Week-2

Mathematics for Data Science - 2

Solutions of System of Linear Equations

Graded Assignment

1 Multiple Select Questions (MSQ)

1. In a particular year, the profit (in lakhs of ₹) of Star Fish company is given by the polynomial $P(x) = ax^2 + bx + c$ where x denotes the number of months since the beginning of the year (i.e., $x = 1$ denotes January, $x = 2$ denotes February, and so on). In January and February the company made a loss of ₹45(in lakhs), and ₹19(in lakhs) respectively, and in March the company made a profit of ₹3(in lakhs). Let the loss be represented by negative of profit.
Choose the correct set of options based on the given information.

   - Option 1: The maximum profit will be in the month of May.
   - **Option 2:** The maximum profit will be in the month of August.
   - **Option 3:** The maximum monthly profit amount is ₹53 lakh.
   - Option 4: The maximum monthly profit amount is ₹35 lakh.

Solution: In January the company made a loss of 45(in lakhs), that is,

$$P(1) = -45 \Rightarrow a+b+c = -45$$

Similarly, $P(2) = -19 \Rightarrow 4a+2b+c = -19$ and

$$P(3) = 3 \Rightarrow 9a+3b+c = 3$$

We have the following set of equations:

$$a+b+c = -45$$
$$4a+2b+c = -19$$
$$9a+3b+c = 3.$$

The augmented matrix is written as:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array}\right]$$

The row reduced echlon form of the above matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{array}\right]$$

Thus, we find that $a=-2, b=32, c=-75$. Therefore, the Polynomial is $P(x)=-2x^2+32x-75$. We have to find the point of global maximum and the corresponding maximum value of the function.

$$P'(x)=-4x+32$$

Critical Point of $P(x)$ is $P'(x)=0 \Rightarrow -4x+32=0 \Rightarrow x=8$

Note that $P''(8)=-4<0$. Therefore, "8" is a point of local maximum. Since "8" is our only critical point, it is a global maximum.

So $P(8)=53$ which is the maximum monthly profit and it happens in the month of August $(x=8)$.

2. If A be a 3×4 matrix and b be a $\overset{3\times 1}{\cancel{1 \times 4}}$ matrix, then choose the set of correct options.

- ◯ **Option 1:** If $(A|b)$ be the augmented matrix and $(A'|b')$ be the matrix obtained from $(A|b)$ after a finite number of elementary row operations then the system $Ax = b$ and the system $A'x = b'$ have the same set of solutions.
- ◯ Option 2: If $(A'|b')$ is the reduced row echelon form of $(A|b)$ then the system $A'x = b'$ has at least one solution.
- ◯ **Option 3:** If $(A'|b')$ is the reduced row echelon form of $(A|b)$, then A' is also in reduced row echelon form.
- ◯ **Option 4:** If $(A'|b')$ is the reduced row echelon form of $(A|b)$ and there is no row such that the only non zero entry lies in the last column of $(A'|b')$ then the system $Ax = b$ has at least one solution.

Solution:

Option 1: We know from the Gauss elimination method that any number of elementary row operation on an augmented matrix does not alter the solutions of $Ax=b$.

Hence, option-1 is correct.

Option-2. Take $(A|b) = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 2 & 4 & 6 & 0 & 1 \\ 3 & 6 & 9 & 0 & 1 \end{array}\right]$

Then $(A'|b') = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The system of $A'x=b'$ has no solution.

Option 3: Observe that if we transform the augmented into reduced row echolon form, the coefficient matrix "A" also get transformed into reduced row echlon form.

Option 4: If $(A'|b')$ is the reduced row echlon form of $(A|b)$ and there is no row such that the only non-zero entry lies in the last column of $(A'|b')$, then all variable can be dependent or there can be at least one variable which will be independent. In both the cases the system of linear equations has at least one solution.

3. Choose the set of correct options

- ◯ **Option 1:** If the sum of all the elements of each row of a matrix A is 0, then A is not invertible.
- ◯ **Option 2:** If E is a matrix of order 3×3 obtained from the identity matrix by a finite number of elementary row operations then E is invertible.
- ◯ Option 3: Any system of linear equations has at least one solution.
- ◯ Option 4: If A is a matrix of order 3×3 and $det(A) = 3$ then $det(Adj(A)) = 3$.
- ◯ **Option 5:** If A is a matrix of order 3×3 and $det(A) = 3$ then $det(Adj(A)) = 9$.

