Chapter 6
Some Continuous Probability Distributions

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6.1 Continuous Uniform Distribution

- Uniform distribution (Rectangular distribution): The density function of the continuous uniform random variable X on the interval [*A*, *B*] is

$$f(x; A, B) = \frac{1}{B-A}, \quad A \le x \le B$$
$$= 0 \qquad \text{elsewhere}$$

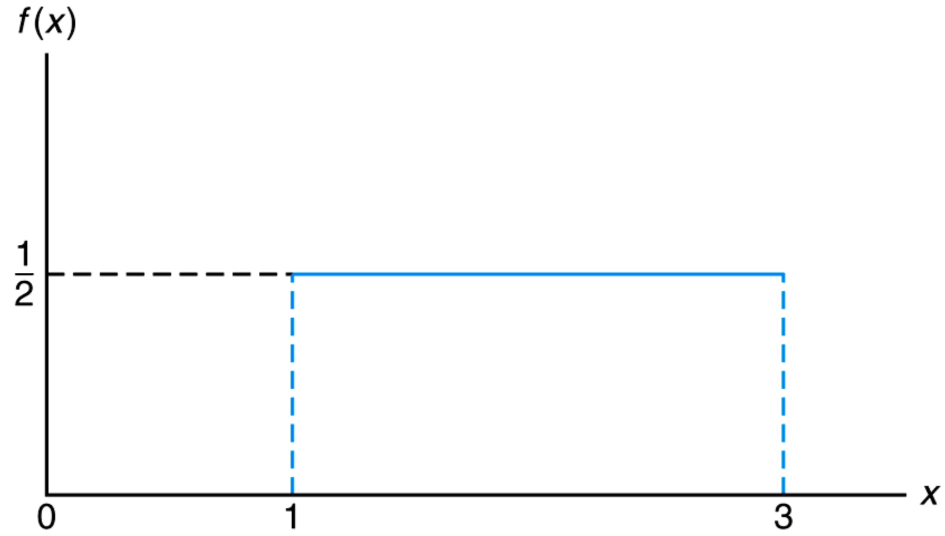


Figure 6.1 The density function for a random variable on the interval [1, 3].

Continuous Uniform Distribution

- Example 6.1: Suppose that a large conference room for a certain company can be reserved for no more than 4 hours. However, the use of the conference room is such that both long and short conferences occur quite often. In fact, it can be assumed that length X of a conference has a uniform distribution on the interval [0, 4].
  (a) What is the probability density function?
  (b) What is the probability that any given conference lasts at least 3 hours?
  - **Solution**

$$\text{(a) } f(x) = \tfrac{1}{4}, \quad 0 \le x \le 4$$
$$= 0 \qquad \text{elsewhere}$$
$$\text{(b) } P(X \ge 3) = \int_3^4 (\tfrac{1}{4})\,dx = \tfrac{1}{4}.$$

Continuous Uniform Distribution

- Theorem 6.1: The mean and variance of the uniform distribution are
  -
$$\mu = \frac{A+B}{2}, \quad \sigma^2 = \frac{(B-A)^2}{12}$$

(Exercise 6.1 p.205)

