1. Let

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

If we know the eigenvalues of A are -4,4,7. Please find an orthogonal matrix Q and diagonal matrix D such that $Q^T A Q = D$. (10%)

1. $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$, $\lambda = -4, 4, 7$

$\lambda = -4$.

$\lambda I - A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & -5 \\ -1 & -9 & -1 \\ -5 & -1 & -5 \end{bmatrix}$

$\begin{cases} -5x_1 - x_2 - 5x_3 = 0. \\ -x_1 - 9x_2 - x_3 = 0. \end{cases} \Rightarrow -x_1 - x_3 = 9x_2 \Rightarrow -5x_1 - 5x_3 = 45x_2$

$\Rightarrow 44x_2 = 0 \Rightarrow x_2 = 0, \ x_1 = -x_3 \Rightarrow x_1 = t, \ x_3 = -t.$

$\begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = -t \end{cases}$, $x = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow W_{-4} = (1, 0, -1) \Rightarrow V_1 = (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}).$

$\lambda = 4$

$\lambda I - A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -5 \\ -1 & -1 & -1 \\ -5 & -1 & 3 \end{bmatrix} \rceil_{\times -1}$

$\Rightarrow 4x_1 + 0x_2 - 4x_3 = 0 \Rightarrow x_1 = x_3 = t., \ -t - x_2 - t = 0 \Rightarrow x_2 = -2t.$

$\begin{cases} x_1 = t \\ x_2 = -2t \\ x_3 = t \end{cases}$, $x = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow W_4 = (1, -2, 1) \Rightarrow V_2 = (\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}).$

$\lambda = 7$

$$\lambda I - A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 6 \end{bmatrix}$$

$\Rightarrow x_2 = \dfrac{x_1 + x_3}{2}$, $11x_1 + 0x_2 - 11x_3 = 0 \Rightarrow x_1 = x_3 = t$, $x_2 = t$.

$$\begin{cases} x_1 = t \\ x_2 = t \\ x_3 = t \end{cases}, \quad x = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow W_7 = (1,1,1) \Rightarrow v_3 = \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right).$$

$$Q = [\, v_1 \mid v_2 \mid v_3 \,] = \boxed{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}}$$

$$D = Q^T A Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{-4}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{6}} & \frac{-8}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\ \frac{7}{\sqrt{3}} & \frac{7}{\sqrt{3}} & \frac{7}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \boxed{\begin{bmatrix} -4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}}.$$

2. Let $A = \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}$

Find the LU decomposition of A.

2.

$$A = \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix} \times \tfrac{1}{3} \qquad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \tfrac{1}{3} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}, \; E_1^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \downarrow \times -2 \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \downarrow \times -2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \; E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} = U \qquad L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\boxed{LU = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}}$$

3. Let

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Find the singular values of A (10%)

3. $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ $A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

$\det(\lambda I - A^TA) = \det\left(\begin{bmatrix} \lambda-8 & -2 \\ -2 & \lambda-5 \end{bmatrix}\right) = (\lambda-8)(\lambda-5)-4$

$= \lambda^2 - 13\lambda + 36 = (\lambda-4)(\lambda-9)$

$\lambda = 4, 9.$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{4} = \boxed{2.}$ $\sigma_2 = \sqrt{\lambda_2} = \sqrt{9} = \boxed{3}$

4. Given a quadratic form $2x_1^2 - 4x_1x_2 - x_2^2$, please make a change of variable $x = Py$ that transforms this quadratic form into one with no cross-product term. (10%)

4. $2x_1^2 - 4x_1x_2 - x_2^2 = [x_1 \ x_2]\underbrace{\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}}_{\to A.}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$

$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-2 & 2 \\ 2 & \lambda+1 \end{bmatrix}\right) = (\lambda-2)(\lambda+1)-4$

