HOMEWORK - 1

Subject:- Information theory. (1 credit)

NAME - ANNU , **ROLL. NO.** - EE21RESCH01010

Degree - PhD in communications & signal processing.

Referrences:- Book by Papulus named "Random Variables & Stochastic Process"

→ **lectures** - MIT lecture series on probability

Exercise 1.5 1. Show that for all (+)ve integers (n, k, l) with $(n > k > l)$, prove that $\frac{n-l}{k-l} \geq \frac{n}{k}$

Solution - Starting with the initial condition -

$$n > k > l$$

$$\Rightarrow n > k \text{ -①} \quad \& \quad k > l \text{ -②}$$

from ① $\quad n > k$

Multiplying by $1/l$ both the sides -

$$\frac{n}{l} > \frac{k}{l}$$

(l is (+)ve ⇒ $1/l$ is also (+)ve ∴ multiplication of (+)ve integer won't change inequality)

$$\Rightarrow \frac{l}{n} < \frac{l}{k} \quad \Rightarrow \frac{l}{n} - 1 < \frac{l}{k} - 1$$

Subtracting 1 both the sides

$$-\left(\frac{l}{n} - 1\right) > -\left(\frac{l}{k} - 1\right)$$

$$\Rightarrow -\left(\frac{l-n}{n}\right) > -\left(\frac{l-k}{k}\right)$$

{ ∵ $n > k > l$
Ex. $n = 3$, $k = 2$ & $l = 1$
$\frac{1}{3} < \frac{1}{2}$
but $-\left(\frac{1}{3} - 1\right) > -\left(\frac{1}{2} - 1\right)$ }

$$\Rightarrow \frac{n-l}{n} > \frac{k-l}{k}$$

$$\Rightarrow (n-l)k > (k-l)n$$

{cross multiplication}

$$\Rightarrow \frac{n-l}{k-l} > \frac{n}{k}$$

$$\Rightarrow \boxed{\frac{n}{k} \leq \frac{n-l}{k-l}}$$

QED//

Scanned with CamScanner

1.5.2 Prove the following bounds for the binomial coefficient :

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{n^k}{k!}\right)$$

Proof - As we know -

$$^nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\Rightarrow \binom{n}{k} = \frac{n(n-1)(n-2)\,---\,(n-(k-1))\,\cancel{(n-k)!}}{k!\,\cancel{(n-k)!}}$$

$$= \frac{n(n-1)(n-2)(n-3)\,---\,(n-(k-1))}{k!} = \frac{\prod_{i=0}^{k-1} n-i}{k!}$$

$$\binom{n}{k} = \frac{\prod_{i=0}^{k-1} n-i}{k(k-1)\,---\,(k-(k-1))} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \quad -(1)$$

$$\therefore \binom{n}{k} = \frac{n(n-1)(n-2)\,---\,(n-(k-1))}{k(k-1)(k-2)\,---\,(k-k+1)} \quad -(2)$$

from Above eqn (1) → we can see each factor in the product (Numerator as well as denominator) is atleast $\frac{n}{k}$ & there are total $(k-1+1=k)$ k factors.

$$\therefore \text{ lower bound, } \frac{n \times n\,---\,(k \text{ times})}{k \times k \times\,--\,(k \text{ times})} = \left(\frac{n}{k}\right)^k$$

$$\therefore k^k > k! \Rightarrow \left(\frac{n}{k}\right)^k \le \frac{n(n-1)(n-2)\,--\,(n-(k-1))}{k(k-1)(k-2)\,---\,(k-k+1)}$$

$$\Rightarrow \boxed{\left(\frac{n}{k}\right)^k \le \binom{n}{k}} \quad [\text{From } (2)] \; - (3)$$

for upper bound :-

$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} = \frac{\prod_{i=0}^{k-1} n-i}{k!}$$

$$\Rightarrow \binom{n}{k} = \frac{n(n-1)(n-2)\,---\,(n-(k-1))}{k!}$$

$$\le \frac{n.n.\,----\,n \;(k \text{ times})}{k!}$$

$$= \frac{n^k}{k!}$$

$$\therefore \boxed{\binom{n}{k} \le \frac{n^k}{k!}} \; (4)$$

→ from (3) & (4) -

$$\boxed{\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{n^k}{k!}\right)}$$ QED

