Probability Theory & Random Processes EE5817

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Outline

Introduction to Probability

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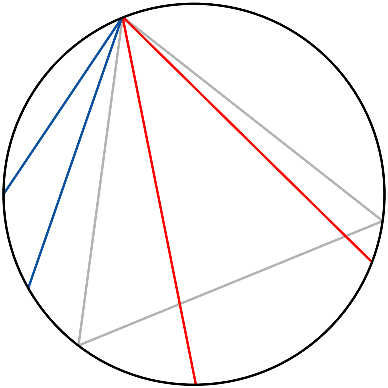
- **Classical Definition:** Probability is number of favourable/ total possible outcomes
- **Frequency-based Definition:** Probability is limit of an events relative frequency in repeated trials
- What happens if the outcome of an event is continous in nature?

Bertrand paradox (probability)

Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle ? **Random endpoints**



[1]https://commons.wikimedia.org/wiki/File:Bertrand1-figure.svg

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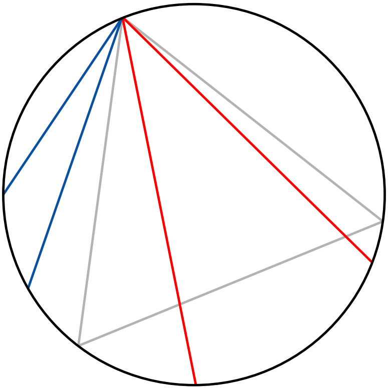

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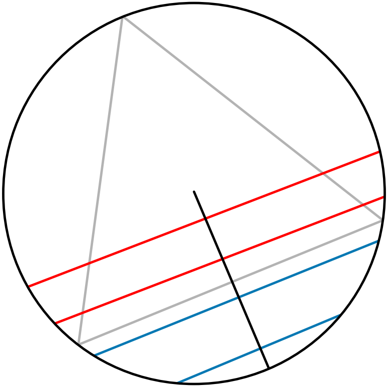

2 https://commons.wikimedia.org/wiki/File:Bertrand2-figure.svg

Bertrand paradox (probability)

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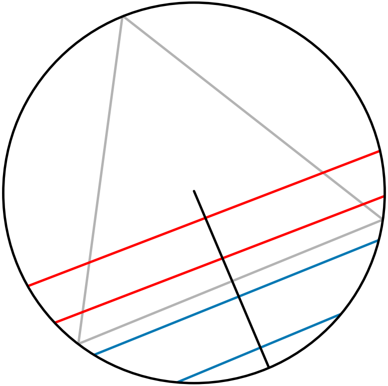

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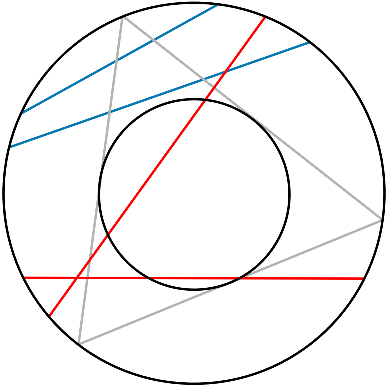
Bertrand paradox (probability)

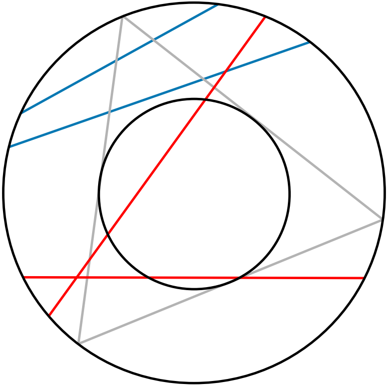
Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle ? **Random midpoint**



[3] https://commons.wikimedia.org/wiki/File:Bertrand3-figure.svg

Bertrand paradox (probability)

Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle ? **Random midpoint** $1/4$



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Axiomatic Definition of Probability

Axiomatic Definition of Probability

- Experiment: e.g., a fair coin is tossed twice
- Sample space: possible set of outcomes of an experiment
  $S = \{\text{HH, HT, TH, TT}\}$
- Event: a subset of possible outcomes under consideration
  $A=\{\text{TT}\}$
- Probability of an event: a number assigned to an event $Pr(A)$ such that it satisfies the following 3 Axioms
- Axiom 1: $Pr(A) \geq 0$
- Axiom 2: $Pr(S) = 1$
- Axiom 3: For every set of disjoint events A_i's,
  $Pr(\cup_i A_i) = \sum_i Pr(A_i)$

