Modelling and performance analysis of *wireless channel* using *Probability*

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Overview

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ABSTRACT

Summary of the paper

In this modern era, Wireless and Telecommunication have become an integral part of each other to provide wireless communication to common man that helps people located in any part of the world communicate easily. Wireless communication technology transmits information over the air using electromagnetic waves like IR (Infrared), RF (Radio Frequency), satellite, etc. For example, GPS, Wi-Fi, satellite television, wireless computer parts, wireless phones that include 3G and 4G networks, and Bluetooth. Therefore,modelling of wireless channels is very much required and their performace must be analysed properly.Fading in wireless channel is very important like Rayleigh fading for channel estimation.

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What Is Probability?

- It is not about finding the exact value produced by a random event but rather accurately characterizing the random event itself and all its possible outcomes
- Two types of random events: *discrete* and *continuous*
- Probability theory involves *mapping* physical phenomena to a mathematical framework such that we can employ a variety of useful tools for analysis
- We can represent a random event via the “black box” model

Random Variables

- Random variable X is a function that maps an outcome ω from the sample space Ω to a real number x
- Probability that discrete random variable X produces an output within $[a, b)$ is given by:

$$P(a \leq X < b) = \sum_{i=a}^{b-1} p_X(i) \tag{1}$$

  where $p_X(i)$ is the probability mass function (PMF)
- Probability that continuous random variable X produces an output within $[a, b)$ is given by:

$$P(a \leq X < b) = \int_a^b f(t)dt \tag{2}$$

  where $f(t)$ is the probability density function (PDF)

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Expectation

- Probability is about characterizing random events and their behavior
- One such mathematical tool is *Expectation*
  - It can be roughly described as a weighted average of all possible outcomes of a random event
- We define Expectation as:

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx \tag{3}$$

- We use Expectation to give us some understanding about the behavior of a random variable

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CDFs and PDFs

- To **perfectly** describe a random event, we use *Cumulative Distribution Functions* (CDFs) and *Probability Density Functions* (PDFs)
- Mathematically, this is described as:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \tag{4}$$

- Here is a useful trick for calculating the probability $P(a \leq X < b)$:

$$P(a \leq X < b) = P(X < b) - P(X < a) = F_X(b) - F_X(a) \tag{5}$$

Important Communications-Related PDFs

- Uniform:

$$f(x) = \begin{cases} \frac{1}{(b-a)}, & a \leq x < b \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

- Gaussian:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-((x-\mu)/\sigma)^2/2} \tag{7}$$

Q-Function

- *Q function* is a convenient way to express right-tail probabilities for Gaussian random variables
- Mathematically, this can be expressed as:

$$Q(x) = 1 - F_X(x) = 1 - P(X \leq x) \tag{8}$$

$$= P(X > x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt \tag{9}$$

Where Does Randomness in Wireless communication Occur?

- Several locations where randomness occurs in the communications system includes:
  - Transmission channel
  - Data generation at information source
  - Clock jitter
  - Processing latency
  - No guiding medium between transmitter and receiver.
  - Multiple signals superpose at the receiver. As a result of destructive interference, strength of signal fades(weakens). This effect is known as fading.
  - ...

Noise Channel Model

Binary Information Source $\rightarrow m[n]$

$\downarrow$

Transmitter $\rightarrow s(t)$

$\downarrow$

Channel $\rightarrow s(t) + n(t)$

$\downarrow$

Receiver $\leftarrow r(t)$

$\downarrow$

Binary Information Sink $\leftarrow \hat{m}[n]$

Gaussian Random Variable

- We frequently use **Gaussian random variables** to model noise contribution to transmitted signal
- Gaussian random variable mathematically expressed as:

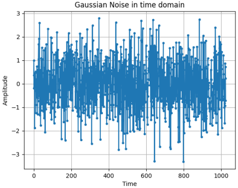
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-((x-\mu)/\sigma)^2/2} \tag{10}$$

where μ is the mean and σ is the standard deviation

AWGN

Additive white Gaussian noise (AWGN) is a basic noise model used in information theory to mimic the effect of many random processes that occur in nature. The modifiers denote specific characteristics:

- Additive because it is added to any noise that might be intrinsic to the information system.
- White refers to the idea that the noise has the same power distribution at every frequency.
- Gaussian because it has a normal distribution in the time domain with an average time domain value of zero.



Gaussian Noise

1. It's a good model for the type of noise that comes from many natural sources, such as thermal vibrations of atoms in the silicon of our receiver's RF components.
2. The central limit theorem tells us that the summation of many random processes will tend to have a Gaussian distribution, even if the individual processes have some other distribution. In other words, when a lot of random things happen and accumulate, the result appears Gaussian. The result will be Gaussian even when individual things are not distributed in this manner.
3. The Gaussian distribution is also called the "Normal" distribution (recall a bell curve). The Gaussian distribution has two parameters: mean and variance. The variance changes how "strong" the noise is. A higher variance will result in larger numbers. It is for this reason that variance defines the noise power. Variance equals standard deviation squared (σ^2).

