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NUFFT and Compressed Sensing

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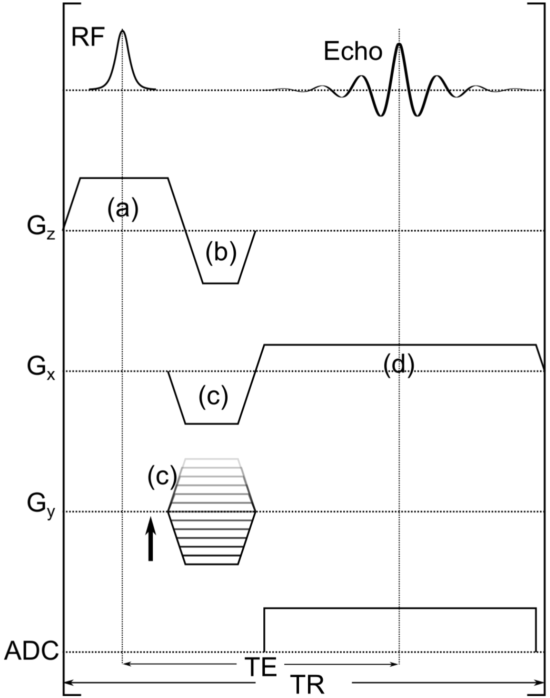
Outline

Non-Uniform FFT (NUFFT)

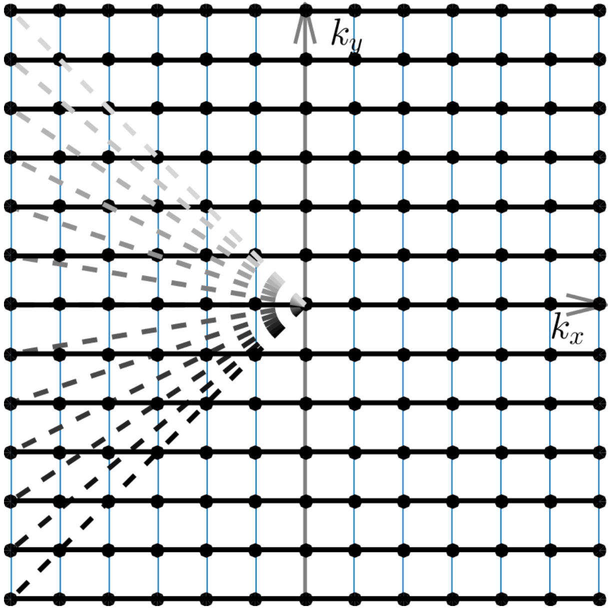
Compressed Sensing (CS)

1 Non-Uniform FFT (NUFFT)

Introduction: Cartesian Sampling (e.g. FLASH) [1]



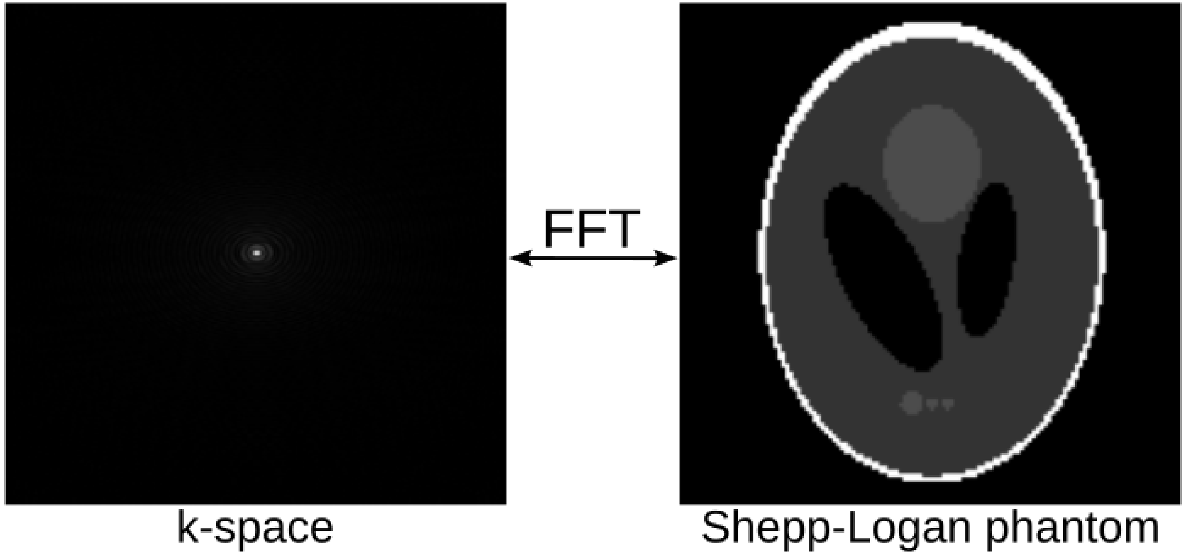
Gradients:
(a) Slice selection
(b) Rewinder
(c) Prephasing
(d) Readout



[1] Haase A, Frahm J, Matthaei D, Hanicke W, Merboldt KD. FLASH imaging. Rapid NMR imaging using low flip-angle pulses. *J Magn Reson* (1986).

