Week-2
Mathematics for Data Science - 2
Solutions of System of Linear Equations
Graded Assignment

1. In a particular year, the profit (in lakhs of ₹) of Star Fish company is given by the polynomial $P(x) = ax^2 + bx + c$ where x denotes the number of months since the beginning of the year (i.e., $x = 1$ denotes January, $x = 2$ denotes February, and so on). In January and February the company made a loss of ₹45(in lakhs), and ₹19(in lakhs) respectively, and in March the company made a profit of ₹3(in lakhs). Let the loss be represented by negative of profit.
Choose the correct set of options based on the given information.

   ○ Option 1: The maximum profit will be in the month of May.
   ○ **Option 2:** The maximum profit will be in the month of August.
   ○ **Option 3:** The maximum monthly profit amount is ₹53 lakh.
   ○ Option 4: The maximum monthly profit amount is ₹35 lakh.

Solution:

In the month of January ($x = 1$), the company made a loss of ₹45 lakh. That is,

$$P(1) = a(1)^2 + b(1) + c = -45 \implies a + b + c = -45 \tag{1}$$

Similarly, for the month of February ($x = 2$) and March ($x = 3$), we have:

$$\begin{aligned} P(2) &= 4a + 2b + c = -19 \\ P(3) &= 9a + 3b + c = 3 \end{aligned} \tag{2}$$

Thus, we have the following set of equations:

$$\begin{aligned} a + b + c &= -45 \\ 4a + 2b + c &= -19 \\ 9a + 3b + c &= 3 \end{aligned} \tag{3}$$

The augmented matrix A using the above equations can be written as:

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array}\right]$$

By reducing A to its reduced row echelon form R, we get:

$$R = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{array}\right]$$

Thus, we have: $a = -2$, $b = 32$, and $c = -75$. By substituting these values in the given polynomial, we get:

$$P(x) = -2x^2 + 32x - 75 \tag{4}$$

Approach 1: The minima/maxima of a quadratic equation is given by:

$$\begin{aligned} maxima &= \frac{-b}{2a} \\ &= \frac{-32}{-4} = 8 \end{aligned}$$

Approach 2: Take the derivative of the polynomial, and set it to zero:

$$\begin{aligned} P'(x) = -4x + 32 = 0 \\ \implies x = 8 \end{aligned}$$

Since $P''(8) = -4 < 0$, $x = 8$ is a point of local maximum. Also, since $x = 8$ is the only critical point, it is the global maximum.

Thus, at $x = 8$ (the month of August), the value of the polynomial is the highest. Evaluating P at $x = 8$, we get: $P(8) = 53$. Hence, **Option 2** and **Option 3** are correct.

2. If A be a 3×4 matrix and b be a 3×1 matrix, then choose the set of correct options.

   ○ **Option 1:** If $(A|b)$ be the augmented matrix and $(A'|b')$ be the matrix obtained from $(A|b)$ after a finite number of elementary row operations then the system $Ax = b$ and the system $A'x = b'$ have the same set of solutions.

   ○ Option 2: If $(A'|b')$ is the reduced row echelon form of $(A|b)$ then the system $A'x = b'$ has at least one solution.

   ○ **Option 3:** If $(A'|b')$ is the reduced row echelon form of $(A|b)$, then A' is also in reduced row echelon form.

   ○ **Option 4:** If $(A'|b')$ is the reduced row echelon form of $(A|b)$ and there is no row such that the only non zero entry lies in the last column of $(A'|b')$ then the system $Ax = b$ has at least one solution.

Solution:

Option 1:
From Gaussian Elimination, we know that any number of elementary row operations on an augmented matrix A does not affect the solution to $Ax = b$. Hence, this option is correct.

Option 2:
To prove this option incorrect, take an example as follows:

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 2 & 4 & 6 & 0 & 1 \\ 3 & 6 & 9 & 0 & 1 \end{array}\right]$$

Then its reduced row echelon form is:

$$[A'|b'] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system $A'x = b'$ does not have a solution. Hence, this option is incorrect.

Option 3: Observe that if we reduce the augmented matrix to reduced row echelon form, then its coefficient matrix also gets reduced to reduced row echelon form. (Check, as an example, the matrix given for option 2.) Hence, this option too is correct.