Bivariate Gaussian

General definition for bivariate Gaussian density with parameters μ_X, μ_Y, σ_X^2, σ_Y^2, and correlation coefficient ρ is given by:

$$f_{XY}(x,y) = \frac{\exp\left(\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}, \tag{11}$$

where *correlation coefficient* is defined as:

$$\rho = E\left[\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]. \tag{12}$$

Multipath Fading in wireless communication

- **Multipath fading:**a propagation phenomenon that results in signals reaching the receiver by two or more paths, which we experience in real-world wireless systems

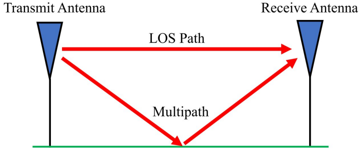


Figure: Multipath fading

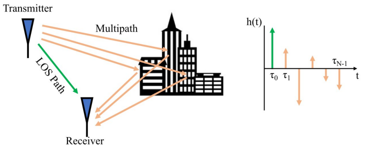


Figure: Multipath fading

Types of Fading in Time domain

There are two types of fading from a time domain perspective

- **Slow Fading:** The channel doesn't change within one packet's worth of data. That is, a deep null during slow fading will wipe out the whole packet.
- **Fast Fading:** The channel changes very quickly compared to the length of one packet. Forward error correction, combined with interleaving, can combat fast fading.

Types of Fading in frequency domain

There are two types of fading from a time domain perspective

- **Frequency Selective Fading:** The constructive/destructive interference changes within the frequency range of the signal. When we have a wideband signal, we span a large range of frequencies. Recall that wavelength determines whether it's constructive or destructive. Well if our signal spans a wide frequency range, it also spans a wide wavelength range (since wavelength is the inverse of frequency). Consequently we can get different channel qualities in different portions of our signal (in the frequency domain). Hence the name frequency selective fading.
- **Flat Fading:**Occurs when the signal's bandwidth is narrow enough that all frequencies experience roughly the same channel. If there is a deep fade then the whole signal will disappear (for the duration of the deep fade).

In the figure 4 below, the red shape shows our signal in the frequency domain, and the black curvy line shows the current channel condition over frequency. Because the narrower signal is experiencing the same channel conditions throughout the whole signal, it's experiencing flat fading. The wider signal is very much experiencing frequency selective fading.

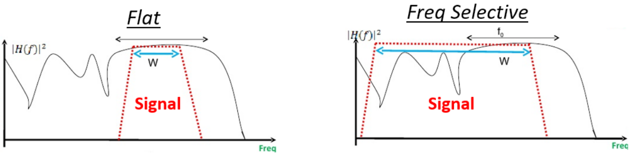


Figure: Multipath fading

Here is an example of a 16 MHz wide signal that is continuously transmitting. There are several moments in the middle where there's a period of time a piece of signal is missing. This example depicts frequency selective fading, which causes holes in the signal that wipe out some frequencies but not others.

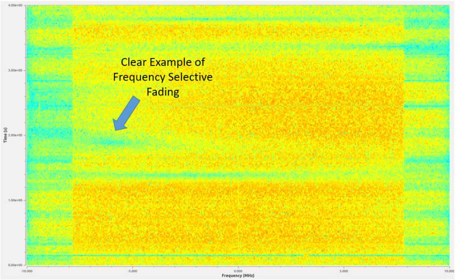


Figure: Multipath fading

Rayleigh Fading

The probability density function of the Rayleigh distribution is

$$f(x;\sigma) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}, \quad x \geq 0, \tag{13}$$

where σ is the scale parameter of the distribution.

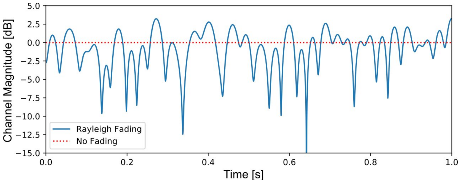


Figure: Rayleigh Fading

The fading channel coefficient h depends on factors like attenuation (a_i) and time delay(τ_i) associated with the channel.

Modelling the distribution of the fading channel coefficient

$$h = \sum_{i=0}^{L-1} a_i \exp\left(-j2\pi f_c \tau_i\right) \tag{14}$$

$$= \sum_{i=0}^{L-1} a_i \cos\left(2\pi f_c \tau_i\right) - j \sum_{i=0}^{L-1} a_i \sin\left(2\pi f_c \tau_i\right) \tag{15}$$

$$= X + jY \tag{16}$$

where X=$\sum_{i=0}^{L-1} a_i \cos\left(2\pi f_c \tau_i\right)$ and Y=$-\sum_{i=0}^{L-1} a_i \sin\left(2\pi f_c \tau_i\right)$

As X and Y are the sums of a large number of independent random variables, by the central limit theorem X and Y can be assumed to be Gaussian distributed random variables.

