Exercise 2.3

Question 3 HW2

NAME - ANNU, ROLL NO - EE21RESCH01010 (1)

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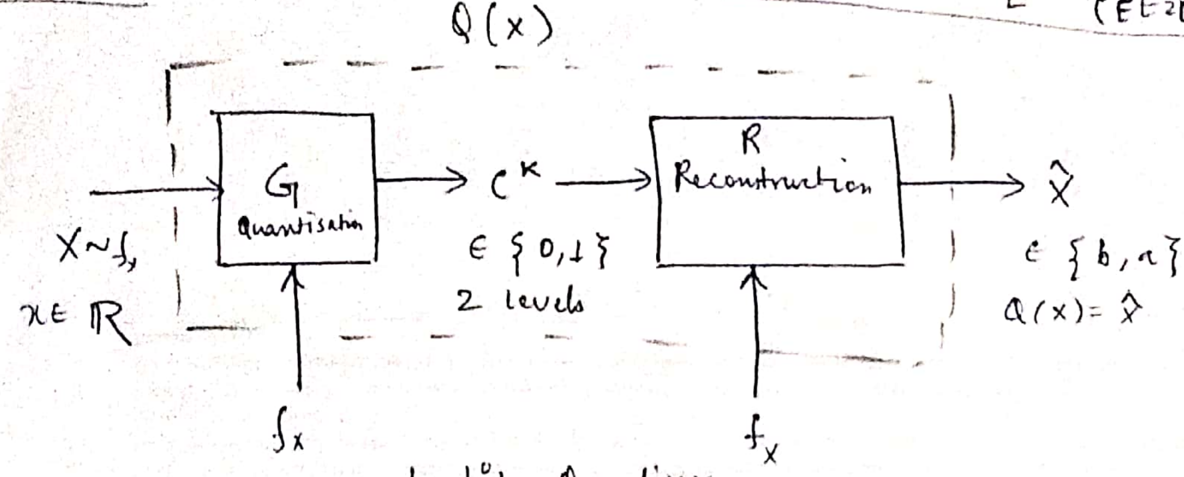


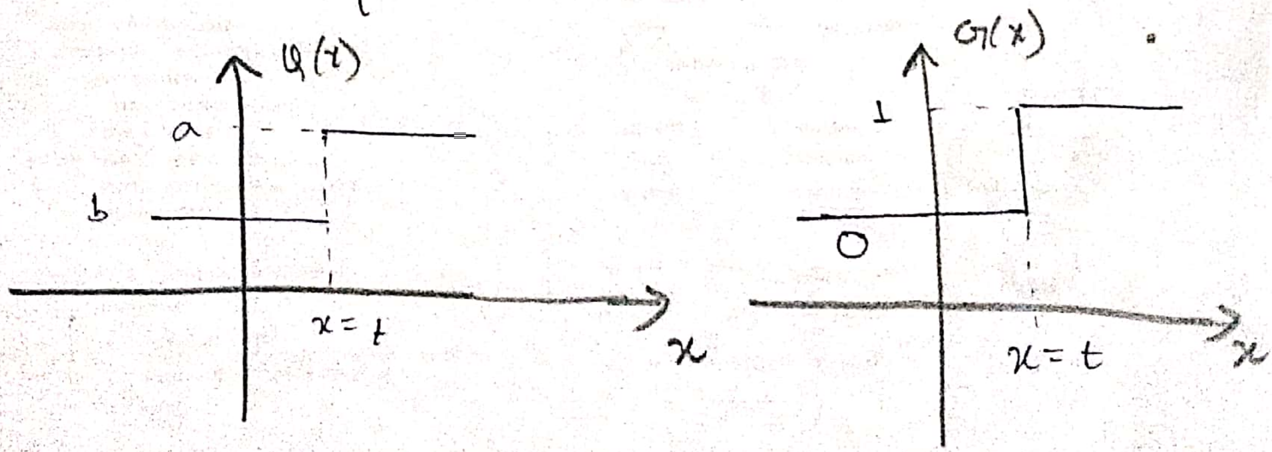
- 1-bit Quantiser = 2^1 levels

$$G(x) = \begin{cases} 0, & \text{if } x \le t \\ 1, & \text{if } x > t \end{cases}$$

Scalar Quantisation is of form:-

$$Q(x) = \begin{cases} a & \text{if } x > t \\ b & \text{if } x \le t \end{cases}$$

$$= \begin{cases} a & \text{if } x > t, \text{ when } G(x) = 1 \\ b & \text{if } x \le t, \text{ when } G(x) = 0 \end{cases}$$



{ Since $b < t < a$ & $a > b$ }

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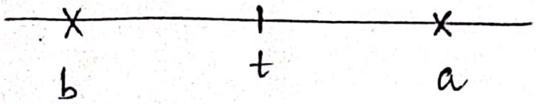
Scalar Quantisation is of form

