Real Analysis (Folland) Exercises

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November 13, 2022

Contents

Chapter 1: Measures

Exercise 2

Show that $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:
(a) the open intervals $\mathcal{E}_1 = \{(a,b) : a < b\}$,
(b) the closed intervals $\mathcal{E}_2 = \{[a,b] : a < b\}$,
(c) the half-open intervals $\mathcal{E}_3 = \{(a,b] : a < b)\}$ or $\epsilon_3 = \{(a,b] : a < b)\}$,
(d) the open rays $\mathcal{E}_5 = \{(a,\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$,
(e) the closed rays $\mathcal{E}_7 = \{[a,\infty)\ a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Proof. Recall lemma 1.1, which states if $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$. It is easy to observe $\mathcal{E}_i \subset \mathcal{B}_{\mathbb{R}}$, therefore $\mathcal{M}(\mathcal{E}_i) \subset \mathcal{B}_{\mathbb{R}}$.

(a) Since every open set can be written as a countable union of intervals, denote $\mathcal{O}$ as the set of all open sets, then $\mathcal{O} \subset \mathcal{M}(\mathcal{E}_1)$, $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_1)$. Hence $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1)$.
(b) Attempt to show $\mathcal{E}_1 \subset \mathcal{M}(\mathcal{E}_2)$. Apparently $(a,b) = \cup_{n=1}^{\infty}[a + 1/n, b - 1/n]$.
(c) $\mathcal{E}_1 \subset \mathcal{M}(\mathcal{E}_3)$ since $(a,b) = \cup_{n=1}^{\infty}(a, b - 1/n]$. The same goes for $\mathcal{E}_4$.
(d) $\mathcal{E}_3 \subset \mathcal{M}(\mathcal{E}_5)$ since $(a,b] = (a,\infty) \cap ((b,\infty))^c$. The same argument goes for $\mathcal{E}_6$.
(e) $\mathcal{E}_4 \subset \mathcal{M}(\mathcal{E}_7)$ since $[a,b) = [a,\infty] \cap ([b,\infty))^c$.
Therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_i)$. □

Exercise 4

An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions.

Proof. If $\mathcal{A}$ is closed under countable increasing unions, for any countable collection of sets $\{F_j\}$ in $\mathcal{A}$, let $E_1 = F_1$, $E_2 = E_1 \cup F_2$, $E_n = E_{n-1} \cup F_n$, then $\{E_j\}$ is an increasing sequence of sets, therefore $\cup_{n=1}^{\infty} E_n = \cup_{j=1}^{\infty} F_j \in \mathcal{A}$. Therefore $\mathcal{A}$ is a σ-algebra. The reverse is trivial. □

Exercise 5

If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

Proof. Let A be the index set of all countable subsets of $\mathcal{E}$. First claim that $\mathcal{B} = \cup_{\alpha \in A}\mathcal{M}(\mathcal{F}_\alpha)$ is a σ-algebra. $\forall\, E \in \mathcal{B}$, $E \in \mathcal{M}(\mathcal{F}_\alpha)$, therefore $E^c \in \mathcal{B}$. Given a countable collection of sets $\{E_j\}$ in $\mathcal{B}$, since $E_j \in \mathcal{M}(F_\alpha)$, E_j must be in at least one $\mathcal{M}(\mathcal{F}_j)$. Let $\mathcal{H} = \cup_{j=1}^{\infty}\mathcal{F}_j$, consider $\mathcal{M}(\mathcal{H})$. Obviously $\{E_j\} \in \mathcal{M}(\mathcal{H})$, therefore $\cup_{j=1}^{\infty} E_j \in \mathcal{M}(\mathcal{H})$. Since $\mathcal{H}$ is also a countable subset of $\mathcal{E}$, $\mathcal{M}(\mathcal{H}) \subset \mathcal{B}$. Therefore $\mathcal{B}$ is indeed a σ-algebra.

It is straightforward that $\mathcal{E} \subset \mathcal{B}$. For the reverse, $\forall\, E \in \mathcal{B}$, E is in some σ-algebra generated by $\mathcal{F}_\alpha$, therefore $E \in \mathcal{M}$. Thus $\mathcal{M} = \mathcal{B}$. □

Exercise 6

Suppose that (X, M, μ) is a measure space. Let $\mathcal{N} = \{N \in M : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subset N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof. Apparently $\overline{\mathcal{M}}$ is closed under countable unions. For any $E \in \mathcal{M}, F \subset N \in \mathcal{N}$, without the loss of generality assume $E \cap N = \varnothing$ (otherwise replace N, F with $N \backslash E$ and $F \backslash E$). Then $E \cup F = (E \cup N) \cap (N^c \cup F)$, $(E \cup F)^c = (E \cup N)^c \cup (N \cap F^c)^c \in \overline{\mathcal{M}}$. Therefore $\overline{\mathcal{M}}$ is a σ-algebra.

Now consider the extension $\overline{\mu}$. Let $\overline{\mu}(E \cup F) = \mu(E)$. This is well-defined since if $E_1 \cup F_1 = E_2 \cup F_2$ then $E_1 \subset E_2 \cup N_2$, $\mu(E_1) \leq \mu(E_2)$, and likewise $\mu(E_1) \geq \mu(E_2)$, thus $\mu(E_1) = \mu(E_2)$. Then $\overline{\mu}(\varnothing) = \overline{\mu}(\varnothing \cup \varnothing) = 0$, and the countable additivity can be likewise easily verified. For the uniqueness, give any other measure $\overline{\mu}'$, $\overline{\mu}'(E \cup F) \leq \overline{\mu}'(E \cup N) \leq \mu(E)$. But $\overline{\mu}'(E \cup F) \geq \overline{\mu}'(E \cup \varnothing) = \mu(E)$, thus $\overline{\mu}' = \overline{\mu}$. □

Exercise 7

If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Proof. Let $\mu' = \sum_1^n a_j \mu_j$. Then $\mu'(\varnothing) = 0$, given any collection disjoint sets $\{E_j\}$ in $\mathcal{M}$, $\mu'(\cup_1^\infty E_j) = \sum_1^n a_j \mu_j(\cup_1^\infty E_j) = \sum_{j=1}^\infty \sum_1^n a_j \mu_j(E_j) = \sum_{j=1}^\infty \mu'(E_j)$, therefore μ' is also a measure. □

Exercise 8

If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\cup_1^\infty E_j) < \infty$.

Proof. Recall

$$\liminf E_j = \bigcup_{j=1}^\infty \bigcap_{i=j}^\infty E_i, \quad \limsup E_j = \bigcap_{j=1}^\infty \bigcup_{i=j}^\infty E_i$$

observe that $\{A_j = \cap_{i=j}^\infty E_j\}$ gives a sequence such that $A_1 \subset A_2 \cdots$, since μ is a measure, $\mu(\liminf E_j) = \mu(\cup_{j=1}^\infty A_j) = \lim_{j\to\infty} \mu(A_j) \leq \liminf \mu(E_j)$. For the second claim, in the same sense let $\{B_j = \cup_{i=j}^\infty E_j\}$, then $\mu(\limsup E_j) = \lim_{j\to\infty} \mu(B_j) \geq \limsup \mu(B_j)$. □

Exercise 9

If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Proof. Since μ is a measure, $\mu(E)+\mu(F) = \mu(E\cap F)+\mu(E\cap F^c)+\mu(E\cap F)+\mu(E^c\cap F) = \mu(E\cup F)+\mu(E\cap F)$. □

Exercise 10

Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then μ_E is also a measure.

Proof. Apparently $\mu_E(\varnothing) = 0$. Given any collection of disjoint sets $\{A_j\}$ in $\mathcal{M}$, $\mu_E(\cup_{j=1}^\infty A_j) = \mu(\cup_{j=1}^\infty A_j \cap E) = \mu(\cup_{j=1}^\infty (A_j \cap E)) = \sum_{j=1}^\infty \mu_E(A_j)$. Therefore μ_E is a measure. □

Exercise 11

A finitely additive measure μ is a measure iff it is continuous from below. If $\mu(X) < \infty$, μ is a measure iff it is countinuous from above.

Proof. Given a finitely additive measure μ, if it is continuous from below, then given a sequence of disjoint sets $\{E_j\}$, let $\{A_j = \cup_{i=1}^j E_i\}$, $\mu(\cup_j E_j) = \mu(\cup_j A_j) = \lim_{n\to\infty} \mu(A_n)$, by finite additivity $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \sum_{i=1}^n \mu(E_i) = \sum_{n=1}^\infty \mu(E_j)$. For the second claim, let $\{B_j = A_j^c\}$, then $\cup_j E_j = \cup_j A_j = (\cap_j (A_j^c))^c = (\cap_j (B_j))^c$, therefore $\mu(\cup_j E_j) + \mu(\cap_j B_j) = \mu(X)$. By continuity from above, $\mu(\cap_j B_j) = \lim_{j\to\infty} \mu(B_j) = \mu(X) - \lim_{j\to\infty} \mu(A_j)$, the rest is the same as the previous argument. □

Exercise 12

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.

(b) Say that $E \sim F$ if $\mu(E \triangle F)$. Then $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) For $E, F \in M$, define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $M/\sim$.

