Probability Theory & Random Processes
EE5817

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Outline

Random Variables

A Random Variable is a mapping from the sample space S to the real line $\mathbb{R}$.

For e.g., a random variable X maps as follows

$\{\text{Tail, Head}\} \rightarrow \{0, 1\}$ or $\{\text{Failure, Success}\} \rightarrow \{0, 1\}$

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Where is probability?

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Where is probability?

Depends on the type of random variable:

- **Discrete random variable:** the image of X is *c*ountable
- **Continous random variable:** the image of X is *u*ncountably infinite

Discrete Random Variable

Probability Mass Function (PMF) of a discrete random variable (r.v.) X is given by $p_X(x_i) = P\{X = x_i\}$.
For e.g., for fair coin toss experiment,

$$p_X(x_i) = P\{X = x_i\} = \begin{cases} 0.5 & x_i = 0, \\ 0.5 & x_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Discrete Random Variable

Cumulative Distribution Function (CDF) of a r.v. X is defined as $F_X(x) = P\{X \leq x\}$.
For e.g., for fair coin toss experiment,

$$F_X(x) = P\{X \leq x\} = \begin{cases} 0 & x < 0, \\ 0.5 & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases}$$

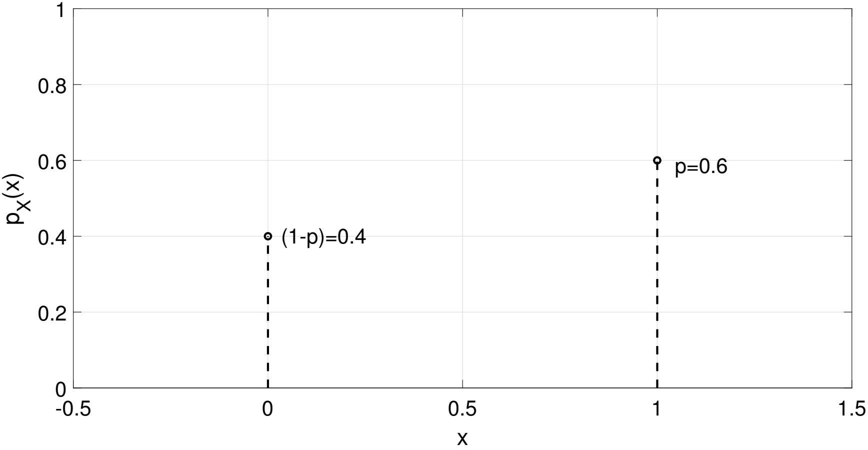
Discrete Random Variable

Bernoulli distribution: Single experiment with binary outcomes

$$p_X(x) = \begin{cases} (1-p) & x = 0, \\ p & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

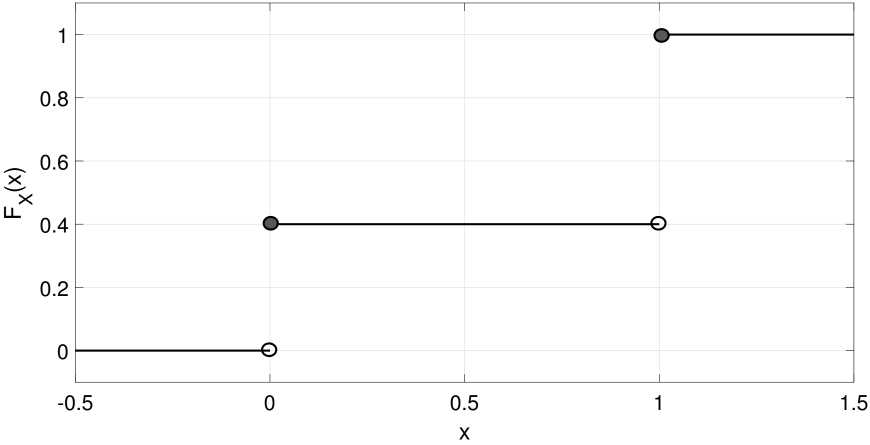
Discrete Random Variable

Bernoulli distribution (PMF):



Discrete Random Variable

Bernoulli distribution (CDF):



CDF

For any r.v. X, its CDF $F_X(x)$ will satisfy the following properties:

- It is non-decreasing
- It is right continous
- $\lim_{x \to -\infty} F_X(x) = 0$
- $\lim_{x \to +\infty} F_X(x) = 1$
- If X is purely discrete, $F_X(x) = \sum_{x_i \leq x} p_X(x_i)$
- If X is continous, $F_X(A) - F_X(B) = P\{B < X \leq A\}$
- Self reading Please check out more properties of CDF

Expectation

Expectation for a discrete r.v. X with pmf $p_X(x)$ is given by
$E[X] = \sum_x x p_X(x)$
Properties of Expectation Operator:

- Linearity, $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$
- Non-multiplicativity, $E[XY]$ not always equal to $E[X]E[Y]$
- **Mean** of a r.v. X is defined as its expected value, $\mu_X = E[X]$
- Self reading Please check out more properties of Expectation

Variance of a R.V.

