1. $A = \begin{bmatrix}1&1&5\\1&5&1\\5&1&1\end{bmatrix}$ eigenvalues of A are $-4, 4, 7$.

$\Rightarrow (\lambda+4)(\lambda-4)(\lambda-7)=0$

① $\lambda=-4$

$$\lambda I - A = \begin{bmatrix}-5&-1&-5\\-1&-9&-1\\-5&-1&-5\end{bmatrix} = A'$$

$$A'x=0 \Rightarrow \begin{cases}-5x_1-x_2-5x_3=0\\-x_1-9x_2-x_3=0\\-5x_1-x_2-5x_3=0\end{cases} \Rightarrow \begin{cases}x_1=s\\x_2=0\\x_3=-s\end{cases}$$

$$x = s\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad W_{-4}=(1,0,-1) \Rightarrow V_1=\left(\tfrac{1}{\sqrt2}, 0, \tfrac{-1}{\sqrt2}\right)$$

② $\lambda=4$

$$\lambda I - A = \begin{bmatrix}3&-1&-5\\-1&-1&-1\\-5&-1&3\end{bmatrix} = A'$$

$$A'x=0 \Rightarrow \begin{cases}3x_1-x_2-5x_3=0\\-x_1-x_2-x_3=0\\-5x_1-x_2+3x_3=0\end{cases} \Rightarrow \begin{cases}x_1=s\\x_2=-2s\\x_3=s\end{cases}$$

$$x = s\begin{bmatrix}1\\-2\\1\end{bmatrix} \quad W_{4}=(1,-2,1) \Rightarrow V_2=\left(\tfrac{1}{\sqrt6}, \tfrac{-2}{\sqrt6}, \tfrac{1}{\sqrt6}\right)$$

③ $\lambda=7$

$$\lambda I - A = \begin{bmatrix}6&-1&-5\\-1&2&-1\\-5&-1&6\end{bmatrix} = A'$$

$$A'x=0 \Rightarrow \begin{cases}6x_1-x_2-5x_3=0\\-x_1+2x_2-x_3=0\\-5x_1-x_2+6x_3=0\end{cases} \Rightarrow \begin{cases}x_1=s\\x_2=s\\x_3=s\end{cases}$$

$$x = s\begin{bmatrix}1\\1\\1\end{bmatrix} \quad W_{7}=(1,1,1) \Rightarrow V_3=\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$

$$Q = \begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&\frac{-2}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{-1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{bmatrix}$$

$$Q^TAQ = \begin{bmatrix}\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt6}&\frac{-2}{\sqrt6}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\end{bmatrix}\begin{bmatrix}1&1&5\\1&5&1\\5&1&1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&\frac{-2}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{-1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{-4}{\sqrt2}&0&\frac{4}{\sqrt2}\\\frac{4}{\sqrt6}&\frac{-8}{\sqrt6}&\frac{4}{\sqrt6}\\\frac{7}{\sqrt3}&\frac{7}{\sqrt3}&\frac{7}{\sqrt3}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&\frac{-2}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{-1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{bmatrix} = \begin{bmatrix}-4&0&0\\0&4&0\\0&0&7\end{bmatrix} \quad \#$$

2. $A = \begin{bmatrix}6&9\\4&5\end{bmatrix}$ Find LU

$$A = \begin{bmatrix}6&9\\4&5\end{bmatrix} \times\tfrac13 \quad E_1=\begin{bmatrix}\frac13&0\\0&1\end{bmatrix} \Rightarrow E_1'=\begin{bmatrix}3&0\\0&1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2&3\\4&5\end{bmatrix} \times(-2)+ \quad E_2=\begin{bmatrix}1&0\\-2&1\end{bmatrix} \Rightarrow E_2'=\begin{bmatrix}1&0\\2&1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2&3\\0&-1\end{bmatrix} = U$$

$$L = E_1^{-1}E_2^{-1} = \begin{bmatrix}3&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix} = \begin{bmatrix}3&2\\0&1\end{bmatrix}$$

