HOMEWORK - 1

Subject :- Information theory. (1 credit)

NAME - ANNU , **ROLL. NO.** - EE21RESCH01010

Degree - PhD in communications & signal processing.

Referrences :- Book by Papulus named "Random Variables & Stochastic process"

→ **lectures** - MIT lecture series on probability

Exercise 1.5 1. Show that for all (+)ve integers (n, k, l) with $(n > k > l)$, prove that $\frac{n-l}{k-l} \geq \frac{n}{k}$

Solution - Starting with the initial condition -

$$n > k > l$$

$$\Rightarrow n > k \text{ -(1) } \& \quad k > l \text{ -(2)}$$

from (1) $\quad n > k$

multiplying by $1/l$ both the sides -

$$\frac{n}{l} > \frac{k}{l}$$

(l is (+)ve ⇒ $1/l$ is also (+)ve ∴ multiplication of (+)ve integer won't change inequality)

$$\Rightarrow \frac{l}{n} < \frac{l}{k} \qquad \Rightarrow \frac{l}{n} - 1 < \frac{l}{k} - 1$$

subtracting 1 both the sides

$$-\left(\frac{l}{n} - 1\right) > -\left(\frac{l}{k} - 1\right)$$

$$\Rightarrow -\left(\frac{l-n}{n}\right) > -\left(\frac{l-k}{k}\right)$$

{ ∵ $n > k > l$
Ex. $n = 3$, $k = 2$ & $l = 1$
$\frac{1}{3} < \frac{1}{2}$
but $-\left(\frac{1}{3} - 1\right) > -\left(\frac{1}{2} - 1\right)$ }

$$\Rightarrow \frac{n-l}{n} > \frac{k-l}{k}$$

$$\Rightarrow (n-l)k > (k-l)n \qquad \text{\{cross multiplication\}}$$

$$\Rightarrow \frac{n-l}{k-l} > \frac{n}{k}$$

$$\Rightarrow \boxed{\frac{n}{k} \leq \frac{n-l}{k-l}}$$

QED//

1.5.2 Prove the following bounds for the binomial coefficient :

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{n^k}{k!}\right)$$

Proof - As we know -

$$^nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\Rightarrow \binom{n}{k} = \frac{n(n-1)(n-2) --- (n-(k-1))\cancel{(n-k)}!}{k!\,\cancel{(n-k)}!}$$

$$= \frac{n(n-1)(n-2)(n-3) --- (n-(k-1))}{k!} = \frac{\prod_{i=0}^{k-1} n-i}{k!}$$

$$\binom{n}{k} = \frac{\prod_{i=0}^{k-1} n-i}{k(k-1) --- (k-(k-1))} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \quad -(1)$$

$$\therefore \binom{n}{k} = \frac{n(n-1)(n-2) --- (n-(k-1))}{k(k-1)(k-2) --- (k-k+1)} \quad -(2)$$

from Above eqn (1) $\rightarrow$ we can see each factor in the product (Numerator as well as denominator) is atleast $\frac{n}{k}$ & there are total $(k-1+1=k)$ k factors.

$$\therefore \text{lower bound, } \frac{n \times n --- (k \text{ times})}{k \times k \times --- (k \text{ times})} = \left(\frac{n}{k}\right)^k$$

$$\because k^k > k! \Rightarrow \left(\frac{n}{k}\right)^k \leq \frac{n(n-1)(n-2) -- (n-(k-1))}{k(k-1)(k-2) --- (k-k+1)}$$

$$\Rightarrow \boxed{\left(\frac{n}{k}\right)^k \leq \binom{n}{k}} \quad [\text{From } (2)] \quad - (3)$$

for upper bound :- $\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} = \frac{\prod_{i=0}^{k-1} n-i}{k!}$

$$\Rightarrow \binom{n}{k} = \frac{n(n-1)(n-2) --- (n-(k-1))}{k!}$$

$$\leq \frac{n.n. ---- n \ (k \text{ times})}{k!}$$

$$= \frac{n^k}{k!}$$

from (3) & (4) -

$$\therefore \boxed{\binom{n}{k} \leq \frac{n^k}{k!}} \quad (4) \longrightarrow \boxed{\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{n^k}{k!}\right)} \quad \text{QED}$$

