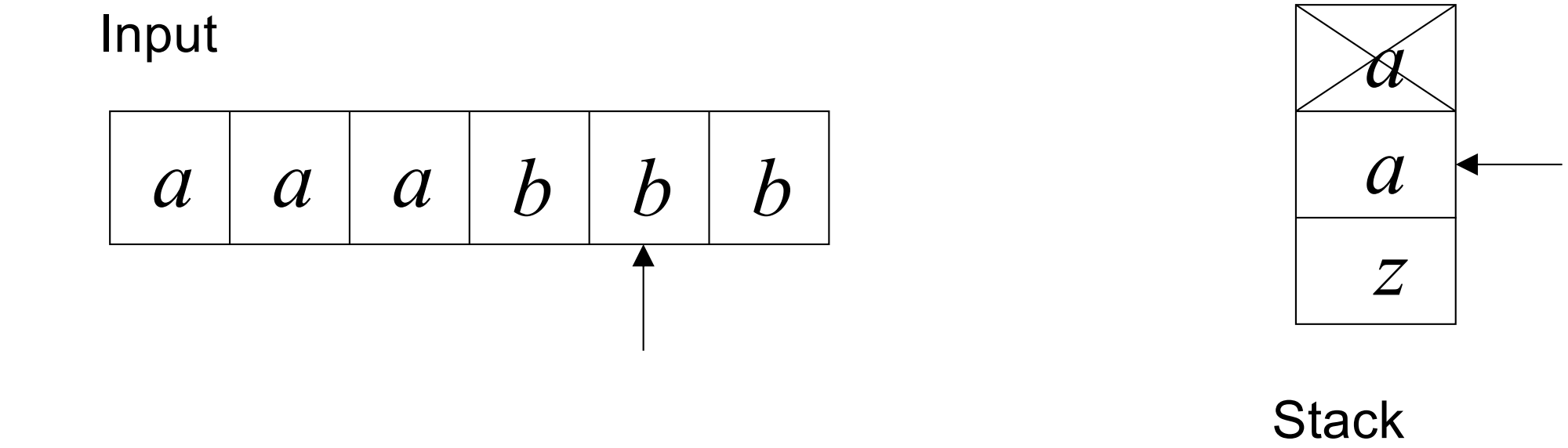
Time 5

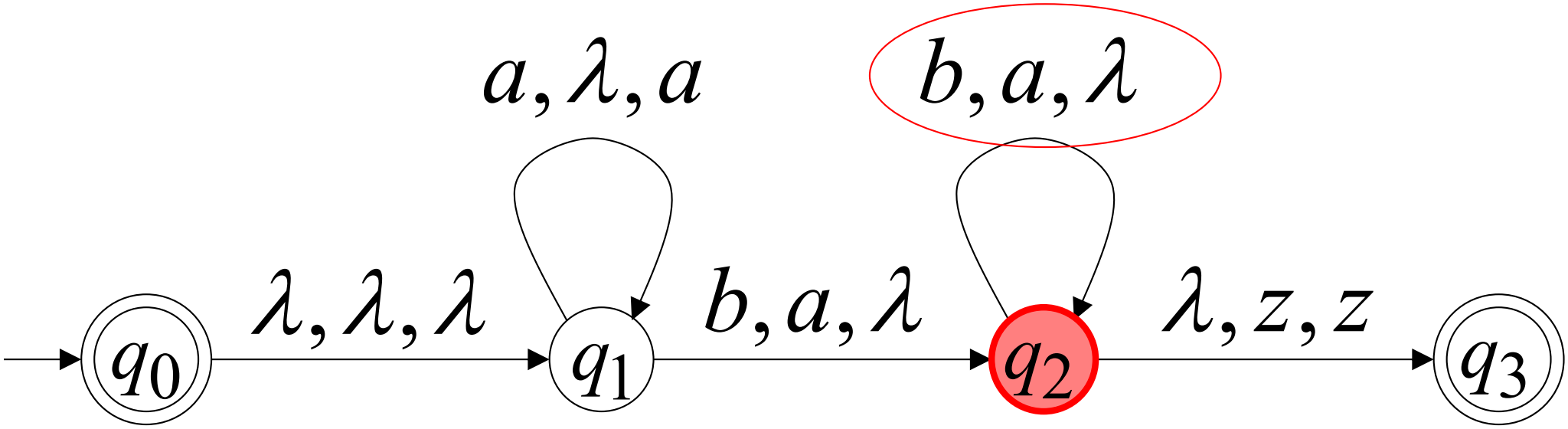






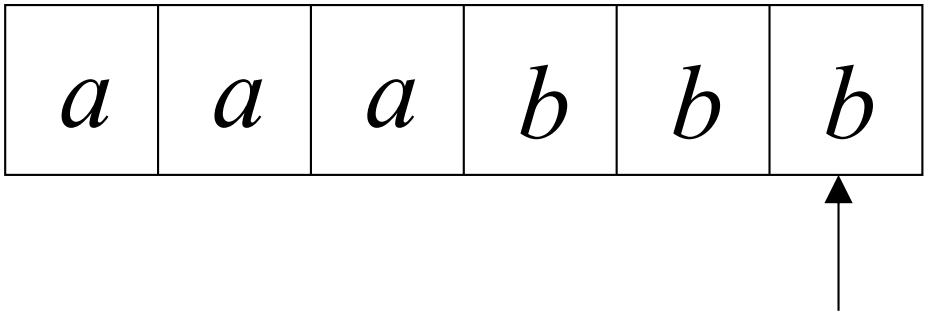
Time 6

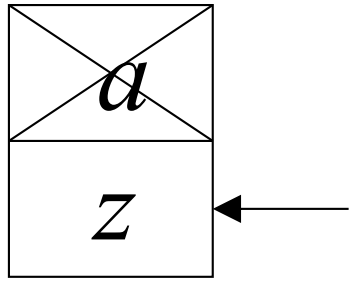




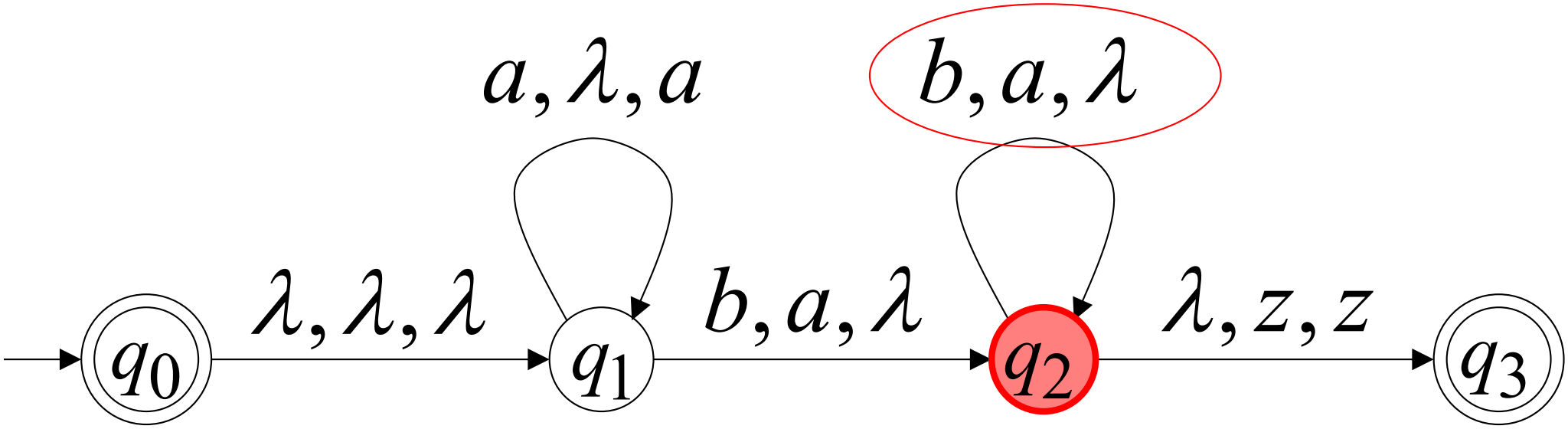
Time 7

Input



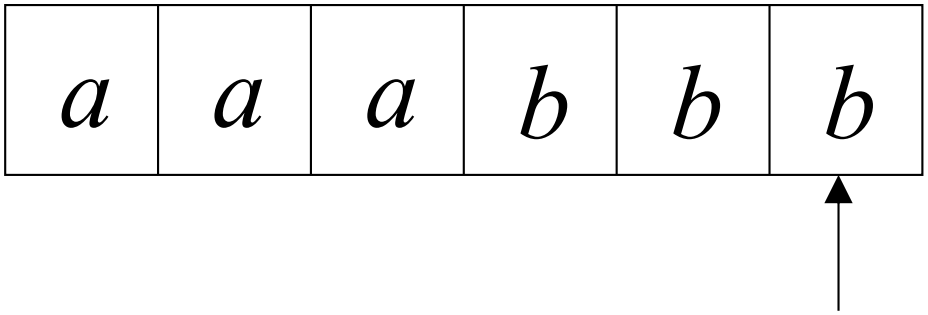


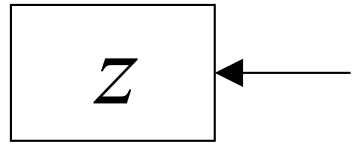
Stack



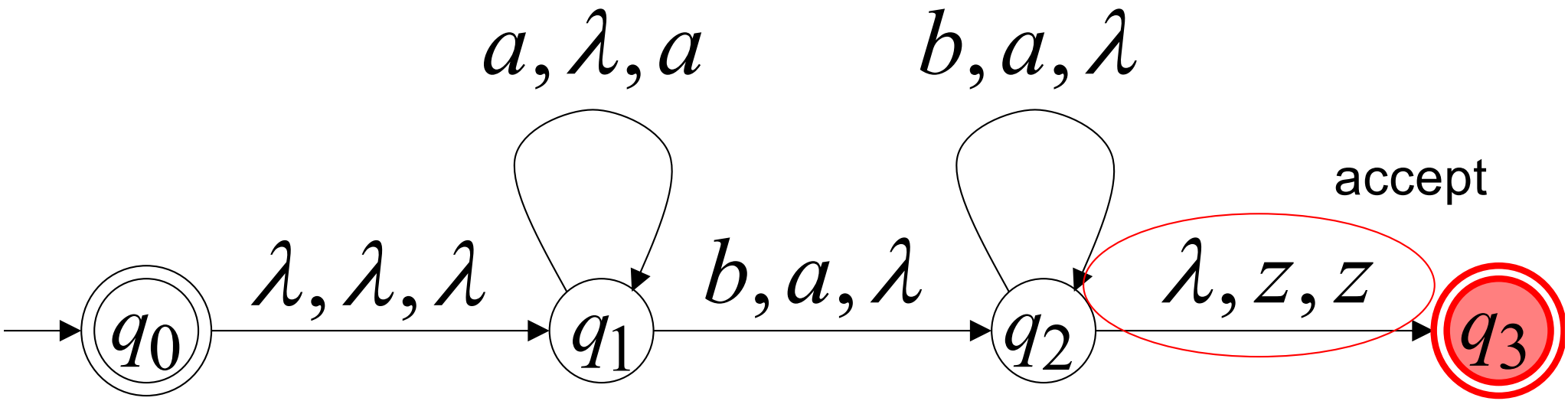
Time 8

Input





Stack



A string is accepted if there is
a computation such that:

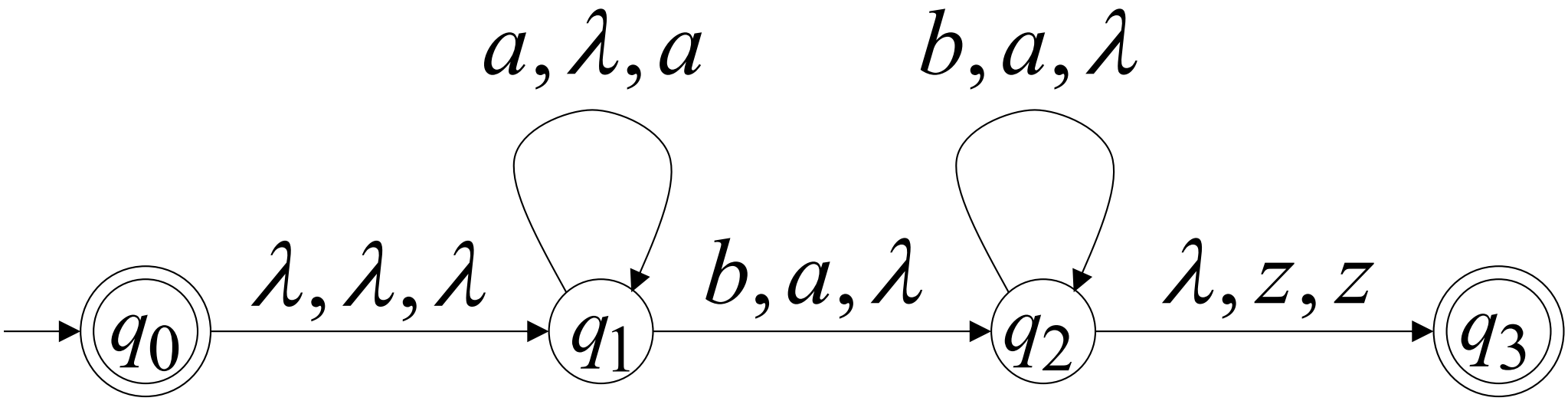
All the input is consumed

AND

The last state is a final state

At the end of the computation,
we do not care about the stack contents

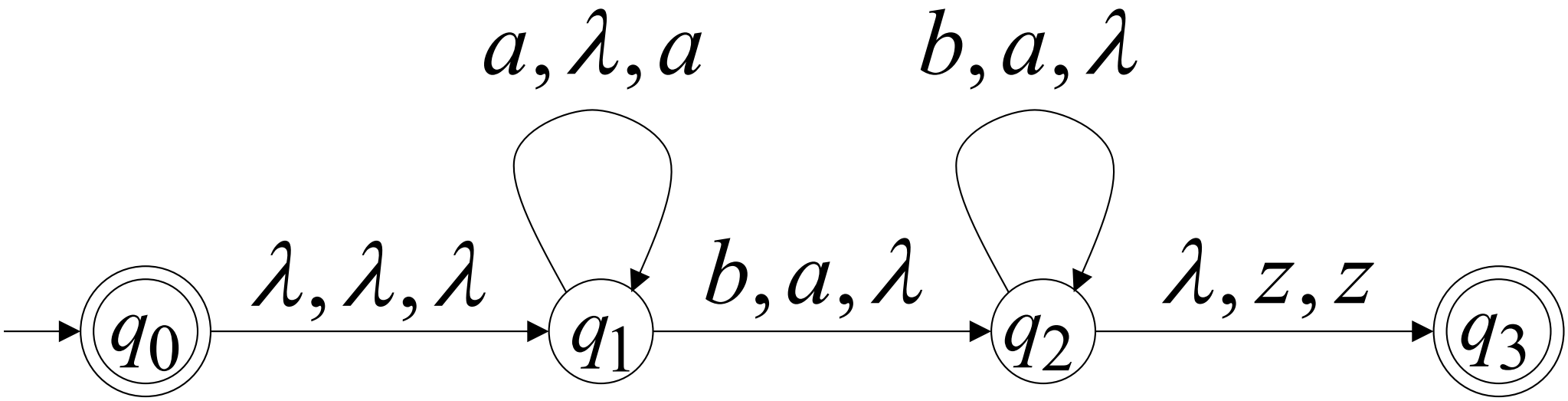
The input string $aaabbb$
is accepted by the NPDA:



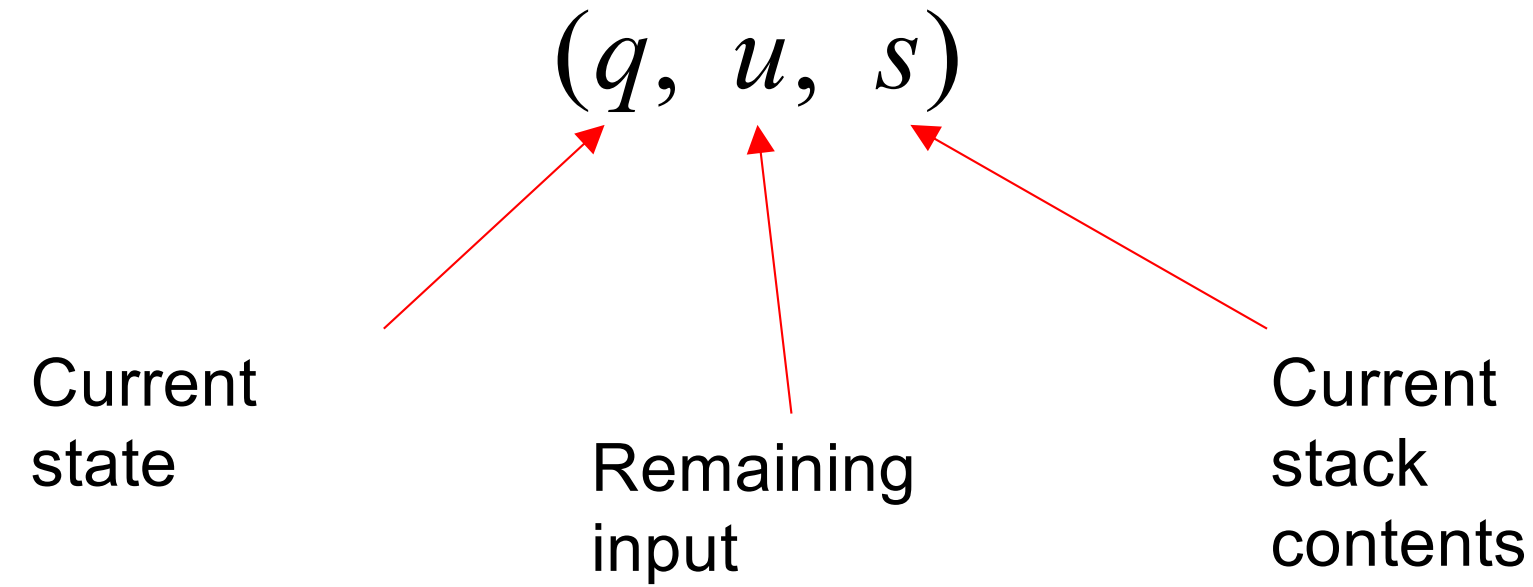
In general,

$$L = \{a^n b^n : n \geq 0\}$$

is the language accepted by the NPDA:



Instantaneous Description (ID)

