

AI1103 Assignment 1

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PDF is given by $f(x) = ce^{-x^4}$, $x \in \mathbb{R}$, where $c = \frac{2}{\Gamma(\frac{1}{4})}$. What is the CDF ?
Answer: Gamma function (Γ) is defined as

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

Similarly, upper incomplete Gamma function is defined as

$$\Gamma(n, y) = \int_y^{\infty} e^{-x} x^{n-1} dx$$

Cumulative Distribution Function F for a Probability Density Function f is defined as

$$F(y) = P(X \leq y) = \int_{-\infty}^y f(x) dx$$

- Case 1: $y \geq 0$

$$\begin{aligned} I &= \int_{-\infty}^y ce^{-x^4} dx \\ &= \int_{-\infty}^0 ce^{-x^4} dx + \int_0^y ce^{-x^4} dx \\ &= \int_0^{\infty} ce^{-x^4} dx + \int_0^y ce^{-x^4} dx \end{aligned}$$

$$\begin{aligned} &t = x^4 \\ \Rightarrow &dt = 4x^3 dx \\ \Rightarrow &dx = \frac{1}{4} t^{-\frac{3}{4}} dt \end{aligned}$$

$$\begin{aligned}
I &= \int_0^\infty \frac{c}{4} e^{t^{(\frac{1}{4}-1)}} dt + \int_0^{y^4} \frac{c}{4} e^{t^{(\frac{1}{4}-1)}} dt \\
&= \int_0^\infty \frac{c}{4} e^{t^{(\frac{1}{4}-1)}} dt + \int_0^\infty \frac{c}{4} e^{t^{(\frac{1}{4}-1)}} dt - \int_{y^4}^\infty \frac{c}{4} e^{t^{(\frac{1}{4}-1)}} dt \\
&= \frac{c}{4} \Gamma\left(\frac{1}{4}\right) + \frac{c}{4} \Gamma\left(\frac{1}{4}\right) - \frac{c}{4} \Gamma\left(\frac{1}{4}, y^4\right) \\
&= \frac{c}{2} \Gamma\left(\frac{1}{4}\right) - \frac{c}{4} \Gamma\left(\frac{1}{4}, y^4\right) \\
&= 1 - \frac{c}{4} \Gamma\left(\frac{1}{4}, y^4\right) \quad (\text{substituting } c)
\end{aligned}$$

- Case 2: $y < 0$

$$\begin{aligned}
I &= \int_{-\infty}^y c e^{-x^4} dx \\
&= \int_{y'}^\infty c e^{-x^4} dx \quad (\text{where } y' = -y) \\
&= \int_{y'^4}^\infty \frac{c}{4} e^{t^{(\frac{1}{4}-1)}} dt \\
&= \frac{c}{4} \Gamma\left(\frac{1}{4}, y'^4\right)
\end{aligned}$$

So,

$$F(y) = \begin{cases} 1 - \frac{c}{4} \Gamma\left(\frac{1}{4}, y^4\right), & \text{if } y \geq 0 \\ \frac{c}{4} \Gamma\left(\frac{1}{4}, y^4\right) & \text{otherwise} \end{cases}$$