(1.5.3) Prove that for all integers ((+)ve integers), $n > k > 0$, we have
$$\frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)} \leq \binom{n}{k} \leq \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)}$$
where $p = k/n$ & $H_2(p) = -p\log_2 p + -(1-p)\log_2(1-p)$

Stirling's approximation -
$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} \leq n! \leq e\, n^{n+1/2} e^{-n} \quad -(1)$$

proof -

as we know -
$$\binom{n}{k} = {}^nC_k = \frac{n!}{k!(n-k)!} = \frac{n!}{(np)!\,(n-np)!} \quad [\because k = np]$$
$$\Rightarrow \binom{n}{k} = \frac{n!}{(np)!\,(n(1-p))!} = \frac{n!}{(np)!\,(nq)!} \quad [\text{let } q = 1-p]$$

Using (1) -
$$n! \leq e\, n^{n+1/2} e^{-n}$$
$$\Rightarrow \frac{n!}{(np)!\,(nq)!} \leq \frac{\cancel{e}\, n^{(n+1/2)} e^{-n}}{\cancel{e}(np)^{(np+1/2)} e^{-np}\; e(nq)^{(nq+1/2)} e^{-nq}}$$
$$\Rightarrow \binom{n}{k} \leq \frac{n^n \cdot n^{1/2}\, e^{-n}}{(np)^{np} \cdot (np)^{1/2} \cdot e^{-np} \cdot e \cdot (nq)^{nq} \cdot (nq)^{1/2} e^{-nq}} \quad [\because a^{(b+c)} = a^b \cdot a^c]$$
$$= \frac{n^n \cdot \cancel{n^{1/2}} \cdot e^{-n}}{n^{np} \cdot p^{np} \cdot \cancel{n^{1/2}} \cdot p^{1/2} e^{-np} \cdot e \cdot (n)^{nq} (q)^{(nq)} n^{1/2} q^{1/2} e^{-nq}} \quad [\because (ab)^c = a^c \cdot b^c]$$
$$\binom{n}{k} \leq \frac{n^n \cdot n^{-np} \cdot n^{-nq} \cdot e^{-n} \cdot e^{np} \cdot e^{nq}}{e\,(n^{1/2} p^{1/2} q^{1/2})\,(p^{np} \cdot q^{nq})} = \frac{n^{n(1-p-q)} \cdot e^{n(-1+p+q)}}{e\sqrt{npq}\,(p^{np} \cdot q^{nq})} \quad -(2)$$
$$\because 1-p-q = 1-p-(1-p) = \cancel{1}\cancel{p}\cancel{-1}+\cancel{p} = 0$$
$$\Rightarrow -1+p+q = 0$$

Scanned with CamScanner

from eqn ② —

$$\therefore \binom{n}{k} \leq \frac{n^{n \cdot 0} \cdot e^{n \cdot 0}}{e\sqrt{npq}\ \ p^{np} \cdot q^{np}} = \frac{1}{e\sqrt{np(1-p)}\ p^{np} \cdot q^{np}} \quad \text{— (4)}$$

Now since entropy $H_2(p)$ is given by —

$$H_2(p) = -[p\log_2 p + [\log_2(1-p)] \cdot [1-p]] \quad (\text{given})$$

$$\Rightarrow H_2(p) = -[\log_2 p^p + \log_2 (1-p)^{(1-p)}]$$

$$= -\log_2 (p^p \cdot (1-p)^{(1-p)}) \quad [\because \log a + \log b = \log ab]$$

$$\Rightarrow H_2(p) = -\log_2 (p^p \cdot q^q)$$

$$\Rightarrow p^p \cdot q^q = 2^{-H_2(p)}$$

taking nth power both sides —

$$\Rightarrow p^{np} \cdot q^{nq} = 2^{-nH_2(p)} \quad \text{— (3)}$$

Now, using equation ③ in eqn ④ —

$$\binom{n}{k} \leq \frac{1}{e\sqrt{np(1-p)}\ 2^{-nH_2(p)}} \quad \text{— (5)}$$

Now, $\frac{1}{e} = 0.3678$

& $\frac{e}{2\pi} = 0.432$

$$\therefore \frac{1}{e} < \frac{e}{2\pi}$$

$$\Rightarrow \frac{1}{e\sqrt{np(1-p)}\ 2^{-nH_2(p)}} < \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}\ 2^{-nH_2(p)}}$$

$\therefore$ from ⑤ —

$$\binom{n}{k} \leq \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}\ 2^{-nH_2(p)}}$$

$$= \frac{e}{2\pi\sqrt{np(1-p)}} 2^{nH_2(p)}$$

$$\therefore \boxed{\binom{n}{k} \leq \frac{e}{2\pi\sqrt{np(1-p)}} 2^{nH_2(p)}} \quad \text{— (a)}$$

upper bound proved.

