Week-4
Mathematics for Data Science - 2
Basis of a vector space, Rank and dimension of a matrix
Graded Assignment

1. Which of the following statements are correct?

 ◯ Option 1: If A and B are two square matrices, then Rank(AB)=Rank(BA).

 ◯ **Option 2:** If A and B are two 2×2 square matrices, then Rank(AB) $\leq$ Rank(B).

 ◯ **Option 3:** If A and B are two 2×2 square matrices, then Rank(AB) $\leq$ Rank(A).

 ◯ **Option 4:** If A and B are $n \times n$ matrices of rank n, then AB has rank n.

Solution:

Option 1:

To prove this option incorrect, consider this example.

$$\text{Let } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{, which means } rank(AB) = 1$$

$$\text{But } BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{, which means } rank(BA) = 0$$

Hence this option is incorrect.

Option 2 and Option 3:

We know: $rank(AB) \leq min\{rank(A), rank(B)\}$.

Hence, both these options are correct.

Option 4: If A and B are both $n \times n$ matrices with $rank = n$, then:

$$\begin{aligned} & det(A) \neq 0 \text{ and } det(B) \neq 0 \\ \implies & det(AB) \neq 0 \qquad \text{...as } det(AB) = det(A)det(B) \end{aligned}$$

Thus AB is also non-invertible. That is, AB also has $rank = n$.

Thus, this option is also correct.

2. Match the vector spaces (with the usual scalar multiplication and vector addition as in $M_{3\times3}(\mathbb{R})$) in column A with their bases in column B and the dimensions of the vector spaces in column C in Table : M2W4GA1.

	Vector space (Column A)		Basis (Column B)		Dimension of the vector space (Column C)
a)	$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \mid x + 2y = z, \text{ and } x, y, z \in \mathbb{R} \right\}$	i)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$	1)	6
b)	$V = \{A \mid A \in M_{3\times3}(\mathbb{R}),$ A is a symmetric matrix $\}$	ii)	$\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$	2)	6
c)	$V = \{A \mid A \in M_{3\times3}(\mathbb{R}),$ A is an upper triangular matrix $\}$	iii)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$	3)	2

Table : M2W4GA1

Choose the correct option.

- ○ Option 1: a → i → 2.
- ○ Option 2: b → i → 2.
- ○ **Option 3:** a → ii → 3.
- ○ Option 4: c → iii → 1

○ **Option 5:** b → iii → 2.

○ **Option 6:** c → i → 1.

Solution:

(a): We are given the following vector space:

$$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \mid x + 2y = z \right\}$$
$$= \left\{ \begin{bmatrix} x & y & x+2y \\ 0 & x+2y & x \\ y & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$
$$= \left\{ x \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

Thus, the set $\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$ spans V and is also linearly independent. Thus, **option 3** is correct.

(b): $V = \{A \in M_{3\times 3}(\mathbb{R}) : A^T = A\}$. Any matrix in V has the form $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$, where $a, b, c, d, e, f \in \mathbb{R}$.

Thus $A = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$ spans V and is also linearly independent.

Hence, **option 5** is correct.

(c):

$V = \{A \in M_{3\times3}(\mathbb{R}) : A$ is an upper triangular matrix$\}$. Then A takes the form $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$, where $a, b, c, d, e, f \in \mathbb{R}$. Now, $A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$ $c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus, the following set spans V and is linearly independent:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Hence, **option 6** is correct.

3. Find the dimension of the vector space

$$V = \{(x, y, z) \mid x, y, z \in \mathbb{R}, x + y + z = 0, z = 0\}$$
$$\textit{Addition: } (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$$
$$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V$$
$$\textit{Scalar multiplication: } c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V,\ c \in \mathbb{R}$$

[Answer: 1]

Solution:

We are given that if a vector $(x, y, z) \in V$, then: $x + y + z = 0$ and $z = 0$ Thus, we have the following:

$$\begin{aligned} & x + y + z = 0 \text{ and } z = 0 \\ \implies & x + y + 0 = 0 \\ \implies & x = -y \end{aligned}$$

Thus, we can write any vector in V as $(x, -x, 0)$. That is, we can have a basis for V as $\{(1, -1, 0)\}$. Since the cardinality of basis for V is 1, the dimension of V is 1.

4. Find the rank of the matrix A, where $A = [a_{ij}]$ is of order 2021×2021 and $a_{i,j} = \min\{i, j\}$, $i, j = 1, 2, \ldots 2021$

[**Hint:** First do it for smaller matrices, such as 3×3 or 4×4.]