Option 4: If $[A'|b']$ is the reduced row echelon form and there is no row such that the only non-zero entry lies in the last column of $[A'|b']$ (that is, in b'), then the last row of the reduced matrix A' should contain a non-zero entry or all the entries of the last row of $[A'|b']$ should be zero. In either case, we have a solution to the system. Hence, option 4 is also correct.

3. Choose the set of correct options

- ◯ **Option 1:** If the sum of all the elements of each row of a matrix A is 0, then A is not invertible.
- ◯ **Option 2:** If E is a matrix of order 3×3 obtained from the identity matrix by a finite number of elementary row operations then E is invertible.
- ◯ Option 3: Any system of linear equations has at least one solution.
- ◯ Option 4: If A is a matrix of order 3×3 and $det(A) = 3$ then $det(Adj(A)) = 3$.
- ◯ **Option 5:** If A is a matrix of order 3×3 and $det(A) = 3$ then $det(Adj(A)) = 9$.

Solution:

Option 1: Consider the matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ such that: } \qquad \begin{aligned} a_{11} + a_{12} + a_{13} = 0 \\ a_{21} + a_{22} + a_{23} = 0 \\ a_{31} + a_{32} + a_{33} = 0 \end{aligned}$$

$$det(A) = det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = det\left(\begin{bmatrix} a_{11} + a_{12} + a_{13} & a_{12} & a_{13} \\ a_{21} + a_{22} + a_{23} & a_{22} & a_{23} \\ a_{31} + a_{32} + a_{33} & a_{32} & a_{33} \end{bmatrix}\right)$$

$$\implies det(A) = det\left(\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}\right) = 0$$

As the determinant of the matrix is 0, the matrix A is not invertible. Hence this option is correct.

Option 2:
$det(I) = 1$. Any matrix obtained by elementary row operations on I will have a determinant as $c\ det(I)$, where c is some non-zero constant.
Thus, $det(E) = c\ det(I) = c$, where $c \neq 0$ Hence, the matrix E is invertible- this option is correct.

Option 3: Consider the following example:

$$\begin{aligned} 2x + 3y = 0 \\ 2x + 3y = 1 \end{aligned}$$

The above system does not have a solution, hence this option is incorrect.

Option 4 and 5:

We know:

$$A^{-1} = \frac{Adj(A)}{det(A)}$$

$$det(A)A^{-1} = Adj(A) \qquad \text{...multiplying both sides by det(A)}$$

$$det(det(A)A^{-1}) = det(Adj(A)) \qquad \text{...take determinant on both sides}$$

$$det(A)^3 det(A^{-1}) = det(Adj(A)) \qquad ...det(cA) = c^3 det(A), \text{ when } A \text{ is a } 3 \times 3 \text{ matrix}$$

$$det(A)^2 det(A) det(A^{-1}) = det(Adj(A))$$

$$det(A)^2 = det(Adj(A)) \qquad ...det(A)det(A^{-1}) = 1$$

$$3^2 = 9$$

Thus, **option 5** is correct and **option 4** is incorrect.

4. Ramya bought 1 comic book, 2 horror books, and 1 novel from a bookshop which cost her ₹1000. Romy bought 2 comic books, 5 horror books, and 1 novel which cost him ₹2000. Farjana bought 4 comic books, 5 horror books, and c novels from a shop which cost her ₹d. If $x_1, x_2,$ and x_3 represent the price of each comic book, horror book, and novel, respectively, then choose the set of correct options.

- ○ **Option 1:** The matrix representation to find x_1, x_2 and x_3 is

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 4 & 5 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ d \end{bmatrix}$$

- ○ **Option 2:** The matrix representation to find x_1, x_2 and x_3 is

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & c \end{bmatrix} = \begin{bmatrix} 1000 & 2000 & d \end{bmatrix}$$

- ○ Option 3: The matrix representation to find x_1, x_2 and x_3 is

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & c \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1000 & 2000 & d \end{bmatrix}$$

- ○ Option 4: If Farjana tries to find $x_1, x_2,$ and x_3 using appropriate matrix representation by taking $c = 2$ and $d = 4000$, then the price of each comic book that she thus arrives at, will not be unique.
- ○ **Option 5:** If $c = 7$ and $d = 4000$, then the price of each comic book cannot be determined from this data.
- ○ **Option 6:** If $c = 7$ and $d = 3000$, then the shopkeeper has made a mistake.
- ○ **Option 7:** If $c = 2$ and $d = 3000$, then the price of each comic book can be determined from the data.

