Probability Theory & Random Processes EE5817

Dr. Abhinav Kumar

Department of Electrical Engineering
Indian Institute of Technology Hyderabad, Telangana, India
Email: abhinavkumar@ee.iith.ac.in

Outline

Continous Random Variables

A Continous Random Variable X has a CDF, $F_X(x)$, that is differentiable almost everywhere.

For e.g., a number uniform randomly selected in the interval $[0, 1]$ has a CDF $F_X(x) = x\, \forall x \in [0, 1]$, 0 for $x < 0$ and 1 otherwise.

A Continous Random Variable

The PDF of a continous random variable X is the derivative of the CDF (if the derivative exists!).

$$f_X(x) = F'_X(x) \text{ alternatively,}$$

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta}$$

For e.g., for X uniformly distributed in the interval $[0, 1]$ the PDF $f_X(x) = 1 \, \forall x \in (0, 1)$ and 0 otherwise.

A Continous Random Variable

Properties of the PDF:

- $f_X(x) \geq 0$
- $\int_{-\infty}^{\infty} f_X(u)du = 1$
- $F_X(x) = \int_{-\infty}^{x} f_X(u)du$
- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)du$
- More generally, $P(X \in \Phi) == \int_\phi f_X(u)du$. For e.g., $P(X \in [a, b] \cup [c, d]) = \int_a^b f_X(u)du + \int_c^d f_X(u)du$

Expectation

Expectation for a continous r.v. X with PDF $f_X(x)$ is given by $E[X] = \int_{-\infty}^{\infty} uf_X(u)du$

- Mean: $\mu_X = E[X]$
- Variance: $\sigma_X^2 = E[X^2] - \mu_X^2$

Uniform Random Variable

Uniform distribution: $X \sim unif(a, b)$

- $$f_X(x) = \begin{cases} 0 & x \leq a, \\ \dfrac{1}{(b-a)} & x \in (a, b), \\ 0 & x \geq b. \end{cases}$$

- $$F_X(x) = \begin{cases} 0 & x \leq a, \\ \dfrac{x-a}{(b-a)} & x \in (a, b), \\ 1 & x \geq b. \end{cases}$$

- $\mu_X = (a+b)/2$
- $\sigma_X^2 = \frac{(b-a)^2}{12}$

Exponential Distribution

Exponential distribution: $X \sim \exp(\lambda)$

- $$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-\lambda x} & x \geq 0. \end{cases}$$
- $$f_X(x) = \begin{cases} 0 & x < 0, \\ \lambda e^{-\lambda x} & x \geq 0. \end{cases}$$
- $\mu_X = \frac{1}{\lambda}$
- $\sigma_X^2 = \frac{1}{\lambda^2}$

Exponential Distribution

Exponential Distribution exhibits memorylessness:

- $P\{X > t_1 + t_2 | X > t_1\} = P\{X > t_2\}$

Standard Gaussian Distribution

Standard Normal distribution: $X \sim N(0,1)$

- $$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5(x)^2}$$
- $\mu_X = \int_{-\infty}^{\infty} x f_X(x) dx = 0$, as $x f_X(x)$ is an odd function of x
- $\sigma_X^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-0.5x^2} dx =$ $\frac{2}{\sqrt{2\pi}} (\int_0^{\infty} x(xe^{-0.5x^2}) dx$ integrating by parts gives $= \frac{2}{\sqrt{2\pi}} (-xe^{-0.5x^2}|_0^{\infty} + \int_0^{\infty} e^{-0.5x^2} dx) = \frac{2}{\sqrt{2\pi}} (0 + \frac{\sqrt{2\pi}}{2}) = 1$
- Self reading: Jacobian and $\int_0^{\infty} e^{-0.5x^2} dx = \frac{\sqrt{2\pi}}{2}$

Standard Gaussian Distribution

Standard Normal distribution: $X \sim N(0,1)$

- $$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5(x)^2}$$
- $\mu_X = 0$
- $\sigma_X^2 = 1$

Standard Gaussian Distribution

Standard Normal distribution: $X \sim N(0,1)$

- $$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5(x)^2}$$
- $\mu_X = 0$
- $\sigma_X^2 = 1$
- $F_X(x) =?$