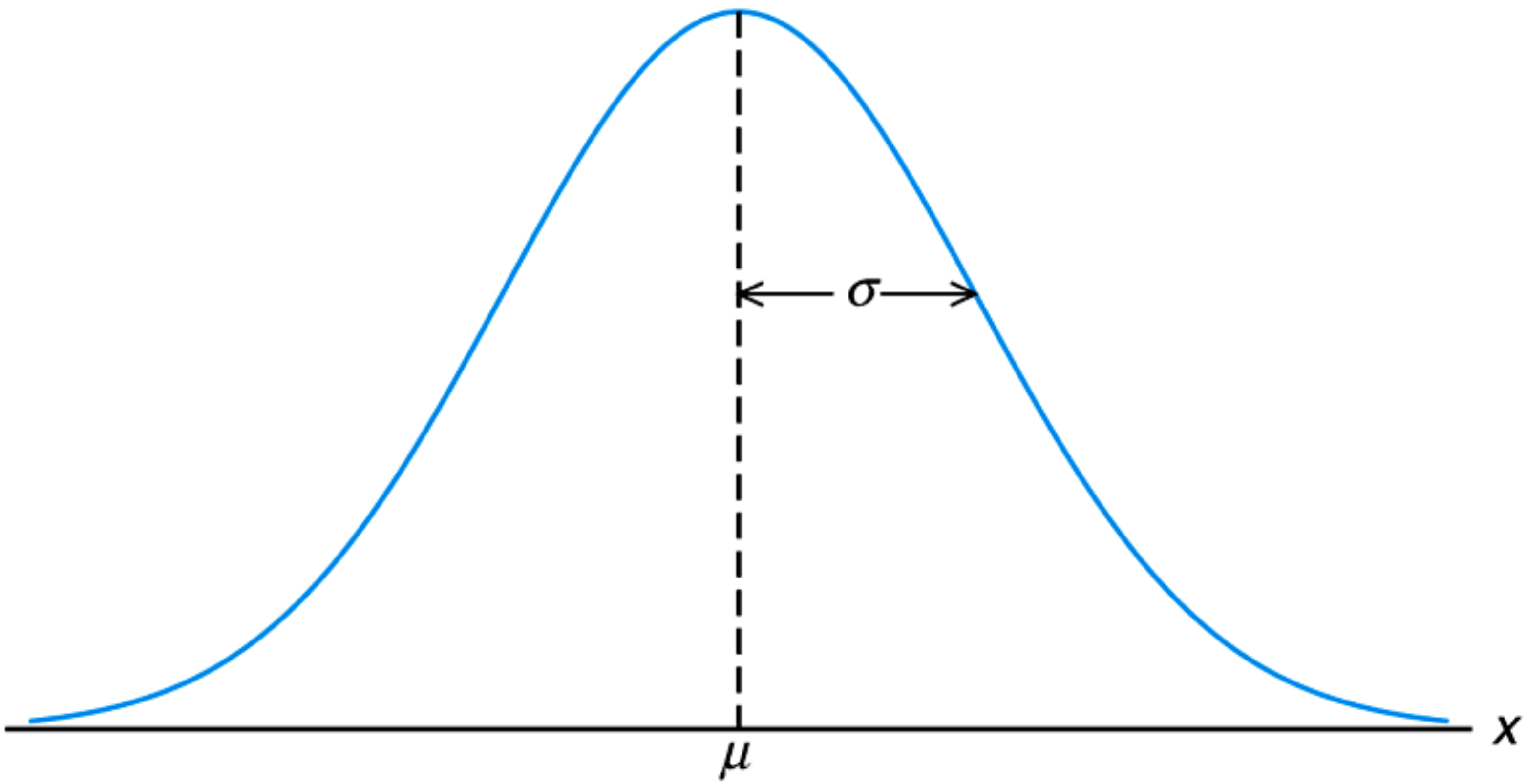
6.2 Normal Distribution

- The most important continuous probability distribution in the entire field of statistics is the normal distribution.
- In 1733, Abraham DeMoivre developed the mathematical equation of the normal curve.
- The normal distribution is often referred to as the Gaussian distribution, in honor of Karl Friedrich Gauss (1777-1855). Who also derived its equation from a study of errors in repeated measurements of the same quantity.
- Normal Distribution: The density function of the normal random variable *X*, with mean μ and variance σ^2, is

$$n(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(1/2)[(x-\mu)/\sigma]^2},\quad -\infty<x<\infty,$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$

Figure 6.2 The normal curve.



Normal Distribution

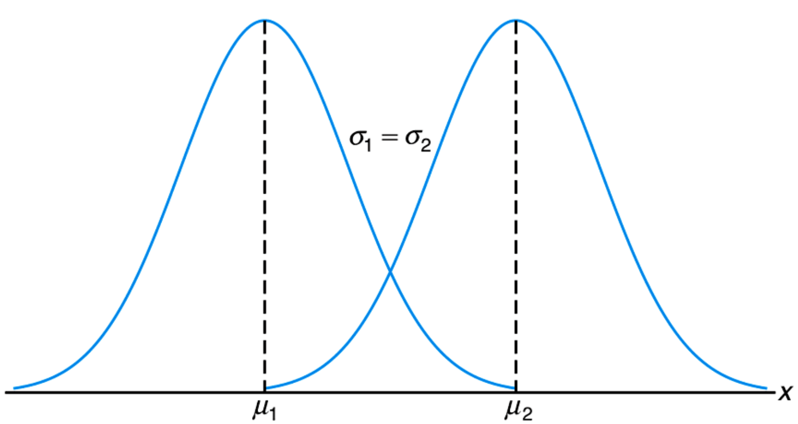


Figure 6.3 Normal curves with $\mu_1 \boxed{} \mu_2$ and $\sigma_1 \boxed{} \sigma_2$

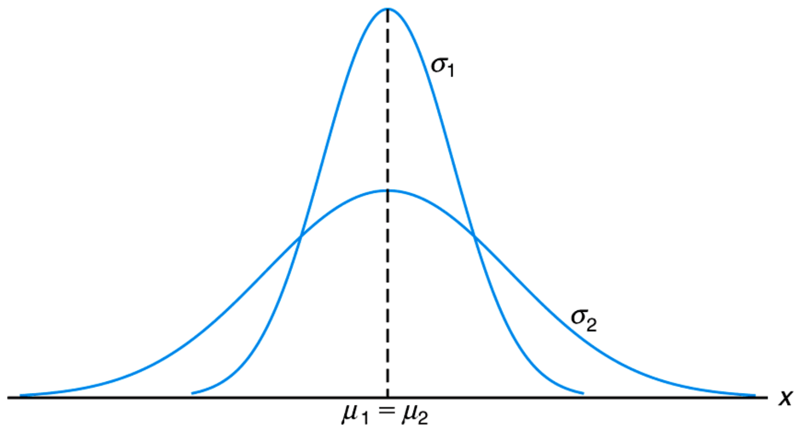


Figure 6.4 Normal curves with $\mu_1 \boxed{} \mu_2$ and $\sigma_1 \boxed{} \sigma_2$