3. $A = \begin{bmatrix}2&-1\\2&2\end{bmatrix}$

$$A^TA = \begin{bmatrix}2&2\\-1&2\end{bmatrix}\begin{bmatrix}2&-1\\2&2\end{bmatrix} = \begin{bmatrix}8&2\\2&5\end{bmatrix}$$

$$AA^T = \begin{bmatrix}2&-1\\2&2\end{bmatrix}\begin{bmatrix}2&2\\-1&2\end{bmatrix} = \begin{bmatrix}5&2\\2&8\end{bmatrix}$$

$$\det(\lambda I - A^TA) = (\lambda-8)(\lambda-5)-4 = \lambda^2-13\lambda+36 = (\lambda-9)(\lambda-4)$$

$\lambda = 4, 9$

$V = 2, 3 \quad k=2 \Rightarrow \Sigma = \begin{bmatrix}2&0\\0&3\end{bmatrix}$

① $\lambda=4$

$$\lambda I - A^TA = \begin{bmatrix}-4&-2\\-2&-1\end{bmatrix} = A'$$

$$A'x=0 \Rightarrow \begin{cases}-4x_1-2x_2=0\\-2x_1-x_2=0\end{cases} \Rightarrow \begin{cases}x_1=t\\x_2=-2t\end{cases}$$

$$x = t\begin{bmatrix}1\\-2\end{bmatrix} \quad V_1=\begin{bmatrix}\frac{1}{\sqrt5}\\\frac{-2}{\sqrt5}\end{bmatrix}$$

② $\lambda=9$

$$\lambda I - A^TA = \begin{bmatrix}1&-2\\-2&4\end{bmatrix} = A'$$

$$A'x=0 \Rightarrow \begin{cases}x_1-2x_2\\-2x_1+4x_2\end{cases} \Rightarrow \begin{cases}x_1=2t\\x_2=t\end{cases}$$

$$x = t\begin{bmatrix}2\\1\end{bmatrix} \quad V_2=\begin{bmatrix}\frac{2}{\sqrt5}\\\frac{1}{\sqrt5}\end{bmatrix}$$

$$\det(\lambda I - AA^T) = (\lambda-5)(\lambda-8)-4 = \lambda^2-13\lambda+36 = (\lambda-4)(\lambda-9)$$

① $\lambda=4$

$$\lambda I - AA^T = \begin{bmatrix}-1&-2\\-2&-4\end{bmatrix} = A'$$

$$A'x=0 \Rightarrow \begin{cases}-x_1-2x_2\\-2x_1-4x_2\end{cases} \Rightarrow \begin{cases}x_1=-2t\\x_2=t\end{cases}$$

$$x = t\begin{bmatrix}-2\\1\end{bmatrix} \quad u_1=\begin{bmatrix}\frac{-2}{\sqrt5}\\\frac{1}{\sqrt5}\end{bmatrix}$$

② $\lambda=9$

$$\lambda I - AA^T = \begin{bmatrix}4&-2\\-2&1\end{bmatrix} = A'$$

$$A'x=0 \Rightarrow \begin{cases}4x_1-2x_2=0\\-2x_1+x_2=0\end{cases} \Rightarrow \begin{cases}x_1=t\\x_2=2t\end{cases}$$

$$x = t\begin{bmatrix}1\\2\end{bmatrix} \quad u_2=\begin{bmatrix}\frac{1}{\sqrt5}\\\frac{2}{\sqrt5}\end{bmatrix}$$

4. $2x_1^2 + 10x_1x_2 + 2x_2^2 = [x_1 \; x_2]\begin{bmatrix}2&5\\5&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$A = \begin{bmatrix}2&5\\5&2\end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix}\lambda-2&-5\\-5&\lambda-2\end{bmatrix}$

$\det(\lambda I - A) = (\lambda-2)(\lambda-2)-25 = \lambda^2-4\lambda-21$
$= (\lambda-7)(\lambda+3)$

