Sections and Chapters

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1 Basic Logic

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1.1 (Remark) Modern Logic

-Proof theory - properties of proofs
- Model theory - properties of interpretations
- Recursion theory - properties of algorithms
- Set theory - structureMaster
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1.2 (Remark) Bootstrapping

- Informal math contains all the real ideas, while formal math contains encoded symbols
- Formal math is mere toy-replica of the informal math
- Properties of the toy can tell us about properties of the real thing
- Formal reasoning (syntax) is much safer than informal reasoning (syntax + semantics)
- The informal theory about a formal theory is called a metatheory
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1.3 (Remark) Notions in the metatheory are informal

- Metamathematics exists outside and independently of our effort to build this or that formal system
- - All its constructs are available to us for use in the analysis of the behaviour of a formal system
- Properties about objects that they must satisfy can only be recognized in the metatheory
- - Formal theories are pre-built systems that are designed to run not knowing about these properties
- The formal theory is just a generator of theorems, and not a parser
- - It cannot remember what it generates nor state the properties of some given string
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1.4 (Definition) Pre-requisite informal notions

- Subset $X \subseteq Y$ for sets X and Y iff for all $x \in X$, $x \in Y$
- Union $X \cup Y$ for sets X and Y iff for all $x \in X$ and $y \in Y$, $x \in X \cup Y$ and $y \in X \cup Y$
- Kleene star S^* is the set of all sequences that can be made from the elements of set S
- The string S from a set V is defined by $S \in V^*$
- String concatenation $A * B$ for strings A and B is defined by A appended with B
- The empty string λ is defined by for all strings S, $S = \lambda * S = S * \lambda$
- String A occurs in string B iff there exists strings C and D, $B = C * A * D$
- Variadic notation $[a_i]_{i=1}^n$ is an abbreviation for $a_1, a_2, a_3, ..., a_n$
- TODO: rules + rule metatheorems, schema subsitution ==

1.5 (Definition) Rule

- The rule Q is a metatheoretical function that takes in a sequence of strings and outputs strings
- The rule with n arguments and an output is called $(n + 1)$-ary
- The rule has an associated function $Arity$ which outputs the number of arguments for a given function or relation
- The function $ArityR$ outputs the arity of given a given rule - Rules must be algorithmic and can be executed within a finite number of steps - The immediate predecessors of a string d on the $(n + 1)$-ary rule Q are $[s_i]_{i=1}^n$ iff $Q([s_i]_{i=1}^n, d)$ holds
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1.6 (Definition) Closed set under a rule

- TODO: use/meaning/intent - wat - The set S is closed under the $(n+1)$-ary rule Q iff for all d where $Q([s_i]_{i=1}^n, d)$, if $\{[s_i]_{i=1}^n\} \subseteq S$, then $d \in S$
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1.7 (Definition) Rule-defined set

- The rule-defined set $Cl(J, R)$ consists of:
- - The set of base objects J
- - J can be treated like a set of rules that takes no arguments
- - The set of rules for generating inductive objects R
- Closure $Cl(J, R)$ is the smallest set that satisfies all of the following:
- - $J \subseteq Cl(J, R)$
- - for all $Q \in R$, $closed(Cl(J, R), Q)$
- Sets built via rules satisfy a smallest qualifier to remove erroneous structures and elements
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1.8 (Metatheorem) Induction on a rule-defined set

- TODO: use/meaning/intent - If a property P holds for all in J, and propagates (true for input (Induction Hypothesis) -¿ true for output (Inductive Step)) through for all in R, then it holds for the entire closure ??? - For any property P and $w \in Cl(J, R)$, P(w) iff - For all $j \in J$, P(j) - For all $r \in R$, P closed blablablablalb TODO fuck this shit - If $J \subseteq T$ (Basis Step) and for any $Q \in R$, T is closed under Q (Inductive Step), then $Cl(J, R) \subseteq T$
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1.9 (Definition) Ambiguous pair

- TODO: use/meaning/intent - ??? - The pair (J, R) is ambiguous if there exists $d \in Cl(J, R)$ satisfies any of the following:
- - There exists $Q \in R$ where $Q([x_i]_{i=1}^{ArityR(Q)})$ and $Q([y_i]_{i=1}^{ArityR(Q)})$ and $\langle [x_i]_{i=1}^{ArityR(Q)} \rangle \neq \langle [y_i]_{i=1}^{ArityR(Q)} \rangle$
- - There exists $P \in R$ and $Q \in R$ where $P([s_i]_{i=1}^n, d)$ and $Q([s_i]_{i=1}^n, d)$ and $P \neq Q$
- - There exists $Q \in R$ where $Q([s_i]_{i=1}^n, d)$ and $d \in J$
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1.10 (Metatheorem) Definition by recursion

TOFIX - TODO: use/meaning/intent - Cl(J, R) from some unambiguous (J, R) has some nice unique mapping - TODO: [ABSTRACTED] - All inductive or recursive definitions must be unambiguous to be well-defined ========================

1.11 (Remark) The state of the metatheory

- The metatheory houses assumptions or axioms about behavior of theories
- Is it necessary to prove the consistency of the metatheory?
- - No because otherwise, it would require a meta-metatheory which would require a meta-meta-metatheory, and so on which would never end
- - The metatheory should be small and simple and close to intuition such that it would not require formalized verification of its consistency
- Is it okay to use infinite sets and induction in the metatheory?
- - This is mostly political, but we should avoid using suspicious inconsistent tools such as full-blown naive set theory
- - There are ways we could simulate infinite sets using safer finite notions like using finite calculations then arbitrarily large finite can be a stand-in for infinite, but it is not worth it
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2 First Order Languages

