Complex Geometry

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October 3, 2022

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1 Local Theory

1.1 Holomorphic Functions of Several Variables

Definition 1.1 (Holomorphic Functions)**.** There is 3 equivalent definitions: Given $U \subset \mathbb{C}$ an open subset. A function $f : U \to \mathbb{C}$ is called holomorphic if

(1) for any $z_0 \in U$ there exists a ball $B_\epsilon(z_0) \subset U$ such that on $B_\epsilon(z_0)$ can be written as a convergent power series, i.e. $\forall z \in B_\epsilon(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

.

(2) f is continuously differentiable and $\forall z_0 \in U$, $\dfrac{\partial f}{\partial \overline{z}} = 0$.

(3) f is continuously differentiable and $\forall\, \overline{B_\epsilon(z_0)} \subset U$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0} \mathrm{d}z$$

Remark 1.1. For a holomorphic function, there are following theorems:

- Maximum Principle. If f is a holomorphic non-constant function defined on connected open set U, then $|f|$ has no local maximum in U.
- Identity Theorem. If $f, g : U \to \mathbb{C}$ are two holomorphic functions on a connected open set $U \subset \mathbb{C}$ such that $f(z) = g(z)$ for all z in a non-empty open subset $V \subset U$, then $f = g$.
- Riemann Extension Theorem. Let $f : B_\epsilon(z_0) \backslash \{z_0\} \to \mathbb{C}$ be a bounded holomorphic function. Then f can be extended to a holomorphic function $f : B_\epsilon(0) \to \mathbb{C}$.
- Riemann Mapping Theorem. Let $U \subset \mathbb{C}$ be a simply connected proper open subset. Then U is biholomorphic to the unit ball.
- Liouville Theorem. Every bounded holomorphic function from $\mathbb{C}$ to itself is constant.
- Residue Theorem. Let $f : B_\epsilon(0) \backslash \{0\} \to \mathbb{C}$ be a holomorphic function. Then f can be expanded in a Laurent series $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ and $a_{-1} = 1/(2\pi i) \int_{|z|=\epsilon/2} f(z) \mathrm{d}z$.

Definition 1.2 (Multi-value Holomorphic Functions)**.** Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \to \mathbb{C}$ be a continuously differentiable function. Then f is said to be holomorphic if f is holomorphic respect to every variable.

Theorem 1.1. Let $f : \overline{B_\epsilon(w)} \to \mathbb{C}$ be a holomorphic function. Then $\forall\, z \in B_\epsilon(w)$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_i - w_i| = \epsilon_i} \frac{f(\xi_1, \dots, \xi_n)}{(\xi_1 - z_1) \dots (\xi_n - z_n)} \mathrm{d}\xi_1 \dots \mathrm{d}\xi_n$$

Proof. This immediately follows by applying the cauthy theorem of a single variable repeatedly. □

Theorem 1.2. Any holomorphic function $f : B_\epsilon(0) \to \mathbb{C}$ can be written as a power series on $B_\epsilon(0)$.

Proof. By the definition of holomorphic function, $f(z_1, w) = \sum_{n=0}^{\infty} a_n(w) z_1^n$. Since

$$a_n(w) = \int_{|\xi - z_1| = \delta} \frac{f(\xi, w)}{\xi - z_1} \mathrm{d}\xi$$

$a_n(w)$ is holomorphic with respect to w. □

Remark 1.2. The Maximum Principle, Identity Theorem and the Liouville Theorem holds for higher dimention. The Riemann Mapping Theorem will fail.

Lemma 1.1. Let $U \subset \mathbb{C}^n$ be an subset and let $V \subset \mathbb{C}$ be an open neighbourhood of the boundary of $B_\epsilon(0) \subset \mathbb{C}$. Assume that $f : V \times U \to \mathbb{C}$ is a holomorphic function. Then

$$g(z) := g(z_1, \dots, z_n) := \int_{|\xi| = \epsilon} f(\xi, z_1, \dots, z_n) \mathrm{d}\xi$$

is a holomorphic function on V.

Proof. Since $\partial B_\epsilon(0)$ is compact, there exists a finite cover $B_{\delta(\xi_i)}(\xi_i)$. Therefore the unit disk is divided by this open cover, denote each segment by $1, \dots, m$. Therefore

$$g(z) = \sum_{i=1}^{m} \int_i f(\xi, z_1, \dots, z_n) \mathrm{d}\xi$$

On each segment f is uniformly continuous since f is on a unit disk. Thus for every segment

$$\int_i f(\xi, z_1, \dots, z_n) \mathrm{d}\xi = \int_i \sum_{k_0, \dots, k_{n+1} = 0}^{\infty} a_{in} (\xi - \xi_i)^{k_0} (z_1 - z_{1_i})^{k_1} \dots (z_n - z_{n_i})^{k_{n+1}} \mathrm{d}\xi$$

the integral and the summation commutes, therefore $g(z)$ can still be written as a power series, hence $g(z)$ is holomorphic. □