① $\lambda = 7$

$\lambda I - A = \begin{bmatrix}5&-5\\-5&5\end{bmatrix} = A'$

$A'x = 0 \Rightarrow \begin{cases}5x_1-5x_2=0\\-5x_1+5x_2=0\end{cases} \Rightarrow \begin{cases}x_1=t\\x_2=t\end{cases}$

$x = t\begin{bmatrix}1\\1\end{bmatrix}$ $\quad u_1 = \begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}$

② $\lambda = -3$

$\lambda I - A = \begin{bmatrix}-5&-5\\-5&-5\end{bmatrix} = A'$

$A'x = 0 \Rightarrow \begin{cases}-5x_1-5x_2=0\\-5x_1-5x_2=0\end{cases} \Rightarrow \begin{cases}x_1=t\\x_2=-t\end{cases}$

$x = t\begin{bmatrix}1\\-1\end{bmatrix}$ $\quad u_2 = \begin{bmatrix}\frac{1}{\sqrt2}\\\frac{-1}{\sqrt2}\end{bmatrix}$

$Q = \begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{bmatrix}$ $\quad X = \begin{bmatrix}X\\Y\end{bmatrix}$ $\quad X = QY$ $\quad Y = \begin{bmatrix}X'\\Y'\end{bmatrix}$

$Q^TAQ = D = \begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{bmatrix}\begin{bmatrix}2&5\\5&2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{bmatrix}$

$= \begin{bmatrix}\frac{7}{\sqrt2}&\frac{7}{\sqrt2}\\\frac{-3}{\sqrt2}&\frac{3}{\sqrt2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{bmatrix} = \begin{bmatrix}7&0\\0&-3\end{bmatrix}$

$X^TAX = Y^TDY = [X' \; Y']\begin{bmatrix}7&0\\0&-3\end{bmatrix}\begin{bmatrix}X'\\Y'\end{bmatrix}$

$= [7(X') \;\; -3(Y')]\begin{bmatrix}X'\\Y'\end{bmatrix}$

$= 7(X')^2 - 3(Y')^2 = 1$

5. $A = \begin{bmatrix}5&4&3\\-1&0&-3\\1&-2&1\end{bmatrix}$ $\det(\lambda I - A) = \lambda^3 - 6\lambda^2 + 32 = P(\lambda)$

$P(A) = A^3 - 6A^2 + 32I$

$A^3 - 6A^2 = -32I$

$\begin{bmatrix}112&84&36\\-48&-20&-36\\48&12&28\end{bmatrix} - 6\begin{bmatrix}24&14&6\\-8&2&-6\\8&2&10\end{bmatrix} = \begin{bmatrix}112-144&84-84&36-36\\-48+48&-20-12&-36+36\\48-48&12-12&28-60\end{bmatrix}$

$= \begin{bmatrix}-32&0&0\\0&-32&0\\0&0&-32\end{bmatrix} = -32I$

6. $A = \begin{bmatrix}7&2\\2&4\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt5}&\frac{-1}{\sqrt5}\\\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\end{bmatrix}\begin{bmatrix}8&0\\0&3\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}&\frac{1}{\sqrt5}\\\frac{-1}{\sqrt5}&\frac{2}{\sqrt5}\end{bmatrix}$

$\det(\lambda I - A) = (\lambda-7)(\lambda-4) - 4$
$= \lambda^2 - 11\lambda + 24 = (\lambda-3)(\lambda-8)$

① $\lambda = 3$

$\lambda I - A = \begin{bmatrix}-4&-2\\-2&-1\end{bmatrix} = A'$

$A'x = 0 \Rightarrow \begin{cases}-4x_1-2x_2=0\\-2x_1-x_2=0\end{cases} \Rightarrow \begin{cases}x_1=t\\x_2=-2t\end{cases}$

$x = t\begin{bmatrix}1\\-2\end{bmatrix}$ $\quad u_1 = \begin{bmatrix}\frac{1}{\sqrt5}\\\frac{-2}{\sqrt5}\end{bmatrix}$