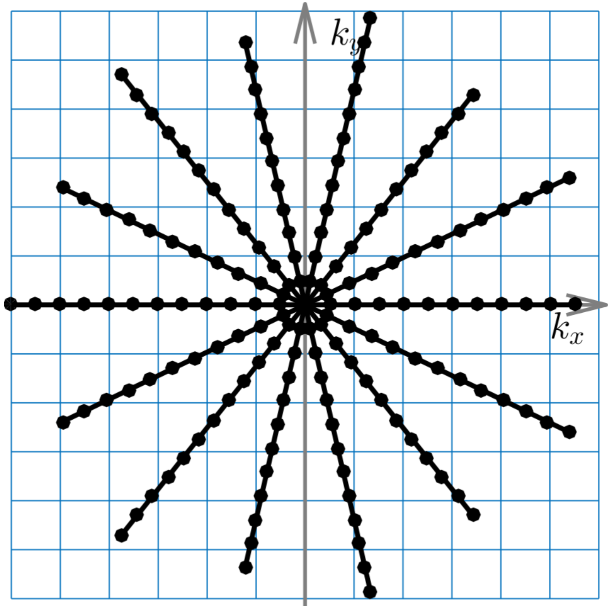
Introduction: Fast Fourier Transform

$$s(k_x, k_y) = \int \rho(x, y) \cdot e^{-i(k_x \cdot x + k_y \cdot y)} \mathrm{d}x\mathrm{d}y \tag{1}$$

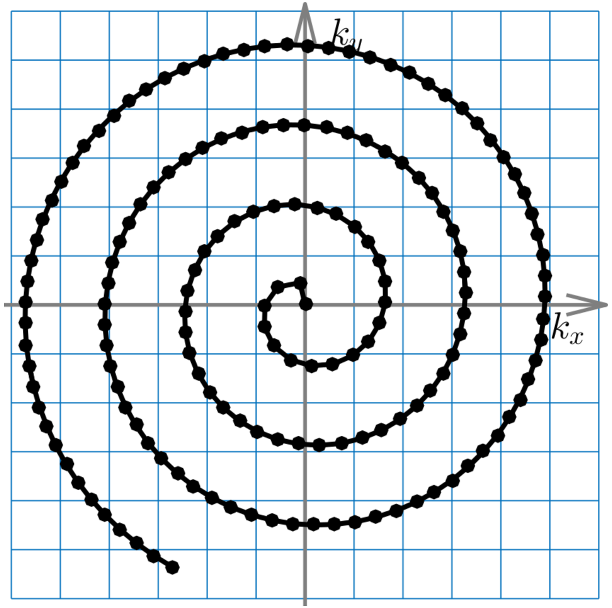


Non-Cartesian Sampling (e.g. Radial [2] and Spiral [3])

Radial



Spiral



[2]Lauterbur PC. Image formation by induced local interactions: Examples employing nuclear magnetic resonance. *Nature* (1973).
[3]Nishimura DG, Irarrazabal P, Meyer C. A velocity k-space analysis of flow effects in echo-planar and spiral imaging. *Magn Reson Med* (1995).

Downsides of non-Cartesian sampling

- Reconstruction is more complicated
- Sensitive to gradient errors (delays, eddy currents)
- Sensitive to B0 errors

Good for PhD theses!

3

Courtesy: Dr. Frank Ong @ UC Berkeley

Image Reconstruction of Non-Cartesian Data Requires NUFFT

NUFFT: Non-Uniform FFT [4], [5], [6]

1. Density compensation
2. Gridding: Convolution with the Kaiser-Bessel window
3. Deapodization & Inverse FFT

[4] OSulllivan J. A fast sinc function gridding algorithm for Fourier inversion in computed tomography. *IEEE Trans Med Imaging* (1985).
[5] Jackson J, Meyer CH, Nishimura DG, Macovski A. Selection of a convolution function for Fourier inversion using gridding. *IEEE Trans Med Imaging* (1991).
[6] Fessler JA, Sutton BP. Nonuniform fast Fourier transforms using min-max interpolation. *IEEE Trans Med Imaging* (2003).

NUFFT Step 1: Density Compensation

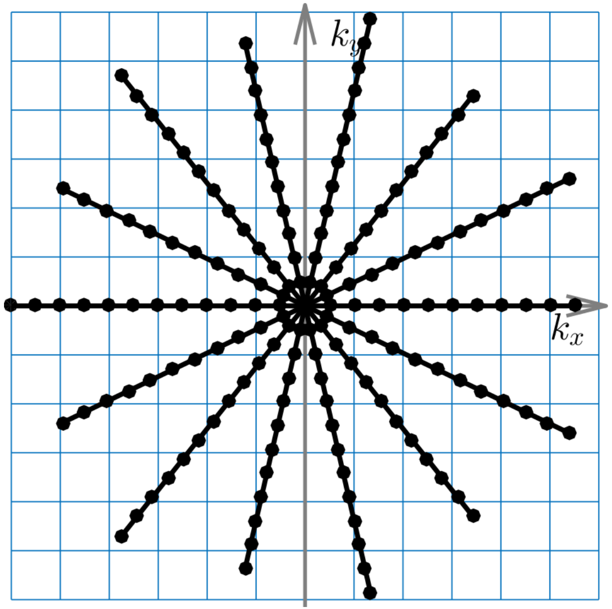
- Non-Cartesian samples are usually not uniformly acquired in k-space.
- e.g. In radial sampling, the central k-space points are more densely acquired than peripheral k-space points.
- Analytical density compensation function (DCF) in radial sampling.