Proof. Recall $E \triangle F = (E \backslash F) \cup (F \backslash E)$.

(a) Since $E \backslash F$ and $F \backslash E$ are disjoint, $\mu(E \triangle F) = \mu(E \backslash F) + \mu(F \backslash E)$. Since $E = (E \backslash F) \cup (E \cap F)$, $F = (F \backslash E) \cup (E \cap F)$, $\mu(E \triangle F) = \mu(E) + \mu(F) - 2\mu(E \cap F)$. Notice that $\mu(E) \geq \mu(E \cap F)$, $\mu(F) \geq \mu(E \cap F)$, therefore when $\mu(E \triangle F) = 0$, $\mu(E) = \mu(F) = \mu(E \cap F)$.

(b) Since "=" is an equivalence relation on $[0, \infty)$, "$\sim$" is obviously also an equivalence relation.

(c) Attempt to verify $\mu(E \triangle G) \leq \mu(E \triangle F) + \mu(F \triangle G)$:

$$\begin{aligned}
\mu(E \triangle G) &= \mu(E \backslash G) + \mu(G \backslash E) \\
&= \mu(E) + \mu(G) - 2\mu(E \cap G) \\
&\leq \mu(E) + 2\mu(F) + \mu(G) - 2\mu(E \cap F) - 2\mu(F \cap G) \\
&= \mu(E \triangle F) + \mu(F \triangle G)
\end{aligned}$$

where the inequality $\mu(F) + \mu(E \cap G) = \mu(F \cap E^c \cap G^c) + \mu(F \cap G^c \cap E) + \mu(F \cap G \cap E^c) + 2\mu(F \cap G \cap E) + \mu(E \cap G \cap F^c) \geq \mu(F \cap E \cap G^c) + \mu(F \cap G \cap E^c) + 2\mu(E \cap F \cap G) = \mu(E \cap F) + \mu(F \cap G)$ is utilized. $\square$

Exercise 14

If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

Proof. Assume that there exists $C > 0$ such that $\forall F \subset E$, $\mu(F) \leq C$, then $\sup\{\mu(F) : F \subset E\} \leq C$. Denote the supremum with S. Then $\forall n \in \mathbb{N}, \exists F_n \subset E$ such that $S - 1/n < \mu(F_n) \leq S$. Since $F' = \cup_{n=1}^{\infty} F_n \subset E$, $\mu(F') = S$. Then consider $E \backslash F'$. Obviously $\mu(E \backslash F') = \infty$. Because μ is semifinite, there exist F'' such that $0 < \mu(F'') < \infty$. Then $\mu(F' \cup F'') > S$, contradiction. Therefore there is no supremum. $\square$

Exercise 15

Given a measure μ on $(X, \mathcal{M})$, define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) : F \subset E \, \text{and} \, \mu(F) < \infty\}$.

(a) μ_0 is a semifinite measure. It is called the semifinite part of μ.

(b) If μ is semifinite, then $\mu = \mu_0$.

(c) There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only values 0 and ∞ such that $\mu = \mu_0 + \nu$.

Proof. (a) First verify that μ_0 is a measure. Obviously $\mu_0(\varnothing) = 0$. Give any collection of disjoint sets $\{E_j\}$, let $E = \cup_{j=1}^{\infty} E_j$. For a measurable set $F \subset E$ and $\mu(F) < \infty$, $\mu(F) = \sum_j \mu(F \cap E_j) \leq \sum_j \mu_0(E_j)$. Since this holds for any subset of E that has finite measure, $\mu_0(E) \leq \sum_j \mu_0(E_j)$. If $\mu_0(E) = \infty$, then the reverse trivially holds. Otherwise $\mu_0(E) < \infty$. Then for each E_j, $\forall \epsilon/2^j$, there exists $F_j \subset E_j$ such that $\mu_0(E_j) - \epsilon/2^j < \mu(F_j) \leq \mu_0(E_j)$. Then $\mu_0(E) \geq \mu(\cup_{j=1}^{\infty} F_j) = \sum_j \mu_0(E_j) - \epsilon$. Therefore $\mu_0(E) = \sum_j \mu_0(E_j)$, μ_0 is a measure.

Given a E such that $\mu_0(E) = \infty$, take any $C > 0$, then $\exists F \subset E$ such that $C < \mu(F) < \infty$. Then $\mu_0(F) = \mu(F)$ is non-zero and finite. Therefore μ_0 is a semifinite measure.

(b) For any $E \in \mathcal{M}$, if $\mu(E) < \infty$, then $\mu(E) = \mu_0(E)$. If $\mu(E) = \infty$, then by Exercise 14 $\mu_0(E) = \infty$. Therefore $\mu = \mu_0$.

(c) Let

$$\nu(E) = \begin{cases} 0, & \text{if } E \text{ is } \sigma\text{-finite} \\ \infty, & \text{otherwise} \end{cases}$$

ν is a measure since the disjoint union of σ-finite sets is still a σ-finite set, and if there is a set that is not σ-finite in the collection the union will also not be σ-finite. Now verify $\mu(E) = \mu_0(E) + \nu(E)$. When E is σ-finite, if $\mu(E)$ is finite, then the quality holds. If $\mu(E)$ is not finite, then by previous exercise $\mu_0(E) = \infty$, the quality still holds. If E is not σ-finite, the quality holds trivially. $\square$

Exercise 16

Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called locally measurable if for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$, $E \cap A \in \mathcal{M}$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called saturated.

(a) If μ is σ-finite, then μ is saturated.

(b) $\widetilde{\mathcal{M}}$ is a σ-algebra.

(c) Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu} = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the saturation of μ.

(d) If μ is complete, so is $\overline{\mu}$.

(e) Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \,\text{and}\, A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

(f) Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $\mathcal{P}(X_1)$ and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\widetilde{M} = \mathcal{P}(X)$, and in the notation of (c) and (e), $\widetilde{\mu} \neq \underline{\mu}$.

Proof. (a) Since μ is σ-finite, there exists a countable collection of disjoint sets $\{E_j\}$ such that $X = \cup_{j=1}^{\infty} E_j$ and $\mu(E_j) \leq \infty$. Therefore $\forall\, E \in \widetilde{\mathcal{M}}$, for each E_j, $E \cap E_j \in \mathcal{M}$. Thus $E = \cup_1^{\infty}(E \cap E_j) \in \mathcal{M}$. Hence $\widetilde{\mathcal{M}} = \mathcal{M}$.

(b) $\forall\, E \in \widetilde{\mathcal{M}}$, $\forall\, A \in \mathcal{M}$ such that $\mu(A) < \infty$, $E^c \cap A = (A \cap (E \cap A)^c) \in \mathcal{M}$, therefore $E^c \in \widetilde{\mathcal{M}}$. Give any countable collection of sets $\{E_j\}$ in $\widetilde{\mathcal{M}}$, for any $A \in \mathcal{M}$ that has finite measure, $(\cup_j E_j) \cap A = \cup_j (E_j \cap A) \in \mathcal{M}$. Thus $\widetilde{\mathcal{M}}$ is a σ-algebra.

(c) First check that $\widetilde{\mu}$ is a measure. Apparently $\widetilde{\mu}(\varnothing) = 0$. Given any countable collection of disjoint sets $\{E_j\}$ in $\widetilde{\mathcal{M}}$, if $E_j \in \mathcal{M}$ for each j, then the additivity trivially holds. If $\exists\, i$ such that $E_i \notin \mathcal{M}$, assume $\cup_j E_j \in \mathcal{M}$. Obviously $\cup_j E_j$ cannot have finite measure. Therefore the equality still holds. Then check that $\widetilde{\mu}$ is saturated. $\forall\, E$, if $\forall\, A \in \widetilde{\mathcal{M}}$ such that $\widetilde{\mu}(A) < \infty$, $A \cap E \in \widetilde{\mathcal{M}}$, then $\mu(A) < \infty$, therefore $\widetilde{\mu}(A \cap E) < \infty$, $A \cap E \in \mathcal{M}$, $E \in \widetilde{\mathcal{M}}$.

(d) $\forall\, N \in \widetilde{\mathcal{M}}$, if $\widetilde{\mu}(N) = 0$, then because μ is complete, $\forall\, F \subset N$, $\widetilde{\mu}(F) = 0$. Therefore $\widetilde{\mu}$ is also complete.

(e) First verify $\underline{\mu}$ is a measure. Obviously $\underline{\mu}(\varnothing) = 0$. Given any countable collection of disjoint sets $\{E_j\}$ in $\widetilde{\mathcal{M}}$, assume they are all finite. $\forall\, E_j$, $\exists\, A_j$ such that $A_j \in \mathcal{M}$, $A_j \subset E_j$, $\underline{\mu}(E_j) - \epsilon/2^j < \mu(A_j) \leq \underline{\mu}(E_j)$. Then $\underline{\mu}(\cup_j E_j) \geq \mu(\cup_j A_j) > \sum_j \underline{\mu}(E_j) - \epsilon$, therefore $\underline{\mu}(\cup_j E_j) \geq \sum_j \underline{\mu}(E_j)$. For the reverse inequality, take $A \in \mathcal{M}$ such that $\underline{\mu}(\cup_j E_j) - \epsilon < \mu(A) \leq \underline{\mu}(\cup_j E_j)$, since $A \subset \cup_j E_j$ and $\mu(A) < \infty$, $A_j = A \cap E_j \in \mathcal{M}$, therefore $\underline{\mu}(\cup_j E_j) - \epsilon < \mu(A) \leq \sum_j \underline{\mu}(E_j)$. Therefore the reverse inequality holds, $\underline{\mu}(\cup_j E_j) = \sum_j \underline{\mu}(E_j)$. For the infinite case, since μ is semifinite, by exercise 14 both inequality hold trivially. If $E \in \mathcal{M}$, then $\mu(E) = \underline{\mu}(E)$ since μ is semifinite. Therefore $\underline{\mu}$ is an extend of μ.

