CHAOS THEORY

How well can we predict the movement of a seemingly irregular system?

Chaos theory is a branch of mathematics that studies chaos. Chaotic systems at the first glance may seem indeterministic but in reality they are just dynamic systems whose apparent random states of disorder and irregularities are governed by underlying patterns and deterministic laws that are highly sensitive to initial conditions. In this report I aim to discuss solving Chaotic Systems such as the motion of three planetary bodies and the motion of a double pendulum using the Runge-Kutta Methods.

THE DOUBLE PENDULUM

The double pendulum does unexpected turns and loops, and the exact direction of the ball after a few bounces is difficult to predict. The motion is controlled by ordinary differential equations, whose solutions are extremely sensitive to the initial conditions. The resulting shape of the trajectories is confusing, although it can be computed rather simply to arbitrarily many decimals (Gutzwiller, 2007). A pendulum with a single arm with one bob has a predictable motion, in which the bob oscillates towards and away from equilibrium position of the. But as soon as another arm and bob is added the kinetic and potential energy of 2 masses affect each other and creates a chaotic motion that is sensitive to initial conditions. Though a general solution for the double pendulum does not exist, the path of the double pendulum can still be predicted using the fourth order Runge-Kutta method.

Physical Background

Suppose there is a bob with mass m, connected to a massless arm with length L, which is hanging from a point. And at the end of that pendulum another pendulum with mass m, connected to a massless arm with length L is connected. The angle of the arms at any given time is given by the angles θ_1 and θ_2 .

The heights of the 2 bobs are given by;

$h_1 = -Lcos\theta_1$ and $h_2 = -L(cos\theta_1 + cos\theta_2)$

And the total potential energy would be,

$U = mgh_1 + mgh_2$, since both bobs have the same mass, m

$U = \ mg(h_1 + h_2) = \ -mgL(2cos\theta_1 + cos\theta_2)$

The velocity of the two bos can be calculated by taking the first time derivative of θ_1 and θ_2. the square of the velocities would be;

$v_1^2 = \ L^2\left(\frac{d\theta_1}{dt}\right)^2$ and the for $v_2^2 = \ L^2\left(\frac{d\theta_1}{dt}\right)^2 + L^2\left(\frac{d\theta_2}{dt}\right)^2 + 2L^2\left(\frac{d\theta_1}{dt}\right)\left(\frac{d\theta_2}{dt}\right)cos(\theta_1 - \theta_2)$

Therefore the total kinetic energy would be,

$$K = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = mL^2\left[v_2^2 = \left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\theta_2}{dt}\right)^2 + \left(\frac{d\theta_1}{dt}\right)\left(\frac{d\theta_2}{dt}\right)cos(\theta_1 - \theta_2)\right]$$

The Lagrangian of the system is;

$$\mathcal{L} = K - U = mL^2\left[v_2^2 = \left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\theta_2}{dt}\right)^2 + \left(\frac{d\theta_1}{dt}\right)\left(\frac{d\theta_2}{dt}\right)cos(\theta_1 - \theta_2)\right] + mgL(2cos\theta_1 + cos\theta_2)$$

Taking the partial derivative of Lagrangian, with respect to $\frac{d\theta_1}{dt}$ and $\frac{d\theta_2}{dt}$ and then taking the time derivative of that partial derivative the Euler-Lagrange equation is obtained which then produces two second order equations:

$$2\left(\frac{d^2\theta_1}{dt^2}\right) + \left(\frac{d^2\theta_2}{dt^2}\right)cos(\theta_1 - \theta_2) + \left(\frac{d\theta_2}{dt}\right)^2 sin(\theta_1 - \theta_2) + 2\frac{g}{L}sin\theta_1 = 0$$

$$\left(\frac{d^2\theta_2}{dt^2}\right) + \left(\frac{d^2\theta_1}{dt^2}\right)cos(\theta_1 - \theta_2) - \left(\frac{d\theta_1}{dt}\right)^2 sin(\theta_1 - \theta_2) + \frac{g}{L}sin\theta_2 = 0$$

Converting these two second order equations into four first order equation using change of variables:

$\omega_1 = \frac{d\theta_1}{dt}$ and $\omega_2 = \frac{d\theta_2}{dt}$

and taking the first time derivative;

$\frac{d\omega_1}{dt} = \frac{d^2\theta_1}{dt^2}$ and $\frac{d\omega_2}{dt} = \frac{d^2\theta_2}{dt^2}$

