Challenging Problem 11-part 3

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Download all latex codes from

https://github.com/annu100/AI5002−Probability−and−Random−variables/tree/main.tex/challenging problems

1 Problem

(UGC/MATH 2018 (June set-a)-Q.106) Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with $E(X_i) = 0$ and $V(X_i) = 1$. Which of the following are true?

1) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 0$ in probability
2) $\frac{1}{n^{3/4}}\sum_{i=1}^{n} X_i \to 0$ in probability
3) $\frac{1}{n^{1/2}}\sum_{i=1}^{n} X_i \to 0$ in probability
4) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 1$ in probability

2 Solution-Part 3

Here $X_1, X_2 \ldots X_n$ are i.i.d random variables and to show the argument of part 3,let us assume these are normal random variables with zero mean and unit variance.
$X_i \sim N(0,1)$

Let $S_n = X_1 + X_2 + X_3 + \ldots X_n$
Now,S_n converges in distribution a normal random variable ,let us say U with zero mean ,Now if we assume $S_n/\sqrt{n}$ converges in probability at all it must be to U.
Now according to Cauchy criterion for convergence in probability,which says

The sequence $(\frac{S_n}{\sqrt{n}})_{n\geq 1}$ converges in probability if and only if
$Pr(|\frac{S_n}{\sqrt{n}} - \frac{S_m}{\sqrt{m}}| \geq \epsilon) \to 0$ for every, $\epsilon \geq 0$,
provided m,n $\to \infty$

$$Pr(|\frac{S_n}{\sqrt{n}} - \frac{S_m}{\sqrt{m}}| \geq \epsilon) \tag{2.0.1}$$
$$= Pr(\frac{S_n}{\sqrt{n}} - \frac{S_m}{\sqrt{m}} \geq \epsilon) \tag{2.0.2}$$
$$+Pr(\frac{S_n}{\sqrt{n}} - \frac{S_m}{\sqrt{m}} \leq -\epsilon) \tag{2.0.3}$$
$$\geq Pr(\frac{S_n}{\sqrt{n}} \geq 2\epsilon, \frac{S_m}{\sqrt{m}} \leq \epsilon) \tag{2.0.4}$$
$$+Pr(\frac{S_n}{\sqrt{n}} \leq -2\epsilon, \frac{S_m}{\sqrt{m}} \geq -\epsilon) \tag{2.0.5}$$
$$\geq Pr(\frac{S_n}{\sqrt{n}} \geq 2\epsilon) \tag{2.0.6}$$
$$+Pr(\frac{S_n}{\sqrt{n}} \leq -2\epsilon) \tag{2.0.7}$$
$$+Pr(\frac{S_m}{\sqrt{m}} \leq \epsilon) \tag{2.0.8}$$
$$+Pr(\frac{S_m}{\sqrt{m}} \geq -\epsilon) - 2 \tag{2.0.9}$$
$$(m,n \to \infty) \to 2(Q(2\epsilon) + Q(-\epsilon) - 1), \tag{2.0.10}$$
$$\tag{2.0.11}$$

where $Q(x) = \int_x^\infty N(0,1)dx$
So,for any

$$\epsilon \geq 0$$

,terms inside brackets of $(Q(2\epsilon) + Q(-\epsilon) - 1)$ is positive.
And so, $\frac{1}{n^{1/2}}\sum_{i=1}^{n} X_i \to 0$ in probability is wrong.
This third option is not correct.