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2.1 (Definition) Formal theory

- TODO: use/meaning/intent - notionmaster - The formal theory T is defined by $T = (V, Wff, Thm)$
- The alphabet V is defined by the set of all symbols allowed in the theory
- The set of all strings $String$ from an alphabet V
- The set of all well-formed formulas Wff is defined by $Wff \subseteq String$
- The set of all theorems Thm is defined by $Thm \subseteq Wff$
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2.2 (Definition) Formal language

- The formal language encodes the notions of a theory
- The formal language L is defined by $L = (V, Term, Wff)$
- $Term$ encodes the objects of a theory
- Wff encodes the statements of a theory
- L is a (one-sorted) language with only one type of variable in V, and then predicates are used to simulate typing
- $Term, Wff, Thm$ can be explicitly provided or they can be rule-defined ======================================

2.3 (Definition) Alphabet

- TODO: use/meaning/intent - writeables - An alphabet V consists of elements from LS and NLS:
- - The set of all logical symbols LS consists of:
- - - Elements from the set of all variables Var
- - - The boolean connectives $\neg, \vee$
- - - The existential quantifier $\exists$
- - - The equality predicate $\equiv$
- - The set of all nonlogical symbols NLS consists of:
- - - Elements from the set of all constants $Const$
- - - Elements from the set of all predicates $Pred$
- - - Elements from the set of all functions $Func$
- NLS can be extended to contain additional symbols
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2.4 (Definition) Set of all terms

- TODO: use/meaning/intent - talkables - The set of all terms $Term$ is defined by $Term = Cl(J_{Term}, R_{Term})$
- The set of all base terms J_{Term} consists of:
- - Variables from Var
- - Constants from $Const$
- The set of term generating rules R_{Term} is defined by the rules:
- - If $f \in Func$ and $\{[t_i]_{i=1}^{Ar(f)}\} \subseteq Term$, then $f[t_i]_{i=1}^{Ar(f)} \in Term$
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2.5 (Definition) Set of all wffs

- TODO: use/meaning/intent - truthables - The set of all wffs Wff is defined by $Wff = Cl(J_{Wff}, R_{Wff})$
- The set of all base (or atomic) wffs J_{Wff} is the set of all predicates that take only terms as arguments and it is defined by the rules:
- - If $\{t, s\} \subseteq Term$, then $(t \equiv s) \in J_{Wff}$
- - If $p \in Pred$ and $\{[t_i]_{i=1}^{Ar(p)}\} \subseteq Term$, then $p[t_i]_{i=1}^{Ar(f)} \in J_{Wff}$
- The set of wff generating rules R_{Wff} is defined by the rules:
- - If $A \in Wff$, then $\neg A \in Wff$
- - If $\{A, B\} \subseteq Wff$, then $\vee AB \in Wff$
- - If $x \in Var$ and $A \in Wff$, then $\exists_x A \in Wff$
- First-order languages are languages that only allows quantification over variables
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2.6 (Definition) Set of all thms

- TODO: use/meaning/intent - truths - The set of all thms Thm is defined by $Thm = Cl(J_{Thm}, R_{Thm})$
- The set of all base (or axiomatic) thms J_{Thm} is defined by $J_{Thm} \subseteq Wff$

- The set of thm generating rules (or rules of inference) R_{Thm} is defined a set of rules where each rule takes in wffs and outputs a wff

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3 Constructs for Thm

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3.1 (Definition) Wff abbreviations

- $(A \vee B)$ is an abbreviation for $\vee AB$
- $\forall_x A$ is an abbreviation for $\neg \exists_x \neg A$
- $(A \wedge B)$ is an abbreviaton for $\neg(\neg A \vee \neg B)$
- $(A \implies B)$ is an abbreviation for $\neg A \vee B$
- $(A \equiv B)$ is an abbreviation for $((A \implies B) \wedge (B \implies A))$
- Precedence:
- - $\neg, \wedge, \vee, \implies, \equiv$
- - Equal precedence invokes right associativity

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3.2 (Definition) Free variable in a term

- TODO: use/meaning/intent - ???? - The variable x is free in a term t iff it x occurs in t

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3.3 (Definition) Closed term

- TODO: use/meaning/intent - ???? - The term t is closed iff it contains no free variable

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3.4 (Definition) Free and bound variable in a wff

- TODO: use/meaning/intent - ???? - The variable x is free in a wff B if it satisfies any of the following:
- - Iff B is an atomic wff and x occurs in B
- - Iff B is $\neg C$ and x is free in C
- - Iff B is $C \vee D$ and (x is free in C or x is free in D)
- - Iff B is $\exists_y(C)$ and $x \neq y$ and x is free in C
- The variable x is bounded in a wff B iff x occurs in B and x is not free in B

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3.5 (Definition) Closed and open wff

- TODO: use/meaning/intent - ??? - The wff C is closed iff it contains no free variable
- The wff C is open iff it contains no quantifier
- Open wffs are also closed

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3.6 (Definition) Variable occurrence notation

- $A[[v_i]_{i=1}^n]$ iff the variables $[v_i]_{i=1}^n$ occur in the wff A
- $A([v_i]_{i=1}^n)$ iff the variables $[v_i]_{i=1}^n$ occur in the wff A and no other variables occur in A

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3.7 (Definition) The set all prime wffs