Joint Pdf of X and Y

$$X, Y \sim \mathcal{N}(\mu, \sigma^2) \tag{17}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp \frac{-(x-\mu)^2}{2\sigma^2} \tag{18}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} exp \frac{-(y-\mu)^2}{2\sigma^2} \tag{19}$$

Assuming X and Y are independent random variables and substituting $\mu = 0$ for simplification

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} exp\left(-(x^2 + y^2)\right) \tag{20}$$

Distribution of the fading channel in terms of its Amplitude and phase using Jacobian

$$h = x + jy = ae^{j\phi} \tag{21}$$

$$a = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x} \tag{22}$$

$$f_{A,\Phi}(a,\phi) = f_{XY}(x,y)\left|J_{XY}\right| \tag{23}$$

$$\left|J_{XY}\right| = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -a\sin\phi & a\cos\phi \end{vmatrix} \tag{24}$$

from (20),(22) and (24) we get

$$f_{A,\Phi}(a,\phi) = \frac{a}{2\pi\sigma^2}\exp\left(\frac{-a^2}{2\sigma^2}\right) \tag{25}$$

Marginal distribution of A

$$f_A(a) = \int_{-\pi}^{\pi} f_{A,\Phi}(a,\phi)\, d\phi \tag{26}$$

$$= \int_{-\pi}^{\pi} \frac{a}{2\pi\sigma^2} \exp\left(\frac{-a^2}{2\sigma^2}\right) d\phi \tag{27}$$

$$= \frac{a}{\sigma^2} \exp\left(\frac{-a^2}{2\sigma^2}\right) \tag{28}$$

Thus the coefficient follows the Rayleigh distribution and is fading in nature. It is therefore called as a Rayleigh fading channel.

Performace Analysis of wireless channels

Outage Probability

As we know, fading channels are characterized by deep fades, i.e, the period when the signal level falls below a certain threshold or certain noise level. During such fades, the user experiences signal outage. We would like to compute the probability, in certain fading channel, that a user will experience signal outage. This is called outage probability. Outage probability can be easily computed if we know the probability distribution characteristics of the fading. $P_{out}(R) = Pr((log(1 + |h|^2 \times SNR) < R)$

Error rate

In digital transmission, the number of bit errors is the number of received bits of a data stream over a communication channel that have been altered due to noise, interference, distortion or bit synchronization errors. The bit error rate (BER) is the number of bit errors per unit time.

Simulation Results-Outage Probability

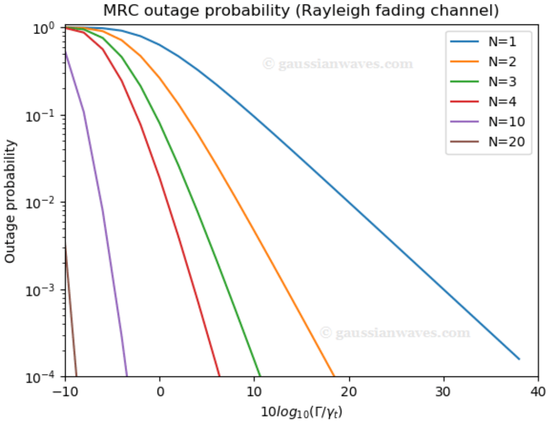


Figure: Outage probability of MRC processing in Rayleigh fading channel

Simulation Results-Error Rate

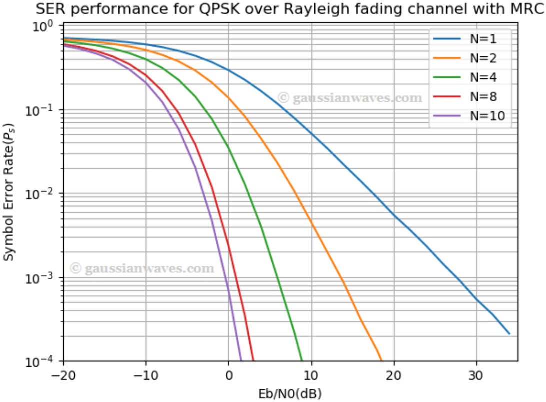


Figure: Symbol error rate Vs EbN0 for QPSK over i.i.d Rayleigh flat fading channel with MRC processing at the receiver

Frequency Domain Perspective of Communication System

- It is sometimes more convenient to study a communication system in the frequency domain rather than the time domain
  - Mathematical analysis is more tractable
  - Operations such as convolution are transformed into simple multiplications
- Physically it makes sense to study wireless transmissions in the frequency domain
  - Digital communications is the transferral of data based on changes of electromagnetic wave characteristics such as frequency, amplitude, and phase

Fourier Transform

- Mathematical relationship between a time domain waveform and weighted sum of sinusoidal components that constitute it, i.e., frequency domain representation
- Translating between time and frequency domains is achieved using the Fourier transform and inverse Fourier transform:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt \tag{29}$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df \tag{30}$$

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Einstein-Wiener-Khinchin Theorem

- In particular, we are interested in the *power spectral density* (PSD) of a signal, which is related to the autocorrelation function via the *Einstein-Wiener-Khintchine* (EWK) Relations:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau \tag{31}$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df \tag{32}$$

- Relating the PSD between input $x(t)$ and output $y(t)$ of a system $h(t)$, we have the following very important result:

$$S_Y(f) = |H(f)|^2 S_X(f) \tag{33}$$

PSD Properties

- Zero frequency scenario:

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$$

- Mean-squared value:

$$E\{x^2(t)\} = \int_{-\infty}^{\infty} S_X(f) df$$

- Non-negative PSD: $S_X(f) \geq 0, \ \forall f$
- Real-valued process: $S_X(-f) = S_X(f)$
- Normalized PSD associated with a PDF:

$$p_X(f) = \frac{S_X(f)}{\int_{-\infty}^{\infty} S_X(f) df}$$

PSD Example

- Find $S_X(f)$ of the following random process:

$$x(t) = A\cos(2\pi f_c t + \Theta),$$

  where Θ is uniformly distributed over the interval $[-\pi, \pi]$.
- First solve for $R_X(\tau)$ by using:

$$R_X(\tau) = E\{x(t+\tau)x(t)\} = \frac{1}{2}A^2\cos(2\pi f_c \tau)$$

- Then solve for the PSD using EWK relations:

$$\begin{aligned} S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} \frac{1}{2}A^2\cos(2\pi f_c\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} \frac{1}{4}A^2\left(e^{j2\pi f_c\tau} + e^{-j2\pi f_c\tau}\right)e^{-j2\pi f\tau}d\tau \\ &= \frac{A^2}{4}(\delta(f-f_c) + \delta(f+f_c)) \end{aligned}$$

Identifying PSD Features

- PSD characteristics can reveal a substantial amount of info regarding a signal
  - Modulation format
  - Pulse shape filtering
  - Transmission bandwidth
- Time-varying behavior indicates network traffic levels
- PSD shape can uniquely identify specific wireless standards

Anatomy of a Typical Digital Communication System

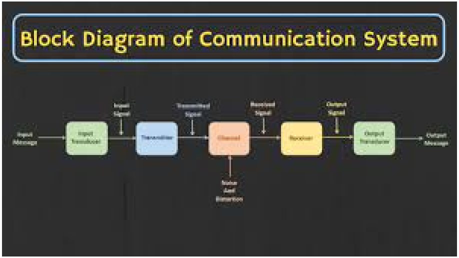

Figure: Digital Communication System Block Diagram

Sending Messages

- Digital communications involves the transmission of a message $m(t)$ and producing a reconstructed version of the message $\hat{m}(t)$ at the output of the receiver
  - Our goal is to have $P(\hat{m}(t) \neq m(t))$ as small as needed for a particular application
  - Probability of Error: $Pe = P(\hat{m}(t) \neq m(t))$
- Various data applications possess different P_e requirements
  - Voice: $Pe \sim 10^{-3}$
  - Data: $Pe \sim 10^{-5} - 10^{-6}$
  - Fiber optics: $Pe \sim 10^{-9}$

Purpose of Source Encoder

- Given $\underline{u}$ = sequence of source symbols and $\underline{v}$ = sequence of source encoded symbols:
  - We should have $v_i \in \underline{v}$ as close to random as possible
  - Components of $\underline{v}$ are uncorrelated, i.e., unrelated
- Why do we perform source encoding?
  - No redundancy in $v_i \in \underline{v}$
  - Do not want to waste channel resources (e.g: power, bandwidth) in the transmission of "predictable symbols"
- Source encoder removes redundant information from the source symbols in order to realize efficient transmission
  - Note that the source needs to be *digital*

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A Source Encoding Example

- Analog TV uses 6 MHz bands to transmit every channel
- Digital TV employing source encoding
  - This can be performed since information is digitally represented
  - Eight digitally encoded TV channels can be fit in one 6 MHz band

Why Perform Channel Coding?

- Channel coding is designed to correct channel transmission errors
  - Achieved via *controlled* introduction of redundancy
  - How is this different than the redundancy removed during source encoding?
- Each vector of source encoded outputs, v_l, $l = 1, 2, ..., 2^k$, are assigned a unique *codeword*
  - Take $v_l = (101010\ldots)$ and assign unique codeword $c_l \in \mathbb{C}$ where $\mathbb{C}$ is a *codebook*
- We have added $N - K = r$ controlled number of bits to the channel encoding process
- The *code rate* of a communications system is equal to the ratio of the number of information bits to the size of the codeword, i.e., Code Rate=k/N

Hamming Distance

- The *Hamming Distance* $d_H(c_i, c_j)$ between two codewords say c_i and c_j is equal to the number of components in which c_i and c_j are different
- We often are looking for the minimum Hamming Distances between codewords, i.e.:

$$d_{H,\min} = \min_{c_i, c_j \in \mathbb{C},\ i \neq j} d_H(c_i, c_j) \tag{34}$$

- Our goal is to maximize the minimum Hamming Distance in a given codebook
- For example:
  - $\{101, 010\} \rightarrow d_{H,min} = 3 \rightarrow$ GOOD!
  - $\{111, 101\} \rightarrow d_{H,min} = 1 \rightarrow$ POOR!

