Challenging problem 8

Annu-EE21RESCH01010

Download latex code from here-

https://github.com/annu100/AI5002−Probability−and−Random−variables/tree/main.tex/challenging problems

I. Challenging problem 8

Let $X_1, X_2, ..., X_n$ be independent Poisson random variables with $E[X_i] = \mu_i$.Find the conditional distribution of $X_1, ..., X_n \Big| \sum_{i=1}^{n} X_i = y$

II. SOLUTIONS

A. Likelihood approach to this problem

Definition of likelihood function-
Given $\vec{x_1} = x_1, \vec{x_2} = x_2, \dots \vec{x_n} = x_n$ the joint $pdf \;\; f_x(x_1, x_2 \dots x_n; y)$ is defined to be likelihood function.

The random variable X_i is distributed as Poisson if the density of X_i is given by
$E[X_i] = \mu_i = \mu$
(let) Also, $Var(X_i) = \mu$
$f(x_i : \mu_i)=\begin{cases} \frac{e^{-\mu_i}\times\mu^{x_i}}{x_i!} & , x_i \geq 0 \\ 0 & , otherwise \end{cases}$
Since $X_1, X_2, ..., X_n$ be independent Poisson random variables-
Likelihood function L is required to be calculated and it is given by

$$L = \frac{e^{-\mu_1} \times \mu^{x_1}}{x_1!} \times \frac{e^{-\mu_2} \times \mu^{x_2}}{x_2!} \cdots \frac{e^{-\mu_n} \times \mu^{x_n}}{x_n!} \tag{1}$$
$$= \frac{e^{-\mu n} \times \mu^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \tag{2}$$
$$\tag{3}$$

So,above case is only defined when
$\sum_{i=1}^{n} X_i = y$ for some y
otherwise L=0.

Here L is the conditional distribution of
$X_1, ..., X_n \Big| \sum_{i=1}^{n} X_i = y$
Also sum of two independent Possion R.V is also R,V with mean as sum of mean of individual random variables which provides for the proof of above expression for L.

B. Estimation theory approach to this problem

Sufficient statistics
A function $T(\vec{x}) = T(x_1, x_2, \dots, x_n)$ is said to be sufficient statistic for y if $T(\vec{x})$ contains all information about y that is contained in the data set i.e given the probability density
$Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \dots \vec{x_n} = x_n; y)$
if $Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \dots \vec{x_n} = x_n; y = T(\vec{x}))$
does not depend on y for all possible values of x_i, then $T(\vec{x})$ must contain all information about y that is contained in sample set $\vec{x}$.

Thus the statistic $T(\vec{x})$ is said to be sufficient if the conditional $p.d.f$
$f(x_1, x_2, \dots x_n; y | T(\vec{x}) = t)$ does not depend on y.
This function is also known as likelihood function.
Now for our problem
$x_1, x_2 \dots x_n$ be i.i.d Poisson distributed ,let us say $P(\mu)$and consider the function
$T(\vec{x}) = T(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \vec{x_i}$.
Then,
$Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \dots \vec{x_n} = x_n; y = T(\vec{x}))= \begin{cases} Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \dots \vec{x_n} = t - (x_1 + x_2 \dots\dots x_{n-1})) & , \\ \text{such that} t = \sum_{i=1}^{n} x_i & \\ 0 & \\ , \text{such that} t \neq \sum_{i=1}^{n} x_i & \end{cases}$
But as we know $T(x) \sim P(n\mu)$.Thus

$$P(T = \sum_{i=1}^{n} x_i) = e^{-\mu} \frac{(n\mu)^t}{(t)!} \tag{4}$$
$$= e^{-\mu} \frac{(n\mu)^{\sum_{i=1}^{n} x_i}}{(\sum_{i=1}^{n} x_i)!} \tag{5}$$

$$\frac{Pr(\vec{x_1} = x_1, \vec{x_2} = x_2 \dots \vec{x_n} = t - (x_1 + x_2 \dots\dots x_{n-1}))}{T(\vec{x}) = T(x_1, x_2, \dots, x_n)} \tag{6}$$

$$= \frac{e^{-\mu}\frac{(\mu)^{x_1}}{(x_1)!} \times e^{-\mu}\frac{(\mu)^{x_2}}{(x_2)!} \cdots e^{-\mu}\frac{(\mu)^{t-(x_1,x_2\dots x_{n-1})}}{(t-(x_1,x_2\dots x_{n-1}))!}}{e^{-n\mu}\frac{(n\mu)^t}{(t)!}} \tag{7}$$

$$= \frac{(\sum_{i=1}^{n} x_i)!}{x_1!, x_2! \dots x_n! n!} \tag{8}$$

$T(\vec{x}) = \sum_{i=1}^{n} x_i$ is sufficient for μ.