Given $\mathrm{ktrj} = k_x + 1i * k_y$,

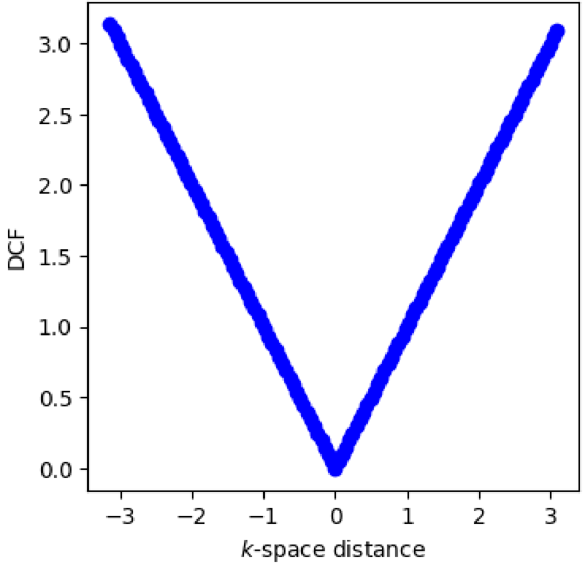
$$\mathrm{DCF} = \sqrt{k_x^2 + k_y^2} \tag{2}$$

DCF in Radial Sampling

Radial

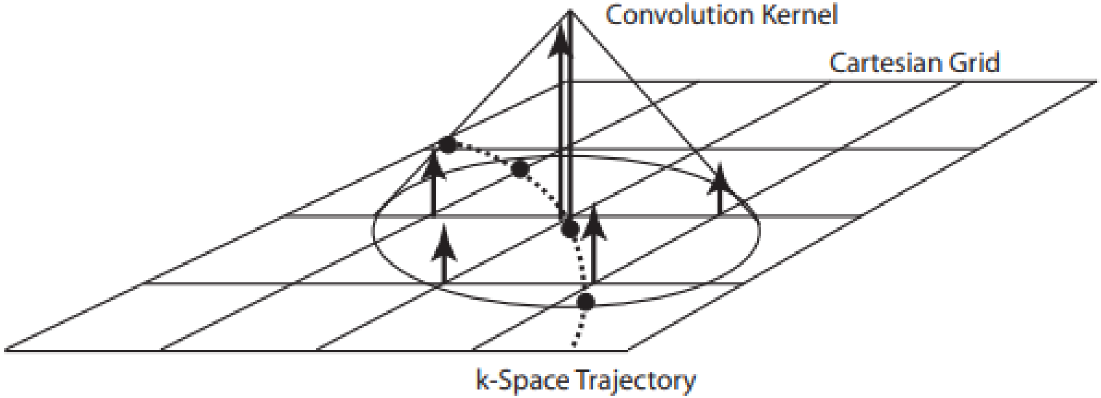


DCF

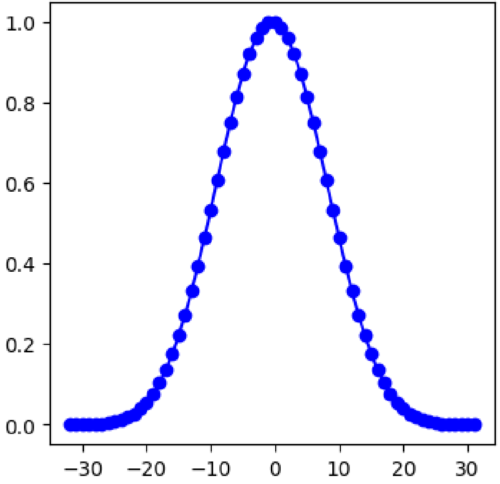


NUFFT Step 2: Gridding (Interpolation) with the Kaiser-Bessel Window

Interpolation [a]



Kaiser-Bessel Window



[a] https://users.fmrib.ox.ac.uk/~karla/reading_group/lecture_notes/AdvRecon_Pauly_read.pdf

NUFFT Step 3: Deapodization

- The gridded signal on Cartesian grids can be written as the convolution between the non-Cartesian signal and the Kaiser-Bessel window,

$$s_{\mathrm{Cart}} = s_{\mathrm{non-Cart}} \circledast W_{\mathrm{kb}} \tag{3}$$

- with FFT, it becomes

$$\mathcal{F}\{s_{\mathrm{Cart}}\} = \mathcal{F}\{s_{\mathrm{non-Cart}}\} \cdot \mathcal{F}\{W_{\mathrm{kb}}\} \tag{4}$$

