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ROLL NO. - EE21RESCH01010

Subject - Channel Coding

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"EE20MTECH12001"

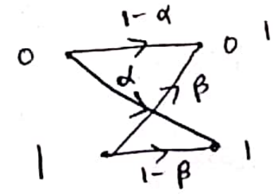
References Used - Thomas book on "Information Theory"
MIT lecture Notes.

Exercise 1.1 Let A be the matrix with (i,j)th entry –

$$A_{ij} = P_{Y|X}(i|j)$$

$$P_{Y|X} = \begin{array}{c|c} & Y \rightarrow \; 0 \quad\quad 1 \\ X \downarrow & \\ 0 & \multirow{2}{*}{$\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$} \\ 1 & \end{array}$$

The matrix A is doubly stochastic –



$$\forall i, \quad \sum_j A_{ij} = 1$$

$$\forall j, \quad \sum_i A_{ij} = 1$$

Let $P_X = (P_0 \ P_1)$, $\sum_{i=0}^{1} P_i = 1$ (sum of probability $= 1$)

Consider the lagrangian :-

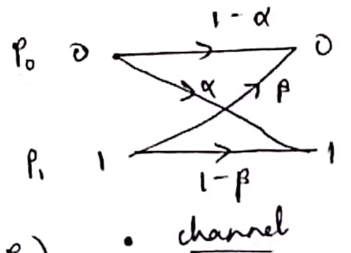
$$f(P_0 \ P_1) = I(X;Y) + \lambda \underbrace{(P_0 + P_1 - 1)}_{\text{constraint } P_0 + P_1 = 1}$$

$$= H(Y) - H(Y|X) + \lambda (P_0 + P_1 - 1)$$

differentiating $f(P_0 \ P_1)$ w.r.t. P_i & equating it to 0, to get value of $P = (P_0 \ P_1)$ so that f is maximized.

$$\boxed{\frac{\partial f}{\partial P_i} = \frac{\partial H(Y)}{\partial P_i} - \frac{\partial H(Y|X)}{\partial P_i} + \lambda} \quad — (1)$$

Consider – $\dfrac{\partial H(Y|X)}{\partial P_i}$



$H(Y|X)$

$= P_0 \, H(1-\alpha, \alpha) + P_1 \, H(\beta, 1-\beta)$

$\because P_1 = 1 - P_0$

$$\therefore H(Y|X) = P_0 \, H(1-\alpha, \alpha) + (1-P_0) H(\beta, 1-\beta)$$

$$= P_0 (1-\alpha) \log_2 \left(\frac{1}{1-\alpha}\right) + P_0 \alpha \log_2 \frac{1}{\alpha} + (1-P_0) \beta \log_2 \frac{1}{\beta} + (1-P_0)(1-\beta) \log_2 \frac{1}{(1-\beta)}$$

$$\frac{\partial H(Y|X)}{\partial P_0} = (1-\alpha) \log_2 \left(\frac{1}{1-\alpha}\right) + \alpha \log_2 \frac{1}{\alpha} - \beta \log_2 \frac{1}{\beta} - (1-\beta) \log_2 \frac{1}{(1-\beta)}$$

$$\frac{\partial H(Y|X)}{\partial P_0} = \log_2 \left[\frac{\beta^{\beta} (1-\beta)^{(1-\beta)}}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}} \right] \quad — (2)$$

Consider. $\dfrac{\partial H(Y)}{\partial P_i}$ ②

$$P_X = P_Y = \begin{pmatrix} P_0 & P_1 \end{pmatrix}$$

where $P_1 = 1 - P_0$

$$P_{Y|X} = \begin{array}{c c} & \overbrace{\begin{array}{cc} 0 \quad & \quad 1 \end{array}}^{Y} \\ X\left\{\begin{array}{c} 0 \\ 1 \end{array}\right. & \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \end{array}$$

$$P_Y(0) = P_0(1-\alpha) + (1-P_0)\beta$$

$$P_Y(1) = \alpha P_0 + (1-\beta)(1-P_0)$$

$$H(Y) = P_Y(0)\log_2 \frac{1}{P_Y(0)} + P_Y(1)\log_2 \frac{1}{P_Y(1)}$$

$$= \left(P_0(1-\alpha) + (1-P_0)\beta\right)\log_2 \frac{1}{\left(P_0(1-\alpha) + (1-P_0)\beta\right)} + \left(\alpha P_0 + (1-\beta)(1-P_0)\right) \times \log_2\left(\frac{1}{\alpha P_0 + (1-\beta)(1-P_0)}\right)$$

$$\frac{\partial H(Y)}{\partial P_0} = (1-\alpha-\beta)\log_2 \frac{1}{\left(P_0(1-\alpha)+(1-P_0)\beta\right)} + \left(P_0(1-\alpha)+(1-P_0)\beta\right)^2 \times \left(\frac{-1(1-\alpha-\beta)}{P_0(1-\alpha)+(1-P_0)\beta}\right)^2 \cdot \frac{1}{\ln 2}$$