Decoding Spheres

- We can use *decoding spheres* (also known as *Hamming spheres*) to make decisions on received information
- Decoding spheres should not overlap $\rightarrow d_{H,\min} = 2t + 1$

A Channel Encoding Example

- Example: A rate $1/3$ repetition code with no source encoding would look like:

$$\begin{aligned} 1 \rightarrow 111 &= c_1 \text{ (1st codeword)} \\ 0 \rightarrow 000 &= c_2 \text{ (2nd codeword)} \\ \therefore C &= \{000, 111\} \end{aligned} \tag{35}$$

- What are the Hamming Distances for the following codeword pairs?
  - d_H(111,000)=3
  - d_H(111,101)=1

Shannon's Channel Coding Theorem (1948)

- Consider a channel with capacity C and we transmit at a fixed code rate K/N which is equal to R_c (a constant)
  - If we increase N, we must increase K to keep R_c equal to a constant
- There exists a code such that for $R_c = K/N < C$ as $N \to \infty$, we have $P_e \to 0$
  - Conversely, for $R_c = K/N \geq C$, no such code exists
- Hence, C is the limit in rate for reliable communications, i.e., C is the **absolute limit** that you cannot go any faster than this amount without causing errors

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Shannon's Channel Capacity

- *Reliability* in digital communications is usually expressed as the Probability of Bit Error measured at the output of the receiver
- It would be nice to know what this capacity is given the transmission bandwidth, B, the received signal-to-noise ratio (SNR)
  - Shannon derived the *information capacity of the channel* to be equal to:

$$C = B\log_2(1 + SNR) \qquad [\text{b/s}] \tag{36}$$

  - The information capacity tells us the achievable data rate, but it does not tell us how to build a transceiver to achieve this

Why Is This Important?

- The information capacity of the channel is useful for the following reasons:
  - This expression provides us with a bound on the achievable data rate given bandwidth B and received SNR, employed in the ratio $\eta = R/C$, where R is the signalling rate and C is the channel capacity
    - As $\eta \rightarrow 1$, the system becomes more efficient
  - The capacity expression provides us with a basis for trade-off analysis between B and SNR
  - The capacity expression can be used for comparing the noise performance of one modulated scheme versus another

Additive White Gaussian Noise Channel

- In many instances, we model the communications channel as an additive white Gaussian noise (AWGN) channel
- The white Gaussian noise has an autocorrelation $R_n(\tau) = R_n(-\tau) = E\{n(t)n(t+\tau)\} = (N_0/2)\delta(t)$

Filtered AWGN Channel

- We know that the power spectral density of white Gaussian noise is equal to:

$$S_N(f) = \mathbb{F}\{R_n(\tau)\} = \int_{-\infty}^{\infty} R_n(\tau)e^{-j2\pi f\tau}d\tau = \frac{N_0}{2} \tag{37}$$

- When this noise is passed through an LTI system with impulse response $h(t)$, the output power spectral density will be defined by the Einstein-Wiener-Khinchin (EWK) Theorem, namely:

$$S_Y(f) = |H(f)|^2 S_N(f) \tag{38}$$

BPAM Modulation

- The process of modulation can be viewed as a mapping of b bits into a symbol
  - The message m_b is a b-vector of binary digits
- For each m_b (there are 2^b values of m_b available), we need a unique signal $s_i(t)$, $1 \leq i \leq 2^b$

Signal Constellation

- Modulation rule:
  - "1" $\rightarrow s_1(t)$
  - "0" $\rightarrow s_2(t)$
- The *bit rate* is defined as $R_b = 1/T$ bps
- The *symbol energy* is defined as $E_s = \int_{-\infty}^{\infty} s^2(t)dt$ [Joules]
- Suppose we have the following modulation rule given the waveform:

$$s(t) = A \cdot [u(t) - u(t - T)] \tag{39}$$

where $u(t)$ is the unit step function, then:

  - "1" $\rightarrow s(t)$
  - "0" $\rightarrow -s(t)$

Average Bit Energy

- The symbol energy is then $E_s{=}E_{-s}{=}A^2T{=}\frac{A^2}{R_b}$
  - Notice how E_s decreases as R_b increases
- We define the energy per bit as:

$$\bar{E}_b = P(1) \cdot \int_{-\infty}^{\infty} s_1^2(t)dt + P(0) \cdot \int_{-\infty}^{\infty} s_2^2(t)dt \tag{40}$$

where $P(1)$ is the probability that the bit is a "1", and $P(0)$ is the probability that the bit is a "0"

- Suppose $s_1(t) = s(t)$ and $s_2(t) = -s(t)$, then:

$$\bar{E}_b = E_s\{P(1) + P(0)\} = E_s = \int_{-\infty}^{\infty} s^2(t)dt = A^2T \tag{41}$$

Euclidean Distance: $s_i(t)$ Versus $s_j(t)$

- We define the *Euclidean Distance* as:

$$d_{ij}^2 = \int_{-\infty}^{\infty} (s_i(t) - s_j(t))^2 dt = E_{\Delta s_{ij}} \tag{42}$$

where $\Delta s_{ij}(t) = s_i(t) - s_j(t)$

- For a signal, $s_i(t)$, used for modulation:

$$d_{min}^2 = \min_{s_i(t), s_j(t), i \neq j} \int_{-\infty}^{\infty} (s_i(t) - s_j(t))^2 dt \tag{43}$$