- Then deapodization is

$$I = \mathcal{F}\{s_{\mathrm{Cart}}\}./\mathcal{F}\{W_{\mathrm{kb}}\} \tag{5}$$

NUFFT Implementations

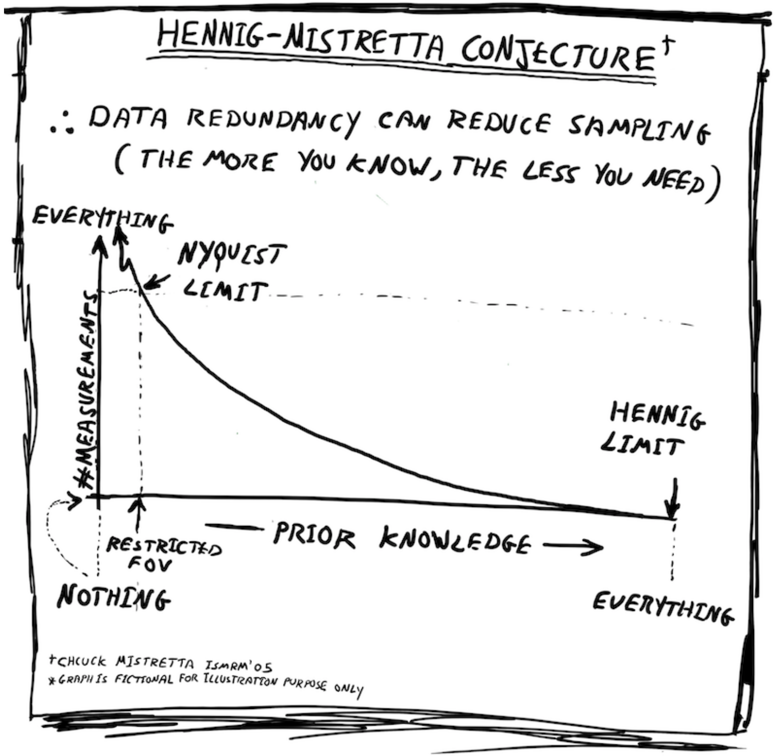
1. Python / PyTorch -
   - sigpy: https://github.com/mikgroup/sigpy
   - torchkbnufft: https://github.com/mmuckley/torchkbnufft
2. C -
   - Gridding Functions: http://mrsrl.stanford.edu/~brian/gridding/
   - BART: https://github.com/mrirecon/bart

NUFFT Exercises

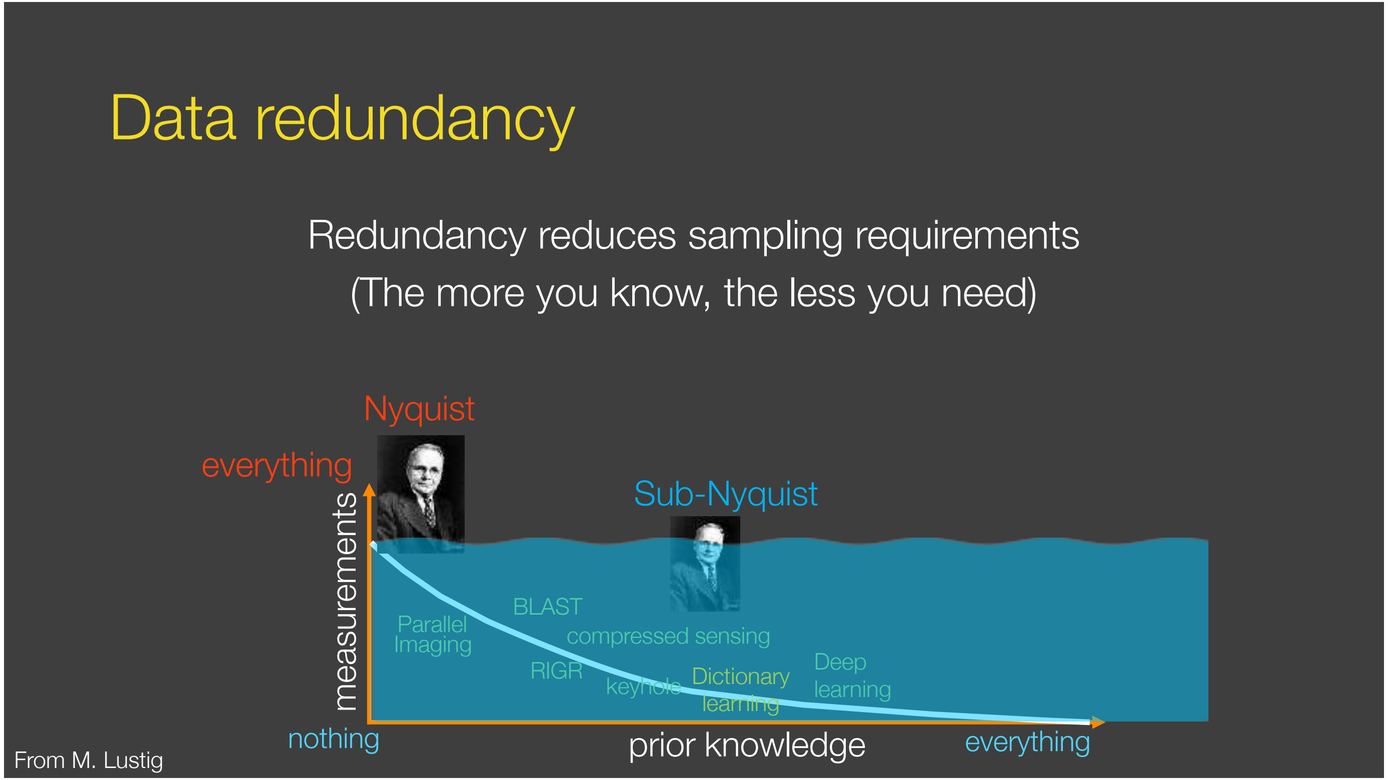
- Double "num_samples" in the function "make_adc"
- Reduce "Nphase", what undersampling artifacts do you see?
- What artifacts do you see in the presence of gradient delays?
- How would you correct for gradient delays in reality?
- What if the density compensation function is unknown?

