AI1103 Assignement 2

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Q59. Suppose that $\begin{pmatrix} X \\ Y \end{pmatrix}$ has a bivariate density $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, where f_1 and f_2 are respectively, the densities of bivariate normal distribution $N(\mu_1, \sum)$, and $N(\mu_2, \sum)$, with $\mu_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mu_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $\sum = \mathbb{I}_2$, the 2×2 identity matrix. Then which of the following is correct ?

a) X and Y are positively correlated
b) X and Y are negatively correlated
c) X and Y are uncorrelated but they are not independent
d) X and Y are independent

Answer:

To check the correlation between X and Y we can calculate the covariance of X and Y. If the value of covariance is $+ve$ then X and Y are positively correlated, if the value is $-ve$ then X and Y are negatively correlated and if the value is 0 then X and Y are independent.

For a bivariate, covariance is defind as

$$cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu y)f(x,y)dxdy \tag{1}$$

The bivariate normal distribution is defined as

$$f(\underline{X}) = \left(2\pi\sqrt{\left|\sum\right|}\right)^{-1} e^{-\frac{1}{2}[(\underline{X}-\underline{\mu})^T \sum^{-1}(\underline{X}-\underline{\mu})]} \tag{2}$$

where

$$\underline{X} = \begin{pmatrix} X \\ Y \end{pmatrix}, \sum = \begin{pmatrix} Var[X] & Cov(X,Y) \\ Cov(X,Y) & Var[Y] \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

ρ is the correlation coefficient.
and standard normal distribution is given by

$$g(\underline{(}X)) = \frac{1}{2\pi}e^{-\frac{1}{2}[\underline{X}^T\underline{X}]} \tag{3}$$

where,

$$\sum = \mathbb{I}_2, \mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the given problem mean can be obtained as such

$$\mathbb{E}_f\left[\begin{pmatrix} X \\ Y \end{pmatrix}\right] = \frac{1}{2}\,\mathbb{E}_{f_1}\left[\begin{pmatrix} X \\ Y \end{pmatrix}\right] + \frac{1}{2}\,\mathbb{E}_{f_2}\left[\begin{pmatrix} X \\ Y \end{pmatrix}\right] = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{aligned} cov(X,Y) &= \mathbb{E}_f\left[(X-\mu_x)(Y-\mu_y)\right] \\ &= \frac{1}{2}\,\mathbb{E}_{f_1}\left[XY\right] + \frac{1}{2}\,\mathbb{E}_{f_2}\left[XY\right] \qquad \mathit{(}\mu_x = 0, \mu_y = 0\mathit{)} \end{aligned} \tag{4}$$

Now,

$$\begin{aligned} \mathbb{E}_{f_1}[X,Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_1dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\frac{1}{2\pi}e^{-\frac{1}{2}[(\underline{X}-\underline{\mu_1})^T(\underline{X}-\underline{\mu_1})]}dxdy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xye^{-\frac{1}{2}\left(\|\underline{X}\|^2+\left\|\underline{\mu_1}\right\|^2-2\underline{X}^T\underline{\mu_1}\right)}dxdy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xye^{-\frac{1}{2}(x^2+y^2+2-2x-2y)}dxdy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xye^{-\frac{1}{2}[(x-1)^2+(y-1)^2]}dxdy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x'+1)(y'+1)e^{-\frac{1}{2}[(x')^2+(y')^2]}dxdy \quad \mathit{(}x' = x-1, y' = y-1\mathit{)} \\ &= \mathbb{E}_g[X'+1, Y'+1] \qquad \textit{(comparing with } (3) \end{aligned}$$

Similarly,

$$\mathbb{E}_{f_2}[X,Y] = \mathbb{E}_g[X\text{"}+1, Y\text{"}+1] \qquad \textit{(X"=X+1, Y"=Y+1)}$$

$$
\begin{aligned}
\mathbb{E}_{f_1}[XY] &= \mathbb{E}_g\left[(X'+1)(Y'+1)\right] \\
&= \mathbb{E}_g[X'Y'] + \mathbb{E}_g[X'] + \mathbb{E}_g[Y'] + \mathbb{E}_g[1] \\
&= 0+0+0+1 \\
&= 1 \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_{f_2}[XY] &= \mathbb{E}_g\left[(X"-1)(Y"-1)\right] \\
&= \mathbb{E}_g[X"Y"] - \mathbb{E}_g[X"] - \mathbb{E}_g[Y"] + \mathbb{E}_g[1] \\
&= 0+0+0+1 \\
&= 1 \qquad (6)
\end{aligned}
$$

Substituting (5) and (6) back in (4)

$$
\begin{aligned}
cov(X,Y) &= \frac{1}{2}\,\mathbb{E}_{f_1}[XY] + \frac{1}{2}\,\mathbb{E}_{f_2}[XY] \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1 \qquad \textit{(X, Y are positively correlated)}
\end{aligned}
$$