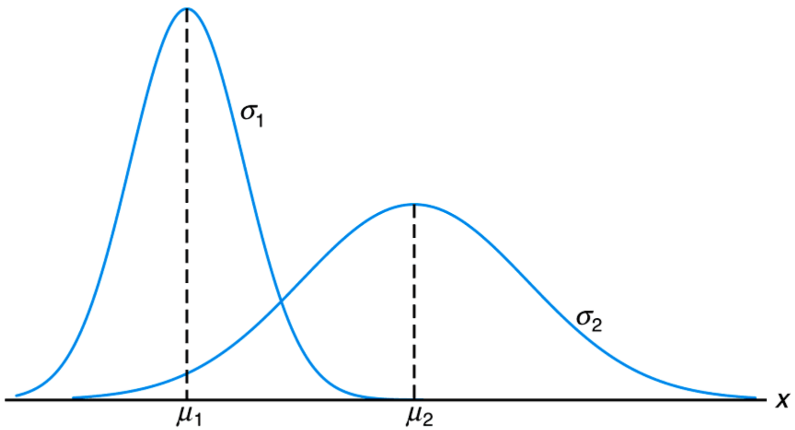


Figure 6.5 Normal curves with $\mu_1 \boxed{} \mu_2$ and $\sigma_1 \boxed{} \mu_2$

Normal Distribution

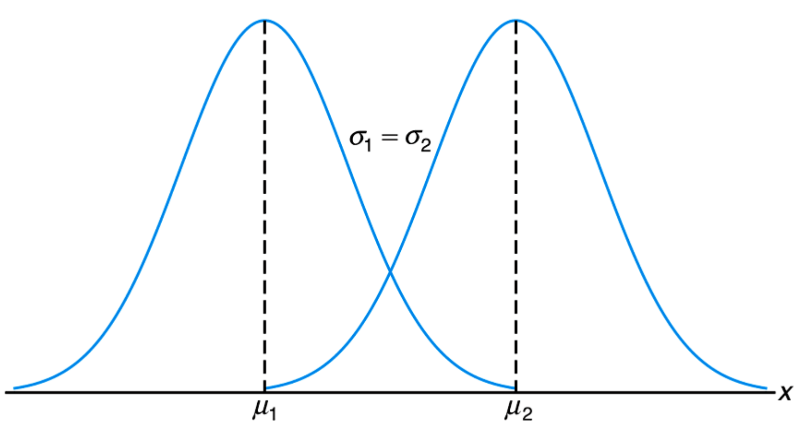


Figure 6.3 Normal curves with $\mu_1 \boxed{<} \mu_2$ and $\sigma_1 \boxed{=} \sigma_2$

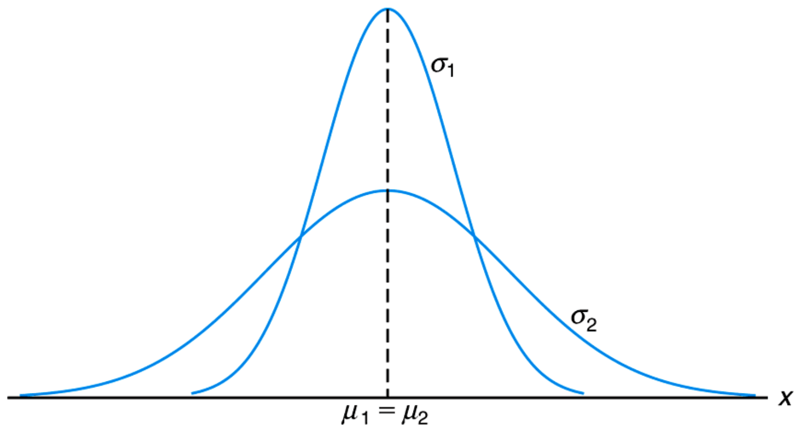


Figure 6.4 Normal curves with $\mu_1 \boxed{=} \mu_2$ and $\sigma_1 \boxed{<} \sigma_2$

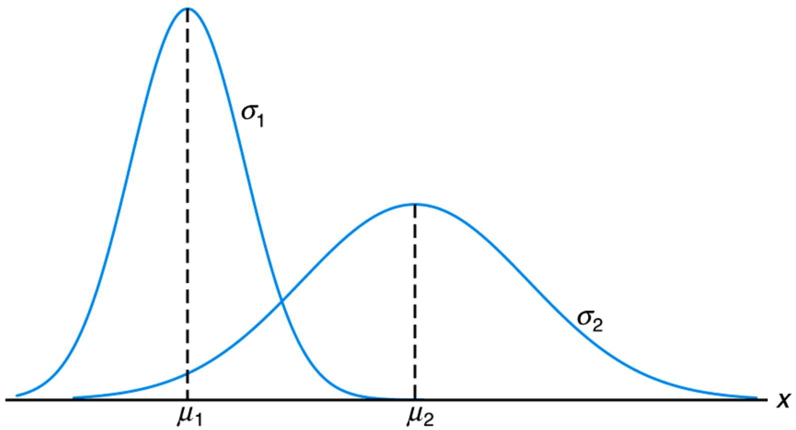


Figure 6.5 Normal curves with $\mu_1 \boxed{<} \mu_2$ and $\sigma_1 \boxed{<} \mu_2$

