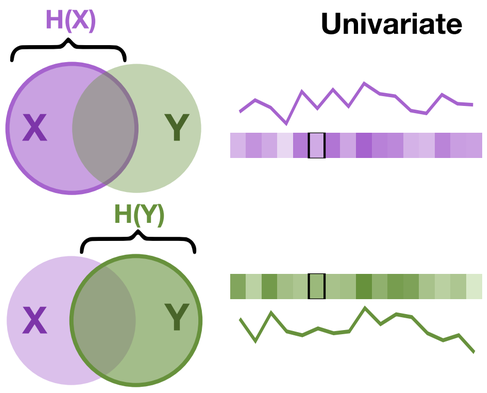
Contemporaneous univariate measures

Marginal entropy

Univariate

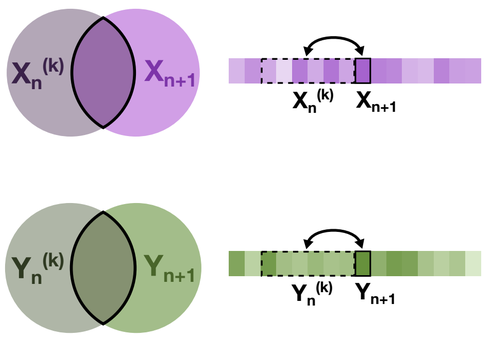


$$H(X) = -\sum_{i=1}^{n} P(x_i)\log_2 P(x_i)$$

$$H(Y) = -\sum_{i=1}^{n} P(y_i)\log_2 P(y_i)$$

Time-dependent univariate measures

Active information storage



$$A_X^{(k)} = I(X_n^{(k)}; X_{n+1})$$

$$A_X(k) = \sum_{x_{n+1},x_n^{(k)} \epsilon X} p(x_{n+1}, x_n^{(k)}) \log \left[\frac{p(x_{n+1}, x_n^{(k)})}{p(x_{n+1})p(x_n^{(k)})} \right]$$

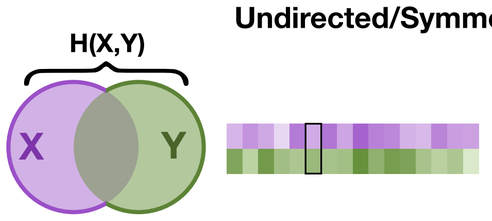
$$A_Y^{(k)} = I(Y_n^{(k)}; Y_{n+1})$$

$$A_Y(k) = \sum_{y_{n+1},y_n^{(k)} \epsilon Y} p(y_{n+1}, y_n^{(k)}) \log \left[\frac{p(y_{n+1}, y_n^{(k)})}{p(y_{n+1})p(y_n^{(k)})} \right]$$

Contemporaneous bivariate measures

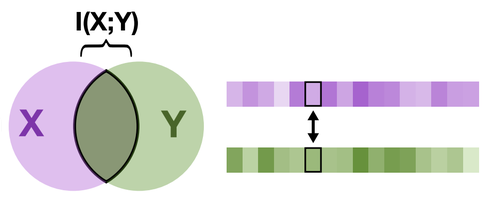
Undirected/Symmetric

Joint entropy (JE)



$$H(X,Y) = -\sum_{x \in X}\sum_{y \in Y} P(x,y)\log P(x,y)$$

Mutual information (MI)



$$I(X;Y) = \sum_{x \epsilon X}\sum_{y \epsilon Y} P(x,y)\log \left[\frac{P(x,y)}{P(x)P(y)} \right]$$

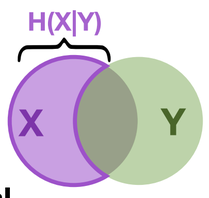
or

$$I(X;Y) = H(X,Y) - H(X|Y) - H(Y|X)$$

Directed/Asymmetric

Conditional entropy (CE)