$= \lambda^2 - \lambda - 6 = \underline{(\lambda-3)(\lambda+2)=0}.$

$\lambda = 3, -2.$

$D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$ $x = \begin{bmatrix} x \\ y \end{bmatrix}$ $Y = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$Y^TDY = [x' \ y']\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = [3x' \ -2y']\begin{bmatrix} x' \\ y' \end{bmatrix}$

$= \boxed{3(x')^2 - 2(y')^2}$

$\lambda = 3$

$$\lambda I - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad x_1 = -2x_2.$$

$$\begin{cases} x_1 = -2t \\ x_2 = t \end{cases}, x = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}, W_3 = (-2, 1) \Rightarrow v_1 = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$\lambda = -2$

$$\lambda I - A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \quad x_2 = 2x_1$$

$$\begin{cases} x_1 = t \\ x_2 = 2t \end{cases}, x = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}, W_{-2} = (1, 2) \Rightarrow v_2 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

5. Let

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Use the Cayley-Hamilton theorem to compute $A^3 - 6A^2$.

5.

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-5 & -4 & -3 \\ 1 & \lambda & 3 \\ -1 & 2 & \lambda-1 \end{bmatrix}\right).$$

$$\lambda^3 - 6\lambda^2 + \cancel{5\lambda}$$

$$= (\lambda-5)\lambda(\lambda-1) + 12 + (-6) - \cancel{3\lambda} + \cancel{4\lambda} - 4 - \cancel{6\lambda} + 30$$

$$= \lambda^3 - 6\lambda^2 + 32 = p(\lambda)$$

$$p(A) = A^3 - 6A^2 + 32I = 0.$$

$$\Rightarrow A^3 - 6A^2 = -32I = \boxed{\begin{bmatrix} -32 & 0 & 0 \\ 0 & -32 & 0 \\ 0 & 0 & -32 \end{bmatrix}}.$$

6. Construct a spectral decomposition of the matrix A that has the orthogonal diagonalization:

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

(10%)

6.

$$\overset{A}{A} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \overset{U}{\left[\underset{u_1}{\begin{matrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{matrix}} \;\; \underset{u_2}{\begin{matrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{matrix}} \right]} \overset{\Sigma}{\begin{bmatrix} \overset{\lambda_1}{8} & 0 \\ 0 & \underset{\lambda_2}{3} \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}$$

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$

$$= 8 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} + 3 \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}.$$

$$\boxed{= 8 \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{4}{5} \end{bmatrix}}$$

7. If the singular value decomposition of matrix A is

$$A = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

Please compute the pseudoinverse of A. (10%)

7,

$$A = \overset{U}{\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}}$$

$$A^+ = V \Sigma^+ U^T$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{6\sqrt{10}} & 0 \\ 0 & \frac{1}{3\sqrt{10}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{18\sqrt{10}} & \frac{-2}{9\sqrt{10}} \\ \frac{2}{18\sqrt{10}} & \frac{-1}{9\sqrt{10}} \\ \frac{2}{18\sqrt{10}} & \frac{2}{9\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{3}{180} - \frac{4}{180} & \frac{1}{180} + \frac{12}{180} \\ \frac{6}{180} - \frac{2}{180} & \frac{2}{180} + \frac{6}{180} \\ \frac{6}{180} + \frac{4}{180} & \frac{2}{180} \frac{-12}{180} \end{bmatrix}$$

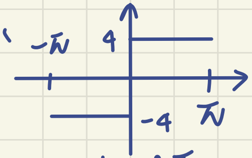
$$= \begin{bmatrix} -\frac{1}{180} & \frac{13}{180} \\ \frac{1}{45} & \frac{2}{45} \\ \frac{1}{18}. & \frac{-1}{18} \end{bmatrix}$$

8. Let

$$f(x) = \begin{cases} -4, & -\pi \le x \le 0 \\ 4, & 0 < x \le \pi \end{cases}$$

Find the Fourier series of the function f. (10%)

8.