Power Efficiency ε_p

- The *Power Efficiency* of a signal set used for modulation is given by the expression:

$$\varepsilon_p = \frac{d_{\min}^2}{\bar{E}_b} \tag{44}$$

- Suppose we would like to find the ε_p for Binary PAM
  - Given the waveforms:

$$\begin{aligned} s_1(t) &= A \cdot [u(t) - u(t-T)] = s(t) \\ s_2(t) &= -A \cdot [u(t) - u(t-T)] = -s(t) \end{aligned} \tag{45}$$

    where $u(t)$ is the unit step function
  - Compute the minimum Euclidean Distance: $d_{\min}^2 = \int_{-\infty}^{\infty} \Delta s_{ij}^2(t)dt = 4A^2T$
  - Compute the average bit energy: $\bar{E}_b = E_s\{P(1) + P(0)\} = E_s = \int_{-\infty}^{\infty} s^2(t)dt = A^2T$
  - Compute the power efficiency: $\varepsilon_P = d_{\min}^2 / \bar{E}_b = 4$

MPAM Modulation Format

- We can expression the M-ary PAM waveform as:

$$s_i(t) = A_i \cdot p(t), \text{ for } i = 1, 2, \ldots, M/2 \tag{46}$$

where $A_i = A(2i - 1)$, $p(t) = u(t) - u(t - T)$, and $u(t)$ is the unit step function

Computing $\varepsilon_{p,\text{M-PAM}}$

- Selecting the d^2_{min} pair $s_1(t) = A \cdot p(t)$ and $s_2(t) = -A \cdot p(t)$
  - Solving $\Delta s(t) = 2A \cdot p(t) \rightarrow d^2_{\text{min}} = 4A^2 T$
- Find $\bar{E}_s$ in general else only positive symbols due to symmetry of signal constellation, which yields:

$$\bar{E}_s = \frac{2}{M} A^2 T \sum_{i=1}^{M/2} (2i-1)^2$$

$$= A^2 T \frac{(M^2-1)}{3} \quad \text{which is simplified via tables}$$

$$\rightarrow \bar{E}_b = \frac{\bar{E}_s}{\log_2(M)} = \frac{A^2 T (2^{2b} - 1)}{3b}$$

- Solving for the power efficiency yields $\varepsilon_{p,\text{M-PAM}} = \frac{12b}{2^{2b}-1}$

Simple Receiver Structure

- M-ary QAM is a popular modulation scheme due to its **simple** receiver structure
  - Each branch employs a $\sqrt{M}$-ary PAM detector

Modulation Format

- M-ary QAM looks like two simultaneous $\sqrt{M}$-ary PAM signal constellations
    - One is acting in the real axis while the other is working on the imaginary axis
    - For instance, a 64-QAM signal constellation can be represented by two 8-PAM signal constellations
- We represent this linear modulation technique as:

$$s_{ij}(t) = A_i \cdot \cos(\omega_c t) + B_j \cdot \sin(\omega_c t) \tag{47}$$

Computing $\varepsilon_{p,\mathrm{M-QAM}}$

- To find the ε_p of M-ary QAM, we need to determine the following:
  - Calculate $d^2_{\min} = \int_0^T \Delta s^2(t)dt = 2A^2T$ using:

$$s_1(t) = A \cdot \cos(\omega_c t) + A \cdot \sin(\omega_c t)$$
$$s_2(t) = 3A \cdot \cos(\omega_c t) + A \cdot \sin(\omega_c t)$$

  - For computing $\bar{E}_s$, use the expression from M-ary PAM and replace M with $\sqrt{M}$ such that $\bar{E}_s = A^2T\frac{M-1}{3}$
    - Solve for $\bar{E}_b = \frac{\bar{E}_s}{\log_2(M)} = A^2T\frac{2^b-1}{3b}$
  - The power efficiency is equal to $\varepsilon_{p,\mathrm{M-QAM}} = \frac{3!b}{2^b-1}$

BPSK Modulation

- Modulation Rule:

$$\begin{aligned} \text{“1”} \rightarrow s_1(t) &= A \cdot \cos(\omega_c t + \theta) \\ \text{“0”} \rightarrow s_2(t) &= -A \cdot \cos(\omega_c t + \theta) \\ &= A \cdot \cos(\omega_c (t) + \theta + \pi) \\ &= -s_1(t) \end{aligned} \tag{48}$$

- As we have seen in the past few slides, determining ε_P is important
  - We will now show that $\varepsilon_p = 4$ is the **optimum**

- Employ the definition for d^2_{min} and solve

$$
\begin{aligned}
d^2_{\text{min}} &= \int_0^T (s_1(t) - s_2(t))^2 dt \\
&= 4A^2 \int_0^T \cos^2(\omega_c t + \theta) dt \\
&= \frac{4A^2 T}{2} + \frac{4A^2}{2} \int_0^T \cos(2\omega_c t + 2\theta) dt \\
&= 2A^2 T \rightarrow \text{double frequency term eliminated}
\end{aligned}
$$