- TODO: use/meaning/intent - atom truth assignable - The set of all base terms $Prime$ consists of:
- Atomic wffs from J_{Wff}
- Inductive wffs under from the $\exists_x A$ rule in R_{Wff}

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3.8 (Definition) Propositional valuation

- TODO: use/meaning/intent - ???? - The valuation v is a function defined by $v : Prime \to \{\bot, \top\}$ - The propositional valuation p_v from the valuation v is defined by $p_v : Wff \to \{\bot, \top\}$ and p_v is an extension of v from $Prime$ to Wff
- If wff A is a prime wff, then $p_v(A) = v(A)$
- If wff A is not a prime wff, then p_v is defined as follows:
- - $p_v(\neg A) = F_\neg(A)$
- - - $F_\neg(\top) = \bot$
- - - $F_\neg(\bot) = \top$
- - $p_v(A \vee B) = F_\vee(A, B)$
- - - $F_\vee(\bot, \bot) = \bot$
- - - $F_\vee(\bot, \top) = \top$
- - - $F_\vee(\top, \bot) = \top$
- - - $F_\vee(\top, \top) = \top$
- This definition relies on the definition of prime wff and parsing to be unambiguous ===================================

3.9 (Definition) Tautology

- TODO: use/meaning/intent - ???? - The wff T is a tautology ($\vDash_{Taut} T$) iff for any valuation v, the propositional valuation $p_v(T) = \top$
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3.10 (Definition) Satisfiable wff

- TODO: use/meaning/intent - ???? - The wff U is wffsatisfiable iff there exists a valuation v where, $p_v(U) = \top$
- v wffsatisfies U
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3.11 (Definition) Satisfiable set

- TODO: use/meaning/intent - ????? - The set of wffs Γ is setsatisfiable iff there exists a valuation v where, for any $A \in \Gamma$, $p_v(A) = \top$
- v setsatisfies Γ
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3.12 (Definition) Unsatisfiable wff

- Inverted version of wffsatisfiable
- The wff U is wffunsatisfiable iff for any valuation v, $p_v(U) = \bot$ ==

3.13 (Definition) Unsatisfiable set

- Inverted version of setsatisfiable
- An set of wffs Γ is setunsatisfiable iff for any valuation v, there exists $A \in \Gamma$ where, $p_v(A) = \bot$
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3.14 (Definition) Tautological implication

- TODO: use/meaning/intent - ?????? - The set of wffs Γ tautologically implies a wff A ($\Gamma \vDash_{Taut} A$) iff for any valuation v, if v setsatisfies Γ, then v wffsatisfies A
- This can also be checked via truth tables ===

3.15 (Remark) Satisfiable and unsatisfiable

- """"Satisfiable" and "unsatisfiable" are terms introduced here in the propositional or Boolean sense. These terms have a more complicated meaning when we decide to "see" the object variables and quantifiers that occur in formulas.""" TODO
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3.16 (Definition) Substitution of a term for a variable in term

- TODO: use/meaning/intent - nifty rule dependency - The substitution ($s[x \leftarrow t]$) of the variable x by the term t in the term s is defined as following rule:
- - If s is x, then the substitution outputs t
- - If s is a constant or a variable that is not x, then the substitution outpust s
- - If s is a function f applied to the terms $[t_i]_{i=1}^{n}$, then the substitution outputs $f[t_i[x \leftarrow t]]_{i=1}^{Arity(f)}$
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3.17 (Metatheorem) Substituted terms are terms

- By induction, this is true for B.S. terms and joining terms I.H. implies joined is also a term I.S.
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3.18 (Definition) Substitution of a term for a variable in wff

- TODO: use/meaning/intent - nifty rule - The substitution ($A[x \leftarrow t]$) of the variable x by the term t in the wff A is defined as following rule:
- - If A is $s \equiv r$ on the terms s and r, then the substitution outputs $s[x \leftarrow t] \equiv r[x \leftarrow t]$
- - If A is a predicate p applied to the terms $[t_i]_{i=1}^{Arity(p)}$, then the substitution outputs $p[t_i[x \leftarrow t]]_{i=1}^{Arity(p)}$
- - If A is $\neg B$ on the wff B, then the substitution outputs $\neg B[x \leftarrow t]$
- - If A is $(B \vee C)$ on the wffs B and C, then the substitution outputs $(B[x \leftarrow t] \vee C[x \leftarrow t])$
- - If A is $\exists_x(B)$ on the wff B, then the substitution outputs A
- - If A is $\exists_y(B)$ on the variable y and the wff B and y is not x and y does not occur in t, then the substitution outputs $\exists_y(B[x \leftarrow t])$
- If x is not quantified over, then all variables in the term t must not be quantified over to allow the wff to preserve its intended meaning
- The substitution is defined iff it satisfies a defined condition in the rule
- - Using the substitution notation immediately implies that the substitution output is defined
- The symbols [,], $\leftarrow$ are symbols in the metatheory and they have the higher precedence compared to formal symbols
- The variable x is substitutable by the term t in the wff A iff $A[x \leftarrow t]$ is defined ==================================

3.19 (Metatheorem) Substituted wffs are wffs

- By induction, this true for B.S (terms) and joining wffs I.H. implies joined is also a wff I.S.
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3.20 (Definition) Simultaneous substitution

- The simultaneous substitution $A[[y_i]_{i=1}^{n} \leftarrow [t_i]_{i=1}^{n}]$ of the variables $[y_i]_{i=1}^{n}$ by the terms $[t_i]_{i=1}^{n}$ in the wff A is an abbreviation for $A[[y_i \leftarrow t_i]]_{i=1}^{n}$
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3.21 (Definition) Schemata