Scanned with CamScanner

(1.5.3) Prove that for all integers ((+)ve integers), $n > k > 0$, we have $\frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)} \leq \binom{n}{k} \leq \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)}$

where $p = k/n$ & $H_2(p) = -p\log_2 p + -(1-p)\log_2(1-p)$

Stirling's approximation -

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} \leq n! \leq e\, n^{n+1/2} e^{-n} \quad \text{-} (1)$$

proof -

as we know -

$$\binom{n}{k} = {}^nC_k = \frac{n!}{k!(n-k)!} = \frac{n!}{(np)!\,(n-np)!} \quad [\because k = np]$$

$$\Rightarrow \binom{n}{k} = \frac{n!}{(np)!\,(n(1-p))!} = \frac{n!}{(np)!\,(nq)!} \quad [\text{let } q = 1-p]$$

Using (1) -

$$n! \leq e^{n+1/2} e^{-n}$$

$$\Rightarrow \frac{n!}{(np)!\,(nq)!} \leq \frac{\cancel{e}\, n^{(n+1/2)} e^{-n}}{\cancel{e}\,(np)^{(np+1/2)} e^{-np}\; e\,(nq)^{(nq+1/2)} e^{-nq}}$$

$$\Rightarrow \binom{n}{k} \leq \frac{n^n \cdot n^{1/2}\, e^{-n}}{(np)^{np} \cdot (np)^{1/2} \cdot e^{-np} \cdot e \cdot (nq)^{nq} \cdot (nq)^{1/2} e^{-nq}} \quad [\because a^{(b+c)} = a^b \cdot a^c]$$

$$= \frac{n^n \cdot \cancel{n^{1/2}} \cdot e^{-n}}{n^{np} \cdot p^{np} \cdot \cancel{n^{1/2}} \cdot p^{1/2} e^{-np} \cdot e \cdot (n)^{nq} (q)^{(nq)} n^{1/2} q^{1/2} e^{-nq}} \quad [\because (ab)^c = a^c \cdot b^c]$$

$$\binom{n}{k} \leq \frac{n^n \cdot n^{-np} \cdot n^{-nq} \cdot e^{-n} \cdot e^{np} \cdot e^{nq}}{e\,(n^{1/2} p^{1/2} q^{1/2})\,(p^{np} \cdot q^{nq})} = \frac{n^{n(1-p-q)} \cdot e^{n(-1+p+q)}}{e\sqrt{npq}\,(p^{np} \cdot q^{nq})} \quad \text{-} (2)$$

$$\because 1-p-q = 1-p-(1-p) = \cancel{1}\cancel{p}\cancel{-1}+\cancel{p} = 0$$

$$\Rightarrow -1+p+q = 0$$

Scanned with CamScanner

from eqn ② –

$$\therefore \binom{n}{k} \leq \frac{n^{n \cdot 0} \cdot e^{n \cdot 0}}{e\sqrt{npq}\; p^{np} \cdot q^{np}} = \frac{1}{e\sqrt{np(1-p)}\; p^{np} \cdot q^{np}} \quad \text{– ④}$$

Now since entropy $H_2(p)$ is given by –

$$H_2(p) = -[p\log_2 p + [\log_2(1-p)] \cdot [1-p]] \quad (\text{given})$$

$$\Rightarrow H_2(p) = -[\log_2 p^p + \log_2 (1-p)^{(1-p)}]$$

$$= -\log_2 (p^p \cdot (1-p)^{(1-p)}) \quad [\because \log a + \log b = \log ab]$$

$$\Rightarrow H_2(p) = -\log_2 (p^p \cdot q^q)$$

$$\Rightarrow p^p \cdot q^q = 2^{-H_2(p)}$$

taking nth power both sides –

$$\Rightarrow p^{np} \cdot q^{nq} = 2^{-nH_2(p)} \quad \text{– ③}$$

Now, using equation ③ in eqn ④ –

$$\binom{n}{k} \leq \frac{1}{e\sqrt{np(1-p)}\; 2^{-nH_2(p)}} \quad \text{– ⑤}$$

Now, $\frac{1}{e} = 0.3678$

& $\frac{e}{2\pi} = 0.432$

$$\therefore \frac{1}{e} < \frac{e}{2\pi}$$

$$\Rightarrow \frac{1}{e\sqrt{np(1-p)}\; 2^{-nH_2(p)}} < \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}\; 2^{-nH_2(p)}}$$