For lower bound - from lower bound of Stirling's eqn - (5)

$$n! \geq \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

$$\binom{n}{k} = \frac{n!}{(np)!\,(nq)!} \geq \frac{\sqrt{2\pi}\; n^{n+1/2}\, e^{-n}}{\sqrt{2\pi}\,(nq)^{nq+1/2} e^{-nq} \sqrt{2\pi}\,(np)^{np+1/2} e^{-np}}$$

$$= \frac{n^n \cdot n^{1/2}\, e^{-n} \cdot e^{nq} \cdot e^{np}}{\sqrt{2\pi}\,(nq)^{nq}\,(nq)^{1/2}\,(np)^{np}\,(np)^{1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{n^n \cdot n^{-nq} \cdot n^{-np} \cdot n^{1/2}\, e^{n(-1+q+p)}}{q^{nq} \cdot p^{np} \cdot n^{1/2} \cdot n^{1/2} \cdot q^{1/2} \cdot p^{1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{n^{1/2} q^{1/2} p^{1/2}} \frac{n^{-n(-1+q+p)} \cdot e^{n(-1+q+p)}}{p^{np}\, q^{nq}} \quad -(6)$$

$\because$ $p^{np} \cdot q^{nq} = 2^{-nH_2(p)}$

& $-1+q+p = 0 \Rightarrow n^{-n \cdot 0} = 1$ & $e^{n \times 0} = 1$

$\therefore$ from (6) -
$$\binom{n}{k} \geq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}} \times \frac{1}{2^{-nH_2(p)}} \quad -(7)$$

$$\binom{n}{k} \geq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)}$$

$\because$ $\frac{1}{\sqrt{2\pi}} = 0.398$ & $\frac{\sqrt{2\pi}}{e^2} = 0.339$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \geq \frac{\sqrt{2\pi}}{e^2} \Rightarrow \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)} \geq \frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)}$$

from (7) -
$$\binom{n}{k} \geq \frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{npq}}\, 2^{nH_2(p)}$$

$$\Rightarrow \boxed{\binom{n}{k} \geq \frac{\sqrt{2\pi}}{e^2} \frac{1}{\sqrt{np(1-p)}}\, 2^{nH_2(p)}} \quad -(b)$$ ~~upper~~ lower bound proved

From (a) & (b) -

$$\boxed{\frac{\sqrt{2\pi}}{e^2} \frac{2^{nH_2(p)}}{\sqrt{np(1-p)}} \leq \binom{n}{k} \leq \binom{n}{k} \leq \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}}\, 2^{nH_2(p)}}$$

QED

Exercise 1.5.4 Use the above to conclude for any

$$0 < p \leq 1/2, \quad \lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} = H_2(p)$$

LHS $= \lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor}$

consider $n =$ integer multiple of p

$$n = mp, \quad m \in Z^+$$

$$\Rightarrow p = \frac{m}{n} \quad — (1)$$

$$\Rightarrow \lfloor np \rfloor = \lfloor n \times \frac{m}{n} = \lfloor m = m \quad (\because m \in Z^+)$$

Now consider LHS -

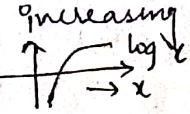
$$\lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor}$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \log_2 \left[\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)} \right] \quad — (2)$$

$\Big\{ \because \binom{n}{k} \leq \frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)}$; $k = np$ (upper bound of previous exer.)

$$\Rightarrow \log_2 \binom{n}{np} \leq \log_2 \left[\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} 2^{nH_2(p)} \right] \Big\}$$

since log function is monotonically increasing function.



