# Linearization at the Transmiter

Consider $x_i$ which is the $i^{th}$ element of $\mathbf{x}$

$$x_i = w_i^H \mathbf{s}, \quad \sigma_{x_i} = E[|x_i|^2] = \sigma_s^2 w_i^H w = w_i^H W$$

where $w_i^H$ is the $i^{th}$ row of $W$

$$\therefore x_i \sim \mathcal{CN}(0, \sigma_{x_i}^2)$$

Let $t_i = Q_{Tx}(x_i)$

where $Q_{Tx}$ is as in (1)

then $t_i = g_i^{Tx} x_i + d_i^{Tx}$

according to bussgang decomposition, where $g_i^{Tx}$, $d_i^{Tx}$ are the bussgang gain & uncorrelated non-gaussian distortion, respectively

$$g_i^{Tx} = E[Q'_{Tx}(x_i)] = \eta_{Tx} E[2\delta(x_i)] = \eta_{Tx} 2p(0) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{\sigma_{x_i}}} = \sqrt{\frac{2}{\pi}} \sigma_x^{-1/2}$$

where $p(x_i)$ is the pdf of $x_i$

Extending this :

$$t_1 = g_1^{Tx} x_1 + d_1^{Tx}$$
$$t_2 = g_2^{Tx} x_2 + d_2^{Tx}, \quad t = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix}$$
$$\vdots$$
$$t_N = g_N^{Tx} x_N + d_N^{Tx}$$

then $t = \begin{bmatrix} g_1^{Tx} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} \quad \begin{bmatrix} d_1^{Tx} \\ d_2^{Tx} \end{bmatrix}$

$$\begin{bmatrix} 0 & g_2^{Tx} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & g_N^{Tx} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_N^{Tx} \end{bmatrix}$$

$$\boxed{t = G_{Tx}\, \mathbf{x} + d_{Tx}}$$

$$G_{Tx} = \begin{bmatrix} g_1^{Tx} & & & \\ & g_2^{Tx} & & \\ & & \ddots & \\ & & & g_N^{Tx} \end{bmatrix} = \sqrt{\frac{2 n_{Tx}}{\pi}} \begin{bmatrix} \sigma_{x_1}^2 & & & \\ 0 & \sigma_{x_2}^2 & & \\ \vdots & & \ddots & \\ 0 & & & \sigma_{x_N}^2 \end{bmatrix}^{-1/2} =$$

$$= \sqrt{\frac{2 n_{Tx}}{\pi}}\, \mathrm{diag}^{-1/2} \left( \begin{bmatrix} \sigma_{x_1}^2 & E[x_1^* x_2] & \cdots & E[x_1^* x_N] \\ E[x_2^* x_1] & \sigma_{x_2}^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ E[x_N^* x] & \cdots & \cdots & \sigma_{x_N}^2 \end{bmatrix} \right) = \sqrt{\frac{2 n_{Tx}}{\pi}}\, \mathrm{diag}^{-1/2}(C_x)$$

$$\boxed{G_{Tx} = \sqrt{\frac{2 n_{Tx}}{\pi}}\, \mathrm{diag}^{-1/2}(C_x)}$$

where $C_x = E[\mathbf{x}\mathbf{x}^H] = WW^H$ as in (8)

* Deriving $C_t = E[tt^H]$

From G. Jacovitti and el. 2002:

$$C_t[k,m] = E[t_k t_m^*] =$$

$$E\left[Q_{Tx}(x_k)\, Q_{Tx}^*(x_m)\right] = \frac{2}{\pi} n_{Tx} \left(\arcsin\left[Re[\rho_{x_k x_m}]\right] + j \arcsin\left[Im[\rho_{x_k x_m}]\right]\right)$$

$$= C_t^R[k,m] + j\, C_t^I[k,m]$$

where $x_k \sim \mathcal{NC}(0, \sigma_{x_k}^2)$, $C_t[k,m]$ is the element of $C_t$ corresponding to row $k$ and column $n$ and

$$\rho_{x_k x_m} = \frac{E[x_k x_m^*]}{\sqrt{E[|x_k|^2]}\sqrt{E[|x_m|^2]}} = \frac{1}{\sqrt{E[|x_k|^2]}\sqrt{E[|x_m|^2]}}\left(Re\left[E[x_k x_m^*]\right] + j\, Im\left[E[x_k x_m^*]\right]\right) = Re[\rho_{x_k x_m}] + Im[\rho_{x_k x_m}]$$