- The schemata is a rule written down as a metaformula that consists of metasymbols
- These metasymbols can be replaced by formal symbols to output a wff called an instance of the schema
- It is a rule that takes in formal symbols, and substitutes them in place of metavariables, then outputs the instance wff
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3.22 (Definition) The set of all axioms

- TODO: use/meaning/intent - assumptions - The set of all axioms J_{Thm} consists of:
- - Elements from the set of logical axioms Λ
- - Elements from the set of nonlogical axioms Γ

3.23 (Definition) The set of all logical axioms

- TODO: use/meaning/intent - assumed logic behavior - The set of all logical axioms Λ is defined by $\Lambda = Cl(J_\Lambda, R_\Lambda)$
- The set of all base logical axioms J_Λ is the set of all tautological wffs
- The set of logical axiom generating rules R_Λ is defined by the rules:
- - Substitution Axiom: If $A \in \Lambda$, then for any term t and variable x substitutable for t, $(A[x \leftarrow t] \implies \exists_x(A)) \in \Lambda$
- - Self-equivalence Axiom: For any variable x, $(x \equiv x) \in \Lambda$
- - Leibniz Axiom: For any (countable) wff A, variable x, terms t and s, $(t \equiv s \implies (A[x \leftarrow t] \equiv A[x \leftarrow s])) \in \Lambda$
- The rules / axioms are what endow the symbols with the intended behavior
- The Leibniz Axiom is still first-order because ??? ==

3.24 (Definition) The set of all nonlogical axioms

- TODO: use/meaning/intent - assumed notions behavior - The set of all nonlogical axioms Γ are the wffs that are assumed to be true or the hypotheses ==

3.25 (Definition) Rules of inference

- TODO: use/meaning/intent - truth preserving - The rules of inference R_{Thm} is defined by thes rules:
- - Primary rules of inference:
- - - Modus Ponens Rule: Given A, $A \implies B$, the output is B
- - - E-Introduction Rule: Given $A \implies B$ and x is not free in B, the output is $\exists_x(A) \implies B$
- - Derived rules of inference:
- - - Rules of inference derived from the proofs made by other rules of inference
- - - The validity of these rules are only provable within the metatheory

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3.26 (Definition) The set of all Gamma-Theorems

- TODO: use/meaning/intent - Provable wffs - The set of all Gamma-Theorems Thm_Γ is defined by $Thm_\Gamma = Cl(J_{Thm}, R_{Thm})$
- The wff A is a Gamma-theorem or $\Gamma \vdash A$ is an abbreviation for $A \in Thm_\Gamma$
- $\Gamma \vdash \Lambda$ is an abbreviation for for all $L \in \Lambda$, $L \in Thm_\Gamma$

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3.27 (Definition) Abstract formal theory

- The formal theory T_L appropriate for the language L is the set of wffs $T_L \subseteq Wff$ that are considered to be correct within a theory
- The formal theory T is defined by $T = Cl(J_{Thm}, R_{Thm})$
- The formal theory T is inconsistent if $T = Wff$
- We often like to find the smallest set of set of axioms Γ for axiomatizing T such that $T = Thm_\Gamma$
- The set of axioms Γ is recognizable if there exists an algorithmic process to decide if $A \in \Gamma$
- The theory T is recursively axiomatized if Γ is recognizable

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4 Constructs and Metatheorems for Provability

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4.1 (Definition) Derivation and stage

- The (J, R)-derivation is the finite sequence of wffs from the construction of $Cl(J, R)$
- The stage X_i is some collection of all wffs from the (J, R)-derivation until step i

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4.2 (Metatheorem) Equivalent ways of generating closures

- $Cl(J, R) = x : xoccursina(J, R) - derivation = \cup X_i$
- TODO: [ABSTRACTED] duh ==

4.3 (Definition) Gamma-proof

- The Gamma-proof of the wff A is some (J, R)-derivation of a $\Gamma - Theorem$ ==

4.4 (Metatheorem) Transitivity of vdash

- If $\Gamma \vdash \Delta$ and $\Delta \vdash B$, then $\Gamma \vdash B$
- The existing Gamma-proofs can be concatinated to form the Gamma-proof for B
- This allows us to re-use collections of previously established theorems ==

4.5 (Metatheorem) Hypothesis strengthening

- If $\Gamma \subseteq \Delta$ and $\Gamma \vdash A$, then $\Delta \vdash A$
- The Gamma-proof for A is also a valid Delta-proof for A since it contains all the required wffs
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4.6 (Metatheorem) Post's Tautology Theorem

- If $\{[A_i]_{i=1}^{n}\} \vDash_{Taut} B$, then $\{[A_i]_{i=1}^{n}\} \vdash B$
- $\vDash_{Taut} [A_i \implies]_{i=1}^{n} B$ ¡(1) from Hypothesis 1 and truth table)¿
- $\vdash [A_i \implies]_{i=1}^{n} B$ ¡(2) from tautology 1 in J_{Thm}¿
- $\{[A_i]_{i=1}^{n}\} \vdash [A_i \implies]_{i=1}^{n} B$ ¡(3) from Hypothesis strenghtening on (2)¿
- $\{[A_i]_{i=1}^{n}\} \vdash B$ ¡(4) from Modus Ponens Rule on (3) n times¿ ==

4.7 (Definition) Provably equivalent wffs in a theory

- The wff A and wff B are provably equivalent in the theory T iff $\Gamma \vdash A \equiv B$ ==