Solution: Let x_1 be the price of one comic book, x_2 be the price of one horror book,

and x_3 be the price of one novel. Then, Ramya's purchase can be given by:

$$x_1 + 2x_2 + 1x_3 = 1000$$

Similarly, Romy and Farjana's purchases can be given by:

$$2x_1 + 5x_2 + x_3 = 2000$$
$$4x_1 + 5x_2 + cx_3 = d$$

Option 1:
The matrix representation $Ax = b$ for the above system is:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 4 & 5 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ d \end{bmatrix}$$

Thus, **Option 1** is correct.

Option 2 and Option 3:

$$\begin{aligned} &\text{If } Ax = b, \text{ then:} \\ &\implies (Ax)^T = b^T && \text{...taking transpose on both sides} \\ &\implies x^T A^T = b^T && \text{...as } (AB)^T = B^T A^T \end{aligned}$$

By taking the transpose on both sides, we can write the equation from option 1 as follows:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & c \end{bmatrix} = \begin{bmatrix} 1000 & 2000 & d \end{bmatrix}$$

Thus, **Option 2** is correct and **Option 3** is incorrect.

Options 4, 5, and 6:
The augmented matrix for the system can be given as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{array}\right]$$

By reducing this matrix to its row echelon form, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & c-7 & d-4000 \end{array}\right]$$

Option 4: If $c = 2$ and $d = 4000$, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right] \implies \text{The system has unique solution.}$$

Thus, **option 4** is incorrect.

Option 5: If $c = 7$ and $d = 4000$, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \text{The system has infinitely many solutions.}$$

which means, we cannot determine the price of each comic and thus **option 5** is correct.

Option 6: If $c = 7$ and $d = 3000$, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1000 \end{array}\right] \implies \text{The system has no solutions.}$$

which means, that the shopkeeper must have made a mistake. Hence, **option 6** is also correct.

Option 7: If $c = 2$ and $d = 3000$, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & -1000 \end{array}\right] \implies \text{The system has a unique solution.}$$

which means, that the price of each comic book can be determined from the given data. Thus, **option 7** is also correct.

5. Consider the system of equations given below:

$$\frac{x^{2021}}{2021} + \frac{y^{2021}}{2021} - \frac{z^{2021}}{2021} = \pi$$

$$\frac{x^{2021}}{2021} + \frac{y^{2021}}{2021} - \frac{z^{2021}}{2021} = e$$

$$\frac{x^{2021}}{2021} - \frac{y^{2021}}{2021} + \frac{z^{2021}}{2021} = 1729.$$

The number of solutions of the above system of equations is [Ans: 0]

Solution:
Consider the first two equations of this system. The left hand side of both the equations is the same, which gives us $\pi = e$, which is not true. Hence, the system has no solutions.

6. Let the reduced row echelon form of a matrix A be

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}.$$

The first, second, third and fourth columns of A are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, respectively. The value of $a + b + c$ is [Ans: 0]

Solution:

Approach 1:
From the question, we have:

$$A = \begin{bmatrix} 1 & 3 & a & -1 \\ 0 & 2 & b & 0 \\ -1 & 1 & c & 0 \end{bmatrix}$$

To reduce this matrix to reduced row echelon form, we do the following elementary row operations in this sequence:

$$R_3 + R_1 \rightarrow \frac{R_2}{2} \rightarrow R_3 - 4R_2 \rightarrow R_1 - 3R_2 \rightarrow \frac{R_3}{-6} \rightarrow R_2 - R_3 \rightarrow R_1 + 3R_3$$

The resultant matrix R is as follows:

$$A = \begin{bmatrix} 1 & 0 & \dfrac{a-b-c}{2} & \dfrac{-1}{2} \\ 0 & 1 & \dfrac{a+b+c}{6} & \dfrac{-1}{6} \\ 0 & 0 & \dfrac{-a+2b-c}{6} & \dfrac{1}{6} \end{bmatrix}$$

Compare this matrix to the matrix as given in the question, we get:

$$\begin{aligned} \frac{a-b-c}{2} &= 0, & & & a-b-c &= 0 \\ \frac{a+b+c}{6} &= 0, & &\implies & a+b+c &= 0 \\ \frac{-a+2b-c}{6} &= 1 & & & -a+2b-c &= 6 \end{aligned}$$

After solving the above system, we get a unique solution as follows: $a = 0$, $b = 2$, $c = -2$.
Thus, we get answer as: $a + b + c = 0 + 2 - 2 = 0$.