Variance of a r.v. X,

$$
\begin{aligned}
\sigma_X^2 &= E[(X - E[X])^2] \\
&= E[X^2 + E[X]^2 - 2XE[X]] \\
&= E[X^2] + E[E[X]^2] - 2E[XE[X]] \\
&= E[X^2] + E[\mu_X^2] - 2E[X\mu_X] \\
&= E[X^2] + \mu_X^2 - 2E[X]\mu_X \\
&= E[X^2] - \mu_X^2
\end{aligned}
$$

Median and Mode of a R.V.

Median is the value of X, denoted by m, at which $P\{X < m\} \geq 0.5$ and $P\{X > m\} \geq 0.5$

Mode is the most fequently occuring value of X,
mode $=\arg\max_x p_X(x)$

Discrete Random Variable

Bernoulli distribution

- $\mu_X = p.1 + (1-p).0 = p$
- $\sigma_X^2 = E[X^2] - p^2 = 0.(1-p) + 1.p - p^2 = (1-p)p$
- Median

$$= \begin{cases} 0 & p < 0.5, \\ [0,1] & p = 0.5, \\ 1 & p > 0.5. \end{cases}$$

- Mode

$$= \begin{cases} 0 & p < 0.5, \\ \{0,1\} & p = 0.5, \\ 1 & p > 0.5. \end{cases}$$

Binomial Distribution

Binomial distribution: with parameters n and p represents the number of successes in a sequence of n independent identical experiments. For a Binomial distributed r.v. X

- $$p_X(k) = \begin{cases} 0 & k < 0, \\ \binom{n}{k} p^k (1-p)^{(n-k)} & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$
- $\mu_X = np$
- Median $= \lceil np \rceil$ or $\lfloor np \rfloor$
- Mode $= \lceil (n+1)p \rceil - 1$ or $\lfloor (n+1)p \rfloor$
- $\sigma_X^2 = np(1-p)$

Geometric Distribution

Geometric distribution: with parameter p represents the number of trial required to obtain the first successes in a sequence of independent trials. For a Geometric distributed r.v. X

- $$p_X(k) = \begin{cases} 0 & k < 0, \\ (1-p)^{(k-1)}\, p & \text{otherwise.} \end{cases}$$
- $\mu_X = \frac{1}{p}$
- Median $= \lceil \frac{-1}{\log_2(1-p)} \rceil$, not unique if the term is an integer
- Mode $= 1$
- $\sigma_X^2 = \frac{(1-p)}{p^2}$

Geometric Distribution

Geometric distribution exhibits memorylessness:

- $P\{X > m + n | X > n\} = P\{X > m\}$
- $F_X(x) = 1 - (1 - p)^k$

Negative Binomial Distribution

Negative Binomial distribution: models the number of failures in a sequence of independent and identically distributed Bernoulli trials before the success r occurs

- $$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, k \geq r$$
- $\mu_X = r\frac{(1-p)}{p}$
- $\sigma_X^2 = r\frac{(1-p)}{p^2}$

Discrete Uniform Distribution

Discrete Uniform Distribution: models the outcome of an experiment with n number of values that equally likely to be observed between integers a and b s.t. $b = a + n - 1$

- $$p_X(k) = \left(\frac{1}{n}\right), k \in \{a, a+1, \ldots, b\}$$
- $\mu_X = \frac{(a+b)}{2}$
- Median $= \frac{(a+b)}{2}$
- $\sigma_X^2 = \frac{(b-a+1)^2-1}{12}$

Poisson Distribution

Poisson distribution: models the probability of a given number of events occurring in a fixed interval with a known constant mean rate and independent of the time since the last event occurs

- $$p_X(k) = \left(\frac{\lambda^k e^{-\lambda}}{k}\right)$$
- $\mu_X = \lambda$
- $\sigma_X^2 = \lambda$

Questions