AI1103 Assignement 1

Hritik Sarkar

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Question 52. Consider the function $f(x)$ defined as $f(x) = ce^{-x^4}, x \in \mathbb{R}$. For what value of c is f a probability density function?

a) $\frac{2}{\Gamma(\frac{1}{4})}$
b) $\frac{4}{\Gamma(\frac{1}{4})}$
c) $\frac{3}{\Gamma(\frac{1}{3})}$
d) $\frac{1}{4\Gamma(4)}$

Answer: Gamma function (Γ) is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

For a function $f(x)$ to be a probability density function the below condition has to be satisfied

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

We can check the condition for the $f(x)$ given.

$$\begin{aligned} I &= \int_{-\infty}^{\infty} ce^{-x^4}dx \\ &= 2c\int_0^\infty e^{-x^4}dx && \textit{($-x^4$ is an even function)} \\ &= \frac{2c}{4}\int_0^\infty e^{-y}y^{-\frac{3}{4}}dy && \textit{(change of variables)} \end{aligned}$$

consider

$$\begin{aligned} & & y &= x^4 \\ \Rightarrow & & dy &= 4x^3dx \\ \Rightarrow & & dx &= \frac{1}{4}y^{-\frac{3}{4}}dy \end{aligned}$$

Continuing from above

$$
\begin{aligned}
I &= \frac{c}{2}\int_0^\infty e^{-y} y^{-\frac{3}{4}} dy \\
&= \frac{c}{2}\int_0^\infty e^{-y} y^{(\frac{1}{4}-1)} dy \\
&= \frac{c}{2}\Gamma(\frac{1}{4}) \qquad \textit{(comparing with the definition of gamma function)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{c}{2}\Gamma(\frac{1}{4}) = 1 \\
\Rightarrow \quad & c = \frac{2}{\Gamma(\frac{1}{4})} \qquad \textit{Ans.}
\end{aligned}
$$