Chapter 2
Probability

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2.1 Sample Space

- In the study of statistics, we are concerned with the presentation and interpretation of chance outcomes.
- Observation: Any recording of information, whether it be numerical or categorical, is referred to observation.
  - 2, 0, 1, 2
  - D, N, D, N, N
- 'Experiment' is used to describe any process that generates a set of data.
  - E.g., tossing of a coin, two possible outcomes, heads and tails
- In most cases the outcome will depend on chance and, thus, cannot be predicted with certainty.
- Definition 2.1: The set of possible outcomes of a statistical experiment is called the **sample space**, represented by ***S***.
- Each outcome in a sample space is called an element, a member, or a sample point.

Sample Space

- Example 2.1
  - Tossing a coin: $S = \{H, T\}$
  - Tossing a die
    - $S_1 = \{1, 2, 3, 4, 5, 6\}$
    - $S_2 = \{\text{even, odd}\}$
- Tree diagram: List the elements of the sample space systematically.
- Example 2.2
  - $S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$
- Example 2.3
  - $S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$

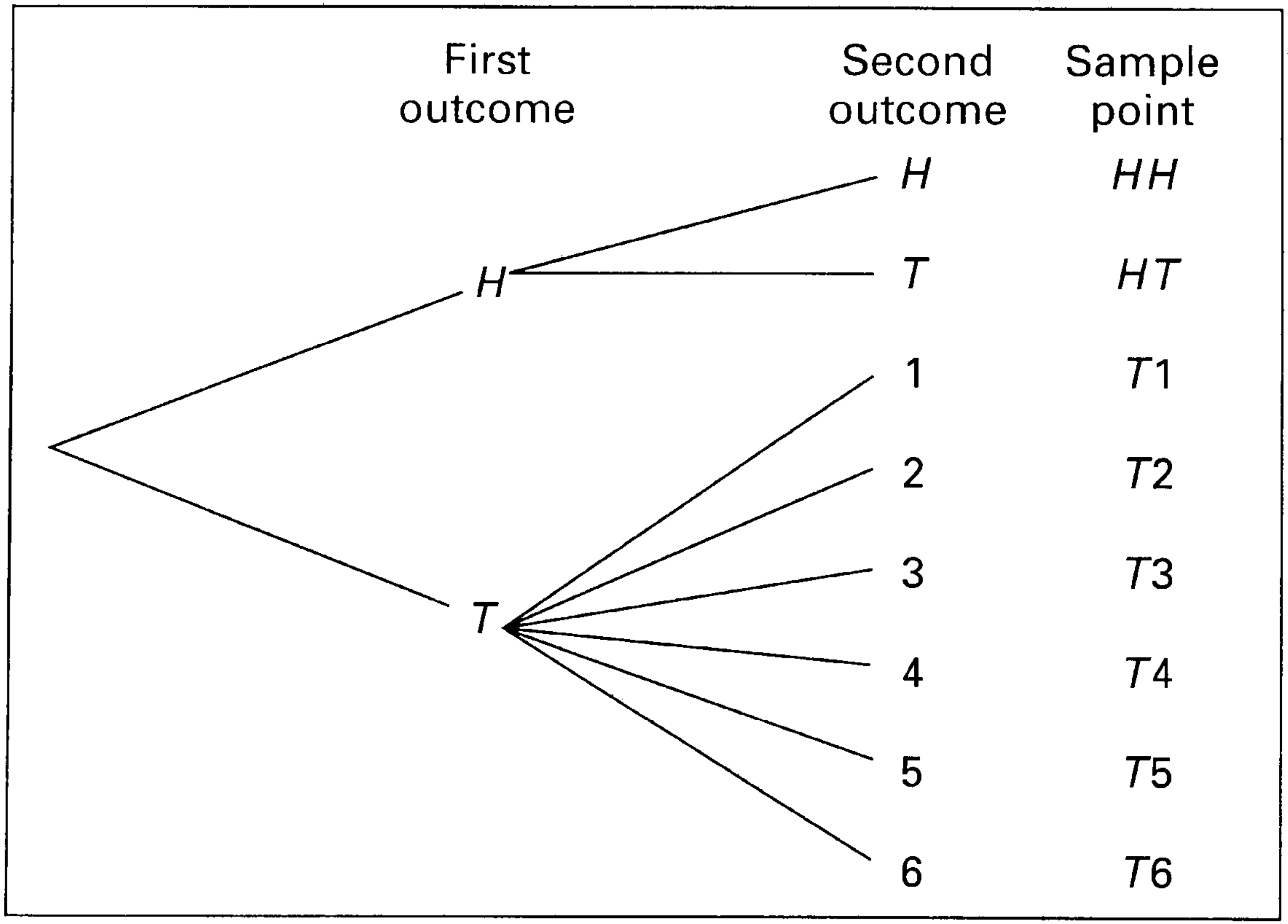

Figure 2.1 Tree diagram for Example 2.2.