- Employ the definition for $\bar{E}_b$ and solve

$$E_{s_1} = \int_0^T s_1^2(t)dt = A^2 \int_0^T \cos^2(\omega_c t + \theta)dt$$

$$= \frac{A^2T}{2} + \frac{A^2}{2}\int_0^T \cos(2\omega_c t + 2\theta)dt$$

$$= \frac{A^2T}{2} \rightarrow \text{double frequency term eliminated}$$

$$E_{s_2} = \frac{A^2T}{2} \quad \therefore \quad \bar{E}_b = P(0)\cdot E_{s_2} + P(1)\cdot E_{s_1} = \frac{A^2T}{2}$$

Solving for $\varepsilon_{p,\text{BPSK}}$: The Finale

- Applying the definition for the power efficiency yields:

$$\varepsilon_{p,\text{BPSK}} = \frac{d_{\text{min}}^2}{\bar{E}_b} = 4 \tag{49}$$

- This is suppose to be the largest possible value for ε_p for a modulation scheme employing all possible signal representations, i.e., $M = 2^b$ waveforms

Alternative Approach to Computing $d^2_{\min}$

- Another way to compute $d^2_{\min}$ is to use the concept of *correlation* such that:

$$d^2_{\min} = \int_0^T (s_2(t) - s_1(t))^2 dt = E_{s_1} + E_{s_2} - 2\rho_{12} \tag{50}$$

where:

$$E_{s_i} = \int_0^T s_i^2(t) dt \quad \text{and} \quad \rho_{12} = \int_0^T s_1(t) s_2(t) dt$$

- In order to get a **large** ε_p, we need to maximize $d^2_{\min}$, which means we want $\rho_{12} < 0$
  - $E_{s_1} = E_{s_2} = E = A^2 T/2 \rightarrow d^2_{\min} = 2(E - \rho_{12}) \rightarrow \rho_{12} = -E$

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An Example

- Show that for the following signal waveforms:

$$s_1(t) = A \cdot \cos(\omega_c t + \theta)$$
$$s_2(t) = 0$$

  the power efficiency is equal to $\varepsilon_p = 2$
  - This represents a 3 dB loss relative to BPSK!!
  - Loss $= 10 \cdot \log_{10} \left(\frac{\varepsilon_{p,\text{BPSK}}}{\varepsilon_{p,\text{other}}} \right)$
- Show that for the following signal waveforms:

$$s_1(t) = A \cdot \cos(\omega_c t + \theta)$$
$$s_2(t) = A \cdot \sin(\omega_c t + \theta)$$

  the power efficiency is equal to $\varepsilon_p = 2$

QPSK Modulation Format

- So far we have seen modulation schemes that consist of just one of two waveforms
  - We will now expand our signal constellation repertoire to four distinct waveforms per modulation scheme
- In QPSK modulation, a signal waveform possesses the following representation:

$$\begin{aligned} s_i(t) = &\pm A \cdot \cos(\omega_c t + \theta) \\ &\pm A \cdot \sin(\omega_c t + \theta) \end{aligned} \tag{51}$$

Computing $\varepsilon_{p,\text{QPSK}}$

- Solving for d^2_{min}, we obtain:

$$d^2_{\text{min}} = \int_0^T \Delta s^2(t)dt = 2A^2T \tag{52}$$

- To find $\bar{E}_b$, we need to average over all the signals, which is equal to (assuming $E_{s_1} = E_{s_2} = E_{s_3} = E_{s_4} = A^2T$):

$$\bar{E}_b = \frac{(E_{s_1} + E_{s_2} + E_{s_3} + E_{s_4})/4}{\log_2(M)} = \frac{A^2T}{2} \tag{53}$$

- Solving for the power efficiency, we get:

$$\varepsilon_{p,\text{QPSK}} = \frac{d^2_{\text{min}}}{\bar{E}_b} = 4 \tag{54}$$

which is the same as BPSK **but** with 2 bits per symbol!!

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MPSK Modulation Format

- Constant distance of signal constellation point to origin
- Consists of *M* equally spaced points on a circle
- General expression for an *M*-ary PSK waveform:

$$s_i(t) = A \cdot \cos\left(\omega_c t + \frac{2\pi i}{M}\right), \text{ for } i = 0, 1, 2, \ldots, M-1 \tag{55}$$

- There are advantages and disadvantages with this modulation scheme
  - As *M* increases, the spacing between signal constellation points decreases $\rightarrow$ error robustness decreases
  - Having information encoded in the phase results in constant envelope modulation, which is:
    - Good for non-linear power amplifiers
    - Robust to amplitude distortion channels

Computing $\varepsilon_{p,\text{M-PSK}}$

- Given $s_1(t) = A \cdot \cos(\omega_c t)$ and $s_2(t) = A \cdot \cos(\omega_c t + 2\pi/M)$
  - Calculate $d^2_{\text{min}} = E_{s_1} + E_{s_2} - 2\rho_{12}$ where:

$$E_{s_i} = \int_0^T s_i^2(t)dt = \frac{A^2 T}{2}, \text{ for } i = 1, 2 \tag{56}$$

and:

$$\rho_{12} = \int_0^T s_1(t)s_2(t)dt = \frac{A^2 T}{2} \cos\left(\frac{2\pi}{M}\right) \tag{57}$$

which yields $d^2_{\text{min}} = A^2 T(1 - \cos(\frac{2\pi}{M}))$

- Given $s_1(t) = A \cdot \cos(\omega_c t)$ and $s_2(t) = A \cdot \cos(\omega_c t + 2\pi/M)$
  - The average bit energy $\bar{E}_b$ is equal to $\bar{E}_b = \frac{\bar{E}_s}{\log_2(M)} = \frac{\bar{E}_s}{b}$, where $\bar{E}_s = A^2 T/2$
  - Using the definition for the power efficiency, we see that $\varepsilon_{p,\mathrm{M-PSK}} = 2b(1 - \cos(\frac{2\pi}{M})) = 4b\sin^2(\frac{\pi}{2^b})$