Accordingly

$$C_t = C_t^R + j\, C_t^I$$

where

$$C_t^R = Re[C_t] \quad \text{and} \quad C_t^I = Im[C_t]$$

From the above we can present $C_t^R$ as:

$$C_t^R = Re[C_t] = \frac{2}{\pi} n_{Tx} \arcsin\left( Re \begin{bmatrix} \frac{E[x_1 x_1^*]}{E[|x_1|^2]} & \cdots & \frac{E[x_1 x_2^*]}{\sqrt{E[|x_1|^2]}\sqrt{E[|x_N|^2]}} \\ \vdots & \frac{E[x_2 x_2^*]}{E[|x_2|^2]} & \\ & & \ddots \\ \frac{E[x_N x_N^*]}{\sqrt{E[|x_N|^2]}\sqrt{E[|x_1|^2]}} & \cdots & \frac{E[x_N x_N^*]}{E[|x_N|^2]} \end{bmatrix} \right)$$

$$= \frac{2}{\pi} n_{Tx} \arcsin(A)$$, where $A$ is presented below:

$$A = \begin{bmatrix} (E[|x_1|^2])^{-1/2} & 0 & \cdots & 0 \\ 0 & (E[|x_2|^2])^{-1/2} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & (E[|x_N|^2])^{-1/2} \end{bmatrix} \begin{bmatrix} Re[E[x_1 x_1^*]] & & Re[E[x_1 x_N^*]] \\ & Re[E[x_2 x_2^*]] & \vdots \\ & \ddots & \\ Re[E[x_1 x_N^*]] & & Re[E[x_N x_N^*]] \end{bmatrix} \begin{bmatrix} E[|x_1|^2] & 0 & \cdots & 0 \\ 0 & E[|x_2|^2] & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & E|x_N| \end{bmatrix}^{-1/2}$$

$$\underbrace{\qquad\qquad}_{\mathrm{diag}^{-1/2}(C_x)} \quad \underbrace{\qquad\qquad}_{\mathrm{Re}[C_x]} \quad \underbrace{\qquad\qquad}_{\mathrm{diag}^{-1/2}(C_x)}$$

$$\therefore \quad C_t^R = \frac{2}{\pi} n_{Tx} \arcsin\left[ \mathrm{diag}^{-1/2}(C_x) \mathrm{Re}[C_x] \mathrm{diag}^{-1/2}(C_x) \right]$$

$C_t^I$ can be derived similarly resulting in :

$$C_t^I = \frac{2}{\pi} n_{Tx} \arcsin\left[ \mathrm{diag}^{-1/2}(C_x) \mathrm{Im}[C_x] \mathrm{diag}^{-1/2}(C_x) \right]$$

and $\quad C_t = C_t^R + j\, C_t^I \quad$ as in (14)

## Linearization at the Reciever

The linearization at the receiver is similar to that of the Tx when $y$ is approximated to be gaussian for large N according the the CLT.