Properties of Probability

Properties of Probability

- $P(A^c) = 1 - P(A)$
- If $A \subset B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

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- $P(A^c) = 1 - P(A)$
- If $A \subset B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  $\rightarrow$ *If* A *and* B *are mutually exclusive, then* $P(A \cap B) = 0$.
  For example, $A = \{TT\}, B = \{HH, HT, TH\}$. $\rightarrow$ *If* A *and* B *are mutually exclusive, then* $P(A \cap B) = 0$.
  For example, $A = \{TT\}, B = \{HH, HT, TH\}$

Set Theory

Self Reading

- Sample space
- A possible outcome
- An event A
- A occurs
- A or B (inclusive)
- A and B
- not A, complement of A
- A and B are mutually exclusive
- A implies B

[4] Joseph K. Blitzstein and Jessica Hwang, "Introduction to Probability',' Chapman & Hall/CRC Texts in Statistical Science, 2014. [Online] https://projects.iq.harvard.edu/stat110

Properties of Probability

Self Reading

- Inclusion–Exclusion Principle
- Binomial Theorem
- Vandermonde Identity
- Simpson's Paradox

Joint and Conditional Probabilities

Joint Probability: For events A and B, $P(AB)$ denotes the joint probability that both events happen

- For e.g., from a roll of a fair six sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
  Event $A = \{\text{an even number is observed}\} \rightarrow P(A) = 1/2$
  Event $B = \{\text{a prime number is observed}\} \rightarrow P(B) = 1/2$
  $P(AB) = P\{\text{An even and prime number is observed}\} = 1/6$

Joint and Conditional Probabilities

Independence:

Events A and B are independent $\Leftrightarrow P(AB) = P(A)P(B)$

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  $S = \{1, 2, 3, 4, 5, 6\}$
  Event $A = \{\text{an odd number is observed}\} \rightarrow P(A) = 1/2$
  Event $B = \{\text{5 or 6 is observed}\} \rightarrow P(B) = 1/3$
  $P(AB) = P(A)P(B) = 1/6$

Events $\{A_1, A_2, \ldots, A_n\}$ are pairwise independent
$\Leftrightarrow P(A_iA_j) = P(A_i)P(A_j) \forall i \neq j, i, j \in \{1, \ldots, n\}$

Joint and Conditional Probabilities

Events $\{A_1, A_2, \ldots, A_n\}$ are independent if every event is independent of any intersection of the other events.

For every $k <= n$ and for every k-element subset of events $\{B_i\}$ of $\{A_1, A_2, \ldots, A_n\}$,

$P\left(\cap_{i=1}^{k} B_i\right) = \prod_{i=1}^{k} P(B_i)$

- For e.g., independently toss three fair coins. Let A_{ij} be the event that coin i and j match, $P(A_{ij}) = 0.5$, $(i, j \in \{1, 2, 3\}, \text{and } i \neq j)$
- Events $\{A_{12}, A_{13}, A_{23}\}$ are pairwise independent as $P(A_{ij} \cap A_{jk}) = P(\text{all coins match}) = 0.25 = (0.5)^2$
- However, the events are not mutually independent as $P(A_{ij} \cap A_{jk} \cap A_{ik}) = P(\text{all coins match}) = 0.25 \neq (0.5)^3$

Joint and Conditional Probabilities

Conditional Probabilities: If A and B are events with $P(A) > 0$, the conditional probability of B given A is
$P(B/A) = P(AB)/P(A)$

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  Event $B = \{\text{5 or 6 is observed}\} \rightarrow P(B) = 1/3$
  $P(B/A) = P(AB)/P(A) = 1/3$

Joint and Conditional Probabilities

Total Probability Theorem: Given set of mutually exclusive events $\{A_1, A_2, \ldots, A_n\}$ s.t. $S = \cup_{i=1}^{n} A_i$, for any event B, $P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B/A_i)P(A_i)$

- For e.g., let a sack contain two coins, coin 1 is a fair coin with $P(H) = 0.5$, while the coin 2 is biased with $P(H) = 0.75$, given that a coin is randomly picked from the sack and tossed, what is the probability of observing a heads.
- $P(H) = P(H/coin1)P(coin1) + P(H/coin2)P(coin2)$
  $= 0.5.0.5 + 0.75.0.5 = 0.625$

Joint and Conditional Probabilities

Bayes Theorem: For events A and B, given that $P(B) > 0$,
$P(A/B) = P(B/A)P(A)/P(B)$

Questions