$\therefore$ from ⑤ –

$$\binom{n}{k} \leq \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}\; 2^{-nH_2(p)}}$$

$$= \frac{e}{2\pi\sqrt{np(1-p)}} 2^{nH_2(p)}$$

$$\therefore \boxed{\binom{n}{k} \leq \frac{e}{2\pi\sqrt{np(1-p)}} 2^{nH_2(p)}}$$

– (a) upper bound proved.

Scanned with CamScanner

For lower bound - from lower bound of stirling's eqn - (5)

$$n! \geq \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

$$\binom{n}{k} = \frac{n!}{(np)!\,(nq)!} \geq \frac{\sqrt{2\pi}\, n^{n+1/2} e^{-n}}{\sqrt{2\pi}\,(nq)^{nq+1/2} e^{-nq}\,\sqrt{2\pi}\,(np)^{np+1/2} e^{-np}}$$

$$= \frac{n^{n} \cdot n^{1/2} e^{-n} \cdot e^{nq} \cdot e^{np}}{\sqrt{2\pi}\,(nq)^{nq}(nq)^{1/2}(np)^{np}(np)^{1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{n^{n} \cdot n^{-nq} \cdot n^{-np} \cdot n^{1/2}\, e^{n(-1+q+p)}}{q^{nq} \cdot p^{np} \cdot n^{1/2} \cdot n^{1/2} \cdot q^{1/2} \cdot p^{1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{n^{1/2} q^{1/2} p^{1/2}} \frac{n^{-n(-1+q+p)} \cdot e^{n(-1+q+p)}}{p^{np}\, q^{nq}} \quad -(6)$$

$\because$ $p^{np} \cdot q^{nq} = 2^{-nH_2(p)}$

& $-1+q+p = 0 \Rightarrow n^{-n\cdot 0} = 1$ & $e^{n\times 0} = 1$

$\therefore$ from (6) — $$\binom{n}{k} \geq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}} \times \frac{1}{2^{-nH_2(p)}} \quad -(7)$$

$$\binom{n}{k} \geq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)}$$

$\because$ $\frac{1}{\sqrt{2\pi}} = 0.398$ & $\frac{\sqrt{2\pi}}{e^2} = 0.339$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \geq \frac{\sqrt{2\pi}}{e^2} \Rightarrow \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)} \geq \frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)}$$

from (7) —

$$\binom{n}{k} \geq \frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)}$$

$$\Rightarrow \boxed{\binom{n}{k} \geq \frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{np(1-p)}}\, 2^{nH_2(p)}} \quad -(b)$$ ~~upper~~ lower bound proved

From (a) & (b) —

$$\boxed{\frac{\sqrt{2\pi}}{e^2} \frac{2^{nH_2(p)}}{\sqrt{np(1-p)}} \leq \binom{n}{k} \leq \binom{n}{k} \leq \frac{e}{2\pi} \frac{2^{nH_2(p)}}{\sqrt{np(1-p)}}}$$

QED

Exercise 1.5.4 Use the above to conclude for any $0 < p \le 1/2$, $\lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} = H_2(p)$ (6)

LHS $= \lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor}$

consider n = integer multiple of p for $0<p\le 1/2$

$n = mp$, $m \in \mathbb{Z}^+$

$\Rightarrow p = \frac{m}{n}$ — (1)

$\Rightarrow \lfloor np \rfloor = \lfloor n \times \frac{m}{n} = \lfloor m = m$ ($\because m \in \mathbb{Z}^+$)

Now consider LHS - for $0<p\le 1/2$

$$\lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor}$$

$$\le \lim_{n\to\infty} \frac{1}{n} \log_2 \left[\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)} \right] \quad \text{— (2)}$$

$\left\{ \because \binom{n}{k} \le \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)} ; k = np \right.$ (upper bound of previous exp.)