[Answer: 2021]

Solution:

Consider a 3×3 matrix as per the question. The matrix is as follows:

$$A_{3\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Reducing it to reduced row echelon form, we get the identity matrix as follows:

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, when $A_{3\times 3}$, $rank(A)$ is 3.

Consider $A_{4\times 4}$. The matrix is as follows:

$$A_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Reducing this matrix to reduced row echelon form as follows:

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1, R_4 \to R_4 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Observe that the inner matrix of R is the same 3×3 matrix that we considered earlier. That is, we can perform the same row operations on this inner matrix to get an identity matrix. Thus, we will get:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can further reduce this matrix to the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $rank(A) = 4$.

Similarly, a 5×5 matrix will contain the above 4×4 matrix, which can be reduced to the identity matrix. A 6×6 matrix will have a 5×5 sub-matrix which can be reduced to the identity matrix and so on. Thus, $rank(A_{2021 \times 2021}) = 2021$.

5. Consider a subspace $W = \{(x_1, x_2, x_3, x_4, x_5) \,|\, x_1 + 2x_5 = 0 \text{ and } x_2 + x_4 = 0\}$ of $\mathbb{R}^5$. The dimension of W is [Ans: 3]

Solution:

We are given that:

$$\begin{aligned} &x_1 + 2x_5 = 0 && \text{and} && x_2 + x_4 = 0 \\ &\implies x_5 = \frac{-x_1}{2} && \text{and} && \implies x_4 = -x_2 \end{aligned} \tag{1}$$

Thus, from (1), we can write any vector $w \in W$ as follows: $(x_1, x_2, x_3, -x_2, \frac{-x_1}{2})$. That is, any vector in W can be given by:

$$w = x_1\left(1, 0, 0, 0, \frac{-1}{2}\right) + x_2(0, 1, 0, -1, 0) + x_3(0, 0, 1, 0, 0)$$

Thus, the following set forms a basis for W:

$$\left\{\left(1, 0, 0, 0, \frac{-1}{2}\right), (0, 1, 0, -1, 0), (0, 0, 1, 0, 0)\right\}$$

Since the cardinality of the above set is 3, the dimension of W is 3.

6. Consider a matrix

$$A = \begin{bmatrix} 2 & -4 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 3 & -7 & 2 & 1 \end{bmatrix}.$$

The rank of the matrix is

Solution:

Reducing A to row echelon form:

$$A = \begin{bmatrix} 2 & -4 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 3 & -7 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & -3 & 1 & 1 \\ 2 & -4 & 1 & 1 \\ 3 & -7 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 3 & -7 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_2 - R_3} \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 1 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The number of non-zero rows is 3. Hence, the $rank(A)$ is 3.

7. Consider two matrices $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 4 \\ 2 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \\ 2 & 4 & 6 \end{bmatrix}$. Then Rank($A$) - Rank ($B$) is [Ans: 0]

Solution:

Observe that:

$$\begin{aligned} & A = B^T \\ \implies & rank(A) = rank(B) \qquad \text{... as } rank(A) = rank(A^T) \\ \implies & rank(A) - rank(B) = 0 \end{aligned}$$

Hence, the correct answer is 0.

8. If V is the subspace spanned by these two vectors $(-1, 0, 1)$ and $(2, 1, -1)$ in $\mathbb{R}^3$, then how many of the following vectors will be in V?
$(1, 1, 0), (5, 2, -3), (8, 5, -4), (-3, 0, 3), (\frac{-5}{6}, \frac{5}{6}, \frac{5}{6}), (-2, 3, 3)$. [Answer: 3]

Solution:

$$\textbf{1:}\quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad \text{Thus, this vector } \in V. \tag{1}$$

$$\textbf{2:}\quad \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad \text{Thus, this vector } \in V. \tag{2}$$

$$\textbf{3:}\quad \begin{bmatrix} 8 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 8 \\ 0 & 1 & 5 \\ 1 & -1 & -4 \end{bmatrix} \qquad \text{...arrange vectors as columns}$$

$$\implies -1(-4 + 5) + 1(10 - 8) \qquad \text{...determinant along 1st column}$$

$$= 1 \neq 0 \qquad \text{...as det } \neq 0\text{, vectors are independent}$$

$$\text{Thus, this vector } \notin V \tag{3}$$

$$\textbf{4:}\quad \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad \text{Thus, this vector } \in V. \tag{4}$$

$$\textbf{5:}\quad \begin{bmatrix} \frac{-5}{6} \\ \frac{5}{6} \\ \frac{5}{6} \end{bmatrix} = \begin{bmatrix} -1 & 2 & \frac{-5}{6} \\ 0 & 1 & \frac{5}{6} \\ 1 & -1 & \frac{5}{6} \end{bmatrix} \qquad \text{...arrange vectors as columns}$$

$$\implies -1(\frac{5}{6}+\frac{5}{6})+1(\frac{10}{6}+\frac{5}{6}) \qquad \text{...determinant along 1st column}$$

$$= \frac{10}{6}+\frac{15}{6}=\frac{5}{6}\neq 0 \qquad \text{...as det } \neq 0\text{, vectors are independent}$$

