Score: ___________ / **35**

PSTAT 5A / **MIDTERM EXAM 2** / Sum. Sess. A 2023 Instructor: **Ethan Marzban**

Name:____ **UCSB NetID:**____________________

First, then Last *NOT your Perm Number!*

Circle Your Section: Olivier 12:30 - 1:20pm Mengrui 2 - 2:50pm Mengrui 3 - 3:50pm

FREE RESPONSE QUESTIONS

Instructions:

- You will have **75 minutes** to complete the entire exam
  - Do not begin working on the exam until instructed to do so.
  - During the final 10 minutes of the exam, we will ask everyone to remain seated until the exam concludes.
- This exam comes in **TWO PARTS**: this is the **FREE RESPONSE** part of the exam.
  - There is a separate booklet containing Multiple Choice questions that should have been distributed to you at the same time as this booklet.
- Write your answers directly in the space provided on this exam booklet.
  - You do not need to write anything on your scantron for this part of the exam.
- Be sure to show all of your work; correct answers with no supporting work will not receive full credit.
- The use of calculators is permitted; the use of any other aids (including notes, laptops, phones, etc.) is strictly prohibited. A list of formulae, as well as a collection of tables, is included with this exam.
- **PLEASE DO NOT DETACH ANY PAGES FROM THIS EXAM.**
- Good Luck!!!

Honor Code: In signing my name below, I certify that all work appearing on this exam is entirely my own and not copied from any external source. I further certify that I have not received any unauthorized aid while taking this exam.

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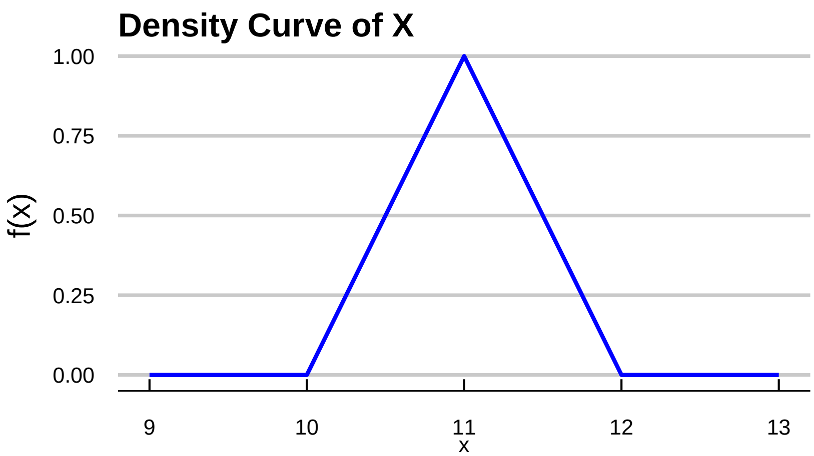
Free Response Questions

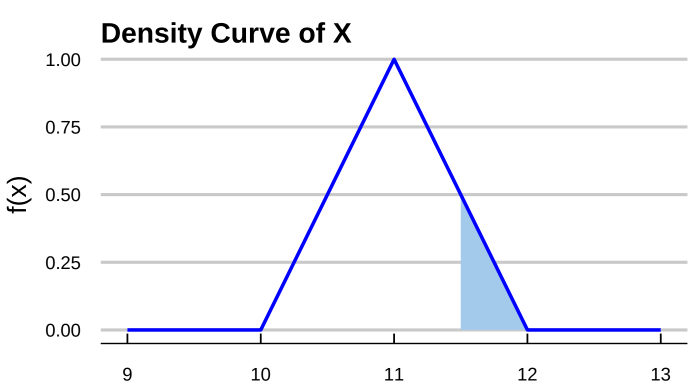
Problem 1. The length of a *GauchoSteel*-brand rod is meant to be 11 feet; due to imperfections in the manufacturing process, however, the length of a randomly-selected *GauchoSteel*-brand rod is actually a random variable X that has the following density curve:



(a) What is the probability that a randomly-selected *GauchoSteel*-brand rod is exactly 11 meters in length? [2pts.]

> **Solution:** Let X denote the length of a randomly-selected *GauchoSteel*-brand rod, so that the quantity we seek can be written as $\mathbb{P}(X = 11)$. Since X is continuous, we know that $\mathbb{P}(X = k) = 0$ for any value of k, meaning the desired probability is simply $\boxed{0}$.

(b) What is the probability that a randomly-selected *GauchoSteel*-brand rod is longer than 11.5 meters? [2pts.]