$\Rightarrow$ odd function $\rightarrow a_n = 0$.

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$.

$= \frac{1}{2\pi}\left(\int_0^{\pi}(4)\,dx + \int_{-\pi}^{0}(-4)\,dx\right)$

$= \frac{1}{2\pi}\left(4x\Big|_0^{\pi} + (-4x)\Big|_{-\pi}^{0}\right)$

$= \frac{1}{2\pi}\left(4\pi - 0 + 0 - (-4)(-\pi)\right)$

$= \frac{1}{2\pi}(4\pi - 4\pi)$

$= 0$.

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx. \end{aligned}}$$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$.

$= \frac{1}{\pi}\left[\int_0^{\pi} 4\sin(nx)\,dx + \int_{-\pi}^{0} -4\sin(nx)\,dx\right]$.

$= \frac{1}{\pi}\left[4\int_0^{\pi}\sin(nx)\,dx + -4\int_{-\pi}^{0}\sin(nx)\,dx\right]$.

$u = nx,\ dx = \frac{1}{n}du$.

$= \frac{1}{\pi}\left[4\int_0^{n\pi}\frac{\sin(u)}{n}\,du + (-4)\int_{-n\pi}^{0}\frac{\sin(u)}{n}\,du\right]$.

$= \frac{1}{\pi}\left[\frac{4}{n}\int_0^{n\pi}\sin(u)\,du + \frac{(-4)}{n}\int_{-n\pi}^{0}\sin(u)\,du\right]$

?

$$\boxed{\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & n = 4k \\ -1 & n = 4k+2 \\ 0 & n: \text{odd}. \end{cases}}$$

$= \frac{1}{\pi}\left[\frac{4}{n}\left([-\cos(u)]\Big|_0^{n\pi}\right) + \frac{(-4)}{n}\left([-\cos(u)]\Big|_{-n\pi}^{0}\right)\right]$

$= \frac{1}{\pi}\left[\frac{4}{n}\left(-\cos(n\pi) + \cos(0)\right) + \frac{(-4)}{n}\left(-\cos(0) + \cos(-n\pi)\right)\right]$

$\cos(n\pi) = (-1)^n \quad \cos'\pi = \cos 180° = -1. \quad \cos 2\pi = \cos 360° = 1.$

$= \frac{1}{\pi}\left[\frac{4}{n}\left(-(-1)^n + 1\right) + \left(\frac{-4}{n}\right)\left(-1 + (-1)^n\right)\right]$

$= \frac{1}{\pi}\cdot\frac{4}{n}\left[(-(-1)^n + 1 - (-1 + (-1)^n\right]$

$= \frac{4}{n\pi}\left[-(-1)^n + 1 + 1 - (-1)^n\right]$

$= \frac{8}{n\pi}\left(-(-1)^n + 1\right)$.

$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$.

$f(x) = 0 + \sum_{n=1}^{\infty}\frac{8}{n\pi}\left(-(-1)^n + 1\right)\sin(nx)$. ~~#~~.

9. Let $f(x) = \frac{1}{2}x^2$ for $-\pi \le x \le \pi$. Use the fact that the Fourier series of f is

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

to compute

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (5%)

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ (5%)

9.

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \boxed{\cos(nx)}.$$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \boxed{(-1)^n}.$ $\cos(\pi) = -1.$

series converge to $\frac{1}{2}[f(-\pi+) + f(\pi-)].$

$f(x) = \frac{1}{2}x^2$ $\quad = \frac{1}{2}\left(\frac{\pi^2}{2} + \frac{\pi^2}{2}\right) = \boxed{\frac{\pi^2}{2}}$

$$\lim_{x\to\pi}\left[\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)\right].$$

$$= \frac{\pi^2}{6} + 2\lim_{x\to\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

$\cos(n\pi) = (-1)^n$

$$= \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi)$$

$$= \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n.$$

$$= \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \boxed{\frac{\pi^2}{2}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} = \left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) \times \frac{1}{2} = \frac{\pi^2}{6} \; \#.$$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \times 1$, $\cos(n\pi) = 1$ $\cos(0) = 1$

series converge to $\frac{1}{2}[f(-0+) + f(0-)]$.

