

Relative phase change during quantum operation *

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Abstract

Quantum operations represented by completely positive maps encompass many of the physical processes and have been very powerful in describing quantum computation and information processing tasks. We introduce the notion of relative phase change for a quantum system undergoing quantum operation. We find that the relative phase shift of a system not only depends on the state of the system, but also depends on the initial state of the ancilla with which it might have interacted in the past. The relative phase change during a sequence of quantum operations is shown to be non-additive in nature. This property can attribute a ‘memory’ to a quantum channel. Also the notion of relative phase shift helps us to define what we call ‘in-phase quantum channels’. We will present the relative phase shift for a qubit undergoing depolarizing channel and complete randomization and discuss their implications.

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1 Introduction

In quantum theory a physical system S is associated with a complex, separable Hilbert space \mathcal{H}^D , D being the dimension of the Hilbert space which can be finite or infinite. An isolated system is described by a pure state which is a normalized vector $|\psi\rangle \in \mathcal{H}$ with $\langle\psi|\psi\rangle = 1$. Two states are equivalent, i.e., $|\psi\rangle \sim |\psi'\rangle$ if $|\psi'\rangle = e^{i\alpha}|\psi\rangle$ with α real. They represent the same physical state of a system. Transformations that are *norm preserving, one-to-one and onto* are unitary, i.e., if $U : |\psi\rangle \rightarrow |\phi\rangle$ satisfying above conditions for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ then $|\phi\rangle = U|\psi\rangle$ with $UU^\dagger = U^\dagger U = I$. The amplitude for such a transformation is given by the *inner product* function $\langle\psi|\phi\rangle = \langle\psi|U|\psi\rangle$ which is a complex number having a modulus and a phase. In the quantum world this phase of the amplitude of a process is what it makes so different from that of a classical process. Though absolute phase has no physical significance, the relative phase of a quantum state with respect to another is an important object of study. In fact, the relative phase is the most crucial quantity in quantum interference which according to Feynman contains the only mysteries [1]. If we have a quantum system S described by a pure state $|\psi(0)\rangle$ at initial time and the state of the system is $|\psi(t)\rangle$ at a later time, then the relative phase shift Φ during such an evolution is given by the argument of the inner product function, i.e.,

$$\Phi = \text{Arg}\langle\psi(0)|\psi(t)\rangle \quad (1)$$

Surprisingly, this simple definition of comparing phase of two distinct states was suggested by Pancharatnam only in 1956 in studying interference of polarization of light rays [2]. He has introduced the notion of two states being *in-phase*. Accordingly, two non-orthogonal states are said to be in phase if the inner-product function $\langle\psi(0)|\psi(t)\rangle$ is real and positive.

In recent years we have learned that the the relative phase shift of a quantum system can be of various origin, namely, it can be purely dynamical or it can be purely geometric or both. The discovery of pure geometric phase shift in the context of light goes back to Pancharatnam. In quantum mechanical context it was discovered by Berry in cyclic adiabatic evolutions [3]. Presently there is an immense interest in holonomy effects in quantum mechanics and has led to many generalizations of the notion of geometric phases [4, 5, 6, 7, 8, 9].

When a pure state evolves in time (the evolution need not be adiabatic, cyclic, unitary, and even Schrödinger type) then it traces an open curve in the Hilbert space, i.e., $\Gamma : t \in [0, t]$ with $|\psi\rangle \rightarrow |\psi(t)\rangle$. If we project the evolution curve by a projection map $\Pi : \Gamma \rightarrow \bar{\Gamma}$, we have an open curve in the projective Hilbert space \mathcal{P} . During the evolution it acquires a relative phase which can be called the total phase. The total phase can always be written as **sum of a dynamical phase**

$$\Phi_D = -i \int dt \left\langle \frac{\psi(t)}{\|\psi(t)\|} \left| \frac{d}{dt} \left(\frac{\psi(t)}{\|\psi(t)\|} \right) \right\rangle \right. \quad (2)$$

and a geometric phase given by

$$\Phi_G = i \int \left\langle \frac{\chi(t)}{\|\chi(t)\|} \left| d \left(\frac{\chi(t)}{\|\chi(t)\|} \right) \right\rangle, \quad (3)$$

where $|\chi(t)\rangle$ is a “**reference-section**” introduced in [8, 9]. In fiber bundle formulation the inner product of “reference-section” with its path derivative gives the *connection one-form* whose line integral provides the generalized geometric phase [8]. It can be shown that the above geometric phase is local gauge invariant, independent of the detailed dynamics, independent of re-parametrization, but depends only on the geometry of the evolution path $\bar{\Gamma}$ in the projective Hilbert space of the quantum system. The above geometric phase can reduce to Berry phase [3], Anandan-Aharonov phase [4], Samuel-Bhandari phase [5], Aitchison-Wanelik phase [6], Mukunda-Simon phase [7] under appropriate limits. In the above generalisation, only the existence of an inner product and continuous nature of time evolution, all that is used.