Theorem 1.3 (Hartogs' Theorem)**.** Suppose ϵ and ϵ' which $\epsilon_i' < \epsilon_i$. When $n > 1$, $\forall\, f : B_\epsilon(0) \backslash \overline{B_\epsilon'(0)} \to \mathbb{C}$ holomorphic, then f can be uniquely extended to a holomorphic map $f : B_\epsilon(0) \to \mathbb{C}$

Proof. Assume $\epsilon = (1, \dots, 1)$, $\epsilon' = (1 - \delta, \dots, 1 - \delta)$. Let

$$\widetilde{f}(z_1, w) = \frac{1}{2\pi i} \int_{|\xi| = 1 - \frac{\delta}{2}} \frac{f(\xi, w)}{\xi - z_1} \mathrm{d}\xi$$

Since $\dfrac{f(\xi, w)}{\xi - z_1}$ is holomorphic on $V \times \{|z_1| < 1 - 3\delta/4\} \times B_{\epsilon_{i>1}}(0)$ where V is a small open neighbourhood containing the integration path. Therefore $\widetilde{f}(z_1, w)$ is holomorphic on $\{|z_1| < 1 - 3\delta/4\} \times B_{\epsilon_{i>1}}(0)$. Because $f(z_1, w) = \widetilde{f}(z_1, w)$ on some openset in $\{1 - 3\delta/4 < |z_1| < 1 - \delta/2\} \times B_{\epsilon_{i>1}}(0)$. Therefore by the identity theorem $f = \widetilde{f}$ on the intersection of their domain. Hence f can be extended to $B_\epsilon(0)$. The uniqueness is also obvious by the identity theorem. □

Theorem 1.4 (Weierstrass Preparation Theorem)**.** Given $f : B_\epsilon(0) \to \mathbb{C}$ a holomorphic function. Assume $f(0) = 0$, $f_0(z_1) \not\equiv 0$. Then there exists a weierstrass polynomial

$$g(z_1, w) = z_1^d + \alpha_1(w) z + 1^{d-1} + \dots + \alpha_d(w)$$

where each $\alpha_i(w)$ is holomorphic and $\alpha_i(0) = 0$ and a holomorphic function h such that $f = gh$ and $h(0) \neq (0)$. Moreover, g is unique.

Proof. Since $f(0)=0$, $f_0(z_1)\not\equiv 0$, there exists a disk $B_{\epsilon_1}(0)$ such that $\forall\, z_1\in B_{\epsilon_1}(0)$, $f_0(z_1)\neq 0$. Take $\epsilon_{i>1}>0$ such that $\forall\,|z_1|=\epsilon_1, |z_i|<\epsilon_i, f(z_1,w)\neq 0$.

Since f is holomorphic,

$$R(w)=\int_{|\xi|=\epsilon_1}\frac{f_w'(\xi)}{f_w(\xi)}\mathrm{d}\xi$$

Obviously $R(w)$ is the number of roots when z_1 is restricted to the disk. Since $R(w)\in\mathbb{Z}$ is holomorphic, $R(w)=d$, where d is the multiplicity at $z=0$. Therefore

$$f(z_1,w)=\prod_{i=1}^{d}(z_1-a_i(w))h_w(z_1)=g_w(z_1)h_w(z_1)$$

where $a_i(w)$ is zeros of f. Therefore for $k=1,\ldots,d$,

$$\sigma_k(w):=\sum_{i=1}^{d}a_i(w)^k=\frac{1}{2\pi i}\int_{|\xi|=\epsilon_1}\xi^k\frac{f_w'(\xi)}{f_w(\xi)}\mathrm{d}\xi$$

By the lemma $\sigma_k(w)$ is holomorphic to w, therefore $a_i(w)$ is holomorphic, $g(z_1,w)$ is holomorphic. Let

$$h(z_1,w)=\int_{|\xi|=\epsilon_1}\frac{f(\xi,w)}{g(\xi,w)(\xi-z_1)}\mathrm{d}\xi$$

it is obvious that $h(z_1,w)$ is holomorphic, and it is equal to $f(z)/g(z)$ when $g(z)$ is non-zero. Therefore $f=gh$. For the uniqueness of g, since $h(0)\neq 0$, f_w and g_w will have the same zeros. Therefore the polynomial is unique. $\square$

Theorem 1.5. Given a open subset $U\subset\mathbb{C}^*$, $f:U\to\mathbb{C}$ holomorphic. Denote the zeros of f with $Z(f)$. $g:U\backslash Z(f)\to\mathbb{C}$ is holomorphic and locally bounded near $Z(f)$, then g can be uniquely extend to $\widetilde{g}:U\to\mathbb{C}$.

Proof. Assume $U=B_\epsilon(0)$ and $f_0(z_1)\neq 0$ for any $0<|z_1|<\epsilon_1$. and $f(z)\neq 0$ for any $|z_1|=\epsilon_1/2$ $\square$