2 Compressed Sensing (CS)

Compressed Sensing



- Candes EJ, Romberg JK, Tao T. Stable signal recovery from incomplete and inaccurate measurements. *Commun Pure Appl Math* (2006).
- Lustig M, Dohono D, Pauly JM. Sparse MRI: The application of compressed sensing for rapid MR imaging. *Magn Reson Med* (2007).
- Block KT, Uecker M, Frahm J. Undersampled radial MRI with multiple coils. Iterative image reconstruction using a total variation constraint. *Magn Reson Med* (2007).
- Comics from `http://people.eecs.berkeley.edu/~mlustig/comics1.html`



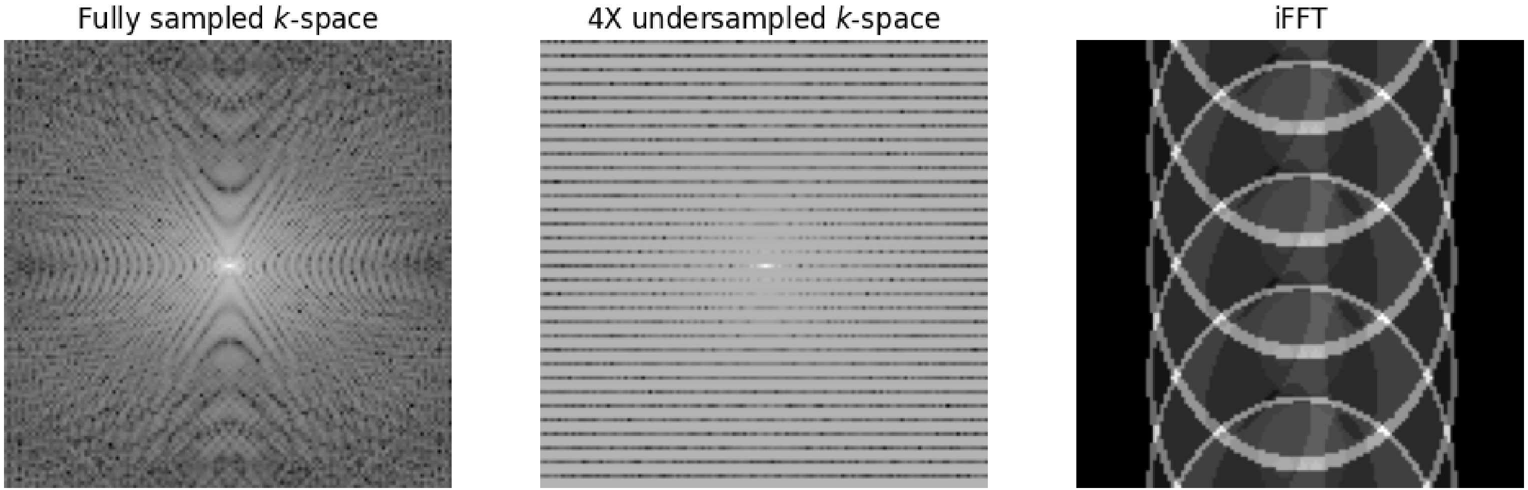
Courtesy: Prof. Dr. Jon Tamir @ UT Austin

What is Nyquist Limit?

Nyquist Sampling Requirement

Given an imaging field-of-view (FOV), the sampling in k-space must satisfy

$$\Delta k \leq 1/\mathrm{FOV} \tag{6}$$



How Compressed Sensing Goes Beyond Nyquist?