Approach 2:

$$A = \begin{bmatrix} 1 & 3 & a & -1 \\ 0 & 2 & b & 0 \\ -1 & 1 & c & 0 \end{bmatrix} \text{ and its reduced row echelon form is } A' = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}.$$

We know $Ax = 0$ and $A'x = 0$ have the same set of solutions. Thus, let $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ be a solution for both the systems. That is, $Ax = 0$ and $A'x = 0$.

From $A'x = 0$, we get:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{-1}{6} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 &= \frac{1}{2}x_4 \\ x_2 &= \frac{1}{6}x_4 \\ x_3 &= \frac{-1}{6}x_4 \end{aligned} \tag{1}$$

From $Ax = 0$, we get:

$$\begin{bmatrix} 1 & 3 & a & -1 \\ 0 & 2 & b & 0 \\ -1 & 1 & c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + 3x_2 + ax_3 - x_4 &= 0 \\ 2x_2 + bx_3 &= 0 \\ -x_1 + x_2 + cx_3 &= 0 \end{aligned} \tag{2}$$

From equations (1) and (2), we get:

$$\begin{aligned} \frac{1}{2}x_4 + \frac{1}{2}x_4 - \frac{a}{6}x_4 - x_4 &= 0 \\ \implies \frac{a}{6}x_4 &= 0 \\ \implies a &= 0 \qquad \text{.. we can choose } x_4 \neq 0 \end{aligned} \tag{3}$$

Similarly, we can find values of b and c, as follows:

$$\begin{aligned} &\frac{1}{3}x_4 - \frac{b}{6}x_4 = 0 \\ \Longrightarrow\ &\frac{x_4}{3} = \frac{bx_4}{6} \\ \Longrightarrow\ &b = 2 && (4) \\ &\frac{-x_4}{2} + \frac{x_4}{6} - \frac{cx_4}{6} = 0 \\ \Longrightarrow\ &c = -2 && (5) \end{aligned}$$

Thus, from (3), (4), and (5) we have $a + b + c = 0 + 2 - 2 = 0$. Hence, the answer is **0**.

7. Three mobile shops- shop A, shop B and shop C, sell three brands of mobile phones: brand R, brand S and brand T. In a week, shop A sold 1 mobile phone of brand R, $3k$ mobile phones of brand S, and $3k+4$ mobile phones of brand T. Shop B sold 1 mobile phone of brand R, $k+4$ mobile phones of brand S, and $4k+2$ mobile phones of brand T. Shop C sold 1 mobile phone of brand R, $2k+2$ mobile phones of brand S, and $3k+4$ mobile phones of brand T (assume, $k \neq 2$). Assume that the price of a given model of a given brand is the same in all the shops. Shop A, shop B, and shop C earned ₹61, ₹65 and ₹66 (in thousands), respectively by selling these three brands of mobile phones. If the price of each mobile phone of brand S is ₹5 (in thousands), then what is the price of each mobile phone of brand T (in thousands)? [Note: Suppose the price comes out to be 20,000, then the answer should be 20]? [Answer: 6]

Solution:

Let x_r, x_s and x_t be the price of mobile brands R, S, and T respectively. Thus, we can write the number of mobiles sold by Shop A in a week as:

$$x_r + 3kx_s + (3k+4)x_t = 61 \tag{1}$$

Similarly, the number of mobiles sold by Shop B and Shop C can be written respectively as follows:

$$\begin{aligned} x_r + (k+4)x_s + (4k+2)x_t &= 65 \qquad (2)\\ x_r + (2k+2)x_s + (3k+4)x_t &= 66 \end{aligned}$$