from (2) -

$$\lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} \leq \lim_{n\to\infty} \frac{1}{n} \left[\log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right) + \log_2 \left(2^{nH_2(p)} \right) \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right) + \frac{1}{n} \log_2 \left(2^{nH_2(p)} \right) \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right) + \frac{1}{\cancel{n}} \times \cancel{n} H_2(p) \right]$$

$$= \lim_{n\to\infty} \left[\underbrace{\frac{1}{n} \log_2 \left(\frac{e}{2\pi} \frac{1}{\sqrt{np(1-p)}} \right)}_{E} + H_2(p) \right]$$

when $n \to \infty \Rightarrow E \to 0$

$$\therefore \lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} \longrightarrow H_2(p)$$

since $\lfloor np \rfloor$ is used $\therefore$ limit leads $=$ sign.

$$\boxed{\lim_{n\to\infty} \frac{1}{n} \log_2 \binom{n}{\lfloor np \rfloor} = H_2(p)} \quad \text{QED}$$

Exercise 1.4 (of the empirical frequency is close to the true probability if the number of samples is large). (7)

Given a sequence $x^n = (x_1, x_2, \dots x_n)$, where each $x_i \in \mathcal{X}$, the relative frequency / Emperical freq. of occurance of $a \in \mathcal{X}$ is defined as:

$$\eta_a(x^n) = \frac{\text{No. of times } a \text{ occurs in } x^n}{n}$$

Consider a sequence of n i.i.d random variables $X_1, X_2 \dots X_n$ over the alphabet $\mathcal{X}$ & having distribution P_X.

1. Prove that for each $a \in \mathcal{X}$ & $\epsilon > 0$,

$$\lim_{n\to\infty} P_r\left[\,|\eta_a(X^n) - P_X(a)| > \epsilon\right] = 0 \quad -(1)$$

Also show that the probability above decays exponentially in n.

Solution: Since $\eta_a(x^n) = \frac{\text{No. of times } a \text{ occurs in } x^n}{\text{length of sequence}} = \frac{k}{n}$ (let)

$=$ empirical frequency

$\therefore$ $(\eta_a(x^n) : a \in \mathcal{X})$ forms a valid probability mass function. & $P_X(a) =$ Probability of occurance of $a \in \mathcal{X}$ in a single trial $x \in \mathcal{X}^n$
$=$ Probability of occurance of a in single trial

Now, considering a sequence of n i.i.d ~~var~~ random variables $X_1, X_2 \dots X_n$ over alphabet $\mathcal{X}$ & having probability distribution P_X. $\Rightarrow$ It is same as Bernoulli R.V. with $n =$ no. of trials $=$ length of sequence $a \in \mathcal{X}$ –

No. of times a occurs in x^n = No. of success (k)

Now, since $X_1, X_2, \dots X_n$ are independent (i.i.d)

$\therefore$ No. of "success" in n-trials $= k$ (let) $=$ No. of times a occurs in x^n.

& $P_X(a) = p$ & $1-p = q$. (let) ; $\epsilon > 0$

considering – $\left|\frac{k}{n} - p\right| > \epsilon$

$\Rightarrow \left(\frac{k}{n} - p\right) > \epsilon \quad -(1)$ & $\left(\frac{k}{n} - p\right) < -\epsilon \quad -(2)$

By (1)

$$\left(\frac{k}{n} - p\right) > \epsilon$$

$$\Rightarrow (k - np) > n\epsilon$$

$\Rightarrow \quad k - n(p+\varepsilon) > 0$

taking exponential both sides−

e^{mx} ; $m>0$

x

for any (+)ve m − $e^{m(k-n(p+\varepsilon))} > e^0$

$\Rightarrow \quad e^{m(k-n(p+\varepsilon))} > 1 \quad -(1)$ ($\because e^{mx}$ is increasing f^n)

$$P\left\{\frac{k}{n} - p > \varepsilon\right\} = P\{k > n\varepsilon + np\} = P\{k > n(p+\varepsilon)\}$$

$$= \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} \binom{n}{k} p^k q^{n-k}$$ [$\because$ k → No. of success in n− bernoulli trials]

$$< \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} \quad -(2)$$ (from (1))

$$\sum_{k=0}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{\lceil n(p+\varepsilon)\rceil} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} + \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k}$$

$$\Rightarrow \sum_{k=\lceil n(p+\varepsilon)\rceil}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} < \sum_{k=0}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k}$$

