Week-6
Mathematics for Data Science - 2
Introduction to Vector Space
Graded Assignment

1 Multiple Select Questions (MSQ)

1. Which of the following sets with the given addition and scalar multiplication operations (scalars are real numbers in every case) do not form vector spaces?

   ○ **Option 1:**

   $V_1 = \{(x,y) | x, y \in \mathbb{R}\}$
   Addition: $(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, 1);\ (x_1,y_1),(x_2,y_2) \in V_1$
   Scalar multiplication: $c(x,y) = (cx, 1);\ (x,y) \in V_1,\ c \in \mathbb{R}$

   ○ **Option 2:**

   $V_2 = \{(x,y) | x, y \in \mathbb{R}\}$
   Addition: $(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2);\ (x_1,y_1),(x_2,y_2) \in V_2$
   Scalar multiplication: $c(x,y) = (cx, 0);\ (x,y) \in V_2,\ c \in \mathbb{R}$

   ○ **Option 3:**

   $V_3 = \{(x,y) | x, y \in \mathbb{R}\}$
   Addition: $(x_1,y_1) + (x_2,y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + y_1 + y_2);$
   $(x_1,y_1),(x_2,y_2) \in V_3$
   Scalar multiplication: $c(x,y) = (cx, cy);\ (x,y) \in V_3,\ c \in \mathbb{R}$

   ○ Option 4:

   $V_4 = \{(x,y,z) | x, y, z \in \mathbb{R}, x + y = z\}$
   Addition: $(x_1,y_1,z_1) + (x_2,y_2,z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
   $(x_1,y_1,z_1),(x_2,y_2,z_2) \in V_4$
   Scalar multiplication: $c(x,y,z) = (cx, cy, cz);\ (x,y,z) \in V_4,\ c \in \mathbb{R}$

Soln. Option 1:
Suppose $v = (a,b) \in V_1$ works as a zero, then

$(x,y) + (a,b) = (x,y)$ for all $x, y \in \mathbb{R}$

$\Rightarrow (x+a, 1) = (x,y)$ for all $x, y \in \mathbb{R}$

$\Rightarrow \quad 1 = y$ for all $y \in \mathbb{R}$ (contradiction)

let $(x, y) = (2, 3)$

$$(2,3) + (a,b) = (2+a, 1) \neq (2,3)$$

as, $\underline{1 \neq 3}$

Hence, there does not exist any vector $(a,b) \in V_1$ which works as a zero in V_1.

So, V_1 is not a vector space.

Option 2: let $c = 1$ and $(x, y) = (2, 3) \in V_2$

$$1 \cdot (2,3) = (2,0) \neq (2,3)$$

So, V_2 is not a vector space.

Option 3: Suppose $v = (a, b) \in V_3$ works as a zero,

then $(2,3) + (a,b)$ must be equal to $(2,3)$.

$$(2,3) + (a,b) = (2,3)$$

$$\Rightarrow (5+a+b, \ 5+a+b) = (2,3)$$

$$\Rightarrow \quad a+b = -3 \text{ and } a+b = -2$$

$$\Rightarrow \quad 3 = 2 \text{ (which is absurd).}$$

Hence, there does not exist any such vector (a, b) in V_3 which can work as the zero vector.

So, V_3 is not a vector space.

Option 4: $V_4 = \{(x, y, z) \mid x, y, z \in \mathbb{R},\ x + y = z\}$

The addition and scalar multiplication are the usual addition and scalar multiplication defined on $\mathbb{R}^3$.

Moreover if (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in V_4$

then, $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$.

Hence, $(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in V_4$$

as, $z_1 + z_2 = (x_1 + y_1) + (x_2 + y_2)$

$$= x_1 + x_2 + y_1 + y_2$$

$$c(x_1, y_1, z_1) = (cx_1, cy_1, cz_1) \in V_4$$

as, $cz_1 = c(x_1 + y_1) = cx_1 + cy_1$

$(0, 0, 0)$ also belongs to V_4. Hence, V_4 is a vector space.

2. Choose the set of correct options

- ○ **Option 1:** If V is a real vector space, then $(\alpha+\beta)(x+y) = \alpha x+\beta y+\alpha y+\beta x$, for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$.
- ○ Option 2: A vector space can have more than one zero vector.
- ○ **Option 3:** (-1, 0 ,0), (-1, 1, -1) and (0, 2, 3) are linearly independent vectors in $\mathbb{R}^3$.
- ○ Option 4: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ are linearly dependent vectors in $M_{2\times 2}(\mathbb{R})$.

