AI1103 Assignement 2

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Q59. Suppose that $\begin{pmatrix} X \\ Y \end{pmatrix}$ has a bivariate density $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, where f_1 and f_2 are respectively, the densities of bivariate normal distribution $N(\mu_1, \sum)$, and $N(\mu_2, \sum)$, with $\mu_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mu_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $\sum = \mathbb{I}_2$, the 2×2 identity matrix. Then which of the following is correct ?

a) X and Y are positively correlated
b) X and Y are negatively correlated
c) X and Y are uncorrelated but they are not independent
d) X and Y are independent

Answer:

To check the correlation between X and Y we can calculate the covariance of X and Y. If the value of covariance is $+ve$ then X and Y are positively correlated, if the value is $-ve$ then X and Y are negatively correlated and if the value is 0 then X and Y are independent.

For a bivariate, covariance is defind as

$$cov(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_x)(y - \mu y) f(x, y) dxdy$$

The bivariate normal distribution is defined as

$$f(\underline{X}) = \left(2\pi\sqrt{\left|\sum\right|}\right)^{-1} e^{-\frac{1}{2}[(\underline{X}-\underline{\mu})^T \sum^{-1}(\underline{X}-\underline{\mu})]}$$

where

$$\underline{X} = \begin{pmatrix} X \\ Y \end{pmatrix}, \sum = \begin{pmatrix} Var[X] & Cov(X,Y) \\ Cov(X,Y) & Var[Y] \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

ρ is the correlation coefficient.

The mean can be obtained as such

$$\mathbb{E}\left[\begin{pmatrix} X \\ Y \end{pmatrix}\right] = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\sum = \mathbb{I}_2$$

so,

$$\left|\sum\right| = 1$$

,

$$\sum{}^{-1} = \mathbb{I}_2$$

Now,

$$\begin{aligned} Cov(X,Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy(\frac{1}{2}f_1 + \frac{1}{2}f_2)dxdy \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_1dxdy + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_2dxdy \end{aligned}$$

consider,

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_1dxdy \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\frac{1}{2\pi}e^{-\frac{1}{2}[(\underline{X}-\underline{\mu_1})^T(\underline{X}-\underline{\mu_1})]}dxdy \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xye^{-\frac{1}{2}\left(\|\underline{X}\|^2+\left\|\underline{\mu_1}\right\|^2-2\underline{X}^T\underline{\mu_1}\right)}dxdy \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xye^{-\frac{1}{2}(x^2+y^2+2-2x-2y)}dxdy \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} xe^{-\frac{1}{2}(x-1)^2}dx\int_{-\infty}^{\infty} ye^{-\frac{1}{2}(y-1)^2}dy \end{aligned}$$

let,

$$I_2 = \int_{-\infty}^{\infty} xe^{-\frac{1}{2}(x-1)^2}$$

observe,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2} dx - I_2 &= \int_{-\infty}^{\infty} (1-x) e^{-\frac{1}{2}(x-1)^2} dx \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-\frac{1}{2}(x-1)^2} \right) dx \\
&= \left[e^{-\frac{1}{2}(x-1)^2} \right]_{-\infty}^{\infty} \\
&= 0
\end{aligned}$$

So,

$$\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2} dx \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du && \textit{(u = x - 1)}
\end{aligned}$$

now,

$$\begin{aligned}
I_2^2 &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv && \textit{(v is a dummy variable)} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2+v^2)} du dv \\
&= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-\frac{1}{2}r^2} r dr d\theta && \textit{(changing to polar co-ordinates)} \\
&= 2\pi
\end{aligned}$$

So,

$$I_2 = \sqrt{(2\pi)}$$

Now going back to the expression of I_1,

$$\begin{aligned}
I_1 &= \frac{1}{4\pi} * \sqrt{(2\pi)} * \sqrt{(2\pi)} && \textit{(integral over y is exactly same)} \\
&= \frac{1}{2}
\end{aligned}$$

Now,

$$\begin{aligned}
Cov(X, Y) &= \frac{1}{2} + \frac{1}{2} && \textit{(second part is similar)} \\
&= 1
\end{aligned}$$

Since $Cov(X, Y)$ is $+ve$, X and Y are positively correlated. (Ans.)