$f(x) = \frac{1}{2}x^2$ $= \frac{1}{2}\left(\frac{0}{2} + \frac{0}{2}\right) = \boxed{0.}$

$$\lim_{x \to 0}\left[\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)\right].$$

$$= \frac{\pi^2}{6} + 2\lim_{x \to 0}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

$$= \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \boxed{0.}$$

$$\Rightarrow \sum_{n=1}^{\infty} = \frac{-\pi^2}{12}.$$

10. Let $f(x) = |x|, -1 \le x \le 1$. Use the fact that the Fourier series of f is

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos[(2n-1)\pi x]$$

to find the Fourier series of another function

$$g(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

10.

$f(x) = |x|.$

$f'(x) = \frac{x}{|x|} = g(x).$

$$\left(\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos[(2n-1)\pi x]\right)'$$

$$= 0 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{\cancel{2}}}\left(+\sin((2n-1)\pi x)\cancel{\pi}\cancel{(2n-1)}\right)$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1).}$$ #.

11. Let

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

use the following two facts to obtain the Fourier series of $g(x) = |x|$ for any x in $[-\pi, \pi]$

(i)The Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{4}{\pi}[\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots]$$

(ii)

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

$$f(x) = g'(x)$$

$$\Rightarrow g(x) = \int f(x)dx = \int [\frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x \cdots)]dx.$$

$$= \frac{4}{\pi}\left(\frac{-1}{1^2}\cos x - \frac{1}{3^2}\cos 3x - \frac{1}{5^2}\cos 5x + \cdots\right) + C$$

$$= \frac{-4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right) + C$$

$$= \frac{-4}{\pi} \times \frac{\pi^2}{8} + C.$$

$$g(0) = 0 = \frac{-4}{\pi} \times \frac{\pi^2}{8} + C$$

$$= \frac{-\pi}{2} + C$$

$$\Rightarrow C = \frac{\pi}{2}$$

$$g(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos[(2n-1)x] \quad \#$$

1. Let

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

If we know the eigenvalues of A are -4,4,7. Please find an orthogonal matrix Q and diagonal matrix D such that $Q^T AQ = D$. (10%)

2. Let

$$A = \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}$$

Find the LU decomposition of A. (10%)

3. Let

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Find the singular values of A (10%)

4. Given a quadratic form $2x_1^2 - 4x_1x_2 - x_2^2$, please make a change of variable $x = Py$ that transforms this quadratic form into one with no cross-product term. (10%)

5. Let

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Use the Cayley-Hamilton theorem to compute $A^3 - 6A^2$.

6. Construct a spectral decomposition of the matrix A that has the orthogonal diagonalization:

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

(10%)

7. If the singular value decomposition of matrix A is

$$A = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

Please compute the pseudoinverse of A. (10%)

8. Let

$$f(x) = \begin{cases} -4, & -\pi \le x \le 0 \\ 4, & 0 < x \le \pi \end{cases}$$

Find the Fourier series of the function f. (10%)

9. Let $f(x) = \frac{1}{2}x^2$ for $-\pi \le x \le \pi$. Use the fact that the Fourier series of f is

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

to compute

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (5%)

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ (5%)

10. Let $f(x) = |x|, -1 \le x \le 1$. Use the fact that the Fourier series of f is

$$\frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos[(2n-1)\pi x]$$

to find the Fourier series of another function

$$g(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

11. Let

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

use the following two facts to obtain the Fourier series of $g(x) = |x|$ for any x in $[-\pi, \pi]$

(i)The Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{4}{\pi}[\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots]$$

(ii)

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$