Soln: Option 1:

$$(\alpha+\beta)(x+y) = (\alpha+\beta)x + (\alpha+\beta)y$$
$$= \alpha x + \beta x + \alpha y + \beta y$$
$$= \alpha x + \beta y + \alpha y + \beta x.$$

Option 2: zero vector is unique in a vector space (V)

Suppose there are two zero vectors in a vector space, say v_1 and v_2.

Hence, $v + v_1 = v$ for all $v \in V$ ——(i)

and $v + v_2 = v$ for all $v \in V$ ——(ii)

Hence, $v_2 + v_1 = v_2$ from (i)

$v_1 + v_2 = v_1$ from (ii)

Hence, $v_1 = v_2$.

Option 3:

$$a(-1,0,0)+b(-1,1,-1)+c(0,2,3)=(0,0,0)$$

$$\Rightarrow (-a-b,\ b+2c,\ -b+3c)=(0,0,0)$$

$$\begin{array}{r|l} b+2c=0 & -a-b=0 \\ -b+3c=0 & \\ \hline 5c=0 & \Rightarrow a+b=0 \\ \Rightarrow c=0 & \Rightarrow a=0 \\ b+2(0)=0 \Rightarrow b=0 & \end{array}$$

Hence, the given vectors are linearly independent.

Option 4:

$$a\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}+b\begin{bmatrix}0 & -2\\ 1 & 0\end{bmatrix}+c\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-a-c & -2b+c\\ b+c & a-c\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow -a-c=0,\quad -2b+c=0,\quad b+c=0,\quad a-c=0$$

$$a + c = 0$$
$$a - c = 0$$
$$2a = 0$$

$\Rightarrow a = 0$ so, $c = 0$ and $b = 0$

Hence, the given vectors of $M_{2\times2}(\mathbb{R})$ are linearly independent.

3. A healthy juice consists of 30 units of protein, 11 units of carbohydrate, 53 units of fat, and 213 units of calcium. A juice maker makes two types of juice, Type A and Type B. Type A consists of banana, milk, and almond, whereas Type B consists of apple, milk, and almond. Table M2W6G1 shows the amount of protein, carbohydrate, fat, and calcium present in each banana, apple, and almond, and in 100 ml of milk.

Items	Protein	Carbohydrate	Fat	Calcium
Banana (1 piece)	2	3	1	5
Apple (1 piece)	1	2	1	6
Almond (1 piece)	6	1	15	1
Milk (100 ml)	4	1	3	100

Table: M2W6G1

Use the above information to choose the correct options.

- ○ **Option 1:** A healthy juice can be prepared with the right quantities of ingredients of Type A, and those quantities are unique.
- ○ Option 2: A healthy juice can be prepared with the right quantities of ingredients of Type B, and those quantities are unique.
- ○ Option 3: A healthy juice can not be prepared with the right quantities of ingredients of Type A.
- ○ **Option 4:** A healthy juice can not be prepared with the right quantities of ingredients of Type B.

Soln.

	Banana	Milk	Almond	Apple
Type A:	x_1	x_2	x_3	0
Type B:	0	y_1	y_2	y_3

	Protein	Carbohydrate	Fat	Calcium
Type A :	$2x_1+4x_2+6x_3$	$3x_1+x_2+x_3$	$x_1+3x_2+15x_3$	$5x_1+100x_2+x_3$
Type B :	$4y_1+6y_2+y_3$	$y_1+y_2+2y_3$	$3y_1+15y_2+y_3$	$100y_1+y_2+6y_3$

Type A:

$$\begin{aligned} 2x_1+4x_2+6x_3 &= 30\\ 3x_1+x_2+x_3 &= 11\\ x_1+3x_2+15x_3 &= 53\\ 5x_1+100x_2+x_3 &= 213 \end{aligned}$$