From (1) and (2), The augmented matrix of the above system of equations can written as:

$$\left[\begin{array}{ccc|c} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{array}\right]$$

The row echelon form of the above matrix comes out to be as follows:

$$\left[\begin{array}{ccc|c} 1 & 3k & 3k+4 & 61 \\ 0 & 1 & 0 & \dfrac{5}{2-k} \\ 0 & 0 & 1 & \dfrac{6}{2-k} \end{array}\right] \tag{3}$$

Since we are given that $k \neq 2$, we can write:

$$x_s = \frac{5}{2-k} \tag{4}$$

$$x_t = \frac{6}{2-k} \tag{5}$$

We are also given that $x_s = 5$ (in thousands). Thus,

$$x_s = \frac{5}{2-k} = 5$$

Solving the above gives us $k = 1$. Substituting this value of k in equation 5, we get $x_t = 6$. Hence, the answer is 6.

8. The number of solutions of the system of equations

$$\begin{aligned} 2x + 3y + 5z &= 1 \\ x + 2y + 3z &= 1 \\ x + y + 2z &= 7 \end{aligned}$$

is [Ans: 0]

Solution:

The matrix representation of the above system can be given as:

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

The augmented matrix can be given by:

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 7 \end{array}\right]$$

Reducing the above matrix gives as:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

It is clear from the last row of the matrix, that the system has no solutions. Hence, the answer is 0.

9. A is the reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 1 & 5 & 11 \\ 2 & 2 & 7 & 8 \\ 3 & 9 & 0 & 0 \end{bmatrix}.$$

Then determinant of A is [Ans: 0]

Solution:

Approach 1:
The reduced row echelon form of the matrix can be given as follows:

$$R = \begin{bmatrix} 1 & 0 & 0 & \dfrac{37}{19} \\ 0 & 1 & 0 & \dfrac{-37}{57} \\ 0 & 0 & 1 & \dfrac{44}{57} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All the entries of the last row are zero. Hence, the determinant will be 0.

Approach 2:
If we perform the following row operation: $R_4 = 3R_1 - R_4$, we get a zero row. Thus, if A is in the reduced row echelon form of the given matrix, it will have at least one row of zeros. Hence, the determinant of the matrix will be 0.

10. If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution of the system of equations

$$\begin{aligned} 7x + 2y + z &= 7 \\ 3y - z &= 2 \\ -3x + 4y - 2z &= 1, \end{aligned}$$

then the value of $x + y + z$ is

Solution:

The matrix representation of the above system of linear equations can be given by:

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$$

Then, the augmented matrix can be given by:

$$\left[\begin{array}{rrr|r} 7 & 2 & 1 & 7 \\ 0 & 3 & -1 & 2 \\ -3 & 4 & -2 & 1 \end{array}\right]$$

Row reduced form of the above matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 16 \end{array}\right] \implies \begin{aligned} x &= -3 \\ y &= 6 \\ z &= 16 \end{aligned}$$

Thus, $x + y + z = 19$.

11. Let $A = [a_1, a_2, a_3, a_4]$ and M denote the reduced row echelon form of $A^T A$. The number of non-zero rows of M is [Ans: 1]

Solution:

$$A = \begin{bmatrix} 2 & 7 & 3 & 9 \end{bmatrix} \implies A^T = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 9 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 9 \end{bmatrix} \begin{bmatrix} 2 & 7 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 14 & 6 & 18 \\ 14 & 49 & 21 & 63 \\ 6 & 21 & 9 & 27 \\ 18 & 63 & 27 & 81 \end{bmatrix}$$

The reduced row echelon form of $A^T A$ is:

$$M = \begin{bmatrix} 1 & \frac{14}{4} & \frac{6}{4} & \frac{18}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the number of non-zero rows of M is 1.