$\Rightarrow \log_2 \binom{n}{np} \le \log_2 \left[\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)} \right] \Big\}$

since log function is monotonically increasing function.

from (2) -

$$\lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} \le \lim_{n\to\infty} \frac{1}{n} \left[\log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right) + \log_2 \left(2^{nH_2(p)}\right) \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right) + \frac{1}{n} \log_2 \left(2^{nH_2(p)}\right) \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right) + \frac{1}{\cancel{n}} \times \cancel{n} H_2(p) \right]$$

$$= \lim_{n\to\infty} \left[\underbrace{\frac{1}{n} \log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right)}_{E} + H_2(p) \right]$$

when $n \to \infty \Rightarrow E \to 0$

$\therefore \lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} \longrightarrow H_2(p)$ since $\lfloor np \rfloor$ is used $\therefore$ limit leads $=$ sign.

$$\boxed{\lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} = H_2(p)}$$

QED

Scanned with CamScanner

Exercise 1.4 (of the empirical frequency is close to the true probability if the number of samples is large). (7)

Given a sequence $x^n = (x_1, x_2, \dots x_n)$, where each $x_i \in \mathcal{X}$, the relative frequency / Empirical freq. of occurrence of $a \in \mathcal{X}$ is defined as:

$$\eta_a(x^n) = \frac{\text{No. of times } a \text{ occurs in } x^n}{n}$$

Consider a sequence of n i.i.d random variables $X_1, X_2 \dots X_n$ over the alphabet $\mathcal{X}$ & having distribution P_X.

1. Prove that for each $a \in \mathcal{X}$ & $\varepsilon > 0$,

$$\lim_{n \to \infty} P_r\left[\left| \eta_a(X^n) - P_X(a) \right| > \varepsilon \right] = 0 \quad \text{--(1)}$$

Also show that the probability above decays exponentially in n.

Solution: Since $\eta_a(x^n) = \dfrac{\text{No. of times } a \text{ occurs in } x^n}{\text{length of sequence}} = \dfrac{k}{n}$ (let)

$= \text{empirical frequency}$

$\therefore (\eta_a(x^n) : a \in \mathcal{X})$ forms a valid probability mass function. & $P_X(a)$ = Probability of occurrence of $a \in \mathcal{X}$ in a single trial $x \in \mathcal{X}^n$ = probability of occurrence of a in single trial

Now, considering a sequence of n i.i.d ~~var~~ random variables $X_1, X_2 \dots X_n$ over alphabet $\mathcal{X}$ & having probability distribution P_X. $\Rightarrow$ It is same as Bernoulli R.V. ~~and~~ with n = no. of trials = length of sequence & $a \in \mathcal{X}$ - No. of times a occurs in x^n = No. of success (k)

Now, since $X_1, X_2, \dots X_n$ are independent (i.i.d)

$\therefore$ No. of "success" in n-trials = ~~X~~ k = No. of times a occurs in x^n. (let)

& $P_X(a) = p$ & $1-p = q$. (let)

considering - $\left| \frac{k}{n} - p \right| > \varepsilon$; $\varepsilon > 0$

$\Rightarrow \left(\frac{k}{n} - p \right) > \varepsilon$ --(1) & $\left(\frac{k}{n} - p \right) < -\varepsilon$ --(2)

By (1)

$$\left(\frac{k}{n} - p \right) > \varepsilon$$

$$\Rightarrow (k - np) > n\varepsilon$$

Scanned with CamScanner

$\Rightarrow \quad k - n(p+\varepsilon) > 0$

taking exponential both sides-

for any (+)ve m - $e^{m(k-n(p+\varepsilon))} > e^{0}$

$\Rightarrow \quad e^{m(k-n(p+\varepsilon))} > 1 \quad -(1) \quad (\because e^{mx}$ is increasing $f^n)$

$$P\left\{\frac{k}{n} - p > \varepsilon\right\} = P\{k > n\varepsilon + np\} = P\{k > n(p+\varepsilon)\}$$

$$= \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} \binom{n}{k} p^k q^{n-k} \quad [\because k \to \text{No. of success in } n\text{- bernoulli trials}]$$

$$< \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} \quad -(2) \quad (\text{from } (1))$$

$$\sum_{k=0}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{\lceil n(p+\varepsilon)\rceil} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} + \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k}$$

$$\Rightarrow \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} < \sum_{k=0}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k}$$