4.8 (Metatheorem) Theorems are provably equivalent

- If $\Gamma \vdash \{A, B\}$, then $\Gamma \vdash (A \equiv B)$
- - $\Gamma \vdash A$ ¡(1) from Hypothesis 1¿
- - $A \vDash_{Taut} B \implies A$ ¡(2) from tautological implication¿
- - $A \vdash B \implies A$ ¡(3) from Post's Tautology Theorem on (2)¿
- - $\Gamma \vdash B \implies A$ ¡(4) from Transitivity of vdash on (1, 3)¿
- - $\Gamma \vdash A \implies B$ ¡(5) Ditto of (1-4) but utilizing Hypothesis 2¿
- - $\Gamma \vdash \{B \implies A, A \implies B\}$ ¡(6) Collection of established theorems on (4, 5)¿
- - $\{A \implies B, B \implies A\} \vDash_{Taut} A \equiv B$ ¡(7) from tautological implication¿
- - $\{A \implies B, B \implies A\} \vdash A \equiv B$ ¡(8) from Post's Tautology Theorem on (7)¿
- - $\Gamma \vdash A \equiv B$ ¡(9) Transitivity of vdash on (6, 8)¿
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4.9 (Metatheorem) A-Introduction

- If the variable x is not free in the wff $\neg A$, then $A \implies B \vdash A \implies \forall_x(B)$
- - $A \implies B \vDash_{Taut} \neg B \implies \neg A$ ¡(1) from tautological implication¿
- - $A \implies B \vdash \neg B \implies \neg A$ ¡(2) from Post's Tautology Theorem on (1)¿
- - $\neg B \implies \neg A \vdash \exists_x(\neg B) \implies \neg A$ ¡(3) from E-Introduction WITH Hypothesis 1¿
- - $\exists_x(\neg B) \implies \neg A \vDash_{Taut} A \implies \neg(\exists_x(\neg B))$ ¡(4) from tautological implication¿ - - $\exists_x(\neg B) \implies \neg A \vdash A \implies \neg(\exists_x(\neg B))$ ¡(5) from Post's Tautology Theorem on (4)¿
- - $A \implies \neg(\exists_x(\neg B)) \vdash A \implies \forall_x(B)$ ¡(6) from abbreviation of forall on (5)¿
- - $A \implies B \vdash A \implies \forall_x(B)$ ¡(8) from Transitivity of vdash on (2, 3, 5, 6)¿
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4.10 (Metatheorem) Specialization

- The succeeding proofs will be less based less on metatheory and based more on derivation
- For any wff A and term t, $\vdash \forall_x(A) \implies A[x \leftarrow t]$
- - $(\neg A)[x \leftarrow t] \implies \exists_x(\neg A)$ ¡(1) from Substitution Axiom¿
- - $\neg \exists_x(\neg A) \implies A[x \leftarrow t]$ ¡(2) from tautological implication and Post's Tautology Theorem on (1)¿
- - $\forall_x(A) \implies A[x \leftarrow t]$ ¡(3) from abbreviation of forall on (2)¿
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4.11 (Metatheorem) Specialization corollary

- For any wff A, $\vdash \forall_x(A) \implies A$
- - $\forall_x(A) \implies A[x \leftarrow x]$ ¡(1) from Specialization¿ - - $\forall_x(A) \implies A$ ¡(2) from definition of substitution on (1)¿
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4.12 (Metatheorem) Generalization

- For any wff A, $\vdash A \implies \forall_x(A)$ - - $A \implies A$ ¡(1) from tautological implication and Post's Tautology Theorem¿
- - $A \implies \forall_x(A)$ ¡(2) A-Introduction on (1) WITH x is not free in A¿
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4.13 (Metatheorem) Generalization corollary

- For any wff A, $\vdash A \equiv \forall_x(A)$
- - $\vdash A \implies \forall_x(A)$ ¡(1) from Generalization¿
- - $\vdash \forall_x(A) \implies \vdash A$ ¡(2) from Specialization corollary¿
- - $\vdash A \equiv \forall_x(A)$ ¡(3) from tautological implication on (1, 2) and Post's Tautology Theorem¿
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4.14 (Metatheorem) CorollaryMaster

- For any wff A, $A \vdash \forall_x(A)$ and $\forall_x(A) \vdash A$
- - $A \vdash \forall_x(A)$ ¡(1) from Modus Ponens Rule on ($A \vdash A$, Generalization corollary)¿
- - $\forall_x(A) \vdash A$ ¡(2) from Modus Ponens Rule on ($\forall_x(A) \vdash \forall_x(A)$, Generalization corollary)¿
- - $\vdash A \equiv \forall_x(A)$ ¡(3) from tautological implication on (1, 2) and Post's Tautology Theorem¿
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4.15 (Definition) Universal closure

- The universal closure of a wff A with free variables $[y_i]_{i=1}^n$ is defined to be $[\forall_{y_i}]_{i=1}^n(A)$
- Any formula deduces and is deduced by its universal closure ¡from CorollaryMaster¿
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4.16 (Metatheorem) Substitution of terms

- For any terms $[t_i]_{i=1}^n$, $A[[x_i]_{i=1}^n] \vdash A[[t_i]_{i=1}^n]$ - - $[\forall_{x_i}]_{i=1}^n(A)$ ¡(1) Generalization on (Hypothesis 1) n times¿ - - $A[[t_i]_{i=1}^n$ ¡(2) Specialization on (1) n times¿ ==