12. If the graph of the polynomial $p(x) = a+bx+cx^2$ passes through the points $(2,4), (3,6), (4,10)$, then the value of $a - b + c$ is [Ans: 10]

Solution:

Given that $p(x) = a+bx+cx^2$ passes through the points $(2,4)$, $(3,6)$, and $(4,10)$, we have:

$$\begin{aligned} p(2) = 4 &\implies a + 2b + 4c = 4 \\ p(3) = 6 &\implies a + 3b + 9c = 6 \\ p(4) = 10 &\implies a + 4b + 16c = 10 \end{aligned}$$

Augmented matrix of the above system is as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 6 \\ 1 & 4 & 16 & 10 \end{array}\right]$$

Row reduced echelon form of the above matrix is:

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \implies \qquad \begin{aligned} a &= 6 \\ b &= -3 \\ c &= 1 \end{aligned}$$

Hence, $a - b + c = 6 - (-3) + 1 = 10$.

Comprehension Type Question:

The network in Figure: M2W2GA1 shows a proposed plan for flow of traffic around a park. All the streets are assumed to be one-way and the arrows denote the direction of flow of traffic. The plan calls for a computerized traffic light at the South Street. Let $2x_1, 3x_2, 2x_3,$ and x_4 denote the average number (per hour) of vehicles expected to pass through the connecting streets (e.g., $2x_1$ denote the average number (per hour) of vehicles expected to pass through the street connecting the North Street and West Street as shown in Figure: M2W2GA1). 400, 1000, 900, and c denote the average number (per hour) of vehicles expected to pass through West, North, East, and South Streets respectively.

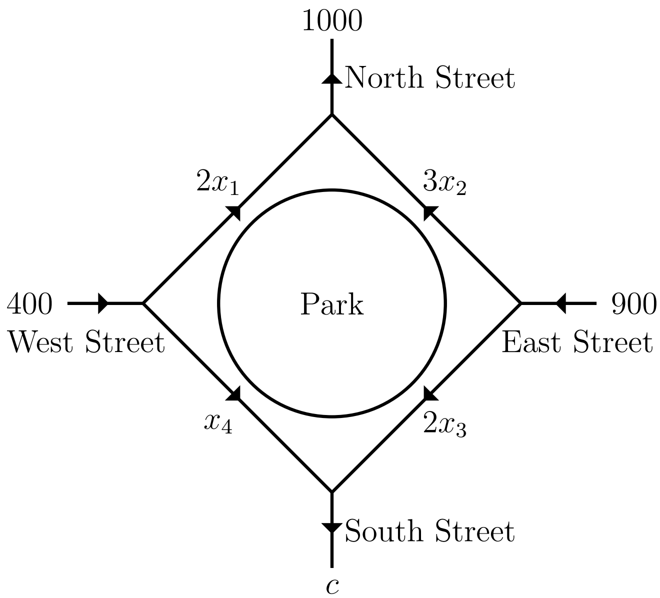

Figure: M2W2GA1

13. Which of the following options are correct?

◯ **Option 1:** The system of equations corresponding to the flow of expected traffic according to the given data above, will be

$$
\begin{aligned}
2x_1 + 3x_2 &= 1000 \\
3x_2 + 2x_3 &= 900 \\
2x_3 + x_4 &= c \\
2x_1 + x_4 &= 400
\end{aligned}
$$

◯ Option 2: The system of equations corresponding to the flow of expected traffic according to the given data above, will be

$$\begin{aligned} 2x_1 + 3x_2 &= 900 \\ 3x_2 + 2x_3 &= 1000 \\ 2x_3 + x_4 &= 400 \\ 2x_1 + x_4 &= c \end{aligned}$$

○ Option 3: The matrix representation of the system of equations corresponding to the flow of expected traffic according to the given data above is

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 900 \\ 1000 \\ 400 \\ c \end{bmatrix}$$

○ **Option 4:** The matrix representation of the system of equations corresponding to the flow of expected traffic according to the given data above is

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1000 \\ 900 \\ c \\ 400 \end{bmatrix}$$

Solution:

The total average number of vehicles expected to pass through the West street is 400. We are also given that:

$2x_1 =$ number of cars passing through the street connecting the West and North Streets
$x_4 =$ number of cars passing through the street connecting the West and South Streets

Thus, we have:

$$\text{Total Vehicles through West st} = 2x_1 + x_4 = 400$$

Similarly, we can arrive at the equations for other streets. We get:

$$\begin{aligned} 2x_3 + x_4 &= c \\ 3x_2 + 2x_3 &= 900 \\ 2x_1 + 3x_2 &= 1000 \end{aligned}$$

The matrix representation of the above system is:

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1000 \\ 900 \\ c \\ 400 \end{bmatrix}$$

Hence, the correct options are **Option 1** and **Option 4**.