From (2) −

$$P\left\{\frac{k}{n} - p > \varepsilon\right\} \le \sum_{k=0}^{n} e^{m(k-n(p+\varepsilon))} \binom{n}{k} p^k q^{n-k} \quad -(3)$$

$$\because (x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r} \quad -(4)$$

from (3) −

$$P\left\{\frac{k}{n} - p > \varepsilon\right\} < \cancel{\sum_{k=0}^{n}} e^{-mn\varepsilon} \sum_{k=0}^{n} \binom{n}{k} (pe^{mq}) (qe^{-mp})^{(n-k)}$$

$$= e^{-mn\varepsilon}(pe^{mq} + qe^{-mp})^n \quad -(5)$$ [using (4)]

Consider − $pe^{mq} + qe^{-mp}$

$$\le p(mq + e^{m^2q^2}) + q(-mp + e^{m^2p^2})$$

$$= pe^{m^2q^2} + qe^{m^2p^2}$$ [$\because e^x \le x + e^{x^2}$ for $x>0$]

$= p e^{m^2 q^2} + q e^{m^2 p^2} \leq e^{m^2}$ $\left[\because e^x \leq x + e^{x^2} \text{ for } x > 0\right]$ (9)

$\Rightarrow e^{-mn\varepsilon}\left[p e^{\lambda^2 q^2} + q e^{\lambda^2 p^2}\right]^n \leq e^{m^2 n} e^{-mn\varepsilon}$

$\Rightarrow$ from eqn (5) —

$$P\left\{\frac{k}{n} - p \geq \varepsilon\right\} \leq e^{m^2 n} e^{-mn\varepsilon}$$

$$\Rightarrow P\left\{\frac{k}{n} - p \geq \varepsilon\right\} \leq e^{m^2 - mn\varepsilon}, \quad \varepsilon > 0 \quad (6)$$

let $g(m) = e^{m^2 - mn\varepsilon}$

let's find min. value of $g(m) \Rightarrow$ min value of $m^2 - mn\varepsilon$

$\Rightarrow f(m) = m^2 n - mn\varepsilon$

$$\frac{d f(m)}{dm} = 2mn - n\varepsilon = 0$$

$$\Rightarrow m = \frac{\varepsilon}{2}$$

$$\frac{d^2 f(m)}{dm^2} = -n\varepsilon < 0 \Rightarrow m = \frac{\varepsilon}{2} \text{ corresponds to min. value of } g(m).$$

$\Rightarrow$ at $m = \frac{\varepsilon}{2}$, $e^{m^2 n - mn\varepsilon}$ is minimum at $\boxed{m = \frac{\varepsilon}{2}}$, $\varepsilon > 0$

$\therefore$ from (6) —

$$P\left\{\frac{k}{n} - p \geq \varepsilon\right\} \leq e^{\left(\frac{\varepsilon}{2}\right)^2 - \left(\frac{\varepsilon}{2}\right)n\varepsilon}$$

$$= e^{-\frac{n\varepsilon^2}{4}}$$

when $m = \varepsilon/2$

$$\Rightarrow \boxed{P\left\{\frac{k}{n} - p \geq \varepsilon\right\} \leq e^{-n\varepsilon^2/4}} \quad (7)$$

Similarly, $P\left\{\frac{k}{n} - p < -\varepsilon\right\} \leq e^{-n\varepsilon^2/4}$ (8)

Adding (7) & (8) —

$$\Rightarrow \boxed{P\left\{\left|\frac{k}{n} - p\right| \geq \varepsilon\right\} \leq 2e^{-n\varepsilon^2/4}}$$

$$\Rightarrow \boxed{P\left\{\left|\eta_a(x^n) - P_X(a)\right| \geq \varepsilon\right\} \leq 2e^{-n\varepsilon^2/4}} \quad \forall\, a \in \mathcal{X} \;\&\; \varepsilon > 0$$

when $n \to \infty$: $P\left\{\left|\eta_a(x^n) - P_X(a)\right| \geq \varepsilon\right\} = 0$ $\left[\because e^{-\infty} = 0\right]$

this shows probability $P\{|\eta_a(x^n) - P_X(a)| \geq \varepsilon\}$ decays exponentially in n.