Solution:

Option 1: Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Such that
$$a_{11}+a_{12}+a_{13}=0$$
$$a_{21}+a_{22}+a_{23}=0$$
$$a_{31}+a_{32}+a_{33}=0$$

$$Det(A) = Det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = Det\begin{bmatrix} a_{11}+a_{12}+a_{13} & a_{12} & a_{13} \\ a_{21}+a_{22}+a_{23} & a_{22} & a_{23} \\ a_{31}+a_{32}+a_{33} & a_{32} & a_{33} \end{bmatrix}$$

$$= Det\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} = 0. \Rightarrow A \text{ is not invertible.}$$

<u>Option 2:</u> $\det(E) = c \det(I)$ where $c \neq 0$

$= c \neq 0$

$\Rightarrow E$ is invertible.

<u>Option 3:</u> No. <u>Ex:</u> $2x_1 + 3x_2 = 0$

$2x_1 + 3x_2 = 1$.

<u>Options 4 and 5.</u> Let A be a square matrix of order 3.

We know $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

$\Rightarrow \det(A) A^{-1} = \operatorname{adj}(A)$

$\Rightarrow \det(\det(A) \cdot A^{-1}) = \det(\operatorname{adj}(A))$

$\Rightarrow \det(A)^3 \det(A^{-1}) = \det(\operatorname{adj}(A))$.

$\Rightarrow \det(A)^3 \det(A)^{-1} = \det(\operatorname{adj}(A))$

$\Rightarrow \underset{\overset{\parallel}{3^2 = 9.}}{\det(A)^2} = \det(\operatorname{adj}(A))$.

$\boxed{\Rightarrow \det(\operatorname{adj}(A)) = 9.}$

4. Ramya bought 1 comic book, 2 horror books, and 1 novel from a bookshop which cost her ₹1000. Romy bought 2 comic books, 5 horror books, and 1 novel which cost him ₹2000. Farjana bought 4 comic books, 5 horror books, and c novels from a shop which cost her ₹d. If x_1, x_2, and x_3 represent the price of each comic book, horror book, and novel, respectively, then choose the set of correct options.

- ○ **Option 1:** The matrix representation to find x_1, x_2 and x_3 is

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 4 & 5 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ d \end{bmatrix}$$

- ○ **Option 2:** The matrix representation to find x_1, x_2 and x_3 is

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & c \end{bmatrix} = \begin{bmatrix} 1000 & 2000 & d \end{bmatrix}$$

- ○ Option 3: The matrix representation to find x_1, x_2 and x_3 is

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & c \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1000 & 2000 & d \end{bmatrix}$$

- ○ Option 4: If Farjana tries to find x_1, x_2, and x_3 using appropriate matrix representation by taking $c = 2$ and $d = 4000$, then the price of each comic book that she thus arrives at, will not be unique.
- ○ **Option 5:** If $c = 7$ and $d = 4000$, then the price of each comic book cannot be determined from this data.
- ○ **Option 6:** If $c = 7$ and $d = 3000$, then the shopkeeper has made a mistake.
- ○ **Option 7:** If $c = 2$ and $d = 3000$, then the price of each comic book can be determined from the data.

Solution: Given x_1, x_2 and x_3 represent the price of each comic book, horror book, and novel, respectively.

So, prices of 1 comic book, 2 horror books, and 1 novel are x_1, $2x_2$ and x_3 respectively and Ramya paid 1000 for these book. This gives us

$$x_1 + 2x_2 + x_3 = 1000$$

Similarly, we can get for Romy and Farjana.

The three equations for the total price of all the books purchased by Ramya, Romy and Farjana respectively as the following:

$$x_1 + 2x_2 + x_3 = 1000$$
$$2x_1 + 5x_2 + x_3 = 2000$$
$$4x_1 + 5x_2 + cx_3 = d$$

Option 1: The matrix representation "$AX=b$" of the above system of linear equation is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 4 & 5 & c \end{bmatrix} \overset{=X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 1000 \\ 2000 \\ d \end{bmatrix} = b.$$

Option 2/3: If $AX = b \Rightarrow (AX)^T = b^T \Rightarrow X^T A^T = b^T$ $\left[\because (AB)^T = B^T A^T\right]$

By taking the transpose on both sides, we can write the above as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & c \end{bmatrix} = \begin{bmatrix} 1000 & 2000 & d \end{bmatrix}$$

* The augmented matrix of the above system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{array}\right]$$

The row echlon form of the above augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & c-7 & d-4000 \end{array}\right]$$

Option-4: If $c=2, d=4000$, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right] \Rightarrow \text{unique solution.}$$

Price of each comic book is unique.