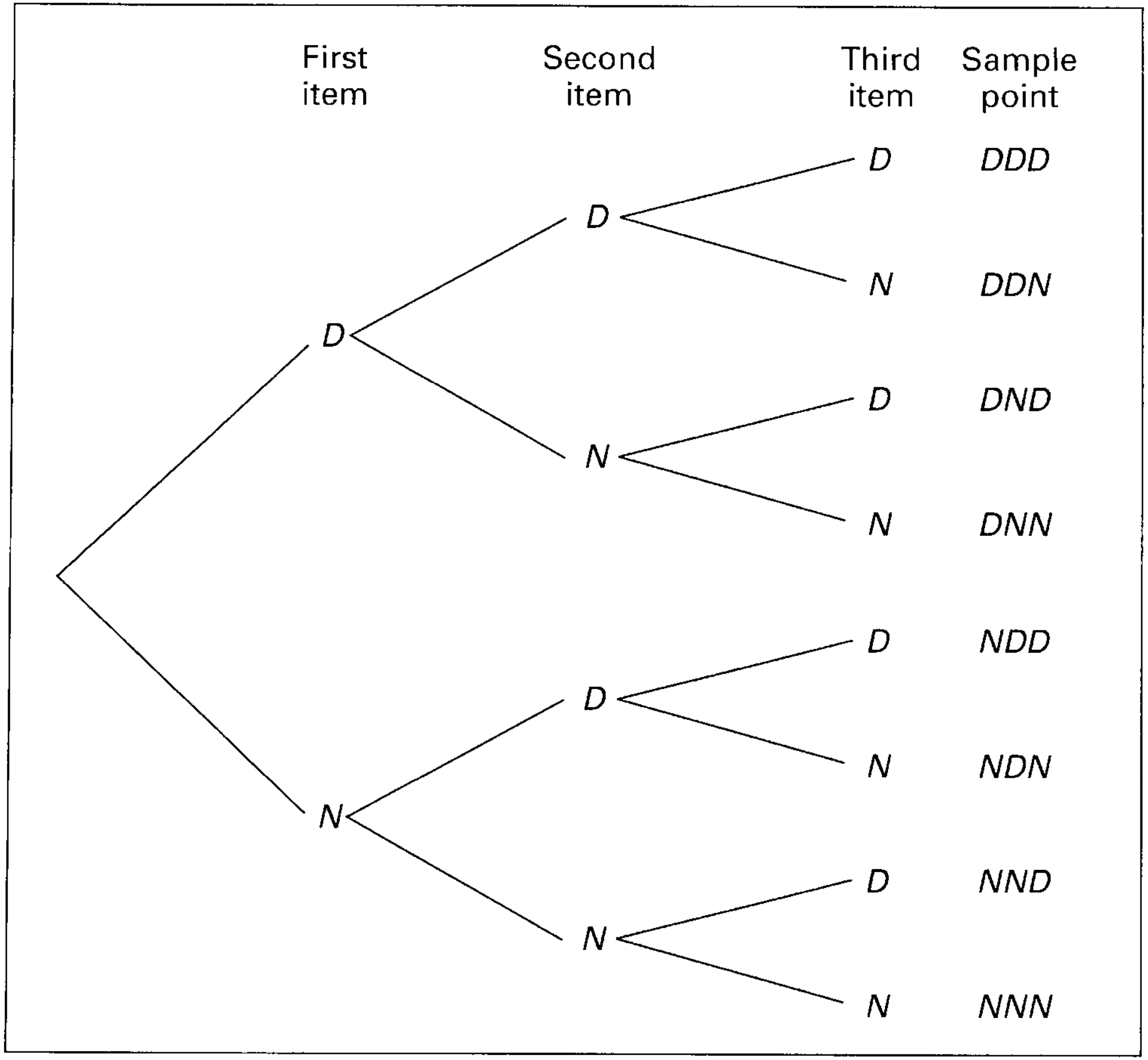

Figure 2.2 Tree diagram for Example 2.3.

Sample Space

- Statement (Rule): Sample spaces with a large or infinite number of sample points are best described by a statement or rule.
  - $S = \{x | x \text{ is a city with population over 1 million}\}$
  - $S = \{(x,y) \mid x^2 + y^2 \leq 4\}$
- The rule method has practical advantages, particularly for the many experiments where a listing becomes a tedious chore (雜物).