14. How many vehicles are expected to pass through the South Street per hour on an average? [Answer: 300]

    Solution:

    The total average number of expected vehicles passing through the South Street is c. Thus, we need to solve for the value of c. The system of equation is as follows:

$$2x_1 + 3x_2 + 0x_3 + 0x_4 = 1000 \tag{1}$$
$$0x_1 + 3x_2 + 2x_3 + 0x_4 = 900 \tag{2}$$
$$0x_1 + 0x_2 + 2x_3 + x_4 = c \tag{3}$$
$$2x_1 + 0x_2 + 0x_3 + x_4 = 400 \tag{4}$$

Subtracting equation (4) from equation (1), we get:

$$0x_1 + 3x_2 + 0x_3 - x_4 = 600 \tag{5}$$

Subtracting equation (5) from equation (2), we get:

$$2x_3 + x_4 = 300 \tag{6}$$

Thus, from (3) and (6),

$$2x_3 + x_4 = c = 300$$

Hence, $c = 300$. Thus, the number of vehicles expected to pass through South Street on average is 300.

15. Match the names of the street in Column A with the maximum and minimum number of vehicles expected to pass through the street on an average (per hour) in Column B and Column C, respectively; in Table M2W2GA1.

	Name of the connecting street		The maximum number of vehicles expected to pass through the street (per hour)		The minimum number of vehicles expected to pass through the street (per hour)
	Column A		Column B		Column C
a)	Connecting West and North street	i)	300	1)	0
b)	Connecting East and North street	ii)	300	2)	100
c)	Connecting East and South street	iii)	400	3)	600
d)	Connecting West and South street	iv)	900	4)	0

Table : M2W2GA1

- ◯ Option 1: d → i → 2
- ◯ Option 2: b → iii → 3.
- ◯ **Option 3:** b → iv → 3.
- ◯ **Option 4:** d → ii → 1.
- ◯ **Option 5:** a → iii → 2.
- ◯ Option 6 : a → iii → 4.
- ◯ **Option 7:** c → i → 4.

Solution:

The system of equations is:

$$2x_1 + 3x_2 + 0x_3 + 0x_4 = 1000 \quad (1)$$
$$0x_1 + 3x_2 + 2x_3 + 0x_4 = 900 \quad (2)$$
$$0x_1 + 0x_2 + 2x_3 + x_4 = 300 \quad (3)$$
$$2x_1 + 0x_2 + 0x_3 + x_4 = 400 \quad (4)$$

From (3), we have: $2x_3 = 300 - x_4$, where $2x_3$ represents the number of vehicles passing through the street connecting the East and the South Streets.

Observe that the average number of vehicles expected to pass through a street cannot be negative. That is, $2x_3$ cannot be negative. Thus, x_4 cannot be greater than 300.

$$x_4 \leq 300$$

which means that the maximum number of vehicles passing through the street that connects West and South Streets(x_4) is 300.

Also, observe that the number of vehicles passing through the street that connects West and South Streets(x_4) can be 0. That is, the minimum number of vehicles can be 0. Thus,

$$0 \leq x_4 \leq 300 \tag{5}$$

Hence, **option 4** is correct.

Similarly, with equation (2), we have:

$$\begin{aligned}
2x_3 &= 300 - x_4 \\
3x_2 &= 900 - 2x_3 \\
&= 900 - 300 + x_4 \\
&= 600 + x_4
\end{aligned}$$

Since x_4 can be at minimum 0, and at maximum 300,

$$600 \leq 3x_2 \leq 900 \tag{6}$$

Hence, **option 3** is correct.

Similarly, the average number of vehicles expected to pass through the West street is 400.

$$2x_1 + x_4 = 400 \qquad \text{(...from Question 13)}$$

Since x_4 can be at minimum 0, and at maximum 300,

$$100 \leq 2x_1 \leq 400 \tag{7}$$

Hence, **option 5** is correct.

Similarly, from equation (3), we have:

$$0 \leq 2x_3 \leq 300$$

Hence, **option 7** is correct.