Normal Distribution

- The properties of the normal curve
  1. The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at $x = \mu$.
  2. The curve is symmetric about a vertical axis through the mean μ.
  3. The curve has its points of inflection at $x = \mu \pm \sigma$ is concave downward if $\mu - \sigma < X < \mu + \sigma$, and is concave upward otherwise.
  4. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
  5. The total area under the curve and above the horizontal axis is equal to 1.

Normal Distribution

- Show that the parameters μ and σ^2 are indeed the mean and the variance of the normal distribution.

$$E(X)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{1}{2}[\frac{(x-\mu)}{\sigma}]^2}dx, \text{Setting } z=\frac{(x-\mu)}{\sigma} \text{ and } dx=\sigma dz$$

$$\Rightarrow E(X)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mu+\sigma z)e^{-\frac{1}{2}z^2}dz$$

$$=\mu\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}z^2}dz}_{1}+\underbrace{\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ze^{-\frac{1}{2}z^2}dz}_{0}=\mu$$

Normal curve:
Mean: μ =0
Variance: σ^2 =1

$$E[(X-\mu)^2]=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2e^{-\frac{1}{2}[\frac{(x-\mu)}{\sigma}]^2}dx$$

$$\text{Setting } z=\frac{(x-\mu)}{\sigma} \text{ and } dx=\sigma dz \Rightarrow E[(X-\mu)^2]=\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^2e^{-\frac{1}{2}z^2}dz$$

$$\text{Integrating by parts } (\int udv=uv-\int vdu) \text{ with } u=z \text{ and } dv=ze^{-\frac{1}{2}z^2},$$

$$\therefore du=dz \text{ and } v=-e^{-\frac{1}{2}z^2} \Rightarrow E[(X-\mu)^2]=\frac{\sigma^2}{\sqrt{2\pi}}(-ze^{-\frac{1}{2}z^2}|_{-\infty}^{\infty}+\underbrace{\int_{-\infty}^{\infty}e^{-\frac{1}{2}z^2}dz}_{1})=\sigma^2(0+1)=\sigma^2$$

6.3 Areas Under the Normal Curve

- The probability of the random variable *X* assuming a value between x_1 and x_2.

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x;\mu,\sigma)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-(1/2)[(x-\mu)/\sigma]^2}dx$$

- The area under the curve between any two ordinates must also depend on the values μ and σ.

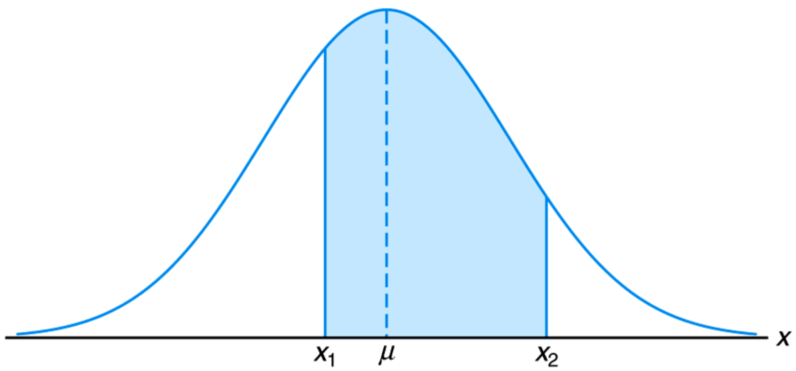


Figure 6.6 $P(x_1 < X < x_2)$ = area of the shaded region

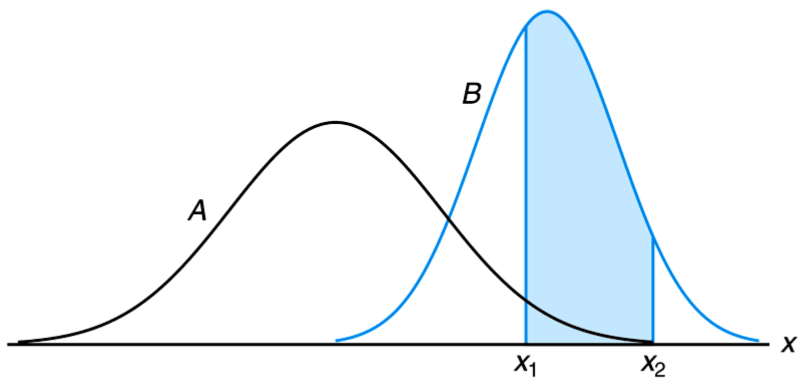


Figure 6.7 $P(x_1 < X < x_2)$ for different normal curves

Areas Under the Normal Curve

- Definition 6.1: The distribution of a normal random variable with mean zero and variance 1 is called a standard normal distribution.