Now check that $\underline{\mu}$ is saturated. $\forall\, E \in \widetilde{\widetilde{\mathcal{M}}}$, $\forall\, A \in \mathcal{M}$ such that $\underline{\mu}(A) < \infty$, $E \cap A \in \widetilde{\mathcal{M}}$. Then $E \cap A = E \cap A \cap A \in \mathcal{M}$.

(f) Since μ_0 is a well-defined measure, it is straightforward that μ is also a measure. $\forall\, A \subset X$, given any B such that $B \in \mathcal{M}$ and $\mu(B) < \infty$, since $B \cap X_1$ is finite, B must be countable. Therefore $B \cap A$ is also countable, $B \cap A \subset \widetilde{\mathcal{M}}$. Therefore $\widetilde{\mathcal{M}} = \mathcal{P}(X)$. Obviously $\widetilde{\mu} \neq \underline{\mu}$, one example may be $\{x_1\} \cup X_2$ where $x_1 \in X_1$. □

Exercise 17

If μ^* is an outer measure on X and $\{A_j\}_1^{\infty}$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\cup_1^{\infty} A_j)) = \sum_1^{\infty} \mu^*(E \cap A_j)$ for any $E \subset X$.

Proof. Since $\cup_j (E \cap A_j) = E \cap (\cup_j A_j)$, $\mu^*(E \cap (\cup_1^{\infty} A_j)) \leq \sum_1^{\infty} \mu^*(E \cap A_j)$. For the reverse inequality, let $B_n = \cup_{i=1}^{n} A_i$. Then $\mu^*(E \cap B_n) = \mu^*(E \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) = \sum_{i=1}^{n} \mu^*(E \cap A_i)$. Since $\mu^*(E \cap B_\infty) \geq \mu^*(E \cap B_n) = \sum_1^n \mu^*(E \cap A_i)$ for any n, $\mu^*(E \cap (\cup_1^{\infty} A_j)) \geq \sum_1^{\infty} \mu^*(E \cap A_j)$. □

Exercise 18

Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \backslash E) = 0$.

(c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Proof. (a) Recall the definition of the outer measure μ^* on X:

$$\mu^*(E) = \inf\{\sum_j \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \cup_j A_j, j = 1, 2, \cdots\}$$

If $\mu^*(E) = \infty$, the inequality holds trivially. Consider the case where $\mu^*(E) < \infty$. Then $\forall \epsilon > 0$, $\exists \{A_j\}$ with $A_j \in \mathcal{A}$ for each j and $E \subset \cup_j A_j$ such that $\mu^*(\cup_j A_j) \leq \sum_j \mu^*(A_j) \leq \mu^*(E) + \epsilon$. Therefore take $A = \cup_j A_j$.

(b) If E is μ^*-measurable, then by the first claim given $\epsilon = 1/k$, $k \in \mathbb{N}$, there exists $A_k \in \mathcal{A}_\sigma$ such that $E \subset A_k$, $\mu^*(A_k) = \mu^*(A \cap E) + 1/k$. Let $B = \cap_k A_k$. It is obvious that $\mu^*(B) = \mu^*(E)$. Therefore $\mu^*(E) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(B)$, $\mu^*(B \backslash E) = 0$.

For the inverse, $\forall A \subset X$, $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A \cap B) + \mu^*(A \cap E^c \cap B^c) + \mu^*(A \cap E^c \cap B) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c)$. $\forall \epsilon > 0$, $\exists C \in \mathcal{A}_\sigma$ such that $\mu^*(C) \leq \mu^*(A) + \epsilon$ and $A \subset C$. By Caratheodory's theorem μ^* is a measure on $\mathcal{M}(\mathcal{A})$, therefore $\mu^*(A) + \epsilon \geq \mu^*(C) = \mu^*(C \cap B) + \mu^*(C \cap B^c) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$. Therefore $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

(c) Notice that only need to prove the forword direction given that $\mu^*(E) = \infty$. Since μ_0 is σ-finite, $\exists \{A_j\} \subset \mathcal{A}$ such that $\mu_0(A_j) < \infty$, $X = \cup_j A_j$. Let $E_j = E \cap A_j$, $\forall \epsilon > 0$, take $B_j \in \mathcal{A}_\sigma$ and $E_j \subset B_j$ such that $\mu^*(B_j) \leq \mu^*(E_j) + \epsilon/2^j$, then $\mu^*(B \backslash E) \leq \mu^*(\cup_j (B_j \backslash E_j)) \leq \sum_j \mu^*(B_j \backslash E_j) = \sum_j (\mu^*(B_j) - \mu^*(E_j)) \leq \epsilon$. Therefore $\mu^*(B \backslash E) = 0$. □

Exercise 19

Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

Proof. If E is μ^*-measurable, then $\mu_0(X) = \mu^*(X) = \mu^*(E) + \mu^*(E^c)$, hence $\mu^*(E) = \mu_*(E)$. For the inverse, given $\mu^*(E) + \mu^*(E^c) = \mu_0(X)$, by exercise 18, $\forall n \in \mathbb{N}, \exists A_n \in \mathcal{A}_\sigma$ such that $E \subset A_n$, $\mu^*(A_n) \leq \mu^*(E) + 1/n$. Let $A = \cap_n A_n$, then $A \in \mathcal{A}_{\sigma\delta}$ with $E \subset A$. Since A_n is μ^*-measurable, $\mu^*(A \cap E^c) \leq \mu^*(A_n \cap E^c) = \mu(E^c) - \mu(A_n^c \cap E) \leq \mu_0(X) - \mu^*(E) - \mu(A_n^c) \leq \mu^*(A_n) - \mu^*(E) \leq 1/n$ for any n, thus $\mu^*(A \cap E^c) = 0$, therefore by exercise 18 E is μ^*-measurable. □

Exercise 20

Let μ^* be an outer measure on X, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, $\overline{\mu} = \mu^*|\mathcal{M}^*$, and μ^+ the outer measure induced by $\overline{\mu}$.

(a) If $E \subset X$, we have $\mu^*(E) \leq \mu^+(E)$, with equality iff there exists $A \in \mathcal{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$.

(b) If μ^* is induced from a premeasure, then $\mu^* = \mu^+$.

(c) If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \mu^+$.

Proof. (a) By the construction of the outer measure, if $\mu^+(E) < \infty$, then $\forall \epsilon > 0, \exists E_j$ with $E_j \in \mathcal{M}^*$ for each j, and $E \subset \cup_j E_j$ such that $\mu^*(E) \leq \sum_j \mu^*(E_j) \leq \mu^+(E) + \epsilon$, therefore $\mu^*(E) \leq \mu^+(E)$. For the second claim, when $\mu^*(E) = \mu^+(E)$, one may take $E_j \in \mathcal{M}^*$ such that $\{E_j\}$ covers E and $\mu^*(E) = \mu^+(E) = \sum_j \mu^*(E_j)$. Thus just take $A = \cup_j E_j$. For the reverse, since A covers E, $\mu^*(E) \leq \mu^+(E) \leq \mu^*(A)$. By $\mu^*(E) = \mu^*(A)$ the equality must be taken.

(b) Since μ^* is induced from a premeasure, by exercise 18, for any $n \in \mathbb{N}$, there exists $A_n \in \mathcal{M}^*$ such that $E \subset A_n$ and $\mu^*(E) \leq \mu^*(A_n) \leq \mu^*(E) + 1/n$. Let $A = \cap_n A_n$, then $A \in \mathcal{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$. By (a) $\mu^*(E) = \mu^+(E)$ for any $E \subset X$.

(c) Since $\mathcal{P}(X) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$, and $\mu^*(\varnothing) = \mu^+(\varnothing) = 0$, let

$$\mu^*(\{0\}) = a, \quad \mu^*(\{1\}) = b, \quad \mu^*(\{0, 1\}) = c$$

because of monotonicity, $0 \leq a \leq c$, $0 \leq b \leq c$. Then by subadditivity, $a + b \geq c$. If $\{0\}$ or $\{1\}$ is μ^*-measurable, then $\mathcal{M}^* = \mathcal{P}(X)$, $\overline{\mu} = \mu^* = \mu^+$. Therefore they must not be μ^*-measurable, $a + b \neq c$. Then $\mu^+(\{0\}) = \mu^+(\{1\}) = c$, $\mu^* \neq \mu^+$. □

Exercise 21

Let μ^* be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\overline{\mu}$ is saturated.