$$+ (\alpha-1-\beta)\log_2 \frac{1}{P_0\alpha + (1-P_0)(1-\beta)} + \frac{\left(P_0\alpha + (1-P_0)(1-\beta)\right)^2 - (\alpha-1+\beta)}{\left(P_0\alpha + (1-P_0)(1-\beta)\right)^2} \cdot \frac{1}{\ln 2}$$

$$= (1-\alpha-\beta)\left[\log_2 \frac{1}{\left(P_0(1-\alpha)+(1-P_0)\beta\right)} - \cancel{\frac{1}{\ln 2}} - \log_2 \frac{1}{P_0\alpha + (1-P_0)(1-\beta)} + \cancel{\frac{1}{\ln 2}}\right]$$

$$\frac{\partial H(Y)}{\partial P_0} = (1-\alpha-\beta)\left[\log_2\left(\frac{P_0\alpha + (1-P_0)(1-\beta)}{P_0(1-\alpha)+(1-P_0)\beta}\right)\right] \quad -(3)$$

from equation (1) -

$$\frac{\partial f}{\partial P_0} = \frac{\partial H(Y)}{\partial P_0} - \frac{\partial H(Y|X)}{\partial P_0} + \lambda$$

Putting values from eqn (2) & eqn (3) in above eqn -

$$\frac{\partial f}{\partial P_0} = (1-\alpha-\beta)\left[\log_2\left(\frac{P_0\alpha+(1-P_0)(1-\beta)}{P_0(1-\alpha)+(1-P_0)\beta}\right)\right] - \log_2\left[\frac{\beta^\beta(1-\beta)^{(1-\beta)}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right] + \lambda$$

As we know, from shannon's theoram,

$$\text{Capacity } C = \max_{P_X} I(X;Y)$$

with lagrangian -

$$C = \max_{P_X} f(P_0\ P_1)$$

So, $\frac{\partial f}{\partial P_0} = 0$, gives

$$\log_2\left(\frac{P_0\alpha+(1-P_0)(1-\beta)}{P_0(1-\alpha)+(1-P_0)\beta}\right)^{1-\alpha-\beta} - \log_2\left[\frac{\beta^\beta(1-\beta)^{(1-\beta)}}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}}\right] = -\lambda$$

$$\log_2\left[\frac{\beta^\beta(1-\beta)^{(1-\beta)}}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}}\cdot\left(\frac{P_0(1-\alpha)+(1-P_0)\beta}{P_0\alpha+(1-P_0)(1-\beta)}\right)^{1-\alpha-\beta}\right] = \lambda$$

let $\lambda = \log_2\tilde{\lambda}$

$$\Rightarrow \boxed{\frac{\beta^\beta(1-\beta)^{(1-\beta)}}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}}\cdot\left(\frac{P_0(1-\alpha)+\beta(1-P_0)}{P_0\alpha+(1-P_0)(1-\beta)}\right)^{1-\alpha-\beta} = \tilde{\lambda}}$$

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let $\left(\frac{\beta^{\beta}(1-\beta)^{(1-\beta)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)^{1-\alpha-\beta} = k$ (constant)

$$k \cdot \left[\frac{P_0(1-\alpha)+\beta(1-P_0)}{P_0\alpha+(1-P_0)(1-\beta)}\right] = \tilde{\lambda}$$

$$kP_0(1-\alpha)+k\beta(1-P_0) = \tilde{\lambda}P_0\alpha+(1-P_0)(1-\beta)\tilde{\lambda}$$

$$P_0\left[k(1-\alpha)-k\beta\right]+k\beta = P_0\left[\tilde{\lambda}\alpha+(\beta-1)\tilde{\lambda}\right]+(1-\beta)\tilde{\lambda}$$

$$P_0\left[k(1-\alpha)-k\beta-\tilde{\lambda}\alpha+(\beta-1)\tilde{\lambda}\right] = k\beta-(1-\beta)\tilde{\lambda}$$

$$\boxed{P_0 = \frac{k\beta-(1-\beta)\tilde{\lambda}}{k(1-\alpha)-k\beta-\tilde{\lambda}\alpha+(\beta-1)\tilde{\lambda}}}$$

$\tilde{\lambda} = \log_2 \lambda$ & λ can be calculated using constraint.