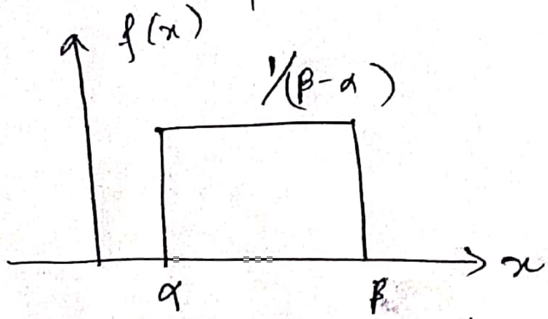
$$\hat{X} = \begin{cases} a, & x > t \\ b, & x \leq t \end{cases}$$



$Q(x) = 0$ $\qquad$ $Q(x) = 1$

Given distribution is uniform & can be defined as

$$f(x) = \begin{cases} \dfrac{1}{(\beta - \alpha)} & \alpha \leq x \leq \beta \quad \text{or} \quad x \in [\alpha, \beta] \end{cases}$$



Mean square error is defined as–

$$MSE = E\,(X - Q(X))^2$$

$$= \int_{-\infty}^{\infty} (x - Q(x))^2 f(x)\,dx$$

$$= \int_{-\infty}^{t} (x-b)^2 f(x)\,dx + \int_{t}^{\infty} (x-a) f(x)\,dx$$

Since $f(x)$ only lies between α & β. (3)

$$\therefore \quad MSE = \int_{\alpha}^{t} (x-b)^2 f(x)\,dx + \int_{t}^{\beta} (x-a)^2 f(x)\,dx$$

$$= \int_{\alpha}^{t} (x-b)^2 \frac{1}{(\beta-\alpha)}\,dx + \int_{t}^{\beta} (x-a)^2 \frac{1}{(\beta-\alpha)}\,dx$$

$$= \frac{1}{(\beta-\alpha)} \left[\frac{(x-b)^3}{3} \right]_{\alpha}^{t} + \frac{1}{(\beta-\alpha)} \left[\frac{(x-a)^3}{3} \right]_{t}^{\beta}$$

$$= \frac{1}{3(\beta-\alpha)} \left[(t-b)^3 - (\alpha-b)^3 \right] + \frac{1}{3(\beta-\alpha)} \left[(\beta-a)^3 - (t-a)^3 \right]$$

$$= \frac{1}{3(\beta-\alpha)} \left[(t-b)^3 - (\alpha-b)^3 + (\beta-a)^3 - (t-a)^3 \right]$$

$$= K(a, b, t)$$

where K is the function of a, b & t.

Now, MSE is expressed as K which is the function of a, b & t

Convex optimisation problem can be formulated as -

Minimise $K(a, b, t)$

such that $b < t < a$.

If $\hat{a}$ is value which minimises $K(a, b, t)$ -

$$\hat{a} = \underset{a}{\operatorname{argmin}}\ K(a, b, t)$$

If $\hat{b}$ is value which minimizes $K(a, b, t)$ -

$$\hat{b} = \underset{b}{\operatorname{argmin}}\ K(a, b, t)$$

If $\hat{t}$ is value which minimizes $K(a, b, t)$ -

$$\hat{t} = \underset{t}{\operatorname{argmin}}\ K(a, b, t)$$

Minimizing over a -

from theory of convex optimisation, partial derivative of $k(a,b,t)$ w.r.t. a must be 0.

i.e. $$\frac{\partial k(a,b,t)}{\partial a} = \frac{\partial}{\partial a}\left[\frac{1}{3(\beta-\alpha)}\left[(t-b)^3-(\alpha-b)^3+(\beta-a)^3 - (t-a)^3\right]\right]$$

$$= \frac{1}{3(\beta-\alpha)}\left[(-1)3(\beta-a)^2 - 3(t-a)^2(-1)\right]$$

equating $\frac{\partial k(a,b,t)}{\partial a} = 0$

$$\Rightarrow 3(\beta-a)^2(-1)+3(t-a)^2 = 0$$

$$\Rightarrow (t-a)^2 = (\beta-a)^2$$

$$\Rightarrow (t-a) = \pm(\beta-a)$$

when (+)ve sign considered -

$$t-a = \beta-a$$

$$\Rightarrow t = \beta \quad \text{(Not possible)}$$

~~tot~~ but since t lies between a & b & thus t also lies between α & β.

when (−)ve sign considered -

$$t-a = -(\beta-a)$$

$$\Rightarrow 2a = t+\beta$$

$$\boxed{a = \frac{t+\beta}{2}} \quad - (1)$$

Minimising over t -

from theory of convex optimisation, partial derivative of $k(a,b,t)$ w.r.t. t must be 0.

Differentiating $k(a,b,t)$ w.r.t. t & equating it to 0 -

$$\frac{\partial k(a,b,t)}{\partial t} = \frac{1}{3(\beta-\alpha)}\left[3(t-b)^2 - 3(t-a)^2\right] = 0$$

$$\Rightarrow (t-b)^2 = (t-a)^2$$

$$(t-b) = \pm(t-a)$$

Considering (+)ve sign -

$$t-b = t-a$$

$$b = a \quad \text{(Not possible as } a > b \text{)}$$

Considering (-)ve sign -

$$t-b = -(t-a)$$

$$t-b = -t+a$$

$$2t = a+b$$

$$\boxed{t = \frac{a+b}{2}} \quad - (2)$$

Minimising over b.