Type B:

$$\begin{aligned} 4y_1+6y_2+y_3 &= 30\\ y_1+y_2+2y_3 &= 11\\ 3y_1+15y_2+y_3 &= 53\\ 100y_1+y_2+6y_3 &= 213 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 6 & 30\\ 3 & 1 & 1 & 11\\ 1 & 3 & 15 & 53\\ 5 & 100 & 1 & 213 \end{array}\right) \xrightarrow{R_1/2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 15\\ 3 & 1 & 1 & 11\\ 1 & 3 & 15 & 53\\ 5 & 100 & 1 & 213 \end{array}\right)$$

$$\xrightarrow{R_2-3R_1,\ R_3-R_1,\ R_4-5R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 15\\ 0 & -5 & -8 & -34\\ 0 & 1 & 12 & 38\\ 0 & 90 & -14 & 138 \end{array}\right) \xrightarrow{-R_2/5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 15\\ 0 & 1 & 8/5 & 34/5\\ 0 & 1 & 12 & 38\\ 0 & 90 & -14 & 138 \end{array}\right)$$

$$\xrightarrow{R_3-R_2,\ R_4-90R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 15\\ 0 & 1 & 8/5 & 34/5\\ 0 & 0 & 52/5 & 156/5\\ 0 & 0 & -158 & -474 \end{array}\right) \xrightarrow[-\frac{1}{158}R_4]{\frac{5}{52}R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 15\\ 0 & 1 & 8/5 & 34/5\\ 0 & 0 & 1 & 3\\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\Big\downarrow R_4 - R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \xleftarrow[R_2 - \frac{8}{5}R_3]{R_1 - 3R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 8/5 & 34/5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Big\downarrow R_1 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, $\left.\begin{array}{l} x_1 = 2 \\ x_2 = 2 \\ x_3 = 3 \end{array}\right\}$ unique Soln.

Hence, option 1 is true.

Solve the system of linear equations corresponding to Type B. Observe that, there is no solution of the system of linear equations.

Hence, option 4 is true.

4) Consider the set of vectors $S = \{(-1,1,5),(2,1,3),(2,1,2),(1,-1,7),(-1,3,-5)\}$ from $\mathbb{R}^3$ and choose the set of correct options.

- ○ Option 1: The singleton set $\{(-1,1,5)\}$ is linearly dependent.
- ○ **Option 2:** If $\alpha, \beta \in S$ and α, β are distinct then $\{\alpha, \beta\}$ is a linearly independent set of vectors.
- ○ **Option 3:** The set {(-1, 1, 5), (2, 1, 3),(-2, 2, 10)} is a linearly dependent set of vectors.
- ○ Option 4: The set S is a linearly independent set of vectors.
- ○ Option 5: The set $\{\alpha, \beta, \gamma\}$ is a linearly dependent set of vectors for any $\alpha, \beta, \gamma \in S$, where all the three are distinct vectors.
- ○ Option 6: The set $\{\alpha, \beta, \gamma, \delta\}$ is a linearly independent set of vectors for any $\alpha, \beta, \gamma, \delta \in S$, where all the four are distinct vectors.
- ○ **Option 7:** The system $AX = b$, where $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 7 \\ -1 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has unique solution.
- ○ **Option 8:** The system $AX = b$, where $A = \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has infinitely many solutions.
- ○ Option 9: The system $AX = b$ where $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & -5 \\ -1 & 1 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has infinitely many solutions.

Soln: Option 1: Any singleton set with non-zero vector is linearly independent.

Option 2: $\alpha, \beta \in S$. $\{\alpha, \beta\}$ is linearly dependent if and only if β is scalar multiple of α.

But in S, no two vectors are scalar multiple of each other. Hence, $\{\alpha, \beta\}$ is linearly independent set of vectors.

Option 3: $2(-1,1,5)+0(2,1,3)-1(-2,2,10)=0$

Hence $\{(-1,1,5),(2,1,3),(-2,2,10)\}$ is linearly dependent.

Option 4: All the vectors are from $\mathbb{R}^3$.

If a set $S \subseteq \mathbb{R}^3$ has cardinality strictly greater that 3, then S must be linearly dependent.

Option 5: Let $\alpha=(-1,1,5)$, $\beta=(2,1,3)$, $\gamma=(2,1,2)$

$$\begin{vmatrix} -1 & 1 & 5 \\ 2 & 1 & 3 \\ 2 & 1 & 2 \end{vmatrix} = -1(2-3)-1(4-6)+5(2-2)$$
$$= 1+2=3 \neq 0$$

Hence, $\{\alpha, \beta, \gamma\}$ is linearly independent.

Option 6: The set $\{\alpha, \beta, \gamma, \delta\}$ has the cardinality 4 which is strictly greater than 3, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}^3$.