Therefore,

$$2\left(\frac{d\omega_1}{dt}\right) + \left(\frac{d\omega_2}{dt}\right)cos(\theta_1 - \theta_2) + (\omega_2)^2 sin(\theta_1 - \theta_2) + 2\frac{g}{L}sin\theta_1 = 0$$

$$\left(\frac{d\omega_2}{dt}\right) + \left(\frac{d\omega_2}{dt}\right)cos(\theta_1 - \theta_2) - (\omega_1)^2 sin(\theta_1 - \theta_2) + \frac{g}{L}sin\theta_2 = 0$$

Rearranging it to one single derivative of ω_1 and ω_2 ;

$$\frac{d\omega_1}{dt} = -\frac{\omega_1^2 sin(2\theta_1 - 2\theta_2) + \omega_2^2 sin(\theta_1 - \theta_2) + g/L\left[sin(\theta_1 - 2\theta_2) + 3sin\theta_1\right]}{3 - cos(2\theta_1 - 2\theta_2)}$$

$\frac{d\omega_2}{dt} = -\frac{4\omega_1^2 sin(\theta_1-\theta_2)+\omega_2^2 sin(2\theta_1-2\theta_2)+2g/L\left[sin(2\theta_1-\theta_2)-sin\theta_2\right]}{3-cos(2\theta_1-2\theta_2)}$ (Newman, 2013, pp. 399-400)

Computational Background

The computational method used to solve the Double Pendulum system is the Runge-Kutta method. In numerical analysis, the Runge–Kutta methods are a family of implicit and explicit iterative methods, integrating systems of Ordinary differential Equations with initial values (DeVries et al., 2010). These methods were developed around 1900 by the German mathematicians Carl Runge and Wilhelm Kutta.

As the order of Runge-Kutta equations increases, it gets more and more accurate in solving differential equations. But the downside is that the equations get more complicated as the order increases. The sweet spot however is the fourth order Runge-Kutta equations, which offers a good balance of high accuracy and equations that are still relatively simple to program. The fourth order Runge-Kutta equations are the following:

$k_1 = hf(x, t)$

$k_2 = hf(x + \frac{1}{2}k_1, t + \frac{1}{2}h)$

$k_3 = hf(x + \frac{1}{2}k_2, t + \frac{1}{2}h)$

$k_4 = hf(x + k_3, t + h)$

$x(t + h) = x(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

The RK4 method is by far the most common method for numerical solution of ordinary differential equations. It is accurate intemrs of h^4 and carries and error of the order h^5. The derivation of these equations is quite complicated. The final equations are relatively simple. These five equations result in a method that is three orders of h more accurate than Euler's method for each step size (Newman, 2013, pp. 336-337).

Program and Plots

Using the 4 first order differential equations obtained in the physical background, the Runge-Kutta method described in the computational background the code '*double_pendulum_with_prompt.py*' is written. Which will ask the user to input the initial condition of the starting angles for the pendulum and plot the trajectory of it over 100 seconds.

Figure 1 and Figure 2 plots the movement of the pendulum as a function of time, for the given initial angles, length of the arm and mass of the bob.

Table 1: Initial Conditions for the Double Pendulum

Pendulum	1	2
Mass, m	1	1
Length, L	0.4	0.4
Angle, θ	0.5 π	0.5 π

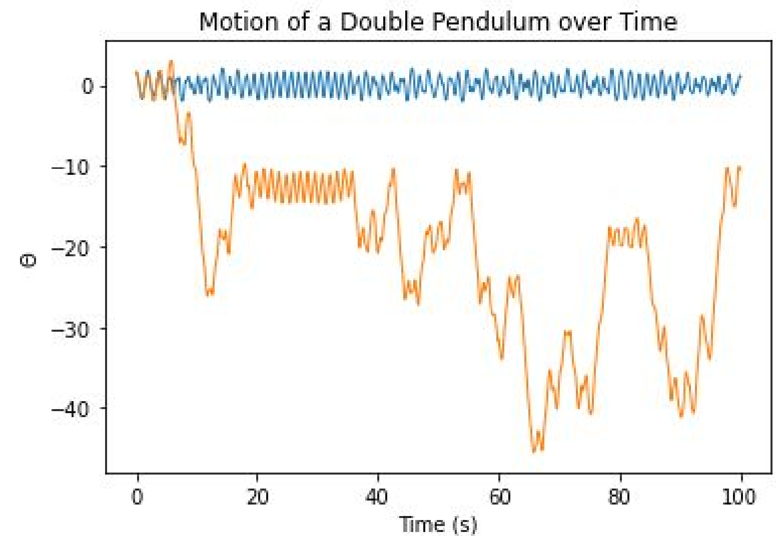


Figure 1: The resulting plot of given initial conditions in Table 1.