From (2) -

$$P\left\{\frac{k}{n} - p > \varepsilon\right\} \le \sum_{k=0}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} \quad -(3)$$

$$\because (x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r} \quad -(4)$$

from (3) -

$$P\left\{\frac{k}{n} - p > \varepsilon\right\} < e^{-mn\varepsilon} \sum_{k=0}^{n} \binom{n}{k} (pe^{mq})^k (qe^{-mp})^{n-k}$$

$$= e^{-mn\varepsilon} (pe^{mq} + qe^{-mp})^n \quad -(5) \quad [\text{using } (4)]$$

Consider - $pe^{mq} + qe^{-mp}$

$$\le p(mq + e^{m^2q^2}) + q(-mp + e^{m^2p^2})$$

$$= pe^{m^2q^2} + qe^{m^2p^2} \qquad [\because e^x \le x + e^{x^2} \text{ for } x > 0]$$

$$= p e^{M^2 q^2} + q e^{M^2 p^2} \leq e^{M^2} \quad \left[\because e^x \leq x + e^{x^2} \text{ for } x > 0\right] \quad (9)$$

$$\Rightarrow e^{-mn\varepsilon}\left[p e^{\lambda^2 q^2} + q e^{\lambda^2 p^2}\right]^n \leq e^{m^2 n} e^{-mn\varepsilon}$$

$\Rightarrow$ from eqn (5) —

$$P\left\{\frac{k}{n} - p \geq \varepsilon\right\} \leq e^{M^2 n} e^{-mn\varepsilon}$$

$$\Rightarrow P\left\{\frac{k}{n} - p \geq \varepsilon\right\} \leq e^{m^2 - mn\varepsilon}, \quad \varepsilon > 0 \quad (6)$$

let $g(m) = e^{m^2 - mn\varepsilon}$

let's find min. value of $g(m) \Rightarrow$ min value of $m^2 - mn\varepsilon$

$$\Rightarrow f(m) = m^2 n - mn\varepsilon$$

$$\frac{df(m)}{dm} = 2mn - n\varepsilon = 0$$

$$\Rightarrow m = \frac{\varepsilon}{2}$$

$$\frac{d^2 f(m)}{dm^2} = -n\varepsilon < 0 \Rightarrow m = \frac{\varepsilon}{2} \text{ corresponds to min. value of } g(m).$$

$\Rightarrow$ at $m = \frac{\varepsilon}{2}$, $e^{m^2 n - mn\varepsilon}$ is minimum at $\boxed{m = \frac{\varepsilon}{2}}$, $\varepsilon > 0$

$\therefore$ from (6) —

$$P\left\{\frac{k}{n} - p > \varepsilon\right\} \leq e^{\left(\frac{\varepsilon}{2}\right)^2 - \left(\frac{\varepsilon}{2}\right)n\varepsilon} = e^{-\frac{n\varepsilon^2}{4}}$$

when $m = \varepsilon/2$

$$\Rightarrow \boxed{P\left\{\frac{k}{n} - p > \varepsilon\right\} \leq e^{-n\varepsilon^2/4}} \quad (7)$$

Similarly, $P\left\{\frac{k}{n} - p < -\varepsilon\right\} \leq e^{-n\varepsilon^2/4}$ — (8)

Adding (7) & (8) —

$$\Rightarrow \boxed{P\left\{\left|\frac{k}{n} - p\right| > \varepsilon\right\} \leq 2e^{-n\varepsilon^2/4}}$$

$$\Rightarrow \boxed{P\left\{\left|\eta_a(x^n) - P_X(a)\right| > \varepsilon\right\} \leq 2e^{-n\varepsilon^2/4}} \quad \forall a \in \mathcal{X} \ \& \ \varepsilon > 0$$

when $n \to \infty$: $P\left\{\left|\eta_a(x^n) - P_X(a)\right| > \varepsilon\right\} = 0 \quad [\because e^{-\infty} = 0]$

this shows probability $P\{|\eta_a(x^n) - P_X(a)| > \varepsilon\}$ decays exponentially in n.

