Assignment 12-Probability and Random Variable

Annu-EE21RESCH01010

Download latex code from here-

https://github.com/annu100/AI5002−Probability−and−Random−variables/tree/main.tex/ASSIGNMENT_12

Download python code from here-

https://github.com/annu100/AI5002−Probability−and−Random−variables/tree/main.py/ASSIGNMENT_12

I. Ugc-Net june 2019-51

Consider a Markov Chain with state space 0,1,2,3,4 and transition matrix-

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ Then $\lim_{n\to\infty} Pr_{23}{}^{n}$ equals

II. SOLUTIONS

To understand this,let us first understand Discrete Time Markov Chain(DTMC).
A stochastic process along with markov property makes a Discrete Time Markov Chain.
S_j here represents a state on our statespace. $Pr(X_{n+1} = S_j|X_n, X_{n-1}, \ldots X_0) = Pr(X_{n+1} = S_j|X_n)$.
Further,the probabilistics are often fixed in time.
Now,Time Homogeneous property/stationary property of MC is-

$$\begin{aligned} Pr(X_{n+1} = S_j|X_n) &= Pr(X_1 = S_j|X_0) && (1)\\ &= Pr(X_2 = S_j|X_1) && (2)\\ &= Pr(X_3 = S_j|X_2) && (3)\\ &= Pr(X_4 = S_j|X_3) && (4)\end{aligned}$$

So.time homogeneous or stationary property means our one step transition probability are independent of time (n).

Now consider the case when n tends to ∞.
As sum of all elements in a row of probability transition matrix.So,
$Pr(X_{n+1} = S_j|X_n = S_i) = Pr_{ij}$

$\sum_{j=1}^{N} Pr(X_{n+1} = S_j|X_n = S_i) = 1$

$\sum_{j=1}^{N} Pij = 1$

Limiting behaviour of discrete time Markov Chain is

$Pr(X_n = S_j)$as n tends to ∞ ,we need to find -
let$\pi_j$be stationary distribution of MC i.e
$\pi_j = Pr(X_n = S_j|X_0 = S_1)$

$$\begin{aligned} \lim_{n\to\infty} Pr(X_n = S_j) &= \lim_{n\to\infty} \sum_i Pr(X_n = S_j|X_0 = S_i) \times Pr(X_0 = S_i) && (5)\\ &= \sum_i \lim_{n\to\infty} Pr(X_n = S_j|X_0 = S_i) \times Pr(X_0 = S_i) && (6)\\ &= \sum_i \pi_j \times Pr(X_0 = S_i) && (7)\\ &= \sum \pi_j \times Pr(X_0 = S_i) && (8)\\ &= \pi_j \times \sum_i Pr(X_0 = S_i) && (9)\\ &= \pi_j && (10)\end{aligned}$$

$\lim_{n\to\infty} Pr(X_n = S_j) = \pi_j$ irrespective of initial distribution i.e $Pr(X_0 = S_j)$.
This is limiting behaviour of DTMC

Stationary distribution
starting with initial distribution for inital state X_0

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \vdots \\ \pi_N \end{bmatrix}$$

$\pi_i = Pr(X_0 = S_i)$
Now for stationary distribution- $\pi^{\intercal} P = \pi$
where π is the stationary distribution.
It is left eigen vector of probability ransition matrix P or eigen vector of $P^{\intercal}$
let $A = P^{\intercal}$ so,$|A - \lambda I| = 0$ for eigen value calculation.
and Eigen vector lies in null space of matrix $A - \lambda I$
As it is 5*5 matrix,using python code for calculation of eigen values and eigen vector ,we get Eigen values are-
λ_1=1 , λ_2=1, λ_3=0.804, λ_4=-0.138, λ_5=0.333
Possible eigenvectors are:
eigenvector 1:

$$\begin{bmatrix} 1.0 \\ 0. \\ -5.445260 \\ 1.4340 \\ -3.16 \end{bmatrix}$$

eigenvector 2:

$$\begin{bmatrix} 0 \\ 0 \\ 4.5 \\ 6.92 \\ -5.26 \end{bmatrix}$$

eigenvector3:

$$\begin{bmatrix} 0 \\ 0 \\ 3.189 \\ -4.89 \\ 6.32 \end{bmatrix}$$

eigenvector4:

$$\begin{bmatrix} 0 \\ 0 \\ 3.189 \\ -4.89 \\ -6.32 \end{bmatrix}$$

Now $\lim_{n\to\infty} {Pr_n}^n$ approaches a matrix which has structure that all rows of matrix are identical and each row will give limiting distribution and this limiting distribution corresponds to stationary distribution,

So,from possible eigen vectors third value will be $\lim_{n\to\infty} {Pr_{23}}^n$.
i.e.1.43,-4.89 or 6.92.
(according to my eigen vector obtained as for one eigen value ,eigen vector is not unique and we can get infinite eigen vectors).