One can also introduce a notion of holonomy to mixed quantal states [10] purely on mathematical ground as well as on physical ground such as in the context of interferometry [11]. Usually one may tend to think that a mixture is an incoherent superposition of quantum states, so there is no notion of ‘relative phase’. However, it is legitimate to ask what is the relative ‘phase difference’ between an initial and final mixtures. Unlike in the case of pure states, there are two types of ‘phases’ for mixed states and both of them go over to the same expression for pure state case [12]. Further differences between Uhlmann phase and Sjöqvist et al phase has been pointed out recently [13, 14]. The geometric phase may be useful in the context of quantum computing as a tool to achieve fault-tolerance [15, 16, 17, 18]. It

may be mentioned that for practical implementations of geometric quantum computing, it is important to understand the behavior of the geometric phase in the presence of decoherence. Possibility of measuring geometric phase in ion traps has also been suggested [19]. Since ion traps have been proposed as suitable devices for quantum information processing tasks, the above proposal may be useful indeed. In another context, a proposal has been made to produce and observe geometric phase shift for a three-level system (a qutrit) in interferometry [20].

In this paper we will mention the generalization of relative phase shift for mixed states undergoing unitary evolution. Using Kraus representation theorem we will define the relative phase shift for a system (either pure or mixed) undergoing a quantum operation, namely, a completely positive (CP) map. We will also prove that the relative phase shift is non-additive in nature, thus attributing a ‘memory’ to a quantum channel. Further, using these notion, one can define what I call “in-phase quantum channel”. Through such a channel a quantum state remains in phase with the initial state. In addition, we will discuss few examples to illustrate these ideas.

2 Phase shift for mixed states

In actual physical situation, the state of a quantum system may not be a pure. This could happen due to variety of reasons. One common cause is unavoidable interaction with the surrounding that leads to loss of coherence and thus the state may become a mixture. Other reason could be that the system of our interest may be a part of a larger system with composite being in an entangled state. That is to say that we do not have access to whole system, but a part of it, thus giving rise to a mixed state.

If we have a mixed state $\rho = \sum_{k=1}^N w_k |k\rangle\langle k|$ that undergoes a unitary time evolution, then it traces a path $\Gamma : t \in [0, t]$ with $\rho \rightarrow \rho(t) = U(t)\rho U^\dagger(t)$. For such a situation a computable definition of relative phase shift was not known. Sjoqvist *et al* [11] have generalized the notion of relative phase shift for a mixed state undergoing a unitary evolution as follows

$$\Phi = \text{ArgTr}[\rho U(t)] = \sum_k w_k \langle k | U(t) | k \rangle. \quad (4)$$

This is the relative phase shift between ρ and $\rho(t)$. That this is a correct prescription was verified using interferometric techniques. By sending a mixed

state as an input through a Mach-Zehnder interferometer with a variable relative $U(1)$ phase χ in one arm of the interferometer (the reference beam) and the other arm (the target beam) is exposed to the unitary operator $U(t)$, it was found that the output of interference pattern oscillates according to

$$I \propto 1 + |\text{Tr}(U(t)\rho)| \cos[\chi - \arg \text{Tr}(U(t)\rho)]. \quad (5)$$

Here, one may notice that the interference fringe produced by varying the phase χ is shifted by $\Phi = \text{ArgTr}(U(t)\rho)$ and that this shift reduces to Pancharatnam's original prescription for pure states. For example, when $\rho = |\psi\rangle\langle\psi|$, we have $\Phi = \text{ArgTr}(U(t)\rho) = \text{ArgTr}\langle\psi(0)|\psi(t)\rangle$. These two facts are the central properties for Φ being a natural generalization of Pancharatnam's relative phase to mixed states undergoing unitary evolution. However, the definition (4) does not hold along an evolution path when $\text{Tr}(U(t)\rho) = 0$. This situation is similar to the pure state case, where Pancharatnam's definition of phase difference also breaks down for two orthogonal states [21, 22].