2.2 Events

- Definition 2.2: An event is a subset of a sample space.
  - E.g., we may be interested in the event A that the outcome when a die is tossed is divisible by 3, $A = \{3, 6\}$.
- Example 2.4
  - Given the sample space $S = \{t \mid t \geq 0\}$, where t is the life in years of a certain electronic component.
  - The event A that the component fails before the end of the fifth year is the subset $A = \{t \mid 0 \leq t < 5\}$.
- Null set, denoted Ø, contains no elements at all.

Events

- Definition 2.3: The complement of an event *A* with respect to *S* is the subset of all elements of *S* that are not in *A*. We denote the complement of *A* by the symbol *A'*.
- Example 2.5
  - Let *R* be the event that a red card is selected from an ordinary deck of 52 playing cards.
  - *S* be the entire deck.
  - *R'* is the event that the card selected from the deck is not a red but a black card.

Events

- Definition 2.4: The intersection of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.
- Example 2.7
  - Let E be the event that a person selected at random in a classroom is majoring in engineering.
  - F is the event that the person is female.
  - The event $E \cap F$ is the set of all female engineering students in the classroom.

Events

- Definition 2.5: Two events A and B are mutually exclusive, or disjoint if $A \cap B = \emptyset$, i.e., if A and B have no elements in common.
- Definition 2.6: The union of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Events

- Venn diagram: The relationship between events and the corresponding sample space can be illustrated graphically by Venn diagram.
- In a Venn diagram, let the sample space be a rectangle and represent events by circles.
- In Figure 2.3
  - $A \cap B$ = regions 1 and 2
  - $A \cup C$ = regions 1, 2, 3, 4, 5, and 7
  - $B' \cap A$ = regions 4 and 7

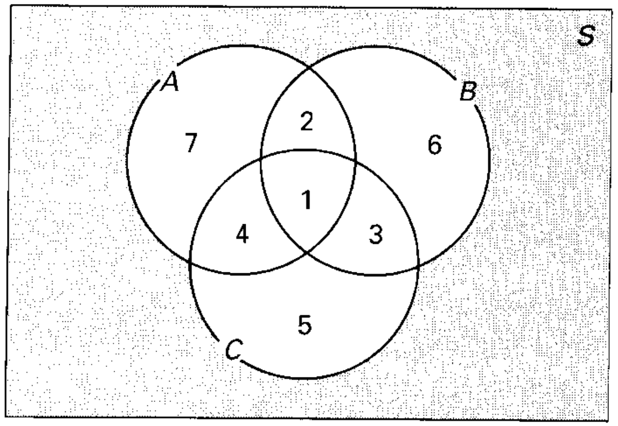

Figure 2.3 Events represented by various regions.

2.3 Counting Sample Points

- Multiplication rule (Rule 2.1): If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.
- Example 2.13
  - Sampling points of a dice thrown twice? $n_1 n_2 = (6)(6) = 36$ possible ways.

Counting Sample Points

- Example 2.14
  - Offers for home buyers with Tudor, rustic, colonial, and traditional exterior styling
  - With ranch, two-story, and split-level floor plans.
  - How many ways can a buyer order one of these homes?