Thus, this vector $\notin V$ (5)

6: $\begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 1 & 3 \\ 1 & -1 & 3 \end{bmatrix}$ ...arrange vectors as columns

$$\implies -1(3+3)+1(6+2) \qquad \text{...determinant along 1st column}$$

$$= 2 \neq 0 \qquad \text{...as det } \neq 0\text{, vectors are independent}$$

Thus, this vector $\notin V$ (6)

Thus, vectors $(1), (2), (4) \in V$, and the others $\notin V$. Hence, the correct answer is 3.

Note: For (1), (2), and (4), we could have created a matrix too and computed the determinant. The determinant would have been 0, and the answer reached would have been the same. That is, without finding the scalar multiples, we could have arrived at the same solution.

9. Find the value of c for which the vector $(-2, 26, c)$ will be in the spanning set of the vectors $(1, 0, 1)$ and $(0, 1, -1)$ in $\mathbb{R}^3$ with usual addition and scalar multiplication.

Solution:

If $\begin{bmatrix} -2 \\ 26 \\ c \end{bmatrix} \in span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$, then these vectors must be linearly dependent.

That is:

$$det\left(\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 26\\ 1 & -1 & c\end{bmatrix}\right) = 0$$

$$= 1(c+26) - 2(0-1) = 0$$
$$= c + 28 = 0$$
$$c = -28$$

Hence, the correct answer is **-28**.

10. Consider the subspace $V = \{(x, y, z) \mid z = 7x + 6y,\ \text{where } x, y, z \in \mathbb{R}\}$ of $\mathbb{R}^3$ and $\text{span}(\{(p, 0, 1), (0, q, 1)\}) = V$. Find the value of $\frac{1}{pq}$.

Solution:

The vectors $\begin{bmatrix}p\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\q\\1\end{bmatrix}$ span V. Thus, the given vectors belong in V.

Also, we are given that $V = \{(x, y, z) \mid z = 7x + 6y\}$.

Thus, we can say:

$1 = 7p \quad \text{and} \quad 1 = 6q$

$p = \dfrac{1}{7} \quad \text{and} \quad q = \dfrac{1}{6}$

That gives us:

$$pq = \frac{1}{6 \times 7} = \frac{1}{42}$$
$$\implies \frac{1}{pq} = 42$$

Thus, the correct answer is **42**.

11. Suppose V is a vector space defined as $V = \{A \mid A \in M_{4\times 4}(\mathbb{R})$, A is an upper triangular matrix, and the sum of the diagonal entries is zero $\}$. What is the cardinality of a basis of V?

Solution:

We are given that $A \in V$ is a 4×4 upper triangular matrix, whose diagonal entries sum to 0. Thus we have:

$$V = \left\{ A \mid A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & x_5 & x_6 & x_7 \\ 0 & 0 & x_8 & x_9 \\ 0 & 0 & 0 & x_{10} \end{bmatrix}, \text{ where } x_1 + x_5 + x_8 + x_{10} = 0 \right\}$$

But note that we can write $x_{10} = -x_1 - x_5 - x_8$

$$= \left\{ A \mid A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & x_5 & x_6 & x_7 \\ 0 & 0 & x_8 & x_9 \\ 0 & 0 & 0 & -x_1 - x_5 - x_8 \end{bmatrix} \right\}$$

Thus the following set can constitute a basis for V:

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

The cardinality of S is 9. Hence, the correct answer is 9.

Comprehension Type Question:

Shubham, Poulami, Asrifa, Raphael, and Vicky bought the same type of pen drives from a shop. The shopkeeper asked them to give ratings on the quality of the pen drives, 1-star being the lowest denoting *not satisfied* and 5-stars being the highest denoting *extremely satisfied*. Whenever a customer gives a rating, say 3-stars rating, then the vector we consider corresponding to that rating is $(0, 0, 1, 0, 0)$. Note that one customer can rate only once and the same rating can be given by more than one customer.
Example: Table M2W4GA2 shows an example of the rating given by the customers mentioned above as: Shubham gave 2-stars rating, Poulami gave 1-star rating, Asrifa gave 1-star rating, Raphael gave 5-stars rating, and Vicky did not give any rating.

	ratings				
Customers	1-star	2-stars	3-stars	4-stars	5-stars
Shubham	0	1	0	0	0
Poulami	1	0	0	0	0
Asrifa	1	0	0	0	0
Raphael	0	0	0	0	1
Vicky	0	0	0	0	0

Table: M2W4GA2

Based on the above information answer the following questions.