Using this definition it was possible to introduce the notion of parallel transport condition, holonomy transformation and connection-form for mixed states undergoing a unitary evolution. The geometric phase for a mixed state undergoing parallel transportation can be expressed as an average connection-form

$$\Phi_G[\Gamma] = \int \sum_k w_k i \langle \chi_k | d\chi_k \rangle, \quad (6)$$

where $|\chi_k\rangle$ is the "reference-section" for the k th pure state component in the ensemble. For unitary time evolution of pure quantum state the above expression reduces to that of (3) obtained in [8] by the present author.

3 Phase shift during quantum operation

An open quantum system described by a mixed state may undergo a non-unitary evolution. Here, we generalize the notion of relative phase shift and geometric phase shift to such scenarios. If the system undergoes a general quantum operation described by a completely positive map, then what would be the relative phase shift? Recently we have made some progress in

understanding the phase change under a CP map [23]. Geometric phase for non-unitary evolution with continuous version of CP map has been independently addressed in [24]. Quantum operation is a very powerful technique in describing many quantum information processing tasks. Most of the physical operations such as attaching an ancilla, tracing out a subsystem, unitary evolution, measurements (such as von Neumann and POVMs) and non-unitary stochastic operations may be modeled by quantum operations [25]. In fact, one can imagine that the mixed states result due to some noise acting on the system which is also represented by a CP map.

Let \mathcal{E} be the quantum operation that maps $\rho \rightarrow \mathcal{E}(\rho) = \sum_{\mu} E_{\mu} \rho E_{\mu}^{\dagger}$, where E_{μ} is the set of Kraus operators satisfying the completeness condition $\sum_{\mu} E_{\mu}^{\dagger} E_{\mu} = I$. The question we would like to answer is *what is the relative phase shift between ρ and $\mathcal{E}(\rho)$?* Since every CP map has a unitary representation in an extended Hilbert space, let us imagine that there is an ancilla with an initial state $|0\rangle\langle 0|$. The combined state of the system and ancilla is $\rho_s \otimes |0\rangle_a\langle 0|$. The combined system undergoes a unitary evolution

$$\rho_s \otimes |0\rangle_a\langle 0| \rightarrow U(\rho_s \otimes |0\rangle_a\langle 0|)U^{\dagger}. \quad (7)$$

Then the evolution of the system is obtained by tracing over this ancilla yielding

$$\mathcal{E}(\rho_s) = \text{Tr}_a[U(\rho_s \otimes |0\rangle_a\langle 0|)U^{\dagger}] = \sum_{\mu} E_{\mu} \rho_s E_{\mu}^{\dagger}, \quad (8)$$

where the Kraus operators are $E_{\mu} = \langle \mu|U|0\rangle_a$ in terms of an orthonormal basis $\{|\mu\rangle\}$, $\mu = 0, \dots, K-1 \geq N$, of the K -dimensional Hilbert space of the ancilla [26]. In fact, it is sufficient that $K = N^2$. The operator elements appearing in (8) are not unique, because different set of Kraus operators may give rise to same quantum operation. This is a completely positive map as it takes density operators into density operators, and also all trivial extensions. Conversely, any CP map has a Kraus representation of the form Eq. (8) [25, 26, 27].

The relative phase shift of the system can be thought of as the relative phase shift of the combined system under unitary operation in an enlarged system. Therefore, tracing out the ancilla gives us

$$\Phi = \text{ArgTr}_{s+a}[U\rho_s \otimes |0\rangle_a\langle 0|]$$

$$= \text{ArgTr}_s[\sum_{\mu} \langle \mu | U | 0 \rangle_a \langle 0 | \mu \rangle \rho_s] = \text{ArgTr}_s[E_0 \rho_s] \quad (9)$$

where we have used the orthogonality $\langle 0 | \mu \rangle = \delta_{0\mu}$. The quantity Φ is a natural definition of relative phase as it shifts the maximum of the interference pattern and reduces to the phase defined in [11] for unitarily evolving mixed states.

Since we have ignored the ancilla, we are legitimate to think that the above phase is the quantum phase shift for the system undergoing a quantum operation. However, the above expression holds true if the initial state of the ancilla is in one of the orthogonal basis state (say) in the state $|0\rangle$. In next section, we will obtain a general expression for the relative phase shift when the initial state of the ancilla is in an arbitrary state.