Proof. Give a set $E \subset X$ such that $\forall\, A$ that is μ^*-measurable, $E \cap A$ is still μ^*-measurable and $\mu^*(A) < \infty$, now show that E is μ^*-measurable. For any $F \subset X$ that $\mu^*(F) < \infty$, $\exists\, \epsilon > 0$ such that $A \in \mathcal{A}_\sigma$ such that $F \subset A$ and

$$\begin{aligned}\mu^*(F) + \epsilon \geq \mu^*(A) &= \mu^*(A \cap (A \cap E)) + \mu^*(A \cap (A \cap E)^c) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^c) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)\end{aligned}$$

therefore E is μ^*-measurable. □

Exercise 22

Let $(X, \mathcal{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\overline{\mu} = \mu^*|\mathcal{M}^*$

(a) If μ is σ-finite, then $\overline{\mu}$ is the completion of μ.

(b) In general, $\overline{\mu}$ is the saturation of the completion of μ.

Proof. (a) Since μ is σ-finite, if $E \in \mathcal{M}^*$ then $\exists\, B \in \mathcal{M}$ such that $E \subset B$ and $\mu^*(B \backslash E) = 0$. Therefore for any $n \in \mathbb{N}$, $\exists\, A_n \in \mathcal{M}$ such that $B \backslash E \subset A_n$, $\mu^*(A_n) \leq 1/n$. Then let $A = \cap_n A_n$, $\mu(A) = 0$, $B \backslash E \subset A$. Therefore $(B \backslash A) \subset E$ and $E \backslash (B \backslash A) \subset A$, $E \subset \overline{\mathcal{M}}$. Therefore $\mathcal{M}^* = \overline{\mathcal{M}}$. Obviously the measure on $\overline{\mathcal{M}}$ is the same as the completion of the measure.

(b) Denote the completion of $(\mu, \mathcal{M})$ with $(\hat{\mu}, \overline{\mathcal{M}})$, and the saturation of the completion $(\widetilde{\mu}, \widetilde{\mathcal{M}})$. First show that $\widetilde{\mathcal{M}} = \mathcal{M}^*$. Give any E that is locally $\hat{\mu}$-measurable, for any $F \subset X$ that $\mu^*(F) < \infty$, exists $A \in \mathcal{M}$ such that $F \subset A$ and $\mu^*(F) + \epsilon \geq \mu(A) = \hat{\mu}(A \cap (A \cap E)) + \hat{\mu}(A \cap (A \cap E)^c) \geq \mu^*(E \cap F) + \mu^*(E^c \cap F)$, therefore E is μ^*-measurable. Conversely, if E is μ^*-measurable, for any $A \in \hat{\mathcal{M}}$ such that $\hat{\mu}(A) < \infty$, obviously $A \in \mathcal{M}^*$, therefore $E \cap A \in \mathcal{M}^*$, $\mu^*(E \cap A) = \hat{\mu}(E \cap A) \leq \infty$. Then by (a), $E \cap A \in \overline{\mathcal{M}}$, therefore E is locally $\hat{\mu}$-measurable.

Now show that $\widetilde{\mu} = \overline{\mu}$. $\forall\, E \in \widetilde{\mathcal{M}}$, if E is in $\overline{\mathcal{M}}$, then $\widetilde{\mu}(E) = \overline{\mu}(E)$ since the extension is unique. If E is not in $\overline{\mathcal{M}}$, then $\widetilde{\mu}(E) = \infty$. If $\mu^*(E) < \infty$, then $E \in \overline{\mathcal{M}}$. Therefore $\widetilde{\mu} = \overline{\mu}$. □

Exercise 23

Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$.

(a) $\mathcal{A}$ is an algebra on $\mathbb{Q}$.

(b) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

(c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\varnothing) = 0$ and $\mu_0(A) = \infty$ for $A \neq \varnothing$. Then μ_0 is a premeasure on A, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Proof. (a) Obviously $\mathbb{Q}$ and $\varnothing$ are in $\mathcal{A}$, and finite unions of elements in $\mathcal{A}$ are still in $\mathcal{A}$. Give $(a, b] \cap \mathbb{Q}$, its completion is $(-\infty, a] \cup (b, \infty] \cap \mathbb{Q}$ is still a finite union, therefore $\mathcal{A}$ is an algebra.

(b) Since for any $a \in \mathbb{Q}$, $\cap_{n=1}^{\infty}(a, a + 1/n] \cap \mathbb{Q} = \{a\}$ and $\mathbb{Q}$ is countable, any subset of $\mathbb{Q}$ may be generated by single point sets. Therefore $\mathcal{M}(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$.

(c) It is easy to see that μ_0 is finitely additive. Two measures that agree with μ_0 when restricted to $\mathcal{A}$ may be given: (1) the counting measure; (2) the outer measure given by μ_0. They will produce different results on $\{0\}$. □

Exercise 24

Let μ be a finite measure on $(X, \mathcal{M})$, and let μ^* be the outer measure induced by μ. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$.

(a) If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

(b) Let $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$, and define the function ν on $\mathcal{M}_E$ defined by $\nu(A \cap E) = \mu(A)$. Then $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Proof. (a) $\mu^*(X\backslash E) = 0$. Therefore $\mu(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(B \cap E) = \mu(B)$, and the reverse inequality is also true in the same sense. Therefore $\mu(A) = \mu(B)$.

(b) Obviously $\varnothing$ and E are in $\mathcal{M}_E$. For any $A \in \mathcal{M}$, the completion of $A \cap E$ in E is still in $\mathcal{M}_E$. $\mathcal{M}_E$ is also closed to countable unions since $\mathcal{M}$ is a σ-algebra. Give any countable collection of disjoint sets $\{A_j \cap E\}$, $\nu(\cup_j A_j \cap E) = \mu(\cup_j A_j)$. Let $B_n = A_n \backslash \cup_1^{n-1} A_n$, then $B_j \cap E = A_j \cap E$. Therefore $\mu(\cup_j A_j) = \sum_j \mu(B_j) = \sum_j \mu(A_j) = \sum_j \nu(A_j \cap E)$. □

Exercise 25

If $E \subset \mathbb{R}$, the following are equivalent:

(a) $E \in \mathcal{M}_m u$.

(b) $E = V \backslash N_1$ where V is a G_δ set and $\mu(N_1) = 0$.

(c) $E = H \cup N_2$ where H is an F_σ set an $\mu(N_2) = 0$.

Proof. Obviously (b) and (c) implies (a). Suppose $E \in \mathcal{M}_\mu$, if $\mu(E) < \infty$, give any positive integer n, according the previous proposition one may select an open set U_n and a compact set K_n such that the error of their measure is within $1/n$. Then by taking the countable union or intersetion one may find such H and V. If $\mu(E) = \infty$, let $E_j = E \cap (a_j, b_j]$. For any $\epsilon > 0$, for each j, one can find U_j such that $E_j \subset U_j$ and $\mu(U_j) \le \mu(E_j) + 2^{-j}\epsilon$. Let $V = \cup_j U_j$, then $\mu(V\backslash E) = \sum_j \mu(U_j \backslash E_j) \le \epsilon$. In the same sense one can find a countable union of compact sets, H, such that $\mu(E\backslash H) = 0$. □

Exercise 26

If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \bigtriangleup A) < \epsilon$.

Proof. By theorem 1.18, give any $\epsilon > 0$ one can find a compact K and an open U such that $\mu(U) - \epsilon \le \mu(E) \le \mu(K) + \epsilon$. Therefore one can find finite union of open intervals $I = \cup_j I_j$ that $K \subset I \subset U$. Then $\mu(E \bigtriangleup I) = \mu(E\backslash I) + \mu(I\backslash E) \le 2\mu(U\backslash K) = 2\epsilon$. □

Exercise 27

Denote the Cantor set C. Show that if $x, y \in C$ and $x < y$, there exists $z \notin C$ such that $x < z < y$.

Proof. If such z does not exist, then x, y must lie in the same interval, which implies $|x - y| < 3^{-n}$ for any n, thus $x = y$, contradiction. Therefore x and y must not lie in the same interval. Hence $\exists N$ such that x and y are seperated at the n-th iteration. Thus just pick any z in the middle third of the interval then $x < z < y$. □

Exercise 28

Let F be increasing and right continuous, and let μ_F be the assiciated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a, b)) = F(b-) - F(a-)$, $\mu_F([a, b]) = F(b) - F(a-)$, and $\mu_F = F(b-) - F(a)$.

Proof. Since $\{a\} = \cap_n [a, a+1/n)$, $\mu_F(\{a\}) = \mu(\cap_n (a-1/n, a]) = \lim_{n\to\infty}(F(a) - F(a-1/n)) = F(a) - F(a-)$. Then $\mu_F([a, b)) = \mu_F((a, b]) + \mu(\{a\}) - \mu(\{b\}) = F(b-) - F(a-)$. The rest can be easily shown with the same argument. □

Exercise 29

Let E be a Lebesgue measurable set.

(a) If $E \subset N$ where N is the nonmeasurable set (taking one element of each equivalence class in $[0,1)/\{x-y \in \mathbb{Q}\}$), then $m(E) = 0$.