> **Solution:** Letting X be defined as above, we seek $\mathbb{P}(X > 11.5)$. This corresponds to the following area underneath the density curve:
>
> 
>
>
> This is a triangle with base length $(12 - 11.5) = (0.5)$ and height (0.5),

meaning its area (and, consequently, the desired probability) is simply

$$\frac{1}{2} \cdot (0.5) \cdot (0.5) = \boxed{\frac{1}{8}}$$

(c) A sample of 10 *GauchoSteel*-brand rods is taken with replacement, and the number of rods longer than 11.5 meters is recorded. What is the probability that this sample contains exactly 4 rods that are longer than 11.5 meters? Be sure to define any new random variables clearly and explicitly, and make sure to check any/all relevant conditions! You do **not** need to report your final answer as a decimal. [4pts.]

Solution: Let Y denote the number of *GauchoSteel*-brand rods, in a sample of 10 rods taken with replacement, that are longer tahn 11.5 meters. We suspect Y to be binomially distributed; to verify this, we check the four Binomial Conditions:

1) Independence Across Trials? Yes, since our sample is taken with replacement.

2) Fixed Number of Trials? Yes; $n = 10$.

3) Well-Defined Notion of Success? Yes; "success" = "given rod is longer than 11.5 meters"

4) Fixed Probability of Success? Yes; $p = 1/8$, as computed in part (a) above.

Since all four conditions are met, we conclude that $Y \sim \mathrm{Bin}(10,\ 1/8)$ and so, using the formula for the probability mass function of the Binomial distribution, we have

$$\mathbb{P}(Y = 4) = \binom{10}{4}\left(\frac{1}{8}\right)^4\left(1 - \frac{1}{8}\right)^{10-4} \approx 0.023$$

Problem 2. Alayah is interested in performing inference on the true average monthly rent (in thousands of dollars) of a 1-bedroom apartment in Santa Barbara. To that effect, she takes a representative sample of 100 1-bedroom apartments in Santa Barbara, and finds that these 100 apartments have a combined average monthly rent of 2.2 thousand dollars per month. From prior studies, she knows that the standard deviation of <u>all</u> monthly rents of 1-bedroom apartments in Santa Barbara is 0.75 thousand dollars.

(a) Define the parameter of interest. [1pts.]

Solution: Let μ denote the true average rent (in thousands of dollars) of a 1-bedroom apartment in Santa Barbara.

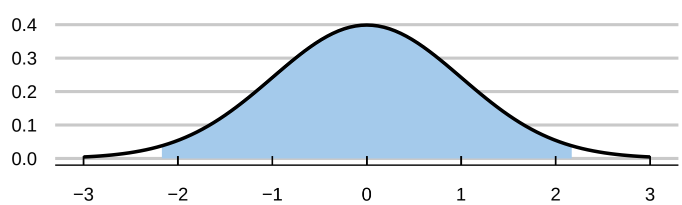
(b) Define the random variable of interest. [1pts.]

> **Solution:** Let $\overline{X}$ denote the average monthly rent, in thousands of dollars, of a representative sample of 100 1-bedroom apartments.

(c) What distribution should Alayah use when making inferences about the true average monthly rent of a 1-bedroom apartment in Santa Barbara? Be sure to check any/all relevant conditions. [3pts.]

> **Solution:** We go through our flowchart:
>
> - **Normal Population?** No; we are not told that the distribution of *all* monthly rents of 1-bedroom apartments in Santa Barbara are normally distributed.
> - **Large Enough Sample?** Yes; $n = 100 \geq 30$ ✓
> - **σ or s?** The value of 0.75 is stated to be the standard deviation of *all* rents, meaning it is σ.
>
> Based on the answers to this question, we use the standard normal distribution.

(d) Construct a 97% confidence interval for the true average monthly rent of a 1-bedroom apartment in Santa Barbara. Be sure to interpret your interval in the context of the problem! [3pts.]

> **Solution:** The general form of a confidence interval for μ, assuming we are using the normal distribution and that we have access to σ, is
>
> $$\overline{x} \pm z^* \cdot \frac{\sigma}{\sqrt{n}}$$
>
> Because we seek to construct a 97% confidence interval, we select the confidence coefficient z^* to be the value such that the following blue area is 97%:
>
> 
>
>
> This tells us that the tails, separately, must have area $(1 - 0.97)/2 = 0.015$ meaning we seek either negative one times the 1.5^{th} percentile, or the $(100 - 1.5) = 98.5^{\text{th}}$ percentile. Either way we find $z^* \approx 2.17$, meaning our confidence interval becomes
>
> $$(2.2) \pm (2.17) \cdot \frac{0.75}{\sqrt{100}} \approx [2.0373\ ,\ 2.3628]$$

One interpretation of this interval is as follows:

> We are 97% confident that the true average monthly rent of a 1-bedroom apartment in santa Barbara is between 2.0373 and 2.3628 thousand dollars.

(e) Would a 95% confidence interval for the true average monthly rent of a 1-bedroom apartment in Santa Barbara be wider or narrower than the interval you constructed in part (d) above? Explain briefly; you do not need to construct the interval. [2pts.]