Since phase information has leaked from the system part, the interference information contained in Eq. (9) is only partial. The remaining part may be uncovered by flipping the state of the environment associated with the reference beam to an orthogonal state $|\mu \neq 0\rangle$. This assumes that we have full control over the ancilla and may be thought of as an extra degree of internal freedom. This transformation may be represented by the operator

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_{s+a} + \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 0 \end{pmatrix} \otimes 1_s \otimes F_{0 \rightarrow \mu}, \quad (10)$$

where the first matrix in each term represents the spatial part and the operator $F_{0 \rightarrow \mu}$ flips $|0\rangle$ to $|\mu \neq 0\rangle$. The phase shift is determined by

$$\begin{aligned} \Phi_{\mu} &= \text{Tr}_{s+a}[U_{s+a} \rho_s |0\rangle \langle 0| F_{0 \rightarrow \mu}^{\dagger}] \\ &= \text{Tr}_{s+a}[U_{s+a} \rho_s |0\rangle \langle \mu|] = \text{Tr}_s[E_{\mu} \rho_s] \end{aligned} \quad (11)$$

for each $\mu = 1, \dots, K-1$. The phase Φ and Φ_{μ} contain maximal information about the interference effect during a quantum operation. For unitarily evolving mixed states one obtains $\nu_{\mu} = \delta_{0\mu}$, due to orthogonality of the ancilla states, and the surviving interference pattern Eq. (9) reduces to that of [11].

The above results can be derived by considering purifications. We may lift ρ_s to a purified state $|\Psi\rangle_{sas'}$ by attaching an ancilla according to

$$|\Psi\rangle_{sas'} = \sum_k \sqrt{w_k} |k\rangle_s |0\rangle_a |k'\rangle_{s'} \quad (12)$$

with $\{|k\rangle_{s'}\}$ a basis in an auxiliary Hilbert space of dimension at least as large as that of the internal Hilbert space. This state is mapped by the operators $U = U_{s+a} \otimes I_{s'}$ and $F = I_s \otimes F_{0 \rightarrow \mu} \otimes I_{s'}$ in the target and reference beam, respectively, i.e.

$$\begin{aligned} |\Psi_{\text{tar}}\rangle_{sas'} &= \sum_k \sqrt{w_k} [U_{s+a} |k\rangle_s \otimes |0\rangle_a] \otimes |k'\rangle_{s'}, \\ |\Psi_{\text{ref}}\rangle_{sas'} &= \sum_k \sqrt{w_k} |k\rangle_s \otimes [F_{0 \rightarrow \mu} |0\rangle_a] \otimes |k\rangle_{s'}, \end{aligned} \quad (13)$$

The Pancharatnam phase difference between these purified states is given by

$$\begin{aligned} \Phi &= \text{Arg} \langle \Psi_{\text{ref}} | \Psi_{\text{tar}} \rangle = \text{Arg} \sum_k w_k \langle k | \langle 0 | F_{0 \rightarrow \mu}^\dagger U_{s+a} | 0 \rangle | k \rangle \\ &= \text{Arg} \sum_k w_k \langle k | E_\mu | k \rangle = \text{Arg} \text{Tr}_s (E_\mu \rho_s) \end{aligned} \quad (14)$$

in agreement with Eqs. (9) and (11). Hence, we can understand the relative phase shift of a mixed state under a completely positive map as a relative phase shift of the purified state in a larger Hilbert space under a unitary evolution.

4 Phase shift and ancilla state

Here, we will discuss how the relative phase shift for a system depends on the choice of the initial state of the ancilla. Suppose that instead of the initial state of the ancilla being $|0\rangle_a |0\rangle$ it is an arbitrary state $|A\rangle = \sum_\mu a_\mu |\mu\rangle$. Then the general expression for the phase shift will be

$$\begin{aligned} \Phi &= \text{Arg} \text{Tr}_{s+a} [U \rho_s \otimes |A\rangle_a \langle A|] \\ &= \text{Arg} \text{Tr}_s \left[\sum_\mu \langle \mu | U | A \rangle \langle A | \mu \rangle \rho_s \right] \end{aligned} \quad (15)$$

Notice that this choice of ancilla state gives rise to another set of Kraus operator and hence a different quantum operation, in general. The Kraus operators are given by $F_\mu = \langle \mu | U | A \rangle$ and the quantum operation is described by $\rho \rightarrow \mathcal{F}(\rho) = \sum_\mu F_\mu \rho F_\mu^\dagger$. The relative phase shift in this case is given by

$$\Phi = \text{Arg} \sum_\mu a_\mu^* \text{Tr}_s [F_\mu \rho_s]. \quad (16)$$

In general (9) and (16) are different. In fact, there is no priori reason to believe that they are same. This shows that the relative phase shift depends on the choice of the initial state of the ancilla.