$$\text{Transformation}: Z=\frac{X-\mu}{\sigma}, z_1=\frac{x_1-\mu}{\sigma}, z_2=\frac{x_2-\mu}{\sigma}$$

$$\therefore P(x_1<X<x_2)=\frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-(1/2)[(x-\mu)/\sigma]^2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{-z^2/2}dz$$

$$=\int_{z_1}^{z_2} n(z;0,1)dz = P(z_1<Z<z_2)$$

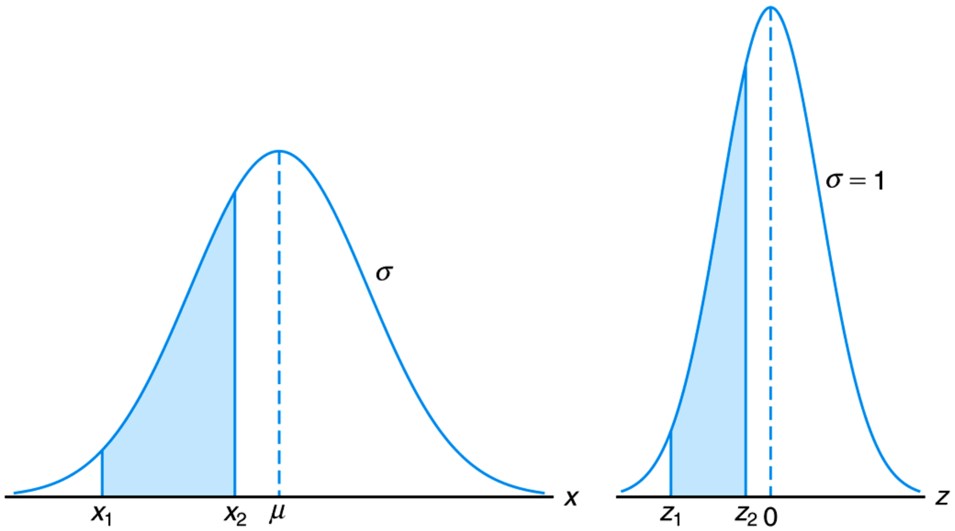


Figure 6.8 The original and transformed normal distributions

Areas Under the Normal Curve

- Example 6.2: Given a standard normal distribution, find the area under the curve that lies (a) to the right of z = 1.84 and (b) between z = -1.97 and z = 0.86.
  - **Solution**

  (a) 1 minus the area to the left of z = 1.84 (Table A.3, p. 755)

  1 – 0.9671 = 0.0329

  (b) The area to the left of z = 0.86 minus the left of z = -1.97

  0.8051 – 0.0244 = 0.7807

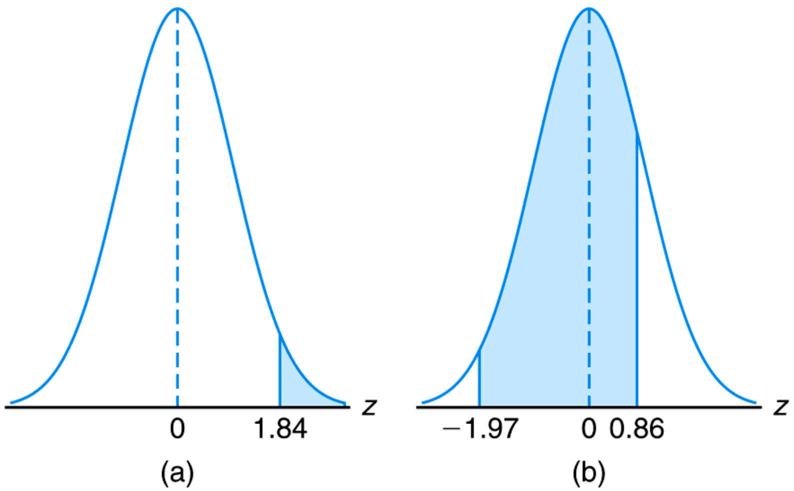


Figure 6.9 Areas for Example 6.2

TABLE A.3 (continued) Areas Under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993

Areas Under the Normal Curve

- Example 6.3: Given a standard normal distribution, find the value of k such that (a) $P(Z > k) = 0.3015$ and (b) $P(k < Z < -0.18) = 0.4197$.
  - **Solution**