(b) If $m(E) > 0$, then E contains a nonmeasurable set.

Proof. (a) Suppose $R = \mathbb{Q} \cap [0, 1)$. Take $E_r = \{x + r : x \in E \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in E \cap [1 - r, 1)\}$. Then each E_r is measurable and a subset of $[0, 1)$. Therefore $1 = m([0, 1)) \ge m(\cup_r E_r) = \sum_r m(E_r) = \sum_r m(E)$, $m(E) = 0$. (b) Because of translation invariance it suffices to consider $E \subset [0, 1]$. Obviously $E = \cup_r E \cap N_r$. Then if each $E \cap N_r$ is measurable, $m(E) = \sum_r m(\cup_r(E \cap N_r)) = \sum_r m((E \cap N))$, therefore $m(E) = 0$, contradiction. □

Exercise 30

If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an interval I such that $m(E \cap I) > \alpha m(I)$.

Proof. Suppose that there exists an α such that for every open interval I, $m(E \cap I) \leq \alpha m(I)$. If E is bounded, then there exists a collection of disjoint open intervals such that $E \subset \cup_k I_k$ with $\sum_k m(I_k) \leq (1+\epsilon)m(E)$ for any $\epsilon > 0$. Then $m(E) = m(\cup_k (E \cap I_k)) \leq \sum_k \alpha m(I_k) \leq \alpha(1+\epsilon)(E)$, contradiction. If E is not bounded, by σ-finiteness, one may write $E = \cup_k E_k$ where $m(E_k) < \infty$ for each k. Take E_i such that $m(E_i) > 0$. Then for any $\alpha < 1$ there is an interval I such that $m(E \cap I) \geq m(E_i \cap I) > \alpha m(I)$. $\square$

Exercise 31

If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0.

Proof. By exercise 30, there is an interval $I = (x_0 - \alpha, x_0 + \alpha)$ such that $m(E \cap I) > 3/4m(I)$. Suppose there is a δ such that $0 \leq \delta < a$ and $\delta \notin E - E$. Then for any pair $x, y \in E$, $x - y \neq \delta$. Let $E_1 = E \cap (x_0 - a, x_0]$, $E_2 = E \cap (x_0, x_0 + a)$. Then $\forall\, x \in E_1$, $x + \delta \in I$ but not in E. Therefore $E_1 + \delta \subset I \backslash E$. Similarly $E_2 - \delta \subset I \backslash E$. Then $m(E \cap I) \leq m(E_1) + m(E_2) \leq 2(m(I) - m(I \cap E)) < 2/3m(E \cap I)$, contradiction. Therefore $\delta \in E - E$ and $-\delta \in E - E$, $(-\alpha, \alpha) \subset E - E$. $\square$

Exercise 33

There exists a Borel set $A \subset [0,1]$ such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0,1]$.

Proof. Enumerate the subintervals of I with rational endpoints. Then construct a series of cantor sets. For I_1, split it into two disjoint intervals with finite measure. Then on each subinterval contruct a Cantor set K_1, K_1', both with finite measure. Next assume that $K_1, \cdots, K_n$ and $K_1', \cdots, K_n'$ are already given for $I_1, \cdots, I_n$. Let $L_n = (K_1 \cup \cdots \cup K_n) \cup (K_1' \cup \cdots \cup K_n')$, then L_n is compact and totally disconnected. Therefore $I_{n+1} \backslash L_n$ must contain some intervals, namely J_{n+1}. Then split J_{n+1} and construct K_{n+1} and K_{n+1}' on each subinterval. Let $K = \cup_n K_n$ and then obviously K_n' is disjoint from K for any n. Since K is the union of some Cantor sets, it is a borel set.

Let I be some subinterval of $[0,1]$. Then there must be some I_n such that $I_n \subset I$. Therefore $K_n, K_n' \in I$. Then $0 < m(K_n \cap I_n) \leq m(K \cap I) < m(K \cap I) + m(K_n') \leq m(I)$. $\square$

Chapter 2: Integration

Let the measurable space be $(X, \mathcal{M})$ for Exercise 1-7.

Exercise 1

Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable iff $f^{-1}(\{\pm\infty\}) \in \mathcal{M}$, and f is measurable on Y.

Proof. If f is measurable then $f^{-1}(\{\pm\infty\}) \in \mathcal{M}$. Give any borel set $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, $f^{-1}(B) \in \mathcal{M}$. Therefore $f^{-1}(B \cap \mathbb{R}) = f^{-1}(B) \cap Y \in \mathcal{M}$, f measurable on Y. Conversely, for any borel set $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, $f^{-1}(B) = f^{-1}((B \cap \mathbb{R}) \cup (B \cap \{\infty, -\infty\})) \in \mathcal{M}$, f measurable. $\square$

Exercise 2

Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.

(a) fg is measurable (where $0 \cdot (\pm\infty) = 0$).

(b) Fix $a \in \overline{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

Proof. (a) It is easy to see that $(fg)^{-1}(\pm\infty) \in \mathcal{M}$. Consider fg on $Y = (fg)^{-1}(\mathbb{R})$. If both f and g are finite, then fg measurable on this domain Y_1. If one of the maps is infinite and the other map is zero, denote this domain with $Y_2 \in \mathcal{M}$. Y_2 is included in the inverse image of 0. Therefore fg is measurable on $Y_1 \cup Y_2 = Y$. Therefore fg is measurable on $\overline{\mathbb{R}}$ by exercise 1.

(b) Obviously $(f+g)^{-1}(\{\pm\infty\}) \in \mathcal{M}$. In the same sense consider $f+g$ on Y. If f and g are both finite, then $f+g$ is measurable on this domain Y_1. Otherwise these two maps produce infinity of different signs and included in the reverse image of a. Therefore $f+g$ is measurable on $\overline{\mathbb{R}}$. $\square$

Exercise 3

If $\{f_n\}$ is a sequence of measurable functions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

Proof. $\forall\, x \in X$, $\lim f_n(x)$ exists if and only if $g_3(x) = g_4(x)$, where $g_3(x) = \limsup f_n(x)$, $g_4(x) = \liminf f_n(x)$. Since f_n is measurable for each n, g_3 and g_4 are measurable, which implies $g_3 - g_4$ is also measurable on both $\mathbb{R}$ and $\overline{\mathbb{R}}$. Therefore $\{x : \lim f_n(x) \text{ exists}\} = (g_3 - g_4)^{-1}(\{0\}) \cup \{g_3^{-1}(\infty)\} \cap \{g_4^{-1}(\infty)\} \cup \{g_3^{-1}(-\infty)\} \cap \{g_4^{-1}(-\infty)\}$ is measurable. $\square$

Exercise 4

If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Proof. $\forall\, r \in \mathbb{R}$, by the definition of real numbers there is a cauthy sequence of increasing rational numbers q_n such that $\lim q_n = r$. Then $f^{-1}((r, \infty]) = f^{-1}(\cap_n (q_n, \infty]) = \cap_n f^{-1}((q_n, \infty]) \in \mathcal{M}$, f measurable. $\square$

Exercise 5

If $X = A \cup B$ where $A, B \in \mathcal{M}$, a function f is measurable on X iff f measurable on both A and B.

Proof. Recall that f is measurable on $A \subset X$ if $f^{-1}(B) \cap A \in \mathcal{M}$ for any set B that is measurable. Therefore obviously f measurable on A and B. Conversely, give any measurable set M, then $f^{-1}(M) \cap A \in \mathcal{M}$, $f^{-1}(M) \cap B \in \mathcal{M}$. Then $f^{-1}(M) \in \mathcal{M}$. $\square$

Exercise 6

The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable.

Solution. Consider any unmeasurable set Y (then it is uncountable), give $f_y = \chi_y$ for any $y \in Y$. Then $\sup_y f_y = \chi_Y$ is not measurable since Y is not measurable. $\square$

Exercise 7

Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathcal{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\cup_{\alpha \in \mathbb{R}} E_\alpha = X$, and $\cap_{\alpha \in \mathbb{R}} E_\alpha = \varnothing$. Then there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \le \alpha$ on E_α and $f(x) \ge \alpha$ on E_α^c for every α.

Solution. Take $f(x) = \inf\{q \in \mathbb{Q} : x \in E_q\}$. Then $\forall\, x \in E_\alpha$, for any rational q that $q > \alpha$, $x \in E_q$. Therefore $f(x) \le \alpha$. Similarly $\forall\, x \in E_\alpha^c$, $x \in E_q^c$ for any rational numbers $q \le a$, therefore $x \notin E_q$, x may only be in some E_q that $q > \alpha$, therefore $f(x) \ge \alpha$. Note that: (1) f is $\mathbb{R}$-valued since $\forall\, x \in X$, $x \in E_q$ for some rational q, therefore $f(x) \le q$; if $f(x) = -\infty$ then $x \in \cap_{\alpha \in \mathbb{R}} E_\alpha$ contradiction. (2) f is $\mathbb{R}$-measurable because $\forall\, \alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty)) = \cup_n f^{-1}([q_n, \infty)) = \cup_n \{x : f(x) \ge q_n\} = \cup_n E_q^c \in \mathcal{M}$ where q_n is some decreasing cauthy sequence of rationals that converges to α. $\square$

Exercise 8

If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is borel measurable.