Alternately, one can express (17) as

$$\Phi = \text{ArgTr}_s[\rho_s \langle A|U|A \rangle] = \text{Arg} \sum_k w_k \langle k|\mathcal{N}_s|k \rangle, \quad (17)$$

where $\mathcal{N}_s = \langle A|U_{s+a}|A \rangle$ is a non-unitary operator acting only on \mathcal{H}_s . It is intriguing that even if the ancilla may be far apart, the quantum operation on the system causing the relative phase shift depends on what would have been the initial state of the ancilla. On the other hand, it is not surprising because after all the quantum operation depend on the choice of the initial state of the ancilla.

One can think that choice of initial state of ancilla is equivalent to choosing a local unitary transformation on the ancilla, i.e., if $|A \rangle = u|0 \rangle = \sum_\mu \langle \mu|u|0 \rangle |\mu \rangle$, then instead of the unitary evolution U of combined system, we have a unitary evolution $U(I \otimes u)$ acting on the system and ancilla. In general, they give rise to two different quantum operations. However, when U and $(I \otimes u)$ commutes, i.e., $U(I \otimes u) = (I \otimes u)U$ then only we have the same quantum operations. From the Theorem [25] of Unitary freedom of operator-sum representation, we know that two quantum operations are equal, i.e., $\mathcal{E} = \mathcal{F}$ iff $\mathcal{F} = \sum_j u_{ij} \mathcal{E}$ with u_{ij} as complex numbers. In general $U(I \otimes u) \neq (I \otimes u)U$ and hence, we have different relative phases. This can provide a physical means to distinguish two quantum operations.

5 Non-additive nature of phase

Suppose we have a quantum system that undergoes a sequence of quantum operations described by CP maps. Let us consider two CP maps \mathcal{E} followed by \mathcal{F} . Thus a density matrix $\rho \rightarrow \mathcal{E}(\rho) = \rho' \rightarrow \mathcal{F}(\rho') = \rho''$. Also imagine that there is a CP map that can transform ρ directly into ρ'' via $\rho \rightarrow \mathcal{G}(\rho) = \rho''$. Let $\{E_\mu\}$, $\{F_\mu\}$, and $\{G_\mu\}$ are the Kraus elements corresponding to quantum operations \mathcal{E} , \mathcal{F} and \mathcal{G} , respectively. Let Φ_{12} be relative phase change between ρ and ρ' , Φ_{23} be the relative phase change between ρ' and ρ'' and Φ_{13} be the relative phase change between ρ and ρ'' . Now it can be shown that the phase shift between ρ and ρ'' is *not* sum of the phase shifts between ρ and ρ' , and ρ' and ρ'' , i.e., $\Phi_{13} \neq \Phi_{12} + \Phi_{23}$.

One can prove this directly using the definition of relative phases of mixed states with CP maps. But it is simple and illustrative to imagine the unitary extension of these sequence of CP maps along with purification of the states of the system that is of interest to us. Let U be the unitary representation for the CP map \mathcal{E} , V be the unitary representation for the CP map \mathcal{F} and VU be the unitary representation for the CP map \mathcal{G} . Let $|\Psi\rangle_{ss'a}$ be the combined pure state of the system ($s + s'$) (after purification) and ancilla. Now in the enlarged Hilbert space we have the sequence of unitary transformations and direct unitary transformation given by

$$\begin{aligned} |\Psi\rangle_{ss'a} &\rightarrow |\Psi'\rangle_{ss'a} = U|\Psi\rangle_{ss'a} \rightarrow |\Psi''\rangle_{ss'a} = V|\Psi'\rangle_{ss'a} \\ |\Psi\rangle_{ss'a} &\rightarrow |\Psi''\rangle_{ss'a} = VU|\Psi\rangle_{ss'a}. \end{aligned} \quad (18)$$