## Tractable Approximation of the MSE

$$\varepsilon = \frac{1}{K} E[\| s - V^H r \|^2] = \frac{1}{K} E[(s - V^H r)^H (s - V^H r)] =$$

$$\frac{1}{K} E[ s^H s - s^H V^H r - r^H V s + r^H V V^H r ] =$$

$$= \frac{1}{K} E[ s^H s - 2\,\mathrm{Re}[s^H V^H r] + r^H V V^H r ] = \frac{1}{K} tr(E[ss^H]) + \frac{1}{K} tr(V^H E[rr^H] V)$$

$$- \frac{2}{K} tr(V^H E[rs^H])$$

$$= 1 + \frac{1}{K} tr(V^H C_r V) - \frac{2}{K} tr(V^H E[rs^H])$$

$$E[rs^H] = E[((G_{Rx}(\sqrt{p} H(G_{Tx} W s + d_{Tx}) + z) + d_{Rx}) s^H]$$

$$= \sqrt{p}\, G_{Rx} H G_{Tx} W E[ss^H]$$

$$\varepsilon = 1 + \frac{1}{K} tr(V^H C_r V) - \frac{2}{K} \sqrt{p}\, tr(Re[G_{Rx} H G_{Tx} W])$$

$$\nabla \varepsilon_V = \frac{1}{K} V^H C_r - \frac{\sqrt{p}}{K} \nabla_V (tr[V^H G_{Rx} H G_{Tx} W + (G_{Rx} H G_{Tx} W)^H V])$$

$$\nabla \varepsilon_V = \frac{2}{K} V^H - \frac{2\sqrt{p}}{K} (G_{Rx} H G_{Tx} W)^H$$

$$\nabla \varepsilon_V = 0 \Rightarrow \boxed{\hat{V} = \sqrt{p}(G_{Rx} H G_{Tx} W) C_r^{-1}}$$

## Channel Model

Discrete physical channel model [15]:

$$H = \sum_{k=1}^{L} \beta_k\, a_R(\phi_{R,k}, \theta_{R,k})\, a_T^H(\phi_{T,k}, \theta_{T,k})$$

$$= A_R(\phi_R, \theta_R)\, H_p\, A_T^H(\phi_T, \theta_T)$$

$$H \quad \cdots \quad (\beta \quad \cdots \quad \beta)$$

$$R_p = \mathrm{diag}(p_1, \ldots, p_L)$$

For a $\sqrt{N} \times \sqrt{N}$ ($\sqrt{M} \times \sqrt{M}$) planar array with half-wavelength spaced antennas placed with broadsides facing each other,

The steering vector is given as:

$$a_x(\phi_x, \theta) = \Big[1, e^{j\pi(\sin\theta_x \cos\phi_x + \sin\theta_x \sin\phi_x)}, \ldots,$$
$$\ldots e^{j\pi(\sin\theta_x \cos\phi_x + (\sqrt{N}-1)\sin\theta_x \sin\phi_x)}, \ldots,$$
$$\ldots e^{j\pi((\sqrt{N}-1)\sin\theta_x \phi_x + (\sqrt{N}-1)\sin\theta_x \sin\phi_x)}\Big]$$

where $x$ can be Tx or Rx & $M = N$.