Exercise 1.4.2 - Use this to prove that

$$\lim_{n\to\infty} P_r\left[\,|\eta_a(x^n) - P_x(a)| > \varepsilon \text{ for some } a \in \mathcal{X}\right] = 0$$

but any $a \in \mathcal{X}$.

This equation does not enforce $\forall\, a \in \mathcal{X}$,

let k = No. of times for some a occurs in x^n, n = length of sequence.

$\Rightarrow \eta_a(x^n) = \dfrac{k}{n}$ & $P_x(a) = p$ (let) & $q = 1-p$ (let)

$$P_r\left[\,|\eta_a(x^n) - P_x(a)| > \varepsilon\right] = P_r\left[\,\left|\frac{k}{n} - p\right| > \varepsilon\right]$$

$$= P_r\left[\left(\frac{k}{n} - p\right)^2 > \varepsilon^2\right]$$

$$= P_r\left[(k-np)^2 > n^2\varepsilon^2\right]$$

consider

$$\sum P_r\left[\,|\eta_a(x^n) - P_x(a)| > \varepsilon\right] = \sum P_r\left[(k-np)^2 > n^2\varepsilon^2\right]$$

where $P_n(k) = P_x\{X = \text{No. of times } a \text{ occurs in } x^n\}$

$= \binom{n}{k} p^k q^{n-k}$

& $P_x(a)$ = probability of occurance of any $a \in \mathcal{X}$ in a single trial $= p$ (let)

$q = 1-p$

$$(k-np)^2 > n^2\varepsilon^2$$

$$\Rightarrow \sum_{k=0}^{n} (k-np)^2 P_n(k) > \sum_{k=0}^{n} n^2\varepsilon^2 P_n(k)$$

$$\Rightarrow \sum_{k=0}^{n} (k-np)^2 P_n(k) > n^2\varepsilon^2 \sum_{k=0}^{n} P_n(k)$$

$$\Rightarrow \sum_{k=0}^{n} (k-np)^2 P_n(k) > n^2\varepsilon^2 \qquad \left[\because \sum_{k=0}^{n} P_n(k) = \text{sum of pmf} = 1\right]$$

$$\Rightarrow \boxed{\sum_{k=0}^{n} k^2 P_n(k) - 2np\sum_{k=0}^{n} k\, P_n(k) + n^2p^2 > n^2\varepsilon^2} \quad \text{--- (1)}$$

calculating $\displaystyle\sum_{k=0}^{n} k\, P_n(k) = \sum_{k=1}^{n} k\, {}^nC_k\, p^k (1-p)^{n-k}$ $[\because k > 0]$

$$= \sum_{k=1}^{n} k \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} q^{n-k}$$

$$= np \sum_{k-1=0}^{n-k} {}^{(n-1)}C_{(k-1)}\, p^{(r-1)} (1-p)^{(n-1)-(r-1)} \quad (1)$$

$$\boxed{\sum_{k=1}^{n} k P_n(k) = np[p+1-p]^{n-1} = np \times 1 = np} \quad -(2)$$

$$\left[\because \sum_{r=0}^{n} {}^nC_r\, p^r (1-p)^{n-r} = (p+1-p)^n \right.$$

$$\left. \text{or } (x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r} \right]$$

$$\sum_{k=0}^{n} k^2 P_n(k) = \sum_{k=1}^{n} k \frac{n!}{(n-k)!\,(k-1)!} p^k q^{n-k}$$

$$= n^2 p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!\,(k-2)!} p^{k-2} q^{n-k}$$

$$+ \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k q^{n-k}$$

$$= n^2 p^2 [p+1-p]^{n-2} + npq$$

$$\boxed{\sum_{k=0}^{n} k^2 P_n(k) = n^2p^2 + npq} \quad -(3)$$

Now putting values from eqn (2) & (3) into eqn (1)–

$$\sum_{k=0}^{n} (k-np)^2 P_n(k) = \sum_{k=0}^{n} k^2 P_n(k) - 2np \sum_{k=0}^{K} k P_n(k) + n^2p^2$$