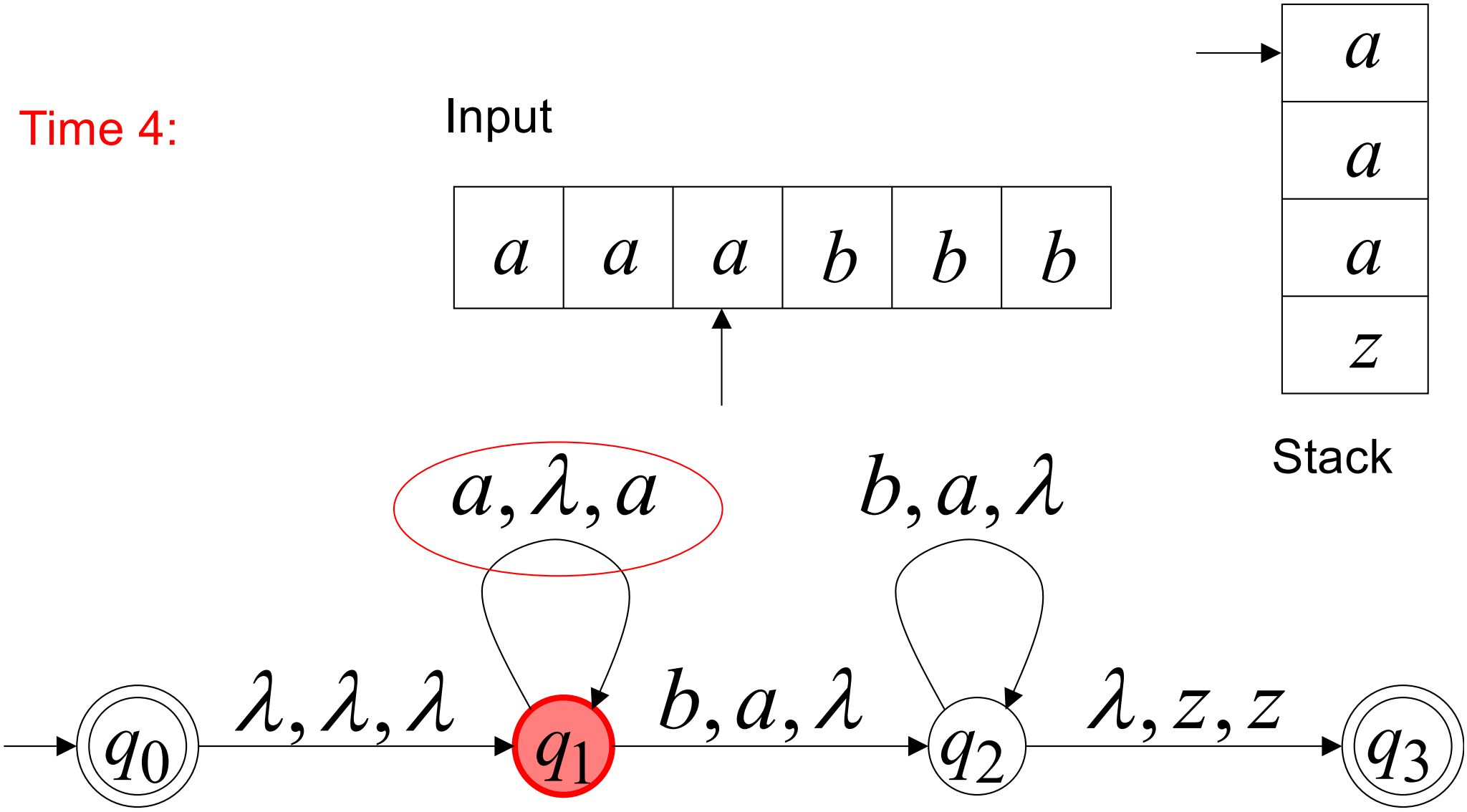


Example:

Instantaneous Description

$$(q_1, bbb, aaaz)$$

Time 4:

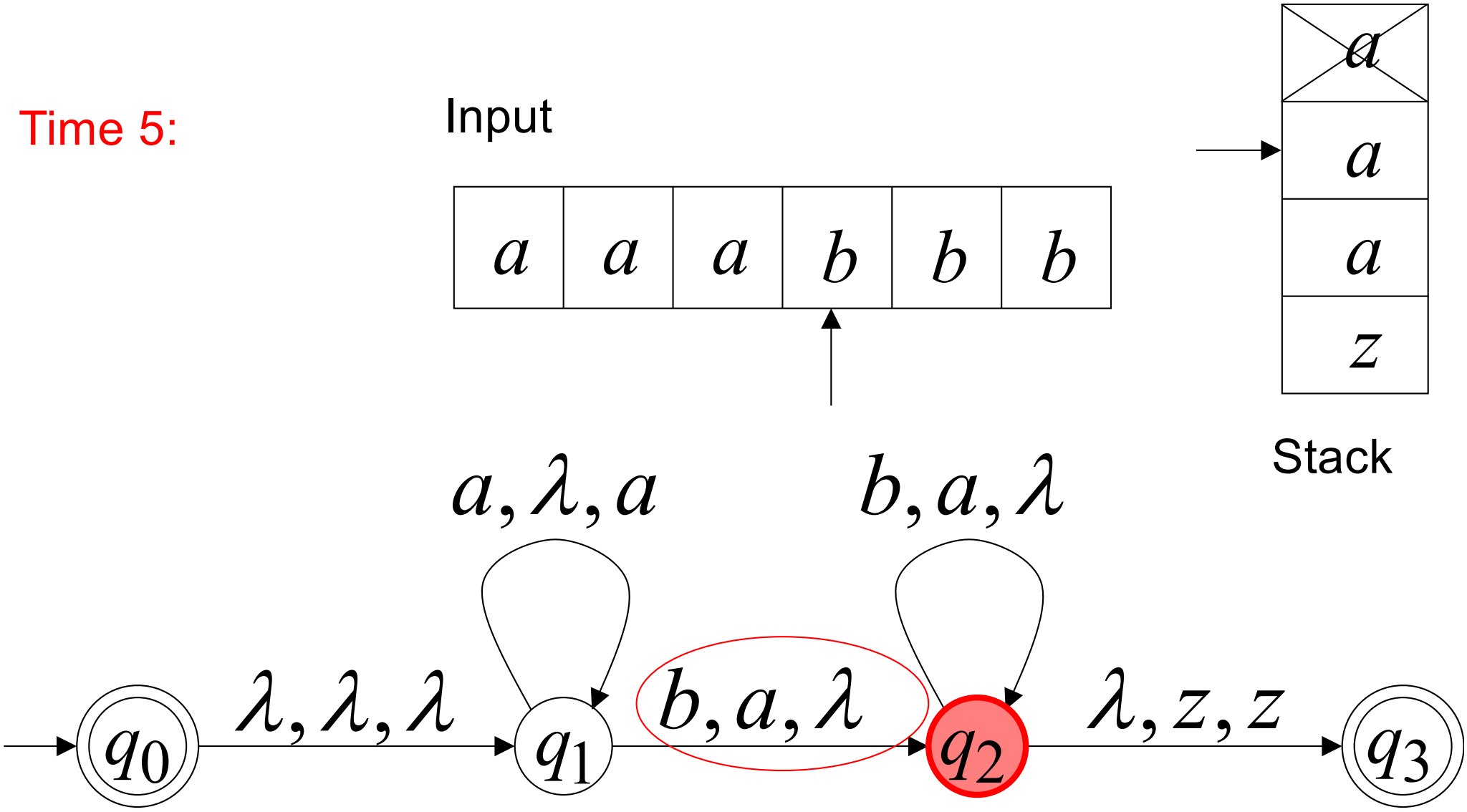


Example:

Instantaneous Description

$$(q_2, bb, aaz)$$

Time 5:



We write:

$$(q_1, bbb, aaaz) \vdash (q_2, bb, aaz)$$

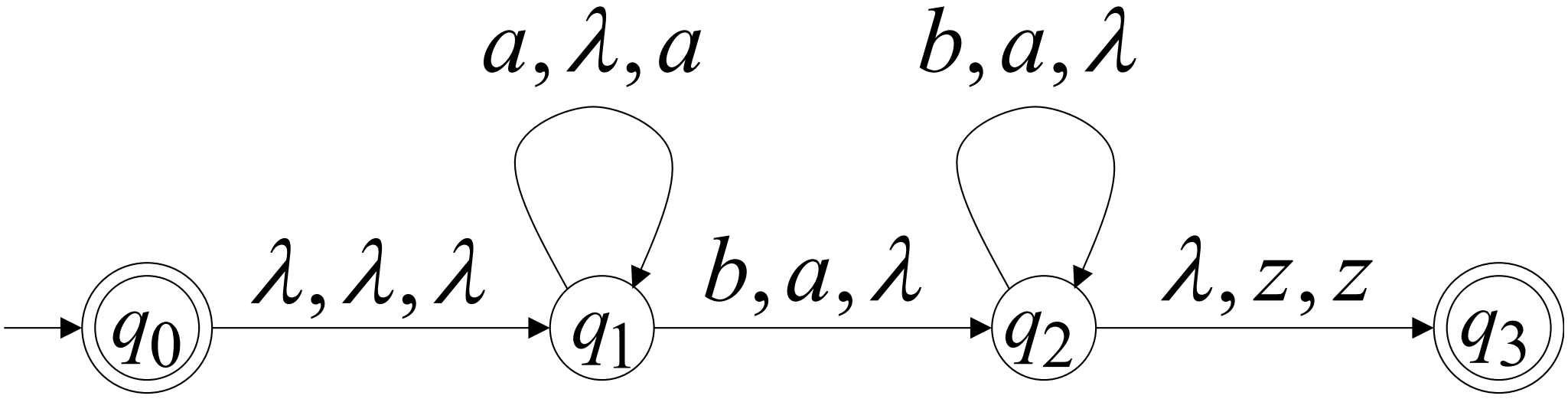
Time 4

Time 5

The symbol denotes a move from one ID to another

A computation:

$$(q_0, aaabbb, z) \vdash (q_1, aaabbb, z) \vdash$$
$$(q_1, aabbb, az) \vdash (q_1, abbb, aaz) \vdash (q_1, bbb, aaaz) \vdash$$
$$(q_2, bb, aaz) \vdash (q_2, b, az) \vdash (q_2, \lambda, z) \vdash (q_3, \lambda, z)$$



$$(q_0, aaabbb, z) \vdash (q_1, aaabbb, z) \vdash$$
$$(q_1, aabbb, az) \vdash (q_1, abbb, aaz) \vdash (q_1, bbb, aaaz) \vdash$$
$$(q_2, bb, aaz) \vdash (q_2, b, az) \vdash (q_2, \lambda, z) \vdash (q_3, \lambda, z)$$

For convenience we write:

$$(q_0, aaabbb, z) \overset{*}{\vdash} (q_3, \lambda, z)$$

Formal Definition

Language $L(M)$ of NPDA M:

$$L(M) = \{w: \ (q_0, w, z) \overset{*}{\vdash} (q_f, \lambda, s)\}$$

Initial state

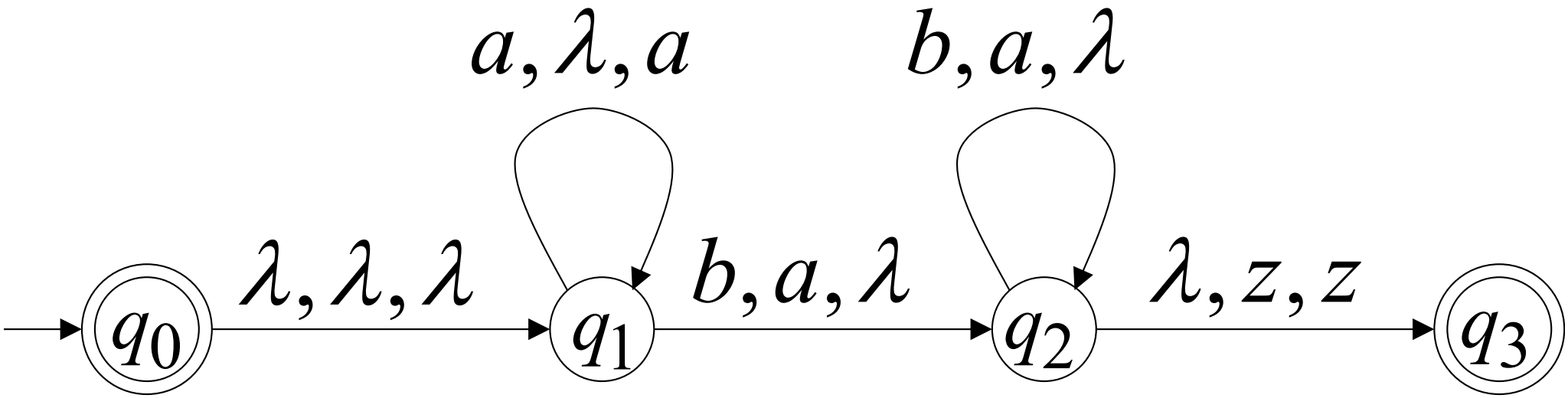
Final state

Example:

$$(q_0, aaabbb, z) \overset{*}{\vdash} (q_3, \lambda, z)$$

$$aaabbb \in L(M)$$

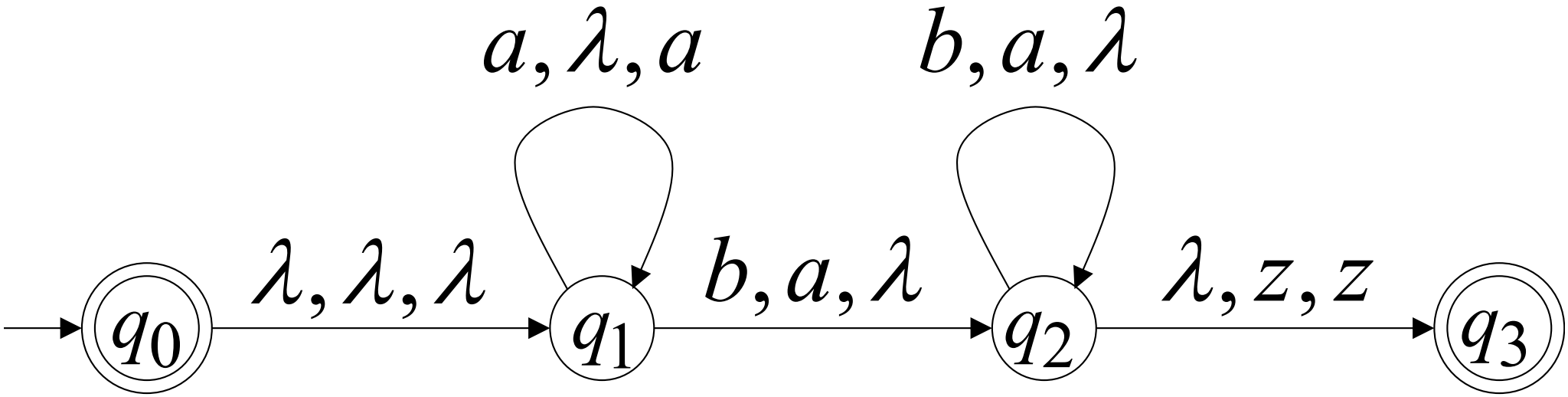
NPDA M :



$$(q_0, a^n b^n, z) \overset{*}{\vdash} (q_3, \lambda, z)$$

$$a^n b^n \in L(M)$$

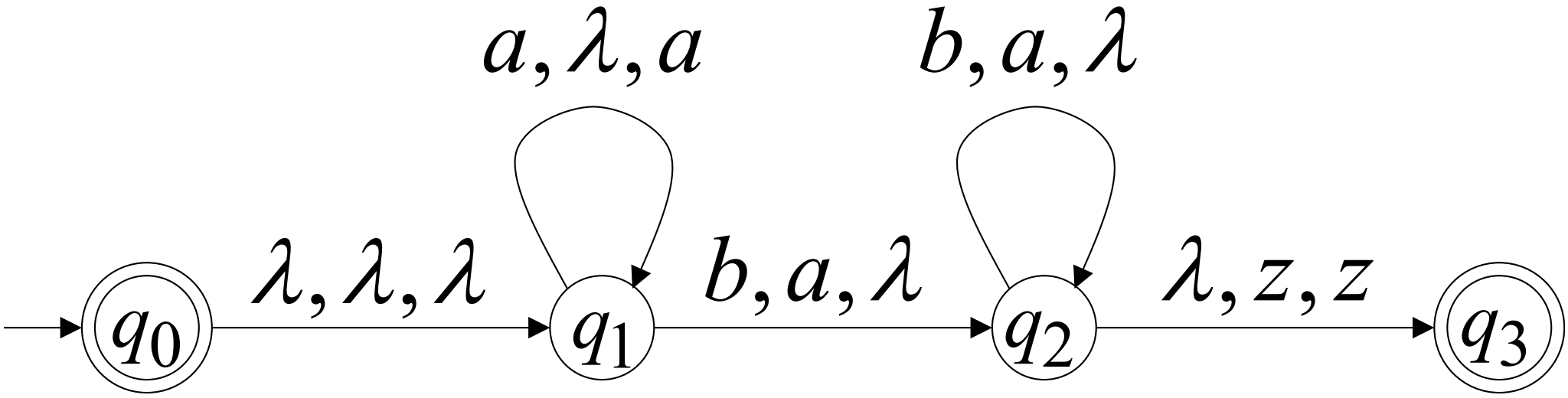
NPDA M :



Therefore:

$$L(M) = \{a^n b^n : n \geq 0\}$$

NPDA M :



Example 7.4

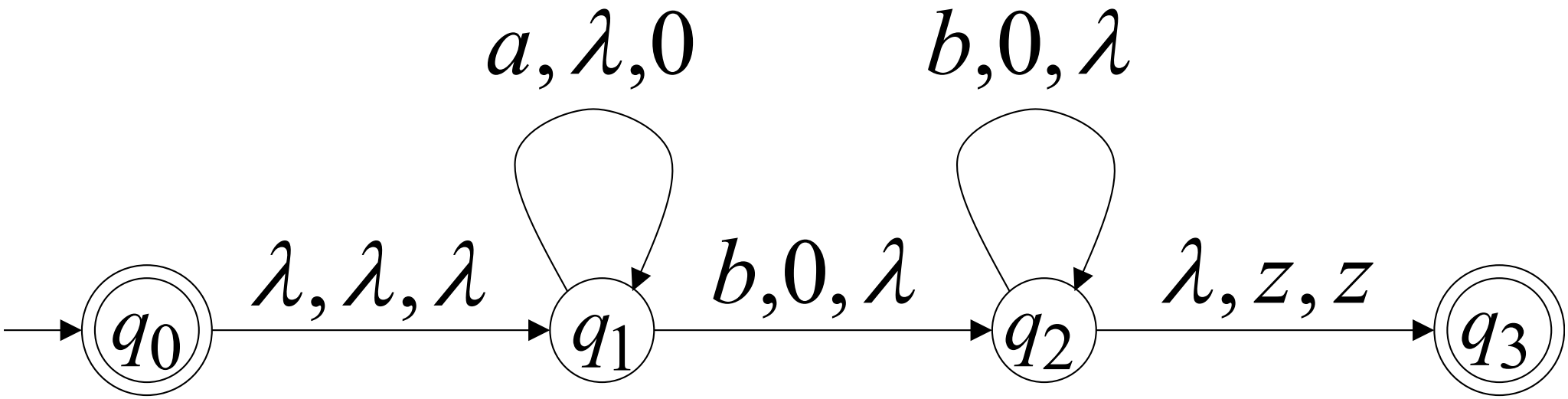
NPDA M

$$L(M) = \{w: \quad n_a = n_b\}$$

Stack

Order of *a* and *b* is don't care

How about more *b*'s than *a*'s in the prefix of *w*?