Scanned with CamScanner

(3) **Exercise 1.4.2** - Use this to prove that

$$\lim_{n\to\infty} P_r\left[\,|\eta_a(x^n) - P_x(a)| > \varepsilon \text{ for some } a \in \mathcal{X}\right] = 0$$

This equation does not enforce ~~of~~ $a \in \mathcal{X}$, but any $a \in \mathcal{X}$.

let k = No. of times for some a occurs in x^n, n = length of sequence.

$\Rightarrow \eta_a(x^n) = \dfrac{k}{n}$ & $q = 1-p$ (let)

& $P_x(a) = p$ (let)

$$P_r\left[\,|\eta_a(x^n) - P_x(a)| > \varepsilon\right] = P_r\left[\,\left|\tfrac{k}{n} - p\right| > \varepsilon\right]$$

$$= P_r\left[\left(\tfrac{k}{n} - p\right)^2 > \varepsilon^2\right]$$

$$= P_r\left[(k-np)^2 > n^2\varepsilon^2\right]$$

consider

$$\sum P_r\left[\,|\eta_a(x^n) - P_x(a)| > \varepsilon\right] = \sum P_r\left[(k-np)^2 > n^2\varepsilon^2\right]$$

$$(k-np)^2 > n^2\varepsilon^2$$

$$\Rightarrow \sum_{k=0}^{n} (k-np)^2 P_n(k) > \sum_{k=0}^{n} n^2\varepsilon^2 P_n(k)$$

where $P_n(k) = P_x\{x = \text{No. of times } a \text{ occurs in } x^n\}$

$= \binom{n}{k} p^k q^{n-k}$

& $P_x(a)$ = probability of occurance of any $a \in \mathcal{X}$ in a single trial $= p$ (let)

$q = 1-p$

$$\Rightarrow \sum_{k=0}^{n} (k-np)^2 P_n(k) > n^2\varepsilon^2 \sum_{k=0}^{n} P_n(k)$$

$$\Rightarrow \sum_{k=0}^{n} (k-np)^2 P_n(k) > n^2\varepsilon^2 \qquad \left[\because \sum_{k=0}^{n} P_n(k) = \text{sum of pmf} = 1\right]$$

$$\Rightarrow \boxed{\sum_{k=0}^{n} k^2 P_n(k) - 2np\sum_{k=0}^{n} k P_n(k) + n^2p^2 > n^2\varepsilon^2} \quad \text{— (1)}$$

calculating

$$\sum_{k=0}^{n} k P_n(k) = \sum_{k=1}^{n} k\, {}^nC_k\, p^k (1-p)^{n-k} \qquad [\because k > 0]$$

$$= \sum_{k=1}^{n} k \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} q^{n-k}$$

Scanned with CamScanner

$$= np \sum_{k-1=0}^{n-k} {}^{(n-1)}C_{(k-1)}\, p^{(r-1)} (1-p)^{(n-1)-(r-1)} \quad (1)$$

$$\boxed{\sum_{k=1}^{n} k P_n(k) = np\,[\,p + 1 - p\,]^{n-1} = np \times 1 = np} \quad -(2)$$

$$\left[\because \sum_{r=0}^{n} {}^{n}C_r\, p^r (1-p)^{n-r} = (p+1-p)^n \right.$$

$$\left. \text{or } (x+y)^n = \sum_{r=0}^{n} {}^{n}C_r\, x^r y^{n-r} \right]$$

$$\sum_{k=0}^{n} k^2 P_n(k) = \sum_{k=1}^{n} k \frac{n!}{(n-k)!\,(k-1)!} p^k q^{n-k}$$

$$= n^2 p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!\,(k-2)!} p^{k-2} q^{n-k}$$

$$+ \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k q^{n-k}$$

$$= n^2 p^2 [\,p + 1 - p\,]^{n-2} + npq$$

$$\boxed{\sum_{k=0}^{n} k^2 P_n(k) = n^2 p^2 + npq} \quad -(3)$$

Now putting values from eqn (2) & (3) into eqn (1) —

$$\sum_{k=0}^{n} (k-np)^2 P_n(k) = \sum_{k=0}^{n} k^2 P_n(k) - 2np \sum_{k=0}^{K} k P_n(k) + n^2 p^2$$