4.17 (Metatheorem) Renaming

- For any wff A and variable z, if z does not occur in A, then $\vdash (\exists_x(A)) \equiv (\exists_z(A[x \leftarrow z]))$ - - $A[x \leftarrow z] \implies \exists_x(A)$ ¡(1) from Substitution Axiom WITH Hypothesis 1 (the variables in z does not occur in A)¿ - - $\exists_z(A[x \leftarrow z]) \implies \exists_x(A)$ ¡(2) from E-Introduction on (1) WITH Hypothesis 1 (z is not free in $\exists_x(A)$)¿ - - $A[x \leftarrow z][z \leftarrow x] \implies \exists_z(A[x \leftarrow z])$ ¡(3) from Substitution Axiom WITH Hypothesis 1 (the variables in x does not occur in $A[x \leftarrow z]$ and the variables in z does not occur in A)¿
- - $\exists_x(A) \implies \exists_z(A[x \leftarrow z])$ ¡(4) from simplification and E-introduction on (3) WITH Hypothesis 1 (x is not free in $\exists_z(A[x \leftarrow z])$)¿
- - $\exists_x(A) \equiv \exists_z(A[x \leftarrow z])$ ¡(5) from tautological implication on (2, 4) and Post's Tautology Theorem¿
- Since the supply of variables are effectively infinite and Renaming states that a wff is provably equivalent to another wff with a dummy variable, then substitutability can be guaranteed by using dummy variables for variables in the substituted term
==

4.18 (Definition) Theory extension

- The theory T' is an extension of the theory T iff $T \subseteq T'$
- The theory T' is aware of the theory T iff $V \subseteq V'$
- Given multiple theories T and T', $\vdash_T A$ is an abbreviation for $A \in T$ and $\vdash_{T'} A$ is an abbreviation for $A \in T'$
- The extension T' of the theory T is conservative iff for any $A \in L$, if?f $\vdash_{T'} A$, then $\vdash_T A$
==

4.19 (Metatheorem) On constants

- Given that $V' = V \cup \{[e_i]_{i=1}^n\}$ and $\Gamma' = \Gamma$, $\vdash_{L'} A[[e_i]_{i=1}^n]$ iff $\vdash_L A[[e_i]_{i=1}^n]$
- TODO: [ABSTRACTED] from dummy renaming ==

4.20 (Metatheorem) Important corollary

- Given $\{[e_i]_{i=1}^n\} \cap \Gamma = \emptyset$, if $\Gamma \vdash A[[e_i]_{i=1}^n]$, then $\Gamma \vdash [x_i]_{i=1}^n$ - TODO: [ABSTRACTED] from dummy renaming, try after deduction theorem ==

4.21 (Metatheorem) Deduction theorem

- For any closed wff A and wff B and set of wffs Γ, if $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \implies B$
- Proof by induction on Thm - Basis cases: - - If $B = A$ - - - $A \implies B$ is tautological Q.E.D - - If $B \neq A$ - - - B ¡(1) must be $\Gamma axiomatic > ---$B $\vdash A \implies B$ ¡(2) from (1) and Post's Tautology Theorem¿ - - - $\Gamma \vdash A \implies B$ ¡(3) from Hypothesis Strengthening on (2) and (1)¿ - Inductive cases: - - Closed under the Modus Ponens Rule: let the inputs be $\Gamma \cup \{A\} \vdash C$ and $\Gamma \cup \{A\} \vdash C \implies B$:
- - - Inductive hypothesis: $\Gamma \vdash A \implies C$ and $\Gamma \vdash A \implies (C \implies B)$
- - - Inductive step:
- - - - $\{A \implies C, A \implies (C \implies B)\} \vDash_{Taut} A \implies B$ ¡(1) tautological implication¿
- - - - $\Gamma \vdash A \implies B$ ¡(2) Post's Tautology Theorem on (1) and Transitivity of vdash¿
- - Closed under the E-Introduction Rule: let the inputs be $\Gamma \cup \{A\} \vdash (C \implies D)$ where x is not free in D
- - - Inductive hypothesis: $\Gamma \vdash A \implies (C \implies D)$
- - - Inductive step: - - - - $A \implies (C \implies D) \vDash_{Taut} C \implies (A \implies D)$ ¡(1) tautological implication¿
- - - - $\exists_x(C) \implies (A \implies D)$ ¡(2) E-Introduction on (1) WITH Hypothesis (A is closed) and x is not free in D¿
- - - - $\exists_x(C) \implies (A \implies D) \vDash_{Taut} A \implies (\exists_x(C) \implies D)$
- - - - $\Gamma \vdash A \implies (\exists_x(C) \implies D)$ ¡(3) from Post's Tautology theorem¿
- The condition on A must be closed in important to enable one to preserve semantic meaning - - Suppose A is not closed like $x \iff y$, then $x \iff y \vdash \forall_x(x \iff y)$ from Generalization, but this is a contradiction since there exists y where $x \neq y$
==

4.22 (Metatheorem) Proof by contradiction

- For any closed wff A, $\Gamma \vdash A$ iff $Thm_{\Gamma \cup \{\neg A\}} = Wff$
- When $\Gamma \vdash A$
- - $\Gamma \cup \{\neg A\} \vdash A$ ¡() Hypothesis Strengthening¿
- - $\Gamma \cup \{\neg A\} \vdash \neg A$ ¡() nonlogical axiomed¿
- - A, $\neg A \vDash_{Taut} B < ()tautological implication >$
$- When \text{Thm}_{\Gamma \cup \{\neg A\}} = Wff$
- - $\Gamma \cup \{\neg A\} \vdash A$ ¡() hypothesis¿
- - $\Gamma \vdash \neg A \implies A$ ¡() Deduction theorem¿
- - $\neg A \implies A \vDash_{Taut} A$ ¡() tautological implication¿
- - $\Gamma \vdash A$ ¡() Post's Tautology Theorem and Deduction theorem¿
- A is closed necessary for Deduction theorem, add WITH in annotation ¡- TODO
==