Y→X

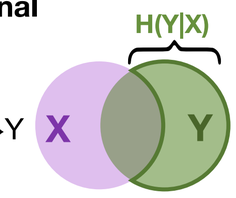


$$H(X|Y) = \sum_{x \epsilon X}\sum_{y \epsilon Y} P(x,y)\log \left[\frac{P(y)}{P(x)P(y)} \right]$$

or

$$H(X|Y) = H(X,Y) - H(Y)$$

X→Y



$$H(Y|X) = \sum_{x \epsilon X}\sum_{y \epsilon Y} P(x,y)\log \left[\frac{P(x)}{P(x)P(y)} \right]$$

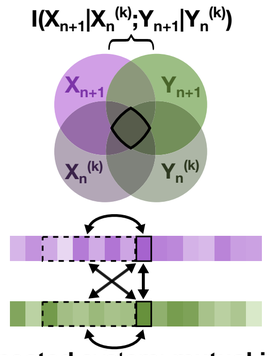
or

$$H(Y|X) = H(X,Y) - H(X)$$

Time-dependent bivariate measures

Undirected/Symmetric

Φ*



Fully connected system: **mutual information** of X and Y, both conditioned on their own pasts

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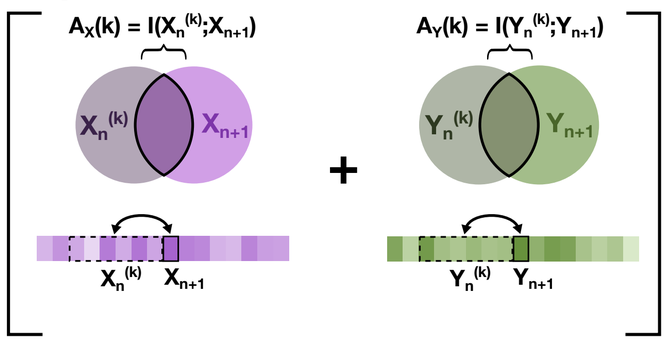
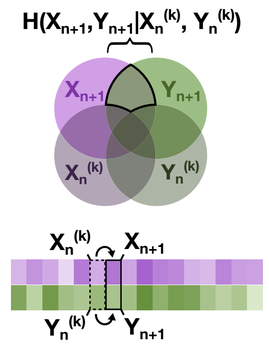


Disconnected system: sum of **AIS of X** plus **AIS of Y** for a given **window length,** *k*

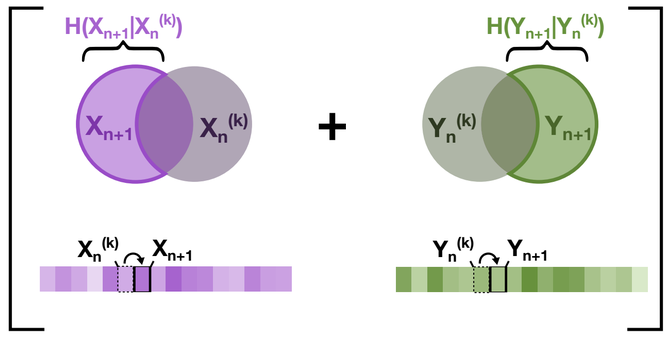
[Equations in terms of (1) entropy terms and (2) probability terms to go here)

Stochastic interaction (SI)



Joint entropy conditioned on the past of X and Y (*k*=1)

vs. or −



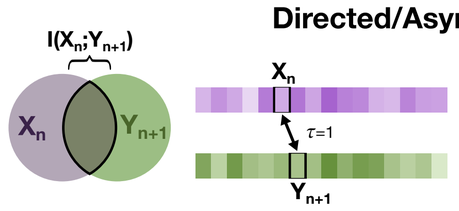
Conditional entropies of X and Y, respectively, each conditioned on its own past (*k*=1)

[Equations in terms of (1) entropy terms and (2) probability terms to go here)

Directed/Asymmetric

Time-lagged mutual information (TLMI)