Proof. Without loss of generality, suppose f is increasing, then f^{-1} is also monotone increasing on $\mathrm{Im} f$. Thus $f^{-1}([a, \infty))$ must be some interval, therefore borel measurable. Hence f is borel measurable. $\square$

Exercise 9

Let $f : [0, 1] \to [0, 1]$ be the cantor function, and let $g(x) = f(x) + x$.

(a) g is a bijection from $[0, 1]$ to $[0, 2]$, and $h = g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$.

(b) If C is the cantor set, $m(g(C)) = 1$.

(c) By Exercise 1.29, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel measurable.

Proof. (a) Obviously g is monotone increasing and continuous, thus $g([0, 1]) = [0, 2]$, g is bijective. Therefore $\forall\, (a, b) \in [0, 1]$, $h^{-1}((a, b)) = g((a, b)) = (g(a), g(b))$, h is open.

(b) Recall $C = [0, 1]\backslash(\cup_k I_k)$. Since g is bijective and $\{I_k\}$ is pairwise disjoint, $g(C) = [0, 2]\backslash g(\cup_k I_k) = [0, 2]\backslash(\cup_k g(I_k))$. By the construction of f, f is constant on I_k. Thus $m(g(I_k)) = m(I_k)$. Therefore

$$m(g(C)) = m([0, 2]) - \sum_k m(I_k) = 1$$

(c) Since $B = g^{-1}(A) \subset g^{-1}(g(C)) = C$, B must be of zero measure because it is contained in some null sets. Since h is continuous hence borel measurable, if B is borel measurable then $A = h^{-1}(B)$ would be borel measurable, contradiction. □

Exercise 10

The following implications are valid iff the measure μ is complete.

(a) If f is measurable then $f = g$ μ-a.e., then g is measurable.

(b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Proof. (a) If μ is complete, then $g - f$ must be measurable since it is only non-zero on some null sets, therefore $g = g - f + f$ is Lebesgue measurable. Conversely, suppose any $N \subset E$ with E a null set. Then let $f = \chi_E$, $g = \chi_{E\backslash N}$. Then $f - g = \chi_N$ must be measurable. Therefore $N = (f - g)^{-1}(\{1\})$ is measurable.

(b) Since f_n is measurable for each n, $\lim f_n$ is measurable, and $\lim f_n = f$ μ-a.e.. If μ is complete, by (a) f is measurable. Conversely, suppose any subset N of a null set, take $f_n = 0$ for each n and $f = \chi_N$, then f is measurable, N must be measurable. □

Exercise 11

Suppose that f is a function on $\mathbb{R} \times \mathbb{R}^k$ such that $f(x, \cdot)$ is borel measurable for each $x \in \mathbb{R}$ and $f(\cdot, y)$ is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows. For $i \in \mathbb{Z}$ let $a_i = i/n$, and for $a_i \le x \le a_{i+1}$ let

$$f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}$$

Then f_n is borel measurable on $\mathbb{R} \times \mathbb{R}^k$ and $f_n \to f$ pointwise; hence f is borel measurable on $\mathbb{R} \times \mathbb{R}^k$. Conclude by induction that every function on $\mathbb{R}^n$ that is continuous in each variable separately is Borel measurable.

Proof. Since $f(x, \cdot) : \mathbb{R}^k \to \mathbb{R}$ and $x - a_i : \mathbb{R} \to \mathbb{R}$ is measurable, $f_n(x, y)$ is measurable. Now show that $f_n \to f$ pointwise. Since

$$\begin{aligned} |f - f_n| &= |f(x, y) - \frac{1}{a_{i+1} - a_i} f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})| \\ &= \frac{1}{a_{i+1} - a_i} |(f(x, y) - f(a_{i+1}, y))(x - a_i) - (f(a_i, y) - f(x, y))(x - a_{i+1})| \end{aligned}$$

Suppose some $\epsilon > 0$, then there is a open neighbourhood $B_\delta(x)$ such that $\forall\, x' \in B_\delta(x)$, $|f(x) - f(x_0)| < \epsilon$. Take n large enough such that $[a_i, a_{i+1}]$ is in that neighbourhood, then

$$|f - f_n| \le \frac{\epsilon}{a_{i+1} - a_i} |(a_{i+1} - a_i)| = \epsilon$$

Since $f_n \to f$, f is borel measurable on $\mathbb{R} \times \mathbb{R}^k$. If $f(x) : \mathbb{R} \to \mathbb{R}$ is continuous, then it is measurable. Assume that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to each variable then it is measurable. Then suppose any function $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. By previous exercise g is measurable. Therefore the proof is done by induction. □

Exercise 13

Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ forall $E \in \mathcal{M}$. However, this need not be true if $\int f = \lim \int f_n = \infty$.

Proof. By Fatou's lemma,

$$\int_E f = \int f\chi_E = \int \liminf f_n\chi_E \leq \liminf \int f_n\chi_E = \liminf \int_E f_n$$

Conversely, write

$$\int f - \int_E f = \int_{E^c} f \leq \liminf \int_{E^c} f_n = \liminf(\int f - \int_E f) = \int f - \limsup \int_E f$$

therefore $\limsup \int_E f_n \leq \int_E f \leq \liminf \int_E f_n$, $\lim \int_E f_n = \int_E f$, the proof is done. For counter-examples, just take $f_n = \chi_{[n,n+1]} + \chi_{(-\infty,0]}$ and $E = [0,\infty)$. $\square$

Exercise 14

If $f \in L^+$, let $\lambda(E) = \int_E f \mathrm{d}\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g \mathrm{d}\lambda = \int fg \mathrm{d}\mu$.

Proof. $\lambda(\varnothing) = 0$. Suppose a collection of disjoint measurable sets $\{E_n\}$, then $\lambda(\cup_n E_n) = \int f\chi_{\cup_n E_n} \mathrm{d}\mu = \sum_n \int f\chi_{E_n} \mathrm{d}\mu = \sum_n \lambda(E_n)$, therefore λ is a measure.

Give $\phi = \sum_i a_i\chi_{E_i}$ a simple function. Then $\int \phi \mathrm{d}\lambda = \sum_i a_i\lambda(E_i) = \int f \sum_i a_i\chi_{E_i} \mathrm{d}\mu = \int f\phi \mathrm{d}\mu$. Now suppose $\{\phi_n\}$ an increasing collection of simple functions that $\phi_n \to g$. Then

$$\int g \mathrm{d}\lambda = \lim \int \phi_n \mathrm{d}\lambda = \lim \int f\phi_n \mathrm{d}\mu = \int fg \mathrm{d}\mu$$

$\square$

Exercise 15

If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

Proof. Obviously $\{f_1 - f_n\}$ increases pointwise to $\{f_1 - f\}$. Therefore by MCT,

$$\lim \int (f_1 - f_n) = \int (f_1 - f)$$

hence

$$\int f = \int f_1 - \int (f_1 - f) = \int f_1 - \lim \int (f_1 - f_n) = \lim f_n$$

where the last equality is because $\int (f_1 - f_n) + \int f_n = \int f_1$. $\square$

Exercise 16

If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > \int f - \epsilon$.

Proof. By the definition of integration, for every $\epsilon > 0$, there exists a simple function ϕ that $\int \phi > \int f - \epsilon$. Write $\phi = \sum_i a_i\chi_{E_i}$ with the standard representation (where $a_i \neq 0$ for each i). Let $E = \cup_i E_i$, then $\int_E f > \int \phi > \int f - \epsilon$. Now show that E is of finite measure. It is obvious that

$$\infty > \int \phi \geq \int \min\{a_i\}\chi_E = \min\{a_i\}\mu(E)$$

therefore $\mu(E) < \infty$. $\square$

Exercise 17

Assume Fatou's Lemma and deduce the monotone convergence theorem.

Proof. Suppose $\{f_n\}$ is a sequence in L^+ such that $f_j \le f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n = \liminf f_n$, then by Fatou's lemma,

$$\int f = \int \liminf f_n \le \liminf \int f_n$$

Conversely,

$$0 = \int \liminf(f - f_n) \le \liminf \int (f - f_n) = \liminf(\int f - \int f_n) = \int f - \limsup \int f_n$$

where $\int(f - f_n) = \int f - \int f_n$ because of $\int(f - f_n + f_n) = \int f_n + \int(f - f_n) = \int f$. Thus $\int f = \lim \int f_n$. □

Exercise 18

Fatou's lemma remains valid if the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that f_n is measurable and $f_n \ge -g$ where $g \in L^+ \cap L^1$.

Proof. Obviously $g_n = f_n + g \ge 0$. Then $\{g_n\}$ is a sequence in L^+. Therefore by Fatou's lemma,

$$\int \liminf g_n = \int \liminf f_n + \int g \le \liminf \int f_n + \int g$$

therefore $\int \liminf f_n \le \liminf \int f_n$. □

Exercise 19

Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.
(a) If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.
(b) If $\mu(X) = \infty$, the conclusions of (a) can fail.