4.23 (Metatheorem) Principle of Explosion

- TODO: same tautological implication gimmick
==

4.24 (Metatheorem) Distributivity of exists and forall

- For any variable x and wff A and wff B, $A \implies B \vdash \exists_x(A) \implies \exists_x(B)$
- For any variable x and wff A and wff B, $A \implies B \vdash \forall_x(A) \implies \forall_x(B)$
==

4.25 (Metatheorem) Leibniz rule

- If $A \equiv B$, then $C[A] \equiv C[B]$ or maybe $C \equiv C'$??
==

4.26 (Metatheorem) Proof by cases

- If $\Gamma \vdash [A_i \vee]_{i=1}^{n}$ and ???? ==

4.27 (Metatheorem) Proof by auxiliary constant

- ???? WIP lazy atm ==

5 Semantics

===

5.1 (Definition) Structure

- TODO: meaning ?? - The structure S appropriate for a language L is defined by $S_L = (M, I)$
- The model M is defined by a non-empty set of concrete objects or the universe
- The intepretation I is a function defined by $I : V \to M$ that safisfies the following:
- For any $a \in Const$, $I(a) \in M$
- For any $f \in Func$, $I(f) : M^{Arity(f)} \to M$ and I is total (defined everywhere)
- For any $p \in Pred$, $I(p) : M^{Arity(p)} \to M$ and $I(p) \subseteq M^{Arity(p)}$
- The interpretation $\bar{I}$ is I but to maps more stuff other than V ==

5.2 (Definition) Extension and restriction of a language

- The language L' is the expansion of the language L iff $LS' = LS$ and $NLS \subseteq NLS'$
- L is the restriction of L'
===

5.3 (Definition) Expansion and reduction of a structure

- The structure S' is an expansion of the structure S iff L' is an extension of L and for all $v \in V$, $I'(v) = I(v)$
- S is the reduct of S'
- The restriction/reduct symbol can be used as $I = I' \restriction L$ or $S = S' \restriction L$
===

5.4 (Definition) Model imported to the language

- TODO: meaning ?? - The imported language $L(S)$ is defined by the language L and the appropriate structure S_L
- The imported alphabet $V(S)$ or S-Alphabet is defined by $V(S) = V \cup M$ - The imported set of all terms $Term(S)$ or S-Term is defined by $V(S)$
- The imported set of all wffs $Wff(S)$ or S-Wff is defined by $Term(S)$ and $V(S)$
- This allows formal substitutions on model objects
- Imported symbols are denoted by $\overline{m}$ and $\bar{I}(\overline{m}) = I(m)$
===

5.5 (Definition) Interpretation on closed S-terms

- TODO: use/meaning/intent - talkables - The interpretation $\bar{I}$ on closed S-terms is the function $\bar{I} : S - Term \to M$ defined by the following:
- If the term is $a \in Const$, then $\bar{I}(a) = I(a)$
- If the term is $f \in Func$ applied to the closed S-terms $[t_i]_{i=1}^{Arity(f)}$, then $\bar{I}\left(f([t_i]_{i=1}^{Arity(f)})\right) = I(f)\left([\bar{I}(t_i)]_{i=1}^{Arity(f)}\right)$
===

5.6 (Definition) Interpretation on closed S-wffs

- TODO: use/meaning/intent - The interpretation I on closed S-wffs is the function $\bar{I} : S - Wff \to \{\top, \bot\}$ defined by the following:
- If the wff is $t \equiv s$ on the closed S-terms t and s, then $\bar{I}(t \equiv s) = \top$ iff $\bar{I}(t) = \bar{I}(s)$
- If the wff is $p \in Pred$ applied to the closed S-terms $[t_i]_{i=1}^{Arity(p)}$, then $\bar{I}\left(p([t_i]_{i=1}^{Arity(p)})\right) = \top$ iff $I(p)\left(\bar{I}(t_i)]_{i=1}^{Arity(p)}\right) = \top$
- If the wff is $\neg A$ on the closed S-wff A, then $\bar{I}(\neg A) = F_{\neg}(\bar{I}(A))$

- If the wff is $A \vee B$ on the closed S-wffs A and B, then $\bar{I}(A \vee B) = F_{\vee}(\bar{I}(A), \bar{I}(B))$
- If the wff is $\exists_x(B)$ on the closed S-wff B and the variable x, then $\bar{I}(\exists_x(B)) = \top$ iff there exists $i \in M$, $\bar{I}(B[x \leftarrow \bar{i}]) = \top$
- We have imported constants from M into L in order to be able to state the semantics of (x)B above in the simple manner we just did (following Shoeneld (1967)). TODO: wat ???
==