② $\lambda = 8$

$\lambda I - A = \begin{bmatrix}1&-2\\-2&4\end{bmatrix} = A'$

$A'x = 0 \Rightarrow \begin{cases}x_1-2x_2=0\\-2x_1+4x_2=0\end{cases} \Rightarrow \begin{cases}x_1=2t\\x_2=t\end{cases}$

$x = t\begin{bmatrix}2\\1\end{bmatrix}$ $\quad u_2 = \begin{bmatrix}\frac{2}{\sqrt5}\\\frac{1}{\sqrt5}\end{bmatrix}$

$u_1u_1^T = \begin{bmatrix}\frac{1}{\sqrt5}\\\frac{-2}{\sqrt5}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt5}&\frac{-2}{\sqrt5}\end{bmatrix} = \begin{bmatrix}\frac15&\frac{-2}{5}\\\frac{-2}{5}&\frac45\end{bmatrix}$

$u_2u_2^T = \begin{bmatrix}\frac{2}{\sqrt5}\\\frac{1}{\sqrt5}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}&\frac{1}{\sqrt5}\end{bmatrix} = \begin{bmatrix}\frac45&\frac25\\\frac25&\frac15\end{bmatrix}$

$A = 3\begin{bmatrix}\frac15&\frac{-2}{5}\\\frac{-2}{5}&\frac45\end{bmatrix} + 8\begin{bmatrix}\frac45&\frac25\\\frac25&\frac15\end{bmatrix}$

7. $A = \begin{bmatrix}\frac{3}{\sqrt{10}}&\frac{1}{\sqrt{10}}\\\frac{1}{\sqrt{10}}&\frac{-3}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}6\sqrt{10}&0&0\\0&3\sqrt{10}&0\end{bmatrix}\begin{bmatrix}\frac13&\frac23&\frac23\\\frac{-2}{3}&\frac{-1}{3}&\frac23\\\frac23&\frac{-2}{3}&\frac13\end{bmatrix}$

$= \begin{bmatrix}18&3&0\\6&-9&0\end{bmatrix}\begin{bmatrix}\frac13&\frac23&\frac23\\\frac{-2}{3}&\frac{-1}{3}&\frac23\\\frac23&\frac{-2}{3}&\frac13\end{bmatrix}$

$= \begin{bmatrix}2&10&14\\8&7&2\end{bmatrix}$

$A^+ = \begin{bmatrix}\frac13&\frac{-2}{3}&\frac23\\\frac23&\frac{-1}{3}&\frac{-2}{3}\\\frac23&\frac{-2}{3}&\frac13\end{bmatrix}\begin{bmatrix}\frac{1}{6\sqrt{10}}&0\\0&\frac{1}{3\sqrt{10}}\\0&0\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{10}}&\frac{1}{\sqrt{10}}\\\frac{1}{\sqrt{10}}&\frac{-3}{\sqrt{10}}\end{bmatrix}$

$= \begin{bmatrix}\frac{1}{18\sqrt{10}}&\frac{2}{9\sqrt{10}}\\\frac{1}{9\sqrt{10}}&\frac{1}{9\sqrt{10}}\\\frac{1}{9\sqrt{10}}&\frac{-2}{9\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{10}}&\frac{1}{\sqrt{10}}\\\frac{1}{\sqrt{10}}&\frac{-3}{\sqrt{10}}\end{bmatrix} = \begin{bmatrix}\frac{7}{180}&\frac{11}{180}\\\frac{2}{45}&\frac19\\\frac{1}{90}&\frac{2}{90}\end{bmatrix}$ #