Proof. (a) Since $f_n \to f$ uniformly, $\exists N$ such that $\forall n \ge N$ and $\forall x \in X$, $|f(x) - f_n(x)| \le 1$. Let $g(x) = |f_N(x)| + 1$, then $f_n \le g$ for each n. Since

$$\int g = \int |f_N(x)| + 1 = \int f_N(x) + \mu(X) < \infty$$

by DCT $f \in L^1(\mu)$ and $\int f_n \to \int f$.
(b) Just take $f_n = (1/n)\chi_{[0,n)}$ □

Exercise 20

If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \le g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$.

Proof. By taking real and imaginary parts, assume f_n and g_n are real. Then $f_n + g_n \ge 0$ and $g_n - f_n \ge 0$. By Fatou's Lemma,

$$\int (f+g) \le \int \liminf(f_n + g_n) \le \liminf \int (f_n + g_n) = \liminf \int f_n + \int g$$

$$\int (g-f) \le \int \liminf(g_n - f_n) \le \liminf \int (g_n - f_n) = \int g - \limsup \int f_n$$

thus $\int f_n \to \int f$. □

Exercise 21

Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f - f_n| \to 0$ iff $\int |f_n| \to \int |f|$.

Proof. Obviously

$$\left|\int |f| - \int |f_n|\right| = \left|\int |f| - |f_n|\right| \leq \int |f - f_n| \to 0$$

Conversely, if $\int |f_n| \to \int |f|$, then by Exercise 20, $\int f_n \to \int f$. Thus $|\int f - \int f_n| = \int |f - f_n| \to 0$. □

Exercise 22

Let μ be a counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorem as statements about infinite series.

Solution. Obviously the measure of a measurable function f on $(\mathbb{N}, \mu)$ is $\int f = \sum_n f(n) = \sum_n a_n$. Therefore by Fatou's lemma, suppose $\{a_{nk}\}$ a sequence of nonnegative numbers, then $\sum_k \liminf_n a_{nk} \leq \liminf_n \sum_k a_{nk}$. By MCT, given a sequence of nonnegative numbers $\{a_{nk}\}$, if $a_{nk} \leq a_{n+1,k}$ for every n and k, and $a_{nk} \to a_k$ for every k, then $\lim_n \sum_k a_{nk} = \sum_k a_k$. The DCT says that for any sequence of complex numbers $\{a_{nk}\}$ such that $|a_{nk}| \leq |g_k|$ for each k, and $a_{nk} \to a_k$ for every k, then $\lim_n \sum_k a_{nk} = \sum_k a_k$. □

Exercise 25

Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.

(a) $g \in L^1(m)$, and in particular $g < \infty$ a.e.

(b) g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

(c) $g^2 < \infty$ a.e., but g^2 not integrable on any interval.

Proof. (a) Observe

$$\int |g| = \int \sum_1^\infty \frac{f(x - r_n)}{2^n} = \sum_1^\infty \frac{1}{2^n} \int f(x - r_n) = \sum_1^\infty \frac{1}{2^{n-1}} < \infty$$

where by MCT,

$$\int f(x - r_n) = \lim_{t \to \infty} \int f(x - r_n) \chi_{(r_n + 1/t, r_n + 1)} = \lim_{t \to \infty} \int_{r_n}^{r_n + 1/t} (x - r_n)^{1/2} \mathrm{d}x = 2$$

therefore $g \in L^1(m)$, and obviously $g < \infty$ a.e.

(b) Suppose $x_0 \in \mathbb{R}$ with g continuous at x_0. Then obviously $g(x_0) < \infty$. For any $\epsilon > 0$ and $0 < \delta < 1$, there exists $r_n \in \mathbb{Q}$ such that $x_0 < r_n < x_0 + \delta$. Let $x' \in (r_n, x_0 + \delta)$ such that

$$g(x_0) + \epsilon < \frac{1}{2^n} f(x' - r_n)$$

then $g(x') \geq \dfrac{1}{2^n} f(x' - r_n) \geq g(x_0) + \epsilon$. Since δ is arbitrary, contradiction. For any interval $(a, b) \subset \mathbb{R}$, take $r_n \in (a, b)$. Then for any ϵ that is sufficiently large, $g(r_n + (\frac{1}{2^n}\epsilon)^2) \geq \epsilon$. Therefore $g(x)$ is unbounded on any interval. If after modification g is no longer unbounded on some interval, take this interval as the same interval (a, b). then $\exists \epsilon > 0$ such that $g(x - r_n) < \epsilon$ for all $x \in (a, b)$, then g is modified on at least $(r_n, r_n + (\frac{1}{2^n}\epsilon)^2)$ which has a non-zero measure, contradiction.

(c) By (a) it immediately follows that $g^2 < \infty$ a.e. For the second part, observe

$$\int g^2 \geq \int \sum_1^\infty \frac{f^2(x - r_n)}{4^n} = \sum_1^\infty \frac{1}{4^n} \int f^2(x - r_n) = \infty$$

where $\int f^2(x - r_n) = \infty$ follows the same argument as (a). □

Exercise 32

Suppose $\mu(X) < \infty$. If f and g are complex valued measurable functions on X, define

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ w.r.t. this metric iff $f_n \to f$ in measure.

Proof. The triangle inequality is obvious since

$$\frac{|f-g|}{1+|f-g|} = 1 - \frac{1}{1+|f-g|}$$

is an increasing function of $|f-g|$. Suppose $\epsilon > 0$. If $f_n \to f$ in measure then for any $\eta > 0$, $\exists\, N$ such that $\forall\, n \geq N$,

$$\mu(E_n = \{x : |f_n(x) - f(x)| > \epsilon\}) < \eta$$

take $\eta = \epsilon$, then

$$\rho(f_n, f) = \int_{E_n} \frac{|f_n - f|}{1+|f_n-f|} + \int_{E_n^c} \frac{|f_n - f|}{1+|f_n-f|} \leq \mu(E_n) + \mu(X)\epsilon = \epsilon(1+\mu(X)) \to 0$$

Conversely suppose $\rho(f_n, f) \to 0$. Then $\forall\, \eta > 0$, $\exists\, N$ such that if $n \geq N$, $\rho(f_n, f) < \eta$. Consequently,

$$\frac{\epsilon}{1+\epsilon}\mu(E_n) \leq \int_{E_n} \frac{|f_n - f|}{1+|f_n-f|} \leq \eta$$

therefore $\forall\, t > 0$, take $\eta = \dfrac{\epsilon t}{1+\epsilon}$, then $\exists\, N$ such that $\mu(E_n) \leq \eta\dfrac{1+\epsilon}{\epsilon} = t$. $\square$

Exercise 33

If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

Proof. Recall that given a sequence of real numbers $\{a_n\}$, there exist a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to L$ for any $\liminf a_n \leq L \leq \limsup a_n$. Then there is a subsequence $\int f_{n_k}$ such that $\lim \int f_{n_k} = \liminf \int f_n$. Obviously $f_{n_k} \to f$ in measure, therefore there is a subsequence $f_{n_{k_i}}$ that converges to f a.e. Therefore by Fatou's Lemma,

$$\int f = \int \liminf_i f_{n_{k_i}} \leq \liminf_i \int f_{n_{k_i}} = \lim_k \int f_{n_k} = \liminf \int f_n$$

$\square$

Exercise 34

Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure,

(1) $\int f = \lim \int f_n$,

(2) $f_n \to f$ in L^1.

Proof. (a) Since $f_n \to f$ in measure iff $\mathrm{Re}(f_n) \to f$ in measure and $\mathrm{Im}(f_n) \to f$ in measure, assume f_n and f are real-valued. Since $f_n \in L^1$ and there is a subsequence of f_n that converges to f a.e., $f \in L^1$. Since $g + f_n$ and $g - f_n$ are non-negative functions, the previous exercise implies that

$$\int g + \int f = \int \liminf(g + f_n) \leq \liminf \int (g + f_n) = \int g + \liminf \int f_n$$

$$\int g - \int f = \int \liminf(g - f_n) \leq \liminf \int (g - f_n) = \int g - \limsup \int f_n$$

therefore $\int f = \lim \int f_n$.

(b) Obviously $|f_n - f|$ converges to 0 in measure. Since $|f_n - f| \leq |f_n| + |f| \leq 2|g| \in L^1$, by (a), $\lim \int |f_n - f| = 0$, $f_n \to f$ in L^1. $\square$

Exercise 35

$f_n \to f$ in measure iff for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \leq \epsilon$ for $n \geq N$.

Proof. For any $\epsilon, \eta > 0$, suppose $\eta < \epsilon$, then $\exists N$ such that $\forall n \geq N$, $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| > \eta\}) < \eta$. The reverse direction is trivial. □

Exercise 36

If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f \in L^1$, then f is a.e. equal to the characteristic function of a measurable set.

Proof. Since $\chi_{E_n} \to f$ in L^1, there exists a subsequence $\chi_{E_{n_k}} \to f$ a.e. Therefore there is a measurable function g such that $g = f$ a.e. Since f and g can only take values 0 or 1, $f = \chi_{g^{-1}\{1\}}$ a.e. □

Exercise 37

Suppose that f_n and f are measurable compelx-valued functions and $\phi : \mathbb{C} \to \mathbb{C}$.