Option 5: If $c=7$, $d=4000$, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{Infinitely many solutions.}$$

one independent variable

The price of each comic book cannot determined from this data.

Option 6: If $c=7$, $d=3000$, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1000 \end{array}\right] \Rightarrow \text{no solution.}$$

non-zero

Zero row

Hence the shopkeeper made a mistake.

Option 7: If $c=2$, $d=3000$ we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & -1000 \end{array}\right] \Rightarrow \text{unique solution.}$$

The price of each comic book can be determined from this data.

2 Numerical Answer Type (NAT):

5. Consider the system of equations given below:

$$\frac{x^{2021}}{2021} + \frac{y^{2021}}{2021} - \frac{z^{2021}}{2021} = \pi \quad \text{——} \; ①$$

$$\frac{x^{2021}}{2021} + \frac{y^{2021}}{2021} - \frac{z^{2021}}{2021} = e \quad \text{——} \; ②$$

$$\frac{x^{2021}}{2021} - \frac{y^{2021}}{2021} + \frac{z^{2021}}{2021} = 1729. \quad \text{——} \; ③$$

The number of solutions of the above system of equations is

Solution: Left side of first two equations are same. This give us $\pi = e$, which is not true. Hence, the system has no solution.

6. Let the reduced row echelon form of a matrix A be

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}.$$

The first, second, third and fourth columns of A are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, respectively. The value of $a+b+c$ is

Solution:

$$A = \begin{bmatrix} 1 & 3 & a & -1 \\ 0 & 2 & b & 0 \\ -1 & 1 & c & 0 \end{bmatrix}$$

To transform this matrix into reduced row echlon form. We do the following row operations in the same sequence:

$$R_3+R_1,\ R_2/2,\ R_3-4R_2,\ R_1-3R_2,\ R_3/-6,\ R_2-R_3,\ R_1+3R_3$$

The transformed matrix is
$$\begin{bmatrix} 1 & 0 & \frac{a-b-c}{2} & -1/2 \\ 0 & 1 & \frac{c+b+a}{6} & -1/6 \\ 0 & 0 & \frac{-c-a+2b}{6} & 1/6 \end{bmatrix}$$

Compare the above matrix with the reduced row echlon form of the matrix A. We get

$$\frac{a-b-c}{2} = 0, \quad \frac{c+b+a}{6} \overset{6}{=} 0, \quad \frac{-c-a+2b}{6} = 1$$

$$\Rightarrow \quad \begin{aligned} a-b-c&=0\\ a+b+c&=0\\ -a+2b-c&=6 \end{aligned}$$

The above system has unique solution, that is,

$$a=0,\ b=2,\ c=-2.$$

Hence, $a+b+c=0$.

Different approach:

$A=\begin{bmatrix} 1 & 3 & a & -1 \\ 0 & 2 & b & 0 \\ -1 & 1 & c & 0 \end{bmatrix}$ and the row reduced echolon of A is $A'=\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 1/6 \end{bmatrix}$.

We know $Ax=0$ and $A'x=0$ have same set of solutions.

Let $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a solution of $Ax=0$ and $A'x=0$.

* From $A'x=0$, we get

$$\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 1/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} x_1 &= \tfrac{1}{2}x_4 \\ x_2 &= \tfrac{1}{6}x_4 \\ x_3 &= -\tfrac{1}{6}x_4 \end{aligned}\right\} \text{— eq. } (1)$$

* From $Ax=0$, we get

$$\begin{bmatrix} 1 & 3 & a & -1 \\ 0 & 2 & b & 0 \\ -1 & 1 & c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1+3x_2+ax_3-x_4 &= 0 \quad \text{— } (2) \\ 2x_2+bx_3 &= 0 \quad \text{— } (3) \\ -x_1+x_2+cx_3 &= 0 \quad \text{— } (4) \end{aligned}$$

From eq. (1) and eq. (2):

$$\tfrac{1}{2}x_4 + \tfrac{1}{2}x_4 - \tfrac{a}{6}x_4 - x_4 = 0 \Rightarrow \tfrac{a}{6}x_4 = 0$$
$$\Rightarrow \boxed{a=0}$$

Note: We can choose a "x" such that $x_4 \neq 0$

From eq (1) and eq (3):

$$\tfrac{1}{3}x_4 - \tfrac{b}{6}x_4 = 0 \Rightarrow \frac{x_4}{3} = \frac{bx_4}{6} \Rightarrow b=2$$

Similarly, from eq (1) and eq (4); we get $c=-2$.