Example 7.4

NPDA M

$$L(M) = \{w: \quad n_a = n_b\}$$

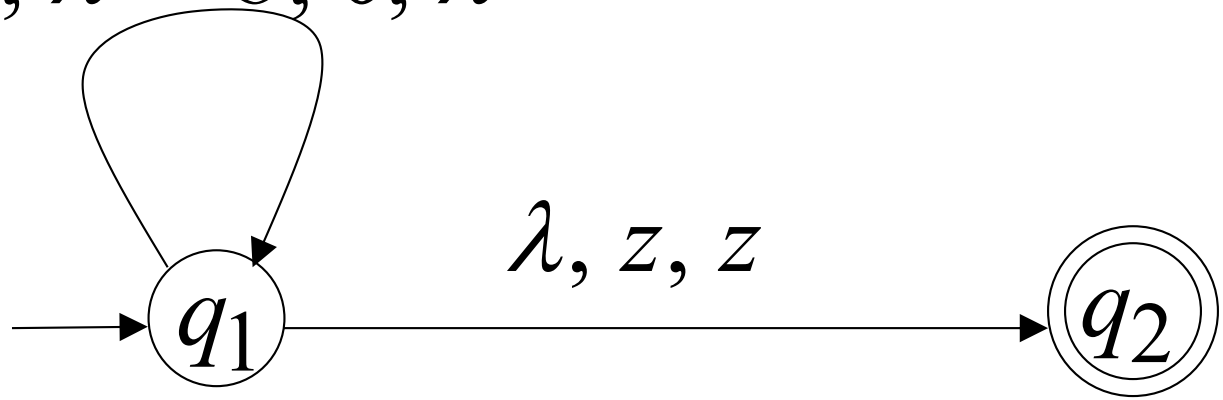
$a, z, 0z \quad b, z, 1z$

$a, 0, 00 \quad b, 1, 11$

$a, 1, \lambda \quad b, 0, \lambda$

Single loop

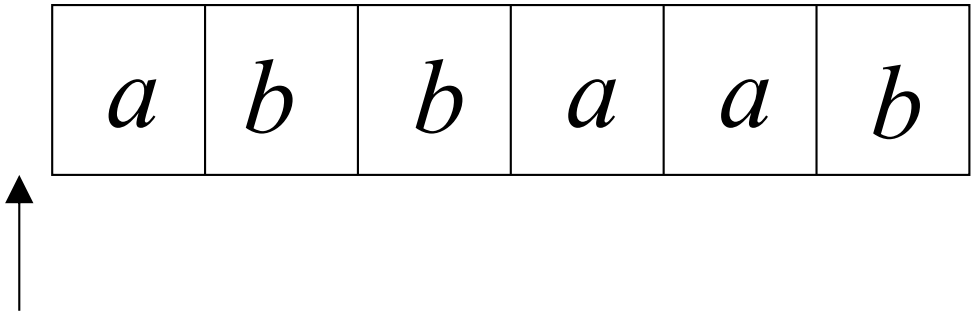
Use negative counter symbol (1)

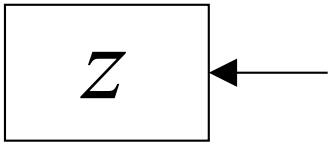


Execution Example:

Time 0

Input





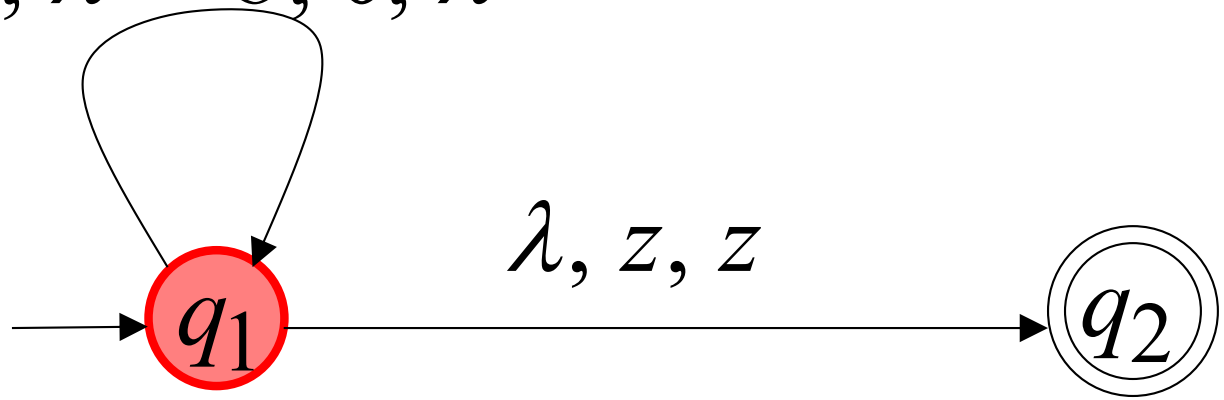
Stack

$a, z, 0z \quad b, z, 1z$

$a, 0, 00 \quad b, 1, 11$

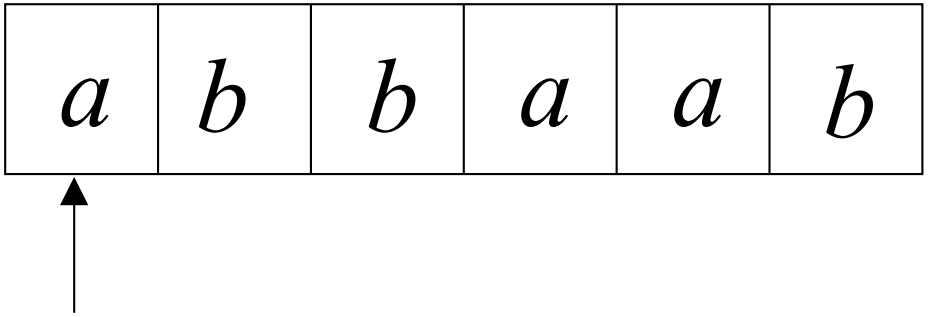
$a, 1, \lambda \quad b, 0, \lambda$

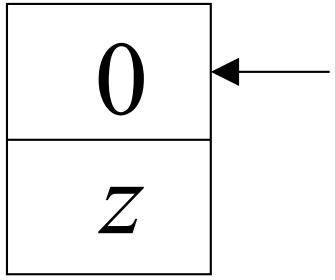
current state



Time 1

Input



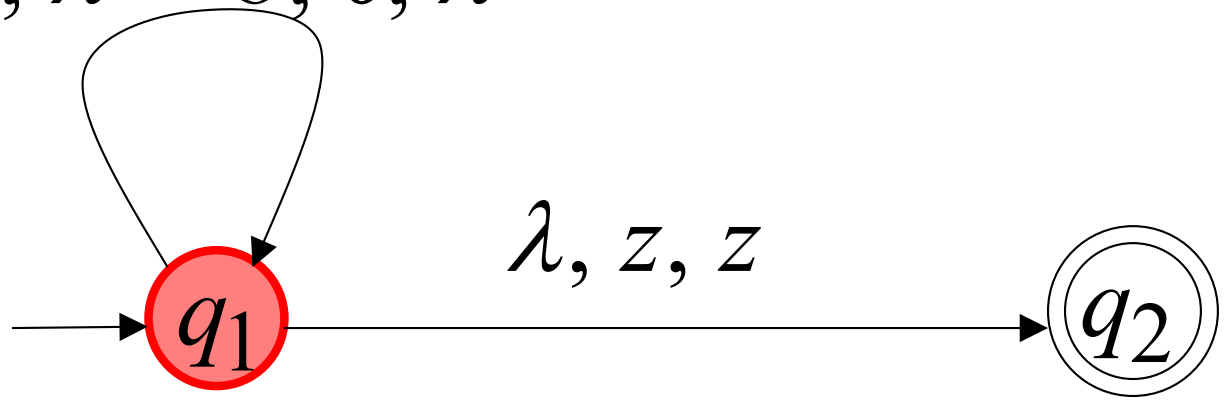


Stack

$a, z, 0z$ $b, z, 1z$

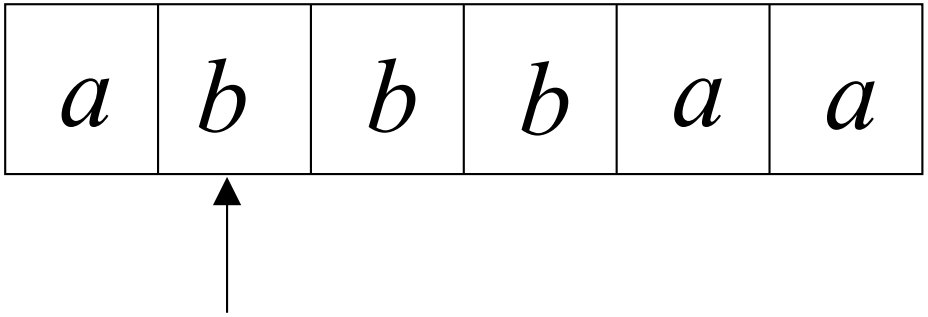
$a, 0, 00$ $b, 1, 11$

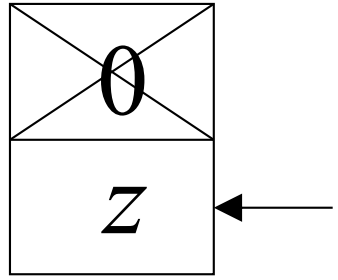
$a, 1, \lambda$ $b, 0, \lambda$



Time 2

Input



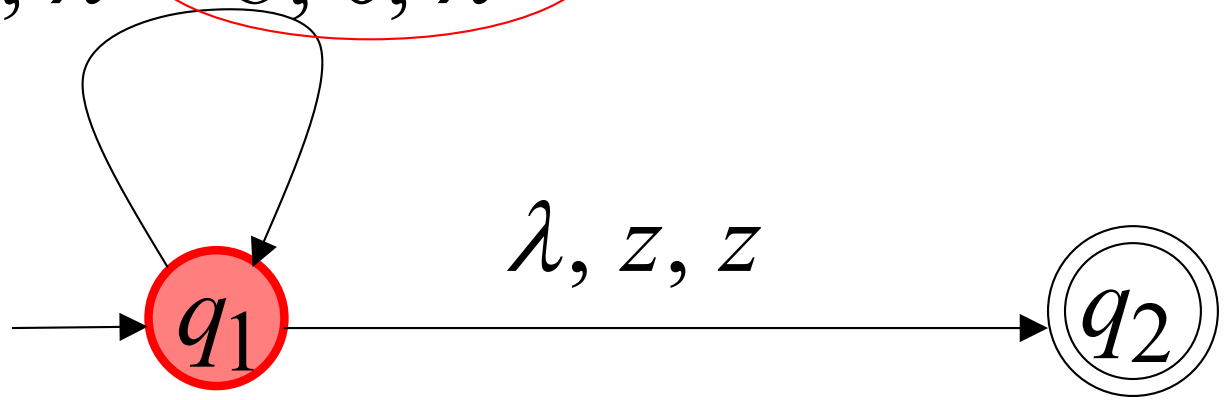


Stack

$a, z, 0z \quad b, z, 1z$

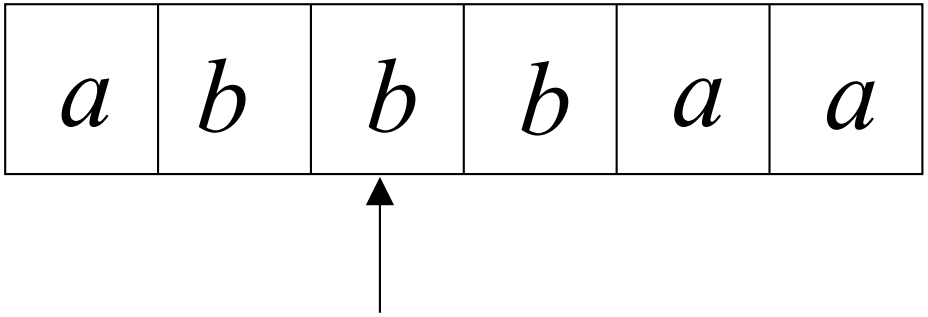
$a, 0, 00 \quad b, 1, 11$

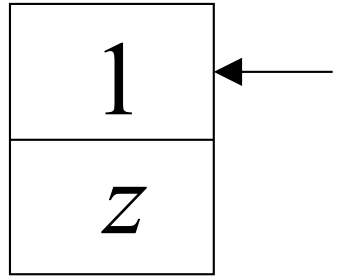
$a, 1, \lambda \quad b, 0, \lambda$



Time 3

Input



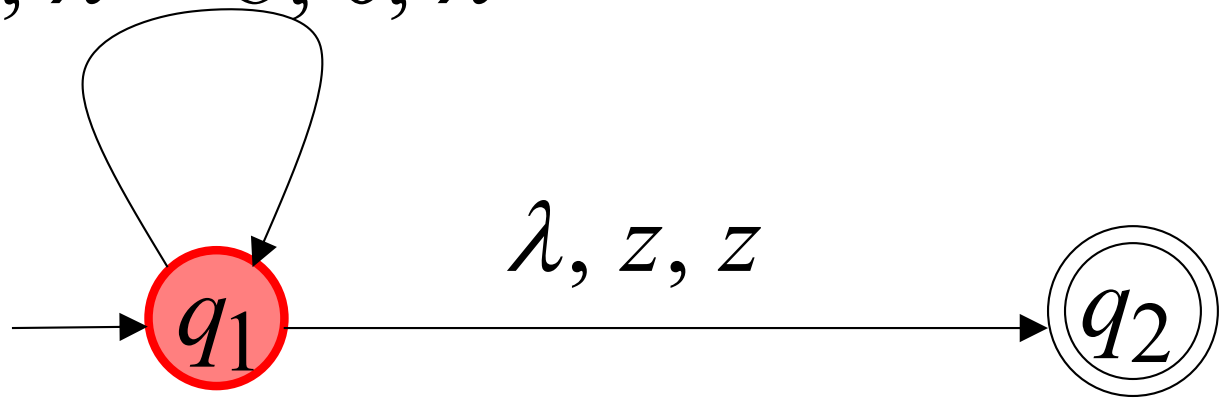


Stack

$a, z, 0z$ $b, z, 1z$

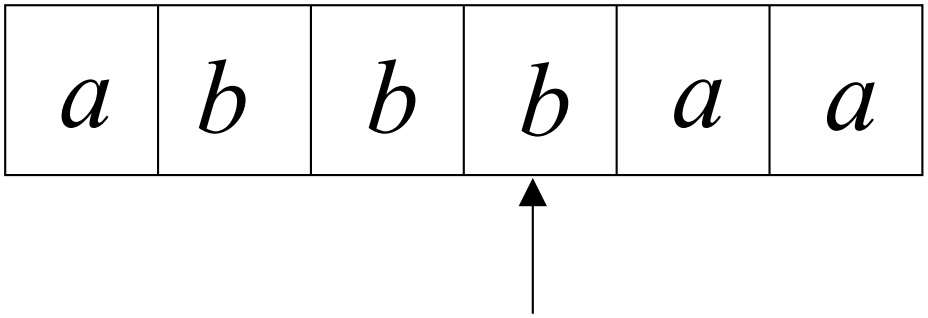
$a, 0, 00$ $b, 1, 11$

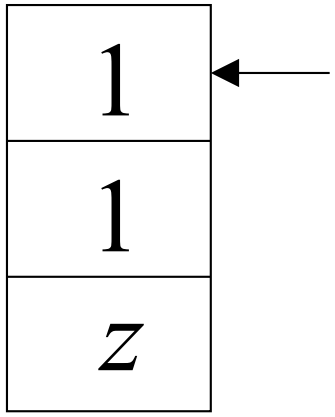
$a, 1, \lambda$ $b, 0, \lambda$



Time 4

Input



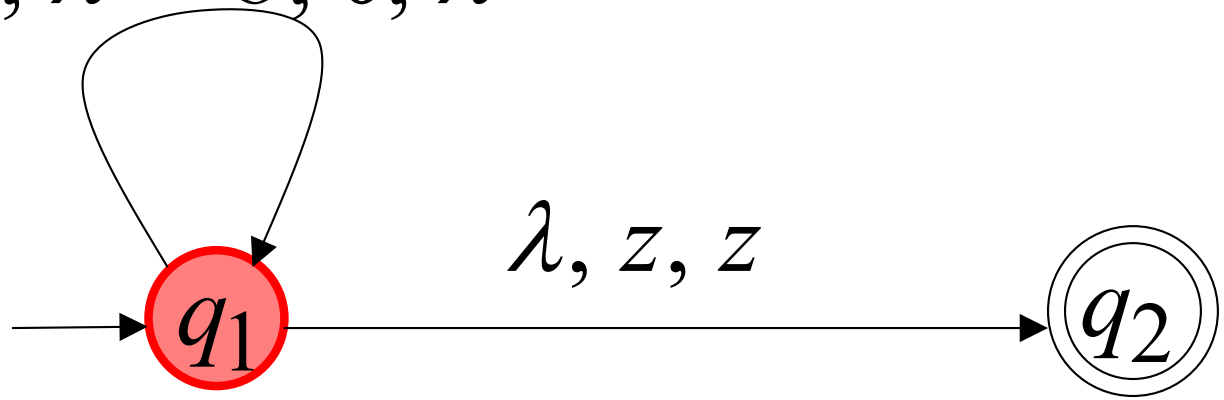


Stack

$a, z, 0z \quad b, z, 1z$

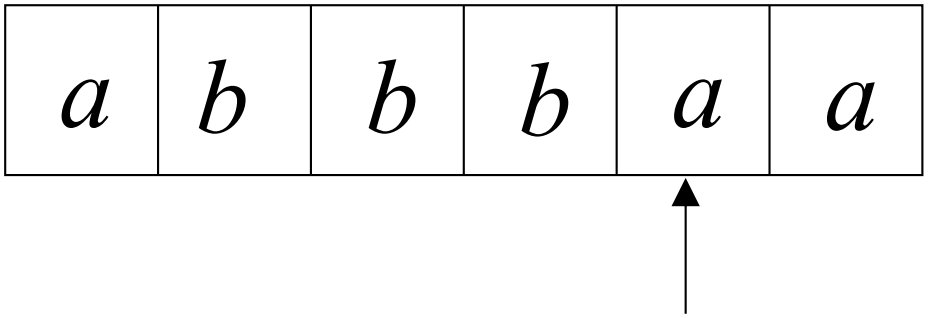
$a, 0, 00 \quad b, 1, 11$

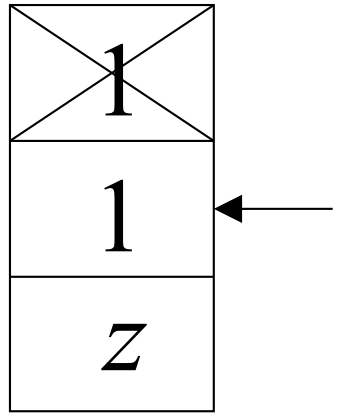
$a, 1, \lambda \quad b, 0, \lambda$



Time 5

Input



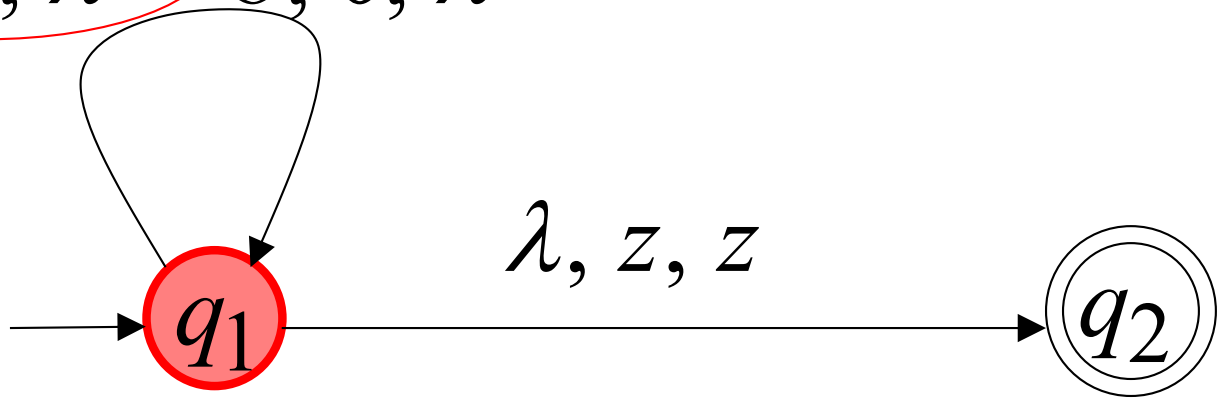


Stack

$a, z, 0z \quad b, z, 1z$

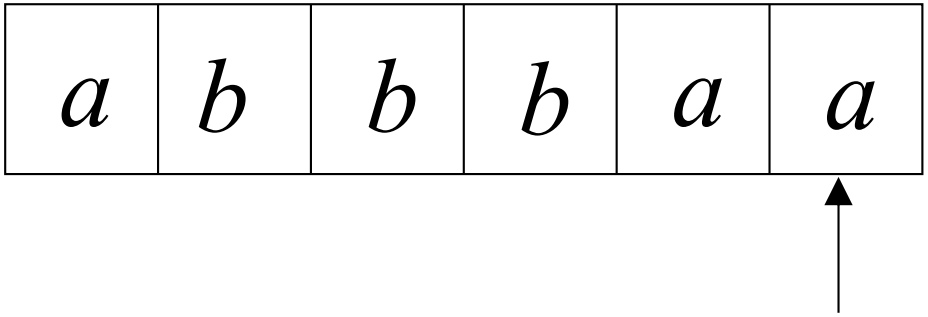
$a, 0, 00 \quad b, 1, 11$

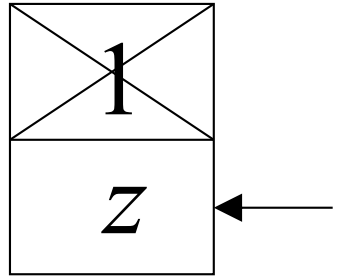
$a, 1, \lambda \quad b, 0, \lambda$



Time 6

Input



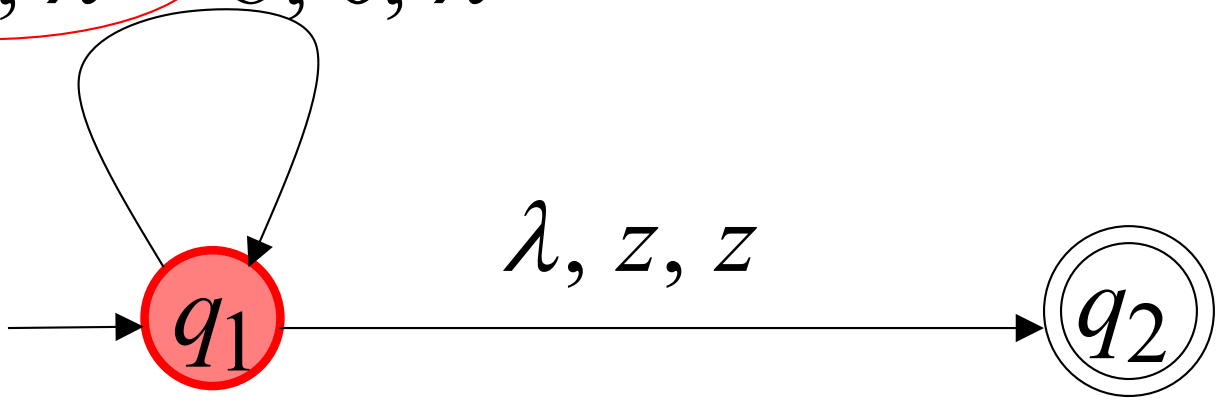


Stack

$a, z, 0z \quad b, z, 1z$

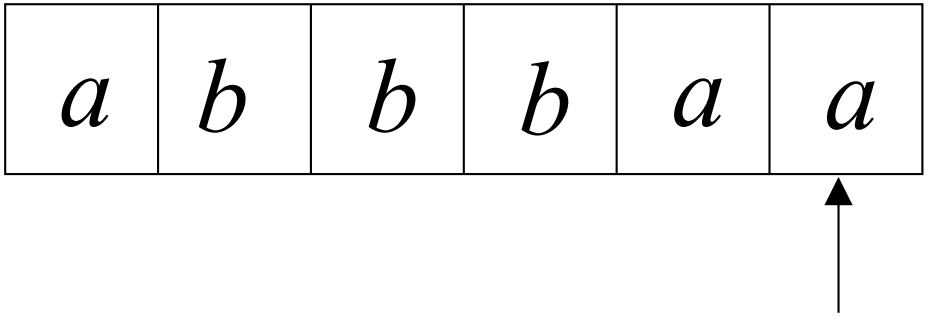
$a, 0, 00 \quad b, 1, 11$

$a, 1, \lambda \quad b, 0, \lambda$



Time 7

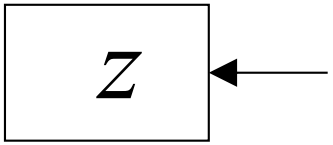
Input



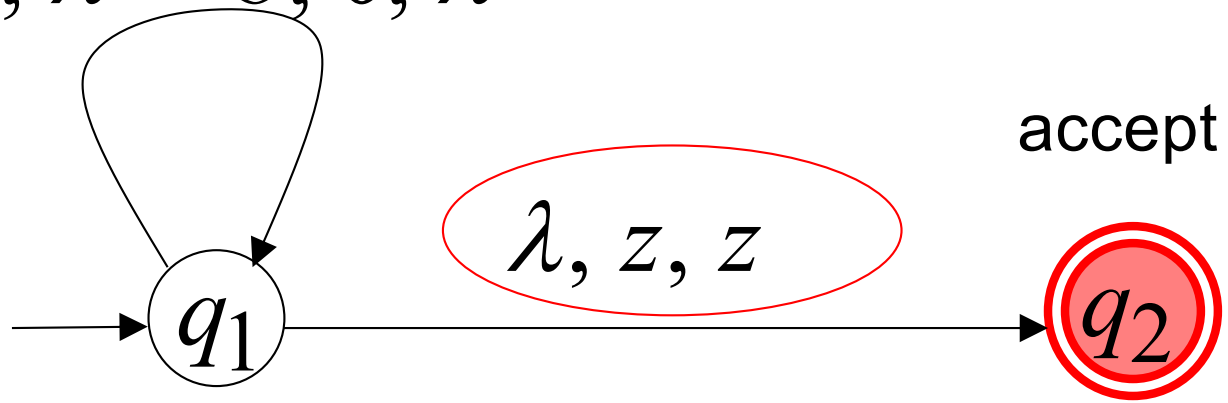
$a, z, 0z \quad b, z, 1z$

$a, 0, 00 \quad b, 1, 11$

$a, 1, \lambda \quad b, 0, \lambda$



Stack

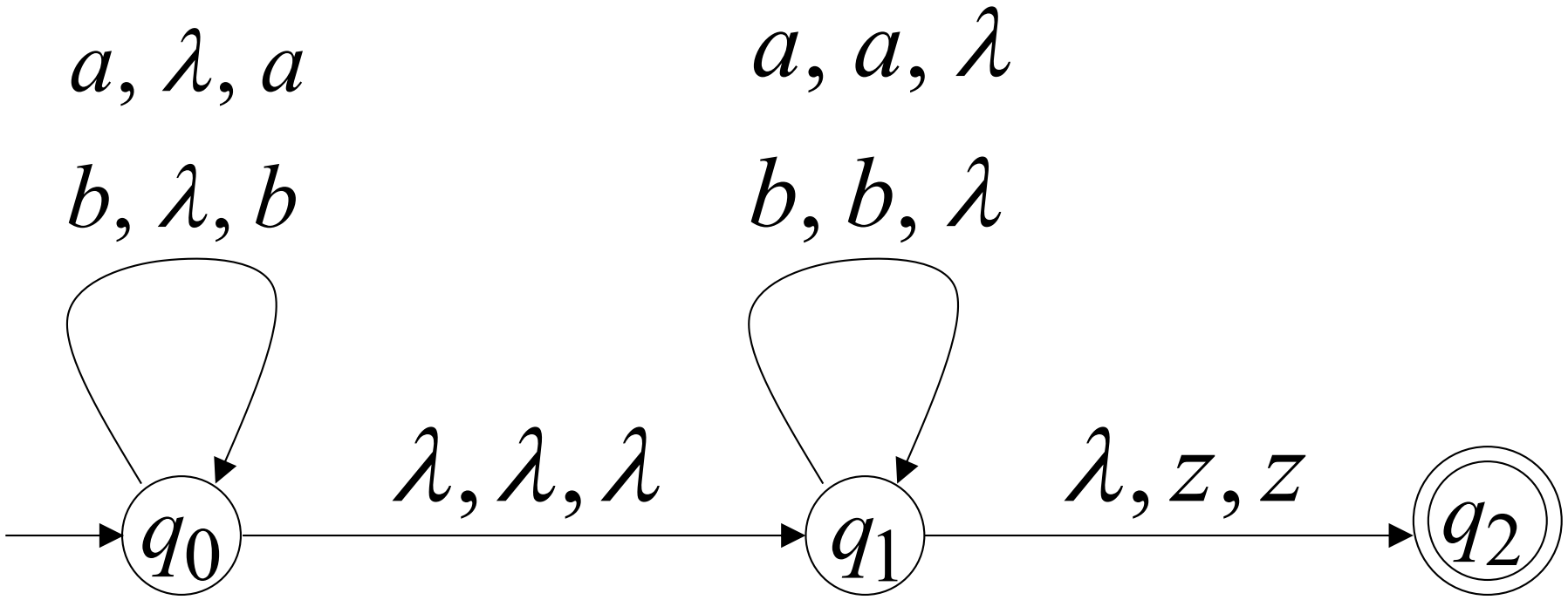


Example 7.5

NPDA M

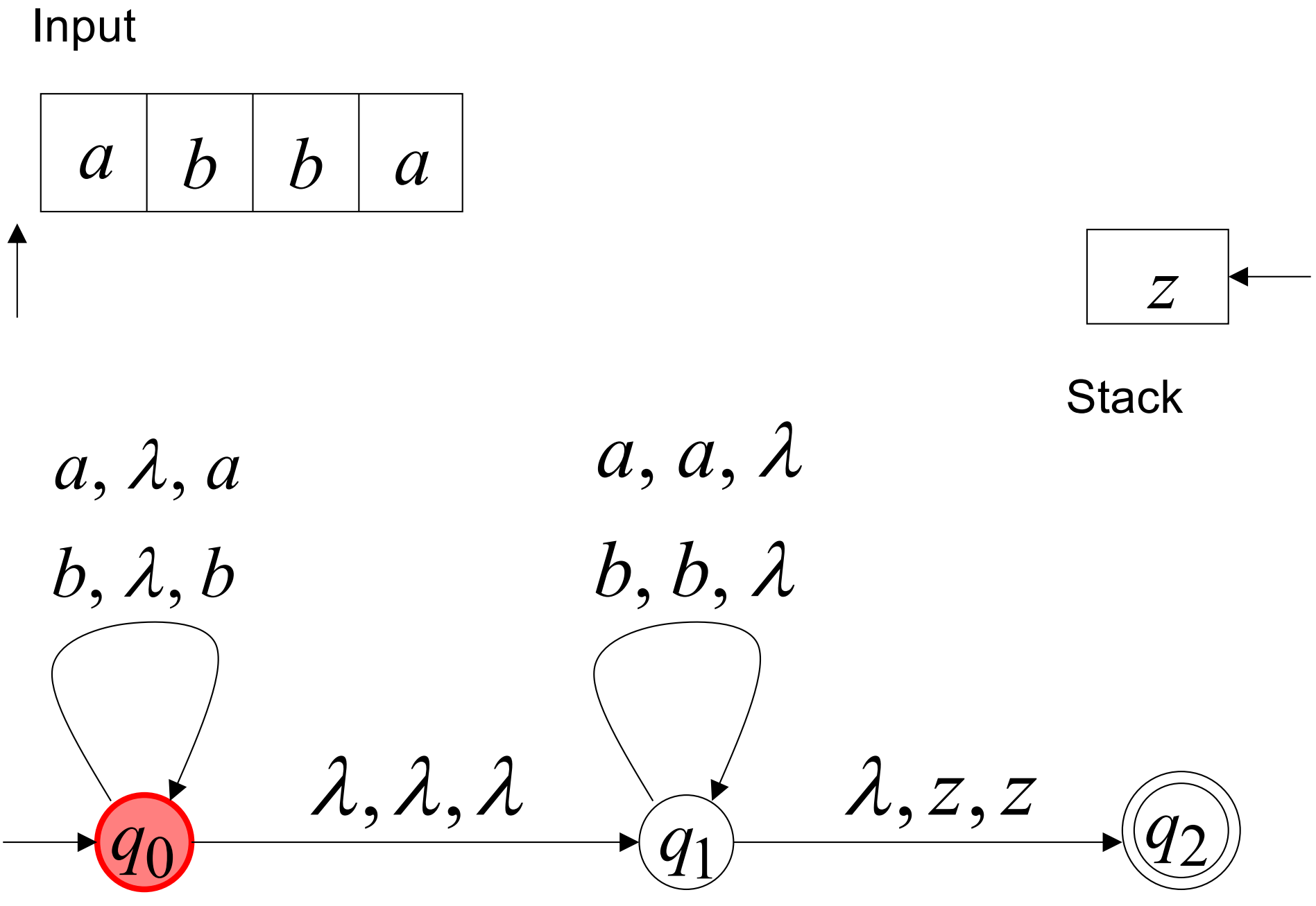
$$L(M) = \{ww^R\}$$

How do we know the middle of the string?



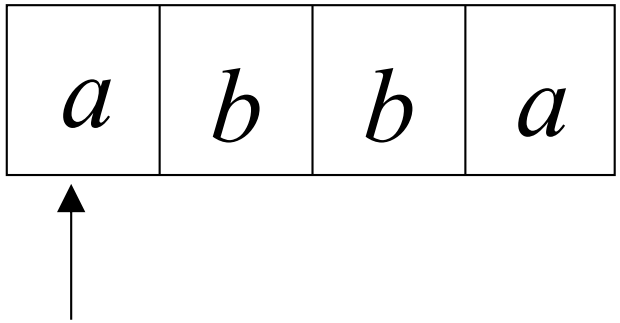
Execution Example:

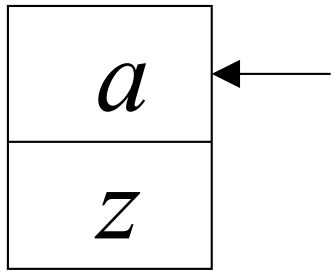
Time 0



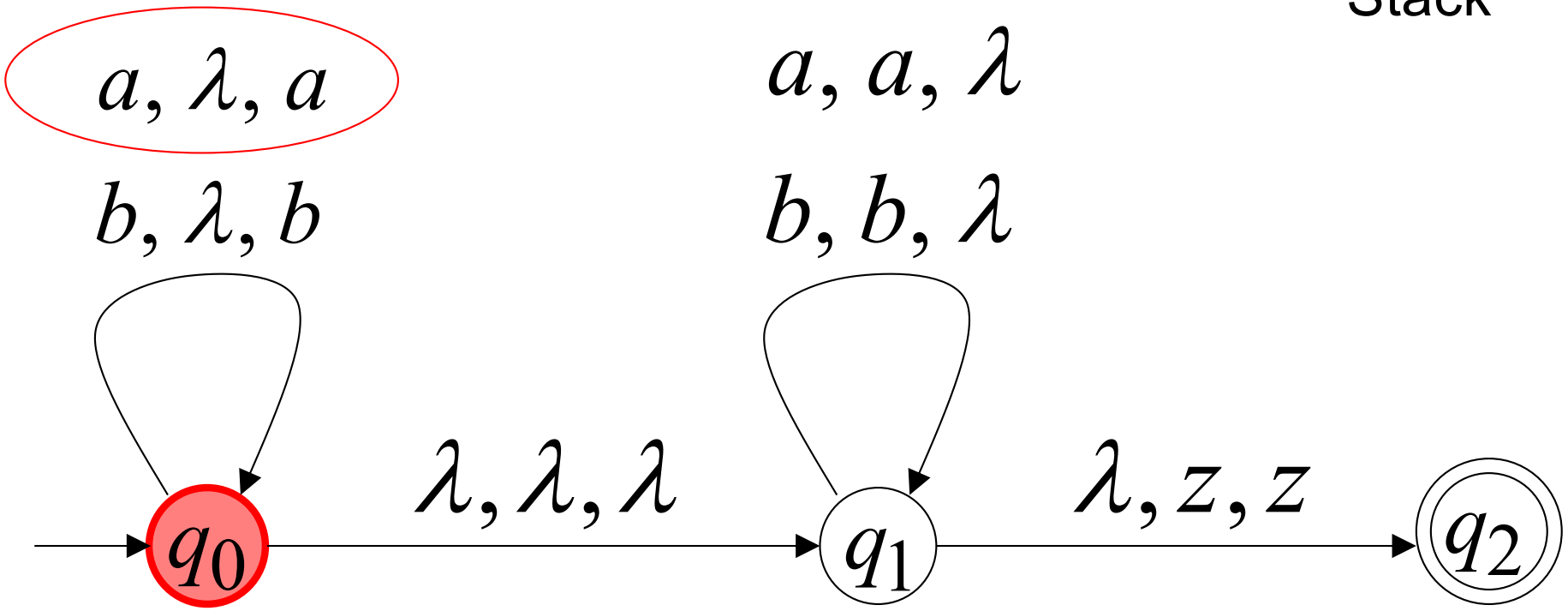
Time 1

Input



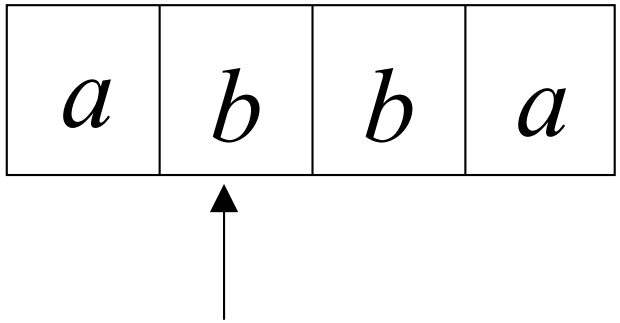


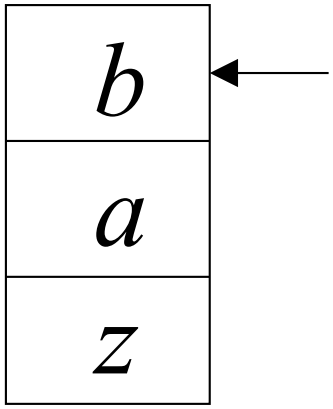
Stack



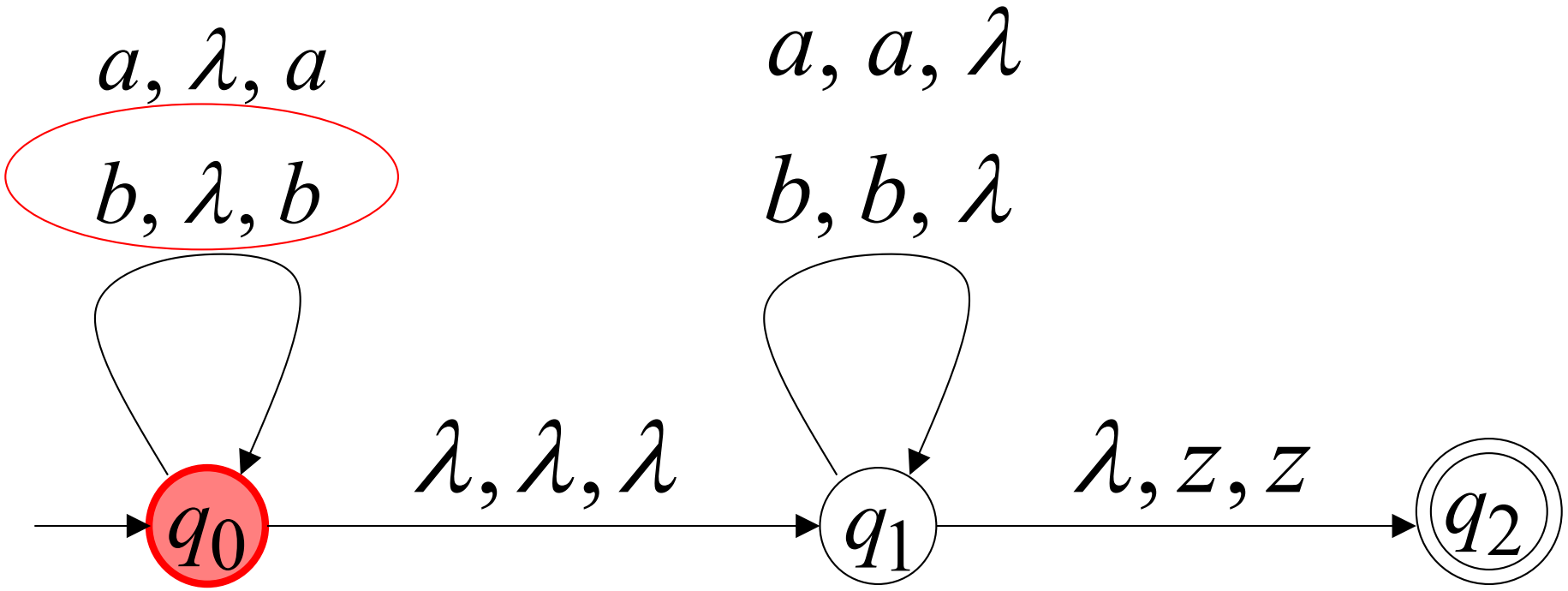
Time 2

Input

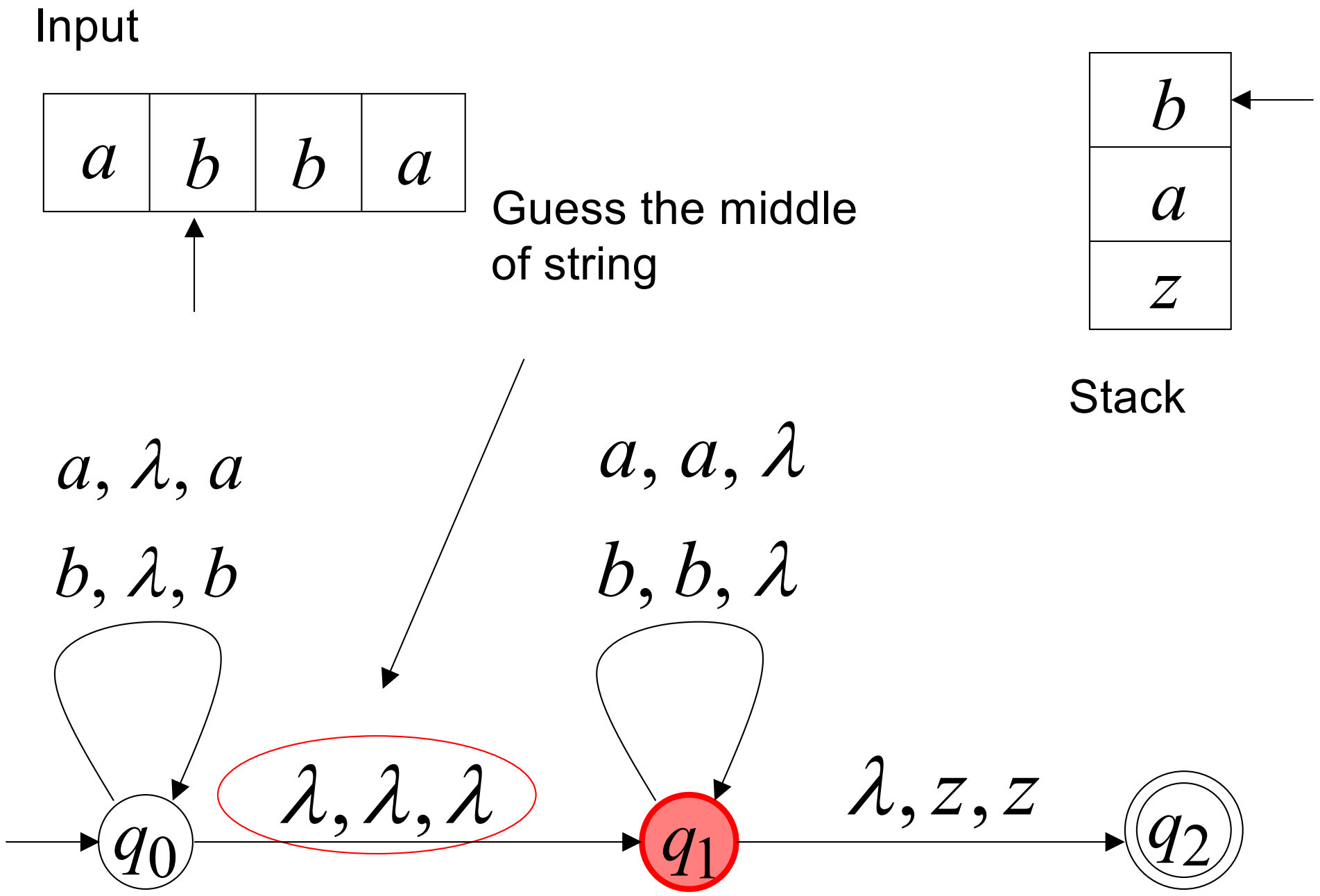




Stack

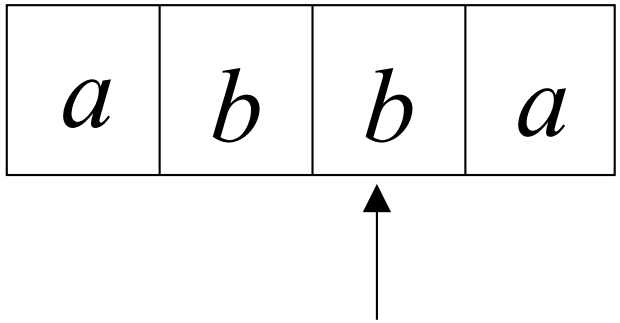


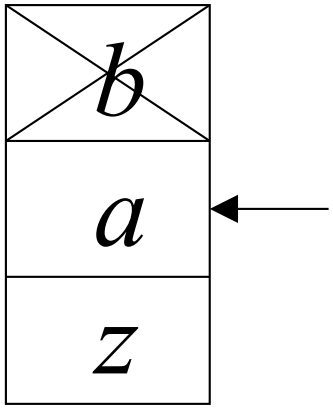
Time 3


Input
a b b a
Guess the middle of string
b
a
z
Stack
a, λ, a
b, λ, b
a, a, λ
b, b, λ
λ, λ, λ
λ, z, z
q0
q1
q2

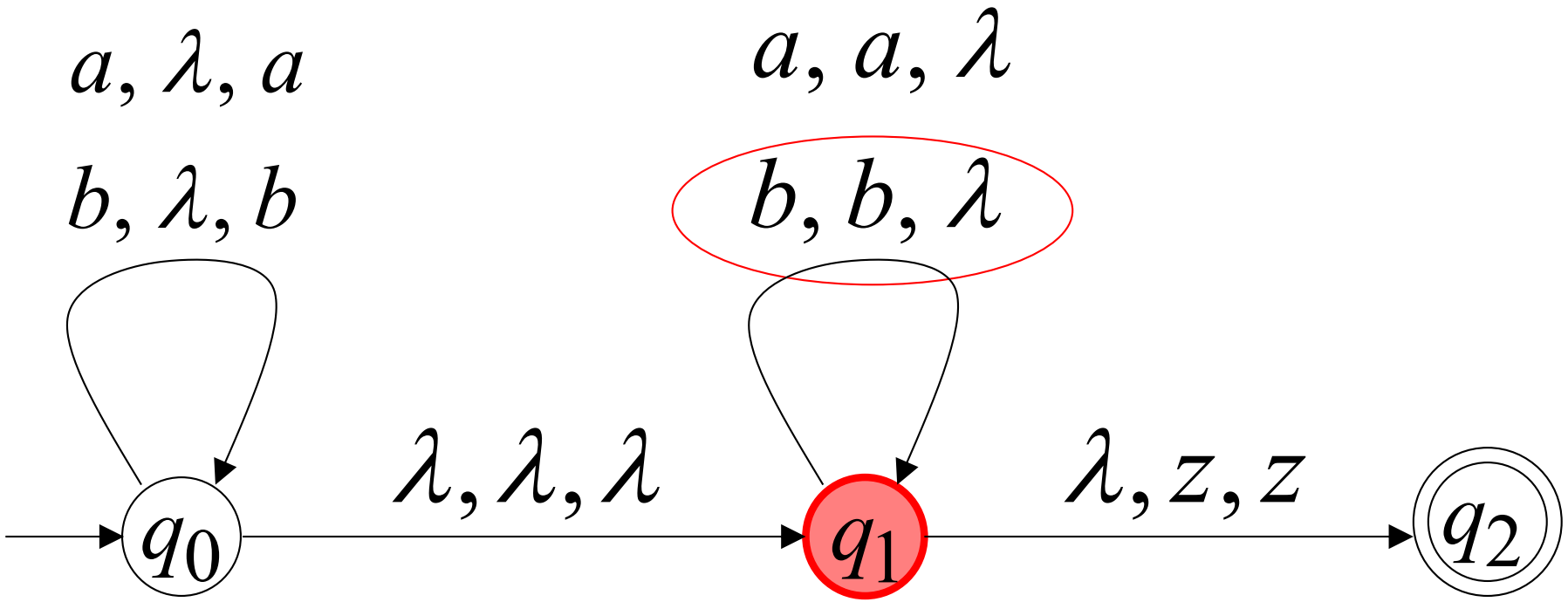
Time 4

Input



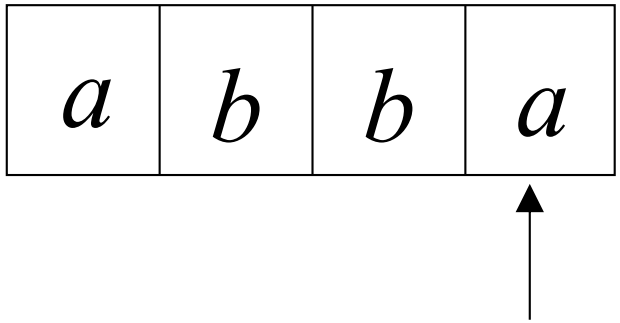


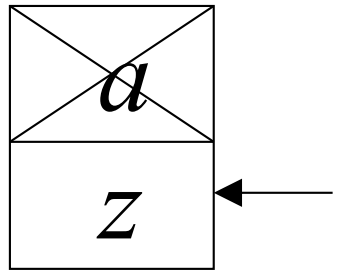
Stack



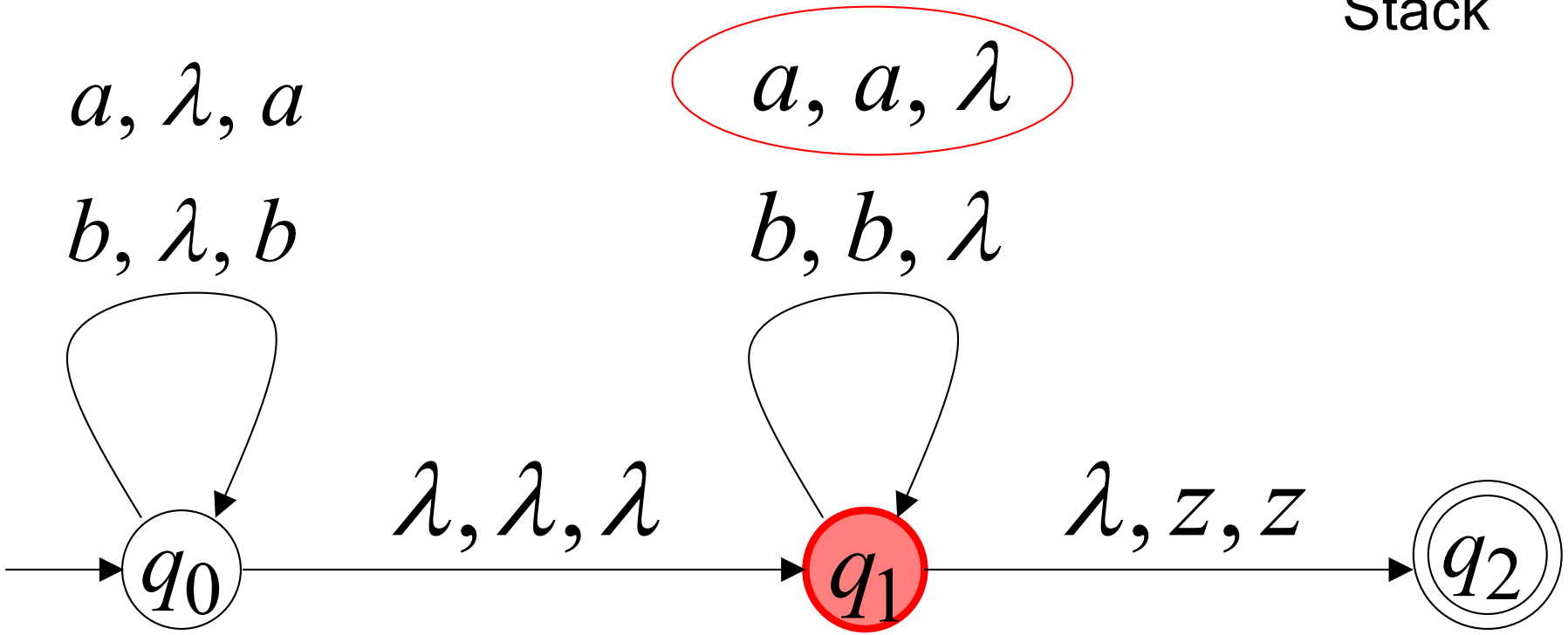
Time 5

Input

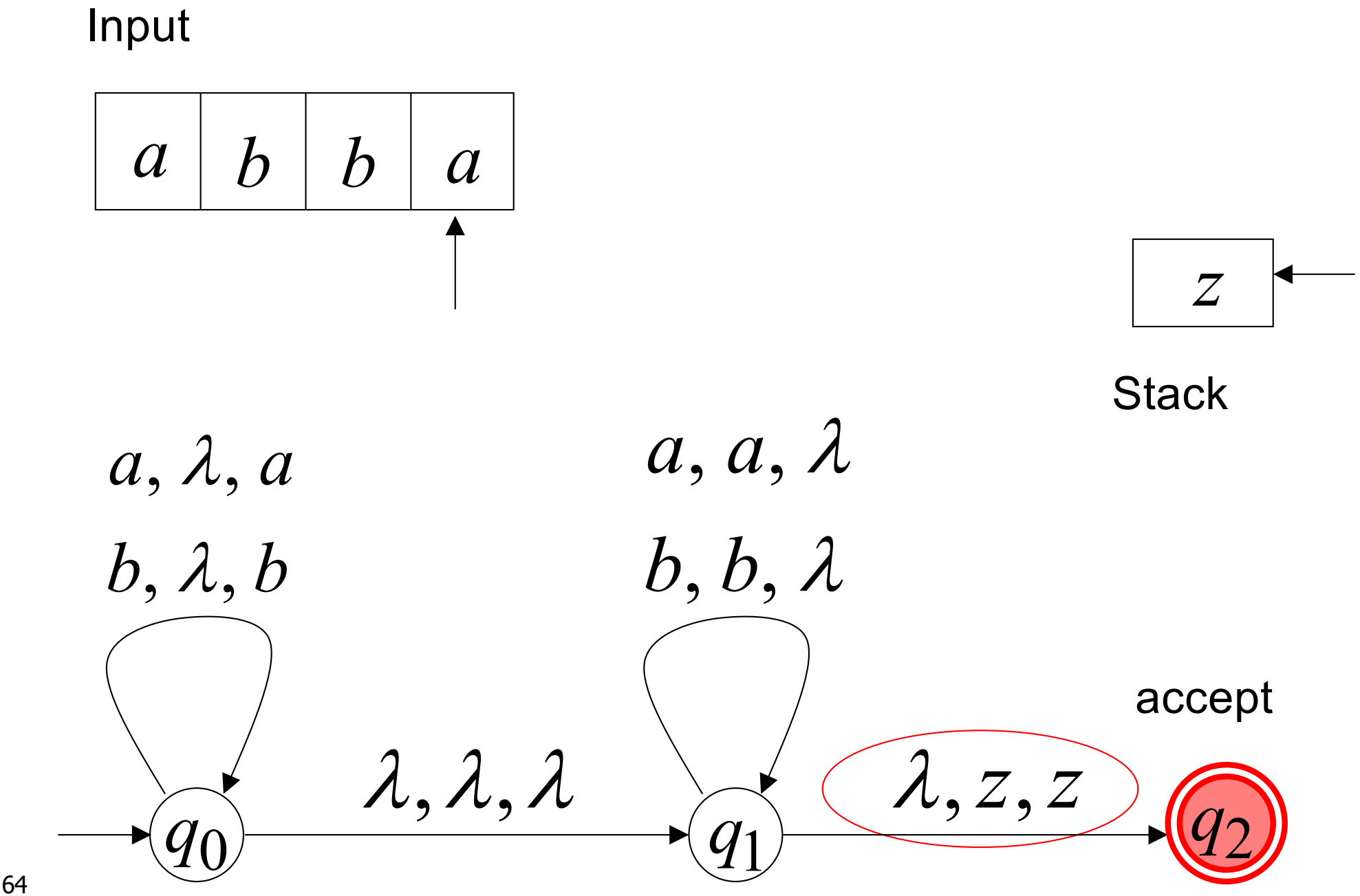




Stack

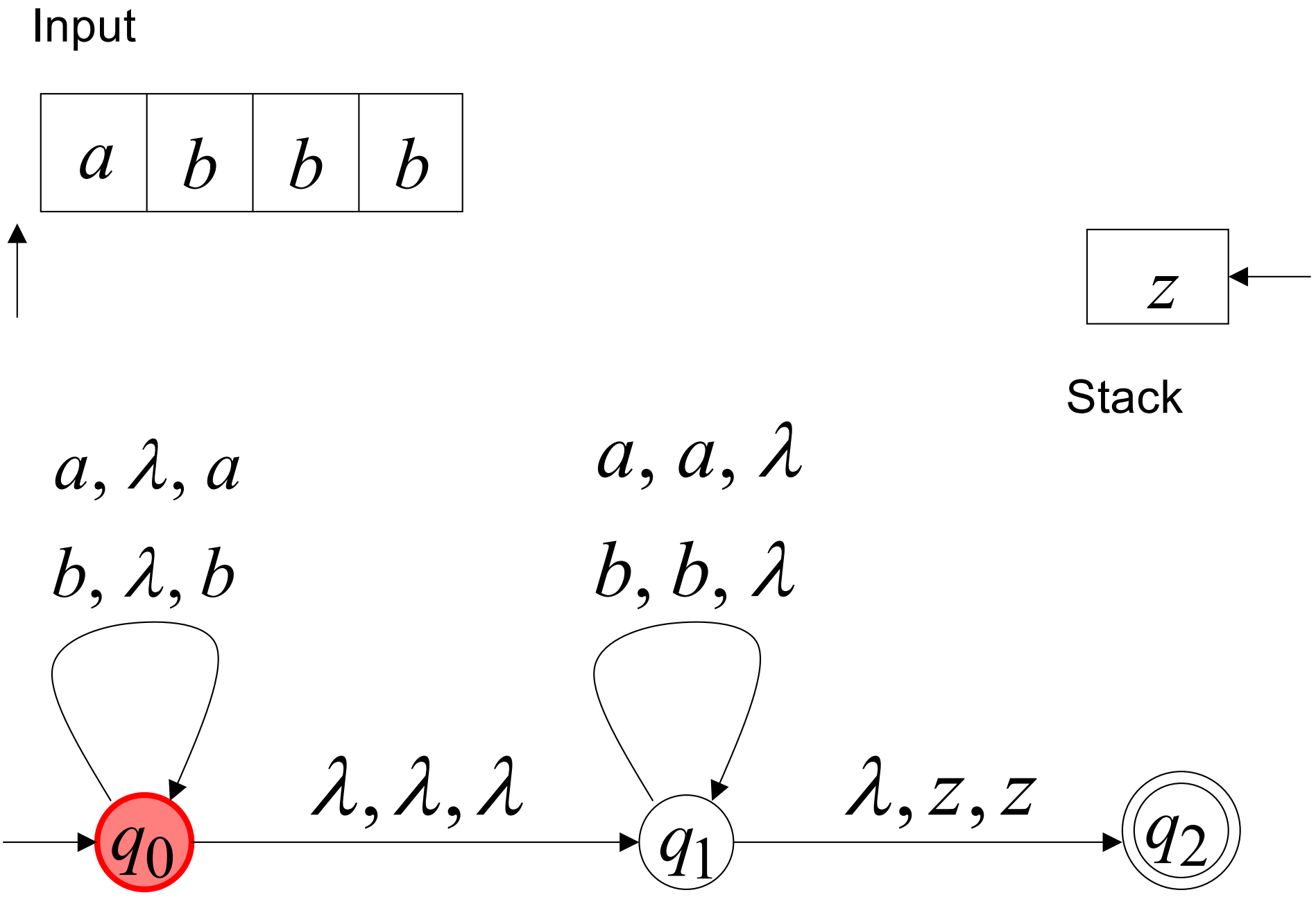


Time 6



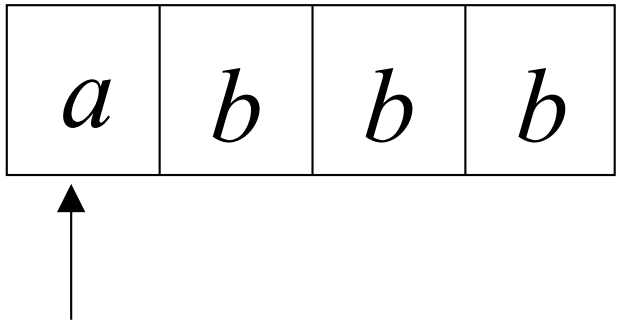
Rejection Example:

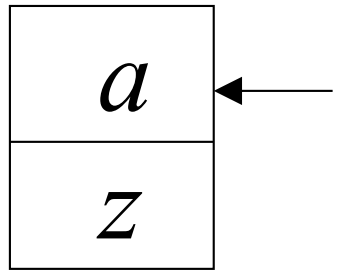
Time 0



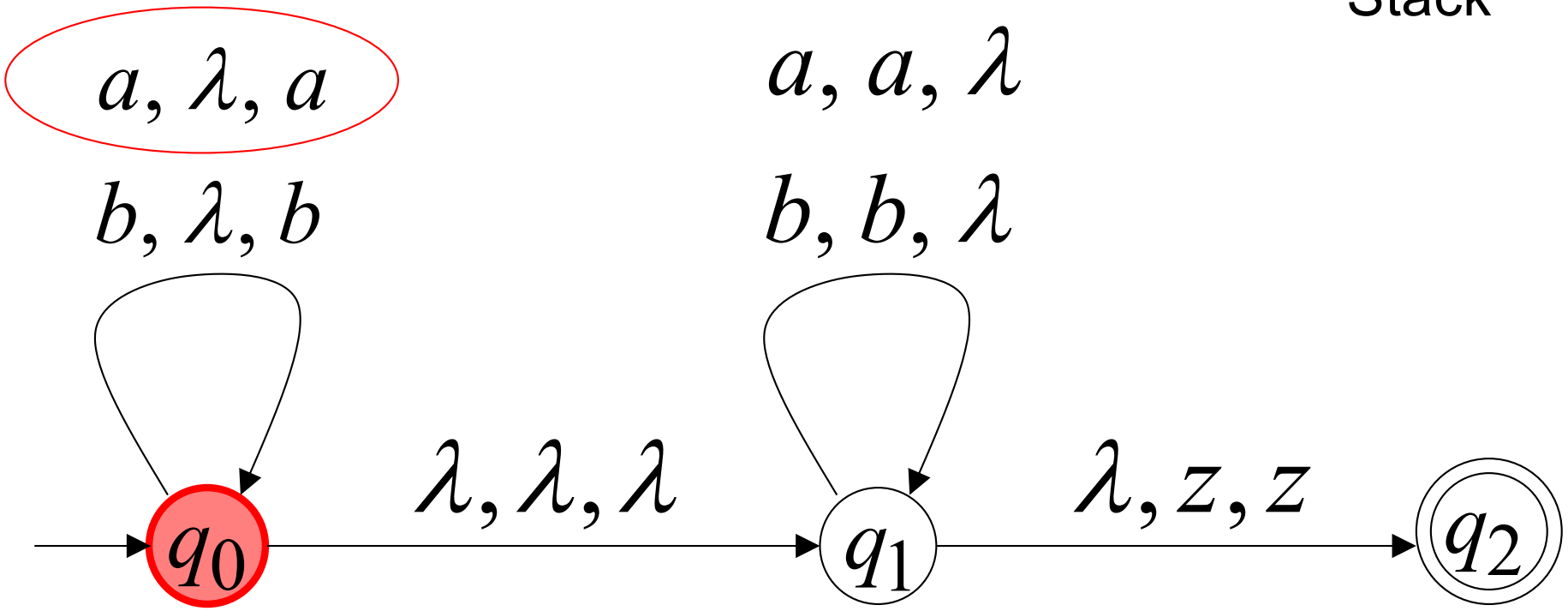
Time 1

Input

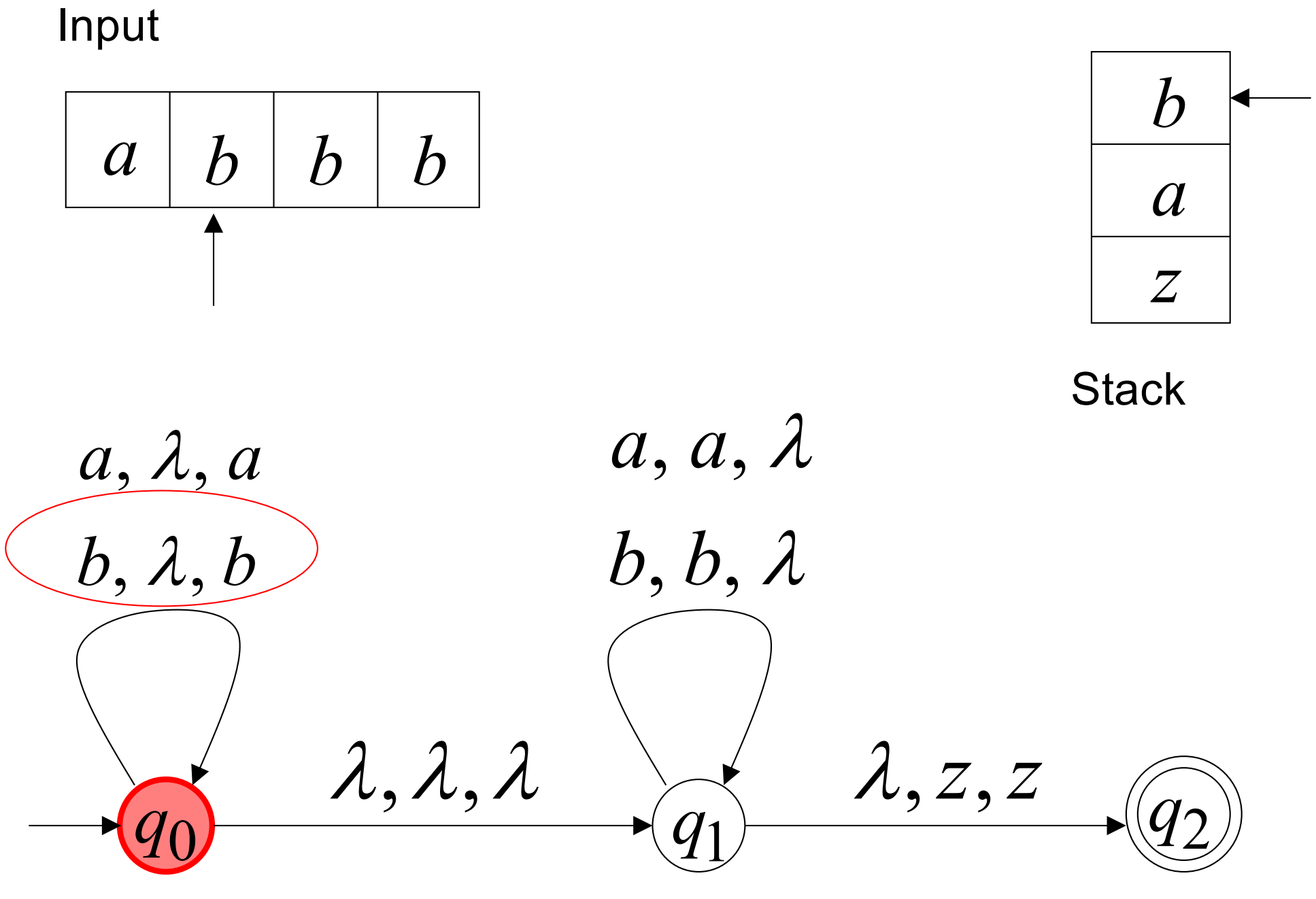




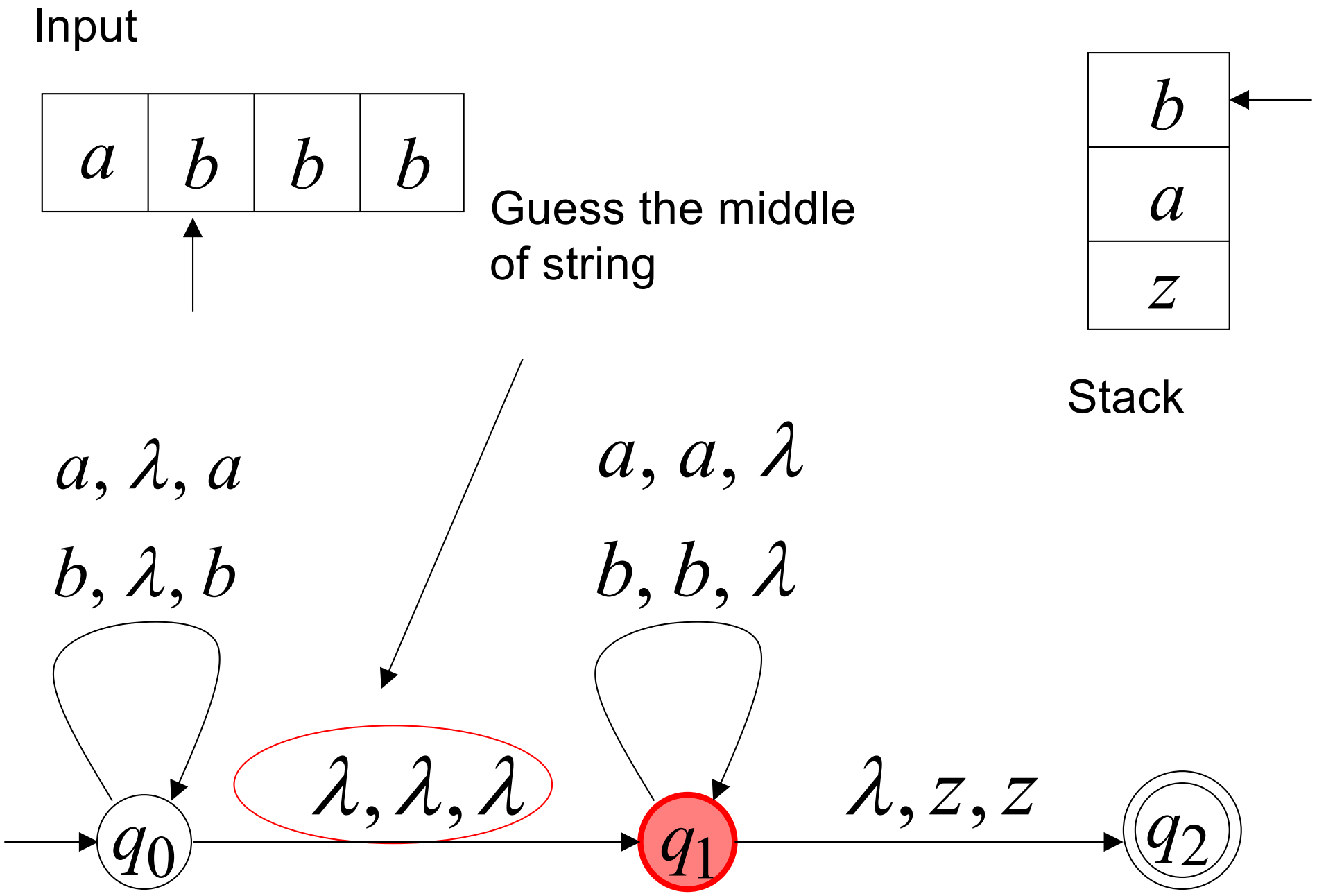
Stack



Time 2

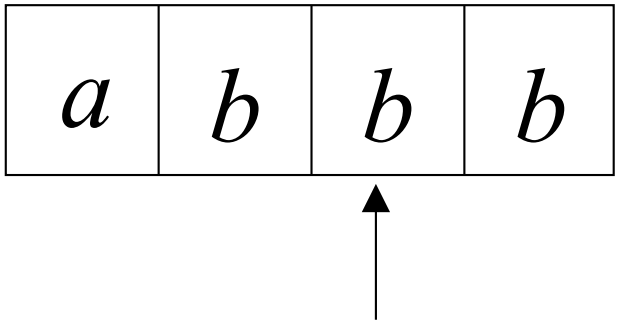


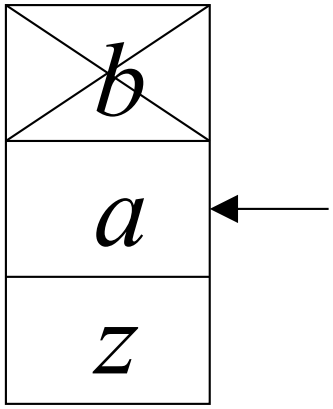
Time 3


Input
a
b
b
b
Guess the middle of string
b
a
z
Stack
a, λ, a
b, λ, b
a, a, λ
b, b, λ
λ, λ, λ
λ, z, z
q0
q1
q2

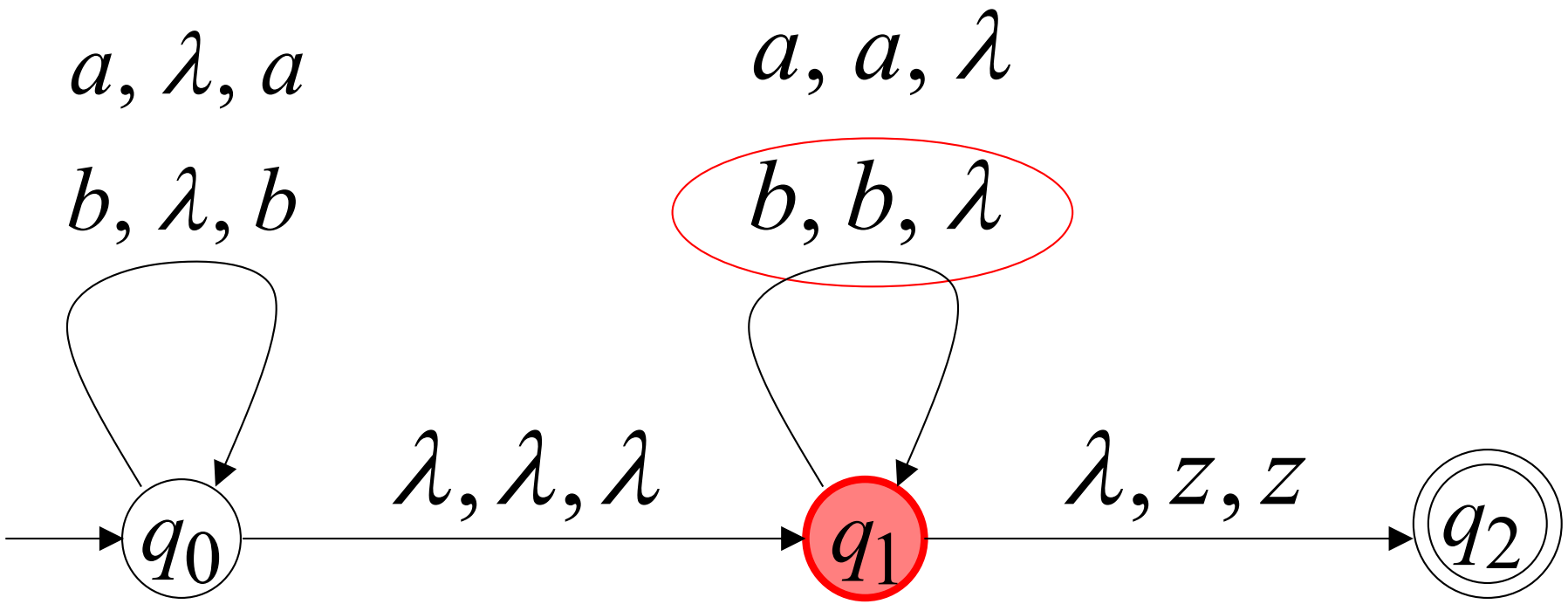
Time 4

Input





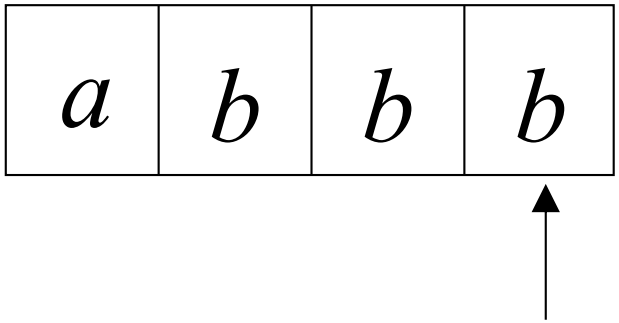
Stack



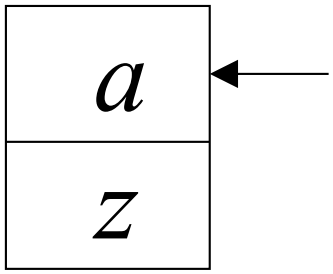
Time 5

There is no possible transition.