The expression which we got can be seen that from both the approaches matches !!!

III. Considering only two random variables X_1 and X_2

A. Likelihood approach to this problem

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Also, $Var(X_i) = \mu$
$f(x_i : \mu_i) = \begin{cases} \frac{e^{-\mu_i} \times \mu^{x_i}}{x_i!} & , x_i \geq 0 \\ 0 & , otherwise \end{cases}$
Since $X_1, X_2, \dots, X_n$ be independent Poisson random variables-
Likelihood function L is required to be calculated and it is given by

$$L = \frac{e^{-\mu_1} \times \mu^{x_1}}{x_1!} \times \frac{e^{-\mu_2} \times \mu^{x_2}}{x_2!} \cdots \frac{e^{-\mu_n} \times \mu^{x_n}}{x_n!} \tag{9}$$

$$= \frac{e^{-2\mu} \times \mu^{\sum_{i=1}^{2} x_i}}{\prod_{i=1}^{2} x_i!} \tag{10}$$

$$\tag{11}$$

So,above case is only defined when
$\sum_{i=1}^{2} X_i = y$ for some y
otherwise L=0.
Here L is the conditional distribution of
$X_1, X_2 \Big| \sum_{i=1}^{2} X_i = y$

Also sum of two independent Possion R.V is also R,V with mean as sum of mean of individual random variables which provides for the proof of above expression for L.

B. Estimation theory approach to this problem

Sufficient statistics

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$Pr(\vec{x_1} = x_1, \vec{x_2} = x_2; y)$
if $Pr(\vec{x_1} = x_1, \vec{x_2} = x_2; y = T(\vec{x}))$
does not depend on y for all possible values of x_i, then $T(\vec{x})$ must contain all information about y that is contained in sample set $\vec{x}$.

Thus the statistic $T(\vec{x})$ is said to be sufficient if the conditional $p.d.f$
$f(x_1, x_2; y | T(\vec{x}) = t)$ does not depend on y.
This function is also known as likelihood function.
Now for our problem
x_1, x_2 be i.i.d Poisson distributed ,let us say $P(\mu)$and consider the function
$T(\vec{x}) = T(x_1, x_2) = \sum_{i=1}^{2} \vec{x_i}$.
Then,
$Pr(\vec{x_1} = x_1, \vec{x_2} = x_2; y = T(\vec{x}))= \begin{cases} Pr(\vec{x_1} = x_1, \vec{x_2} = t - (x_1)) \quad , \\ \text{such that} t = \sum_{i=1}^{2} x_i \\ 0 \\ , \text{such that} t \neq \sum_{i=1}^{2} x_i \end{cases}$
But as we know $T(x) \sim P(2\mu)$.Thus

$$P(T = \sum_{i=1}^{2} x_i) = e^{-\mu}\frac{(2\mu)^t}{(t)!} \tag{12}$$

$$= e^{-\mu}\frac{(2\mu)^{\sum_{i=1}^{2} x_i}}{(\sum_{i=1}^{2} x_i)!} \tag{13}$$

$$\frac{Pr(\vec{x_1} = x_1 \vec{x_2} = t - (x_1))}{T(\vec{x}) = T(x_1, x_2)} \tag{14}$$

$$= \frac{e^{-\mu}\frac{(\mu)^{x_1}}{(x_1)!} \times e^{-\mu}\frac{(\mu)^{t-(x_1)}}{(t-(x_1)!}}{e^{-2\mu}\frac{(2\mu)^t}{(t)!}} \tag{15}$$

$$= \frac{(\sum_{i=1}^{2} x_i)!}{x_1!, x_2! 2!} \tag{16}$$

$T(\vec{x}) = \sum_{i=1}^{2} x_i$ is sufficient for μ.