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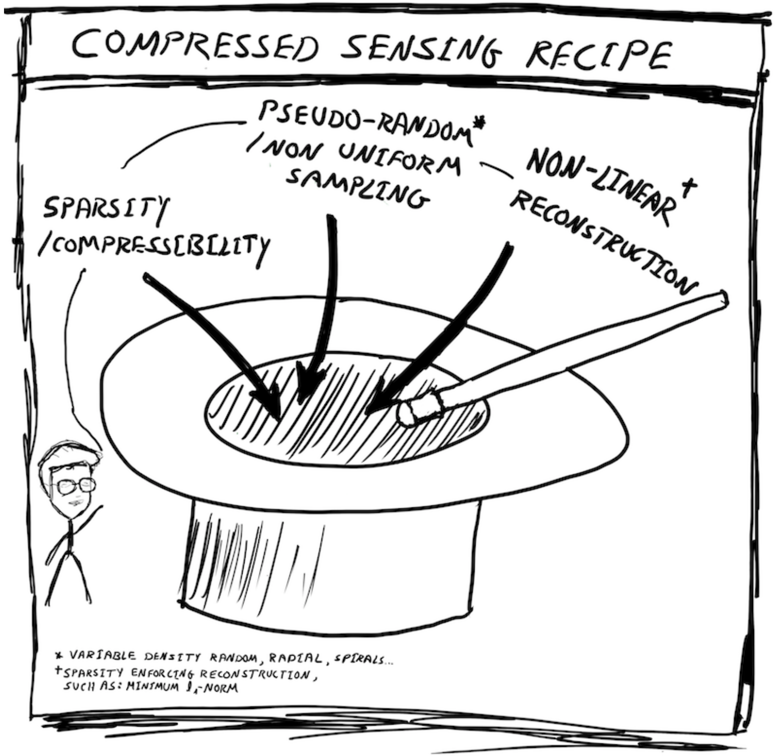
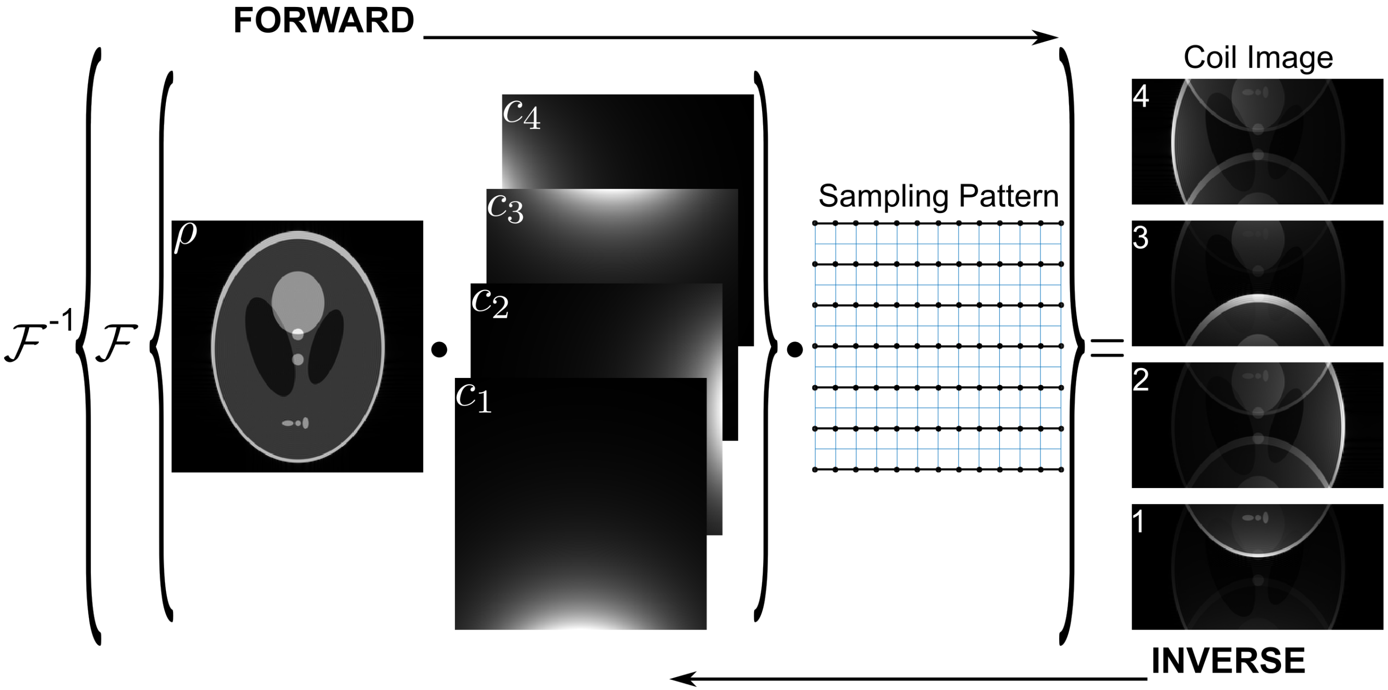


Before Compressed Sensing: Parallel Imaging as SENSE [7]



[7] Pruessmann KP, Weiger M, Scheidegger MB, Boesiger P. SENSE: sensitivity encoding for fast MRI. *Magn Reson Med* (1999).

Parallel Imaging as SENSE: Solving a Linear Inverse Problem

Objective function:

$$\mathrm{argmin}_x \|y - \mathcal{P}\mathcal{F}\mathcal{S}x\|_2^2 \tag{7}$$

- Solver: gradient descent, or conjugate gradient, or ADAM, ...
- Data update via gradient method, e.g. FISTA [8]:

$$x^{(t+1)} = x^{(t)} + \alpha \cdot \mathcal{S}^H \mathcal{F}^{-1} \mathcal{P}^H (y - \mathcal{P}\mathcal{F}\mathcal{S}x^{(t)})$$

[8] Beck A, Teboulle M. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J Imaging Sciences* (2009).