Hence, the set is linearly dependent.

Option 7:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 7 \\ -1 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & -1 & 7 & 0 \\ -1 & 1 & 5 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 7 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & 1 & 5 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 2R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 7 & 0 \\ 0 & 3 & -12 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 3 & -11 & 0 \end{array}\right] \xrightarrow{R_2/3} \left[\begin{array}{ccc|c} 1 & -1 & 7 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 3 & -11 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - 3R_1} \left[\begin{array}{ccc|c} 1 & -1 & 7 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3/12} \left[\begin{array}{ccc|c} 1 & -1 & 7 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - R_3} \left[\begin{array}{ccc|c} 1 & -1 & 7 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_2 = x_3 = 0$

Hence, the Soln. is unique.

Option 8 :

$$\begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 3 & -5 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ -2 & -1 & -3 & 0 \end{array}\right]$$

$$\Big\downarrow R_2 + 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xleftarrow{R_2/-7} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & -7 & 7 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 + 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left.\begin{array}{r} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = x_3 \end{array}$$

Solutions of system of linear equations:

$$\left\{ \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix}, \quad x_3 \in \mathbb{R} \right\}$$

Hence, there are infinite number of solutions.

Option 9:

$$\begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & -5 \\ -1 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ -1 & 3 & -5 & 0 \\ -1 & 1 & 5 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} -1 & 3 & -5 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & 1 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & 1 & 5 & 0 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 7 & -8 & 0 \\ 0 & -2 & 10 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & -2 & 10 & 0 \\ 0 & 7 & -8 & 0 \end{array}\right] \xrightarrow{R_2/-2} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 7 & -8 & 0 \end{array}\right] \xrightarrow{R_3-7R_2} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 27 & 0 \end{array}\right]$$

$$\xrightarrow{R_3/27} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = x_2 = x_3 = 0$

(unique solution)

2 Numerical Answer Type (NAT):

5) 1. Consider the set of three vectors $S = \{(c, -1, -2), (1, 0, -1), (-1, -3, c)\}$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. For which value of c will the above set S be linearly dependent? [Answer: 2.5]

Soln:

$$\begin{vmatrix} c & 1 & -1 \\ -1 & 0 & -3 \\ -2 & -1 & c \end{vmatrix} = c(-3) - 1(-c-6) - 1(1)$$

$$= -3c + c + 6 - 1$$

$$= -2c + 5$$

The vectors are linearly dependent if and only if: $-2c + 5 = 0$

$$\Rightarrow c = 2.5$$

1) Consider the set of three vectors $S = \{(7,7,2), (8,7,4), (5,7,c)\}$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. If S is a linearly independent set, then the value of c can not be equal to [Answer: -2]

Soln. If S is a linearly dependent set, then

$$\begin{vmatrix} 7 & 8 & 5 \\ 7 & 7 & 7 \\ 2 & 4 & c \end{vmatrix} = 0$$

$$\Rightarrow 7(7c - 28) - 7(8c - 20) + 2(56 - 35) = 0$$

$$\Rightarrow 49c - 196 - 56c + 140 + 42 = 0$$

$$\Rightarrow -7c - 196 + 182 = 0$$

$$\Rightarrow -7c - 14 = 0$$

$$\Rightarrow 7c + 14 = 0$$

$$\Rightarrow c = -2$$

Hence, if S is linearly independent set, then the value of c cannot be -2.

3 Comprehension Type Question:

In genetics, a classic example of dominance is the inheritance of shape of seeds in peas. Peas may be round (associated with genotype R) or wrinkled (associated with genotype r). In this case, three combinations of genotypes are possible: RR, rr, and Rr. The RR individuals have round peas and the rr individuals have wrinkled peas. In Rr individuals the R genotype masks the presence of the r genotype, so these individuals also have round peas.

Consider the crossing of RR with RR. This always gives the genotype RR, therefore the probabilities of an offspring to be RR, Rr, and rr respectively are equal to 1, 0, and 0. Next, consider the crossing of Rr with RR. The offspring will have equal chances to be of genotype RR and genotype Rr, therefore the probabilities of RR, Rr, and rr repectively are 1/2, 1/2, and 0. Finally, consider the crossing of rr with RR. This always results in genotype Rr. Therefore, the probabilities of genotypes RR, Rr, and rr are 0, 1, and 0, respectively.