$$= n^2 p^2 + npq - 2np \cdot np + n^2 p^2 = npq.$$

$$\sum_{k=0}^{n} (k-np)^2 P_n(k) = \sum_{|k-np| \le n\varepsilon} (k-np)^2 P_n(k)$$

$$+ \sum_{|k-np| > n\varepsilon} (k-np)^2 P_n(k)$$

$$\geq \sum_{|k-np| > n\varepsilon} (k-np)^2 P_n(k) > n^2 \varepsilon^2 \sum_{|k-np| > n\varepsilon} P_n(k)$$

$$= n^2 \varepsilon^2 P\{\, |k-np| > n\varepsilon \,\}$$

$$\therefore P\left(\left| \frac{k}{n} - p \right| > \varepsilon \text{, for some } a \in \mathcal{X} \right) < \frac{pq}{n\varepsilon^2}$$

$$\Rightarrow \boxed{P\left(|\eta_a(x^n) - P_X(a)| > \varepsilon \text{ for some } a \in \mathcal{X} \right) < \frac{P_X(a)\,[1 - P_X(a)]}{n\varepsilon^2}}$$

for some $a \in \mathcal{X}$ & $\varepsilon > 0$

$$\boxed{\text{when } n \to \infty \quad P\left(|\eta_a(x^n) - P_X(a)| > \varepsilon \text{ for some } a \in \mathcal{X} \right) = 0} \quad \text{QED.}$$

Scanned with CamScanner

Exercise 1.4.3

using program – ~~$n = 1000$~~

~~$\sum_{a \in X} |\eta_a(x^n) - P_X(a)| = 0.0176667$~~

Ans for $n = 10{,}000$ – $\sum_{a \in X} |\eta_a(x^n) - P_X(a)| = 0.003333$ $[X \in \{\frac{1}{3}, \frac{2}{3}\}]$

Exercise 1.2. Consider n independent tosses of a biased coin which lands heads with probability p.

1. Find an expression for the probability that the number of heads is exactly k. forms bernoulli i.i.d.

Probability for getting head $= p$

probability for getting tail $= 1-p$

~~$P(X$~~ $= P_r(\text{No. of heads is } k) = P(X=k)$

$\therefore \boxed{P(X=k) = {}^nC_k \, p^k (1-p)^{n-k}}$ required expression

Exercise 1.2.2

find an expression for the probability that the number of heads is atmost m.

$P_r(\text{No. of heads is at most } m) = P(X \leq m)$

$$f_X(m) = \boxed{P(X \leq m) = \sum_{k=0}^{m} {}^nC_k \, p^k (1-p)^{n-k}} = f_X(m) \text{ required expression}$$

Exercise 1.2.3

$n = 17$, $m = 6$, $p = 0.382477 2289$

$$P(X \leq m) = f_X(m) = \sum_{k=0}^{6} {}^{17}C_k \cdot p^k (1-p)^{17-k}$$

$\boxed{P(X \leq 6) = 0.5075903667914 87}$ {using computer program}

Exercise 1.2.4 – finding bounds for the probability calculated in 1.2.3.

Expectation for bernoulli R.V –

$$\mu = nE[X_1] = n[1 \times p + (1-p) \times 0] = np$$
$$= 17 \times 0.3824772289$$
$$\approx 6.502$$

Scanned with CamScanner

Now from Markov's inequality :-

$Pr[X \geq m] \leq \frac{\mu}{m+1} \Rightarrow$ ~~$P[X \geq$~~ $P[X \geq m+1] \leq \frac{\mu}{m+1}$

(13)

$Pr[X \leq m] = 1 - P[X \geq m+1]$

$P[X \leq m] \geq 1 - \frac{\mu}{m+1} = 1 - \frac{np}{m+1}$

$P[X \leq 6] \geq 1 - \frac{6.502}{7} = 0.07112 67298$

$\therefore \boxed{P[X \leq 6] \geq 0.07112}$ (lower bound)

$\because 0.50759 > 0.07112 \Rightarrow$ Markov inequality satisfied.