What is Linear Inverse Problem and What is Gradient?

Objective function:

$$\text{argmin}_x \|2 - x\|_2^2 \tag{8}$$

Given initial guess $x_{\text{prev}} = 0$ and learning rate $\alpha = 0.1$,

Iteration	x_{prev}	$\text{grad} = 2 * (2 - x)$	$x_{\text{curr}} = x_{\text{prev}} + \alpha * \text{grad}$
0	0	4	0.4
1	0.4	3.2	0.72
⋮	⋮	⋮	⋮
47	1.9999	0.0001	2

What is Linear Inverse Problem and What is Gradient?

```python
x_prev = 0

for n in range(50):
    g = 2 * (2 - x_prev)
    x_curr = x_prev + 0.1 * g
    print(' iter %2d x_prev %6.4f grad %6.4f x_curr %6.4f'%(n, x_prev, g, x_curr))
    x_prev = x_curr
```

Inverse Problems can Become Under-determined / Ill-posed

Objective function:

$$\operatorname{argmin}_{x_1,x_2} \|2 - x_1 - x_2\|_2^2 \tag{9}$$

- You can't find an unique solution for this function.

Inverse Problems can Become Under-determined / Ill-posed

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- ..., however, when there is prior knowledge, e.g.

$$\text{s.t. } x_1 = x_2 \tag{10}$$

Inverse Problems can Become Under-determined / Ill-posed

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- You can't find an unique solution for this function.
- ..., however, when there is prior knowledge, e.g.

$$\text{s.t. } x_1 = x_2 \tag{10}$$

- We can easily solve it, $x_1 = x_2 = 1$.

We can write this toy example in a matrix format

Objective function:

$$\operatorname{argmin}_{x_1,x_2} \|2 - x_1 - x_2\|_2^2 \tag{11}$$

It is equivalent to

$$\operatorname{argmin}_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \tag{12}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

Does it look similar to the inverse problem in MRI?

$$\operatorname{argmin}_{x} \|y - \mathcal{PFS}x\|_2^2 \tag{13}$$

Prior is the Game Changer in Compressed Sensing

- Prof. Dr. Gary Golver from Stanford:
  "We eventually only need to measure one point in k-space!!!"

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  "What is going on here???"

Prior is the Game Changer in Compressed Sensing

- Prof. Dr. Gary Golver from Stanford:
  "We eventually only need to measure one point in k-space!!!"
- People like me:
  "What is going on here???"
- Prof. Dr. Mike Lustig @ UC Berkeley:
  "Sparsifying transformation changes the game!!!"

$$\mathrm{argmin}_x ||y - \mathcal{P}\mathcal{F}\mathcal{S}x||_2^2 + \lambda||\mathit{\Phi} x||_1 \tag{14}$$

where $\mathit{\Phi}$ is wavelet transform, total variation, ...

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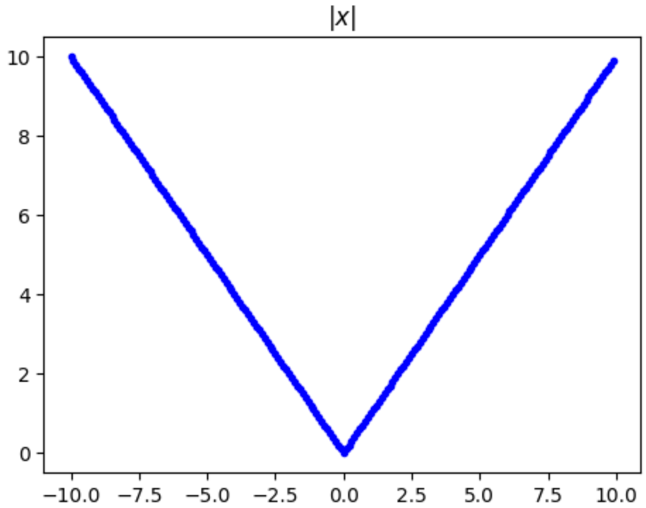
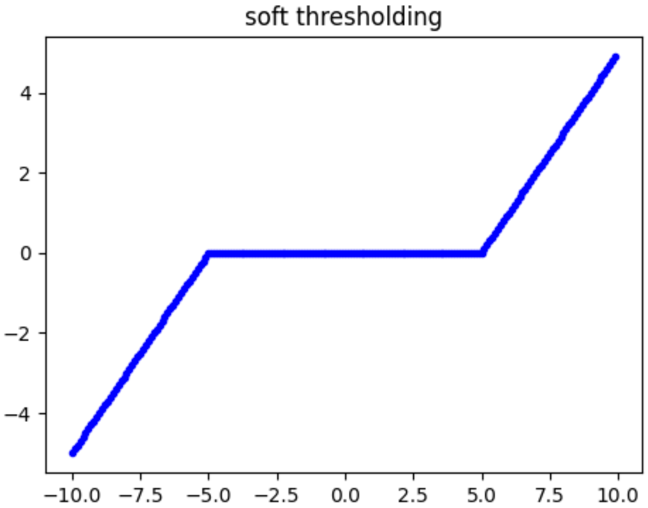
How to Solve this Problem?