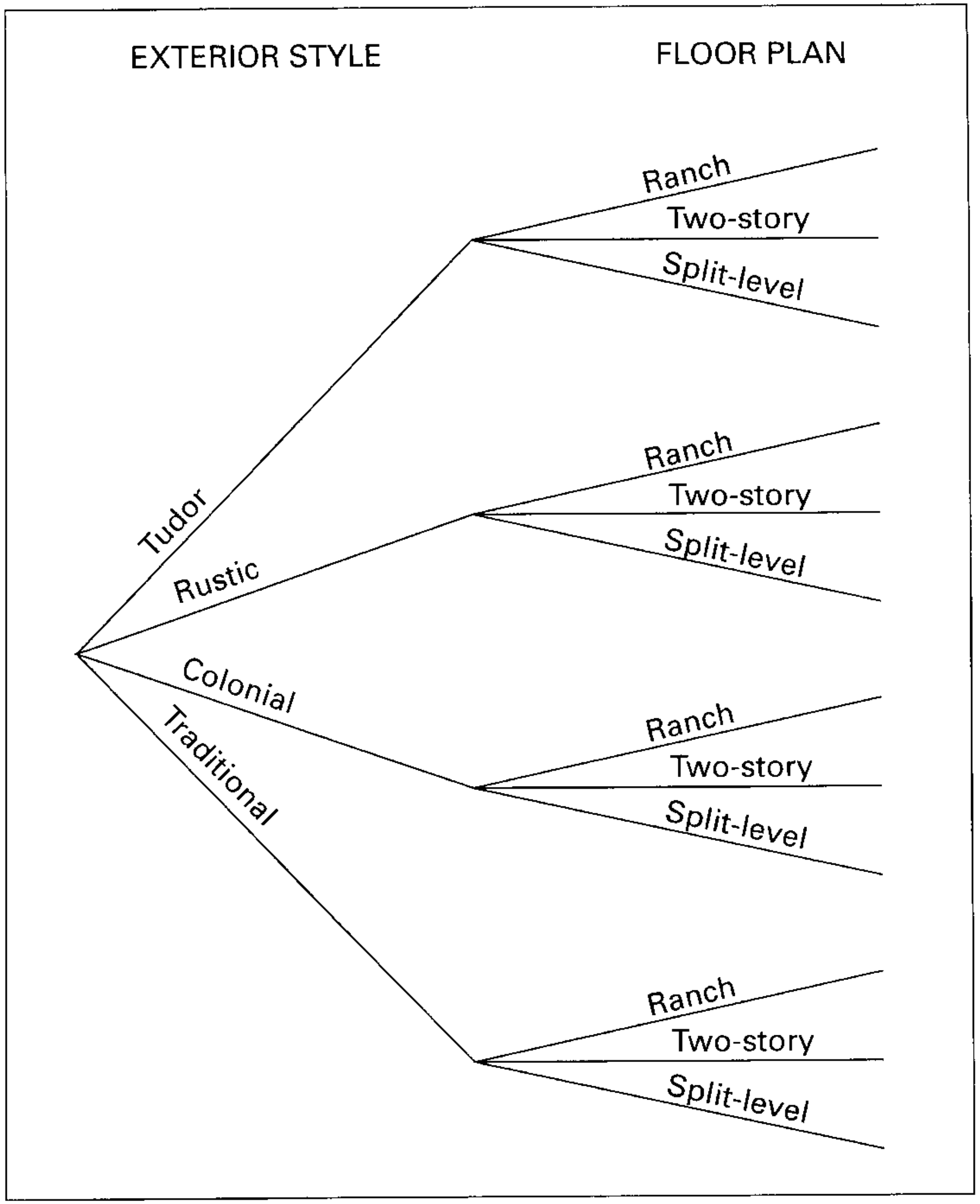

Figure 2.6 Tree diagram for Example 2.14.

Counting Sample Points

- Rule 2.2:
  - If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \ldots n_k$ ways.

Counting Sample Points

- Example 2.16
  - Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?
  - $n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$

Counting Sample Points

- Example 2.17
  - How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6 and 9 if each digit can be used only once?
  - Unit digit 0
    - $n_1n_2n_3n_4 = (1)(5)(4)(3) = 60$
  - Unit digit 2, 6
    - $n_1n_2n_3n_4 = (2)(4)(4)(3) = 96$
  - Total numbers = 60 + 96 = 156

Counting Sample Points

- A permutation is an arrangement of all or part of a set of objects.
- A combination is the number of ways of selecting *r* objects from *n* without regard to order.
- Definition 2.8
  – For any non-negative integer n, n!, called "n factorial," is defined as n! = n(n-1)…(2)(1),
  with special case 0! = 1.

Counting Sample Points

- The number of permutations of n distinct objects is $n!$.
- The number of permutations of n distinct objects taken r at a time is $P(n,r) = n \times (n-1) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$.
- The number of permutations of n distinct objects arranged in a circle is $\frac{n!}{n} = (n-1)!$.
- The number of permutations of n things of which n_1 are of one kind, n_2 of a second kind, …, and n_k of a kth kind is $\frac{n!}{n_1!n_2!\ldots n_k!}$.
- The number of arrangements of a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is
  $$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}, \text{ where } n_1 + n_2 + \cdots + n_r = n.$$
- The number of combinations of n distinct objects taken r at a time is $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Counting Sample Points

- Example 2.21
  - In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?
  - $$\binom{7}{3,2,2} = \frac{7!}{3!2!2!} = 210$$
- Example 2.23
  - How many difference letter arrangements can be made from the letters in the word STATISTICS?

$$\binom{10}{3,3,2,1,1} = \frac{10!}{3!3!2!1!1!} = 50400$$

2.4 Probability of Event

- Perhaps it was man’s unquenchable (不能遏制的) thirst for gambling that led to the early development of probability theory.
- What do we mean when we make the statements
  - John will probably win the tennis match.
  - I have a fifty-fifty chance of getting an even number when a die is tossed.
  - I am not likely to win at bingo tonight.
  - Most of our graduating class will likely be married within 3 years.
- In each case, we are expressing an outcome of which we are not certain, but owing to past information or from an understanding of the structure of the experiment, we have some degree of confidence in the validity of the statement.

