AI1103 Assignement 1

Hritik Sarkar

January 17, 2022

PDF is given by $f(x) = ce^{-x^4}, x \in \mathbb{R}$, where $c = \frac{2}{\Gamma(\frac{1}{4})}$. What is the CDF ?
Answer: Gamma function (Γ) is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Similarly, upper incomplete Gamma function is defined as

$$\Gamma(n, y) = \int_y^\infty e^{-x} x^{n-1} dx$$

Cumulative Distribution Function F for a Probability Density Function f is defined as

$$F(y) = P(X \leq y) = \int_{-\infty}^{y} f(x) dx$$

- Case 1: $y \geq 0$

$$\begin{aligned} I &= \int_{-\infty}^{y} ce^{-x^4} dx \\ &= \int_{-\infty}^{0} ce^{-x^4} dx + \int_0^y ce^{-x^4} dx \\ &= \int_0^\infty ce^{-x^4} dx + \int_0^y ce^{-x^4} dx \end{aligned}$$

$$\begin{aligned} & & t &= x^4 \\ \Rightarrow & & dt &= 4x^3 dx \\ \Rightarrow & & dx &= \frac{1}{4} t^{-\frac{3}{4}} dt \end{aligned}$$

$$
\begin{aligned}
I &= \int_0^\infty \frac{c}{4} e^{-t} t^{(\frac{1}{4}-1)} dt + \int_0^{y^4} \frac{c}{4} e^{-t} t^{(\frac{1}{4}-1)} dt \\
&= \int_0^\infty \frac{c}{4} e^{-t} t^{(\frac{1}{4}-1)} dt + \int_0^\infty \frac{c}{4} e^{-t} t^{(\frac{1}{4}-1)} dt - \int_{y^4}^\infty \frac{c}{4} e^{-t} t^{(\frac{1}{4}-1)} dt \\
&= \frac{c}{4}\Gamma(\frac{1}{4}) + \frac{c}{4}\Gamma(\frac{1}{4}) - \frac{c}{4}\Gamma(\frac{1}{4}, y^4) \\
&= \frac{c}{2}\Gamma(\frac{1}{4}) - \frac{c}{4}\Gamma(\frac{1}{4}, y^4) \\
&= 1 - \frac{c}{4}\Gamma(\frac{1}{4}, y^4) && \textit{(substituting c)}
\end{aligned}
$$

- Case 2: $y < 0$

$$
\begin{aligned}
I &= \int_{-\infty}^{y} c e^{-x^4} dx \\
&= \int_{y'}^{\infty} c e^{-x^4} dx && \textit{(where y' = -y)} \\
&= \int_{y'^4}^{\infty} \frac{c}{4} e^{t} t^{(\frac{1}{4}-1)} dt \\
&= \frac{c}{4}\Gamma(\frac{1}{4}, y'^4)
\end{aligned}
$$

So,

$$
F(y) = \begin{cases} 1 - \frac{c}{4}\Gamma(\frac{1}{4}, y^4), & \text{if } y \geq 0 \\ \\ \frac{c}{4}\Gamma(\frac{1}{4}, y^4) & \text{otherwise} \end{cases}
$$

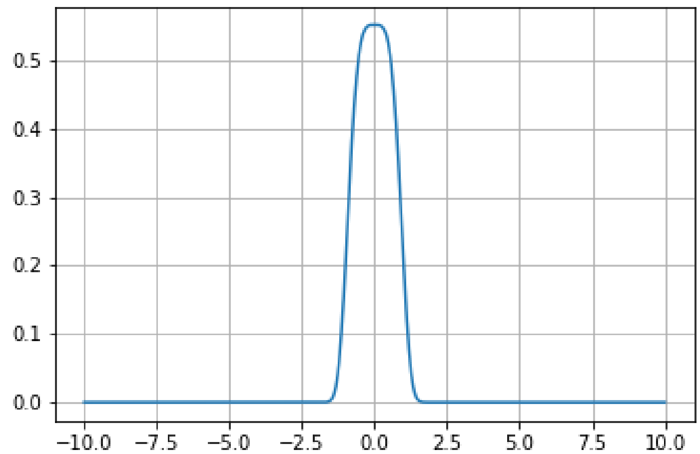

Figure 1: PDF

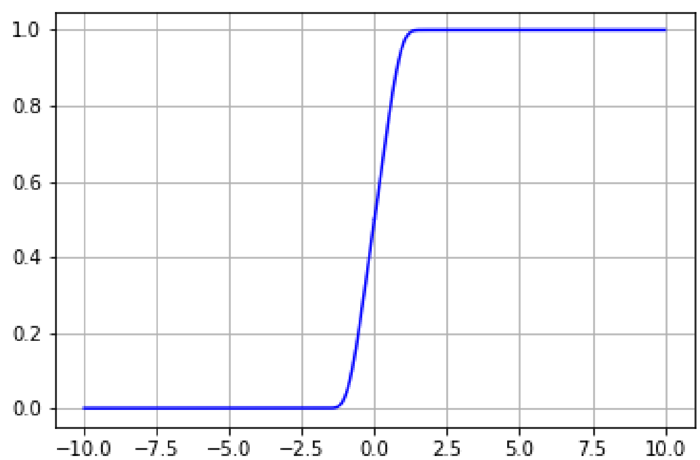

Figure 2: CDF