.

The following table summarizes these facts :

Parents' genotypes	Genotypes of offspring		
	RR	Rr	rr
RR-RR	1	0	0
RR-Rr	$\frac{1}{2}$	$\frac{1}{2}$	0
RR-rr	0	1	0

Table: M2W6G3

The matrix representing this observation is given by $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and the initial distribution vector (1×3 matrix) is denoted by $X_0 = (X_0^1, X_0^2, X_0^3)$, where X_0^1 denotes the distribution of RR, X_0^2 denotes the distribution of Rr, and X_0^3 denotes the distribution of rr. For any positive integer n, the distribution vector after n generations (i.e., at $t = n$) is denoted by X_n and is given by the equation $X_{n-1}P = X_n$.

Using the above information, answer the following questions.

7) Suppose, in an experiment, 100 pairs of parents with genotype combinations RR-RR, 100 pairs of parents with genotype combinations RR-Rr, and 200 pairs of parents with genotype combination RR-rr are taken to observe the genotypes of their offspring. Suppose from crossing of each pair of parents a single offspring is produced. Find the set of correct options from the following. (MSQ)

- ◯ Option 1: There will be at least 200 offspring with wrinkled peas.
- ◯ **Option 2:** There will be no offspring with wrinkled peas.

- ◯ Option 3: There will be no offspring with round peas.
- ◯ **Option 4:** All the offspring will have round peas.
- ◯ Option 5: All the offspring will have wrinkled peas.
- ◯ **Option 6:** There will be at least 100 offspring with combination of genotypes RR.
- ◯ **Option 7:** There will be at least 200 offspring with combination of genotypes Rr.

Soln: Only those offsprings with rr genotypes are wrinkled peas. But the probability of genotypes of an offspring to be rr is 0 in any of the given combination. So no offsprings will have wrinkled peas. Whereas, all the offsprings will have round peas.

So options 2 and 4 are correct.

When the parents' genotype combination is RR - RR then the probability of genotypes of an offspring to be RR is 1. i.e in those cases, the offspring must have the genotype RR. In our experiment we have 100 pairs of genotype RR-RR. So, there will be atleast 100 offsprings with genotype RR (There may be some more coming from the genotype combination RR-Rr).

Similarly an off~~s~~spring of Rr geneotype is guranteed if the combination of parents' genotype is RR-rr and we have 200 such pairs.

So, there will be atleast 200 offsprings with genotype Rr.

8): Which of the following options are correct?

option 1 : The rows of the matrix P form a linearly independent set.

option 2 : $\{(\frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0)\}$ is a linearly independent set.

option 3 : The columns of the matrix P form a linearly dependent set.

option 4 : $\det(P)$ is a non-zero number.

option 1 : Rows of matrix P are $(1,0,0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0,1,0)$.

Let $a, b, c \in \mathbb{R} \ni$

$$a(1,0,0) + b(\tfrac{1}{2}, \tfrac{1}{2}, 0) + c(0,1,0) = (0,0,0)$$

$$\Rightarrow (a + \tfrac{b}{2}, \tfrac{b}{2} + c, 0) = (0,0,0)$$

$$\Rightarrow a + \tfrac{b}{2} = 0, \quad \tfrac{b}{2} + c = 0$$

Observe that the above system of linear equations has infinitely many solution.

Another way, $(\frac{1}{2}, \frac{1}{2}, 0) = \frac{1}{2}(1,0,0) + \frac{1}{2}(0,1,0)$

Hence $(1,0,0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0,1,0)$ are linearly dependent vectors.

option 2 : let $a, b \in \mathbb{R} \ni$

$$a(\tfrac{1}{2}, 0, 0) + b(0, \tfrac{1}{2}, 0) = (0,0,0)$$

$$\Rightarrow (\tfrac{a}{2}, \tfrac{b}{2}, 0) = (0,0,0)$$

$$\Rightarrow a = 0, \quad b = 0$$

Hence $(\frac{1}{2}, 0, 0)$, $(0, \frac{1}{2}, 0)$ are linearly independent.

option 3 : columns of matrix P are $(1, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, 1)$, $(0,0,0)$ in which zero vector is available. Hence vectors are depedent.

option 4 : Observe $\det(P) = 0$.