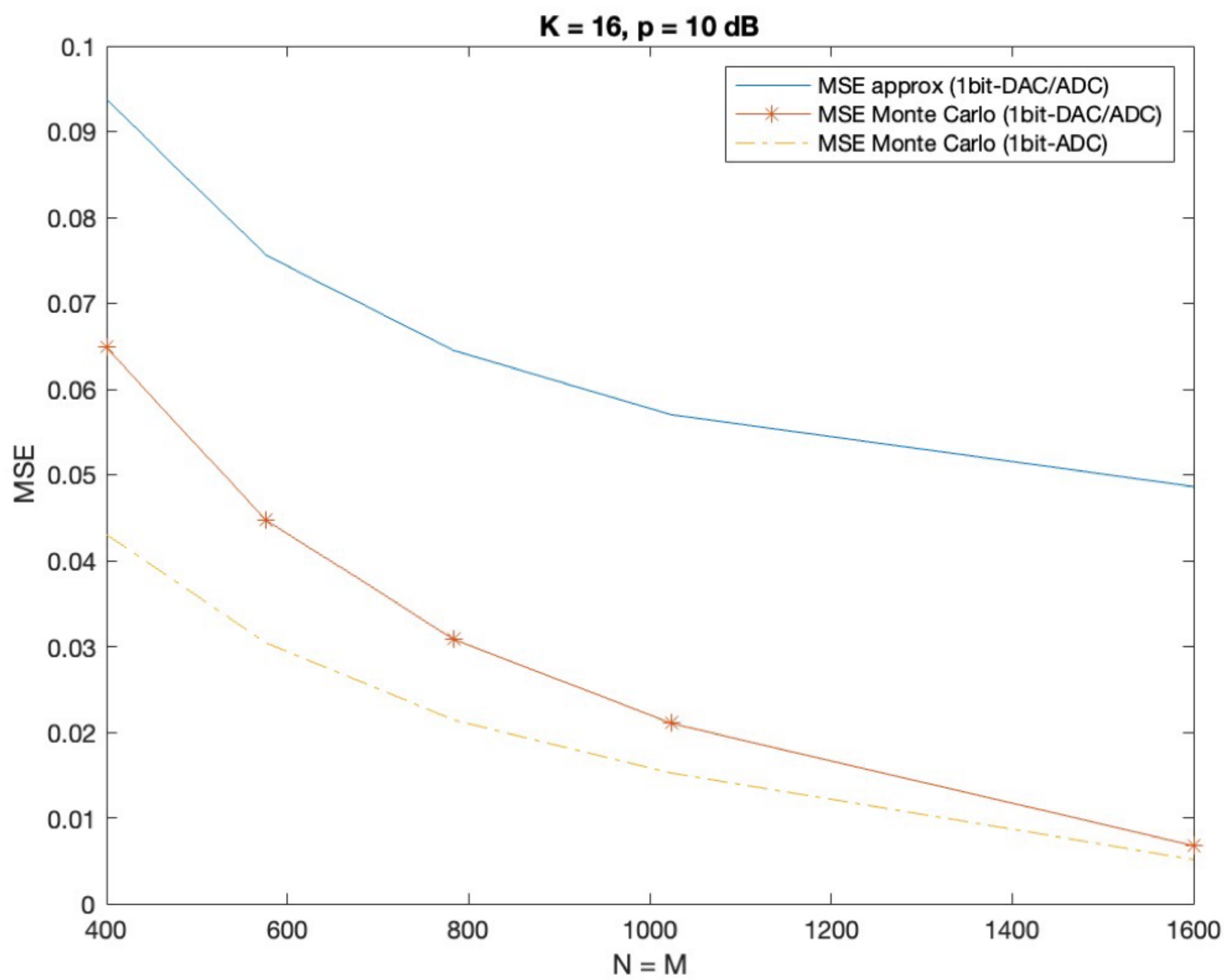
## Simulation Results

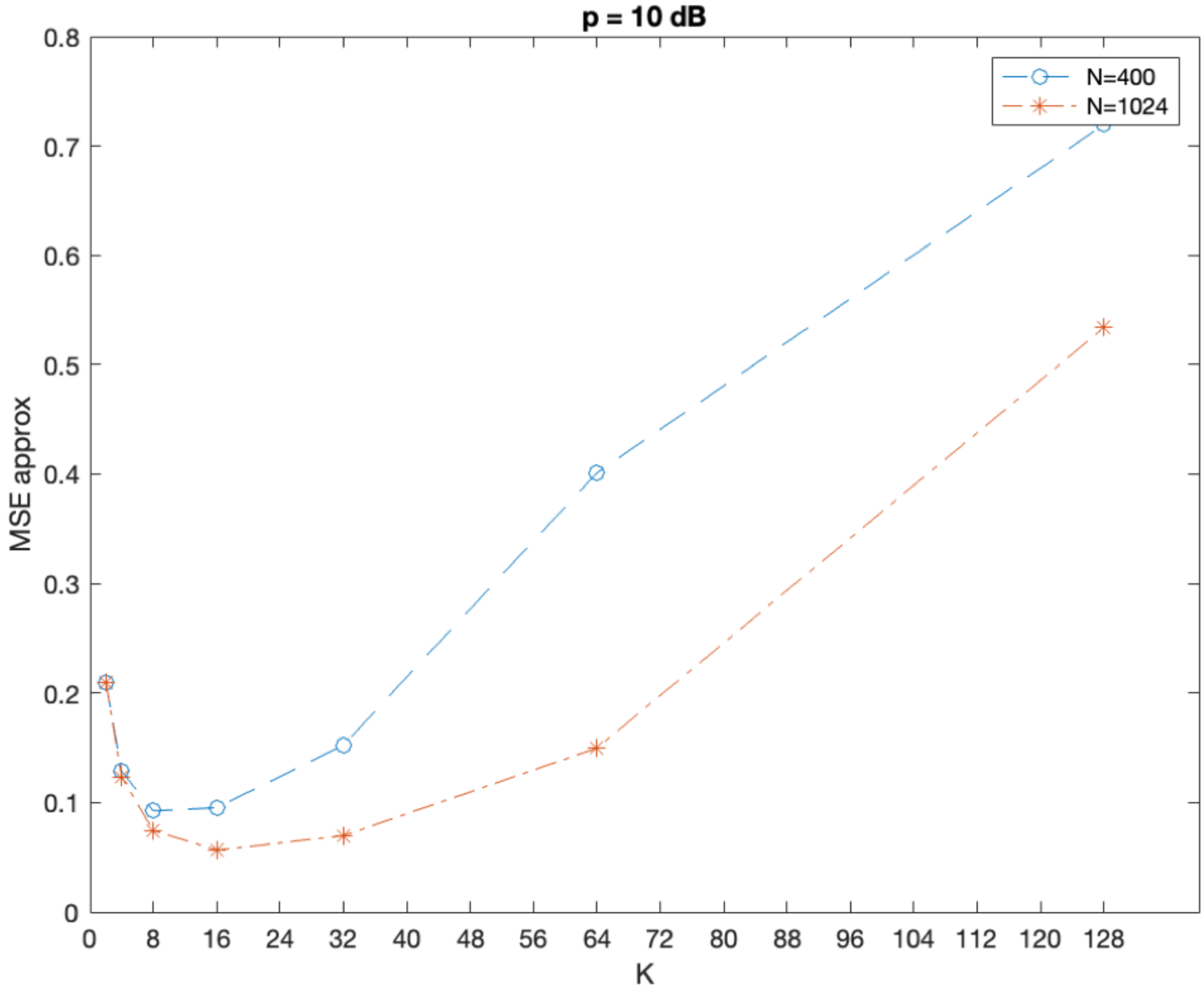
The system is simulated to generate Figure 2, 4, and 5.

For Figure 2, I only simulate $10^2$ independent channel realizations. Also for Monte Carlo simulations I simulated $2 \cdot 10^3$ as with $10^3$, the sample autocorrelation $C_{rr}$ is badly conditioned yielding problems with $C_{rr}^{-1}$.