Hence, $a+b+c=0$.

7. Three mobile shops- shop A, shop B and shop C, sell three brands of mobile phones: brand R, brand S and brand T. In a week, shop A sold 1 mobile phone of brand R, $3k$ mobile phones of brand S, and $3k+4$ mobile phones of brand T. Shop B sold 1 mobile phone of brand R, $k+4$ mobile phones of brand S, and $4k+2$ mobile phones of brand T. Shop C sold 1 mobile phone of brand R, $2k+2$ mobile phones of brand S, and $3k+4$ mobile phones of brand T (assume, $k \neq 2$). Assume that the price of a given model of a given brand is the same in all the shops. Shop A, shop B, and shop C earned ₹61, ₹65 and ₹66 (in thousands), respectively by selling these three brands of mobile phones. If the price of each mobile phone of brand S is ₹5 (in thousands), then what is the price of each mobile phone of brand T (in thousands)? [Note: Suppose the price comes out to be 20,000, then the answer should be 20]

Solution: Let x_R, x_S, x_T be the price of each mobile brand R, S, T respectively.

So, In a week, Shop A earned $(x_R + 3Kx_S + (3K+4)x_T)$ by selling 1 mobile phone of brand R, 3K mobiles of brand S, and 3K+4 mobile phones of brand T which is equal to 61 (in thousands). Similarly we can calculate amount in a week for the shop B and C.

The system of equations representating the total earings of the shops A, B, C can be represented by the following.

$$x_R + 3Kx_S + (3K+4)x_T = 61$$

$$x_R + (K+4)x_S + (4K+2)x_T = 65$$

$$x_R + (2K+2)x_S + (3K+4)x_T = 66$$

So augmented matrix of the above System of linear equation is $\left[\begin{array}{ccc|c} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{array}\right]$

Row echolon form of the above matrix is

$$\left[\begin{array}{ccc|c} 1 & 3k & 3k+4 & 61 \\ 0 & 1 & 0 & 5/2-k \\ 0 & 0 & 1 & 6/2-k \end{array}\right]$$

Note that $k \neq 2$. Then from the above matrix, we can write $x_S = \frac{5}{2-k}$, $x_T = 6/2-k$.

Since the price of each mobile phone of brand S is 5(in thousands), i.e, $x_S = 5$.

After substituting the value $x_S = 5$ in the equation $x_S = 5/2-k$, we got $k=1$. Therefore, $x_T = 6$.

Hence, the price of each mobile phone of brand T is 6(in thousands).

8. The number of solutions of the system of equations

$$\begin{aligned} 2x + 3y + 5z &= 1 \\ x + 2y + 3z &= 1 \\ x + y + 2z &= 7 \end{aligned}$$

is

Solution: The matrix representation of the above system of linear equations is given by

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

Now, the augmented matrix of the above system of linear equations is

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 7 \end{array}\right]$$

Row reduced echelon matrix of the above matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \text{no solution.}$$

zero nonzero

Hence the number of solution of the above system "0".

9. A is the reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 1 & 5 & 11 \\ 2 & 2 & 7 & 8 \\ 3 & 9 & 0 & 0 \end{bmatrix}.$$

Then determinant of A is

Solution: The row reduced echelon form of the above matrix is $M = \begin{bmatrix} 1 & 0 & 0 & \frac{37}{19} \\ 0 & 1 & 0 & -\frac{37}{57} \\ 0 & 0 & 1 & \frac{44}{57} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. All the entries in the Fourth row are zero. Therefore $\det(M) = 0$.