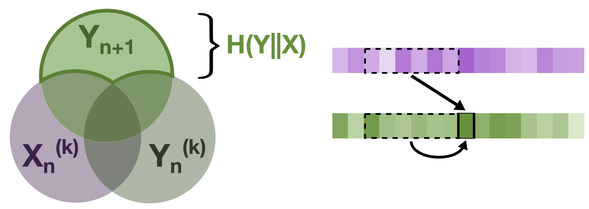
X⟶Y



$$I(X_n; Y_{n+1}) = \sum_{x_n \in X_n}\sum_{y_{n+1} \in Y_{n+1}} P(x_n, y_{n+1})\log \left[\frac{P(x_n, y_{n+1})}{P(x_n)P(y_{n+1})} \right]$$

Causally conditioned entropy (CCE)

X⟶Y

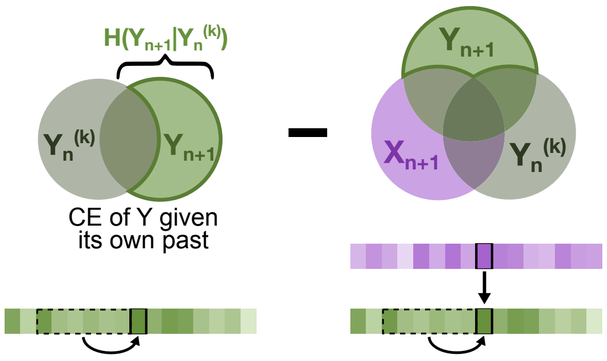


$$H(Y||X)(k) = \sum_{i=1}^{n} H(Y_{n+1}|Y_n^{(k)}, X_n^{(k)})$$

$$H(Y||X)(k) = -\sum_{y_{n+1},y_n^{(k)},x_n^{(k)} \epsilon X,Y} p(y_{n+1}, y_n^{(k)}, x_n^{(k)})\log \left[\frac{p(y_{n+1}|y_n^{(k)}, x_n^{(k)})}{p(y_n^{(k)}, x_n^{(k)})} \right]$$

Directed information (DI)

X⟶Y



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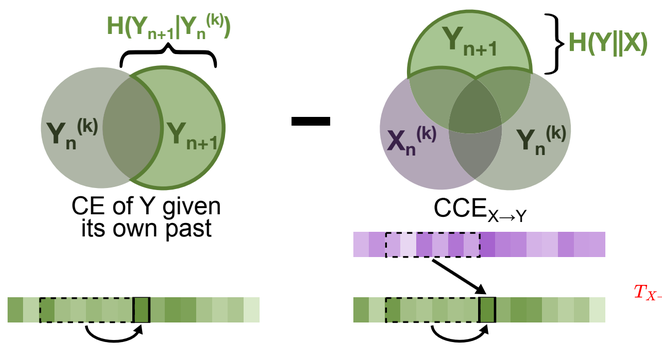
[Equations in terms of (1) entropy terms and (2) probability terms to go here)

Transfer entropy (TE)

X⟶Y



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$$T_{X \to Y} = I(X_n^{(k)}; Y_{n+1}|Y_n^{(k)})$$
$$= H(Y_{n+1}|Y_n^{(k)}) - H(Y||X)$$

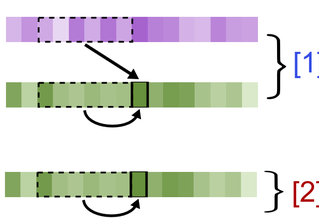
$$T_{X \to Y} = \sum_{y_{n+1},y_n^{(k)},x_n^{(k)} \epsilon X,Y} p(y_{n+1}, y_n^{(k)}, x_n^{(k)})\log \left[\frac{p(y_{n+1}|y_n^{(k)}, x_n^{(k)})}{p(y_{n+1}|y_n^{(k)})} \right]$$

Granger causality

X⟶Y

Fit linear regression models

Compute log-ratio of residual variance between models [1] and [2]



$$GC_{X \to Y} = \log \left[\frac{\sum_{y_{n+1} \epsilon Y_{n+1}} (\hat{y}_{n+1} - y_{n+1})^2}{\sum_{y_{n+1} \epsilon Y_{n+1}} (\hat{y}_{n+1} - y_{n+1})^2} \right]$$

*Note: Granger causality is equivalent to transfer entropy with a Gaussian density estimator