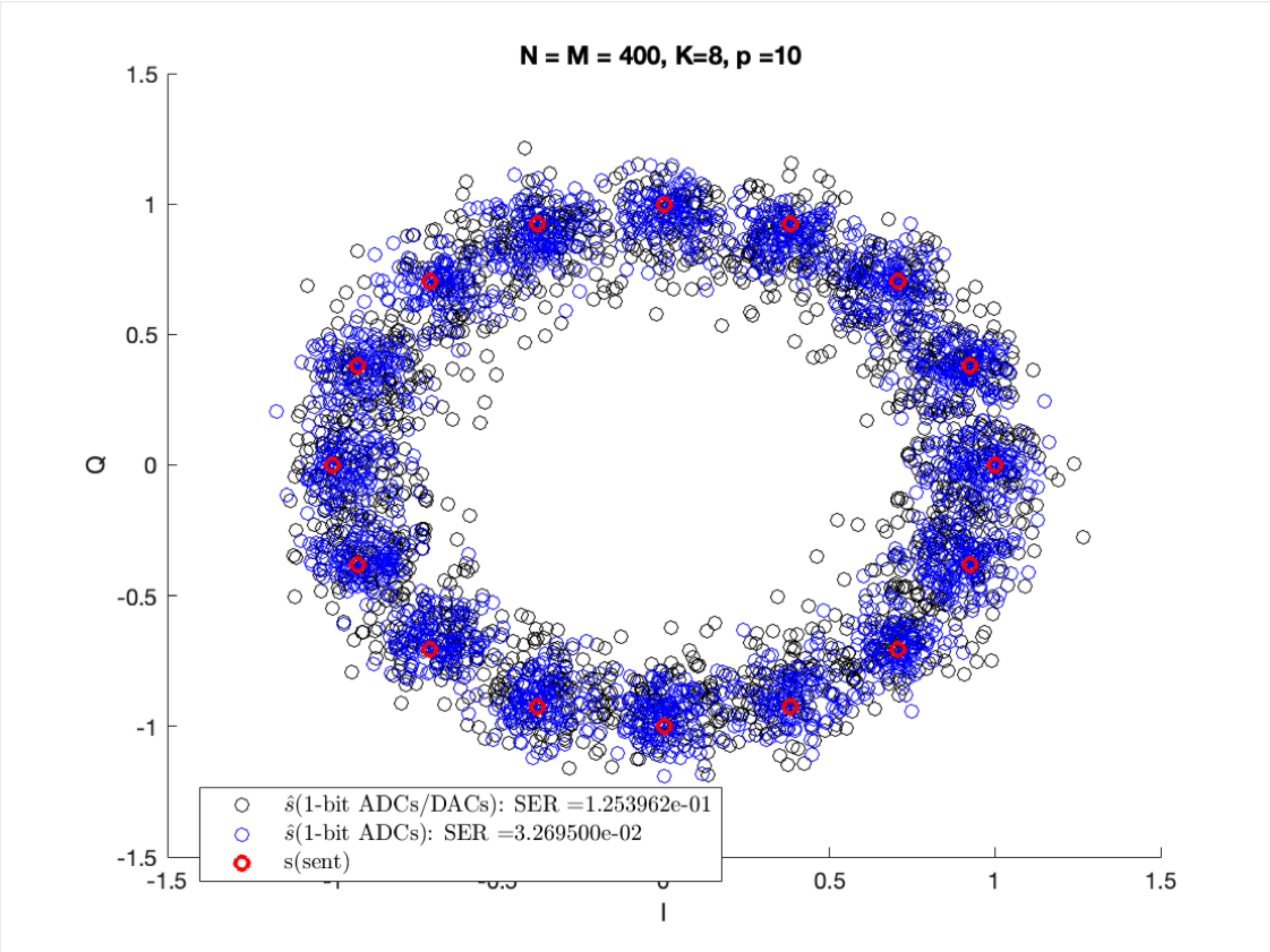
For Figure 4, I only generate two curves corresponding to $N = 400, 1024$.