  (a) $P(Z < k) = 1 - P(Z > k) = 1 - 0.3015 = 0.6985 \Rightarrow k = 0.52$

  (b) $P(Z < -0.18) - P(Z < k) = 0.4286 - P(Z < k) = 0.4197$

  $\Rightarrow k = -2.37$

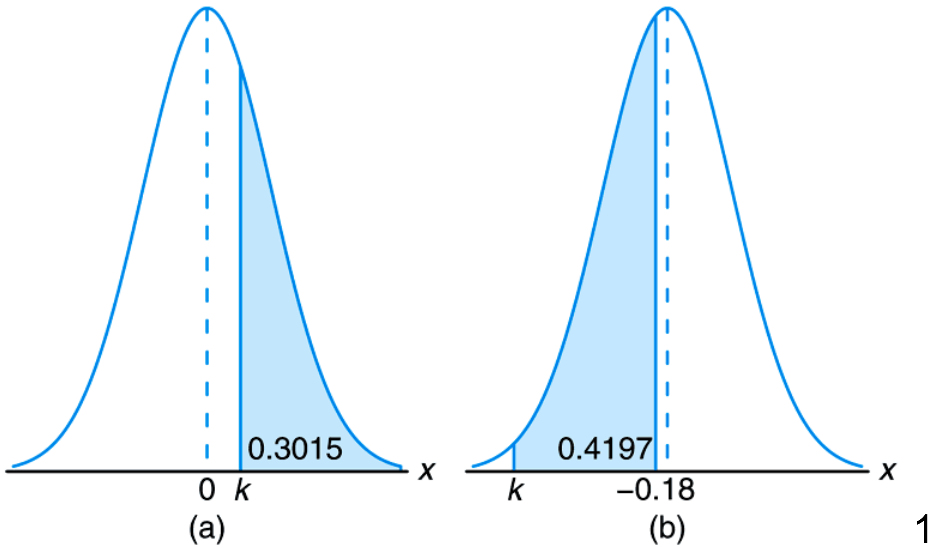


Figure 6.10 Areas for Example 6.3

Areas Under the Normal Curve

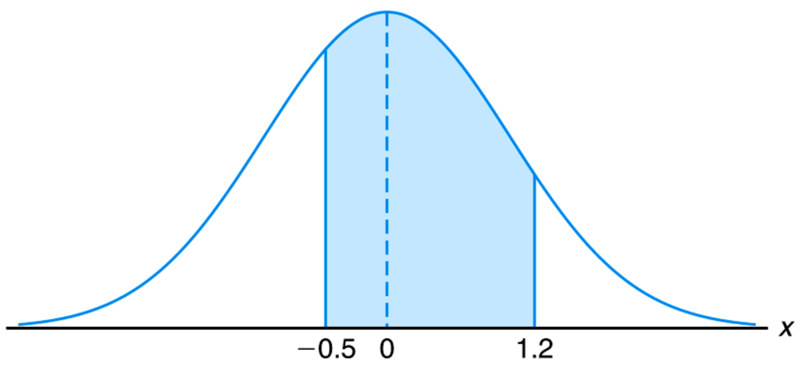
- Example 6.4: Given a random variable *X* having a normal distribution with μ= 50 and σ= 10, find the probability that *X* assumes a value between 45 and 62.
  - **Solution**

    x_1 = 45 and x_2 = 62 => $z_1 = \frac{45-50}{10} = -0.5$ and $z_2 = \frac{62-50}{10} = 1.2$

    $P(45 < X < 62) = P(-0.5 < Z < 1.2) = P(Z < 1.2) - P(Z < -0.5)$

    $= 0.8849 - 0.3085 = 0.5764$

Figure 6.11 Area for Example 6.4



Areas Under the Normal Curve

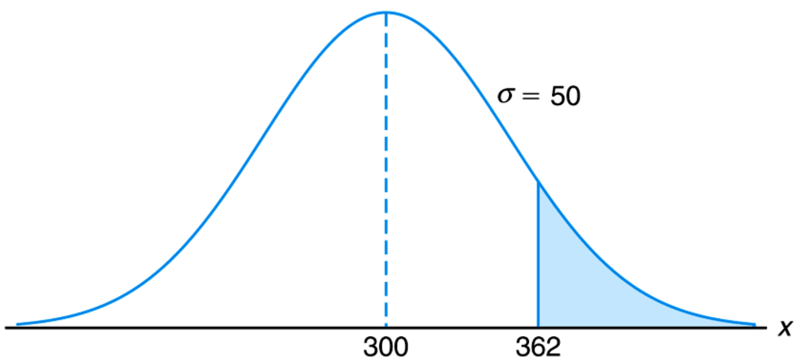
- Example 6.5: Given that a normal distribution with μ= 300 and σ= 50, find the probability that *X* assumes a value greater than 362.
  - **Solution**

$$z = \frac{362-300}{50} = 1.24$$

$P(X > 362) = P(Z > 1.24) = 1 - P(Z < 1.24)$

$= 1 – 0.8925 = 0.1075$

Figure 6.12 Area for Example 6.5



6.4 Applications of the Normal Distribution

- Example 6.7: A certain type of storage battery lasts, on average, 3.0 years, with a standard deviation of 0.5 year. Assuming that the battery lives are normally distributed, find the probability that a given battery will last less than 2.3 years.
  - **Solution**