Standard Gaussian Distribution

Standard Normal distribution: $X \sim N(0,1)$

- $$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5(x)^2}$$
- $\mu_X = 0$
- $\sigma_X^2 = 1$
- $F_X(x) = 1 - Q(x)$, where, $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-0.5(x)^2} dx$
- Self Reading: Properties of Q function

Gaussian Distribution

Normal distribution: $X \sim N(\mu, \sigma^2)$

- $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5\frac{(x-\mu)^2}{\sigma^2}}$$
- $$F_X(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$$
- $\mu_X = \mu$
- $\sigma_X^2 = \sigma^2$

Gaussian Distribution

Normal distribution: $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5\frac{(x-\mu)^2}{\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} (\sigma y + \mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5y^2} \sigma dy \text{ ,where , } \left(y = \frac{x-\mu}{\sigma} \right) \\
&= 0 + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5y^2} \sigma dy \text{ , first part is an odd function} \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5y^2} \sigma dy \text{ , as integration of PDF=1} \\
&= \mu
\end{aligned}
$$

Gaussian Distribution

Normal distribution: $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
\sigma_X^2 &= E[(x-\mu)^2] \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5\frac{(x-\mu)^2}{\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \sigma^2 y^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5y^2} \sigma dy \text{ ,where , } \left(y = \frac{x-\mu}{\sigma}\right) \\
&= \sigma^2 \int_{-\infty}^{\infty} \frac{y^2}{\sqrt{2\pi}} e^{-0.5y^2} dy \\
&= \sigma^2
\end{aligned}
$$

Moments

Moments (if they exist!)

- n$^{\text{th}}$ order moment, $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$
- n$^{\text{th}}$ order central moment, $E[(X-\mu_X)^n] = \int_{-\infty}^{\infty} (x-\mu_X)^n f_X(x)dx$
- n$^{\text{th}}$ order absolute moment, $E[|X|^n] = \int_{-\infty}^{\infty} |x|^n f_X(x)dx$
- n$^{\text{th}}$ order absolute central moment, $E[|X-\mu_X|^n] = \int_{-\infty}^{\infty} |x-\mu_X|^n f_X(x)dx$
- n$^{\text{th}}$ order standardized moment, $E\left[\left(\frac{X-\mu_X}{\sigma}\right)^n\right] = \int_{-\infty}^{\infty} \left(\frac{X-\mu_X}{\sigma}\right)^n f_X(x)dx$

Moments

Moments (if they exist!)

- Mean, $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$
- Variance, $E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$
- Skewness, $E\left[\left(\frac{X - \mu_X}{\sigma}\right)^3\right] = \int_{-\infty}^{\infty} \left(\frac{X - \mu_X}{\sigma}\right)^3 f_X(x) dx$
- Kurtosis, $E\left[\left(\frac{X - \mu_X}{\sigma}\right)^4\right] = \int_{-\infty}^{\infty} \left(\frac{X - \mu_X}{\sigma}\right)^4 f_X(x) dx$
- Excess Kurtosis,
  $E\left[\left(\frac{X - \mu_X}{\sigma}\right)^4\right] - 3 = \int_{-\infty}^{\infty} \left(\frac{X - \mu_X}{\sigma}\right)^4 f_X(x) dx - 3$

Moments

Symmetry around Mean, (if mean exist!)

- $X - \mu$ has the same distribution as $\mu - X$
- Mean and Median will coincide
- All odd central moments of a symmetric distribution are zero

Moments

Moment Generating Function (MGF)

- MGF$=E[e^{sx}]$
- Characteristic Function $\Phi_X(\omega) = E[e^{j\omega x}]$
- Probability Generating Function$= E(t^X)$

Moment Generating Function (MGF)

$$
\begin{aligned}
e^{sx} &= 1 + sx + \frac{s^2x^2}{2!} + \frac{s^3x^3}{3!} + \ldots + \frac{s^nx^n}{n!} + \ldots \\
MGF &= E[e^{sX}] \\
&= E\left[1 + sX + \frac{s^2X^2}{2!} + \ldots + \frac{s^nX^n}{n!} + \ldots\right] \\
&= 1 + sE[X] + \frac{s^2E[X^2]}{2!} + \ldots + \frac{s^nE[X^n]}{n!} + \ldots \\
\left.\frac{dE[e^{sX}]}{ds}\right|_{s=0} &= E[X] \\
\left.\frac{d^nE[e^{sX}]}{ds^n}\right|_{s=0} &= E[X^n]
\end{aligned}
$$