$$= n^2p^2 + npq - 2np \cdot np + n^2p^2 = npq.$$

$$\sum_{k=0}^{n} (k-np)^2 P_n(k) = \sum_{|k-np| \le n\varepsilon} (k-np)^2 P_n(k)$$

$$+ \sum_{|k-np| > n\varepsilon} (k-np)^2 P_n(k)$$

$$\geq \sum_{|k-np|>n\varepsilon} (k-np)^2 P_n(k) > n^2\varepsilon^2 \sum_{|k-np|>n\varepsilon} P_n(k)$$

$$= n^2 \varepsilon^2 P\{|k-np| > n\varepsilon\}$$

$$\therefore P\left(\left|\frac{k}{n} - p\right| > \varepsilon, \text{ for some } a \in \mathcal{X}\right) < \frac{pq}{n\varepsilon^2}$$

$$\Rightarrow \boxed{P\left(|\eta_a(x^n) - P_x(a)| > \varepsilon \text{ for some } a \in \mathcal{X}\right) < \frac{P_x(a)[1-P_x(a)]}{n\varepsilon^2}}$$

for some $a \in \mathcal{X}$ & $\varepsilon > 0$

$$\boxed{\text{when } n \to \infty \quad P\left(|\eta_a(x^n) - P_x(a)| > \varepsilon \text{ for some } a \in \mathcal{X}\right) = 0} \quad \text{QED.}$$

Scanned with CamScanner

Exercise 1.4.3

using program – $n = 1000$

$$\sum_{a \in X} |\eta_a(x^n) - P_Y(a)| = \underline{0.0176667}.$$

Exercise 1.2.1 Consider n independent tosses of a biased coin which lands heads with probability p.

1. Find an expression for the probability that the number of heads is exactly k. forms bernaulli i.i.d.

Probability for getting head $= p$

probability for getting tail $= 1-p$

$\cancel{P(X=)}$ $Pr($ No. of heads is $k) = P(X=k)$

$$\therefore \boxed{P(X=k) = {}^nC_k \, p^k \, (1-p)^{n-k}}$$ required expression

Exercise 1.2.2

Find an expression for the probability that the number of heads is atmost m.

$Pr($ No. of heads is atmost $m) = P(X \le m)$

$$f_X(m) = \boxed{P(X \le m) = \sum_{k=0}^{m} {}^nC_k \, p^k \, (1-p)^{n-k}} = f_X(m) \text{ required expression}$$

Exercise 1.2.3

$n = 17$, $m = 6$, $p = 0.382477 2289$

$$P(X \le m) = f_X(m) = \sum_{k=0}^{6} {}^{17}C_k \cdot p^k \, (1-p)^{17-k}$$

$$\boxed{P(X \le 6) = 0.50759036679 1487}$$ {using computer program}

Exercise 1.2.4 – finding bounds for the probability calculated in 1.2.3.

Expectation for bernaulli R.V –

$$\mu = nE[x_1] = n[1 \times p + (1-p) \times 0] = np$$

$$= 17 \times 0.3824772289$$

$$\simeq 6.502$$

Now, from Markov's Inequality –

$$P_r[X \geq m] \leq \frac{\mu}{a} \Rightarrow P_r[X > m] < \frac{\mu}{a}$$

$$\Rightarrow P_r[X \leq m] = 1 - p[X > m] \geq 1 - \frac{\mu}{m}$$

$$P[X \leq 6] \geq 1 - \frac{\mu}{m} = -0.0836854818833347$$

Probability of getting atmost 6 heads –

$$\boxed{P[X \leq 6] \geq -0.08368548188333347}$$

(lower bound) $0.507597 > -0.08$ ⇒ satisfies

Using chernoff bound – for bernaulli's trial

lower tail $P(X \leq (1-\delta)\mu) \leq e^{-\frac{\mu\delta^2}{2}} \quad \forall \; 0 < \delta < 1$

$$P(X \leq m) = ?$$

$$m = (1-\delta)\mu \Rightarrow \quad 6 = (1-\delta) \times np$$

$$6 = (1-\delta)\, 6.502$$

$$\Rightarrow \quad 6 = 6.502 - 6.502\delta$$

$$\Rightarrow \quad 6.502\delta = 6.502 - 6$$

$$\Rightarrow \quad \delta = \frac{0.502}{6.502} = 0.077$$

$$\therefore \; P[X \leq 6] \leq e^{-\frac{\mu\delta^2}{2}} = 0.98079938515023 48$$

$$\boxed{P[X \leq 6] \leq 0.98079938515023 48}$$ (upper bound)