9) : Suppose $X_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Find out the correct set of correct options. (MSQ)

- ○ Option 1: X_0 and X_1 are linearly dependent.
- ○ **Option 2:** X_0 and X_1 are linearly independent.
- ○ Option 3: The set $\{X_0, X_1, X_2\}$ is a linearly dependent set.
- ○ **Option 4:** The set $\{X_0, X_1, X_2\}$ is a linearly independent set.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad X_{n-1} P = X_n$$

$$X_0 = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$X_0 P = X_1$$

$$\Rightarrow \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/3 + 1/6 & 1/6 + 1/3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 & 0 \end{pmatrix} = X_1$$

$$X_1 P = X_2$$

$$\Rightarrow \begin{pmatrix} 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 + 1/4 & 1/4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/4 & 0 \end{pmatrix} = X_2$$

X_1 is not a scalar multiple of X_0. So $\{X_0, X_1\}$ is linearly independent.

$$\begin{vmatrix} 1/3 & 1/2 & 3/4 \\ 1/3 & 1/2 & 1/4 \\ 1/3 & 0 & 0 \end{vmatrix} = \frac{1}{3}\left(\frac{1}{8} - \frac{3}{8}\right)$$

(Expanding with respect to 3rd Row)

$$= \frac{1}{3}\left(-\frac{2}{8}\right) = -\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)$$

$$= -\frac{1}{12} \neq 0$$

Hence, $\{x_0, x_1, x_2\}$ is linearly independent.

10): Suppose $X_0 = (a, b, 1-a-b)$, where $0 \le a, b, a+b \le 1$. If the set $\{X_0, X_2\}$ is linearly dependent, then what is the value of b? [Answer: 0]

$$X_0 P = X_1$$

$$\Rightarrow \begin{pmatrix} a & b & 1-a-b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a + b/2 & b/2 + 1 - a - b & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a + b/2 & 1 - a - b/2 & 0 \end{pmatrix}$$

$$= X_1$$

$$X_1 P = X_2$$

$$\Rightarrow \begin{pmatrix} a + b/2 & 1 - a - b/2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a + b/2 + 1/2 - \frac{a}{2} - \frac{b}{4} & 1/2 - a/2 - b/4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 + a/2 + b/4 & 1/2 - a/2 - b/4 & 0 \end{pmatrix} = X_2$$

If X_0 and X_2 are linearly dependent, then

X_2 is scalar multiple of X_0.

i.e. $X_2 = cX_0$ for some $c \in \mathbb{R}$

$$\left(\tfrac{1}{2}+\tfrac{a}{2}+\tfrac{b}{4} \quad \tfrac{1}{2}-\tfrac{a}{2}-\tfrac{b}{4} \quad 0\right) = c\left(a \quad b \quad 1-a-b\right)$$

$$c - ca - cb = 0$$

$$c(1-a-b) = 0$$

either $c=0$ or $\underline{a+b=1}$

If $c=0$ then, $\tfrac{1}{2}+\tfrac{a}{2}+\tfrac{b}{4} = 0$

$\tfrac{1}{2}-\tfrac{a}{2}-\tfrac{b}{4} = 0$

$\Rightarrow \tfrac{1}{4} = 0$ (which is absurd)

So, c cannot be 0.

So, $a+b=1$

$\Rightarrow a = 1-b$

We also have

$$ca = 1/2 + a/2 + b/4 \quad \text{and} \quad cb = 1/2 - a/2 - b/4$$

$$\Rightarrow c(1-b) = 1/2 + \frac{1-b}{2} + \frac{b}{4}$$

$$\Rightarrow c(1-b) = \frac{2+2-2b+b}{4}$$

$$\Rightarrow c(1-b) = \frac{4-b}{4}$$

$$\Rightarrow 4c - 4bc = 4-b \longrightarrow (1)$$

$$\Rightarrow cb = \frac{1}{2} - \frac{1-b}{2} - \frac{b}{4}$$

$$\Rightarrow cb = \frac{2-2+2b-b}{4}$$

$$\Rightarrow cb = \frac{b}{4}$$

$$\Rightarrow \underline{4bc = b} \longrightarrow (2)$$

Substituting $4bc = b$ in (1) we get,

$$4c - b = 4-b$$

$$\Rightarrow 4c = 4$$

$$\Rightarrow c = 1$$

Now from (2) we get

$$4b = b$$

$$\Rightarrow 3b = 0$$

$$\Rightarrow \underline{\underline{b = 0}}$$