Using chernoff bound for bernoulli's trial -

~~$P_r(X \leq (1-\delta) \geq np(1+\delta))$~~ $P(X \leq (1-\delta)\mu) \leq e^{-\frac{\mu\delta^2}{2}}$

$\forall \; 0 < \delta < 1$

where $X = \sum_{i=1}^{n} X_i$

$P(X \leq m) = ?$

$m = (1-\delta)\mu \Rightarrow 6 = (1-\delta)np$

$\Rightarrow 6 = 6.502 - 6.502\delta$

$\Rightarrow \delta = \frac{0.502}{6.502} = 0.077$

$\therefore P[X \leq 6] \leq e^{-\frac{\mu\delta^2}{2}} = 0.98111815$

$= 0.98111815$

$\therefore \boxed{P[X \leq 6] \leq 0.98111815}$

$\because 0.50759 < 0.98111815$

Chernoff bound satisfies.

Chebysev bound - $P(|X-\mu| \geq a) \leq \frac{Var(X)}{a^2} = \frac{E[(X-\mu)^2]}{a^2}$

$P(X \leq m) = 1 - P(X \geq m+1) = 1 - P[X^2 \geq (m+1)^2]$

Scanned with CamScanner

$$= 1 - \frac{E[X^2]}{(m+1)^2} \cancel{= 1 - \frac{\mu}{7^2}} \cancel{= 1 - \frac{60.502}{49}}$$

$$P(X \geq m+1) = P(X - np > m+1-np)$$

$$= P(|X-np| > (m+1-np))$$

$$\leq \frac{Var(X)}{(m+1-np)}$$

$$= \frac{np(1-p)}{(m+1-np)}$$

$\because X - np > m+1-np$
$X - np < -m-1+np$
$X < 2np-m-1$
$X \geq 0.$

$$\therefore P(X \leq m) = 1 - P(X \geq m+1)$$

$$\geq 1 - \frac{np(1-p)}{(m+1-np)}$$

$$= -15.1974 \quad \text{(Using program)}$$

$$\therefore \boxed{P(X \leq 6) \geq -15.1974} \quad \text{(Using chebysev)}$$

$\because 0.50759 > -15.1979 \Rightarrow$ this chebysev inequality follows

$$\therefore \boxed{0.07112 \leq P[X \leq 6] \leq 0.98111}$$

1.2.5 $n = N = 1002$ & $m = M = 408$

& $p = 0.3624772289$

$P(X \leq m) =$ probability of getting atmost m heads.

$$= \sum_{k=0}^{m} {}^{n}C_k \, p^k (1-p)^{n-k}$$

$$= F_X(m) = F_X(408)$$

$$\boxed{P(X \leq 408) = 0.9492697985} \quad \text{(Using program)} \quad \text{—(1)}$$

Using Markov's Inequality $\mu = np = 313.23$

$\mu = np = 313.23$

$Pr[X \geq m] \leq \frac{\mu}{m} \Rightarrow Pr[X \geq m+1] \leq \frac{\mu}{m+1}$

$\therefore P[X \leq m] = 1 - P[X \geq m+1]$

$\geq 1 - \frac{\mu}{m+1}$

$\Rightarrow \boxed{P[X \leq 408] \geq 0.0629}$ (Using program)

Using chernoff bound for bernolli's trial -

$P(X \leq (1-\delta)\mu) \leq e^{-\frac{\mu \delta^2}{2}} \quad \forall \; 0 < \delta < 1$ | here $X = \sum_{i=0}^{n} X_i$

$P(X \leq m) = ?$

$m = (1-\delta)\mu \Rightarrow \delta = \frac{\mu - m}{\mu}$

$\delta = 0.646$

$\boxed{P(X \leq m) \leq 0.9894685}$

Using chebyshev - $P(|X-\mu| \geq a) \leq \frac{Var(X)}{a^2} = \frac{E[(X-\mu)^2]}{a^2}$

$P(X \leq m) = 1 - P(X \geq m+1)$

$P(X \geq m+1) = P(X - np > m+1-np)$

$\leq \frac{Var(X)}{m+1-np} = \frac{np(1-p)}{(m+1-np)}$

$\therefore P(X \leq m) \geq 1 - \frac{np(1-p)}{(m+1-np)}$

$\boxed{P[X \leq 408] \geq -0.99769}$

Scanned with CamScanner