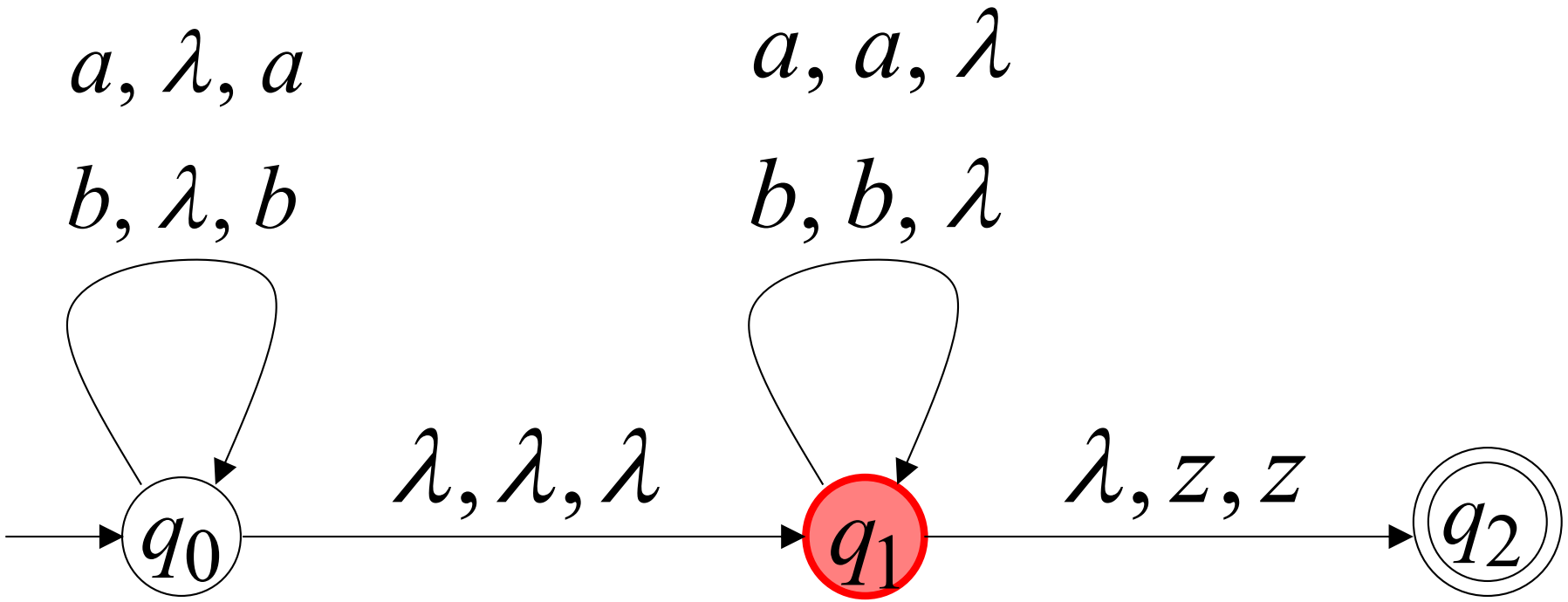
Input



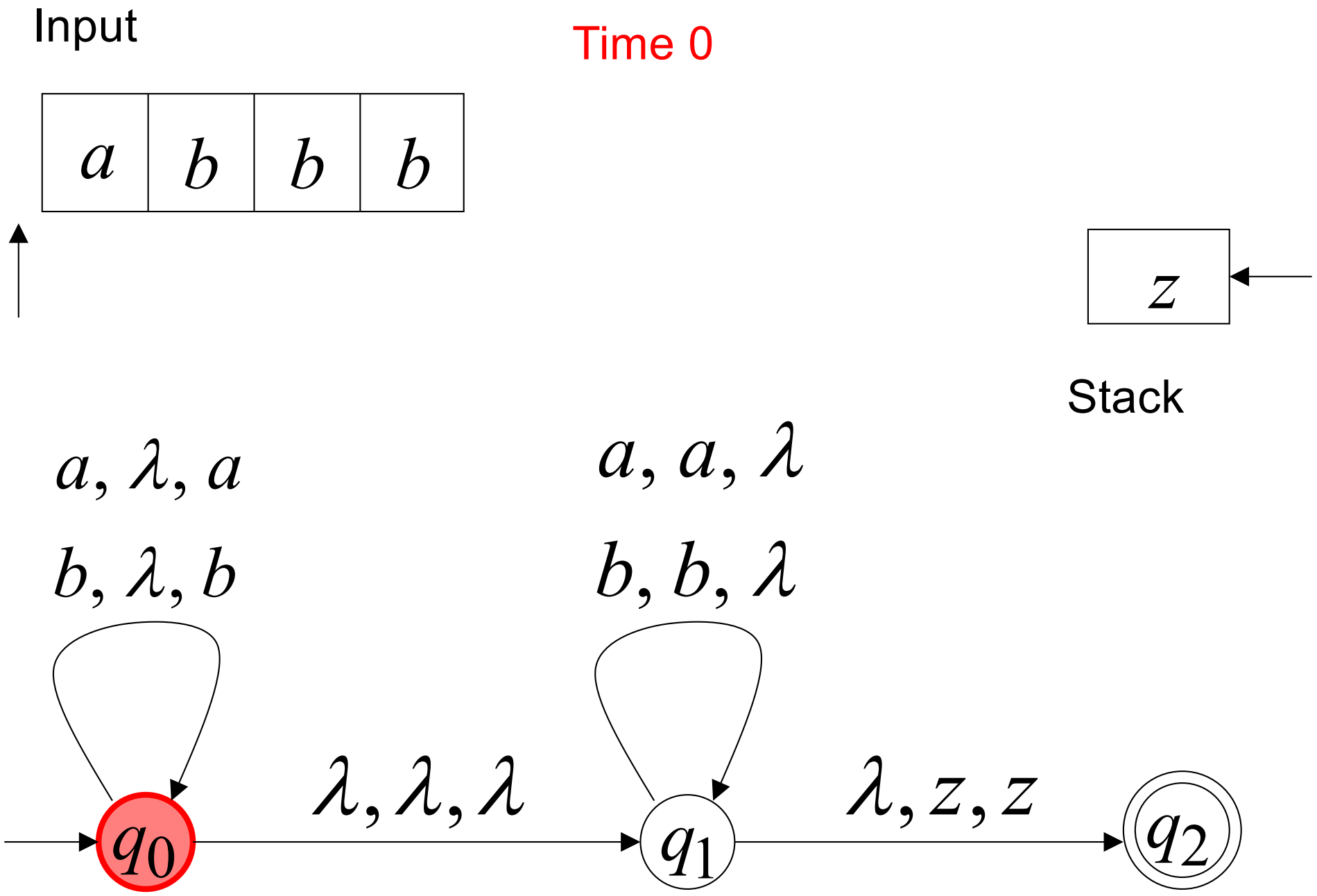
Input is not consumed



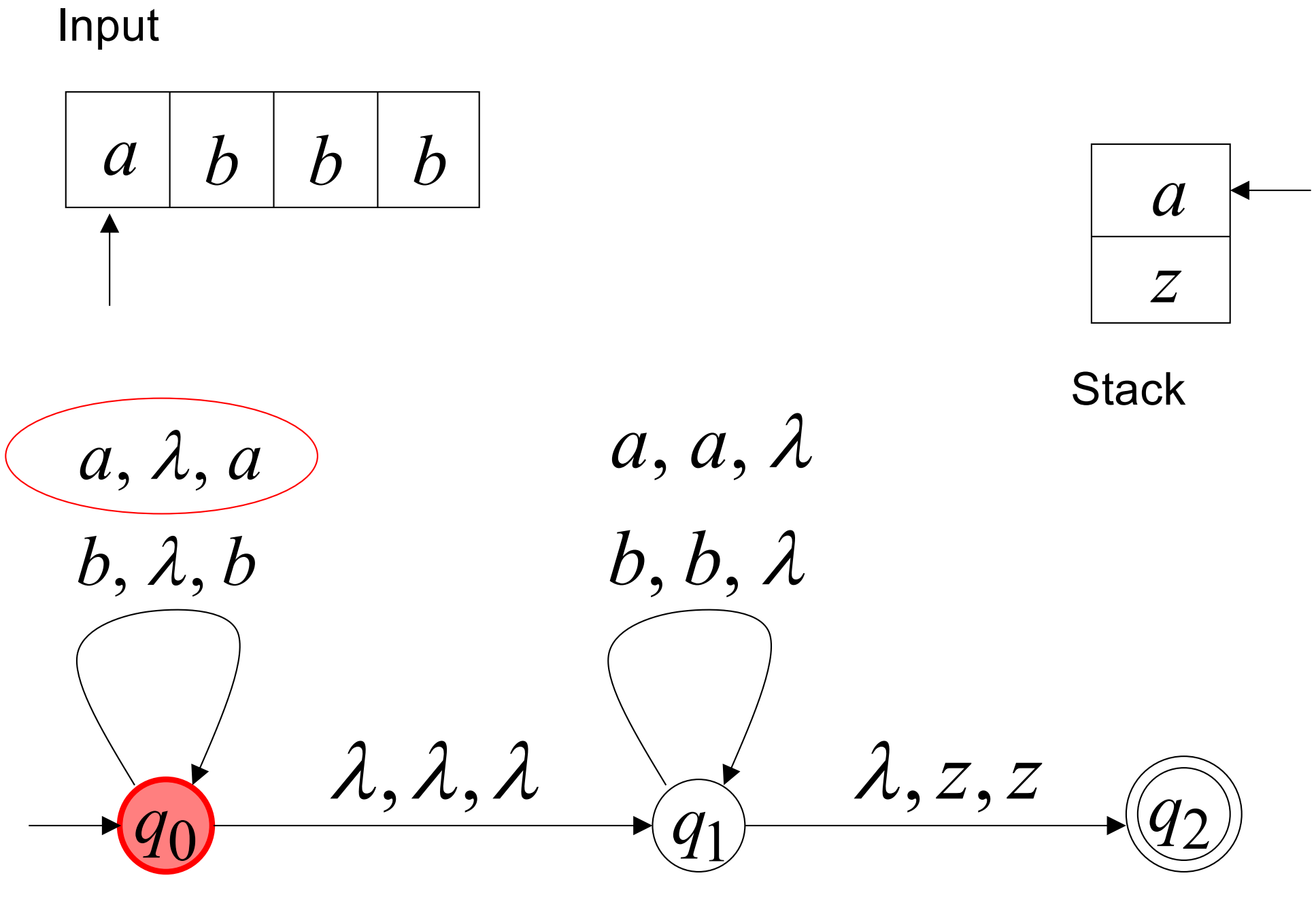
Stack



Another computation on same string:

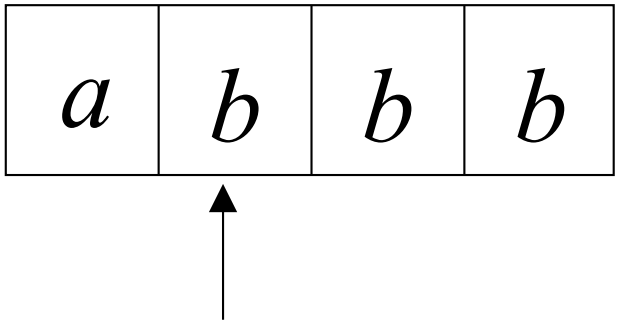


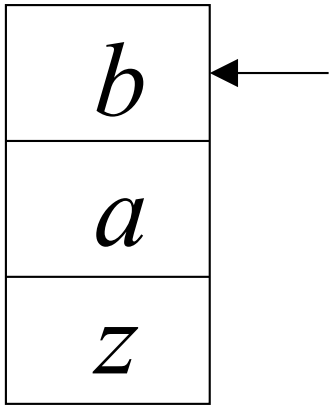
Time 1



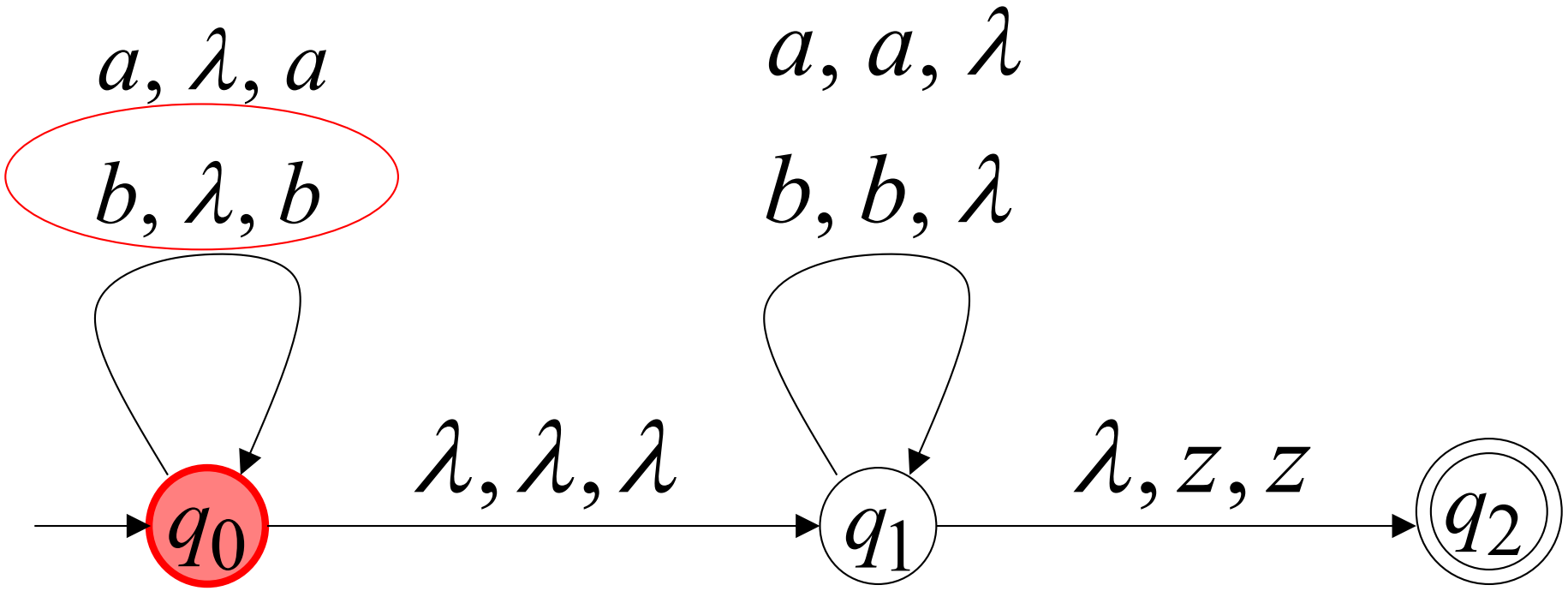
Time 2

Input



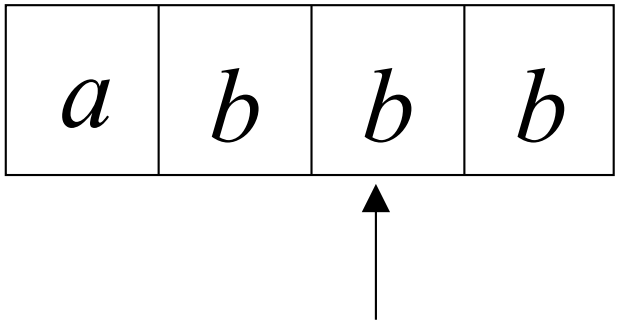


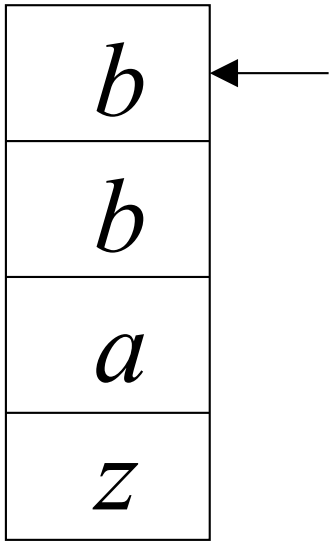
Stack



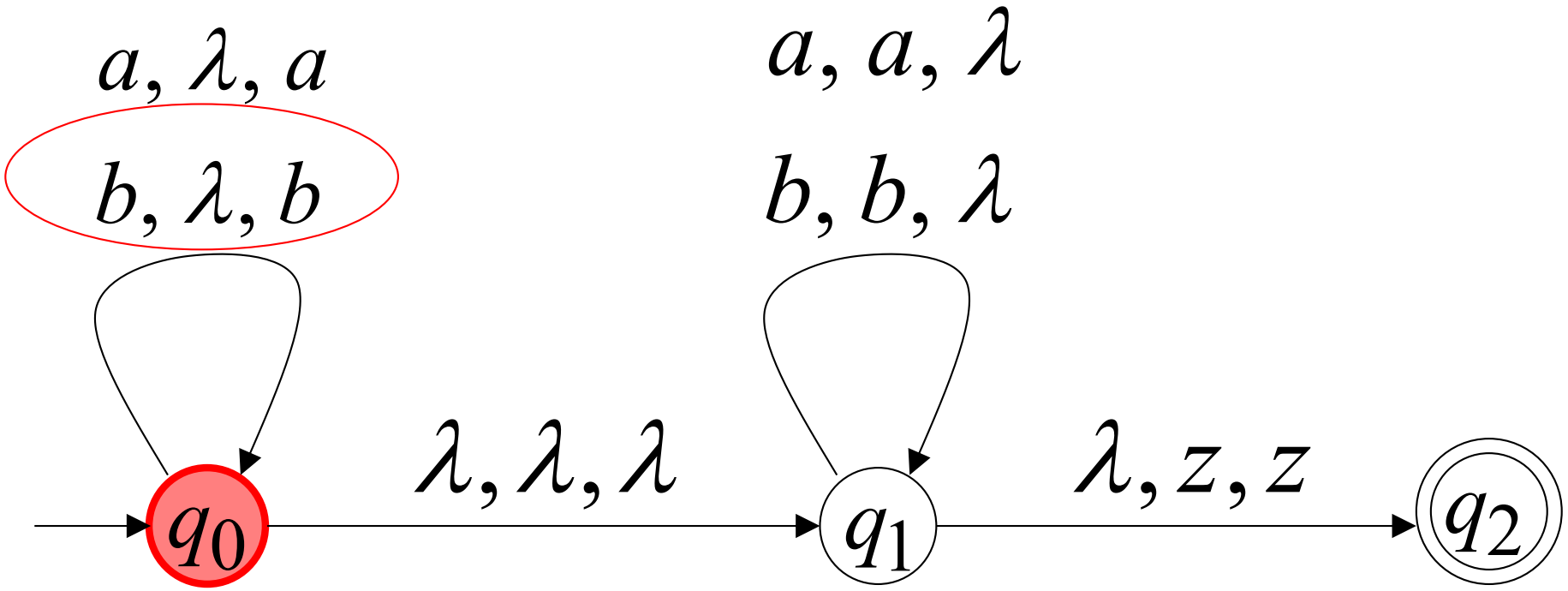
Time 3

Input



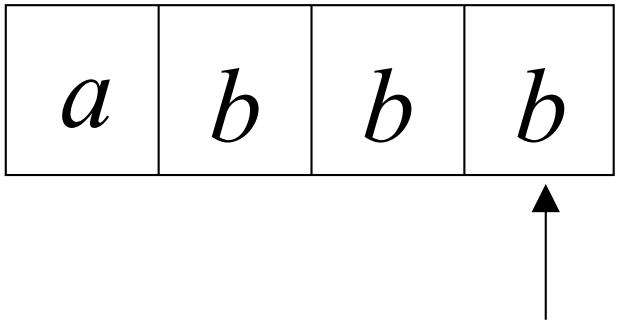


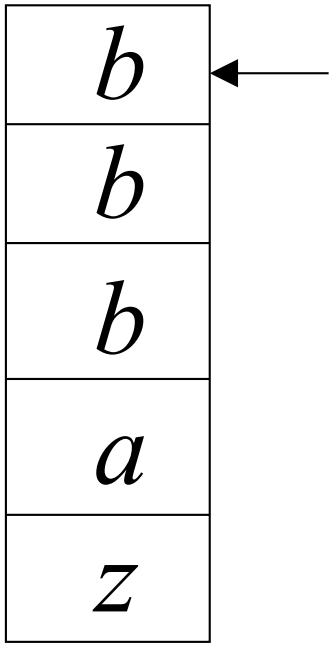
Stack



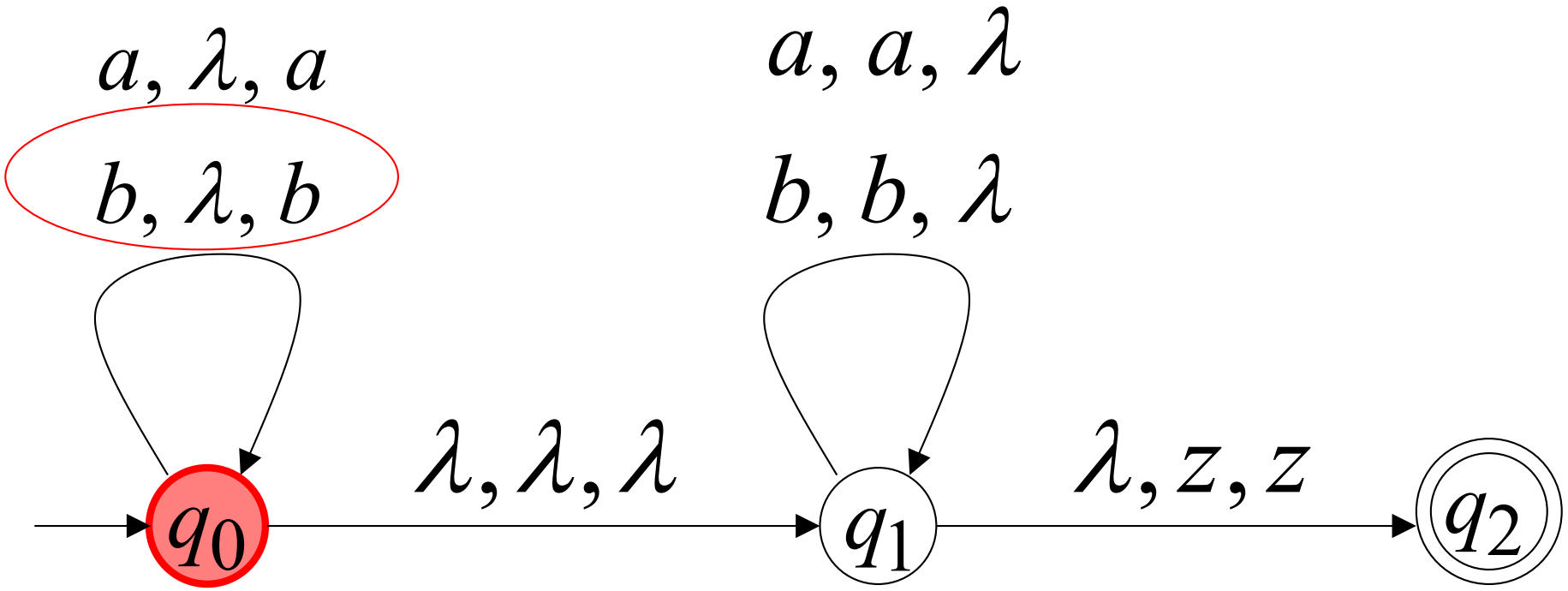
Time 4

Input





Stack



Time 5

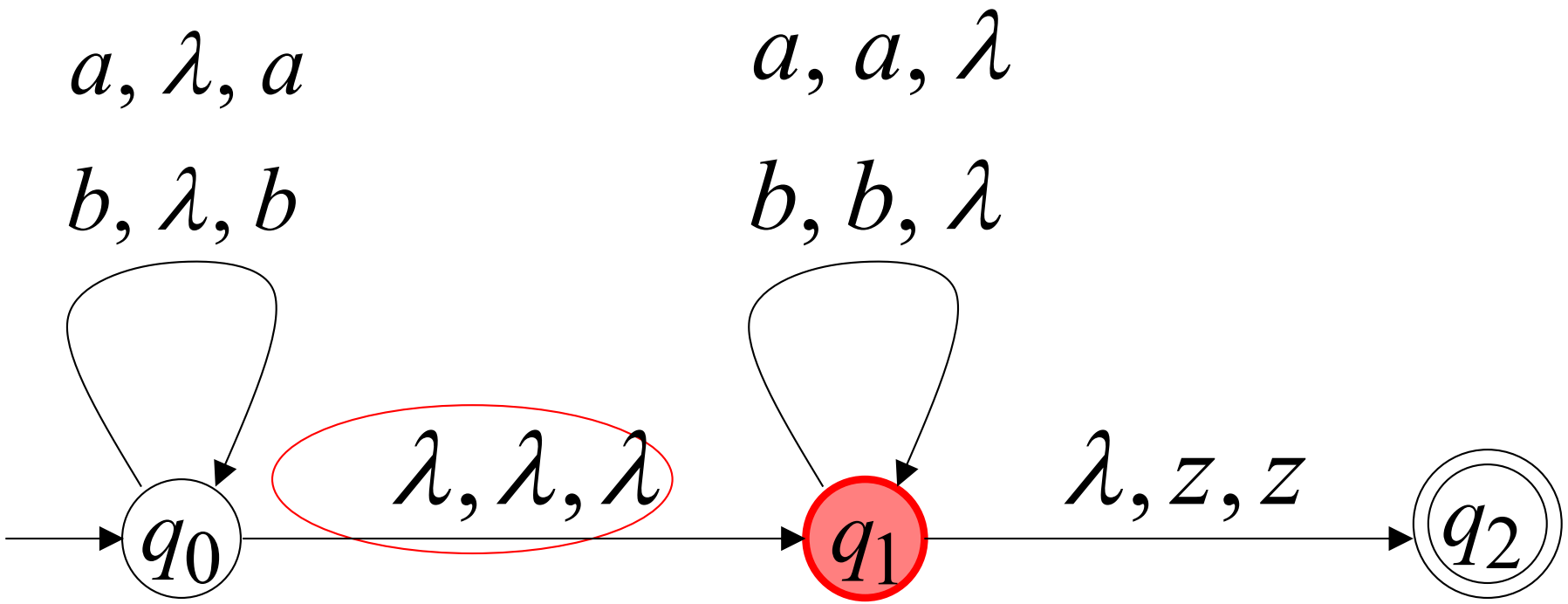
Input

a	b	b	b

No final state is reached

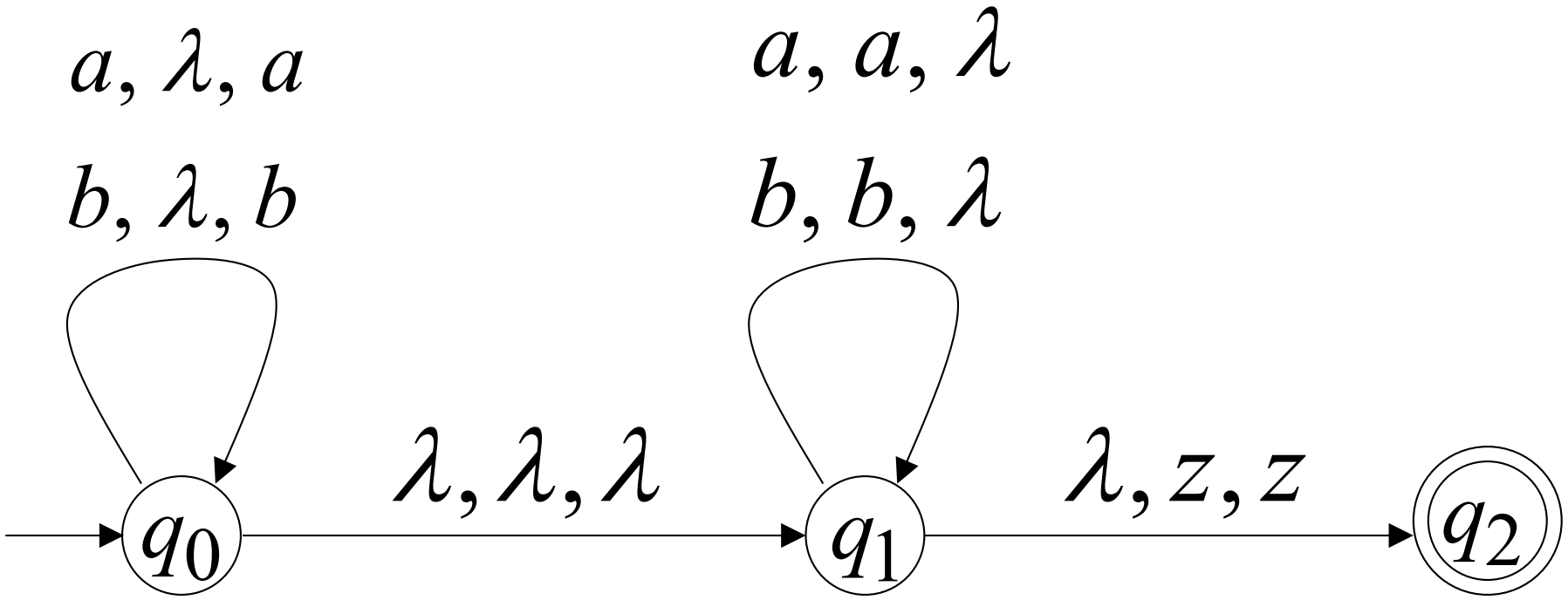
b
b
b
a
z

Stack



There is no computation
that accepts string $abbb$

$$abbb \notin L(M)$$



A string is rejected if there is
NO computation such that:

All the input is consumed

AND

The last state is a final state

At the end of the computation,
we do not care about the stack contents

In other words, a string is rejected
if in every computation with this string:

The input cannot be consumed

OR

The input is consumed and the last state is not a final state

OR

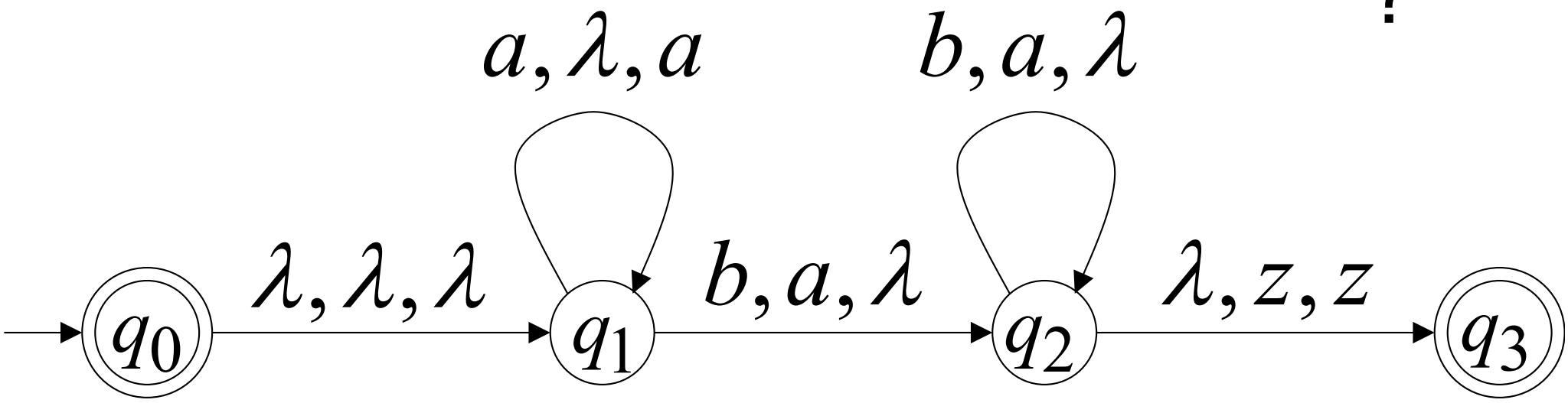
The stack head moves below the bottom of the stack

Another NPDA example

NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$

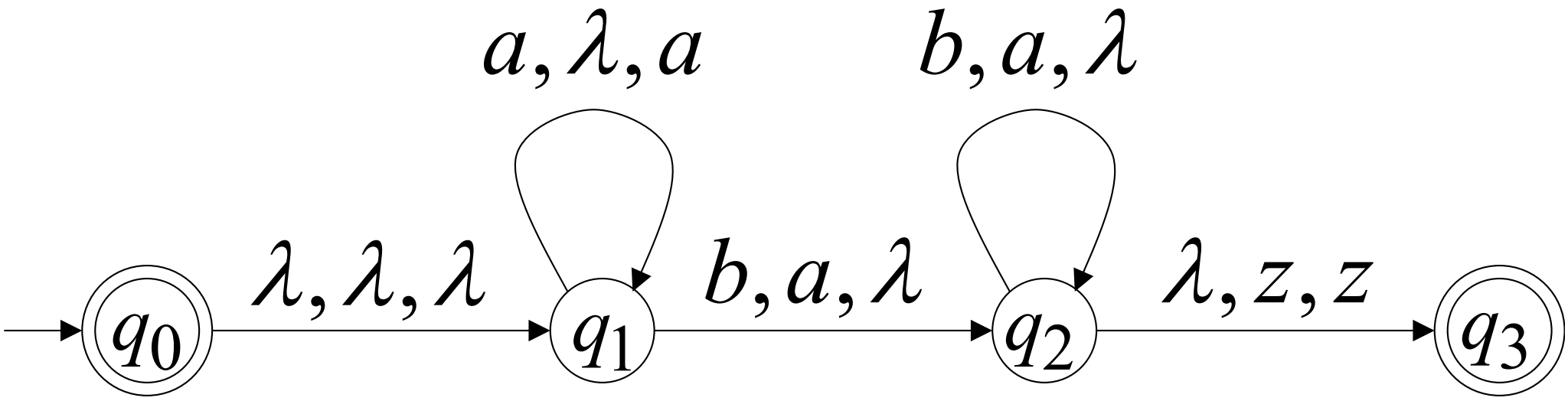
?



Another NPDA example

NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$



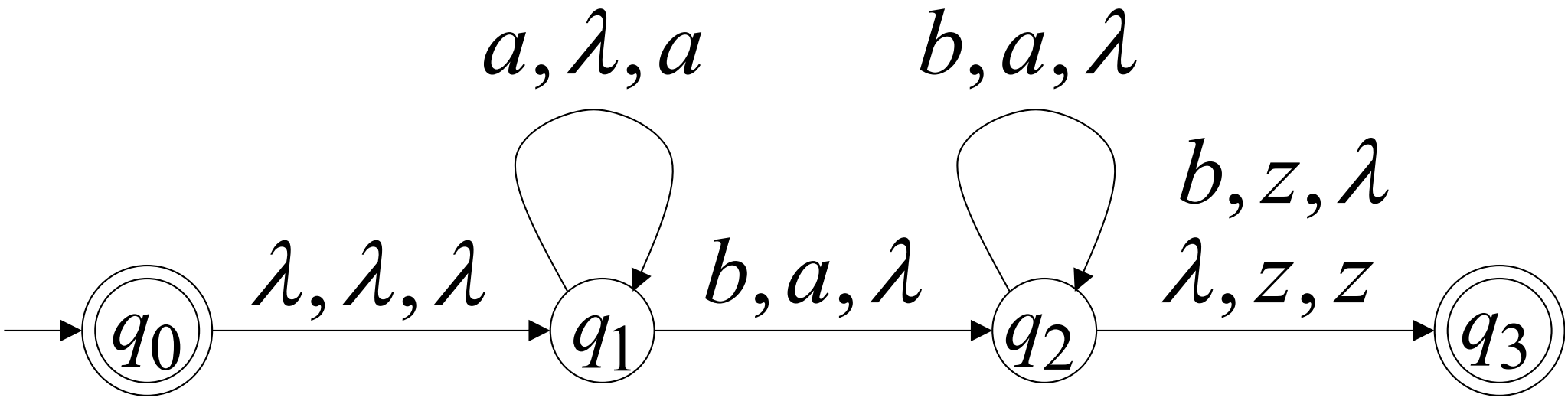
$n = m$

$n >= (m-1)$ is true

Another NPDA example

NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$



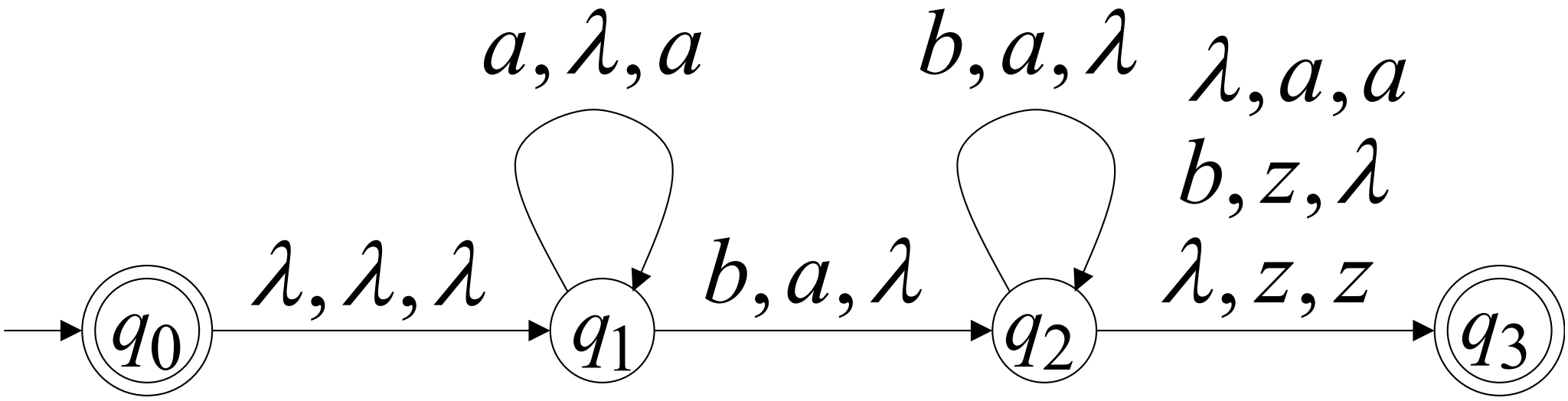
$$n = m-1$$

$$n >= (m-1) \text{ is true}$$

Another NPDA example

NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$



$n >= (m+1)$

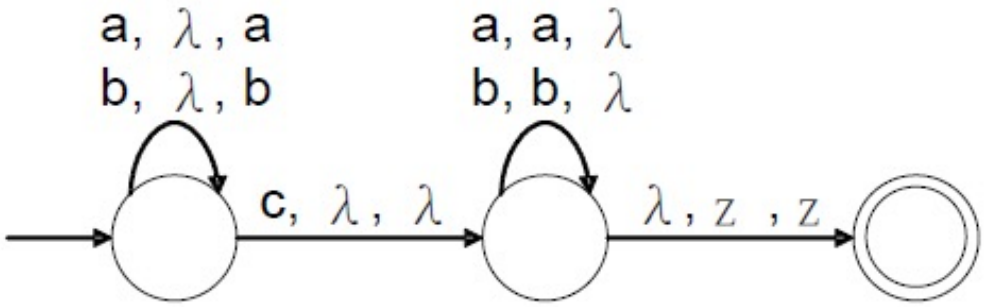
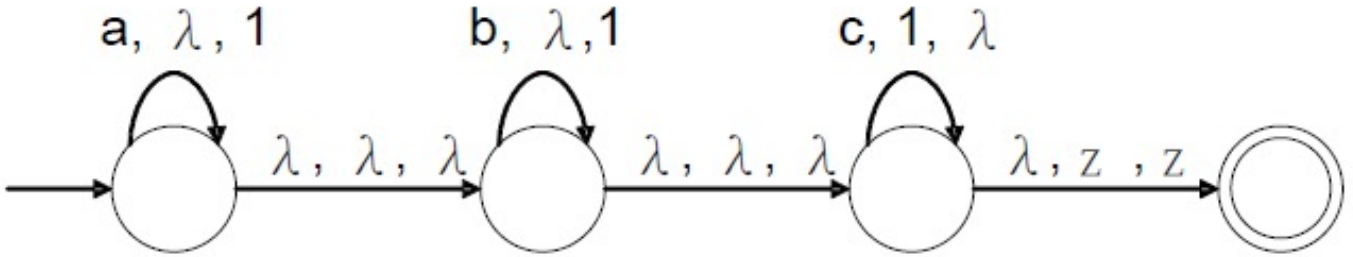
$n >= (m-1)$ is true

Questions?

Quiz

- $L=\{wcw^R : w \in \{a,b\}^*\}$
- $L=\{a^n b^m c^{n+m} : n \geq 0, m \geq 0\}$

(b)



(c)