Practice Problems on Group Theory

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problem 1:

Let G be a group of order pq, where p and q are distinct primes. Prove that G is abelian.

Answer: True. Abelian.

proof:

By Sylow's theorem:

Let n_p = number of Sylow p-subgroups of G. Then $n_p = 1 \pmod p$ and $n_p \mid q$.

Since q is prime $n_p = 1$ or q. But $n_p \equiv 1 \pmod p$

If $n_p = q$, then $q \equiv 1 \pmod p$, impossible if $p < q$. So, $n_p = 1$.

Sylow p-subgroup is unique and normal.

Call them P (order p) and Q (order q). Since both are normal and have co prime orders.

$$G = P \times Q.$$

The direct product of cyclic groups of coprime order is cyclic, and all cyclic groups are abelian. Therefore, G is abelian.

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problem 2 :

Prove that if G is a group of order p^2, where p is prime, then G is abelian if and only if it has $p+1$ subgroups of order p.

Answer : True.

~~Explanation~~

Proof : If G is abelian of order p^2, then by ~~clasifica~~ classification of infinite abelian groups:

$$G \cong \mathbb{Z}_{p^2} \text{ or } \mathbb{Z}_p \times \mathbb{Z}_p$$

$\mathbb{Z}_{p^2}$ has 1 subgroup of order p.

$\mathbb{Z}_p \times \mathbb{Z}_p$ has $p+1$ subgroups of order p.

Suppose, G has $p+1$ subgroups of order p. Then G cannot be ~~at~~ cyclic (cyclic gives 1 subgroup).

A non-cyclic group of order p^2 is abelian, isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Hence condition is necessary and sufficient.

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3. Let G be a finite group and H be a proper subgroup of G. Prove that ~~all~~ the union of all conjugates of H cannot be equal to G. $H \neq G$

<u>Answer</u>: False.

<u>proof:</u>

Consider H proper, $H \neq G$.

The union of all conjugates of H is

$$\bigcup_{g \in G} gHg^{-1}$$

Each conjugate has size $|H| < |G|$. Each element in the union belongs to atleast one conjugate.

If union $= G$, then every element lies in some conjugate. But consider Sylow p-subgroup. P of G that is not contained in ~~G~~ H. The union cannot cover all of G because G is strictly bigger than H and its conjugates.

Therefore, union of all conjugates of a proper subgroup cannot be whole G.

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problem 4 :

Let G be a group and N be a normal subgroup of G. If G/N is cyclic and N is cyclic, prove that G is abelian.

Ans: True.

Proof : Let $N = \langle a \rangle$ and $G/N = \langle bN \rangle$

~~any $x \in G$~~ Any element ~~for~~ in G is of the form $b^k n$ for $n \in N$.

Since N is normal and cyclic (hence abelian), and G/N is cyclic (hence abelian), commutators in G reduce to the identity : ~~for,~~

for, $x = b^k n_1$ ~~say for integers k, m~~

$y = b^m n_2$

$\cancel{n_1 = n^m}$

$\cancel{n_2 = n^k}$

$xy = yx$

because power of b commute modulo N and elements of N commute. Thus, G is abelian.

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Problem 5: Prove that,
In any group G, the set of elements of finite order form a subgroup of G.

Ans: True.

Let $T = \{g \in G \mid g \text{ has finite order}\}$

- identity $e \in T$.
- closure: if $a, b \in T$ with orders m and n, then $(ab)^{mn} = a^{mn} b^{mn} = ee = e$.
- Inverse: if $a \in T$, $a^m = e = (a^{-1})^m = e$.

Therefore T is a ~~gr~~ subgroup of G. (the torsion subgroup). ~~(the~~

problem 6:

Let G be a finite group and p be smallest prime diving $|G|$. Prove that ~~Aa~~ any subgroup of index p ~~is~~ in G is normal.

Ans: True.

Let, $|G| = n$, p the smallest prime diving n, and $H \leq G$ with $[G:H] = p$. The left ~~coset~~ coset action gives a homomorphism $\phi : G \to S_p$ (since $|G/H| = p$), but The image $\phi(G)$ has order dividing $p!$, but since p is the smallest prime diving $n = |G|$,

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and $\ker\phi \leq H$, the index argument shows $\phi(G)$ is a p-group, implying $|\phi(G)| = 1$ or p. In either case, $\ker\phi$ is the core of H, but the minimal index forces H to be normal.

7. Let G be a group and $a, b \in G$. Prove that if $a^4 = b^2$ and $ab = ba$, then $(ab)^6 = e$.

$$(ab)^6 = a^6 b^6 = a^4 a^2 b^2 b^4 = (a^4)(b^2)\, a^2 b^4$$

given $a^4 = b^2$

$$\Rightarrow (ab)^6 = b^2 b^2 a^2 b^4 = b^4 a^2 b^4$$

since $ab = ba$, ~~comp~~

commuting powers,

$$b^4 a^2 b^4 = a^2 b^8$$

Now, $b^2 = a^4$

$$\Rightarrow b^8 = (b^2)^4 = (a^4)^4 = a^{16}$$

combining with a^2 $\Rightarrow a^2 a^{16} = a^{18}$.

To get e, in finite group we assume orders align $(ab)^6 = e$

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(8) Let G be a group and H be a subgroup of G. prove that if $[G:H]=n$, then for any $x \in G$, $x^n \in N$.

Ans: True.

Proof:

If $[G:H]=n$, consider the action of G on the left cosets G/H by multiplication, including $\phi: G \to S_n$.

For $x \in G$, the order of xH in G/H divides n (by Lagrange, since ~~$[G/H]$~~ $= |G/H| = n$.)

Thus, $(xH)^n = H$, so $x^n \in H$.

9. ~~If~~ Let G be a finite group and p be a prime number. If G has exactly one subgroup of order p^k for each $k \le n$, where p^n divides $|G|$, prove that G has normal sylow p-subgroup.

Answer: True.

proof: Let $|G| = p^n m$ $(p \nmid m)$. The Sylow p-subgroups have order p^n. If there is exactly one subgroup of order p^n (given for $k=n$), then the number of sylow p-subgroups $n_p = 1$.

A unique Sylow p-subgroup is normal.

(as conjugates are also Sylow p-subgroups, so invariance under conjugation holds).

10:

Let G be a finite group and H be a subgroup of G. Prove that if $|G| = pm$ where p is prime and p does not divide m, and $|H| = p^n$, then H is normal in G.

Answer: True.

Proof:

Since $|G| = pm$, $p \nmid m$, the Sylow p-subgroup has order p.

By sylow's theorems, n_p divides m and $n_p \equiv 1 \pmod{p}$.

As $p \nmid m$, $n_p = 1$. Thus, there is a unique subgroup P of order p, normal in G.

Since $|H| = p^n$, and $p^n \leq p$, we have $n \leq 1$.

So:

* If $n=0$, $H = \{e\}$, normal.
* If $n=1$, $H = P$, normal.
* If $n > 1$, $p^n > p$, impossible since $p^n \nmid pm$.

Thus, H is normal.

H is either trivial or the unique Sylow p-subgroup. Both Normal.