Moment Generating Function (MGF)

Some examples:

$$
\begin{aligned}
\text{Gaussian} &= e^{\mu s + 0.5\sigma^2 s^2} \\
\text{Exponential} &= \frac{\lambda}{\lambda - s}, \ \text{for } s < \lambda \\
\text{Uniform} &= \begin{cases} \dfrac{e^{sb} - e^{sa}}{s(b-a)} \text{for } s \neq 0, \\ 1 \ s = 0. \end{cases}
\end{aligned}
$$

Characteristic Function

$$\begin{aligned}
\Phi(\omega) &= E[e^{j\omega X}] \\
&= E\left[1 + j\omega X + \frac{j^2\omega^2 X^2}{2!} + \ldots + \frac{(j\omega X)^n}{n!} + \ldots\right] \\
\left.\frac{d^n E[e^{j\omega X}]}{d\omega^n}\right|_{\omega=0} &= j^{-n} E[X^n]
\end{aligned}$$

Characteristic Function

Some examples:

$$\begin{aligned}
\text{Gaussian} &= e^{j\mu\omega - 0.5\sigma^2\omega^2} \\
\text{Exponential} &= \frac{\lambda}{\lambda - j\omega} \\
\text{Uniform} &= \begin{cases} \dfrac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)} \text{for } \omega \neq 0, \\ 1\, \omega = 0. \end{cases}
\end{aligned}$$

Probability Generating Function

$$
\begin{aligned}
G_X(z) &= E(z^X) \\
&= \sum_{x=0}^{\infty} P_X(x) z^x, \text{ for nonnegative X} \\
&= p_0 + p_1 z + p_2 z^2 + \ldots + p_n z^n + \ldots \\
P(X = n) &= \left(\frac{1}{n!}\right) \frac{d^n E(z^X)}{dz^n}\bigg|_{z=0}
\end{aligned}
$$

Self Reading: $E(X(X-1)(X-2)\ldots(X-k+1)) = \frac{d^k E(z^X)}{dz^k}\Big|_{z=1}$

Probability Generating Function

Some examples:

$$
\begin{aligned}
\text{Bernoulli} &= (pz+q) \\
\text{Binomial} &= (pz+q)^n \\
\text{Discrete Uniform Distribution} &= \frac{z^a - z^{b+1}}{n(1-z)} \\
\text{Poisson} &= e^{\lambda(z-1)}
\end{aligned}
$$

Probability Generating Function

Geometric: def. 1, number of trials at which first success is observed with PMF, $P(X = k) = pq^{k-1}$

$$\begin{aligned} \text{PGF} &= \sum_{k=1}^{\infty} pq^{k-1} z^k \\ &= pz \sum_{k=0}^{\infty} (qz)^k \\ &= \frac{pz}{(1 - qz)} \text{ for } |z| < \frac{1}{q} \end{aligned}$$

Probability Generating Function

Geometric: def. 2, number of failures/trials after which first success is observed with PMF, $P(X = k) = pq^k$

$$\begin{aligned} \text{PGF} &= \sum_{k=0}^{\infty} pq^k z^k \\ &= p\sum_{k=0}^{\infty}(qz)^k \\ &= \frac{p}{(1-qz)} \text{ for } |z| < \frac{1}{q} \end{aligned}$$

Probability Generating Function

Negative Binomial:

$$
\begin{aligned}
\text{PGF} &= \sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (q)^{k-r} z^k \\
&= (pz)^r \sum_{k=r}^{\infty} \binom{k-1}{k-r} (zq)^{k-r} \\
&= (pz)^r \sum_{k-r=0}^{\infty} \binom{k-r+r-1}{k-r} (zq)^{k-r} \\
&= (pz)^r \sum_{n=0}^{\infty} \binom{n+r-1}{n} (zq)^n \\
&= \left(\frac{pz}{1-qz}\right)^r \quad \text{for } |z| < \frac{1}{q}
\end{aligned}
$$

Questions