Solution: We know that, in general, higher confidence levels correspond to wider confidence intervals; conversely, lower confidence levels correspond to narrower confidence intervals. Since 95% is less than 97%, we would expect a 95% confidence interval to be narrower than the 97% confidence interval constructed in part (d) above.

Problem 3. In the field of Psychology, a Reaction Time Test is used to measure the time it takes a given person to respond to a specific stimulus; for example, how long it takes a person to press a button once the button has lit up. Suppose that for a particular stimulus, response times of randomly-selected individuals follow a normal distribution centered at 3 seconds with a standard deviation of 0.5 seconds. A person is selected at random, administered the stimulus, and their reaction time is recorded.

(a) Define the random variable of interest, and call it X. [1pts.]

Solution: Let X denote the reaction time (in seconds) of a randomly-selected individual.

(b) What is the probability that a randomly-selected person has a reaction time between 2.5 seconds and 3.7 seconds? [3pts.]

Solution: From the problem statement, we are told that $X \sim \mathcal{N}(3,\ 0.5)$. We seek $\mathbb{P}(2.5 \leq X \leq 3.7)$, which can be computed as:

$$\begin{aligned}
\mathbb{P}(2.5 \leq X \leq 3.7) &= \mathbb{P}(X \leq 3.7) - \mathbb{P}(X \leq 2.5) \\
&= \mathbb{P}\left(\frac{X-3}{0.5} \leq \frac{3.7-3}{0.5} \right) - \mathbb{P}\left(\frac{X-3}{0.5} \leq \frac{2.5-3}{0.5} \right) \\
&= \mathbb{P}\left(\frac{X-3}{0.5} \leq 1.4 \right) - \mathbb{P}\left(\frac{X-3}{0.5} \leq -1 \right) \\
&= 0.9192 - 0.1587 = 0.7605
\end{aligned}$$

(c) Can you foresee any potential difficulties in modeling response times using a normal distribution? Specifically, think in terms of state spaces. [1pts.]

Solution: The state space of a normally-distributed random variable contains negative values, whereas reaction times cannot be negative.

(d) What sort of plot would be best-suited for assessing whether or not a set of reaction times could plausibly have been drawn from a normal distribution? [1pts.]

Solution: A QQ-plot is best suited for this.

Problem 4. According to the World Bank, only 54.2% of households in Ethiopia live with access to electricity. To test these claims, a sociologist takes a representative sample of 130 Ethiopian households, and observes that 67 of these households live with access to electricity. Suppose that the sociologist wants to test the World Bank's claims against a two-sided alternative, at a 5% level of significance.

(a) Define the parameter of interest, and call it p. [1pts.]

> **Solution:** Let p denote the true proportion of households in Ethiopia that live with access to electricity.

(b) Define the random variable of interest, and call it $\widehat{P}$. [1pts.]

> **Solution:** Let $\widehat{P}$ denote the proportion of Ethiopian households, in a representative sample of 130, that have access to electricity.

(c) State the null and alternative hypotheses in terms of p. [2pts.]

> **Solution:** The null hypothesis is that $p = 0.542$. We are told to adopt a two-sided alternative, meaning we take $H_A : \ p \neq 0.542$, and so our hypotheses become
>
> $$\left[\begin{array}{rl} H_0 : & p = 0.542 \\ H_A : & p \neq 0.542 \end{array} \right.$$

(d) Compute the observed value of the test statistic. [2pts.]

> **Solution:**
>
> $$\mathrm{ts} = \frac{\widehat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{\left(\frac{67}{130}\right) - 0.542}{\sqrt{\frac{0.542*(1-0.542)}{130}}} \approx -0.61$$

(e) Compute the critical value of the test. Be sure to check any/all relevant conditions first! [3pts.]

Solution: We would like to use the normal distribution to find the critical value; before we do so, however, we need to ensure that $\widehat{P}$ is normally distributed under the null, which only occurs when:

1) $np_0 = 130 \cdot \left(\frac{67}{130}\right) = 67 \geq 10$ ✓

2) $n(1 - p_0) = 130 \cdot \left(\frac{130-67}{130}\right) = 63 \geq 10$ ✓

Since both conditions are satisfied, we can use the normal distribution to find the critical value. We are using a 5% level of significance, which enables us to directly recall that the critical value will be 1.96.

(f) Now, perform the test and interpret your conclusions in the context of the problem. [2pts.]

Solution: We reject the null only when |TS| exceeds the critical value. Here, $|\text{ts}| = |-0.61| = 0.61 < 1.96$, meaning we fail to reject the null:

> At a 5% level of significance, there was insufficient evidence to reject the World Bank's claims that the true proportion of Ethiopian households with access to electricity is 54.2%, in favor of the alternative that the true proportion is *not* 54.2%.