5.7 (Definition) S-instance of a structure + misc

- The S-instance A' of the wff A on the structure S is defined by the free variables $[x_i]_{i=1}^n$ of A and the imported constants $[\bar{y}_i]_{i=1}^n$ and $A' = A[[x_i \leftarrow y_i]]_{i=1}^n$
- The wff A is valid for the structure S or S is a model of $A \vDash_S A$ iff for any instance A' of A, $\bar{I}(A') = \top$
- The structure S is a model of the set of wffs $\Gamma \vDash_S \Gamma$ iff for any $A \in \Gamma$, $\vDash_S A$
- The wff A is logically valid $\vDash A$ iff for any structure S appropriate for the language, $\vDash_S A$
- The set of wffs Γ is Osatisfiable iff there exists a structure S, $\vDash_S \Gamma$
- The set of wffs Γ is finitely Osatisfiable iff for any finite subset Δ, Δ is Osatisfiable
- - Osatisfiable means the notion of satisfiable extended to open wffs
- The set of wffs Γ logically implies the wff A $\Gamma \vDash A$ iff for any structure S, if $\vDash_S \Gamma$, then $\vDash_S A$
==

5.8 (Definition) Soundness

- The theory T is sound iff for any $A \in Wff$, if $\Gamma \vdash A$, then $\Gamma \vDash A$
- The pure theory PT is sound iff for any $A \in Wff$, if $\vdash_{PT} A$, then $\vDash A$
==

5.9 (Metatheorem) Substitution swapping on terms

- For any term s and term t and constant a and variable x and variable y, if $y \neq x$ and y does not occur in A, then $s[x \leftarrow t][y \leftarrow a] \equiv s[y \leftarrow a][x \leftarrow t]$
- Proof by induction on the term s - Basis cases: - - If s is x, then $s[x \leftarrow t][y \leftarrow a] \equiv t \equiv s[y \leftarrow a][x \leftarrow t]$
- - If s is y, then $s[x \leftarrow t][y \leftarrow a] \equiv a \equiv s[y \leftarrow a][x \leftarrow t]$
- - If s is the variable z and $x \neq z \neq y$, then $s[x \leftarrow t][y \leftarrow a] \equiv z \equiv s[y \leftarrow a][x \leftarrow t]$
- - If s is the constant b, then $s[x \leftarrow t][y \leftarrow a] \equiv b \equiv s[y \leftarrow a][x \leftarrow t]$
- Inductive cases: - - Closed under term generating rule: Let s be $f([r_i]_{i=1}^{Arity(f)})$
- - - Inductive hypothesis: $[(r_i[x \leftarrow t][y \leftarrow a] \equiv r_i[y \leftarrow a][x \leftarrow t])]_{i=1}^{Arity(f)}$
- - - Inductive step:
- - - - $f([r_i[x \leftarrow t][y \leftarrow a]]_{i=1}^{Arity(f)}) \equiv f([r_i[y \leftarrow a][x \leftarrow t]]_{i=1}^{Arity(f)})$ ¡(1) Inductive hypothesis¿
- - - - $s[x \leftarrow t][y \leftarrow a] \equiv$ ¡(2) from (1) and Transitivity of equiv (tautological implication and Post's Tautology Theorem)¿
==

5.10 (Metatheorem) Substitution swapping on wffs

- For any wff A and term t and constant a and variable x and variable y, if $y \neq x$ and y does not occur in A, then $A[x \leftarrow t][y \leftarrow a] \equiv A[y \leftarrow a][x \leftarrow t]$
- Proof by induction of the wff A
- Basis cases:
- - If A is $r \equiv s$ on the terms r and s, then $A[x \leftarrow t][y \leftarrow a] \equiv (r[x \leftarrow t][y \leftarrow a] \equiv s[x \leftarrow t][y \leftarrow a]) \equiv (r[y \leftarrow a][x \leftarrow t] \equiv s[y \leftarrow a][x \leftarrow t]) \equiv A[y \leftarrow a][x \leftarrow t]$ from Import swapping on terms
- - If A is $p([r_i]_{i=1}^{Arity(p)})$ on the the predicate p and the terms $[r_i]_{i=1}^{Arity(p)}$, then $A[x \leftarrow t][y \leftarrow a] = p([r_i[x \leftarrow t][y \leftarrow a]]_{i=1}^{Arity(p)}) \equiv p([r_i[y \leftarrow a][x \leftarrow t]]_{i=1}^{Arity(p)}) \equiv A[y \leftarrow a][x \leftarrow t]$ from Import swapping on terms
- Inductive cases:
- - Closed under lnot rule: Let A be $\neg B$
- - - Inductive hypothesis: $(B[x \leftarrow t][y \leftarrow a] \equiv B)$
- - - Inductive step:
- - - - $f([r_i[x \leftarrow t][y \leftarrow a]]_{i=1}^{Arity(f)}) \equiv f([r_i[y \leftarrow a][x \leftarrow t]]_{i=1}^{Arity(f)})$ ¡(1) Inductive hypothesis¿
- - - - $s[x \leftarrow t][y \leftarrow a] \equiv$ ¡(2) from (1) and Transitivity of equiv (tautological implication and Post's Tautology Theorem)¿
==

5.11 (Metatheorem) Soundness

- ==

- - Closed under the E-Introduction Rule: let the inputs be $\Gamma \cup \{A\} \vdash (C \implies D)$ where x is not free in D
- - - Inductive hypothesis: $\Gamma \vdash A \implies (C \implies D)$
- - - Inductive step: - - - - $A \implies (C \implies D) \vDash_{Taut} C \implies (A \implies D)$ ¡(1) tautological implication¿
- - - - $\exists_x(C) \implies (A \implies D)$ ¡(2) E-Introduction on (1) WITH Hypothesis (A is closed) and x is not free in D¿
- - - - $\exists_x(C) \implies (A \implies D) \vDash_{Taut} A \implies (\exists_x(C) \implies D)$
- - - - $\Gamma \vdash A \implies (\exists_x(C) \implies D)$ ¡(3) from Post's Tautology theorem¿