(a) If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e.

(b) If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$, uniformly, almost uniformly, or in measure, respectively.

(c) There are counterexamples when the continuity assumptions on ϕ are not satisfied.

Proof. (a) Let $x \in X$ be a point where f_n converges to f. Then

$$\lim_{n\to\infty} \phi(f_n(x)) = \phi(\lim_{n\to\infty} f_n(x)) = \phi(f(x))$$

so $\phi \circ f_n \to \phi \circ f$ a.e.

(b) Suppose $f_n \to f$ uniformly, $\forall \epsilon > 0$, $\exists N$ such that $|f_n - f| < \epsilon$ for $n \geq N$. Since ϕ is also uniformly continuous, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|\phi(f_n) - \phi(f)| < \epsilon$ for any $|f_n - f| < \delta$. Therefore $\phi \circ f_n \to \phi \circ f$ uniformly. The same argument applys for the almost uniform case. If $f_n \to f$ in measure, since ϕ is uniformly continuous, $\exists \eta$,

$$\{x : |\phi(f_n(x)) - \phi(f(x))| < \epsilon\} \subset \{x : |f_n(x) - f(x)| < \eta\}$$

the proof is done since $\mu(\{x : |f_n(x) - f(x)| < \eta\}) \to 0$

(c) Give $f_n = e^{-n}$, $f_n \to f$ uniformly, suppose $\phi = \ln x$, then $\phi \circ f_n = -n$, which is anywhere divergent. □

Exercise 38

Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) $f_n + g_n \to f + g$ in measure.

(b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Proof. (a) Let $\epsilon > 0$, then $\exists N_f, N_g$ such that $\mu(\{x : |f_n - f| \geq \epsilon/2\}) < \epsilon/2$ for $n > N_f$ and likewise for g. When n is large enough, since $|(f_n + g_n) - (f + g)| \leq |f_n - f| + |g_n - g|$,

$$\{x : |(f_n + g_n) - (f + g)| \geq \epsilon\} \subset \{x : |f_n - f| \geq \epsilon/2\} \cup \{x : |g_n - g| \geq \epsilon/2\}$$

therefore $\mu(\{x : |(f_n + g_n) - (f + g)| \geq \epsilon\}) \to 0$.

(b) Likewise define ϵ, N_f, N_g. Since $|f_n g_n - fg| \leq |f_n - f||g_n - g| + |f||g_n - g| + |g||f_n - f|$,

$$\{x : |fg - f_n g_n| > \epsilon\} \subset \{x : |f_n - f||g_n - g| > \epsilon/3\} \cup \{x : |f_n - f||g| > \epsilon/3\} \cup \{x : |f||g_n - g| > \epsilon/3\}$$

It is obvious that $\mu(\{x : |f_n - f||g_n - g| > \epsilon/3\}) \to 0$. To show $\mu(\{x : |f||g_n - g| > \epsilon/3\}) \to 0$, claim that for any $\eta > 0$, $\exists N \in \mathbb{N}$ such that $\mu(\{x : |f| > N\}) < \eta$. Let $E_n = \{x : |f| > n\}$, then E_n is a decreasing sequence of sets. Since $\mu(X) < \infty$, and $|f|$ can only take on finite values which implies $\cap_n E_n = \varnothing$, by convergence from below, $\mu(E_n) \to 0$, which verifies the claim. Since

$$\{x : |f||g_n - g| > \epsilon/3\} \subset \{x : |f| > N\} \cup \{x : |g_n - g| < \epsilon/3N\}$$

for each N, there is

$$\mu(\{x : |f_n - f||g_n - g| > \epsilon/3\}) \leq \mu(\{x : |f| > N\}) + \mu(\{x : |g_n - g| > \epsilon/3N\})$$

therefore $\forall \nu > 0$, take N and n such that $\mu(\{x : |f| > N\}) < \nu/2$ and $\mu(\{x : |g_n - g| > \epsilon/3N\} < \nu/2$, it can be seen that $\mu(\{x : |f||g_n - g| > \epsilon/3\}) \to 0$, similarly $\mu(\{x : |g||f_n - f| > \epsilon/3\}) \to 0$, the proof is done. $\square$

Exercise 39

If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

Proof. Since $f_n \to f$ almost uniformly, $\forall n \in \mathbb{N}$, $\exists E_n \subset X$ such that $\mu(E_n) < 1/n$ and $f_n \to f$ uniformly on E_n^c. Then obviously $E = \cap_n E_n$ has zero measure by continuity from below, and $f_n \to f$ on E^c. Therefore $f_n \to f$ a.e.

$\forall \epsilon > 0$, take $E \subset X$ such that $f_n \to f$ uniformly on E^c and $\mu(E) < \epsilon$. Then $\forall \eta > 0$, $\exists N$ such that if $n > N$

$$\{x : |f_n - f| > \eta\} \subset E$$

therefore $f_n \to f$ in measure. $\square$

Exercise 40

In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$".

Proof. Without loss of generality, assume $f_n \to f$ for all $x \in X$. For $k, n \in \mathbb{N}$, let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x : |f_m - f| \geq k^{-1}\}$$

then for fixed k, E_n is a decreasing sequence. For $x \in X$, if $x \in E_1(k)$, then $\exists m$ such that $|f_m - f| \geq 1/k$. Therefore $1/k \leq |f_m + f| \leq 2g$, $\int 1/2k\chi_{E_1(k)} = 1/2k\mu(E_1(k)) \leq \int g$. Since $g \in L^1$, $\mu(E_1(k)) < \infty$. Therefore by continuity from below, $\mu(E_n(k)) \to 0$. Given $\epsilon > 0$ and $k \in \mathbb{N}$, choose n_k so large that $\mu(E_{n_k}(k)) \leq \epsilon 2^{-k}$, and let $E = \cup_k E_{n_k}(k)$. Then $\mu(E) \leq \epsilon$, and $|f_n - f| \leq 1/k$ for $n > n_k$ and $x \in E^c$. $\square$

Exercise 41

If μ is σ-finite and $f_n \to f$ a.e., there exist measurable $E_1, E_2, \cdots \subset X$ such that $\mu((\cup_1^\infty E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.

Proof. Suppose $\mu(X) < \infty$, then by Egoroff's theorem, for each $k \in \mathbb{N}$, $\exists E_k$ such that $\mu(E_k^c) < 1/k$ and $f_n \to f$ uniformly on E_k. Let $F_n = \cup_1^n E_k$, then F_n^c is a decreasing sequence, therefore

$$\mu\left(\left(\bigcup_1^\infty E_j\right)^c\right) = \mu\left(\left(\bigcup_1^\infty F_j\right)^c\right) = \mu\left(\bigcap_1^\infty F_j\right) = 0$$

and $f_n \to f$ uniformly on each E_j.

Since μ is σ-finite, $X = X_1 \cup X_2 \cdots$ each with finite measure. Therefore for each i, there exists $\{E_k^i\}$ such that $\mu(X_i \backslash (\cup_k E_k^i)) = 0$ and $f_n \to f$ uniformly on each E_k^i. Since

$$\mu\left(\left(\bigcup_{i,k} E_k^i\right)^c\right) \leq \mu\left(\bigcup_i \left(X_i \backslash \bigcup_k E_k^i\right)\right) = 0$$

$\{E_k^i\}$ gives the desired sequence. $\square$

Exercise 42

Let μ be the counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

Proof. Suppose $f_n \to f$ in measure. Then $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that if $n > N$,

$$\mu(\{x : |f_n - f| > \epsilon\}) < 1/2$$

therefore $|f_n - f| < \epsilon$ for each $x \in \mathbb{N}$, hence $f_n \to f$ uniformly. Conversely, if $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that if $n > N$, $|f_n - f| < \epsilon$ for each $x \in \mathbb{N}$, then $\mu(\{x : |f_n - f| > \epsilon\}) = 0$. □

Exercise 44

If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a, b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.

Proof. For each $n \in \mathbb{N}$, let $E_n = f^{-1}(B_n(0))$. Then

$$\lim \mu(E_n) = \mu(\cup_n E_n) = \mu([a, b])$$

therefore $\exists\, m \in \mathbb{N}$ such that $\mu([a, b]) - \mu(E_m) \leq \epsilon/3$. Then $|f\chi_{E_m}| \leq m\chi_{[a,b]}$, thus $g \in L^1$. Hence by theorem 2.26 there is a sequence of continuous functions $g_j \to f\chi_{E_m}$. By corollary 2.32, there is a subsequence $g_{j_i} \to f\chi_{E_m}$ a.e. By Egoroff's theorem, there exists $F \subset E_m$ such that $g_{j_i} \to f\chi_{E_m}$ uniformly on $E_m \backslash F$ and $\mu(F) < \epsilon/3$. By theorem 1.18, there exists a compact set E such that $E \subset E_m \backslash F$ and $\mu(E) > \mu(E_m \backslash F) + \epsilon/3$. Therefore $f\chi_E$ is continuous, and

$$\mu(E^c) = \mu(E_m^c) + \mu(E_m \backslash E) \leq \epsilon/3 + \mu(E_m \backslash F) + \mu(E_m \backslash F \backslash E) \leq \epsilon$$

□