Special case :- (without lagrangian)

$$C = \max . I(X;Y)$$
$$= \max\left[H(Y)-H(Y|X)\right]$$

$$\frac{\partial I(X;Y)}{\partial P_0} = 0$$

$$\frac{\partial H(Y)}{\partial P_0} = \frac{\partial H(Y|X)}{\partial P_0}$$

$$(1-\alpha-\beta)\left[\log_2\left(\frac{P_0\alpha+(1-P_0)(1-\beta)}{P_0(1-\alpha)+(1-P_0)\beta}\right)\right] = \log_2\left[\frac{\beta^{\beta}(1-\beta)^{(1-\beta)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right]$$

$$\frac{P_0\alpha+(1-P_0)(1-\beta)}{P_0(1-\alpha)+(1-P_0)\beta} = \left(\frac{\beta^{\beta}(1-\beta)^{(1-\beta)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)^{\frac{1}{1-\alpha-\beta}}$$

let $\left(\frac{\beta^{\beta}(1-\beta)^{(1-\beta)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)^{\frac{1}{1-\alpha-\beta}} = k$

$$\Rightarrow \frac{P_0\alpha+(1-P_0)(1-\beta)}{P_0(1-\alpha)+(1-P_0)\beta} = k$$

$$\Rightarrow P_0\alpha+1-\beta-P_0+P_0\beta = kP_0-kP_0\alpha+k\beta-P_0\beta k$$

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$$P_0(\alpha+\beta-1) + 1-\beta = P_0(k - k\alpha - k\beta) + k\beta$$

$$P_0(k-k\alpha-k\beta) - P_0(\alpha+\beta-1) = 1-\beta-k\beta$$

$$P_0(k-k\alpha-k\beta-\alpha-\beta+1) = 1-\beta(k+1)$$

$$P_0 = \frac{1-\beta(k+1)}{k-k\alpha-k\beta-\alpha-\beta+1}$$

$$\boxed{P_0 = \frac{1-\beta(k+1)}{(1-\alpha-\beta)(k+1)}}$$

Results - data given

P_0by1 = 0.5982945793 = β
P_1by0 = 0.700720551З = α

Transitional probability Matrix

$$P_{Y|X} = \begin{bmatrix} 0.29927 & 0.7007 \\ 0.59829 & 0.401705 \end{bmatrix}$$

$$P_X = [0.50546295 \quad 0.49453705]$$

$$P_Y = [0.44715351 \quad 0.55284649]$$

$$H_Y = 0.991926$$

$$H_{YbyX} = 0.92567386$$

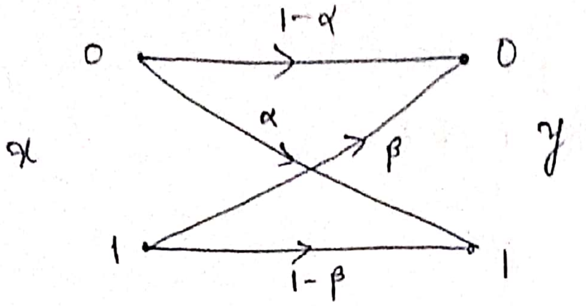
$$\text{Capacity} = 0.066252890643933З6$$

ML decoder estimation Derivation

ML decoder estimation is given by –

$$\hat{x}_{ML} = \arg\max_{x} P_{Y|X}(y|x)$$

If channel is given by –



$$P_{Y|X} = \begin{array}{cc|cc} & & y=0 & 1 \\ \hline x & 0 & 1-\alpha & \alpha \\ & 1 & \beta & 1-\beta \end{array}$$

$\hat{x}_{ML}$ is determined by fixing y & finding symbol x such that likelihood $P_{Y|X}$ corresponding to x is maximum & assign $\hat{x}$ to that symbol x.

let us assume –

$y = 0$ (fix):

$$\hat{x}_{ML} = \begin{cases} 0 & \text{if } P_{Y|X}(0|0) > P_{Y|X}(0|1) \\ & \Rightarrow 1-\alpha > \beta \text{ or } \alpha+\beta < 1 \\ 1 & \text{if } P_{Y|X}(0|0) < P_{Y|X}(0|1) \\ & \Rightarrow 1-\alpha < \beta \text{ or } \alpha+\beta > 1 \end{cases}$$

when $y = 1$:

$$\hat{x}_{ML} = \begin{cases} 0 & \text{if } P_{Y|X}(1|0) > P_{Y|X}(1|1) \\ & \alpha > 1-\beta \text{ or } \alpha+\beta > 1 \\ 1 & \text{if } P_{Y|X}(1|0) < P_{Y|X}(1|1) \\ & \alpha < 1-\beta \text{ or } \alpha+\beta < 1 \end{cases}$$

This can be written as-

$$\boxed{\hat{x}_{ML} = \begin{cases} y & \text{if } \alpha+\beta < 1 \\ \bar{y} & \text{if } \alpha+\beta > 1 \end{cases}}$$

for my problem -

$\alpha = P_{1 by 0} = 0.7007205513$

$\beta = P_{0 by 1} = 0.5982945793$

$\alpha+\beta \approx 1.298 > 1$

$\therefore \hat{x}_{ML} = \{ \bar{y}$

$$= \begin{cases} 0 & \text{if } y = 1 \\ 1 & \text{if } y = 0 \end{cases}$$