Initial velocity is 0 and the program runs for 10000 steps.

Table 2: Initial Conditions for the Double Pendulum

Pendulum	1	2
Mass, m	1	1
Length, L	0.4	0.4
Angle, θ	0.45 π	0.54 π

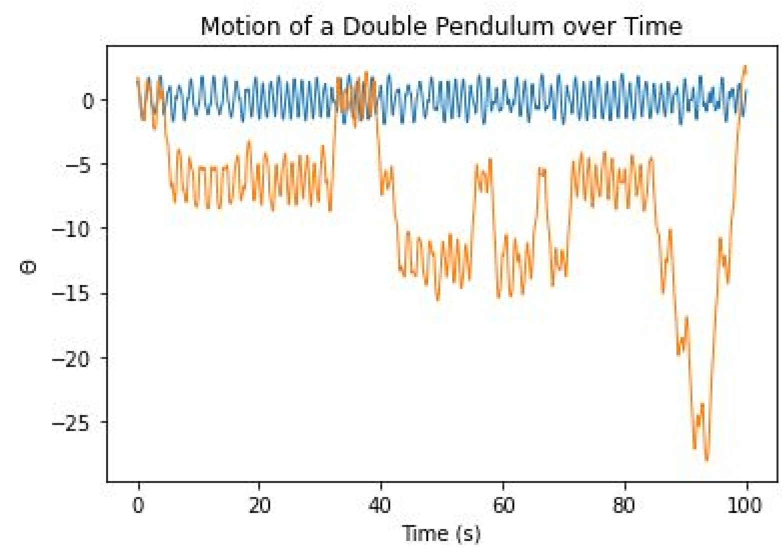


Figure 2: The resulting plot of given initial conditions in Table 2.

Initial velocity is 0 and the program runs for 10000 steps.

Because the double pendulum is a chaotic system the solution would be unpredictable but still deterministic, the Runge-Kutta method, solves these differential equations and gives a possible solution for almost any initial condition. As seen in figure 1 and figure 2 even a small change in the initial angles results in a different plot.

THREE BODY PROBLEM

The three-body problem is taking the initial positions and initial velocities of 3 point masses and using that to solve their subsequent motions according Newton's law of motion and Newton's law of gravitational forces. Unlike a two-body problem there isn't a general solution, since the three bodies generate a dynamical system that is chaotic for most initial conditions, computational methods are required to solve it. Adaptive Runge Kutta method will be used to solve the three body problem.

Physical Background

Suppose we have a planetary system with three stars A, B, and C. Using Newton's 2nd Law of motion, we can find the total force applied on A by B and C, using the following equations of motion (Chenciner, 2007);

Force applied by B, $F_B = \frac{Gm_A m_B}{|r_b - r_a|^2}(r_b - r_a)$

Force applied by B, $F_C = \frac{Gm_A m_c}{|r_c - r_a|^2}(r_c - r_a)$

Adding them up we get the total force on A,

$$F_A = F_B + F_C = \frac{Gm_A m_B}{|r_b - r_a|^2}(r_b - r_a) + \frac{Gm_A m_c}{|r_c - r_a|^2}(r_c - r_a)$$

We know from Newton's law of gravitational motion, the total force on A would be,

$$F_A = m_A a_A = \frac{Gm_A m_B}{|r_B - r_A|^2}(r_B - r_A) + \frac{Gm_A m_C}{|r_C - r_A|^2}(r_C - r_A)$$

Rewriting it as a nonlinear second order equation (second time derivative of the position vector),

$$\frac{d^2 r_A}{dt^2} = \frac{Gm_B}{|r_B - r_A|^2}(r_B - r_A) + \frac{Gm_C}{|r_C - r_A|^2}(r_C - r_A)$$