Probability of Event

- The likelihood of the occurrence of an event resulting from a statistical experiment is evaluated by means of a set of real numbers called weights or probabilities, and ranged from 0 to 1.
- To every point in the sample space we assign a probability such that the sum of all probabilities is 1.
- In many experiments, such as tossing a coin or a die, all the sample points have the same chance of occurring and are assigned equal probabilities.
- For points outside the sample space, i.e., for simple events that cannot possibly occur, we assign a probability of zero.

Probability of Event

- Definition 2.9: The probability of an event A is the sum of the weights of all sample points in A.

$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1$$

- If A_1, A_2, A_3,...is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) \cdots$$

- Example 2.24: A coin is tossed twice. What is the probability that at least one head occur?

$$S = \{HH, HT, TH, TT\}, A = \{HH, HT, TH\}, \text{ and } P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Probability of Event

- If an experiment can result in any one of *N* different equally likely outcomes, and if exactly *n* of these outcomes correspond to event A, then the <u>probability of event A</u> is $P(A) = \frac{n}{N}$.
- Example 2.28: In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

$$P(C) = \frac{C(4,2) \times C(4,3)}{C(52,5)} = \frac{\frac{4!}{2!2!} \times \frac{4!}{3!1!}}{\frac{52!}{5!47!}} = \frac{24}{2{,}598{,}960} \approx 0.9 \times 10^{-5}$$

Probability of Event

- If the outcomes of an experiment are not equally likely to occur, the probabilities must be assigned based on prior knowledge or experimental evidence.
- According to the relative frequency definition of probability, the true probabilities would be the fractions of events that occur in the long run.
- The use of intuition, personal beliefs, and other indirect information in arriving at probabilities is referred to as the subjective definition of probability.

2.5 Additive Rules

- Theorem 2.7: If *A* and *B* are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

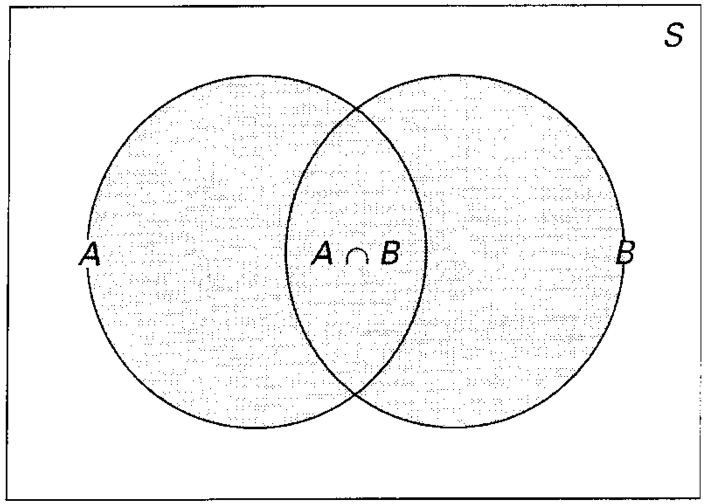

Figure 2.7 Additive rule of probability.

- Corollary 2.1: If *A* and *B* are <u>mutually exclusive</u>, then

$$P(A \cup B) = P(A) + P(B).$$

Additive Rules

- Corollary 2.2: If A_1, A_2, …, A_n, are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

- Corollary 2.3: If A_1, A_2, …, A_n, is a partition of a sample space S, then

$$\begin{aligned} P(A_1 \cup A_2 \cup \cdots \cup A_n) &= P(A_1) + P(A_2) + \cdots + P(A_n) \\ &= P(S) \\ &= 1. \end{aligned}$$

- Theorem 2.8: For three events A, B, and C,

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ + P(A \cap B \cap C). \end{aligned}$$

Additive Rules

- Theorem 2.9: If *A* and *A*’ are complementary events, then *P*(*A*)+*P*(*A*’)=1
  - Proof

$$\because A \cup A' = S, \text{ and } A \text{ and } A' \text{ are disjoint}$$
$$\therefore 1 = P(S) = P(A \cup A') = P(A) + P(A')$$

Additive Rules

- Example 2.32
  - If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?
  - E: at least 5 cars serviced.
  - $P(E) = 1 - P(E') = 1 - (0.12 + 0.19) = 0.69$

Exercise

- 2.38, 2.63, 2.72
- Due 3/31/2015