Objective function:

$$\mathrm{argmin}_x \|y - \mathcal{PFS}x\|_2^2 + \lambda\|\Phi x\|_1 \quad (15)$$

- the regularization term $\lambda\|\Phi x\|_1$ is not differentiable;
- soft thresholding $\mathcal{T}_\lambda(x) = (x-\lambda)_+ \mathrm{sign}(x)$



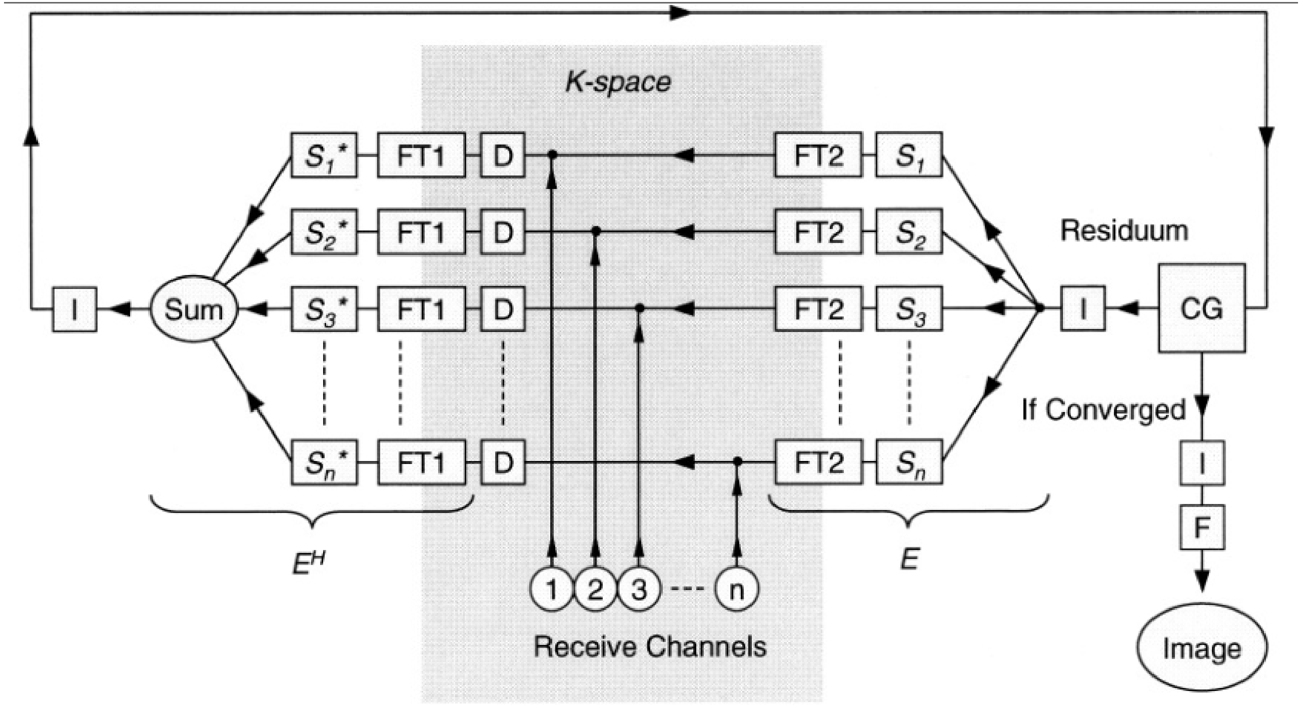


Compressed Sensing: Iterative Soft Thresholding

1. Gradient update:
$$x^{(t+1)} = x^{(t)} + \alpha \cdot \mathcal{S}^H \mathcal{F}^{-1} \mathcal{P}^H (y - \mathcal{P}\mathcal{F}\mathcal{S}x^{(t)})$$

2. Soft thresholding:
$$x^{(t+1)} = \Phi^{-1} \mathcal{T}_\lambda (\Phi x^{(t+1)})$$

What is Gradient Update in Parallel MRI? [9]



[9] Pruessmann KP, Weiger M, Börnert P, Boesiger P. Advances in sensitivity encoding with arbitrary k-space trajectories. *Magn Reson Med* (2001).

Gradient Update in Parallel MRI: CG SENSE

- density correction D:
  $D = 1/d\mathbf{k}$, $\mathbf{k}$ is the relative density of the k-space sampling pattern at the position $\vec{k}$.

- this changes the system equation:
  $(E^H D E)\mathbf{v} = E^H D \mathbf{m}$

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- this speeds up the iterative procedure in practice.

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- Exercise 1: run the gradient step many times without soft thresholding, are you able to get a similar reconstruction as the NUFFT reconstruction with density compensation?

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  $(E^H DE)\mathbf{v} = E^H D\mathbf{m}$
- this speeds up the iterative procedure in practice.
- Exercise 1: run the gradient step many times without soft thresholding, are you able to get a similar reconstruction as the NUFFT reconstruction with density compensation?
- Exercise 2: try to tune the soft thresholding parameter, and show how it affects the reconstruction

Thank You for Your Attention!

- If you are interested in MRI (pulse sequence and image reconstruction), please feel free to reach out: zhengguo.tan@gmail.com