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Nakagami distribution

The **Nakagami distribution** or the **Nakagami-*m* distribution** is a probability distribution related to the gamma distribution. The family of Nakagami distributions has two parameters: a shape parameter $m \geq 1/2$ and a second parameter controlling spread $\Omega > 0$.

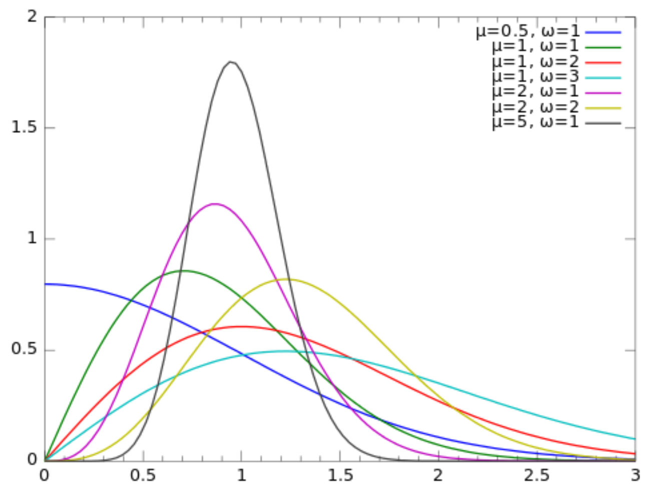
Contents

Characterization

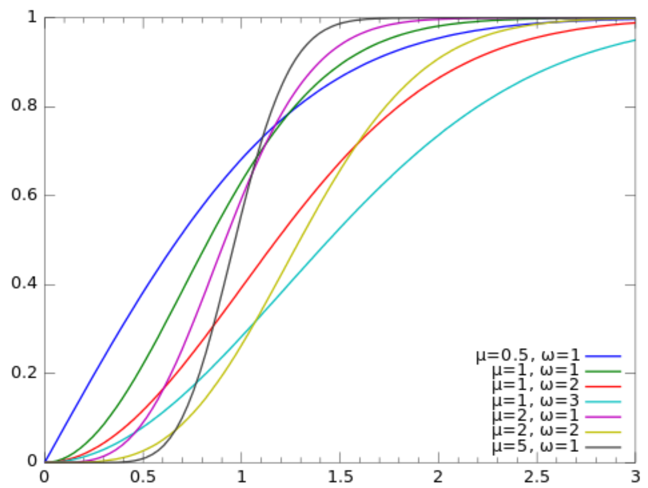
Its probability density function (pdf) is[1]

Nakagami

Probability density function



Cumulative distribution function



Parameters	m or $\mu \geq 0.5$ shape (real) Ω or $\omega > 0$ spread (real)
Support	$x > 0$
PDF	$\frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} \exp\left(-\frac{m}{\Omega} x^2\right)$
CDF	$\frac{\gamma\left(m, \frac{m}{\Omega} x^2\right)}{\Gamma(m)}$
Mean	$\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{1/2}$
Median	No simple closed form
Mode	$\frac{\sqrt{2}}{2} \left(\frac{(2m-1)\Omega}{m}\right)^{1/2}$
Variance	

$$\Omega\left(1-\frac{1}{m}\left(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}\right)^2\right)$$

$$f(x;\,m,\Omega)=\frac{2m^m}{\Gamma(m)\Omega^m}x^{2m-1}\exp\left(-\frac{m}{\Omega}x^2\right),\forall x\geq 0.$$

where $(m\geq 1/2,\text{ and }\Omega>0)$

Its cumulative distribution function is[1]

$$F(x;\,m,\Omega)=P\left(m,\frac{m}{\Omega}x^2\right)$$

where P is the regularized (lower) incomplete gamma function.

Parametrization

The parameters m and Ω are[2]

$$m=\frac{\left(\operatorname{E}\left[X^2\right]\right)^2}{\operatorname{Var}\left[X^2\right]},$$

and

$$\Omega=\operatorname{E}\left[X^2\right].$$

Parameter estimation

An alternative way of fitting the distribution is to re-parametrize Ω and m as $\sigma = \Omega/m$ and m.[3]

Given independent observations $X_1=x_1,\ldots,X_n=x_n$ from the Nakagami distribution, the likelihood function is

$$L(\sigma,m)=\left(\frac{2}{\Gamma(m)\sigma^m}\right)^n\left(\prod_{i=1}^n x_i\right)^{2m-1}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{\sigma}\right).$$

Its logarithm is

$$\ell(\sigma,m)=\log L(\sigma,m)=-n\log\Gamma(m)-nm\log\sigma+(2m-1)\sum_{i=1}^n\log x_i-\frac{\sum_{i=1}^n x_i^2}{\sigma}.$$

Therefore

$$\frac{\partial\ell}{\partial\sigma}=\frac{-nm\sigma+\sum_{i=1}^n x_i^2}{\sigma^2}\quad\text{and}\quad\frac{\partial\ell}{\partial m}=-n\frac{\Gamma'(m)}{\Gamma(m)}-n\log\sigma+2\sum_{i=1}^n\log x_i.$$

These derivatives vanish only when

$$\sigma = \frac{\sum_{i=1}^{n} x_i^2}{nm}$$

and the value of m for which the derivative with respect to m vanishes is found by numerical methods including the Newton–Raphson method.

It can be shown that at the critical point a global maximum is attained, so the critical point is the maximum-likelihood estimate of (m,σ). Because of the equivariance of maximum-likelihood estimation, one then obtains the MLE for Ω as well.

Generation

The Nakagami distribution is related to the gamma distribution. In particular, given a random variable $Y \sim \textrm{Gamma}(k, \theta)$, it is possible to obtain a random variable $X \sim \textrm{Nakagami}(m, \Omega)$, by setting $k = m$, $\theta = \Omega/m$, and taking the square root of Y:

$$X = \sqrt{Y}.$$

Alternatively, the Nakagami distribution $f(y;\, m, \Omega)$ can be generated from the chi distribution with parameter k set to $2m$ and then following it by a scaling transformation of random variables. That is, a Nakagami random variable X is generated by a simple scaling transformation on a Chi-distributed random variable $Y \sim \chi(2m)$ as below.

$$X = \sqrt{(\Omega/2m)Y}.$$

For a Chi-distribution, the degrees of freedom $2m$ must be an integer, but for Nakagami the m can be any real number greater than 1/2. This is the critical difference and accordingly, Nakagami-m is viewed as a generalization of Chi-distribution, similar to a gamma distribution being considered as a generalization of Chi-squared distributions.

History and applications

The Nakagami distribution is relatively new, being first proposed in 1960.[4] It has been used to model attenuation of wireless signals traversing multiple paths [5] and to study the impact of fading channels on wireless communications.[6]

Related distributions

- Restricting *m* to the unit interval (*q* = *m*; 0 < *q* < 1) defines the **Nakagami-*q*** distribution, also known as **Hoyt distribution**.[7][8][9]

> "The radius around the true mean in a bivariate normal random variable, re-written in polar coordinates (radius and angle), follows a Hoyt distribution. Equivalently, the modulus of a complex normal random variable does."

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