10. If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution of the system of equations

$$\begin{aligned} 7x + 2y + z &= 7 \\ 3y - z &= 2 \\ -3x + 4y - 2z &= 1, \end{aligned}$$

then the value of $x + y + z$ is

Solution: The matrix repxesentation of the above System linear equations is

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$$

Now the augmented matrix is

$$\left[\begin{array}{ccc|c} 7 & 2 & 1 & 7 \\ 0 & 3 & -1 & 2 \\ -3 & 4 & -2 & 1 \end{array}\right]$$

Row reduced echelon form of the above matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 16 \end{array}\right] \Rightarrow x = -3,\ y = 6,\ z = 16$$

Hence, $x + y + z = 19$.

11. Let $A = [2, 7, 3, 9]$ and M denote the reduced row echelon form of $A^T A$. The number of non-zero rows of M is

Solution: $A = [2, 7, 3, 9] \Rightarrow A^T = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 9 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 9 \end{bmatrix} [2, 7, 3, 9] = \begin{bmatrix} 4 & 14 & 6 & 18 \\ 14 & 49 & 21 & 63 \\ 6 & 21 & 9 & 27 \\ 18 & 63 & 27 & 81 \end{bmatrix}$$

The row reduced echelon of $A^T A$ is

$$M = \begin{bmatrix} 1 & 14/4 & 6/4 & 18/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows of M is 1.

12. If the graph of the polynomial $p(x) = a+bx+cx^2$ passes through the points $(2,4)$, $(3,6)$, $(4,10)$, then the value of $a - b + c$ is

Solution: Given that $p(x)=a+bx+cx^2$ passes through $(2,4), (3,6), (4,10)$.

$$P(2)=4 \Rightarrow a+2b+4c=4$$
$$P(3)=6 \Rightarrow a+3b+9c=6$$
$$P(4)=10 \Rightarrow a+4b+16c=10$$

Augmented matrix of the above system of linear equations is

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 6 \\ 1 & 4 & 16 & 10 \end{array}\right]$$

Row reduced echelon form of the above matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow a=6, b=-3, c=1.$$

Hence, $a-b+c=10$.

3 Comprehension Type Question:

The network in Figure: M2W5GA1 shows a proposed plan for flow of traffic around a park. All the streets are assumed to be one-way and the arrows denote the direction of flow of traffic. The plan calls for a computerized traffic light at the South Street. Let $2x_1, 3x_2, 2x_3,$ and x_4 denote the average number (per hour) of vehicles expected to pass through the connecting streets (e.g., $2x_1$ denote the average number (per hour) of vehicles expected to pass through the street connecting the North Street and West Street as shown in Figure: M2W5GA1). 400, 1000, 900, and c denote the average number (per hour) of vehicles expected to pass through West, North, East, and South Streets respectively.

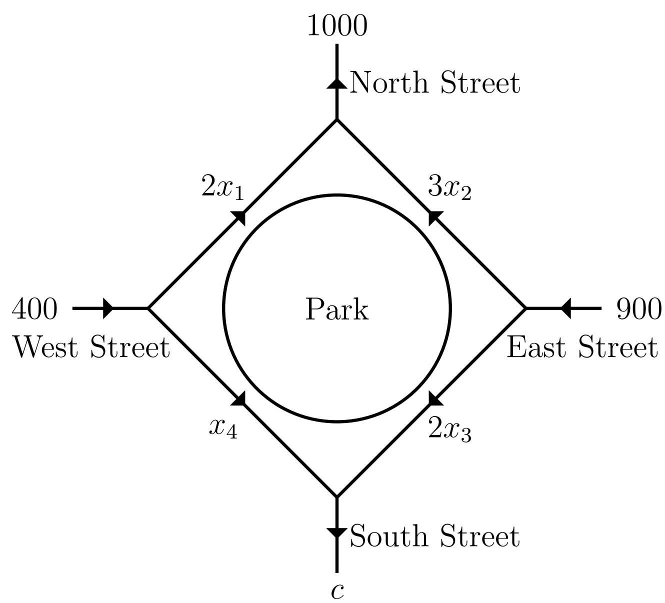

Figure: M2W5GA1

13. Which of the following options are correct?

○ **Option 1:** The system of equations corresponding to the flow of expected traffic according to the given data above, will be

$$\begin{aligned} 2x_1 + 3x_2 &= 1000 \\ 3x_2 + 2x_3 &= 900 \\ 2x_3 + x_4 &= c \\ 2x_1 + x_4 &= 400 \end{aligned}$$