The relative phase shifts, respectively, are given by

$$\begin{aligned} \Phi_{12} &= \text{Arg}[{}_{ss'a}\langle\Psi|\Psi'\rangle_{ss'a}] \\ \Phi_{23} &= \text{Arg}[{}_{ss'a}\langle\Psi'|\Psi''\rangle_{ss'a}] \\ \Phi_{13} &= \text{Arg}[{}_{ss'a}\langle\Psi|\Psi''\rangle_{ss'a}]. \end{aligned} \quad (19)$$

Let us calculate the quantity $\Phi_{12} + \Phi_{23} - \Phi_{13}$ which is nothing but phase difference between the relative phases acquired in sequence of quantum operations and the relative phase in direct quantum operation. It is given by

$$\Phi_{12} + \Phi_{23} - \Phi_{13} = \text{Arg}\Delta^{(3)} = \text{Arg}\langle\Psi|\Psi'\rangle\langle\Psi'|\Psi''\rangle\langle\Psi''|\Psi\rangle. \quad (20)$$

The object in the rhs is nothing but the argument of a three-point Bargmann invariant $\Delta^{(3)}$ which is a complex number, in general, and it is non-zero. Hence, the relative phase shift during a sequence of quantum operations is non-additive in nature. In terms of Kraus operators and density matrices one can express (20) as

$$\Phi_{12} + \Phi_{23} - \Phi_{13} = \text{Arg}\text{Tr}_s(\rho_s E_0)\text{Tr}_s(\rho'_s F_0)\text{Tr}_s(\rho''_s G_0). \quad (21)$$

This implies that a quantum system undergoing a sequence of CP maps remembers its history through these relative phases. This property might be explored further in assigning memory to quantum channels. For example, if some quantum alphabets are send across a channel, then by looking at the relative phase shift one can know if these states have undergone desired quantum operation or there has been some unwanted operations in between.

6 In-phase quantum channel

Our notion of relative phase shift helps us to define what is called a “in-phase” quantum channel, i.e., a channel through which if quantum signals are sent then there will be no relative phase shift of the output states with respect to the input states. This definition is in the spirit of “in-phase” condition of Pancharatnam for a pure quantum state where it does not acquire any phase if the inner-product between the initial and final state is real and positive.

Let ρ_i be a set of alphabets of pure state density operators and p_i be the probability distributions with the mixed state $\rho = \sum_i p_i \rho_i$ such that $\{p_i, \rho_i\}$ denote the ensemble of input states. The quantum channel having a unitary representation is “in-phase” channel if there exists an initial state of the ancilla $|A\rangle$ such that the quantity $\text{Tr}_s[\rho_s \langle A|U|A\rangle]$ is real and positive.

In fact one can use the above condition and obtain the parallel transport condition for mixed states undergoing a CP map. This definition will tell us how well a channel preserves phase of the signals. To illustrate this, let us consider the depolarization channel [27] acting on a qubit in the initial state $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \sigma)$, where $\mathbf{r} = (x, y, z)$ is the Bloch vector with the length $|\mathbf{r}| \leq 1$, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the standard Pauli matrices, and I is the 2×2 unit matrix. We can model this with the Kraus operators

$$\begin{aligned} m_0 &= \sqrt{1-p} I, & m_1 &= \sqrt{p/3} \sigma_x, \\ m_2 &= \sqrt{p/3} \sigma_y, & m_3 &= \sqrt{p/3} \sigma_z \end{aligned} \quad (22)$$

that map $\rho \rightarrow \rho' = \frac{1}{2}(I + \mathbf{r}' \cdot \sigma)$. Here, m_1 , m_2 , and m_3 , correspond to bit flip, both bit and phase flip, and phase flip, respectively. Here, p is the probability that one of these errors occurs and it determines the shrinking factor $|\mathbf{r}'|/|\mathbf{r}| = (1 - 4p/3)$ of the Bloch vector. If the qubit is exposed to the depolarization channel then we have

$$\text{Tr}_s(\rho_s \langle A|U|A\rangle) = \text{Tr}_s(\rho_s E_0) = \sqrt{1-p}. \quad (23)$$

This quantity is real and positive, hence the depolarizing channel is a in-phase channel. The relative phase shift is zero. Thus the channel only reduces the visibility by the factor $\sqrt{1-p}$. The absence of phase shifts can be understood from the fact that the depolarization channel only shrinks the length of the Bloch vector.