$$z = \frac{2.3-3}{0.5} = -1.4$$

$$P(X < 2.3) = P(Z < -1.4) = 0.0808$$

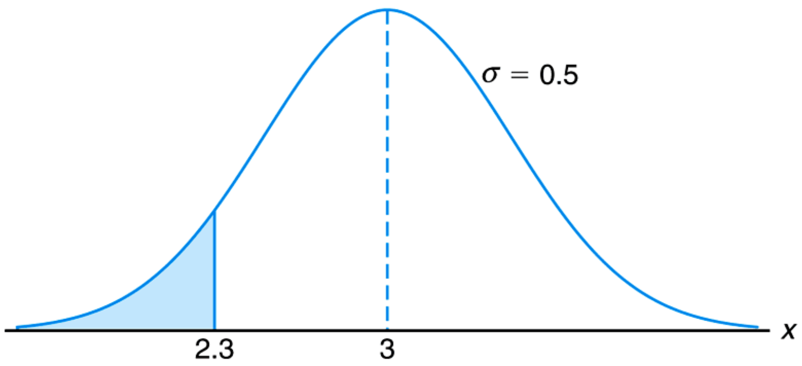


Figure 6.14 Area for Example 6.7

Applications of the Normal Distribution

- Example 6.8: An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.
  - **Solution**

$$z_1 = \frac{778-800}{40} = -0.55 \text{ and } z_2 = \frac{834-800}{40} = 0.85$$

$$\begin{aligned} P(778 < X < 834) &= P(-0.55 < Z < 0.85) \\ &= P(Z < 0.85) - P(Z < -.055) \\ &= 0.8023 - 0.2912 = 0.5111 \end{aligned}$$

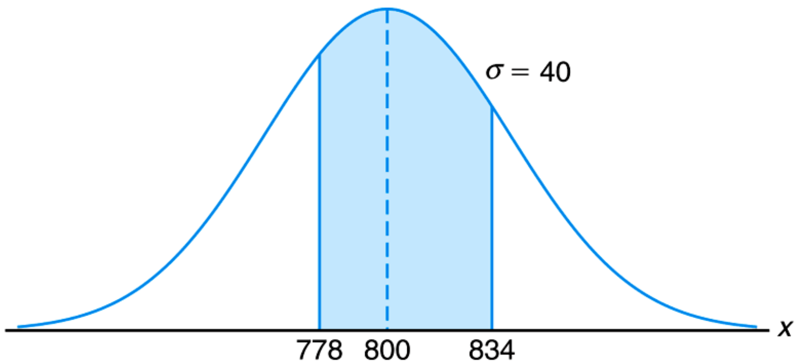


Figure 6.15 Area for Example 6.8

Applications of the Normal Distribution

- Example 6.9: In an industrial process the diameter of a ball bearing is an important component part. The buyer sets specifications on the diameter to be 3.0 ± 0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that in the process the diameter of a ball bearing has a normal distribution with mean 3.0 and standard deviation σ= 0.005. On the average, how many manufactured ball bearings will be scrapped?.
  - **Solution**

$$z_1 = \frac{2.99-3.0}{0.005} = -2.0 \text{ and } z_2 = \frac{3.01-3.0}{0.005} = 2.0$$

$$\begin{aligned} P(2.99 < X < 3.01) &= P(-2.0 < Z < 2.0) \\ &= 1 - 2P(Z < -2.0) \\ &= 1 - 2 \times 0.0228 = 0.9544 \end{aligned}$$

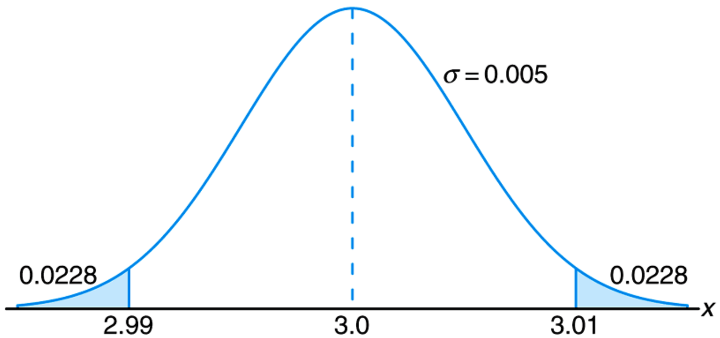


Figure 6.16 Area for Example 6.9

Applications of the Normal Distribution

- Example 6.10: Gauges (測量儀器;錶) are used to reject all components where a certain dimension is not within the specification 1.50 ± *d*. It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.2. Determine the value *d* such that the specifications cover 95% of the measurements.
  - **Solution**

$$\because P(-1.96 < Z < 1.96) = 0.95$$

$$\therefore 1.96 = \frac{(1.50+d)-1.50}{0.2} \Rightarrow d = 0.2 \times 1.96 = 0.392$$