⇒ $(0.50759 \leq 0.980)$ ⇒ satisfies

Using chebysev bound –

$$P[X \leq m] = 1 - P[X < m] = 1 - P[X^2 < m^2]$$

$$= 1 - P[|X| < m]$$

$$\geq 1 - \frac{E[X^2]}{m^2} = \cancel{1 - P[X - \mu < m - \mu]}$$

$$\cancel{1 - P[|X|]}$$

$$|X| < m \Rightarrow X < m \;\&\; X > -m \Rightarrow X < m \;\&\; X > 0 \;(\because X \in \mathbb{Z}^+)$$

$$= 1 - \frac{np}{m^2}$$

$$\left[\because \text{ Using chebysev } \; P[|X-\mu| > a] \leq \frac{E[(X-\mu)^2]}{a}\right]$$

$$= 1 - \frac{6.502}{6^2}$$

$$= 1 - 0.1806111 = -0.8193888$$

$$\boxed{P[X \leq 6] \geq -0.8193888}$$

$$0.50759 > -0.8193888$$

Actual probability ≈ 0.507

It satisfies all bounds.

1.2.5 Exercise 1.2.5

$n = N = 1002$ & $m = M = 408$

& $p = 0.382477 2289$.

$P(X \le m)$ = Probability of getting at most m heads

$= \sum_{k=0}^{m} {}^{n}C_k \, p^k (1-p)^{n-k}$

$= F_X(m) = F_X(408) = \sum_{k=0}^{408} {}^{1002}C_{k} (p)^k (1-p)^{n-k}$

$\boxed{P(X \le 408) = 0.9492697985}$ — ① (using program)

Using Markov's Inequality :- $\mu = np = 383.23$

$Pr[X \ge m] \le \frac{\mu}{a} \Rightarrow Pr[X > m] < \frac{\mu}{a}$

$\Rightarrow Pr[X \le m] = 1 - P[X > m] \ge 1 - \frac{\mu}{m}$

$\Rightarrow P[X \le 408] \ge 1 - \frac{\mu}{a} = 0.06068092314264717$

$\therefore \boxed{P[X \le 408] \ge 0.0606809}$ (lower bound)

$\because 0.9492697985 > 0.0606809$ (using ①)

$\Rightarrow$ it satisfies this bound

Using chernoff bound for bernoulli's trial -

$P(X \le (1-\delta)\mu) \le e^{-\frac{\mu \delta^2}{2}}$ $\forall$ $0 < \delta < 1$

$P(X \le M) = ?$

$m = (1-\delta)\mu \Rightarrow \delta\mu = \mu - m \Rightarrow \delta = \frac{\mu - m}{\mu}$

$\boxed{\delta = 0.646}$

$\therefore P(X \le m) \le e^{-\frac{383.23 \times (0.646)^2}{2}}$

$= 0.449468511270317 9$

$\therefore \boxed{P(X \le 408) \le 0.449468512}$

$\because 0.949269 > 0.4494$ (using)

$\Rightarrow$ chernoff bound satisfies //

Using Chebysev bound -

$P[X \le m] = 1 - P[X \not< m] = 1 - P[X^2 < m^2]$

$= 1 - P[|X| \le m]$

$\geq 1 - \frac{E[X^2]}{m^2}$

$\{ \because |X| < m$
$\Rightarrow X < m \;\&\; X > -m$
$\Rightarrow X < m \;\&\; X > 0 \; (\because X \in Z^+)$

$\{ \because$ Using chebysev -
$P[|X - \mu| > a] \le E\frac{[(X-\mu)^2]}{a} \}$

$\therefore P[X \le 408] \geq 1 - \frac{E[X]}{m^2}$ $[\because E[X^2] = E[X]]$

$= 1 - \frac{np}{m^2}$

$= 1 - \frac{383.23}{408 \times 408} = -0.99769$

$\Rightarrow \boxed{P[X \le 408] \geq -0.99769}$

$\because 0.94926 \geq -0.99769$

$\Rightarrow$ This chebysev bound also satisfies.