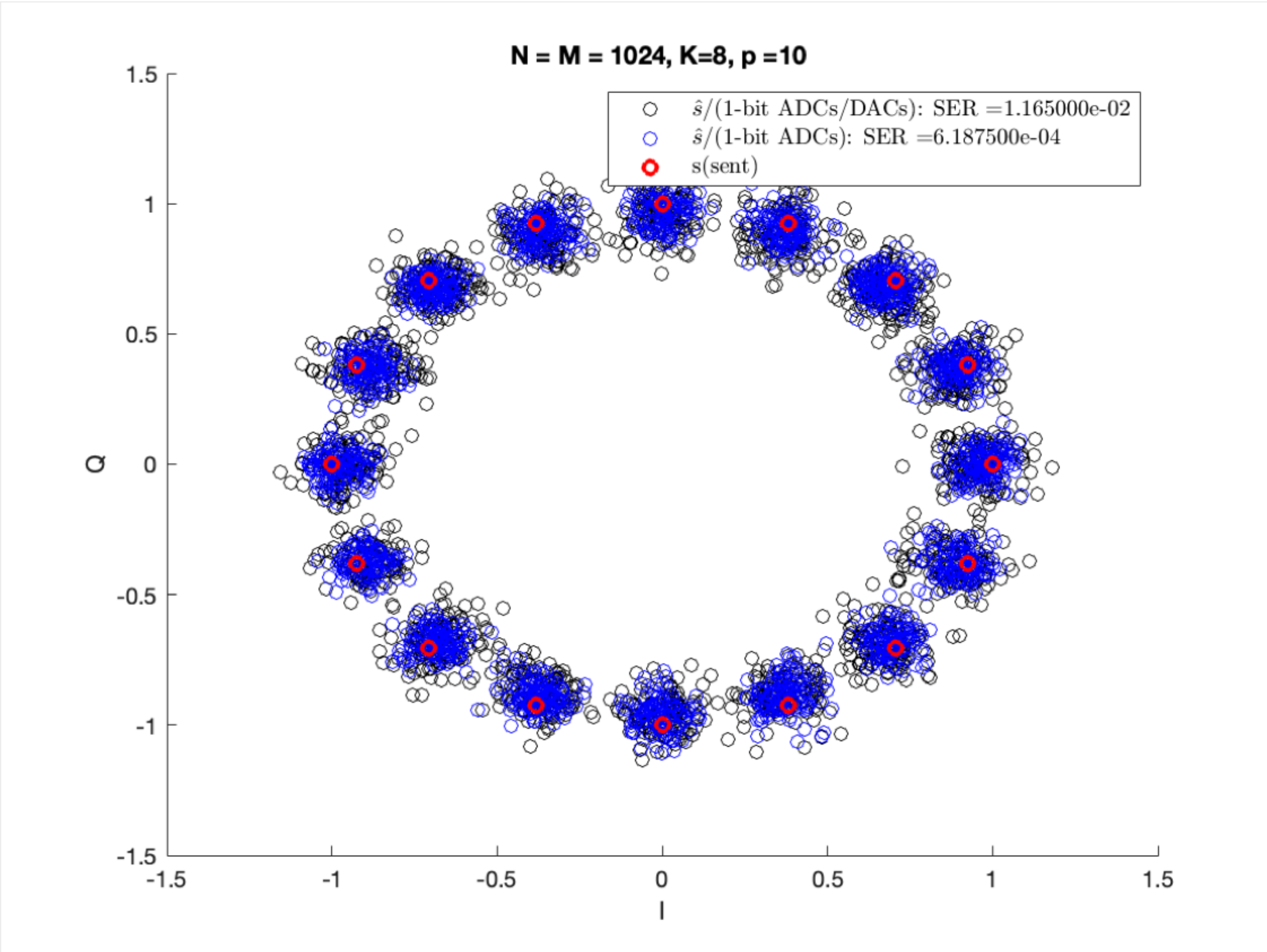
While the numbers of my simulation don't exactly match those in the paper. The curves and constellations are similar in behavior. The code is the attahed.

