Proof that $(\mathbb{Q}, +, \cdot)$ is a field.

Let $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, \cancel{p \neq} \ q \neq 0\}$ be the set of rational numbers, where each rational number is understood as the equivalence class of the pair (p, q) under the relation $\frac{p}{q} = \frac{p'}{q'} \iff pq' = p'q$.

Define addition and multiplication by the usual fraction rules.

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.$$

We verify the field axioms.

① Well-definedness.

If $\frac{p}{q} = \frac{p'}{q'}$ and $\frac{r}{s} = \frac{r'}{s'}$ (so, $pq' = p'q$ and $r's' = r's$) then

$$\frac{ps + rq}{qs} = \frac{p's' + r'q'}{q's'} \quad \text{and} \quad \frac{pr}{qs} = \frac{p'r'}{q's'},$$

because multiplying the numerators and denominators out and using the equalities $pq' = p'q$ and $rs' = r's$, shows the cross-products agree. Thus addition and multiplication depend only on the equivalence classe so the operations are well-defined on $\mathbb{Q}$.

② Closure

If $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ (with $q, s \neq 0$), then $ps+rq$ and pr are integers and $qs \neq 0$, so both $\frac{ps+rq}{qs}$ and $\frac{pr}{qs}$ are in $\mathbb{Q}$. Hence, $\mathbb{Q}$ is closed under $+$ and $\cdot$.

③ Associativity of + and ·

Associativity follows from associativity in $\mathbb{Z}$ and the fraction formulas. For example, for addition

$$\left(\frac{p}{q}+\frac{r}{s}\right)+\frac{t}{u} = \frac{ps+rq}{qs} + \frac{t}{u} = \frac{(ps+rq)u+tqs}{qsu}$$

$$= \frac{psu+rqu+tqs}{qsu}$$

and

$$\frac{p}{q}+\left(\frac{r}{s}+\frac{t}{u}\right) = \frac{p}{q}+\frac{ru+ts}{su} = \frac{p(su)+q(ru+ts)}{qsu}$$

$$= \frac{psu+rqu+tqs}{qsu}$$

so, the two expressions are equal. A similar (and simpler) check works for multiplication.

④ Commutativity of + and ·

Commutativity comes from commutativity in $\mathbb{Z}$:

$$\frac{p}{q}+\frac{r}{s} = \frac{ps+rq}{qs} = \frac{rq+ps}{sq} = \frac{r}{s}+\frac{p}{q},$$

and $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}=\frac{rp}{sq}=\frac{r}{s}\cdot\frac{p}{q}$.

⑤ Identities

⊛ Additive identity : $0=\frac{0}{1}$ satisfies $\frac{p}{q}+\frac{0}{1}=\frac{p\cdot 1+0\cdot q}{q\cdot 1}$

$=\frac{p}{q}$.

⊛ Multiplicative identity :

$1=\frac{1}{1}$ satifies $\frac{p}{q}\cdot\frac{1}{1}=\frac{p\cdot 1}{q\cdot 1}=\frac{p}{q}$.

Both identities lie in $\mathbb{Q}$.

⑥ Additive inverses :

For any $\frac{p}{q}\in\mathbb{Q}$, the element $-\frac{p}{q}=\frac{-p}{q}$ is in $\mathbb{Q}$ and

$$\frac{p}{q}+\frac{-p}{q}=\frac{pq+(-p)q}{q^2}=\frac{0}{q^2}=0.$$

So every element has an additive inverse.

⑦ Multiplicative inverses (for nonzero elements)

if $\frac{p}{q}\in\mathbb{Q}$ and $\frac{p}{q}\neq 0$, then $p\neq 0$. The multiplicative inverse is

$$\left(\frac{p}{q}\right)^{-1}=\frac{q}{p},$$

which lies in $\mathbb{Q}$ because $p\in\mathbb{Z}\setminus\{0\}$. Indeed,

$$\frac{p}{q}\cdot\frac{q}{p}=\frac{pq}{qp}=1.$$

(Well-definedness : if $\frac{p}{q} = \frac{p'}{q'}$ with $p, p' \neq 0$, then $pq' = p'q$ implies $\frac{q}{p} = \frac{q'}{p'}$).

⑧ Distributive Law

For any $\frac{p}{q}, \frac{r}{s}, \frac{t}{u} \in \mathbb{Q}$

$$\frac{p}{q}\left(\frac{r}{s}+\frac{t}{u}\right) = \frac{p}{q} \cdot \frac{ru+ts}{su} = \frac{p(ru+ts)}{qsu}$$

$$= \frac{pr}{qs} + \frac{pt}{qu} = \frac{pru+pts}{qsu}$$

So multiplication distributes over addition.

All field axioms (closure, associativity, commutativity of both operations, existence of additive and multiplicative identities, existence of additive inverses for every ~~moment~~ element, existence of multiplicative inverses for every nonzero element, and distributivity) are satisfied.

Therefore $(\mathbb{Q}, +, \cdot)$ is a field.