Similarly for B and C we get;

$$\frac{d^2 r_B}{dt^2} = \frac{Gm_A}{|r_A - r_B|^2}(r_A - r_B) + \frac{Gm_C}{|r_C - r_B|^2}(r_C - r_B)$$

$$\frac{d^2 r_C}{dt^2} = \frac{Gm_A}{|r_A - r_C|^2}(r_A - r_C) + \frac{Gm_B}{|r_B - r_C|^2}(r_B - r_C)$$

Suppose r is a position vector where, $\overline{r} = (x, y)$

So we get 6 second order equations;

$$\frac{d^2 r_{Ax}}{dt^2} = \frac{Gm_B}{|r_B - r_A|^2}(r_{Bx} - r_{Ax}) + \frac{Gm_C}{|r_C - r_A|^2}(r_{Cx} - r_{Ax})$$

$$\frac{d^2 r_{Ay}}{dt^2} = \frac{Gm_B}{|r_B - r_A|^2}(r_{By} - r_{Ay}) + \frac{Gm_C}{|r_C - r_A|^2}(r_{Cy} - r_{Ay})$$

$$\frac{d^2 r_{Bx}}{dt^2} = \frac{Gm_A}{|r_A - r_B|^2}(r_{Ax} - r_{Bx}) + \frac{Gm_C}{|r_C - r_B|^2}(r_{Cx} - r_{Bx})$$

$$\frac{d^2 r_{By}}{dt^2} = \frac{Gm_A}{|r_A - r_B|^2}(r_{Ay} - r_{By}) + \frac{Gm_C}{|r_C - r_B|^2}(r_{Cy} - r_{By})$$

$$\frac{d^2 r_{Cx}}{dt^2} = \frac{Gm_A}{|r_A - r_C|^2}(r_{Ax} - r_{Cx}) + \frac{Gm_B}{|r_B - r_C|^2}(r_{Bx} - r_{Cx})$$

$$\frac{d^2 r_{Cy}}{dt^2} = \frac{Gm_A}{|r_A - r_C|^2}(r_{Ay} - r_{Cy}) + \frac{Gm_B}{|r_B - r_C|^2}(r_{By} - r_{Cy})$$

Computational Background

The computational method used is the Adaptive Runge-Kutta Method, which is an improved alternative to Euler's methods. The fourth-order Runge-Kutta method is accurate to fourth order but the error on the method is fifth order. That is, the size of the error on a single step is ch^5. So if starting at time t and two steps of size ch^5 is done, then the error will be roughly $2ch^5$. When a single large step of size $2h$ is implemented the error is $c(2h)^5 = 32ch^5$. Using this idea of varying step sizes estimated error is obtained and denoted as, $\varepsilon = ch^5 = (x_1 - x_2)/30$. The goal here is to make the steps sizes as close to the target accuracy as possible. When ε is better than the target, it means that the step sizes are smaller than it needs to be, and when ε is worse than the target, the step cannot be accepted as it does not meet target accuracy, so a smaller step is taken. Suppose the perfect step size would be, h', and the target accuracy per unit time for the calculation is δ then the error on a single step would be $\varepsilon' = h'\delta$ where $h' = h\rho^{1/4}$ and $\rho = \frac{30h\delta}{|x_1 - x_2|}$. If $\rho > 1$ then the accuracy of the Runge-Kutta step is better than target, so for that iteration the value is stored and we move onto time $(t + 2h)$, where the new h' is obtained using ρ. these cycle is repeated till $\rho < 1$, at which point the step sizes have become too large to meet target accuracy. And once that happens, the current step is repeated with a smaller step size. To summarize, at the end of each step, depending on the value of ρ, stepsize is increased and the next step is implemented or the step size is decreased and the current step is repeated.

For x and y direction the error estimate would be different;

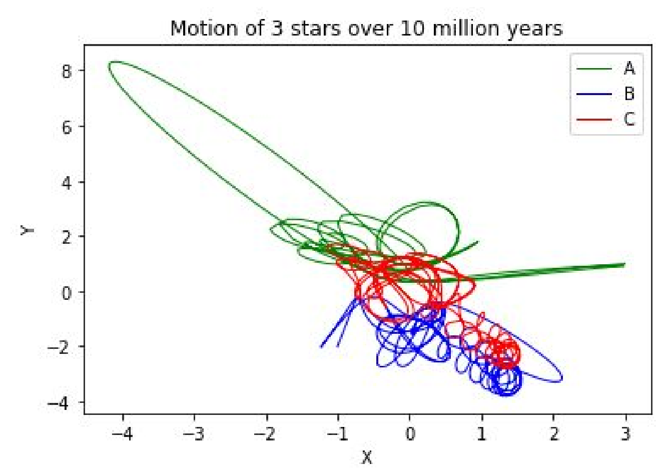
$\varepsilon_x = (x_1 - x_2)/30$ and $\varepsilon_y = (y_1 - y_2)/30$