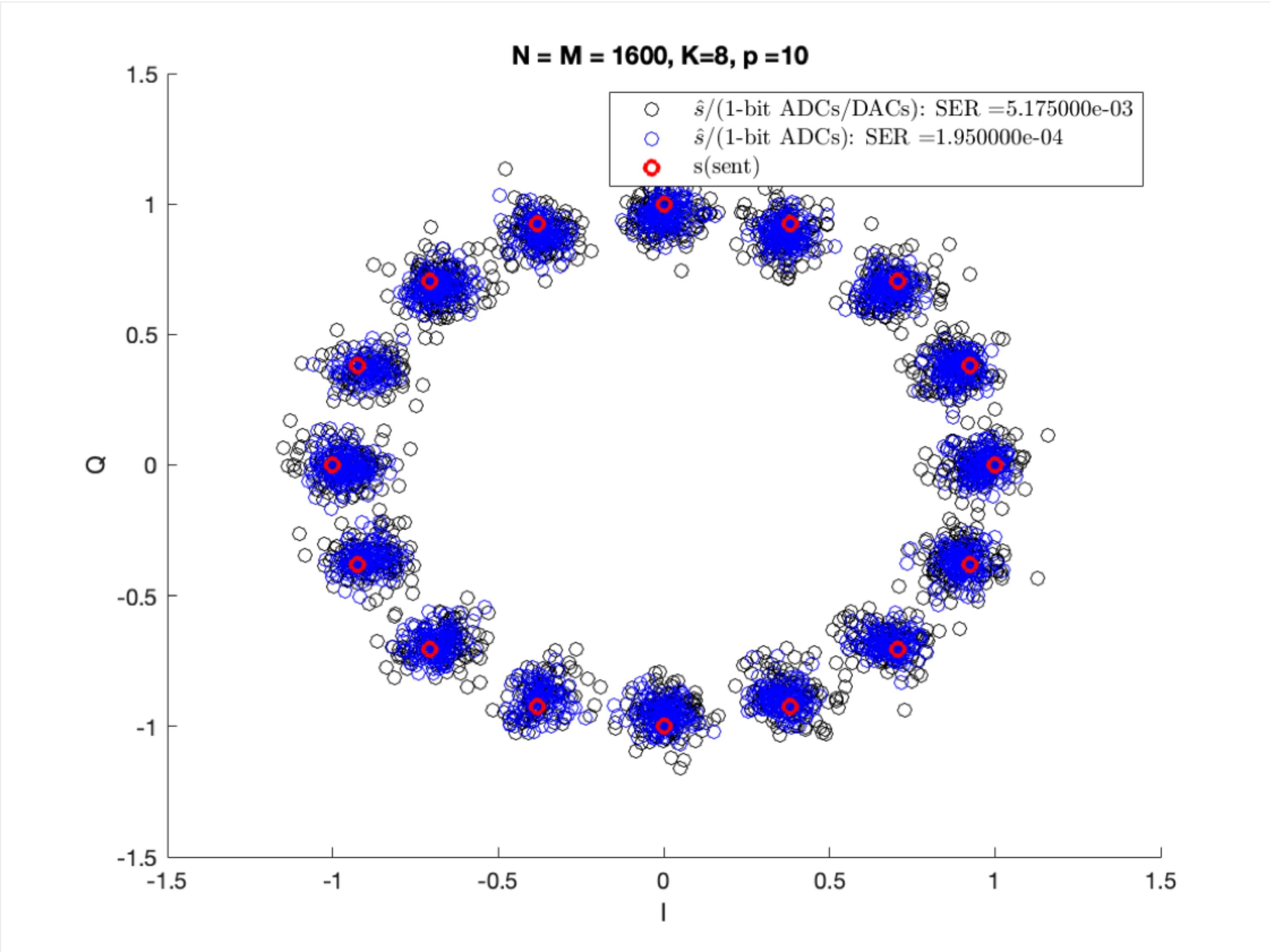




p = 10 dB
MSE approx
K
N=400
N=1024
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
8
16
24
32
40
48
56
64
72
80
88
96
104
112
120
128


N = M = 400, K=8, p =10
1.5
1
0.5
0
-0.5
-1
-1.5
Q
-1.5
-1
-0.5
0
0.5
1
1.5
I
$\hat{s}$(1-bit ADCs/DACs): SER =1.253962e-01
$\hat{s}$(1-bit ADCs): SER =3.269500e-02
s(sent)


N = M = 1024, K=8, p =10
$\hat{s}$/(1-bit ADCs/DACs): SER =1.165000e-02
$\hat{s}$/(1-bit ADCs): SER =6.187500e-04
s(sent)
Q
I
1.5
1
0.5
0
-0.5
-1
-1.5



N = M = 1600, K=8, p =10
$\hat{s}$/(1-bit ADCs/DACs): SER =5.175000e-03
$\hat{s}$/(1-bit ADCs): SER =1.950000e-04
s(sent)
Q
I
1.5
1
0.5
0
-0.5
-1
-1.5