8. $f(x) = \begin{cases} -4, & -\pi \le x \le 0 \\ 4, & 0 < x \le \pi \end{cases}$

odd function ⇒ 沒有 cos 項

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0}(-4)\,dx$$
$$= \frac{1}{2\pi}(-4x)\Big|_{-\pi}^{0}$$
$$= \frac{-4\pi}{2\pi} = -2$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-4)\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{\pi} 4\cdot\sin\left(\frac{n\pi x}{2}\right)dx\right]$$
$$= \frac{1}{\pi}\left\{\left[(-4x)\cdot\cos\left(\frac{n\pi x}{2}\right)\left(\frac{2}{n\pi}\right)\right]\Big|_{-\pi}^{0} + \left[4x\cos\left(\frac{n\pi x}{2}\right)\left(\frac{2}{n\pi}\right)\Big|_{0}^{\pi}\right]\right\}$$
$$= \frac{1}{\pi}\cdot\frac{8}{n\pi}\left\{\left[(-x)\cdot\cos\left(\frac{n\pi x}{2}\right)\Big|_{-\pi}^{0}\right] + \left[x\cos\left(\frac{n\pi x}{2}\right)\Big|_{0}^{\pi}\right]\right\}$$
$$= \frac{8}{n\pi^2}\left[\left(1-\cos\left(\frac{n\pi^2}{2}\right)\right) + \left(1-\cos\left(\frac{n\pi^2}{2}\right)\right)\right]$$
$$= \frac{8}{n\pi^2}\left[2 - 2\cos\left(\frac{n\pi^2}{2}\right)\right] = \frac{16-16\cos\left(\frac{n\pi^2}{2}\right)}{n\pi^2}$$

The Fourier Series is $-2 + \dfrac{16-16\cos\left(\frac{n\pi^2}{2}\right)}{n\pi^2}$ #

9. $f(x) = \frac{1}{2}x^2 \quad (-\pi \le x \le \pi)$

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

(a) $\sum_{n=1}^{\infty}\frac{1}{n^2}$

At $x=\pi$ the Fourier Series can converge to $\frac{1}{2}[f(-\pi^+) + f(\pi^-)] = \frac{1}{2}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \frac{\pi^2}{2}$

ie. $\lim_{x\to\pi} f(x) = \frac{\pi^2}{2} \Rightarrow \lim_{x\to\pi} \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi) = \frac{\pi^2}{2}$

$$\Rightarrow \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi) = \frac{\pi^2}{2}$$
$$\Rightarrow \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{2}$$
$$\Rightarrow 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2} - \frac{\pi^2}{6} = \frac{\pi^2}{3}$$
$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$ #

(b) At $x=0$ the Fourier Series can converge to $\frac{1}{2}[f(0^+) + f(0^-)] = 0$

ie. $\lim_{x\to 0} f(x) = 0 \Rightarrow \lim_{x\to 0}\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi) = 0$

$$\Rightarrow \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(0) = 0$$
$$\Rightarrow \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 0$$
$$\Rightarrow 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{6}$$
$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^2}{n^2} = \frac{-\pi^2}{12}$$ #

10. $\because f(x) = |x|$ is continuous

$f'(x)$ is $\begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$ is piecewise continuous

$\left(f'(x) = \frac{|x|}{x}\right)$

$f(-1) = 1 = f(1)$

$$\therefore g(x) = f'(x) = \left\{\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos[(2n-1)\pi x]\right\}'$$
$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin[(2n-1)\pi x]$$ #

11. $f(x) = \frac{|x|}{x} = g'(x)$

$$\Rightarrow g(x) = \int f(x)\,dx$$
$$= \int \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$
$$= \frac{\pi}{4}\left[-\cos x + \frac{1}{3^2}(-\cos 3x) + \frac{1}{5^2}(-\cos 5x) + \cdots\right] + C$$
$$= \frac{-\pi}{4}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) + C$$

$$g(0) = \frac{-\pi}{4}\left(\underbrace{1 + \frac{1}{9} + \frac{1}{25} + \cdots}_{\to\ \frac{\pi^2}{8}}\right) + C$$

$$\therefore C = -\left(\frac{-4}{\pi}\times\frac{\pi^2}{8}\right) = \frac{\pi}{2}$$