○ Option 2: The system of equations corresponding to the flow of expected traffic according to the given data above, will be

$$\begin{aligned} 2x_1 + 3x_2 &= 900 \\ 3x_2 + 2x_3 &= 1000 \\ 2x_3 + x_4 &= 400 \\ 2x_1 + x_4 &= c \end{aligned}$$

○ Option 3: The matrix representation of the system of equations corresponding to the flow of expected traffic according to the given data above is

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 900 \\ 1000 \\ 400 \\ c \end{bmatrix}$$

○ **Option 4:** The matrix representation of the system of equations corresponding to the flow of expected traffic according to the given data above is

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1000 \\ 900 \\ c \\ 400 \end{bmatrix}$$

Solution: The average number of vehicles expected to pass through West Street is 400. $2x_1$ and x_4 are the average number of vechicels expected to pass through the street connecting West and North and West and South streets respectively.

Hence, $2x_1 + x_4 = 400$.

Similarly, we arrive at the other three equations

$$2x_3 + x_4 = c$$
$$3x_2 + 2x_3 = 900$$
$$2x_1 + 3x_2 = 1000.$$

* Matrix representation of the above system is

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1000 \\ 900 \\ c \\ 400 \end{bmatrix}.$$

14. How many vehicles are expected to pass through the South Street per hour on an average?

Solution: According to the question we need to find the value of c.

The system of linear equations is:

$$2x_1+3x_2=1000 \quad \text{—} \ (1)$$
$$3x_2+2x_3=900 \quad \text{—} \ (2)$$
$$2x_3+x_4=c \quad \text{—} \ (3)$$
$$2x_1+x_4=400 \quad \text{—} \ (4)$$

eq(4) − eq(3): $2x_1-2x_3=400-c$

eq(1) − eq(2): $2x_1-2x_3=100$

$$\Rightarrow 400-c=100$$
$$\Rightarrow \boxed{c=300}$$

15. Match the names of the street in Column A with the maximum and minimum number of vehicles expected to pass through the street on an average (per hour) in Column B and Column C, respectively; in Table M1W5GA1.

	Name of the connecting street		The maximum number of vehicles expected to pass through the street (per hour)		The minimum number of vehicles expected to pass through the street (per hour)
	Column A		Column B		Column C
a)	Connecting West and North street	i)	300	1)	0
b)	Connecting East and North street	ii)	300	2)	100
c)	Connecting East and South street	iii)	400	3)	600
d)	Connecting West and South street	iv)	900	4)	0

Table : M1W5GA1

- ◯ Option 1: d → i → 2
- ◯ Option 2: b → iii → 3.
- ◯ **Option 3:** b → iv → 3.
- ◯ **Option 4:** d → ii → 1.
- ◯ **Option 5:** a → iii → 2.
- ◯ Option 6 : a → iii → 4.
- ◯ **Option 7:** c → i → 4.

Solution: The system of linear equations is:

$$2x_1+3x_2=1000 \quad — (1)$$

$$3x_2+2x_3=900 \quad — (2)$$

$$2x_3+x_4=300 \quad — (3)$$

$$2x_1+x_4=400. \quad — (4)$$

From equation (3): $2x_3 = 300 - x_4$.

Observe that the average number of vechicels expected to pass can not be negative, that means x_4 can not be greater than 300, that is, $x_4 \le 300$.

Therefore, the maximum number of vechicels expected to pass through West Street to South street is 300. But the minimum number of vechicels expected to pass through West to South can be 0.

$$\text{So, } 0 \le x_4 \le 300.$$

* equation (3) $\Rightarrow 2x_3 = 300 - x_4$

equation (2) $\Rightarrow 3x_2 = 900 - 2x_3$.

$$= 900 - 300 + x_4$$

$$= 600 + x_4$$

Therefore, $600 \le 3x_2 \le 900$, (as $0 \le x_4 \le 300$).

* Now, since the average number of vechicles expected to pass through West Street to South Street is 300, so the remaining 100 vehicles pass through West Street to North Street that means the minimum number of vehicles expected to pass through West to North is 100, i.e, $100 < 2x_1$.

From equation (4): $2x_1 = 400 - x_4 \Rightarrow 2x_1 \le 400$

So $100 \le 2x_4 \le 400$.

* From eq (3): $2x_3 = 300 - x_4$

$\Rightarrow 0 \le 2x_3 \le 300$.