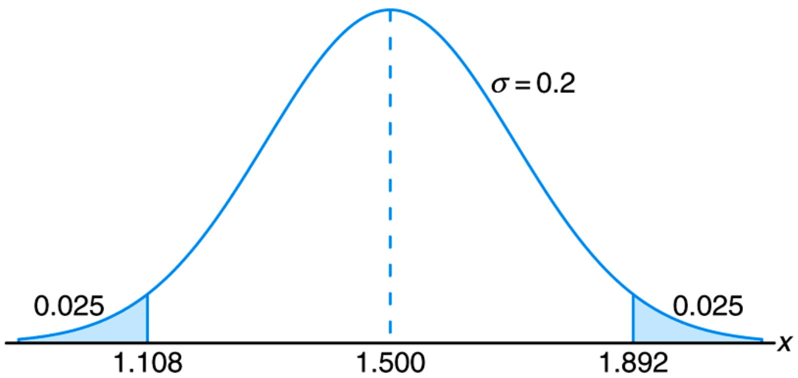


Figure 6.17 Specifications for Example 6.10

Applications of the Normal Distribution

- Example 6.11: A certain machine makes electrical resistors having a mean resistance of 40 ohms and a standard deviation of 2 ohms. Assuming that the resistance follows a normal distribution and can be measured to any degree of accuracy, what percentage of resistors will have a resistance exceeding 43 ohms?
  - **Solution**

$$z = \frac{43-40}{2} = 1.5$$

$$\because P(X > 43) = P(Z > 1.5) = 1 - P(Z < 1.5) = 1 - .09332 = 0.0668$$

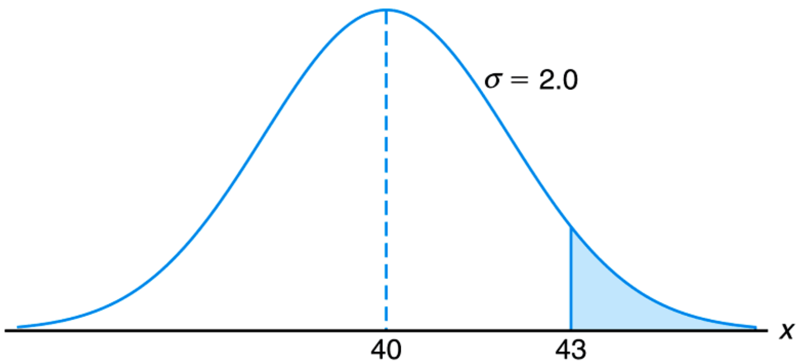


Figure 6.18 Area for Example 6.11

Applications of the Normal Distribution

- Example 6.13: The average grade for an exam is 74, and the standard deviation is 7. If 12% of the class are given A's, and the grades are curved to follow a normal distribution, what is the lowest possible A and the highest possible B?
  - **Solution**

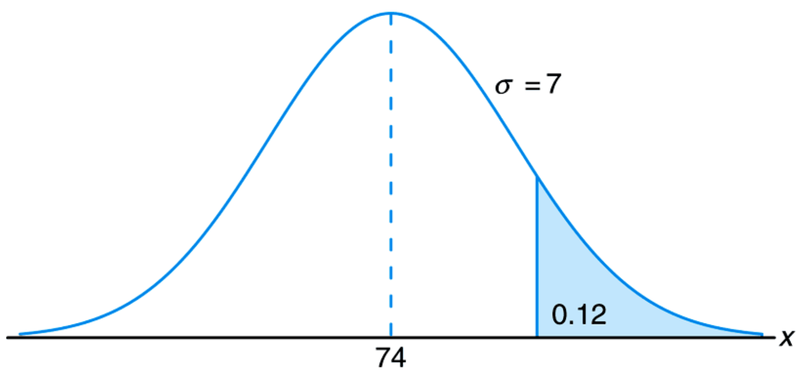


Figure 6.20 Area for Example 6.13

Applications of the Normal Distribution

- Example 6.13: The average grade for an exam is 74, and the standard deviation is 7. If 12% of the class are given A's, and the grades are curved to follow a normal distribution, what is the lowest possible A and the highest possible B?
  - **Solution**

$$\because 1 - 0.12 = 0.88 = P(Z < 1.175)$$

$$\therefore 1.175 = \frac{x-74}{7} => x = 7 \times 1.175 + 74 = 82.225$$

The lowest A is 83 and the highest B is 82.

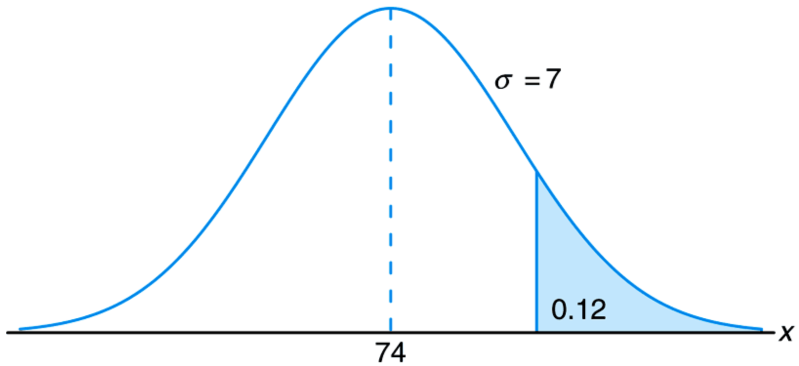


Figure 6.20 Area for Example 6.13