From theory of convex optimisation, partial derivative of $k(a,b,t)$ w.r.t. b must be 0.

Differentiating $k(a,b,t)$ w.r.t b & equating it to 0 -

$$\frac{\partial k(a,b,t)}{\partial b} = \frac{1}{3(\beta-\alpha)}\left[3(t-b)^2(-1) - 3(\alpha-b)^2(-1)\right]$$

$$\Rightarrow -3(t-b)^2 + 3(\alpha-b)^2 = 0$$

$$\Rightarrow (\alpha-b)^2 = (t-b)^2$$

$$\alpha - b = \pm(t-b)$$

Considering (+)ve sign -

$$\alpha - b = t-b$$

$$\alpha = t \quad \text{(Not possible)}$$

Considering (-ve) sign -

$$\alpha - b = -t + b$$

$$2b = \alpha + t$$

$$\boxed{b = \frac{\alpha + t}{2}} \quad - (3)$$

from (1), (2), (3) -

$$a = \frac{t + \beta}{2} \quad - (1)$$

$$t = \frac{a + b}{2} \quad - (2)$$

$$b = \frac{\alpha + t}{2} \quad - (3)$$

putting value of a, b from (1), (3) in eqn (2) -

$$t = \frac{a + b}{2} = \frac{\frac{t + \beta}{2} + \frac{\alpha + t}{2}}{2}$$

$$t = \frac{2t + \alpha + \beta}{4}$$

$$4t = 2t + \alpha + \beta$$

$$2t = \alpha + \beta$$

$$\boxed{t = \frac{\alpha + \beta}{2}}$$

putting value of t in (1) -

$$a = \frac{\frac{\alpha + \beta}{2} + \beta}{2} = \frac{\alpha + \beta + 2\beta}{4}$$

$$\boxed{a = \frac{\alpha + 3\beta}{4}}$$

putting value of t in (3) -

$$b = \frac{\alpha + \frac{\alpha + \beta}{2}}{2} = \frac{2\alpha + \alpha + \beta}{4} \Rightarrow \boxed{b = \frac{3\alpha + \beta}{4}}$$

$$Q(x) = \begin{cases} a = \dfrac{\alpha + 3\beta}{4}, & x > \left(\dfrac{\alpha+\beta}{2}\right) \\[2em] b = \dfrac{\beta + 3\alpha}{4}, & x \leq \left(\dfrac{\alpha+\beta}{2}\right) \end{cases}$$

Above are values of a, b & t such that MSE is minimised.
Minimised error is - (putting values of a, b & t) -

$$MSE = K(a,b,t)$$

$$= \frac{1}{3(\beta-\alpha)}\left[(t-b)^3 - (\alpha-b)^3 + (\beta-a)^3 - (t-a)^3\right]$$

$$= \frac{1}{3(\beta-\alpha)}\left[\left(\frac{\alpha+\beta}{2} - \frac{\beta-3\alpha}{4}\right)^3 - \left(\alpha - \frac{\beta-3\alpha}{4}\right)^3 + \left(\beta - \frac{\alpha-3\beta}{4}\right)^3 - \left(\frac{\alpha+\beta}{2} - \frac{\alpha-3\beta}{4}\right)^3\right]$$

$$= \frac{1}{3(\beta-\alpha)}\left[\left(\frac{\beta-\alpha}{4}\right)^3 - \left(\frac{\alpha-4\beta}{4}\right)^3 + \left(\frac{\beta-4\alpha}{4}\right)^3 - \left(\frac{\alpha-\beta}{4}\right)^3\right]$$

Exercise 2.2 - LZ78 parsing for the sequences

- Python program is submitted

Sequence 1 - d d c b a b b d a b d d b b d

phrases - d, dc, b, a, bb, da, bd, db, bd

encoding (0,d), (1,c), (0,b), (0,a), (3,b), (1,a),
(3,d), (1,b), (7,'')

Since bd is already phrased at 7th phrase

~~Sequence 2 - badababaaddaddd cb~~

~~phrases - b, a, d, ab, aba, ad, da, dd, dc, b~~

sequence 2 - badabaaddadddcb

phrases - b, a, d, ab, aa, dd, ad, ddc, b

encoding - (0,b), (0,a), (0,d), (2,b), (2,a), (3,d), (2,d), (6,c), (1,'')

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Input sequence 3 - b d d d d d b d b a d d c a d

phrases:- b, d, dd, ddb, db, a, ddc, ad

encoding - (0, b), (0, d), (2, d), (3, b), (2, b), (0, a), (3, c), (6, d)

Although for this, a separate python program is submitted!!

Exercise 2.1 - python program is submitted

Encoded arithmetic code for sequence 1 -

.11001111011101111

arithmetic code = 0.8104216905477135

Encoded arithmetic code for sequence 2 -

.0010011001010111

arithmetic code = 0.14977251046942670 6

Exercise 2.3 -

Mean square error = 2.5057219121114

for given input sequence (uniformly distributed array) -

a = 9.948270

b = - 0.296898

t = 2.32568

α = 7.570655

β = -2.91948

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