Challenging problem 8

Annu-EE21RESCH01010

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https://github.com/annu100/AI5002−Probability−and−Random−variables/tree/main.tex/challenging problems

I. Challenging problem 8

Let $X_1, X_2, ..., X_n$ be independent Poisson random variables with $E[X_i] = \mu_i$.Find the conditional distribution of $X_1, ..., X_n \Big| \sum_{i=1}^{n} X_i = y$

II. Solutions

A. Likelihood approach to this problem

Definitionof likelihood function-
Given $\vec{x_1} = x_1, \vec{x_2} = x_2, \ldots \vec{x_n} = x_n$ the joint $pdf \;\; f_x(x_1, x_2 \ldots x_n; y)$ is defined to be likelihood function.

The random variable X_i is distributed as Poisson if the density of X_i is given by
$E[X_i] = \mu_i$
Also, $X_i = \mu$
$f(x_i : \mu_i) = \begin{cases} \frac{e^{-\mu_i} \times \mu^{x_i}}{x_i!} & , x_i \geq 0 \\ 0 & , otherwise \end{cases}$
Since $X_1, X_2, ..., X_n$ be independent Poisson random variables-
Likelihood function L is required to be calculated and it is given by

$$L = \frac{e^{-\mu_1} \times \mu^{x_1}}{x_1!} \times \frac{e^{-\mu_2} \times \mu^{x_2}}{x_2!} \cdots \frac{e^{-\mu_n} \times \mu^{x_n}}{x_n!} \tag{1}$$

$$= \frac{e^{-\mu_n} \times \mu^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \tag{2}$$

$$\tag{3}$$

So,above case is only defined when
$\sum_{i=1}^{n} X_i = y$ for some y
otherwise L=0.

Here L is the conditional distribution of
$X_1, ..., X_n \Big| \sum_{i=1}^{n} X_i = y$
Also sum of two independent Possion R.V is also R,V with mean as sum of mean of individual random variables which provides for the proof of above expression for L.

B. Estimation theory approach to this problem

Sufficient statistics
A function $T(\vec{x}) = T(x_1, x_2, \ldots, x_n)$ is said to be sufficient statistic for y if $T(\vec{x})$ contains all information about y that is contained in the data set i.e given the probability density
$Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \ldots \vec{x_n} = x_n; y)$
if $Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \ldots \vec{x_n} = x_n; y = T(\vec{x}))$
does not depend on y for all possible values of x_i, then $T(\vec{x})$ must contain all information about y that is contained in sample set $\vec{x}$.

Thus the statistic $T(\vec{x})$ is said to be sufficient if the conditional $p.d.f$
$f(x_1, x_2, \ldots x_n; y | T(\vec{x}) = t)$ does not depend on y.
This function is also known as likelihood function.
Now for our problem
$x_1, x_2 \ldots x_n$ be i.i.d Poisson distributed ,let us say $P(\mu)$and consider the function
$T(\vec{x}) = T(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \vec{x_i}$.
Then,
$Pr(\vec{x_1} = x_1, x_2 = x_2 \ldots \vec{x_n} = x_n; y = T(\vec{x}) = \begin{cases} \text{Pr}(=\text{x}_1, \vec{x_2} = x_2 \ldots \vec{x_n} = t - (x_1 + x_2 \ldots \ldots x_{n-1})) & \text{, such that } t = \sum_{i=1}^{n} x_i \\ 0 & \text{, such that } t \neq \sum_{i=1}^{n} x_i \end{cases}$
But as we know $T(x) \sim P(n\mu)$.Thus

$$P(T = \sum_{i=1}^{n} x_i) = e^{-\mu} \frac{(n\mu)^t}{(t)!} \tag{4}$$

$$= e^{-\mu} \frac{(n\mu)^{\sum_{i=1}^{n} x_i}}{(\sum_{i=1}^{n} x_i)!} \tag{5}$$

$$\frac{Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \ldots x_n = t - (x_1 + x_2 \ldots\ldots x_{n-1}))}{T(\vec{x}) = T(x_1, x2, \ldots, x_n)} \tag{6}$$

$$= \frac{e^{-\mu} \frac{(\mu)^{x_1}}{(x_1)!} \times e^{-\mu} \frac{(\mu)^{x_2}}{(x_2)!} \cdots e^{-\mu} \frac{(\mu)^{t-(x_1, x_2 \ldots x_{n-1})}}{(t-(x_1, x_2 \ldots x_{n-1}))!}}{e^{-n\mu} \frac{(n\mu)^t}{(t)!}} \tag{7}$$

$$= \frac{(\sum_{i=1}^{n} x_i)!}{x_1!, x_2! \ldots x_n! n!} \tag{8}$$

$T(\vec{x}) = \sum_{i=1}^{n} x_i$ is sufficient for μ.

The expression which we got can be seen that from both the approaches w matches !!