For euclidean space it can be written as $\varepsilon = \frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{30}$ (Newman, 2013, pp. 357-360)

When the target accuracy is met a local extrapolator is used to better estimate the position, $r = (r_1 - r_2)/15$ (Israel et al., 2002).

Program and Plots

Using the equations derived in the physics background and the Adaptive Runge-Kutta method described in the computational background the code '*3_body_with_promt.py*' is written. Which will ask the user for initial position values for stars A, B and C as well as the number of steps the program will run for. Using the following values the trajectory of the three planetary bodies over 10 million years is plotted. Figure 2 and figure 3 plots the movement of a 3 star system with the following initial condition :

<table>
<tr><td colspan="4">Table 3: Initial conditions of a system with 3 planetary bodies (Newman, 2013, p. 401)</td><td rowspan="5">
</td></tr>
<tr><td>Stars</td><td>A</td><td>B</td><td>C</td></tr>
<tr><td>Mass</td><td>150</td><td>200</td><td>250</td></tr>
<tr><td>X- position</td><td>3</td><td>-1</td><td>-1</td></tr>
<tr><td>Y- position</td><td>1</td><td>-2</td><td>1</td></tr>
<tr><td></td><td></td><td></td><td></td><td>Figure 3: the resulting plot from the given initial condition in Table 3.</td></tr>
</table>

With the initial velocity in both x and y direction being 0, for 400500 steps.

Table 4: Initial conditions of a system with 3 planetary bodies

Stars	A	B	C
Mass	150	200	250
X- position	3.1	-1.2	-1.6
Y- position	1.5	-2.5	1

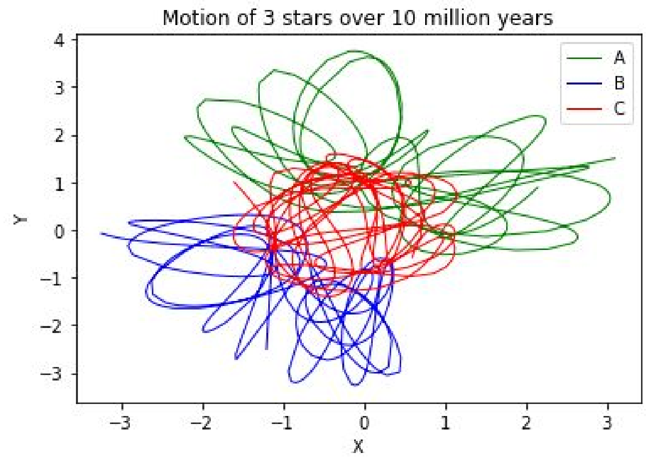


Figure 4: the resulting plot from the given initial condition in Table 4.

With the initial velocity in both x and y direction being 0, for 400500 steps.

Because the three body system is a chaotic system the solution would be unpredictable but still deterministic, the Adaptive Runge-Kutta method, solves these differential equations and gives a possible solution for almost any initial condition. As seen in figure 1 and figure 2 even a small change in the initial angles results in a different plot.

DISCUSSION

While the more traditional science relies on predictability, the seeming unpredictable nature of chaotic systems is rather fascinating. This fascination with disorder has led to the birth of Chaos theory; A study of the randomness of a complex chaotic system, finding the underlying patterns, interconnectedness, repetition, similarity, fractals and self-organization over a period of time. Looking at the first order equations for the double pendulum we can see that there is two degrees of freedom, which results in the chaotic motion we are trying to describe. Same with the three planetary bodies which have six degrees of freedom resulting in the chaotic motion. The theory of chaos was first introduced by Edward Lorenze, an American Meteorologist, who discovered that complicated systems with multiple degrees of freedom are significantly impacted even when the initial condition is changed slightly. However hypothetically, if given the initial condition of these systems were known exactly down to infinite decimal places, chaotic systems can be solved with pinpoint accuracy (Laplace's Demon).

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