7 Examples

7.1 Phase shift under conditional unitary operator

In this section we will describe the relative phase shift during a quantum operation whose unitary representation is a conditional one in the extended Hilbert space. Let the initial state of the system is $\rho_s = \sum_k p_k \rho_{k_s}$ with each $\rho_k = |\psi_k\rangle\langle\psi_k|$ and ancilla state is $\rho_a = |A\rangle\langle A|$. The conditional unitary operator may be written as $U = \sum_i P_i \otimes u_i$ with $P_i = |i\rangle\langle i|$ as one-dimensional projectors in \mathcal{H}_s and u_i as unitary operators in \mathcal{H}_a . One can calculate the relative phase shift either from the Kraus operators which are given by $F_\mu = \sum_i P_i \langle \mu | u_i | A \rangle$. Alternately, we can calculate it from the expression (17) as

$$\begin{aligned} \Phi &= \text{ArgTr}_s[\rho_s \langle A | U | A \rangle] = \text{Arg} \sum_i \text{Tr}_s(\rho_s P_i) \langle A | u_i | A \rangle \\ &= \tan^{-1} \left[\frac{\sum_k p_k \sum_i |c_i^{(k)}|^2 \text{Im} \langle A | u_i | A \rangle}{\sum_k p_k \sum_i |c_i^{(k)}|^2 \text{Re} \langle A | u_i | A \rangle} \right], \end{aligned} \quad (24)$$

where we have used $|\psi_k\rangle = \sum_i c_i^{(k)} |i\rangle$. Thus, to have a non-zero phase shift it must hold that at least for some $i = l$, $\text{Im} \langle A | u_l | A \rangle \neq 0$ and all others may be zero. If for all i , the condition $\text{Im} \langle A | u_i | A \rangle = 0$ holds then this channel will be a “in-phase” quantum channel.

7.2 Phase change during randomization

Here, we will discuss the relative phase shift when a qubit undergoes randomization, i.e., a pure qubit state becomes completely mixed via a quantum operation \mathcal{E} as given by

$$|\psi\rangle\langle\psi| \rightarrow \mathcal{E}(|\psi\rangle\langle\psi|) = \frac{I}{2}, \quad (25)$$

where the Kraus operators are $\{E_\mu\} = \{\frac{I}{2}, \frac{\sigma_x}{2}, \frac{i\sigma_y}{2}, \frac{\sigma_z}{2}\}$. A unitary representation of the above quantum operation is

$$U = I \otimes P_0 + \sigma_x \otimes P_1 + i\sigma_y \otimes P_2 + \sigma_z \otimes P_3 \quad (26)$$

with initial state of the ancilla being $|A\rangle = \frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle) \in \mathcal{H}^4$. Again using the expression (17) one can calculate the relative phase shift as

$$\begin{aligned}\Phi &= \text{ArgTr}_s[\rho_s\langle A|U|A\rangle] \\ &= \tan^{-1}\left[\frac{1 + 2\text{Im}(\alpha\beta^*)}{1 + 2\text{Re}(\alpha\beta^*) + (|\alpha|^2 - |\beta|^2)}\right].\end{aligned}\quad (27)$$

Contrary to the usual believe that during a complete randomization, the phase of a qubit undergoes random changes and becomes maximally mixed, there is indeed a definite relative phase change during randomization. This could be useful, for example, in understanding the relative phase shift in quantum teleportation channel where a pure quantum state become completely randomized after Bell-state measurement but before sending the classical communication. This will be explored further in future.

8 Conclusion

In this work, starting with general notion of relative phases in quantum mechanics, we have presented a generalization of the notion of relative phase shift when a quantum system undergoes a quantum operation described by completely positive maps. In the enlarged Hilbert space (so called ‘Church of the large Hilbert space’) where everything is pure, the notion of relative phase shift coincides with the Pancharatnam phase shift. We have shown how this phase shift during quantum operation depends on the initial state of the ancilla. Further, We have shown how the relative phase shift during a sequence of quantum operation is non-additive in nature. This allows us to introduce the notion of “in-phase quantum channels” and we gave one example of such a channel. It is hoped that this ideas will be useful in the context of quantum information and communication and trigger new experiments on relative phases for quantal systems exposed to environmental interactions. Much more work needs to be done to understand the differential geometric structure of geometric phase under CP map and its unitary equivalence.

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