12. Choose the set of correct options from the following.
    - ◯ **Option 1:** If Shubham and Poulami both gave 2-stars rating, Asrifa and Raphael both gave 1-star rating, and Vicky gave 3-stars rating, then the dimension of the vector space spanned by the row vectors corresponding to the ratings is 3.
    - ◯ **Option 2:** If Shubham and Vicky both gave 3-stars rating, Poulami gave 2-stars rating, Asrifa gave 5-stars rating, and Raphael gave 1-star rating then

the dimension of the vector space spanned by the row vectors corresponding to the ratings is 4.

- ◯ **Option 3:** If Shubham gave 5-stars rating, Poulami gave 3-stars rating, Asrifa gave 1-star rating, Raphael gave 2-stars rating, and Vicky gave 4-stars rating then the dimension of the vector space spanned by the row vectors corresponding to the ratings is 5.
- ◯ Option 4: If Shubham, Poulami, and Asrifa, all gave 2-stars rating, and Raphael and Vicky gave 1-star rating then the dimension of the vector space spanned by the row vectors corresponding to the ratings is 1.

Solution:

Option 1:

If Shubham and Poulami both gave 2-stars rating, then both their ratings vectors would be $(0, 1, 0, 0, 0)$. Similarly, the ratings vectors of Asrifa and Raphael are $(1, 0, 0, 0, 0)$, and the ratings vector of Vicky would be $(0, 0, 1, 0, 0)$.

We can arrange these ratings vectors in the matrix A as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Approach 1: If we reduce the matrix to reduced row echelon form, we get:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $rank(A) = 3$. But note that this gives us the dimension of space spanned by the columns of A. However, Column Rank = Row Rank. Thus, the dimension of the space spanned by the rows of A is also 3. Hence, this option is correct.

Approach 2:
Observe that $R_1 = R_2$ and $R_3 = R_4$ in A. Also observe that R_1, R_3, and R_5 form a linearly independent set. They form a basis for the space spanned by the rows of A. Thus, the vector space spanned by rows of $A = span(R_1, R_3, R_5)$. Since, the cardinality of the basis is 3, the dimension of the space is also 3. Hence, this option is correct.

Option 2: Similar to option 1, if we create a matrix from the ratings vectors, we get the following matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Approach 1:
Similar to option 1, if we reduce the matrix A to its reduced row echelon form we get:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the $rank(A) = 4$. Hence, the dimension of the space spanned by rows of A is 4. Thus, this option is also correct.

Approach 2:
Similar to option 1, $R_1 = R_5$. Hence, the rows R_1, R_2, R_3, R_4 form a linearly independent set. The span of the rows would be $span(R_1, R_2, R_3, R_4)$. Thus, the dimension is 4. Hence, this option is also correct.

Option 3: Arranging the ratings vector into a matrix, we get the matrix as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Observe that simply by swapping the rows we can arrive at the reduced row echelon form of this matrix, which is a 5×5 identity matrix. Since the rank of this matrix is 5, the dimension will also be 5. Hence, this option is also correct.

Option 4:
Arranging the ratings vector into a matrix, we get the matrix as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reducing this matrix to reduced row echelon form gives us:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 2. Hence, this option is incorrect.

13. Consider the matrix A corresponding to the ratings given in Table M2W4GA2, which is given as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following statements about the matrix A are true?

- ◯ Option 1: The set $\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0)\}$ spans the nullspace of A.
- ◯ **Option 2:** The set $\{(0, 0, 1, 0, 0), (0, 0, 0, 1, 0)\}$ spans the nullspace of A.
- ◯ **Option 3:** Nullity of A is 2.
- ◯ Option 4: Nullity of A is 3.

Solution:

We are given the following matrix A:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reducing this matrix to reduced row echelon form, we get:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that if a vector is of the form $(0, 0, x_1, x_2, 0)$, then it lies in the nullspace of A. Thus, we can write the nullspace as follows:

$$N(A) = \left(x_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$N(A) = span \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since cardinality of this set is 2, the dimension of the nullspace of A is also 2. Hence, **option 2** and **option 3** are correct.

14. What will be the rank of the matrix A defined in the above question? [Ans: 3]

Solution:
We have the reduced row echelon form of the matrix from the above question. The number of non-zero rows of this matrix is 3. Hence, the $rank(A) = 3$.