Probability Theory & Random Processes EE5817

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Outline

1. Transformation of a Random Variable
2. Examples
3. Generalizations

Affine Transformations

$$
\begin{aligned}
Y &= aX + b \\
\rightarrow F_Y(y) &= P(Y \leq y) \\
&= P(aX + b \leq y) \\
&= P\left(X \leq \frac{y-b}{a}\right) \text{ if } a > 0 \\
&= F_X\left(\frac{y-b}{a}\right) \text{ if } a > 0 \\
\rightarrow f_Y(y) &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \text{ if } a > 0
\end{aligned}
$$

Affine Transformations

$$
\begin{aligned}
Y &= aX + b \text{ if } a < 0 \\
\rightarrow F_Y(y) &= P(Y \leq y) \\
&= P(aX + b \leq y) \\
&= P\left(X \geq \frac{y-b}{a}\right) \\
&= 1 - F_X\left(\frac{y-b}{a}\right) \\
\rightarrow f_Y(y) &= \frac{-1}{a} f_X\left(\frac{y-b}{a}\right)
\end{aligned}
$$

Affine Transformations

$$
\begin{aligned}
Y &= aX + b \\
\rightarrow f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \\
E[Y] &= E[aX + b] \\
&= a\mu_X + b \\
E[(Y-\mu_Y)^2] &= E[(aX+b)^2 - (a\mu_X + b)^2] \\
&= E[(aX)^2 - (a\mu_X)^2] \\
&= a^2\sigma_X^2
\end{aligned}
$$

Example for Affine Transformation

$X \sim N(0,1)$, then $Y = \sigma X + \mu$ is Gaussian with mean μ and variance σ^2
$Y \sim N(\mu, \sigma^2)$, then $X = \frac{1}{\sigma}(Y - \mu)$ is Gaussian with mean 0 and variance 1

Square

$$
\begin{aligned}
Y &= X^2 \\
\rightarrow F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= P\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) \\
&= F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) \\
\rightarrow f_Y(y) &= \frac{1}{2\sqrt{y}} f_X\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}} f_X\left(-\sqrt{y}\right)
\end{aligned}
$$

Chi-square

$Y \sim N(0,1)$, then $Z = Y^2$ is Chi-square distributed with one degree of freedom

$$
\begin{aligned}
f_Z(z) &= \frac{1}{\sqrt{2\pi z}} \exp\left(\frac{-z}{2}\right) \\
\mu_Z &= 1 \\
\sigma_Z^2 &= 2 \\
\text{Skewness} &= \sqrt{8} \\
\text{Excess kurtosis} &= 12 \\
\text{MGF} &= (1-2s)^{-0.5} \text{ for} s < 0.5 \\
\text{Characteristic function} &= (1-2j\omega)^{-0.5}
\end{aligned}
$$

Self Reading: Start with the distribution of Z as Chi-square and derive the distribution of $Y = \sqrt{Z}$

Log-normal distribution

$X \sim N(\mu, \sigma^2)$ **and** $X = \log(Y)$**, then** Y **is Log-Normal distributed**

$$
\begin{aligned}
f_Y(y) &= \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(y)-\mu)^2}{2\sigma^2}\right) \\
\mu_Y &= \exp(\mu + 0.5\sigma^2) \\
\sigma_Y^2 &= [\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2) \\
\text{Skewness} &= [\exp(\sigma^2) + 2]\sqrt{\exp(\sigma^2) - 1} \\
\text{Excess kurtosis} &= \exp(4\sigma^2) + 2\exp(3\sigma^2) + 3\exp(2\sigma^2) - 6
\end{aligned}
$$

Self Reading: Median, Mode, MGF, CF

Self reading

Self Reading:

$X \sim \mathit{Uniform}(0,1)$, then $Y = (\frac{x}{\alpha})^{\frac{1}{\beta}}$ will have Pareto distribution

$X \sim \mathit{Exponential}(\lambda)$, then $Y = \sqrt{X}$ will have Rayleigh distribution

Discrete Random Variable

$Y = g(X)$, **and both are discrete then**

$$P(Y = y_i) = \sum_{x_i \in g^{-1}(y_i)} P(X = x_i)$$

Continous RV

Let $Y = g(X)$, with $g(.)$ being a one-one and increasing function over the range of X. Then,

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(g(X) \leq y) \\
&= P\left(X \leq g^{-1}(y)\right) \\
&= F_X\left(g^{-1}(y)\right) \text{ using chain rule,} \\
\to f_Y(y) &= \frac{d}{dy} F_X\left(g^{-1}(y)\right) = f_X(g^{-1}(y)) \frac{d}{dy}\left(g^{-1}(y)\right)
\end{aligned}$$

Continous RV

Let $Y = g(X)$, with $g(.)$ being a one-one and decreasing function over the range of X. Then,

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(g(X) \leq y) \\
&= P\left(X \geq g^{-1}(y)\right) \\
&= 1 - F_X\left(g^{-1}(y)\right) \text{ using chain rule,} \\
\rightarrow f_Y(y) &= -\frac{d}{dy} F_X\left(g^{-1}(y)\right) = -f_X(g^{-1}(y)) \frac{d}{dy}\left(g^{-1}(y)\right)
\end{aligned}
$$

Continous RV

For $Y = g(X)$, with $g(.)$ being a one-one and monotonic function over the range of X.

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} \left(g^{-1}(y)\right) \right|$$

Self Reading: What if $g(.)$ is many-to-one

Uniform Distribution

Let X a continuous RV with $F_X(x)$ strictly increasing. Then, $F_X^{-1}()$ exists.

$$
\begin{aligned}
\text{Let } U &= F_X(X), \text{then for } u \in [0,1] \\
F_U(u) &= P(U \leq u) \\
&= P(F_X(X) \leq u) \\
&= P(X \leq F_X^{-1}(u)) \\
&= F_X(F_X^{-1}(u)) \\
&= u
\end{aligned}
$$

Uniform Distribution

Alternatively, for $U \sim Uniform(0,1)$ **Let,** $X = F_X^{-1}(U)$**, then,**

$$\begin{aligned} P(X \leq x) &= P(F_X^{-1}(U) \leq x) \\ &= P(U \leq F_X(x)) \\ &= F_U(F_X(x)) \\ &= F_X(x) \end{aligned}$$

Uniform Distribution

Some examples:

$$F_X(x) = \begin{cases} 0 \text{ if } x < a, \\ \dfrac{(x-a)}{(b-a)} \text{ if } a \leq x \leq b, \\ 1 \text{ if } x > b. \end{cases}$$

$$\text{Then, } u = \frac{(x-a)}{(b-a)}, \quad (b-a)u + a = x = F_X^{-1}(u)$$

Uniform Distribution

Some examples:

$$F_X(x) = \begin{cases} 0 \text{ if } x < 0, \\ 1 - \exp\left(\frac{-x}{\beta}\right) \text{ if } x \geq 0. \end{cases}$$

$$\text{Then, } U = 1 - \exp\left(\frac{-X}{\beta}\right), \quad x = -\beta \log(1-u)$$

Questions