Overall Comparison of ε_p

- To determine how much power efficiency we are losing relative to $\varepsilon_{p,\text{QPSK}}$, we use $\delta\text{SNR} = 10 \cdot \log_{10}(\frac{\varepsilon_{p,\text{QPSK}}}{\varepsilon_{p,\text{other}}})$

Table: δSNR Values of Various Modulation Schemes.

M	b	M-ASK	M-PSK	M-QAM
2	1	0	0	0
4	2	4	0	0
8	3	8.45	3.5	(??)
16	4	13.27	8.17	4.0
32	5	18.34	13.41	(??)
64	6	24.4	18.4	8.45

Observations of ε_p

- Two dimensional modulation is better than one dimensional modulation
- All modulation schemes studied are linear modulation schemes→similar receiver complexity

Signal Vectors

- Let $\phi_j(t)$ be an orthnormal set of functions on the time interval $[0, T]$ such that:
$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$
- Let $s_i(t)$ be the modulation signal that we can represent in terms of the orthonormal functions:
$$s_i(t) = \sum_{k=1}^{N} s_{ik}\phi_k(t) \tag{58}$$
which can be represented by the vector:
$$s_i(t) \rightarrow \mathbf{s}_i = (s_{i1},\ s_{i2},\ s_{i3}, \ldots s_{iN}) \tag{59}$$

Vector Representation of $s_i(t)$

- Let us look at a sample vector representation of $s_i(t)$ in three dimensional space using basis functions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$

- To find s_{ik}, solve:

$$\int_0^T s_i(t)\phi_l(t)dt = \sum_{k=1}^{N} s_{ik} \int_0^T \phi_k(t)\phi_l(t)dt = s_{il} \tag{60}$$

- The vector dot product between $s_i(t)$ and $s_j(t)$ is equal to:

$$\int_0^T s_i(t)s_j(t)dt = \mathbf{s}_i \cdot \mathbf{s}_j = \rho_{ij} \tag{61}$$

while the energy of a signal $s_i(t)$ is equal to:

$$E_{s_i} = \int_0^T s_i^2(t)dt = \mathbf{s}_i \cdot \mathbf{s}_i = \|\mathbf{s}_i\|^2 \tag{62}$$

Euclidean Distance

- To compute the *Euclidean Distance* using signal space vectors, we need to solve:

$$\begin{aligned} d_{\min}^2 &= \int_0^T \Delta s_{ij}^2(t)dt = \int_0^T (s_i(t) - s_j(t))^2 dt \\ &= \|\mathbf{s}_i - \mathbf{s}_j\|^2 = (\mathbf{s}_i - \mathbf{s}_j) \cdot (\mathbf{s}_i - \mathbf{s}_j) \\ &= E_{s_i} + E_{s_j} - 2\rho_{ij} \end{aligned}$$

where:

$$\rho_{ij} = \int_0^T s_i(t)s_j(t)dt = \mathbf{s}_i \cdot \mathbf{s}_j \tag{63}$$

Solving for the Power Efficiency

- Choose orthonormal basis functions $\phi_i(t)$, $i = 1, 2, \ldots, k$, where k is the dimension of the signal space
- Find $\mathbf{s}_i$, $i = 1, 2, \ldots, M$ where $\mathbf{s}_i = (s_{i1},\ s_{i2}, \ldots s_{ik})$ and $s_{ij} = \int_0^T s_i(t)\phi_j(t)dt$
- Consequently, solve for:

$$d_{\min}^2 = \min_{i \neq j} \|\mathbf{s}_i - \mathbf{s}_j\|^2$$

$$\bar{E}_s = \frac{1}{M}\sum_{i=1}^{M} \|\mathbf{s}_i\|^2$$

$$\bar{E}_b = \bar{E}_s / \log_2(M)$$

$$\varepsilon_p = d_{\min}^2 / \bar{E}_b$$

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Conclusion

1. This paper presentation has considered wireless communication in a general fading environment, which can be reduced to other types of fading environments like Rayleigh.
2. Obtained closed-form expressions for PDF and CDF of SIR for the interference-limited system case and closed-form expressions for first-order statistics (PDF, CDF, moments of various order) of newly introduced composite fading/shadowing model allow simple unconstrained analysis and accurate wireless system planning and performance evaluation.
3. Some of the performance measures are evaluated and discussed in the presentation.

Reference

IEEE paper on Modelling and performance analysis of Wireless channel Performance analysis of wireless communication system in general fading environment subjected to shadowing and interference

THANK YOU.......